Discrete Mathematics

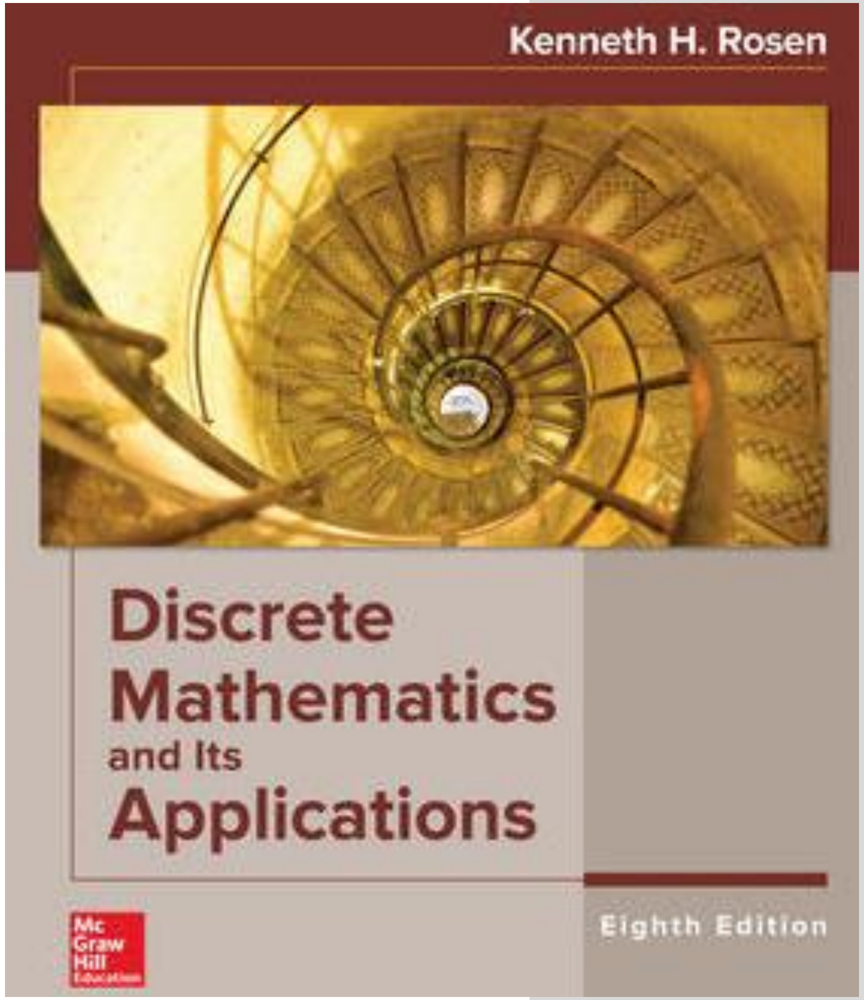
Induction

Shin Hong

Motivation: Climbing Infinite Stairs

- Suppose that a building has an infinite number of floors each of which has upstairs to the next floor and downstairs to the lower floor.

- How to teach someone to reach an *n*-th floor?
  - Teach him/her to the first floor where the stair starts
  - Teach him/her how to climb stairs from one floor to the next floor



Principle of Mathematical Induction

To prove P, define P with subproblems such that $P = P(1) \land P(2) \land \cdots$

And then, to prove that $P(n)$ holds for all positive integers n, we complete these steps:

- *Basis Step*
  Prove that $P(1)$ is true.
- *Inductive Step*
  Prove that $P(k) \rightarrow P(k+1)$ is true for all positive integers k

To prove the inductive step, assuming the *inductive hypothesis* that $P(k)$ holds for an arbitrary integer k, show that $P(k+1)$ must be true.

Important Points About Using Mathematical Induction

- Mathematical induction can be expressed as the rule of inference where the domain is the set of positive integers

