# Probability Distribution

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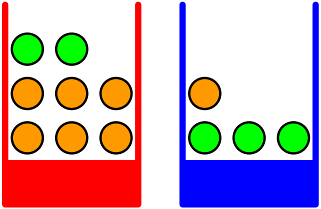
# Outline

# Simple Example

- Uncertainty is a key concept in the fields of pattern recognition and machine learning
- Probability theory provides a consistent framework for the quantification and manipulation of uncertainty and forms one of the central foundations for our study



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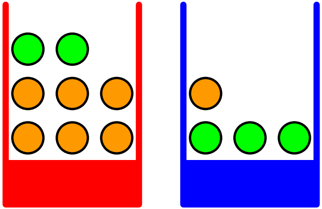

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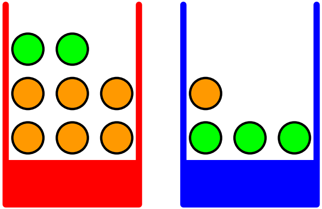

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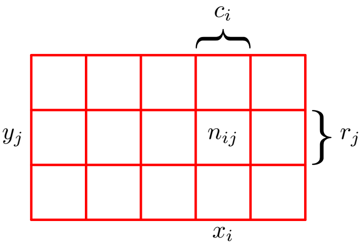
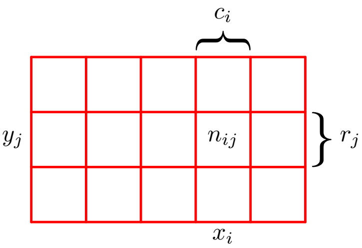
# A General Example




- two random variables $X$ and $Y$
- suppose $X$ can take any of the values $(x_i)_{i=1}^{M}$
- suppose that $Y$ can take the values $(y_j)_{j=1}^{L}$

- consider a total of N trials in which we sample both of the variables $X$ and $Y$
- let $n_{ij}$ be the number of such trials in which $X = x_i$ and $Y = y_j$
- let $r_j$ be the number of trials in which $Y = y_j$
- let $c_i$ be the number of trials in which $X = x_i$

# A General Example (cont'd)

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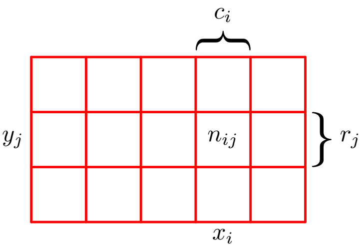


- $P(X = x_i) = c_i/N$
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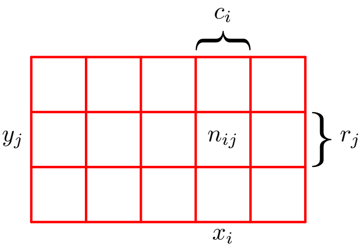


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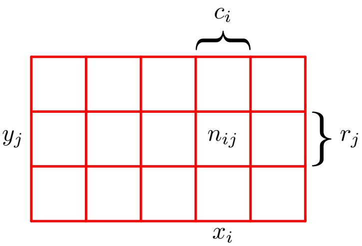


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## The rules of probability

1. sum rule: $P(X = x_i) = \sum_{j=1}^{L} P(X = x_i, Y = y_j)$ [a]
2. product rule:
   $P(X = x_i, Y = y_j) = \frac{n_{ij}}{N} = \frac{n_{ij}}{c_i} \cdot \frac{c_i}{N}$

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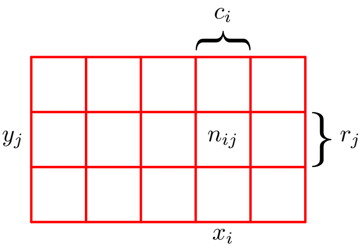


