Discrete Mathematics

Counting

Shin Hong

Basic Counting Principles: The Product Rule

The Product Rule: A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the first task and n_2 ways to do the second task. Then there are $n_1 \cdot n_2$ ways to do the procedure.

E.g. How many binary strings of length seven are there?

$2^7 = 128$ because each of the seven bits is either a 0 or a 1

Counting Functions

- How many functions are there from a set with m elements to a set with n elements?
  - Answer: n^m such functions

- How many one-to-one functions are there from a set with m elements to one with n elements?
  - Answer: $n(n-1)(n-2)\cdots(n-m+1)$ such functions.

Basic Counting Principles: The Sum Rule

- If a task can be done either in one of n_1 ways or in one of n_2 ways to do the second task, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

- **E.g.**: The CSEE department must choose either a CS faculty or EE faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 CS faculties and 83 EE faculties, and no one is in CS and EE at the same time.
  - Answer: 37 + 83 = 120 possible ways to pick a representative.

The Sum Rule in terms of sets.

- The sum rule can be phrased in terms of sets.

  $|A \cup B| = |A| + |B|$ as long as A and B are disjoint sets.

- Or more generally,

  $$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

  when $A_i \cap A_j = \emptyset$ for all i, j.

Combining the Sum and Product Rule

- Question
  Suppose statement labels in a programming language can be either a single lowercase alphabet letter or such a letter followed by a digit. Find the number of possible labels.

```
k = (j + 13) / 27
loop:
  if k > 10 then goto out
  k = k + 1
  i = 3 * k - 1
  goto loop
out:
if ((k==1) || (k==2)) j = 2 * k -
if ((k==3) || (k==5)) j = 3 * k +
if (k==4) j = 4 * k -1
```

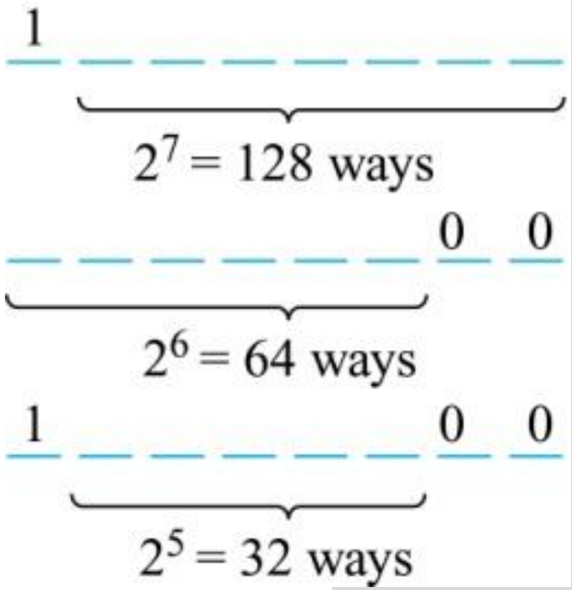
- Answer
  Use the product rule. $26 + 26 \cdot 10 = 286$

Subtraction Rule

- If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.
  - Also known as, the *principle of inclusion-exclusion*

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting Bit Strings



- **Example**: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

- **Solution**: Use the subtraction rule.
  - Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
  - Number of bit strings of length eight that start with bits 00: $2^6 = 64$
  - Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$
  - Hence, the number is $128 + 64 - 32 = 160$.

Counting Passwords

- **Question**. A password must be six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?
- **Answer.** Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.
  - $P = P_6 + P_7 + P_8$
  - $P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560$.
  - $P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920$.
  - $P_8 = 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 = 2{,}612{,}282{,}842{,}880$.
  - Consequently, $P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360$.

