

Discrete Mathematics

# Counting

Shin Hong

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# Basic Counting Principles: The Product Rule

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**The Product Rule:** A procedure can be broken down into a sequence of two tasks. There are  $n_1$  ways to do the first task and  $n_2$  ways to do the second task. Then there are  $n_1 \cdot n_2$  ways to do the procedure.

**E.g.** How many binary strings of length seven are there?

$2^7 = 128$  because each of the seven bits is either a 0 or a 1

# Counting Functions

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- How many functions are there from a set with  $m$  elements to a set with  $n$  elements?
  - Answer:  $n^m$  such functions
- How many one-to-one functions are there from a set with  $m$  elements to one with  $n$  elements?
  - Answer:  $n(n-1)(n-2)\cdots(n-m+1)$  such functions.

# Basic Counting Principles: The Sum Rule

- If a task can be done either in one of  $n_1$  ways or in one of  $n_2$  ways to do the second task, where none of the set of  $n_1$  ways is the same as any of the  $n_2$  ways, then there are  $n_1 + n_2$  ways to do the task.
- **E.g.:** The CSEE department must choose either a CS faculty or EE faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 CS faculties and 83 EE faculties, and no one is in CS and EE at the same time.
  - Answer:  $37 + 83 = 120$  possible ways to pick a representative.

# The Sum Rule in terms of sets.

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- The sum rule can be phrased in terms of sets.

$|A \cup B| = |A| + |B|$  as long as  $A$  and  $B$  are disjoint sets.

- Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

when  $A_i \cap A_j = \emptyset$  for all  $i, j$ .

# Combining the Sum and Product Rule

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- Question

Suppose statement labels in a programming language can be either a single lowercase alphabet letter or such a letter followed by a digit. Find the number of possible labels.

- Answer

Use the product rule.  $26 + 26 \cdot 10 = 286$

```
01 k = (j + 13) / 27
02 loop:
03     if k > 10 then goto out
04     k = k + 1
05     i = 3 * k - 1
06     goto loop
07 out:
08 if ((k==1) || (k==2)) j = 2 * k -
09 if ((k==3) || (k==5)) j = 3 * k +
10 if (k==4) i = 4 * k - 1
```

# Subtraction Rule

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- If a task can be done either in one of  $n_1$  ways or in one of  $n_2$  ways, then the total number of ways to do the task is  $n_1 + n_2$  minus the number of ways to do the task that are common to the two different ways.
  - Also known as, the *principle of inclusion-exclusion*

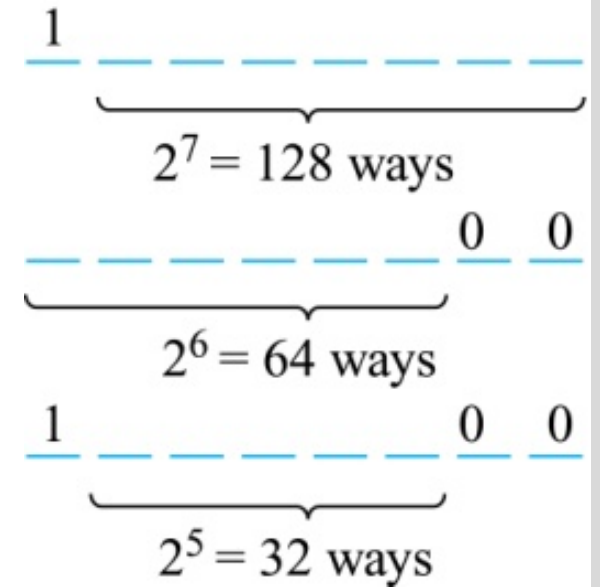
$$|A \cup B| = |A| + |B| - |A \cap B|$$

# Counting Bit Strings

- **Example:** How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

- **Solution:** Use the subtraction rule.

- Number of bit strings of length eight that start with a 1 bit:  $2^7 = 128$
- Number of bit strings of length eight that start with bits 00:  $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 :  $2^5 = 32$
- Hence, the number is  $128 + 64 - 32 = 160$ .





