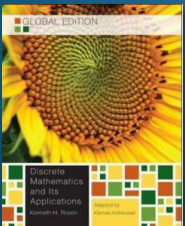


# Relations

## Chapter 9



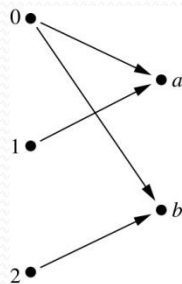
- Taken from the instructor's resource of *Discrete Mathematics and Its Applications*, 7/e
- Edited by Shin Hong [hongshin@handong.edu](mailto:hongshin@handong.edu)

# Binary Relations

**Definition:** A *binary relation*  $R$  from a set  $A$  to a set  $B$  is a subset  $R \subseteq A \times B$ .

**Example:**

- Let  $A = \{0,1,2\}$  and  $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a) , (2, b)\}$  is a relation from  $A$  to  $B$ .
- We can represent relations from a set  $A$  to a set  $B$  graphically or using a table:



$R$	$a$	$b$
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where exactly one element of  $B$  is related to each element of  $A$ .

# Binary Relation on a Set

**Definition:** A binary relation  $R$  on a set  $A$  is a subset of  $A \times A$  or a relation from  $A$  to  $A$ .

**Example:**

- Suppose that  $A = \{a, b, c\}$ . Then  $R = \{(a, a), (a, b), (a, c)\}$  is a relation on  $A$ .
- Let  $A = \{1, 2, 3, 4\}$ . The ordered pairs in the relation  $R = \{(a, b) \mid a \text{ divides } b\}$  are  $(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3)$ , and  $(4, 4)$ .

# Binary Relation on a Set (*cont.*)

**Question:** How many relations are there on a set  $A$ ?

**Solution:**

Since  $A \times A$  has  $n^2$  elements when  $A$  has  $n$  elements, and a set with  $m$  elements has  $2^m$  subsets, there are  $2^{|A|^2}$  subsets of  $A \times A$ .

Therefore, there are  $2^{|A|^2}$  relations on a set  $A$ .

# Binary Relations on a Set (*cont.*)

**Example:** Consider these relations on the set of integers:

$$R_1 = \{(a,b) \mid a \leq b\},$$

$$R_4 = \{(a,b) \mid a = b\},$$

$$R_2 = \{(a,b) \mid a > b\},$$

$$R_5 = \{(a,b) \mid a = b + 1\},$$

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_6 = \{(a,b) \mid a + b \leq 3\}.$$

Which of these relations contain each of the pairs

$(1,1)$ ,  $(1,2)$ ,  $(2,1)$ ,  $(1,-1)$ , and  $(2,2)$ ?

**Solution:** Checking the conditions that define each relation, we see that the pair  $(1,1)$  is in  $R_1$ ,  $R_3$ ,  $R_4$ , and  $R_6$ ;  $(1,2)$  is in  $R_1$  and  $R_6$ ;  $(2,1)$  is in  $R_2$ ,  $R_5$ , and  $R_6$ ;  $(1,-1)$  is in  $R_2$ ,  $R_3$ , and  $R_6$ ;  $(2,2)$  is in  $R_1$ ,  $R_3$ , and  $R_4$ .

# Reflexive Relations

$R$  is *reflexive* iff  $(a,a) \in R$  for every element  $a \in A$ .

- $R$  is reflexive if and only if  $\forall x \in A ((x,x) \in R)$
- If  $A = \emptyset$  then the empty relation is reflexive vacuously.

## reflexive relations:

$$R_1 = \{(a,b) \mid a \leq b\},$$

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a,b) \mid a = b\}.$$

## non reflexive relations :

$$R_2 = \{(a,b) \mid a > b\} \text{ (note that } 3 \not> 3),$$

$$R_5 = \{(a,b) \mid a = b + 1\} \text{ (note that } 3 \neq 3 + 1),$$

$$R_6 = \{(a,b) \mid a + b \leq 3\} \text{ (note that } 4 + 4 \not\leq 3).$$

# Symmetric Relations

$R$  is *symmetric* iff  $(b,a) \in R$  whenever  $(a,b) \in R$  for all  $a,b \in A$ .

$$- \forall x \in A ( \forall y \in A ( (x,y) \in R \longrightarrow (y,x) \in R ) )$$

## Example

**symmetric:**

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a,b) \mid a = b\},$$

$$R_6 = \{(a,b) \mid a + b \leq 3\}.$$

**not symmetric:**

$$R_1 = \{(a,b) \mid a \leq b\} \text{ (note that } 3 \leq 4, \text{ but } 4 \not\leq 3),$$

$$R_2 = \{(a,b) \mid a > b\} \text{ (note that } 4 > 3, \text{ but } 3 \not> 4),$$

$$R_5 = \{(a,b) \mid a = b + 1\} \text{ (note that } 4 = 3 + 1, \text{ but } 3 \neq 4 + 1).$$

# Antisymmetric Relations

- $R$  on a set  $A$  is **antisymmetric** when  $(a,b) \in R$  and  $(b,a) \in R$  only if  $a = b$  for all  $a \in A, b \in A$
- **Example**
  - antisymmetric:
    - $R_1 = \{(a,b) \mid a \leq b\},$
    - $R_2 = \{(a,b) \mid a > b\},$
    - $R_4 = \{(a,b) \mid a = b\},$
    - $R_5 = \{(a,b) \mid a = b + 1\}.$
  - not antisymmetric:
    - $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$   
(note that both  $(1,-1)$  and  $(-1,1)$  belong to  $R_3$ ),
    - $R_6 = \{(a,b) \mid a + b \leq 3\}$  (note that both  $(1,2)$  and  $(2,1)$  belong to  $R_6$ ).