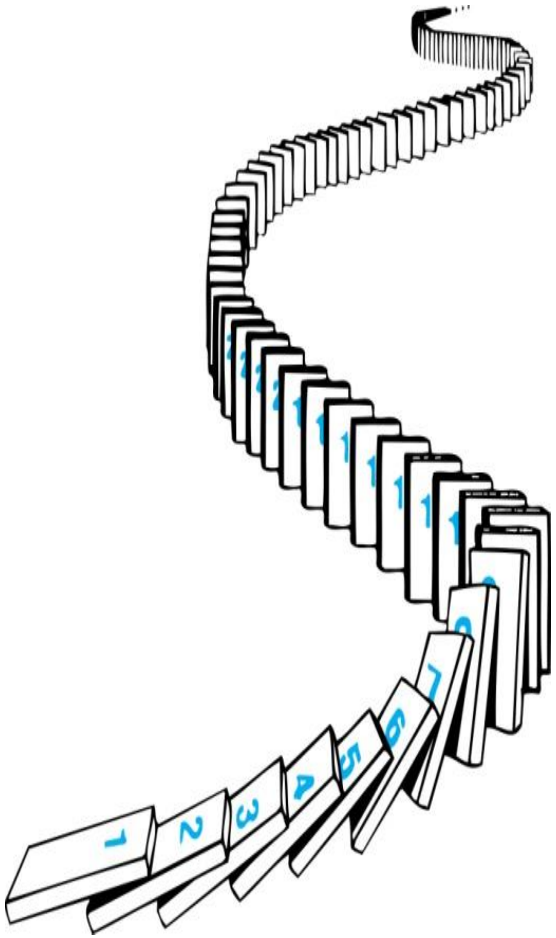
$$(P(1) \land \forall k\,(P(k) \rightarrow P(k+1))) \rightarrow \forall n\, P(n)$$

- Note that it does not assume that $\forall k(P(k))$ is true. Rather, what it asserts is that $P(k+1)$ must be true if $P(k)$ is true
- Note also that a mathematical induction does not necessary to start at integer 1

Remembering How Mathematical Induction Works

Consider an infinite sequence of dominoes, labeled 1,2,3, …, where each domino is standing.

Let $P(n)$ be the proposition that the nth domino is knocked over.



We know that the first domino is knocked down, i.e., $P(1)$ is true .

We also know that if whenever the k-th domino is knocked over, it knocks over the $(k + 1)$st domino, i.e, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.

Hence, all dominos are knocked over.

$P(n)$ is true for all positive integers n.

Validity of Mathematical Induction

- The well ordering property states that every non-empty set of positive integers has a least element

- Mathematical induction is valid because of the well ordering property of positive integers
  - Assume that there is a set S which is a set of positive integers *n* for which P(*n*) is false
  - By the well-ordering property, *S* has a least element, say *m*.
  - We know that *m* cannot be 1 since $P(1)$ holds.
  - Since *m* is positive and greater than 1, $m - 1$ must be a positive integer.
  - Since $m - 1 < m$, it is not in S, so $P(m - 1)$ must be true.
  - But then, since the conditional $P(k) \rightarrow P(k + 1)$ for every positive integer *k* holds, $P(m)$ must also be true. This contradicts $P(m)$ being false.
  - Hence, $P(n)$ must be true for every positive integer *n*.

Proving a Summation Formula by Mathematical Induction

Show that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Solution

- Basis step: *P*(1) is true since 1(1 + 1)/2 = 1.
- Inductive step: Assume that *P*(*k*) holds for 1 < *k*: $\displaystyle\sum_{i=1}^{k} = \frac{k(k+1)}{2}$

  Under this assumption,

$$
\begin{aligned}
1 + 2 + \ldots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

Conjecturing and Proving Correct a Summation Formula

Prove that the sum of the first n positive odd integers is n^2

- Basis step

  $P(1)$ is true since $1^2 = 1$
- Induction step

  $P(k) \to P(k + 1)$ for every positive integer k

  Inductive Hypothesis: $1 + 3 + 5 + \cdots + (2k - 1) = k^2$

$$
\begin{aligned}
1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= [1 + 3 + 5 + \cdots + (2k - 1)] + (2k + 1) \\
&= k^2 + (2k + 1) \quad (\textit{by the inductive hypothesis}) \\
&= k^2 + 2k + 1 \\
&= (k + 1)^2
\end{aligned}
$$

- Hence, we have shown that $P(k + 1)$ follows from $P(k)$.
  Therefore the sum of the first n positive odd integers is n^2.

Proving Inequalities

Prove that $n < 2^n$ for all positive integers *n*.

Solution

Let $P(i)$ be a statement that $i < 2^i$ such that $P = P(1) \wedge P(2) \wedge \ldots$

- Basis step:
  $P(1)$ is true since $1 < 2^1 = 2$.
- Inductive step:
  Assume $P(k)$ holds, i.e., $k < 2^k$, for an arbitrary positive integer *k*.
  Then, $P(k + 1)$ which asserts that $k+1 < 2^{k+1}$ holds because as follows:

$$k + 1 < 2^k + 1 < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

Therefore $n < 2^n$ holds for all positive integers *n*.

Proving Inequalities

- Prove that $2^n < n!$ for every integer $n \geq 4$

- Solution: Let $P(n)$ be the proposition that $2^n < n!$
  - Basis step:
    $P(4)$ is true since $2^4 = 16 < 4! = 24$.