Division Rule

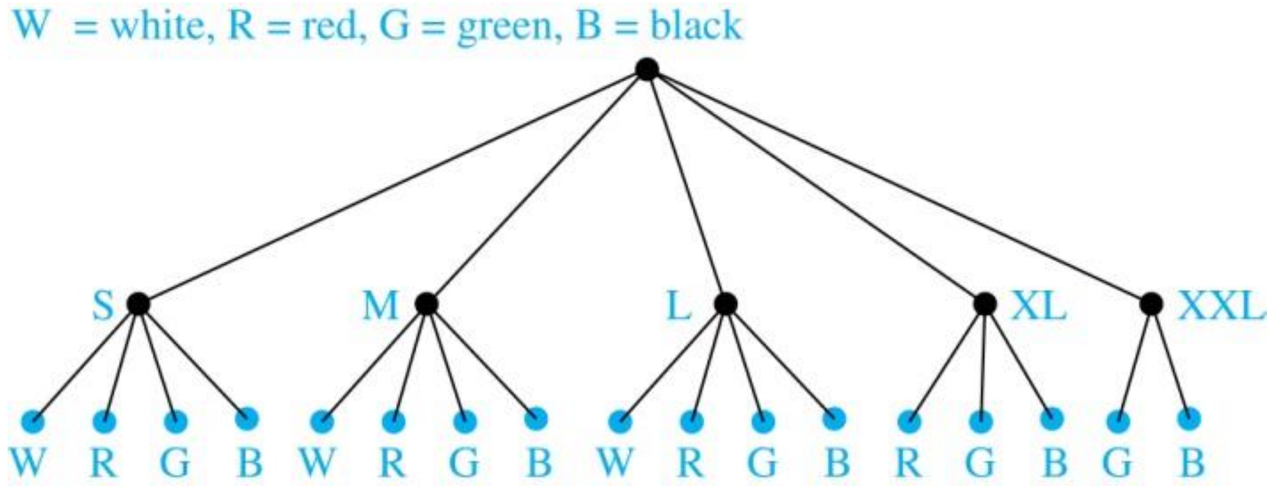
- If a task can be done by a procedure that generates *n* ways, and there are *d* ways of the procedure correspond to the same result, then there are *n*/*d* results of the task.

- **Question**: How many cases are there to seat 4 people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?

- **Answer**: The number the seats around the table from 1 to 4 proceeding clockwise is 4! = 24. But since two seatings are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating. Therefore, by the division rule, there are 24/4 = 6 different seating arrangements.

Tree Diagrams

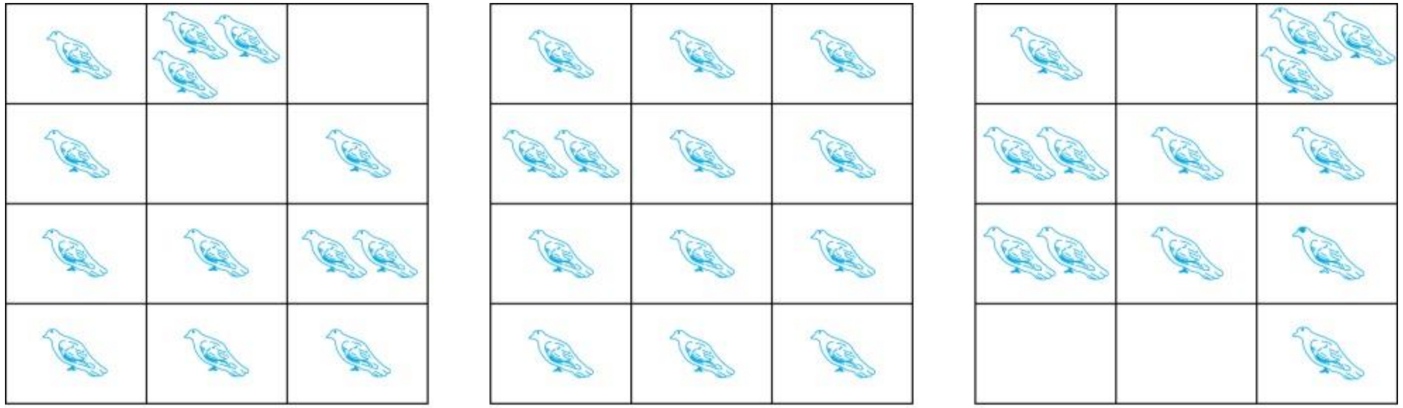
- **Tree Diagrams**: We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.
- **Example**: Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of stores that the campus book store needs to stock to have one of each size and color available?
- **Solution**: Draw the tree diagram.



- The store must stock 17 T-shirts.

The Pigeonhole Principle

- If a flock of 13 pigeons roosts in a set of 12 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



- **Pigeonhole Principle**: If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.
- **Proof**: We use a proof by contraposition. Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k. This contradicts the statement that we have $k + 1$ objects.