# Transitive Relations

- $R$  on a set  $A$  is **transitive** if whenever  $(a,b) \in R$  and  $(b,c) \in R$ , then  $(a,c) \in R$ , for all  $a,b,c \in A$ .
  - $\forall x \forall y \forall z [(x,y) \in R \wedge (y,z) \in R \rightarrow (x,z) \in R]$
- **Examples of transitive relations:**
  - $R_1 = \{(a,b) \mid a \leq b\},$
  - $R_2 = \{(a,b) \mid a > b\},$
  - $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$
  - $R_4 = \{(a,b) \mid a = b\}.$

# Combining Relations

- Given two relations  $R_1$  and  $R_2$ , we can combine them using basic set operations to form new relations such as  $R_1 \cup R_2$ ,  $R_1 \cap R_2$ ,  $R_1 - R_2$ , and  $R_2 - R_1$ .
- Example:** Let  $A = \{1,2,3\}$  and  $B = \{1,2,3,4\}$ . The relations  $R_1 = \{(1,1), (2,2), (3,3)\}$  and  $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$  can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\} \qquad R_1 - R_2 = \{(2,2), (3,3)\}$$

$$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$$

# Composition

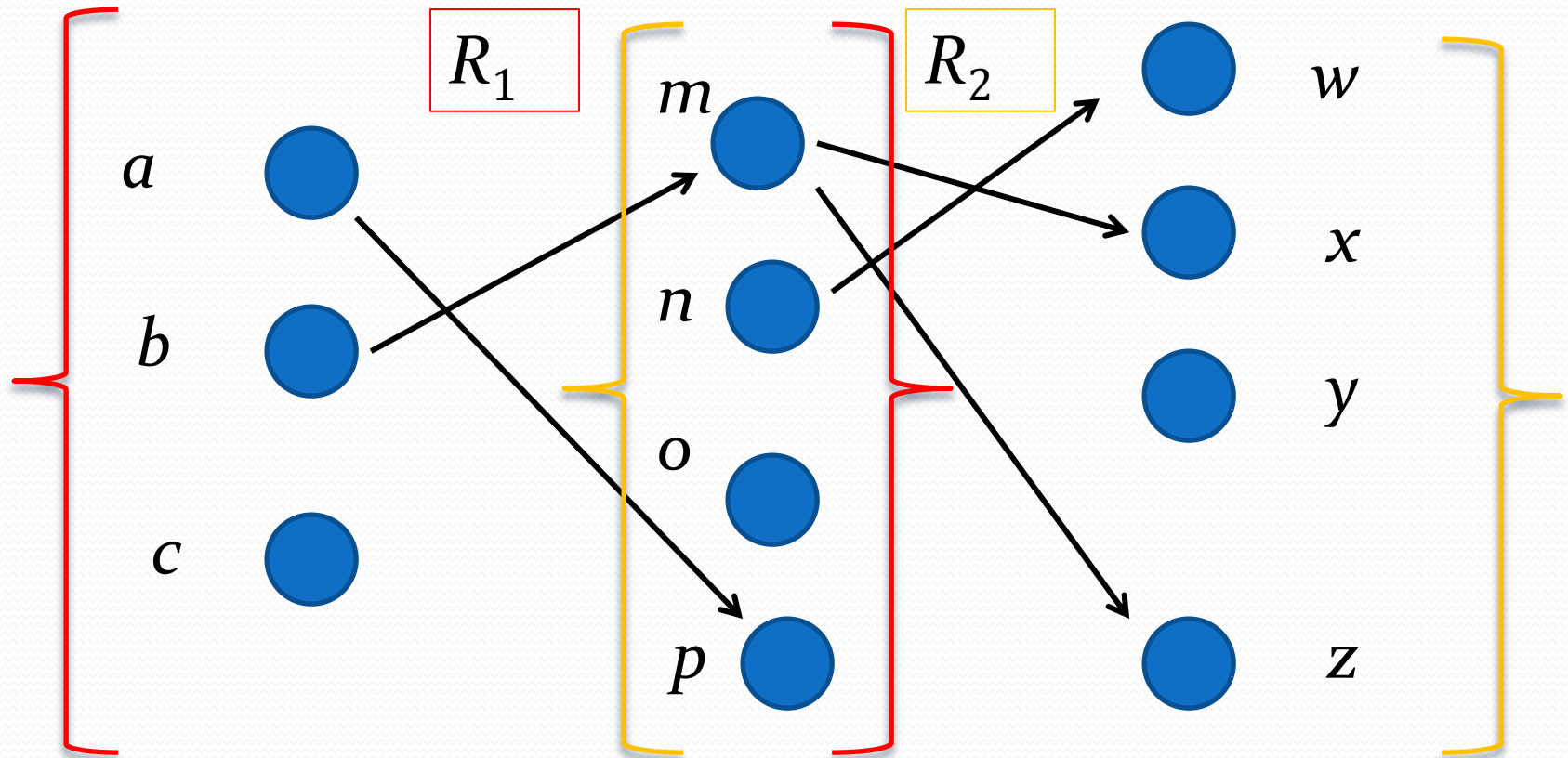
Suppose

- $R_1$  is a relation from a set  $A$  to a set  $B$ .
- $R_2$  is a relation from  $B$  to a set  $C$ .

The *composition* (or *composite*) of  $R_2$  with  $R_1$ , is a relation from  $A$  to  $C$  where

- if  $(x,y)$  is a member of  $R_1$  and  $(y,z)$  is a member of  $R_2$ , then  $(x,z)$  is a member of  $R_2 \circ R_1$ .