# Counting Passwords

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- **Question.** A password must be six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?
- **Answer.** Let  $P$  be the total number of passwords, and let  $P_6$ ,  $P_7$ , and  $P_8$  be the passwords of length 6, 7, and 8.
  - $P = P_6 + P_7 + P_8$
  - $P_6 = 36^6 - 26^6 = 2,176,782,336 - 308,915,776 = 1,867,866,560.$
  - $P_7 = 36^7 - 26^7 = 78,364,164,096 - 8,031,810,176 = 70,332,353,920.$
  - $P_8 = 36^8 - 26^8 = 2,821,109,907,456 - 208,827,064,576 = 2,612,282,842,880.$
  - Consequently,  $P = P_6 + P_7 + P_8 = 2,684,483,063,360.$

# Division Rule

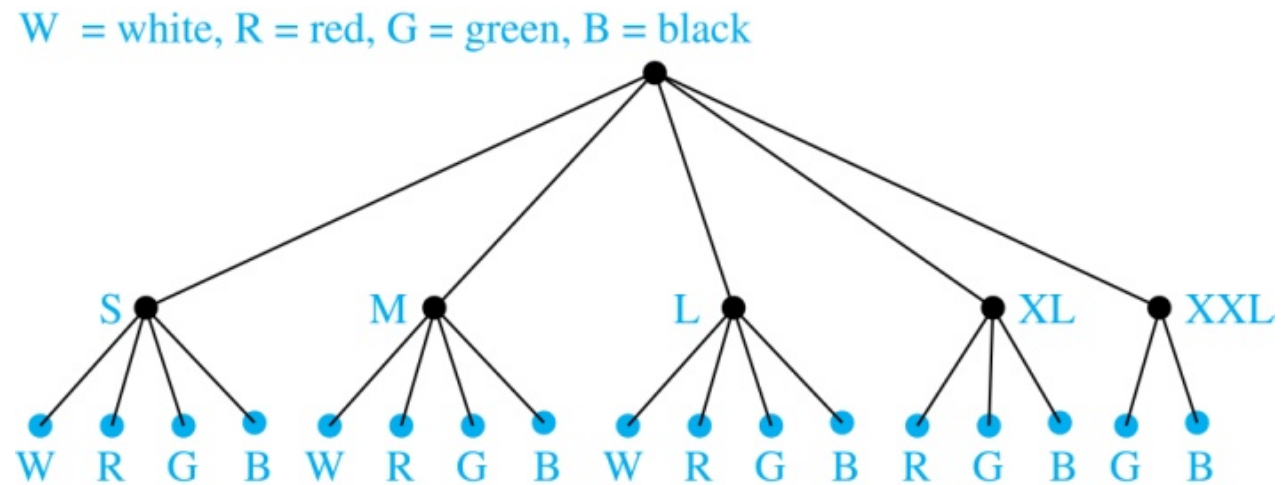
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- If a task can be done by a procedure that generates  $n$  ways, and there are  $d$  ways of the procedure correspond to the same result, then there are  $n/d$  results of the task.
- **Question:** How many cases are there to seat 4 people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?
- **Answer:** The number the seats around the table from 1 to 4 proceeding clockwise is  $4! = 24$ . But since two seatings are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating. Therefore, by the division rule, there are  $24/4 = 6$  different seating arrangements.

# Tree Diagrams

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- **Tree Diagrams:** We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.
- **Example:** Suppose that “I Love Discrete Math” T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of stores that the campus book store needs to stock to have one of each size and color available?
- **Solution:** Draw the tree diagram.