The Pigeonhole Principle

- **Corollary 1**: A function f from a set with $k + 1$ elements to a set with k elements is not one-to-one.

- **Proof**: Use the pigeonhole principle.
  Create a box for each element y in the co-domain of f .
  Put in the box for y all of the elements x from the domain such that $f(x) = y$.
  Because there are $k + 1$ elements and only k boxes, at least one box has two or more elements.
  Hence, f can't be one-to-one.

Pigeonhole Principle

- **Question**: Show that for every integer *n* there is a multiple of *n* that has only 0s and 1s in its decimal expansion.
- **Solution**: Let *n* be a positive integer. Consider the $n + 1$ integers $1, 11, 111, \ldots, 11\cdots1$ (i.e., '1'$^{n+1}$). There are *n* possible remainders when an integer is divided by *n*. By the pigeonhole principle, when each of the $n + 1$ integers is divided by *n*, at least two must have the same remainder. Subtract the smaller from the larger and the result is a multiple of *n* that has only 0s and 1s in its decimal expansion.

The Generalized Pigeonhole Principle

- **The Generalized Pigeonhole Principle**: If *N* objects are placed into *k* boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.
- **Proof**: We use a proof by contraposition. Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects. Then the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

where the inequality $\lceil N/k \rceil < \lceil N/k \rceil + 1$ has been used.
This is a contradiction because there are a total of n objects.

- **Example**: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

The Generalized Pigeonhole Principle

- **Question.** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

- **Answer.** We assume four boxes; one for each suit. Using the generalized pigeonhole principle, at least one box contains at least $\lceil N/4 \rceil$ cards. At least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is $N = 2 \cdot 4 + 1 = 9$.

Permutation

- A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an *r-permuation*.
  - e.g. Let $S = \{1,2,3\}$.
    The ordered arrangement $3,1,2$ is a permutation of S.
    The ordered arrangement $3,2$ is a 2-permutation of S.
- The number of *r*-permuatations of a set with *n* elements is denoted by $P(n,r)$.
  - The 2-permutations of $S = \{1,2,3\}$ are $1,2$; $1,3$; $2,1$; $2,3$; $3,1$; and $3,2$.
    Hence, $P(3,2) = 6$.

A Formula for the Number of Permutations

- **Theorem 1**: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

  $$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1)$$

  r-permutations of a set with n distinct elements.

- **Proof**: Use the product rule. The first element can be chosen in n ways. The second in $n - 1$ ways, and so on until there are $(n - (r - 1))$ ways to choose the last element. Note that $P(n,0) = 1$, since there is only one way to order zero elements.
- **Corollary 1**: If n and r are integers with $1 \leq r \leq n$, then

$$P(n,r) = \frac{n!}{(n-r)!}$$

Solving Counting Problems by Counting Permutations

- **Example.** Suppose that a saleswoman has to visit 8 different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

- **Solution**: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems by Counting Permutations

- **Example**. How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

- **Solution**. We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Combinations

- An *r-combination* of elements of a set is an unordered selection of *r* elements from the set. Thus, an *r*-combination is simply a subset of the set with *r* elements.
  - The number of *r*-combinations of a set with n distinct elements is denoted by $C(n, r)$.
  - The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*.

- Example. Let *S* be the set $\{a, b, c, d\}$. How many 2-combinations from S exist?
  - ~~Answer: $C(4,2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.~~

Combinations

Theorem 2: The number of *r*-combinations of a set with *n* elements, where $n \geq r \geq 0$, equals

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

Proof: By the product rule $P(n, r) = C(n,r) \cdot P(r,r)$. Therefore,

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}.$$

Combinations

- **Corollary** 2: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

- **Proof**: From Theorem 2, it follows that

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

and

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} \; .$$

Combinations

- How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?
- **Solution**: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$\begin{aligned} C(52,5) &= \frac{52!}{5!47!} \\ &= \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 26\cdot 17\cdot 10\cdot 49\cdot 12 = 2,598,960 \end{aligned}$$

- The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960.$$

Combinations

- **Example**: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.
- **Solution**: By Theorem 2, the number of combinations is

$$C(10,5) = \frac{10!}{5!5!} = 252.$$

- **Example**: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?
- **Solution**: By Theorem 2, the number of possible crews is

$$C(30,6) = \frac{30!}{6!24!} = \frac{30\cdot 29\cdot 28\cdot 27\cdot 26\cdot 25}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 593,775\ .$$