# Representing the Composition of a Relation



$$R_2 \circ R_1 = \{(b, x), (b, z)\}$$

# Powers of a Relation

**Definition:** Let  $R$  be a binary relation on  $A$ . Then the powers  $R^n$  of the relation  $R$  can be defined inductively by:

- Basis Step:  $R^1 = R$
- Inductive Step:  $R^{n+1} = R^n \circ R$

*(see the slides for Section 9.3 for further insights)*

The powers of a transitive relation are subsets of the relation. This is established by the following theorem:

**Theorem 1:** The relation  $R$  on a set  $A$  is transitive iff  $R^n \subseteq R$  for  $n = 1, 2, 3, \dots$

*(see the text for a proof via mathematical induction)*

# Representing Relations Using Matrices

- A relation between finite sets can be represented using a zero-one matrix.
- A relation  $R \subseteq A \times B$  is represented by the matrix  $M_R = [m_{ij}]$ , where
$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R, \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$
- The matrix representing  $R$  has a 1 as its  $(i,j)$  entry when  $a_i$  is related to  $b_j$  and a 0 if  $a_i$  is not related to  $b_j$ .

# Example

**Example 1:** Suppose that  $A = \{1,2,3\}$  and  $B = \{1,2\}$ . Let  $R$  be the relation from  $A$  to  $B$  containing  $(a,b)$  if  $a \in A$ ,  $b \in B$ , and  $a > b$ . What is the matrix representing  $R$  (assuming the ordering of elements is the same as the increasing numerical order)?

**Solution:** Because  $R = \{(2,1), (3,1), (3,2)\}$ , the matrix is

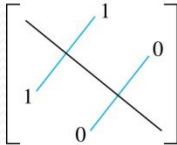
$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

# Matrices of Relations on Sets

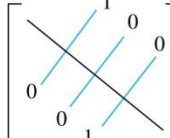
- If  $R$  is a reflexive relation, all the elements on the main diagonal of  $M_R$  are equal to 1.

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

- $R$  is a symmetric relation, if and only if  $m_{ij} = 1$  whenever  $m_{ji} = 1$ .  $R$  is an antisymmetric relation, if and only if  $m_{ij} = 0$  or  $m_{ji} = 0$  when  $i \neq j$ .



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Symmetric

(b) Antisymmetric



# Example of a Relation on a Set

**Example 3:** Suppose that the relation  $R$  on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

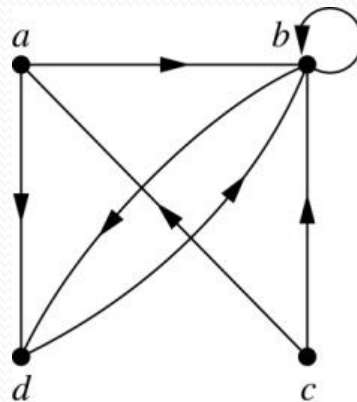
Is  $R$  reflexive, symmetric, and/or antisymmetric?

**Solution:** Because all the diagonal elements are equal to 1,  $R$  is reflexive. Because  $M_R$  is symmetric,  $R$  is symmetric and not antisymmetric because both  $m_{1,2}$  and  $m_{2,1}$  are 1.

# Representing Relations Using Digraphs

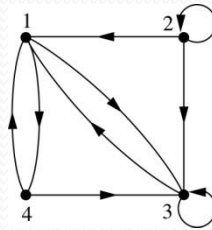
- A *directed graph*, or *digraph*, consists of a set  $V$  of *vertices* (or *nodes*) together with a set  $E$  of ordered pairs of elements of  $V$  called *edges* (or *arcs*). The vertex  $a$  is called the *initial vertex* of the edge  $(a,b)$ , and the vertex  $b$  is called the *terminal vertex* of this edge.
  - An edge of the form  $(a,a)$  is called a *loop*.

**Example 7:** A drawing of the directed graph with vertices  $a$ ,  $b$ ,  $c$ , and  $d$ , and edges  $(a, b)$ ,  $(a, d)$ ,  $(b, b)$ ,  $(b, d)$ ,  $(c, a)$ ,  $(c, b)$ , and  $(d, b)$  is shown here.



# Examples of Digraphs Representing Relations

**Example 8:** What are the ordered pairs in the relation represented by this directed graph?



**Solution:** The ordered pairs in the relation are  $(1, 3)$ ,  $(1, 4)$ ,  $(2, 1)$ ,  $(2, 2)$ ,  $(2, 3)$ ,  $(3, 1)$ ,  $(3, 3)$ ,  $(4, 1)$ , and  $(4, 3)$

# Determining which Properties a Relation has from its Digraph

- *Reflexivity*: A loop must be present at all vertices in the graph.
- *Symmetry*: If  $(x,y)$  is an edge, then so is  $(y,x)$ .
- *Antisymmetry*: If  $(x,y)$  with  $x \neq y$  is an edge, then  $(y,x)$  is not an edge.
- *Transitivity*: If  $(x,y)$  and  $(y,z)$  are edges, then so is  $(x,z)$ .

# Equivalence Relations

- A relation on a set  $A$  is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

**Definition 2:** Two elements  $a$ , and  $b$  that are related by an equivalence relation are called *equivalent*. The notation  $a \sim b$  is often used to denote that  $a$  and  $b$  are equivalent elements with respect to a particular equivalence relation.

# Example

Suppose that  $R$  is the relation on the set of strings of English letters such that  $(a, b) \in R$  if and only if  $l(a) = l(b)$ , where  $l(x)$  is the length of the string  $x$ . Is  $R$  an equivalence relation?