  - Inductive step:
    Assume that $P(k)$ holds, i.e., $2^k < k!$ for an arbitrary integer $k \geq 4$.

    $$2 \cdot 2^k < 2 \cdot k! < (k+1)k! = (k+1)!$$

  Therefore, $2^n < n!$ holds, for every integer $n \geq 4$.

  Note that here the basis step is $P(4)$, since $P(0)$, $P(1)$, $P(2)$, and $P(3)$ are all false.

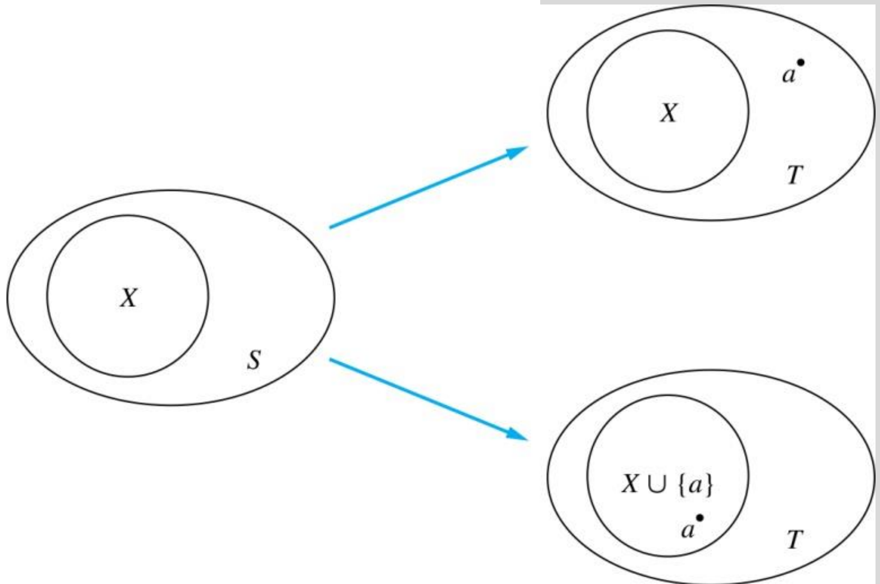
Number of Subsets of a Finite Set

Show that if S is a finite set with n elements, where n is a non-negative integer, S has 2^n subsets

Proof. Let $P(n)$ be the proposition that $|\mathcal{P}(S)|$ for a set S where $|S| = n$ is 2^n

- Basis step
  $P(0)$ is true, because {} has only itself as a subset and $2^0 = 1$.

- Inductive step: assume $P(k)$ is true for an arbitrary non-negative integer k
  - Let S' be a set with $k + 1$ elements. Then $S' = S \cup \{a\}$, where $a \in S'$ and $S = S' - \{a\}$. Hence $|S| = k$.
  - For each subset R of S, there are exactly two subsets of S', i.e., R and $R \cup \{a\}$
  - By the inductive hypothesis S has 2^k subsets. Since there are two subsets of T for each subset of S, the number of subsets of T is $2 \cdot 2^k = 2^{k+1}$

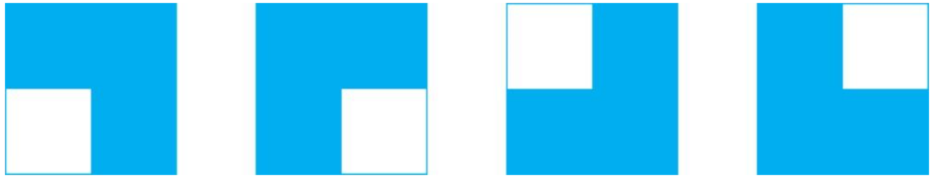


Tiling Checkerboards

- Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes.

  A right triomino is an L-shaped tile which covers three squares at a time.

- **Proof.** Let $P(n)$ be the proposition that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes
  - Basis step: P(1) is true, because each of the four 2×2 checkerboards with one square removed can be tiled using one right triomino.

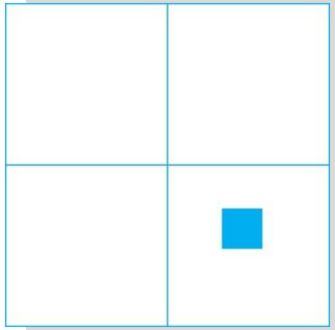


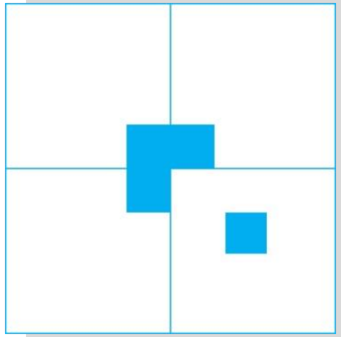
  - Inductive step: Assume that $P(k)$ is true for every $2^k \times 2^k$ checkerboard, for some positive integer k.