Powers of Binomial Expressions

- A *binomial* expression is the sum of two terms, such as $x + y$
  - a term can be a product of a constant and a variable
- Find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer
  - e.g. $(x + y)^3 = (x + y)\ (x + y)\ (x + y)$. What are the coefficients of x^3, x^2y, $x\,y^2$ and y^3 ?
    - To obtain x^3 , an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
    - To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $C(3,2)$ ways to do this and so the coefficient of x^2y is 3.
    - To obtain xy^2, an x must be chosen from one of the sums and a y from the other two. There are $C(3,1)$ ways to do this and so the coefficient of xy^2 is 3.
    - To obtain y^3 , a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.

Binomial Theorem

- **Binomial Theorem**: Let x and y be variables, and n a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

- **Proof**: We use combinatorial reasoning . The terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$ for C(n, n-j) for j = 0,1,2,…,n. To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ xs from the n sums. Therefore, the coefficient of $x^{n-j}y^j$ is which equals C(n, n-j).

Using the Binomial Theorem

- **Example**: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

- **Solution**: We view the expression as $(2x + (-3y))^{25}$.
  By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^{j}.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!}2^{12}3^{13}.$$

Pascal's Identity

- **Pascal's Identity**: If *n* and *k* are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

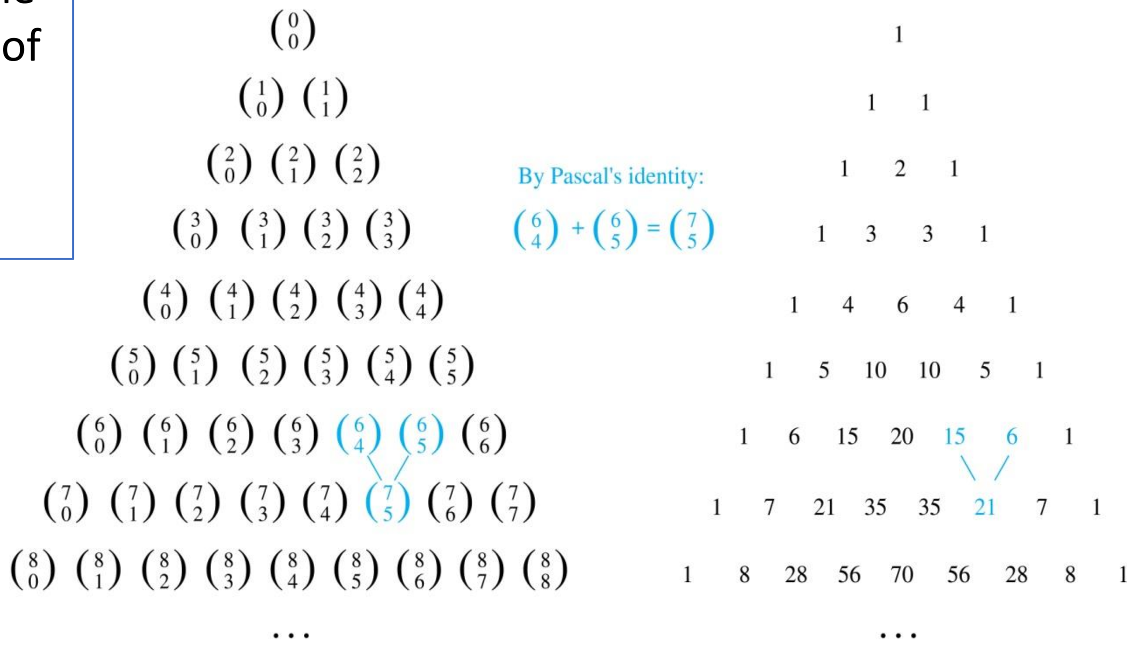
- **Proof.** Let *T* be a set where $|T| = n + 1$, $a \in T$, and $S = T - \{a\}$. There are $C(n+1, k)$ of the *T* subsets containing *k* elements. Each of these subsets either:
  - contains *a* with $k-1$ other elements, or
  - contains *k* elements of *S* and not *a*.

  There are
  - $C(n, k-1)$ subsets of *k* elements that contain *a*
  - $C(n,k)$ subsets of *k* elements of *T* that do not contain *a*, because there are subsets of k elements of S.