## Solution

- *Reflexivity*: as  $l(a) = l(a)$ , it follows  $(a, a) \in R$  for every string  $a$ .
- *Symmetry*:  $(b, a) \in R$  if  $(a, b) \in R$  because  $l(a) = l(b)$ ,  $l(b) = l(a)$
- *Transitivity*:  $(a, c) \in R$  if  $(a, b) \in R \wedge (b, c) \in R$  because, if  $l(a) = l(b)$  and  $l(b) = l(c)$  then  $l(a) = l(c)$

# Congruence Modulo $m$

**Example:** Let  $m$  be an integer with  $m > 1$ . Show that the relation  
$$R = \{(a,b) \mid a \equiv b \pmod{m}\}$$
  
is an equivalence relation on the set of integers.

**Solution:** Recall that  $a \equiv b \pmod{m}$  if and only if  $m$  divides  $a - b$ .

- *Reflexivity:*  $a \equiv a \pmod{m}$  since  $a - a = 0$  is divisible by  $m$  since  $0 = 0 \cdot m$ .
- *Symmetry:* Suppose that  $a \equiv b \pmod{m}$ . Then  $a - b$  is divisible by  $m$ , and so  $a - b = km$ , where  $k$  is an integer. It follows that  $b - a = (-k)m$ , so  $b \equiv a \pmod{m}$ .
- *Transitivity:* Suppose that  $a \equiv b \pmod{m}$  and  $b \equiv c \pmod{m}$ . Then  $m$  divides both  $a - b$  and  $b - c$ . Hence, there are integers  $k$  and  $l$  with  $a - b = km$  and  $b - c = lm$ . We obtain by adding the equations:  
$$a - c = (a - b) + (b - c) = km + lm = (k + l)m.$$

Therefore,  $a \equiv c \pmod{m}$ .

# Divides

**Example:** Show that the “divides” relation on the set of positive integers is not an equivalence relation.

**Solution:** The properties of reflexivity, and transitivity do hold, but there relation is not symmetric. Hence, “divides” is not an equivalence relation.

- *Reflexivity:*  $a \mid a$  for all  $a$ .
- *Not Symmetric:* For example,  $2 \mid 4$ , but  $4 \nmid 2$ . Hence, the relation is not symmetric.
- *Transitivity:* Suppose that  $a$  divides  $b$  and  $b$  divides  $c$ . Then there are positive integers  $k$  and  $l$  such that  $b = ak$  and  $c = bl$ . Hence,  $c = a(kl)$ , so  $a$  divides  $c$ . Therefore, the relation is transitive.



# Equivalence Classes

**Definition 3:** Let  $R$  be an equivalence relation on a set  $A$ . The set of all elements that are related to an element  $a$  of  $A$  is called the *equivalence class* of  $a$ . The equivalence class of  $a$  with respect to  $R$  is denoted by  $[a]_R$ .

When only one relation is under consideration, we can write  $[a]$ , without the subscript  $R$ , for this equivalence class.

Note that  $[a]_R = \{s | (a, s) \in R\}$ .

- If  $b \in [a]_R$ , then  $b$  is called a representative of this equivalence class. Any element of a class can be used as a representative of the class.
- The equivalence classes of the relation congruence modulo  $m$  are called the *congruence classes modulo  $m$* . The congruence class of an integer  $a$  modulo  $m$  is denoted by  $[a]_m$ , so  $[a]_m = \{\dots, a-2m, a-m, a, a+m, a+2m, \dots\}$ . For example,

$$[0]_4 = \{\dots, -8, -4, 0, 4, 8, \dots\}$$

$$[1]_4 = \{\dots, -7, -3, 1, 5, 9, \dots\}$$

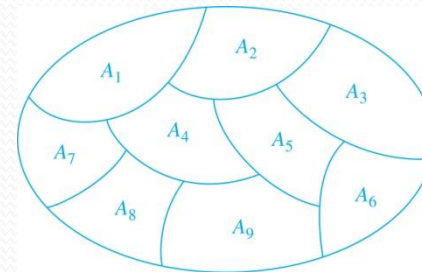
$$[2]_4 = \{\dots, -6, -2, 2, 6, 10, \dots\}$$

$$[3]_4 = \{\dots, -5, -1, 3, 7, 11, \dots\}$$

# Partition of a Set

**Definition:** A *partition* of a set  $S$  is a collection of disjoint nonempty subsets of  $S$  that have  $S$  as their union. In other words, the collection of subsets  $A_i$ , where  $i \in I$  (where  $I$  is an index set), forms a partition of  $S$  if and only if

- $A_i \neq \emptyset$  for  $i \in I$ ,
- $A_i \cap A_j = \emptyset$  when  $i \neq j$ ,
- and  $\bigcup_{i \in I} A_i = S$ .



A Partition of a Set

# An Equivalence Relation Partitions a Set

- Let  $R$  be an equivalence relation on a set  $A$ . The union of all the equivalence classes of  $R$  is all of  $A$ , since an element  $a$  of  $A$  is in its own equivalence class  $[a]_R$ . In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

- From Theorem 1, it follows that these equivalence classes are either equal or disjoint, so  $[a]_R \cap [b]_R = \emptyset$  when  $[a]_R \neq [b]_R$ .
- Therefore, the equivalence classes form a partition of  $A$ , because they split  $A$  into disjoint subsets.

# An Equivalence Relation Partitions a Set

**Theorem 2:** Let  $R$  be an equivalence relation on a set  $S$ . Then the equivalence classes of  $R$  form a partition of  $S$ . Conversely, given a partition  $\{A_i \mid i \in I\}$  of the set  $S$ , there is an equivalence relation  $R$  that has the sets  $A_i$ ,  $i \in I$ , as its equivalence classes.

**Proof:** We have already shown the first part of the theorem.

For the second part, assume that  $\{A_i \mid i \in I\}$  is a partition of  $S$ . Let  $R$  be the relation on  $S$  consisting of the pairs  $(x, y)$  where  $x$  and  $y$  belong to the same subset  $A_i$  in the partition. We must show that  $R$  satisfies the properties of an equivalence relation.