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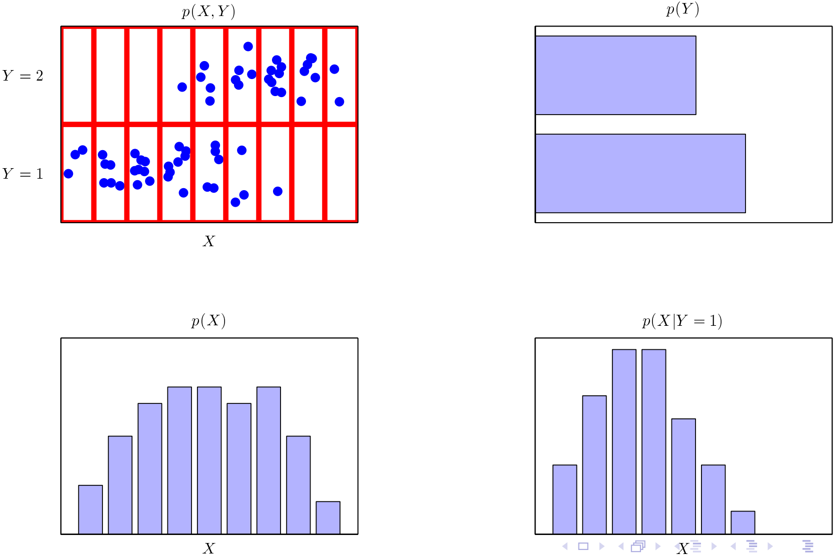
[a] $P(X = x_i)$ is sometimes called the marginal probability

[b] We can derive the Bayes's Theorem.

# An Illustration



# Example: revisit

Assume: $p(B = r) = 4/10, \;\; p(B = b) = 6/10$

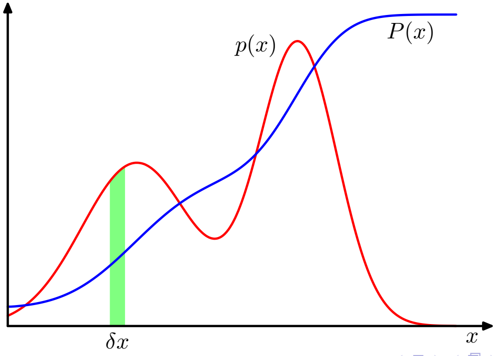
$$
\begin{aligned}
&p(F = a|B = b) = ?\\
&p(F = a) = ?\\
&p(B = r|F = o) = ?
\end{aligned}
$$

# Probability Density

Considering probabilities with respect to continuous variables

## Informal definition

If the probability of a real-valued variable $x$ falling in the interval $(x, x + \delta x)$ is given by $p(x)\delta x$ for $\delta x \to 0$, then $p(x)$ is called the probability density over $x$



# Probability Density (cont'd)

The probability that $x$ will lie in an interval $(a, b)$ is given by

$$P(x \in (a, b)) = \int_a^b p(x)dx$$

## Cumulative distribution function

The probability that $x$ lies in the interval $(-\infty, z)$ is given by

$$P(z) = \int_{-\infty}^{z} p(x)dx$$

Note that If $x$ is a discrete variable, then $p(x)$ is sometimes called a probability mass function

# Expectations

The average value of some function $f(x)$ under a probability distribution $p(x)$ is called the expectation of $f(x)$, denoted by $\mathcal{E}[f]$

## Expectation

- For a discrete distribution,

$$\mathcal{E}[f] = \sum_x p(x)f(x)$$

- For a continuous distribution,

$$\mathcal{E}[f] = \int p(x)f(x)dx$$

# Expectations (cont'd)

In both the continuous and discrete cases, if given a finite number N of points drawn from the probability distribution or probability density, then we can approximate it as a finite sum over these points

$$\mathcal{E}[f] \cong \frac{1}{N}\sum_{n=1}^{N} f(x_n)$$

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## How about expectations of functions of several variables

$\mathcal{E}_x[f(x, y)]$: the average of the function $f(x, y)$ with respect to the distribution of $x$