  Hence, $C(n+1, k) = C(n, k-1) + C(n, k)$

Pascal's Triangle

The *n*th row in the triangle consists of the binomial coefficients $\binom{n}{k}$, k = 0,1,….,n.



(a) (b)

By Pascal's identity, adding two adjacent bionomial coefficients results is the binomial coefficient in the next row between these two coefficients.

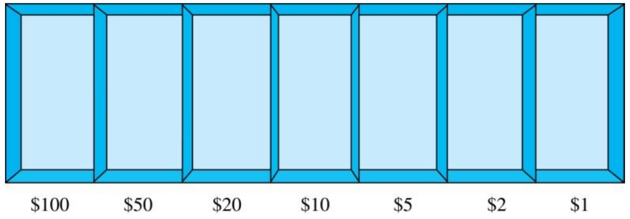
Permutations with Repetition

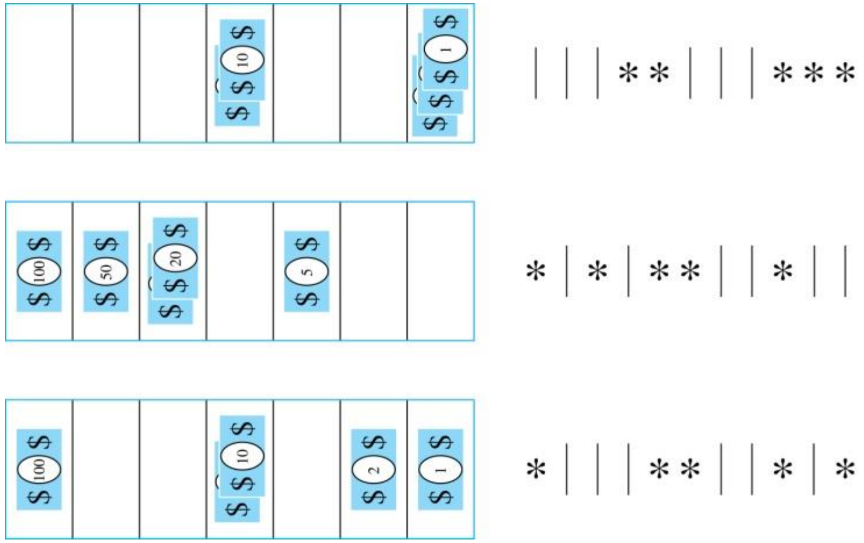
- **Theorem** 1: The number of *r*-permutations of a set of *n* objects with repetition allowed is n^r.

- **Ex.** How many strings of length *r* can be formed from the uppercase letters of the English alphabet?
  - The number of such strings is 26^r, which is the number of *r*-permutations of a set with 26 elements.

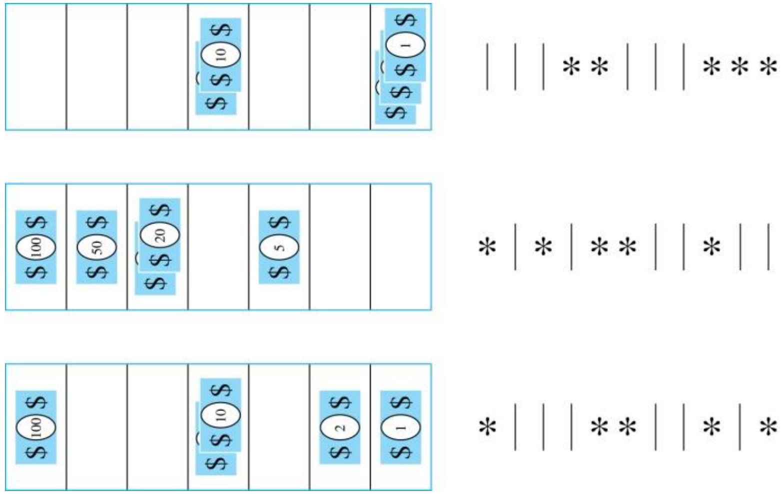
Combinations with Repetition

- How many ways are there to select five bills from a box containing at least five of each of the following denominations: $\$1$, $\$2$, $\$5$, $\$10$, $\$20$, $\$50$, and $\$100$?
- **Solution**: Place the selected bills in the appropriate position of a cash box illustrated below:





Combinations with Repetition



- The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row.
- This is the number of unordered selections of 5 objects from a set of 11. Hence, there are $C(11,5) = \frac{11!}{5!6!} = 462$ ways to choose five bills with seven types of bills.