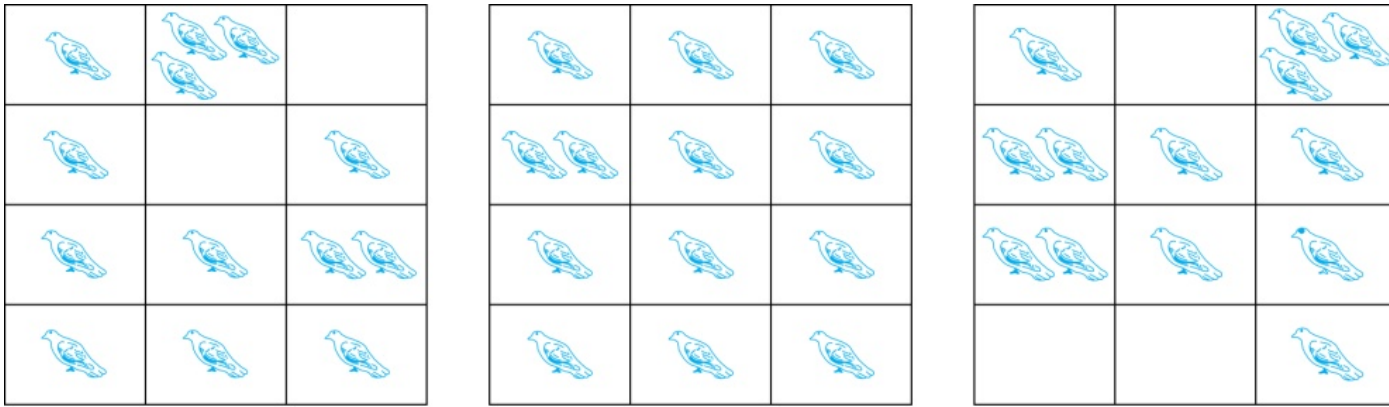


- The store must stock 17 T-shirts.

# The Pigeonhole Principle

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- If a flock of 13 pigeons roosts in a set of 12 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



- **Pigeonhole Principle:** If  $k$  is a positive integer and  $k + 1$  objects are placed into  $k$  boxes, then at least one box contains two or more objects.
- **Proof:** We use a proof by contraposition. Suppose none of the  $k$  boxes has more than one object. Then the total number of objects would be at most  $k$ . This contradicts the statement that we have  $k + 1$  objects.

# The Pigeonhole Principle

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- **Corollary 1:** A function  $f$  from a set with  $k + 1$  elements to a set with  $k$  elements is not one-to-one.
- **Proof:** Use the pigeonhole principle.  
Create a box for each element  $y$  in the co-domain of  $f$ .  
Put in the box for  $y$  all of the elements  $x$  from the domain such that  $f(x) = y$ .  
Because there are  $k + 1$  elements and only  $k$  boxes, at least one box has two or more elements.  
Hence,  $f$  can't be one-to-one.

# Pigeonhole Principle

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- **Question:** Show that for every integer  $n$  there is a multiple of  $n$  that has only 0s and 1s in its decimal expansion.
- **Solution:** Let  $n$  be a positive integer. Consider the  $n + 1$  integers  $1, 11, 111, \dots, 11\dots1$  (i.e.,  $1^{n+1}$ ). There are  $n$  possible remainders when an integer is divided by  $n$ . By the pigeonhole principle, when each of the  $n + 1$  integers is divided by  $n$ , at least two must have the same remainder. Subtract the smaller from the larger and the result is a multiple of  $n$  that has only 0s and 1s in its decimal expansion.

# The Generalized Pigeonhole Principle

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- **The Generalized Pigeonhole Principle:** If  $N$  objects are placed into  $k$  boxes, then there is at least one box containing at least  $\lceil N/k \rceil$  objects.
- **Proof:** We use a proof by contraposition. Suppose that none of the boxes contains more than  $\lceil N/k \rceil - 1$  objects. Then the total number of objects is at most

$$k \left( \left\lceil \frac{N}{k} \right\rceil - 1 \right) < k \left( \left( \frac{N}{k} + 1 \right) - 1 \right) = N,$$

where the inequality  $\lceil N/k \rceil < \lceil N/k \rceil + 1$  has been used.

This is a contradiction because there are a total of  $n$  objects.

- **Example:** Among 100 people there are at least  $\lceil 100/12 \rceil = 9$  who were born in the same month.

# The Generalized Pigeonhole Principle

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- **Question.** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?
- **Answer.** We assume four boxes; one for each suit. Using the generalized pigeonhole principle, at least one box contains at least  $\lceil N/4 \rceil$  cards. At least three cards of one suit are selected if  $\lceil N/4 \rceil \geq 3$ . The smallest integer  $N$  such that  $\lceil N/4 \rceil \geq 3$  is  $N = 2 \cdot 4 + 1 = 9$ .