## Conditional expectation

$$\mathcal{E}_x[f|y] =$$

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# Variance

The variance of $f(x)$

$$var[f] = \mathcal{E}\left[(f(x) - \mathcal{E}[f(x)])^2\right] = \mathcal{E}[f(x)^2] - \mathcal{E}[f(x)]^2$$

## Covariance

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## Covariance

- For two random variables $x, y$

$$cov[x, y] = \mathcal{E}_{x,y}\left[\{x - \mathcal{E}[x]\}\right]\left[\{y - \mathcal{E}[y]\}\right] = \mathcal{E}_{x,y}[xy] - \mathcal{E}[x]\mathcal{E}[y]$$

- For two vectors of random variables $\mathbf{x}$ and $\mathbf{y}$,

$$cov[\mathbf{x}, \mathbf{y}] = \mathcal{E}_{\mathbf{x},\mathbf{y}}\left[\{\mathbf{x} - \mathcal{E}[\mathbf{x}]\}\right]\left[\{\mathbf{y}^\top - \mathcal{E}[\mathbf{y}^\top]\}\right] = \mathcal{E}_{\mathbf{x},\mathbf{y}}[\mathbf{x}\mathbf{y}^\top] - \mathcal{E}[\mathbf{x}]\mathcal{E}[\mathbf{y}^\top]$$

# Bernoulli Distribution

Consider a single binary random variable $x \in \{0, 1\})$

- The probability of $x = 1$ will be denoted by the parameter $\mu$ so that $p(x = 1|\mu) = \mu$, where $0 \leq \mu \leq 1$
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- easily it follows that $p(x = 0|\mu) = 1 - \mu$
- the probability distribution over $x$ can be written in the form

$$Bern(x|\mu) = \mu^x(1 - \mu)^{1-x}$$

- easily to verify that this distribution is normalized and that it has mean and variance given by

$$\mathcal{E}[x] = \mu$$

$$var[x] = \mu(1 - \mu)$$

# Bernoulli Distribution (cont'd)

Suppose we have a dataset $\mathcal{D} = \{x_1, \cdots, x_N\}$ of observed values of $x$

## Likelihood function

$$p(\mathcal{D}|\mu) =$$

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# Overfitting of ML

If we denote the number of observations of $x = 1$ (heads) within this data set by $m$, then

$$\mu_{ML} = \frac{1}{N}\sum_{n=1}^{N} x_n = \frac{m}{N}$$

Example: Suppose now flip the coin 5 (N=5) times, and happen to observe 5 (m=5) heads. Then, $\mu_{ML} = 1$. What does it mean?

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The ML solution would predict that all future observations should give heads.

# Binomial Distribution

Consider an extreme example of the over-fitting associated with maximum likelihood

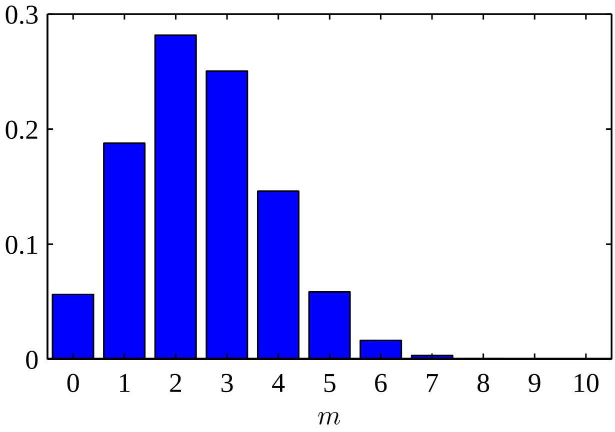
- binomial distribution: the distribution of the number $m$ of observations of $x = 1$, given that the dataset has size $N$: $\mu^m(1-\mu)^{N-m}$
- If we work the distribution of the number $m$ of observations of $x = 1$ given that the dataset has size $N$, we can obtain the binomial distribution

$$\mathrm{Bin}(m|N,\mu) = \underbrace{\begin{pmatrix} N \\ m \end{pmatrix}}_{\frac{N!}{(N-m)!m!}}{}^{a} \mu^m(1-\mu)^{N-m}$$

[a] The number of ways of choosing $m$ objects out of a total of $N$ identical objects.