- *Reflexivity:* For every  $a \in S$ ,  $(a, a) \in R$ , because  $a$  is in the same subset as itself.
- *Symmetry:* If  $(a, b) \in R$ , then  $b$  and  $a$  are in the same subset of the partition, so  $(b, a) \in R$ .
- *Transitivity:* If  $(a, b) \in R$  and  $(b, c) \in R$ , then  $a$  and  $b$  are in the same subset of the partition, as are  $b$  and  $c$ . Since the subsets are disjoint and  $b$  belongs to both, the two subsets of the partition must be identical. Therefore,  $(a, c) \in R$  since  $a$  and  $c$  belong to the same subset of the partition.

# Partial Orderings

**Definition 1:** A relation  $R$  on a set  $S$  is called a *partial ordering*, or *partial order*, if it is reflexive, antisymmetric, and transitive. A set together with a partial ordering  $R$  is called a *partially ordered set*, or *poset*, and is denoted by  $(S, R)$ . Members of  $S$  are called *elements* of the poset.

# Partial Orderings (*continued*)

**Example 1:** Show that the “greater than or equal” relation ( $\geq$ ) is a partial ordering on the set of integers.

- *Reflexivity:*  $a \geq a$  for every integer  $a$ .
- *Antisymmetry:* If  $a \geq b$  and  $b \geq a$ , then  $a = b$ .
- *Transitivity:* If  $a \geq b$  and  $b \geq c$ , then  $a \geq c$ .

These properties all follow from the order axioms for the integers.  
(See Appendix 1).

# Partial Orderings (*continued*)

**Example 2:** Show that the divisibility relation ( $|$ ) is a partial ordering on the set of integers.

- *Reflexivity:*  $a | a$  for all integers  $a$ . (see Example 9 in Section 9.1)
- *Antisymmetry:* If  $a$  and  $b$  are positive integers with  $a | b$  and  $b | a$ , then  $a = b$ . (see Example 12 in Section 9.1)
- *Transitivity:* Suppose that  $a$  divides  $b$  and  $b$  divides  $c$ . Then there are positive integers  $k$  and  $l$  such that  $b = ak$  and  $c = bl$ . Hence,  $c = a(kl)$ , so  $a$  divides  $c$ . Therefore, the relation is transitive.
- $(\mathbb{Z}^+, |)$  is a poset.

# Partial Orderings (*continued*)

**Example 3:** Show that the inclusion relation ( $\subseteq$ ) is a partial ordering on the power set of a set  $S$ .

- *Reflexivity:*  $A \subseteq A$  whenever  $A$  is a subset of  $S$ .
- *Antisymmetry:* If  $A$  and  $B$  are positive integers with  $A \subseteq B$  and  $B \subseteq A$ , then  $A = B$ .
- *Transitivity:* If  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

The properties all follow from the definition of set inclusion.



# Comparability

**Definition 2:** The elements  $a$  and  $b$  of a poset  $(S, \preceq)$  are *comparable* if either  $a \preceq b$  or  $b \preceq a$ . When  $a$  and  $b$  are elements of  $S$  so that neither  $a \preceq b$  nor  $b \preceq a$ , then  $a$  and  $b$  are called *incomparable*.

The symbol  $\preceq$  is used to denote the relation in any poset.

**Definition 3:** If  $(S, \preceq)$  is a poset and every two elements of  $S$  are comparable,  $S$  is called a *totally ordered* or *linearly ordered set*, and  $\preceq$  is called a *total order* or a *linear order*. A totally ordered set is also called a *chain*.

**Definition 4:**  $(S, \preceq)$  is a well-ordered set if it is a poset such that  $\preceq$  is a total ordering and every nonempty subset of  $S$  has a least element.

# Lexicographic Order

**Definition:** Given two posets  $(A_1, \preceq_1)$  and  $(A_2, \preceq_2)$ , the *lexicographic ordering* on  $A_1 \times A_2$  is defined by specifying that  $(a_1, a_2)$  is less than  $(b_1, b_2)$ , that is,

$$(a_1, a_2) < (b_1, b_2),$$

either if  $a_1 <_1 b_1$  or if  $a_1 = b_1$  and  $a_2 <_2 b_2$ .

- This definition can be easily extended to a lexicographic ordering on strings (*see text*).

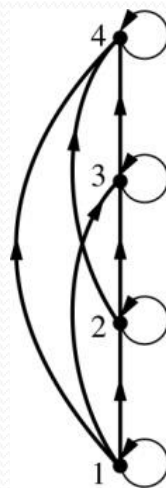
**Example:** Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- *discreet* < *discrete*, because these strings differ in the seventh position and  $e < t$ .
- *discreet* < *discreetness*, because the first eight letters agree, but the second string is longer.

# Hasse Diagrams

**Definition:** A *Hasse diagram* is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.

- The loops due to the reflexive property are deleted.
- The edges due to the transitive property are deleted
- Arrows are omitted because of antisymmetry



(a)

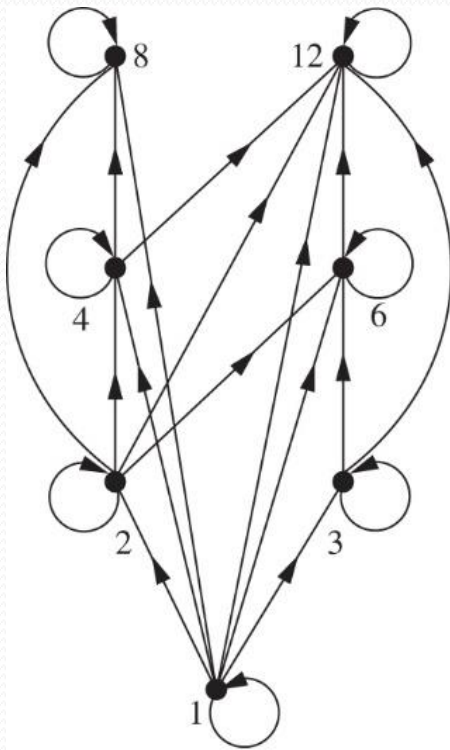


(b)