# Permutation

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- A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of  $r$  elements of a set is called an  *$r$ -permutation*.
  - e.g. Let  $S = \{1,2,3\}$ .
    - The ordered arrangement 3,1,2 is a permutation of  $S$ .
    - The ordered arrangement 3,2 is a 2-permutation of  $S$ .
- The number of  $r$ -permutations of a set with  $n$  elements is denoted by  $P(n,r)$ .
  - The 2-permutations of  $S = \{1,2,3\}$  are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2.  
Hence,  $P(3,2) = 6$ .

# A Formula for the Number of Permutations

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- **Theorem 1:** If  $n$  is a positive integer and  $r$  is an integer with  $1 \leq r \leq n$ , then there are

$$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1)$$

$r$ -permutations of a set with  $n$  distinct elements.

- **Proof:** Use the product rule. The first element can be chosen in  $n$  ways. The second in  $n - 1$  ways, and so on until there are  $(n - (r - 1))$  ways to choose the last element. Note that  $P(n, 0) = 1$ , since there is only one way to order zero elements.
- **Corollary 1:** If  $n$  and  $r$  are integers with  $1 \leq r \leq n$ , then

$$P(n, r) = \frac{n!}{(n-r)!}$$

# Solving Counting Problems by Counting Permutations

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- **Example.** Suppose that a saleswoman has to visit 8 different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?
- **Solution:** The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

# Solving Counting Problems by Counting Permutations

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- **Example.** How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?
- **Solution.** We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

# Combinations

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- An *r-combination* of elements of a set is an unordered selection of  $r$  elements from the set. Thus, an  $r$ -combination is simply a subset of the set with  $r$  elements.
  - The number of  $r$ -combinations of a set with  $n$  distinct elements is denoted by  $C(n, r)$ .
  - The notation  $\binom{n}{r}$  is also used and is called a *binomial coefficient*.
- Example. Let  $S$  be the set  $\{a, b, c, d\}$ . How many 2-combinations from  $S$  exist?
  - ~~Answer:  $C(4, 2) = 6$  because the 2-combinations of  $\{a, b, c, d\}$  are the six subsets  $\{a, b\}$ ,  $\{a, c\}$ ,  $\{a, d\}$ ,  $\{b, c\}$ ,  $\{b, d\}$ , and  $\{c, d\}$ .~~

# Combinations

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**Theorem 2:** The number of  $r$ -combinations of a set with  $n$  elements, where  $n \geq r \geq 0$ , equals

$$C(n, r) = \frac{n!}{(n-r)!r!}.$$

**Proof:** By the product rule  $P(n, r) = C(n, r) \cdot P(r, r)$ . Therefore,

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}.$$

# Combinations

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- **Corollary 2:** Let  $n$  and  $r$  be nonnegative integers with  $r \leq n$ . Then  $C(n, r) = C(n, n - r)$ .

- **Proof:** From Theorem 2, it follows that

and 
$$C(n, r) = \frac{n!}{(n-r)!r!}$$

$$C(n, n - r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} .$$

# Combinations

- How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?
- **Solution:** Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$\begin{aligned} C(52, 5) &= \frac{52!}{5!47!} \\ &= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960 \end{aligned}$$

- The different ways to select 47 cards from 52 is

$$C(52, 47) = \frac{52!}{47!5!} = C(52, 5) = 2,598,960.$$



# Combinations

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- **Example:** How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

- **Solution:** By Theorem 2, the number of combinations is

$$C(10, 5) = \frac{10!}{5!5!} = 252.$$

- **Example:** A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

- **Solution:** By Theorem 2, the number of possible crews is

$$C(30, 6) = \frac{30!}{6!24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775 .$$

# Powers of Binomial Expressions

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- A *binomial* expression is the sum of two terms, such as  $x + y$ 
  - a term can be a product of a constant and a variable
- Find the coefficients in the expansion of  $(x + y)^n$  where  $n$  is a positive integer
  - e.g.  $(x + y)^3 = (x + y)(x + y)(x + y)$ . What are the coefficients of  $x^3$ ,  $x^2y$ ,  $xy^2$  and  $y^3$  ?
    - To obtain  $x^3$ , an  $x$  must be chosen from each of the sums.  
There is only one way to do this. So, the coefficient of  $x^3$  is 1.
    - To obtain  $x^2y$ , an  $x$  must be chosen from two of the sums and a  $y$  from the other.  
There are  $C(3,2)$  ways to do this and so the coefficient of  $x^2y$  is 3.
    - To obtain  $xy^2$ , an  $x$  must be chosen from one of the sums and a  $y$  from the other two. There are  $C(3,1)$  ways to do this and so the coefficient of  $xy^2$  is 3.
    - To obtain  $y^3$ , a  $y$  must be chosen from each of the sums.  
There is only one way to do this. So, the coefficient of  $y^3$  is 1.