# Binomial Distribution (cont'd)

Histogram plot of the binomial distribution as a function of $m$ for $N = 10$ and $\mu = 0.25$



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We therefore choose a prior, called the beta distribution, given by

$$\mathrm{Beta}(\mu|a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\mu^{a-1}(1-\mu)^{b-1}$$

# The Beta Distribution (cont'd)

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$$\int_0^1 \mathrm{Beta}(\mu|a, b) d\mu = 1$$

The mean and variance of the beta distribution are given by

$$\mathcal{E}[\mu] = \frac{a}{a+b}$$

$$\mathrm{var}[\mu] = \frac{ab}{(a+b)^2(a+b+1)}$$

The parameters $a$ and $b$ are often called hyperparameters

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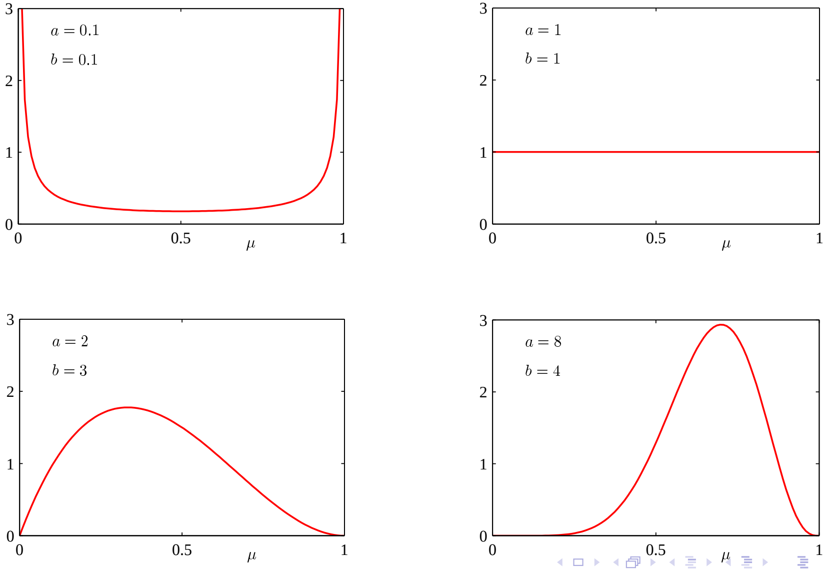

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Keeping only the factors that depend on $\mu$, this posterior distribution has the form

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where $l = N - m$

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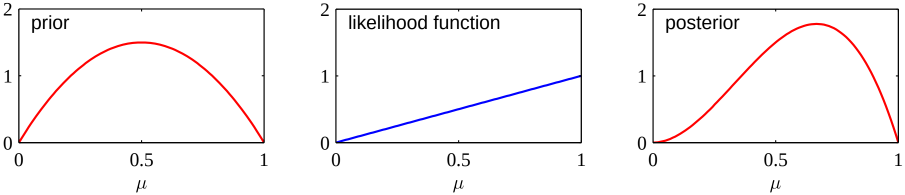
$$p(\mu|m, l, a, b) = \frac{\Gamma(m+a+l+b)}{\Gamma(m+a)\Gamma(l+b)}\mu^{m+a-1}(1-\mu)^{l+b-1}$$

# Illustration

The prior is given by a beta distribution with parameters $a = 2, b = 2$, and the likelihood function, given by binomial distribution with $N = m = 1$, corresponds to a single observation of $x = 1$




We can see that the posterior is given by a beta distribution with parameters $a = 3, b = 2$

> We can interpret $a, b$ in the prior as an effective number of observations of $x = 1$ and $x = 0$, respectively

