Discrete Mathematics

Solving Puzzles with SAT Solver

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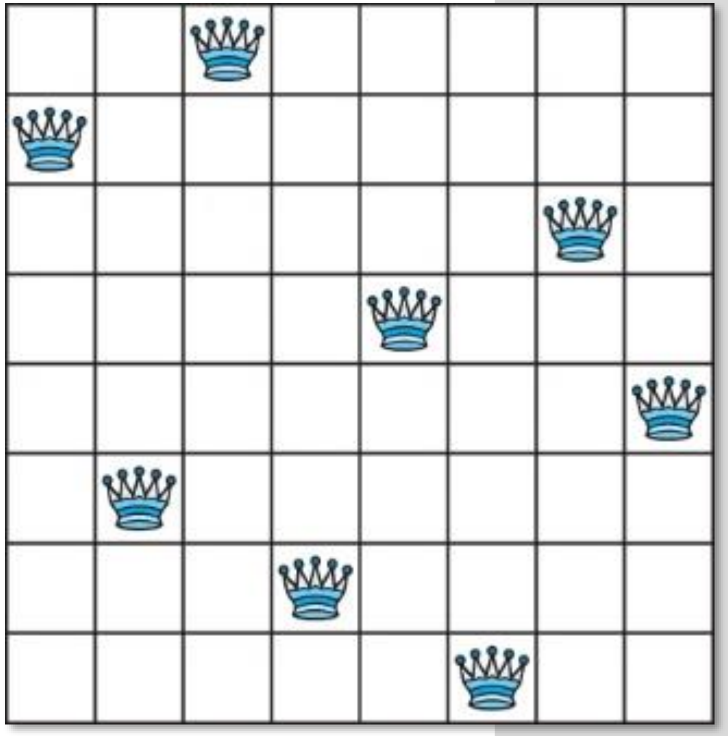
Application: N-Queen Problem



- Problem
  - Place N Queens on a NxN grid, while not placing two Queens on the same vertical, horizontal or diagonal line
- Modeling
  - Proposition $p_{i,j}$ indicates whether a Queen is placed at the i-th row and at the j-th column

$$Q_1 = \bigwedge_{i=1..n} \bigvee_{j=1..n} p_{i,j}$$

$$Q_2 = \bigwedge_{i=1..n} \bigwedge_{j=1..n-1} \bigwedge_{k=j+1..n} \neg(p_{i,j} \wedge p_{i,k})$$

$$Q_3 = \bigwedge_{j=1..n} \bigvee_{i=1..n} p_{i,j}$$

$$Q_4 = \bigwedge_{i=1..n} \bigwedge_{j=1..n-1} \bigwedge_{k=j+1..n} \neg(p_{j,i} \wedge p_{k,i})$$

$$Q_5 = \bigwedge_{i=2..n} \bigwedge_{j=1..n-1} \bigwedge_{k=1..\min(i-1,n-j)} \neg(p_{i,j} \wedge p_{i-k,j+k})$$

$$Q_6 = \bigwedge_{i=1..n-1} \bigwedge_{j=1..n-1} \bigwedge_{k=1..\min(n-i,n-j)} \neg(p_{i,j} \wedge p_{i+k,j+k})$$

$$Q_1 \wedge Q_2 \wedge Q_3 \wedge Q_4 \wedge Q_5 \wedge Q_6$$

SAT/SMT Solver Z3 (1/2)

- Microsoft Z3: https://github.com/Z3Prover/z3
  - Tutorial: https://rise4fun.com/z3/tutorial

- Input example: $(p \to q) \wedge (q \to \neg p) \wedge \neg(p \vee q)$

```
$ cat prop.txt
; prop.txt
(declare-const p Bool)
(declare-const q Bool)
(assert (and (=> p q)
             (=> q (not p))
             (not (or p q))))
(check-sat)
(get-model)
```

```
$ z3 prop.txt
sat
(model
  (define-fun q () Bool
    false)
  (define-fun p () Bool
    false)
)
```

SMT Solver Z3 (2/2)

- Microsoft Z3: https://github.com/Z3Prover/z3
  - Tutorial: https://rise4fun.com/z3/tutorial

- Input example: $\exists a \exists b \exists c\ (a > b + 2\ \wedge\ a = 2c + 10\ \wedge\ b + c \leq 1000)$

```
$ lia.txt
(declare-const a Int)
(declare-const b Int)
(declare-const c Int)
(assert (> a (+ b 2)))
(assert (= a (+ (* 2 c) 10)))
(assert (<= (+ c b) 1000))
(check-sat)
(get-model)
```

```
$ z3 lia.txt
sat
(model
   (define-fun b () Int
     0)
   (define-fun c () Int
     0)
   (define-fun a () Int
     10)
)
```

Application: Sudoku Puzzle (1/3)

- A Sudoku puzzle is represented as a 9x9 grid with nine 3x3 subgrids called subgrids
- Each cell has a number in 1 to 9
- The puzzle is solved by assigning a number to each cell so that every row, every column, and every of a subgrid contains each of the 9 numbers

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

Application: Sudoku Puzzle (2/3)

- $p(i,j,n)$ holds when row i and column j has n

- each cell is assigned with exactly one number

$$\bigwedge_{i=1}^{9}\bigwedge_{j=1}^{9}\bigwedge_{n=1}^{8}\bigwedge_{m=n+1}^{9}\neg(p(i,j,n)\wedge p(i,j,m))$$

$$\bigwedge_{i=1}^{9}\bigwedge_{j=1}^{9}\bigvee_{n=1}^{9}p(i,j,n)$$

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

- each pre-assigned cell contains the given number

  - E.g., $p(1,2,2)\wedge p(1,3,9)\wedge$ ... $\wedge\, p(2,4,5)\wedge\cdots p(9,8,6)$

Application: Sudoku Puzzle (3/3)

- each row has every number between 1 and 9

$$\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i, j, n)$$

- each column has every number between 1 and 9

$$\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i, j, n)$$

- each subgrid has every number between 1 and 9

$$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r + i, 3s + j, n)$$

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	