Tiling Checkerboards

Inductive Hypothesis: Every $2^k \times 2^k$ checkerboard, for some positive integer *k*, with one square removed can be tiled using right triominoes.

- Consider a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed. Split this checkerboard into four checkerboards of size $2^k \times 2^k$,by dividing it in half in both directions.
- Remove a square from one of the four $2^k \times 2^k$ checkerboards. By the inductive hypothesis, this board can be tiled. Also by the inductive hypothesis, the other three boards can be tiled with the square from the corner of the center of the original board removed. We can then cover the three adjacent squares with a triominoe.
- Hence, the entire $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed can be tiled using right triominoes. ◀





Strong Induction

- To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, complete two steps:
  - Basis step
    Verify that the proposition $P(1)$ is true.
  - Inductive step
    Show the $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k + 1)$ holds for all positive integers k

Strong induction is sometimes called the *second principle of mathematical induction* or *complete induction*.

Which Form of Induction Should Be Used?

- We can always use strong induction instead of mathematical induction. But there is no reason to use it if it is simpler to use mathematical induction
- In fact, the principles of mathematical induction, strong induction, and the well-ordering property are all equivalent
- Sometimes it is clear how to proceed using one of the three methods, but not the other two.

Proof using Strong Induction

- Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.
- Solution: Let $P(n)$ be the proposition that postage of n cents can be formed using 4-cent and 5-cent stamps.
  - Basis step: $P(12)$, $P(13)$, $P(14)$, and $P(15)$ hold.
    - $P(12)$ uses three 4-cent stamps.
    - $P(13)$ uses two 4-cent stamps and one 5-cent stamp.
    - $P(14)$ uses one 4-cent stamp and two 5-cent stamps.
    - $P(15)$ uses three 5-cent stamps.
  - Inductive step: The inductive hypothesis states that $P(j)$ holds for $12 \leq j \leq k$, where $15 \leq k$. Using the inductive hypothesis, $P(k-3)$ holds for $12 \leq k-3$. To form postage of $(k+1)$ cents, add a 4-cent stamp to the postage for $(k-3)$ cents.

  Hence, $P(n)$ holds for all $n \geq 12$.

Proof of Same Example using Mathematical Induction

- Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

- Proof: Let *P*(*n*) be the proposition that postage of *n* cents can be formed using 4-cent and 5-cent stamps
  - Basis step: Postage of 12 cents can be formed using three 4-cent stamps.
  - Inductive step: The inductive hypothesis *P*(*k*) for any positive integer *k* is that postage of *k* cents can be formed using 4-cent and 5-cent stamps. To show P(*k* + 1) where $k \geq 12$, we consider two cases:
    - If at least one 4-cent stamp has been used, then a 4-cent stamp can be replaced with a 5-cent stamp to yield a total of k + 1 cents.
    - Otherwise, no 4-cent stamp have been used and at least three 5-cent stamps were used. Three 5-cent stamps can be replaced by four 4-cent stamps to yield a total of k + 1 cents.

Ex. Triangulation of Polygons

- A polygon is a closed geometric figure consisting of a sequence of line segements s_1 to s_n called *sides*
  - An end point of a side is called a vertex

- A polygon is simple when no two non-consecutive sides intersect.
  - A simple polygon divides the plane into interior and exterior

- A polygon is called *convex* if every line connecting two points in the interior lies entirely in the interior.

- A diagonal is a line connecting two non-consecutive verticies
  - An interior diagonal lies in the interior entirely

Ex. Triangulation of Polygons

- Triangulation is a process to divide a polygon into triangles by adding nonintersecting diagonals
- Theorem. A simple polygon with n sides for $3 \leq n$ can be triangulated into n - 2 triangles
  - Lemma. A simple polygon with at least 4 sides has an interior diagonal
  - Proof. T(n) is a proposition that a simple polygon with n sides can be triangulated into n - 2 triangles
    - Basis: T(3) holds, obviously.
    - Induction step: for T(i+1)
      - By Induction hypotheses, T(j) holds for $3 \leq j \leq i$
      - By the lemma, a simple polygon with i+1 sides has an interior diagonal that divides the polygon into another two simple polygons Q and R.
      - Each of Q and R can be triangulated since the number of sides in Q or R is less than i+1.