# Binomial Theorem

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- **Binomial Theorem:** Let  $x$  and  $y$  be variables, and  $n$  a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

- **Proof:** We use combinatorial reasoning. The terms in the expansion of  $(x+y)^n$  are of the form  $x^{n-j} y^j$  for  $\binom{n}{j}$  for  $j = 0, 1, 2, \dots, n$ . To form the term  $x^{n-j} y^j$ , it is necessary to choose  $n-j$   $x$ s from the  $n$  sums. Therefore, the coefficient of  $x^{n-j} y^j$  is which equals  $\binom{n}{j}$ .

# Using the Binomial Theorem

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- **Example:** What is the coefficient of  $x^{12}y^{13}$  in the expansion of  $(2x - 3y)^{25}$ ?

- **Solution:** We view the expression as  $(2x + (-3y))^{25}$ .

By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j.$$

Consequently, the coefficient of  $x^{12}y^{13}$  in the expansion is obtained when  $j = 13$ .

$$\binom{25}{13} 2^{12} (-3)^{13} = -\frac{25!}{13!12!} 2^{12} 3^{13}.$$

# Pascal's Identity

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- **Pascal's Identity:** If  $n$  and  $k$  are integers with  $n \geq k \geq 0$ , then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

- **Proof.** Let  $T$  be a set where  $|T| = n + 1$ ,  $a \in T$ , and  $S = T - \{a\}$ . There are  $C(n+1, k)$  of the  $T$  subsets containing  $k$  elements. Each of these subsets either:
  - contains  $a$  with  $k - 1$  other elements, or
  - contains  $k$  elements of  $S$  and not  $a$ .

There are

- $C(n, k-1)$  subsets of  $k$  elements that contain  $a$
- $C(n, k)$  subsets of  $k$  elements of  $T$  that do not contain  $a$ , because there are subsets of  $k$  elements of  $S$ .

Hence,  $C(n+1, k) = C(n, k-1) + C(n, k)$

# Pascal's Triangle

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The  $n$ th row in the triangle consists of the binomial coefficients  $k = 0, 1, \dots, n$ .  $\binom{n}{k}$

$$\begin{array}{c}
 \binom{0}{0} \\
 \binom{1}{0} \binom{1}{1} \\
 \binom{2}{0} \binom{2}{1} \binom{2}{2} \\
 \binom{3}{0} \binom{3}{1} \binom{3}{2} \binom{3}{3} \\
 \binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4} \\
 \binom{5}{0} \binom{5}{1} \binom{5}{2} \binom{5}{3} \binom{5}{4} \binom{5}{5} \\
 \binom{6}{0} \binom{6}{1} \binom{6}{2} \binom{6}{3} \binom{6}{4} \binom{6}{5} \binom{6}{6} \\
 \binom{7}{0} \binom{7}{1} \binom{7}{2} \binom{7}{3} \binom{7}{4} \binom{7}{5} \binom{7}{6} \binom{7}{7} \\
 \binom{8}{0} \binom{8}{1} \binom{8}{2} \binom{8}{3} \binom{8}{4} \binom{8}{5} \binom{8}{6} \binom{8}{7} \binom{8}{8} \\
 \dots \\
 \text{(a)}
 \end{array}$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$\begin{array}{c}
 1 \\
 1 \quad 1 \\
 1 \quad 2 \quad 1 \\
 1 \quad 3 \quad 3 \quad 1 \\
 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
 1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
 1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
 \dots \\
 \text{(b)}
 \end{array}$$

By Pascal's identity, adding two adjacent binomial coefficients results in the binomial coefficient in the next row between these two coefficients.

Counting  
(Chapter 6)

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