Combinations with Repetition

- **Question.** How many solutions does the equation

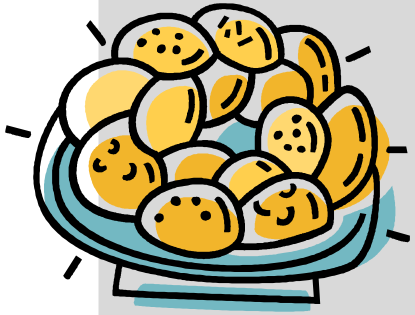
$$x_1 + x_2 + x_3 = 11$$

have, where x_1 , x_2 and x_3 are nonnegative integers?

- **Solution.** Each solution corresponds to a way to select 11 items from a set with three elements; x_1 elements of type one, x_2 of type two, and x_3 of type three.

$$C(3 + 11 - 1, 11) = C(13, 11) = C(13, 2) = \frac{13 \cdot 12}{1 \cdot 2} = 78$$

Combinations with Repetition



- **Example**: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?
- **Solution**: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements.
  By Theorem 2, the number of ways to choose six cookies from the four kinds is

$$C(9,6) = C(9,3) = \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3} = 84$$

Permutations with Indistinguishable Objects

- How many different strings can be made by reordering the letters of the word *SUCCESS*.
- **Solution**: There are seven possible positions for the three *S*'s, two C's, one U, and one E.
  - The three S's can be placed in *C*(7,3) different ways, leaving four positions free.
  - The two C's can be placed in *C*(4,2) different ways, leaving two positions free.
  - The U can be placed in *C*(2,1) different ways, leaving one position free.
  - The E can be placed in *C*(1,1) way.

By the product rule, the number of different strings is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!} = 420.$$

Permutations with Indistinguishable Objects

- **Theorem 3**: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type k, is: $$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

- **Proof**: By the product rule the total number of permutations is:

  $C(n, n_1)\ C(n - n_1, n_2) \cdots C(n - n_1 - n_2 - \cdots - n_k, n_k)$ since:
  - The n_1 objects of type one can be placed in the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions
  - Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions
  - Continue in this fashion, until n_k objects of type k are placed in $C(n - n_1 - n_2 \cdots - n_k, n_k)$ ways

  The product can be manipulated into the desired result as follows

$$\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2!)}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!} = \frac{n!}{n_1!n_2!\cdots n_k!}.$$

Distributing Objects into Boxes (1/4)

- Distinguishable *n* objects and distinguishable *k* boxes
  - There are $n!/(n_1!n_2! \cdots n_k!)$ ways to distribute *n* distinguishable objects into *k* distinguishable boxes.
  - Example: There are $52!/(5!5!5!5!32!)$ ways to distribute hands of 5 cards each to four players.

Distributing Objects into Boxes (2/4)

- *n* indistinguishable *n* objects and *k* distinguishable boxes
  - There are $C(n + k - 1, k - 1)$ ways to place *r* indistinguishable objects into *n* distinguishable boxes.
  - Proof based on one-to-one correspondence between *n*-combinations from a set with *k*-elements when repetition is allowed and the ways to place *n* indistinguishable objects into *k* distinguishable boxes.
  - Example: There are $C(8 + 10 - 1, 10) = C(17,10) = 19{,}448$ ways to place 10 indistinguishable objects into 8 distinguishable boxes.

Distributing Objects into Boxes (3/4)

- Distinguishable objects and indistinguishable boxes.
  - Example: There are 14 ways to put four employees into three indistinguishable offices (see Example 10).
  - There is no simple closed formula for the number of ways to distribute n distinguishable objects into *j* indistinguishable boxes.

Distributing Objects into Boxes (4/4)

- Indistinguishable *objects* and indistinguishable boxes
  - Example: There are 9 ways to pack six copies of the same book into four identical boxes (*see Example* 11).
  - The number of ways of distributing *n* indistinguishable objects into *k* indistinguishable boxes equals the number of ways to write *n* as the sum of at most *k* positive integers in increasing order.
  - No simple closed formula exists for this number.