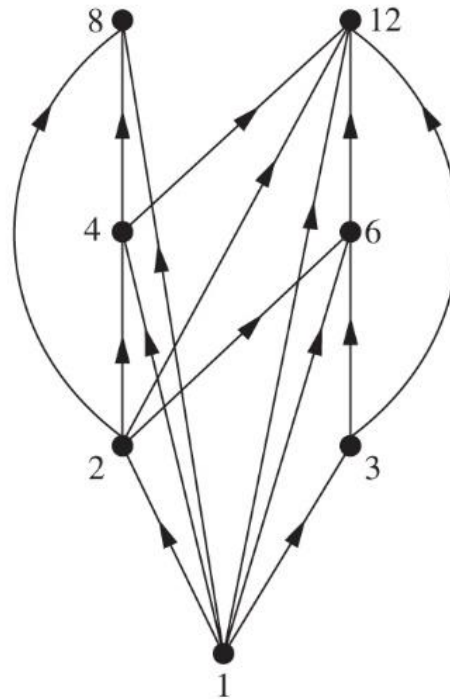


(c)

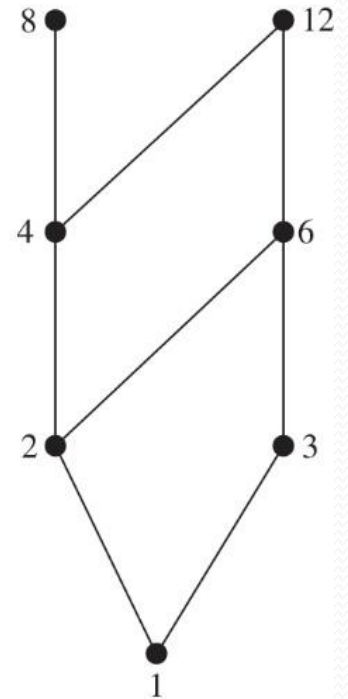
# Example



(a)



(b)



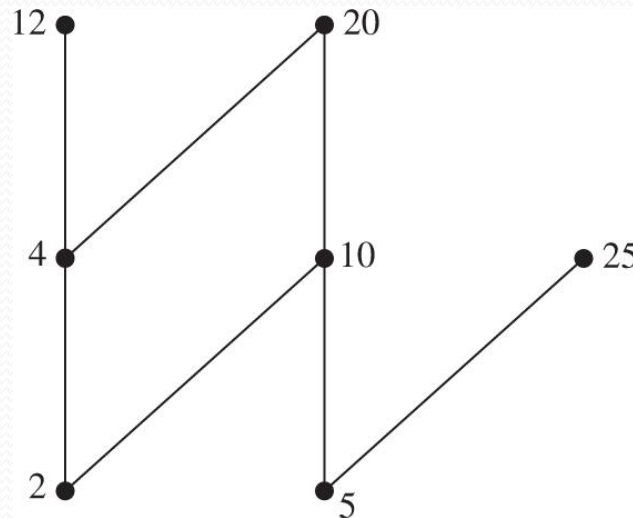
(c)

# Procedure for Constructing a Hasse Diagram

- To represent a finite poset  $(S, \preceq)$  using a Hasse diagram, start with the directed graph of the relation:
  - Remove the loops  $(a, a)$  present at every vertex due to the reflexive property.
  - Remove all edges  $(x, y)$  for which there is an element  $z \in S$  such that  $x \prec z$  and  $z \prec y$ . These are the edges that must be present due to the transitive property.
  - Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

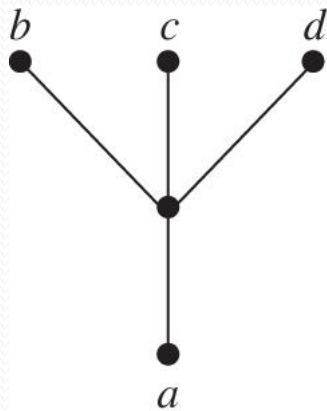
# Maximal and Minimal Elements

- For a poset  $(S, \preceq)$ ,  $a \in S$  is **maximal** if there exists no  $b \in S$  such that  $a < b$
- For a poset  $(S, \preceq)$ ,  $a \in S$  is **minimal** if there exists no  $b \in S$  such that  $b < a$
- Examples

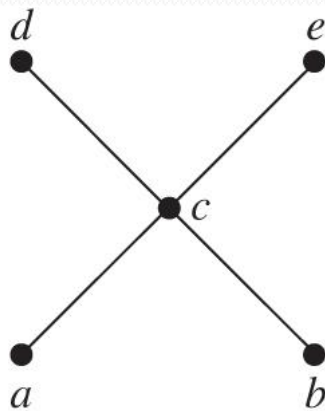


# Greatest and Least Elements

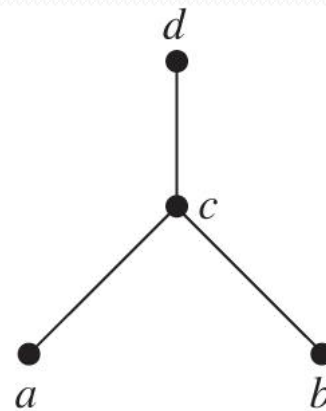
- For a poset  $(S, \preceq)$ ,  $a \in S$  is **greatest** if for all  $b \in S$ ,  $b \preceq a$
- For a poset  $(S, \preceq)$ ,  $a \in S$  is **least** if for all  $b \in S$ ,  $a \preceq b$
- The greatest/least element is unique if there exists



