Discrete Mathematics

Proof

Shin Hong

17 Sep 2020

Proof

- A theorem is an important proposition that can be shown true
 - a theorem (or fact) is a proposition that is true
 - a lemma is a less important proposition that is true (usullay a part of a theorem)
 - a corollary is a theorem directly established from a main theorem
- A proof is a valid argument that establishes the truth of a theorem
 - a proof includes axioms (postulates) which are statement known, assumed, or believed to be true
 - a proof include a conclusion from valid assertions by a valid inference rule
 - a proof can include proven theorems
 - a proof can include premises

• A conjecture is a statement proposed to be true (yet) without a proof

Ch I. Proof

Discrete Math.

Proving Methods

- Direct proof
- Proof by contraposition
- Proof by contradiction
- Exhaustive proof
- Existence proof

Ch I. Proof

Discrete Math.

Direct Proofs

- A direct proof of $p \rightarrow q$ is constructed as follows:
 - first step is the assumption that p is true
 - subsequent steps are constructed by rules of inferences
 - final step shows that q is true under the assumption
- Example: prove that n^2 is odd if n is an odd integer
- 1. x is odd iff there is a positive integer y s.t. x=2y -1 theorem
- 2. *n* is odd premise
- 3. there is a positive integer m s.t. n = 2m 1 MP 1L,2
- 4. $n^2 = (2m-1)^2 = 4m^2 4m + 1$ arithem.
- 5. $n^2 = 2(2m^2 2m) + 1$ arithem.
- 6. n^2 is odd MP IR,5

Proof by Contraposition

- Use the fact that $p \to q$ is equivalent to $\neg q \to \neg p$
- Example: prove that an integer n is odd if 3n+2 is odd = if n is not an odd integer, then 3n+2 is not odd (CP)
 - I. an integer is not odd iff the integer is even
 - 2. *n* is not an odd integer
 - 3. *n* is even
 - 4. if x is even, there is an integer y s.t. x = 2y
 - 5. there is an integer m s.t. n = 2m
 - 6. 3n+2 = 3(2m) + 2 = 2(3m + 1)
 - 7. if there is an integer y s.t. x = 2y, x is even
 - 8. 3n+2 is even

theorem

premise

MP 2, I

theorem

MP 3, 4

arithm.

theorem

MP 6,7

Ch I. Proof

Discrete Math.

Proof by Contradiction

- Proving p by contradiction
 - 1. show that $\neg p \rightarrow q$ is true for a statement q
 - 2. show that q is not true (i.e., unsatisfiable or contradiction)
 - 3. conclude that p is true (i.e., Modus Tollens 1, 2)
- Example: prove that $\sqrt{2}$ is irrational.
 - I. assume that $\sqrt{2}$ is rational
 - 2. there are two integers a and b such that $\sqrt{2} = \frac{a}{b}$, and a and b have no common factor
 - 3. $2b^2 = a^2$
 - 4. a^2 is even, and there is an integer c such that 2c = a
 - 5. $2b^2 = (2c)^2 = 4c^2$
 - 6. b^2 is even as $b^2 = 2c^2$
 - 7. a and b have a common factor as 2
 - 8. it is a contradiction that statements 2 and 7 hold at the same time

Ch I. Proof

Discrete Math.

Exhaustive Proof

- An exhaustive proof of a conditional statement $p \to q$ first divides the condition into multiple cases (i.e., $p = p_1 \lor p_2 ... \lor p_n$) and then proves that the conclusion holds for every case (i.e., $\bigwedge_{i=1}^n p_i \to q$)
- Proof by cases
 - A proof must cover all possible cases that arise in a theorem exhaustively.
- Ex. Prove that if n is an integer, then $n^2 \ge n$ holds.
 - Case I. n = 0: it is shown that $n^2 \ge n$ as $0^2 \ge 0$.
 - Case $2 \cdot n > 0 : n^2 \ge n$ as $n^2 \ge n \times 1$ and $n \ge 1$
 - Case 3. $n < 0 : n^2 \ge n$ as $n^2 \ge 0 > n$

Ch I. Proof

Discrete Math.

Existence Proof

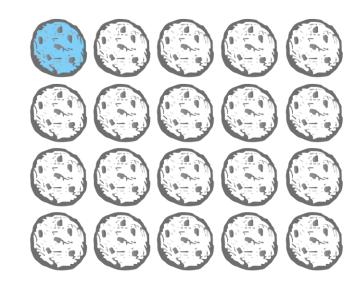
- An existence proof is to assert $\exists x P(x)$
- Strategies
 - Constructive proof: give a concrete case x that P(x) holds
 - Nonconstructive proof: e.g., prove that $\neg \exists x P(x)$ is false
- Ex I (constructive proof). show there is a positive integer that can be written as the sum of cubics of two positive integers in two different ways.
 - Proof. $1729 = 10^3 + 9^3 = 12^3 + 1^3$
- Ex2 (nonconstructive proof). show that there are two irrational numbers x and y such that x^y is a rational number.
 - $\sqrt{2}$ is an irrational number.
 - Case I. $\sqrt{2}^{\sqrt{2}}$ is rational : the claimed statement holds for $x=\sqrt{2}$ and $y=\sqrt{2}$
 - Case 2. $\sqrt{2}^{\sqrt{2}}$ is irrational : the claimed statement holds for $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$

because
$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2$$
 is a rational number

Ch I. Proof

Discrete Math.

Example. Chomp Game



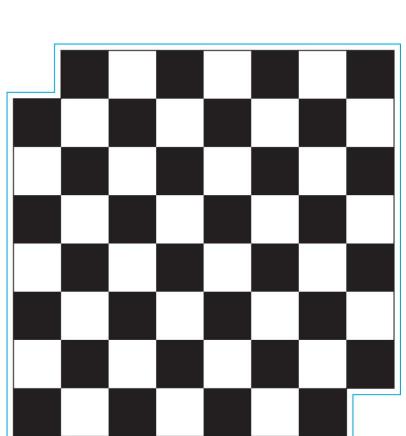
- Game setting
 - There are $m \times n$ cookies arranged as a m-by-n grid for m > 1 and n > 1
 - Turn-over game with two players: Player 1, Player 2, and repeat
 - In each turn, a player must pick a remaining cookie. Then the player takes the picked one and all the ones to the right and/or below the picked on
 - The loser is one who takes the cookie at the top left corner (i.e., the poisoned); the other one is the winner
- Theorem. There is no winning strategy for Player 2
 - There is no way that in any circumstance, Player 2 picks a cookie such that Player 2 has a chance to win the game over Player I in all subsequent cases

Ch I. Proof

Discrete Math.

Creative Proof Idea

• Is there any way to cover a 8x8 grid with the upper left and lower right squares removes using two-dominos?



Ch I. Proof

Discrete Math.