

Discrete Mathematics

# Propositional Logic

Shin Hong

# Textbook coverage

- Sec. 1.1     Propositional logic
- Sec. 1.3.   Propositional equivalence
- Sec. 1.2     Applications of propositional logic

# Logic

3

- Logic, or a logic system, is a set of rules to specify and derive a certain kind of statements
  - to achieve clarity and correctness in an argument
- A logic system has the syntactic and the semantic aspects
  - syntax: symbolic structure of the statements
  - semantics: a relation between symbolic structures and meaning

# Propositional Logic

4

- A statement in the propositional logic consists one or multiple propositions connected with logical operators
- A proposition is a declarative sentence that is either true or false
  - $1 + 1 = 2$
  - *Vancouver is the capital of Canada*
  - ~~$1 + 2 + 3$~~
  - ~~$x + 1 = 2$~~
- A propositional variable is a symbol that represents a propositional statement
  - the value of a propositional variable is either true or false
  - the value is definitive within a statement

# Propositional Logic

5

- An atomic proposition is a proposition that cannot be expressed in term of simpler terms
- A compound proposition is formed with other propositions and logical operators
  - logical operators (connectives): negation, disjunction, conjunction, XOR, implication, etc.
  - E.g., The negation of  $p$  for a proposition  $p$ , denoted as  $\neg p$ , is the proposition that is true only when  $p$  is false.

- Formal grammar

$$P := A \mid C$$
$$A := p \mid q \mid r \mid \dots$$
$$C := \neg P \mid (P) \mid P \vee P \mid P \wedge P \mid \dots$$

# Evaluation

6

- A propositional statement with propositional variables may have different evaluations (truth values) depending on the values of each propositional variable

- ex.  $p \vee (q \wedge r)$

- An assignment (model or valuation) of a propositional statement is a combination of truth values of the propositional variables

- e.g.,  $\phi_1 = (p: T, q: T, r: T)$  or  $\llbracket p \rrbracket_{\phi_1} = T, \llbracket q \rrbracket_{\phi_1} = T, \llbracket r \rrbracket_{\phi_1} = T$

$\phi_2 = (p: F, q: T, r: F)$  or  $\llbracket p \rrbracket_{\phi_2} = F, \llbracket q \rrbracket_{\phi_2} = T, \llbracket r \rrbracket_{\phi_2} = F$

$$\phi_1 \models p \vee (q \wedge r)$$

$$\phi_2 \not\models p \vee (q \wedge r)$$

# Implication (Conditional Statement)

7

- An implication is a logical connective such that  $p \rightarrow q$  evaluates to True when  $q$  is true if  $p$  is true
  - $p \rightarrow q$  is equivalent with  $\neg p \vee q$
  - used to state a condition
    - examples
      - if you do not take midterm, you get F
      - if you are in the Handong campus, you are in Pohang
      - $x < y \rightarrow x < y + 1$
      - $(2 + 3 = 4) \rightarrow (1 + 2 = 4)$
- The converse of  $p \rightarrow q$  is  $q \rightarrow p$ .
- The inverse of  $p \rightarrow q$  is  $\neg p \rightarrow \neg q$ .
- The contrapositive of  $p \rightarrow q$  is  $\neg q \rightarrow \neg p$ .

# Equivalence

8

- The condition that two propositions  $p$  and  $q$  evaluate to the same can be expressed as  $(p \rightarrow q) \wedge (q \rightarrow p)$ , or simply  $p \leftrightarrow q$ 
  - have the same truth value for every assignment
  - a statement  $p \leftrightarrow q$  refers as  $p$  if and only if  $q$  (or simply  $p$  iff  $q$ )



# Example

9

<i>Equivalence</i>	<i>Name</i>
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws

- De Morgan's law:

$$\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$$

$$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$$

$p$	$q$	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Propositional  
Logic

Discrete Math.

2020-09-04

# Propositional Satisfiability

10

- A proposition  $p$  is **satisfiable** if there exists an assignment that makes  $p$  true
- A proposition  $p$  is **unsatisfiable** if  $p$  is not satisfiable
  - A unsatisfiable proposition is called as *contradiction*
- A proposition  $p$  is **valid** if  $p$  is true for all assignments
  - A valid proposition is called as tautology
  - E.g., if  $x = y$ , then  $x = y$   
*I just want to live while I am alive - Bon Jovi*

# Logic Puzzle: Knight or Knaves

- An island has two kinds of inhabitants, *knight*s, who always tell the truth, and *knave*s, who always lie.
- You go to the island and meet A and B.
  - A says “B is a knight.”
  - B says “The two of us are of opposite types.”

What are the types of A and B?



Propositional  
Logic

Discrete Math.

2020-09-04

# Logic Puzzle: Treasure

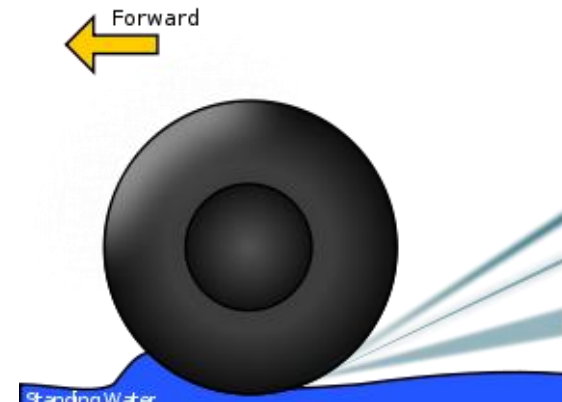


- There are 3 trunks only one of which contains a treasure.
- Trunk 1 and Trunk 2 are inscribed with “This trunk is empty” and Trunk 3 is inscribed with “Treasure is in Trunk 2”.
- You know that only one of the three inscriptions is true.
- Where’s the treasure?

# System Requirement Analysis

13

- Logic-based languages (formal languages) are powerful tools for specifying and analyzing software requirements rigorously
- E.g., Lufthansa A320 Airbus accident at Warsaw in 1993
  - Requirement: Turn on reverse thrust when airplane is running on runway for landing
  - System design specification (adopted)
    - SET REVERSE\_THRUST AS ON IFF (MODE IS LANDING ) AND (ALTITUDE IS 0)
    - SET MODE AS LANDING IFF  
NOT(VELOCITY IS 0) AND NOT(LAND\_GEAR\_ANG IS 0)



Propositional  
Logic

Discrete Math.

2020-09-04