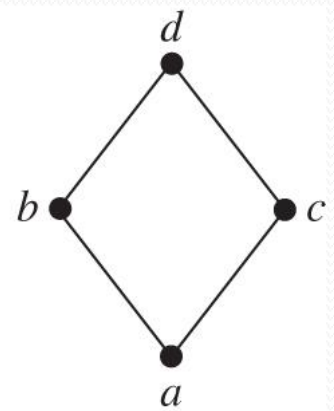
(a)



(b)



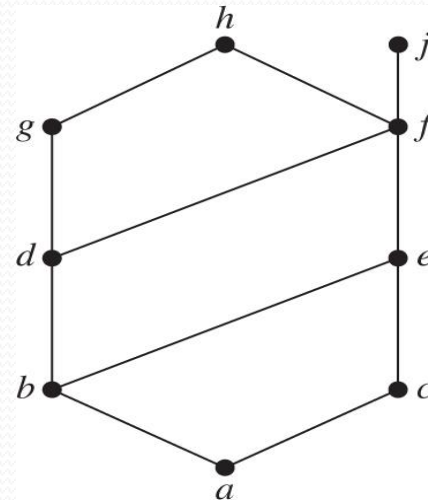
(c)



(d)

# Upper and Lower Bound of Subset

- For a poset  $(S, \preceq)$ ,  $a \in S$  is **upper bound** of elements  $S' \subseteq S$  if for all  $s' \in S'$ ,  $s' \preceq a$ 
  - there may exist multiple upper bounds for a given set of elements
- The element  $x$  is the **least upper bound** of a set of elements  $A$  (i.e.,  $\text{lub}(A)$ ) if  $x$  is the upper bound of  $A$  and is less than every other upper bound of  $A$ 
  - the least upper bound is unique if it exists for a set of elements
- Example
  - upper bounds of  $\{a, b, c\}$  ?
  - upper bound of  $\{j, h\}$ ?
  - lower bound of  $\{a, c, d, f\}$  ?





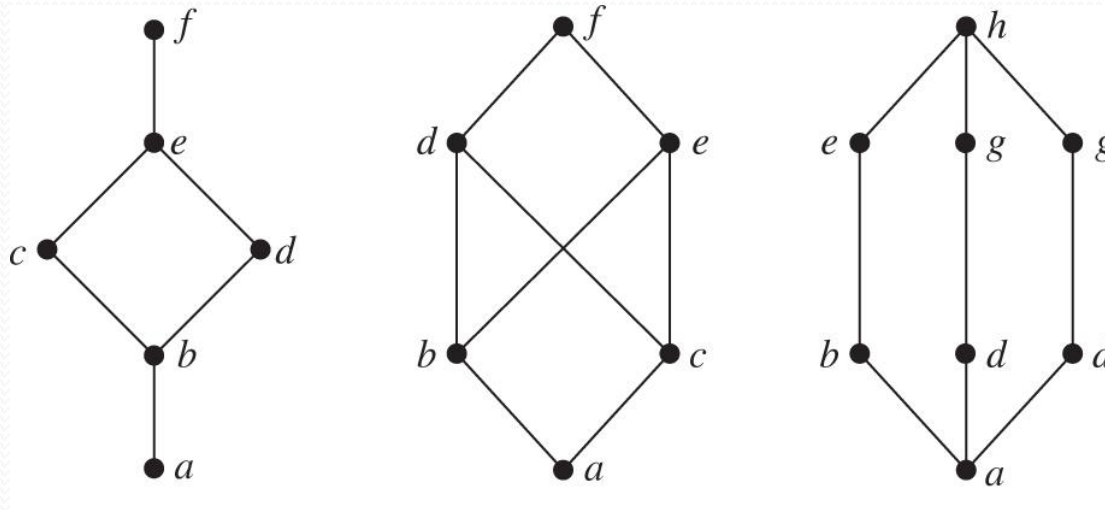
# Example

Find the greatest lower bound and the least upper bound of  $\{3, 9, 12\}$  and  $\{1, 2, 4, 5, 10\}$  if there exists, in  $(\mathbf{Z}^+, |)$

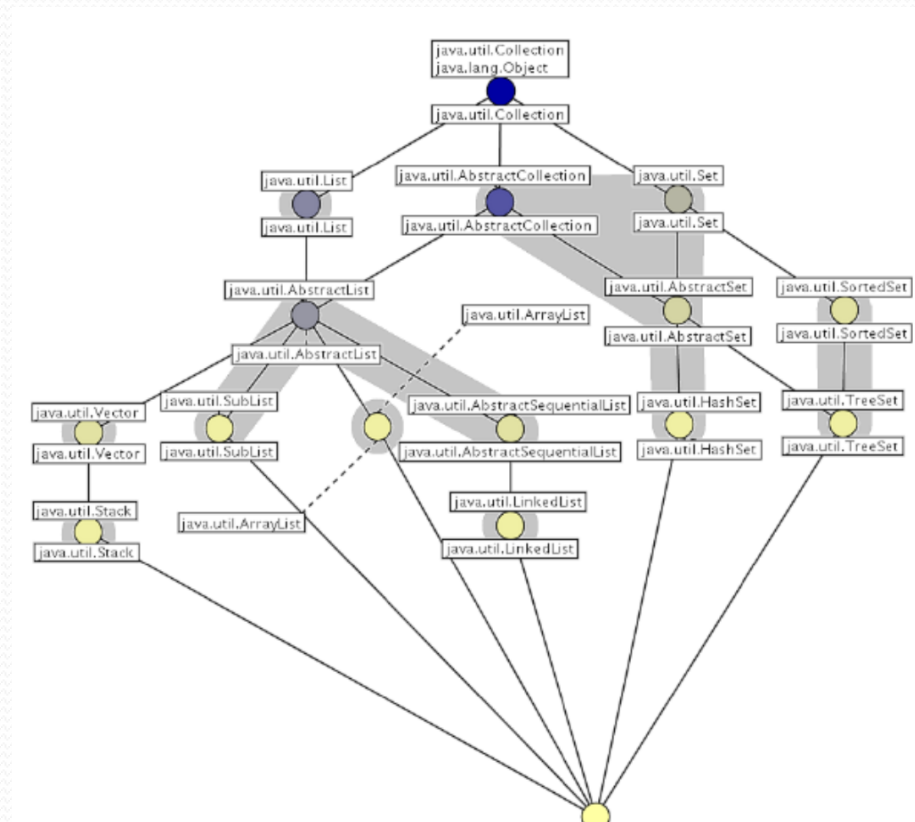
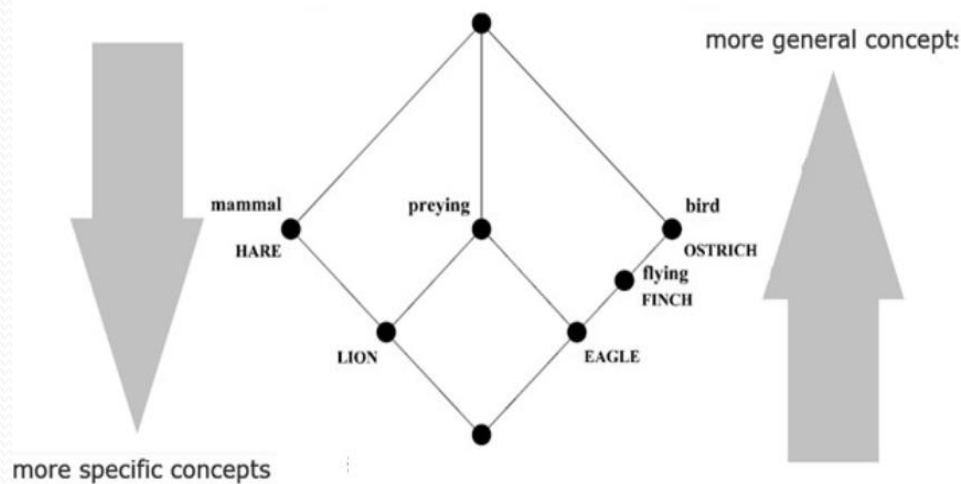
- $a \mid b$  if  $b$  is divisible by  $a$

# Lattices

- A poset is a **lattice** when there exists a least upper bound and a greatest lower bound for every pair of elements
  - used in many applications such as information flow and typing
- Examples
  - Is the poset  $(\mathbb{Z}^+, |)$  a lattice?
  - which one in the figure is a poset?



# Lattice as Knowledge Representation



# Topological Sorting

- A total ordering  $(S, \triangleleft)$  is compatible with a poset  $(S, \preceq)$  when  $a \triangleleft b$  if  $a \preceq b$
- The topological sorting is to find a compatible total ordering from a given partially ordered set
  - take a minimal element at a time until no element remains

**Input:** a poset  $(S, \preceq)$

$k \leftarrow 1$

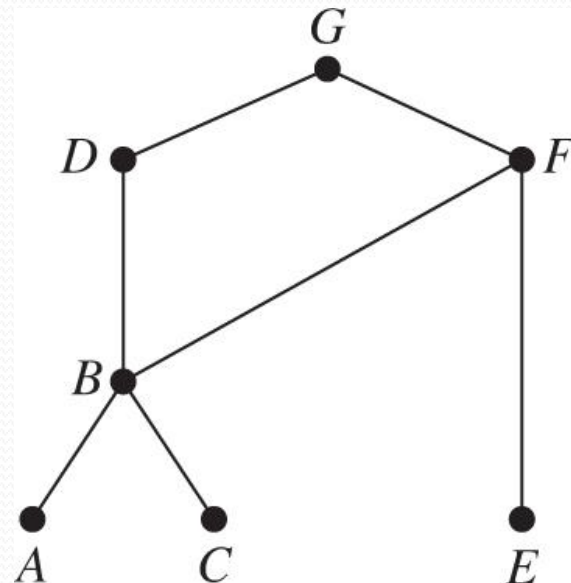
**while**  $S \neq \emptyset$

$a_k \leftarrow$  a minimal element of  $S$

$S \leftarrow S \setminus \{a_k\}$

$k \leftarrow k + 1$

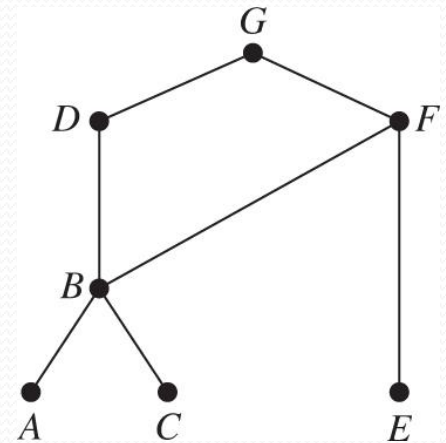
**return**  $\langle a_1, a_2, \dots, a_{|S|} \rangle$



# Example

A development project equires completion of Tasks A to G where some of these have dependencies such that some tasks can be started only after other tasks are finished.

If the dependency is given as a partial ordering among the tasks, how can we find an order in which all tasks can be carried out to complete the project?



Minimal element chosen     A	C	B	E	F	D	G