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$$\begin{aligned} P(x=1|\mathcal{D}) &= \int_0^1 p(x=1|\mu)p(\mu|\mathcal{D})d\mu \\ &= \int_0^1 \mu p(\mu|\mathcal{D})d\mu \\ &= \end{aligned}$$

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# Introduction

- Consider a discrete variable that can take one of possible $K$ values
- Convenient representation with a vector where one element equals 1, others 0, e.g., $\mathbf{x} = (0, 0, 1, 0, 0, 0)^\top$
- If denoting the probability of $x_k = 1$ by the parameter $\mu_k$, then the distribution of $\mathbf{x}$ is given

$$p(\mathbf{x}|\boldsymbol{\mu}) = \prod_{k=1}^{K} \mu_k^{x_k}$$

where $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_K)^\top$, s.t., $\mu_k \geq 0$ and $\sum_k \mu_k = 1$

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$$\sum_{\mathbf{x}} p(\mathbf{x}|\boldsymbol{\mu}) = \sum_{k=1}^{K} \mu_k = 1$$

and that

$$E[\mathbf{x}|\boldsymbol{\mu}] = \sum_{\mathbf{x}} p(\mathbf{x}|\boldsymbol{\mu})\mathbf{x} = (\mu_1, \cdots, \mu_K)^\top = \boldsymbol{\mu}$$

- the likelihood function

$$p(\mathcal{D}|\boldsymbol{\mu}) = \prod_{n=1}^{N} \prod_{k=1}^{K} \mu_k^{x_{nk}} =$$

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- the likelihood function

$$p(\mathcal{D}|\boldsymbol{\mu}) = \prod_{n=1}^{N}\prod_{k=1}^{K} \mu_k^{x_{nk}} = \prod_{k=1}^{K} \mu_k^{(\sum_n x_{nk})} = \prod_{k=1}^{K} \mu_k^{m_k}\ ^{a}$$

[a] The number of observations of $x_k = 1$, and $m_k = \sum_n x_{nk}$

# Maximum Likelihood Estimator

By a Lagrange multiplier $\lambda$ and maximizing

$$\sum_{k=1}^{K} m_k \ln \mu_k + \lambda \left( \sum_{k=1}^{K} \mu_k - 1 \right)$$

$$\Rightarrow \mu_k^{\mathrm{ML}} = \frac{m_k}{N}$$

# Multinomial Distribution

Consider the joint distribution of the quantities $m_1, \cdots, m_K$, the multinomial distribution takes the form

$$\mathrm{Mult}(m_1, \cdots, m_K | \boldsymbol{\mu}, N) = \underbrace{\begin{pmatrix} N \\ m_1 m_2 \cdots m_K \end{pmatrix}}_{\frac{N!}{m_1! m_2! \cdots m_K!}} \prod_{k=1}^{K} \mu^{m_k}$$

The variables $m_k$ are subject to the constraint $\sum_{k=1}^{K} m_k = N$

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$$p(\boldsymbol{\mu}|\boldsymbol{\alpha}) \propto \prod_{k=1}^{K} \mu_k^{\alpha_k - 1}, \quad \text{s.t.} \begin{cases} 0 \leq \mu_k \leq 1 \\ \sum_k \mu_k = 1 \end{cases}$$

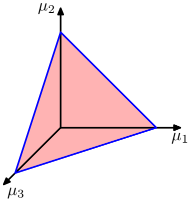
- The normalized form the distribution by

$$\mathrm{Dir}(\boldsymbol{\mu}|\boldsymbol{\alpha}) = \frac{\Gamma(\alpha_0 {}^{a})}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_K)} \prod_{k=1}^{K} \mu_k^{\alpha_k - 1}$$

This is called the Dirichlet distribution.

[a] $\alpha_0 = \sum_{k=1}^{K} \alpha_k$

# Dirichlet Distribution (cont'd)

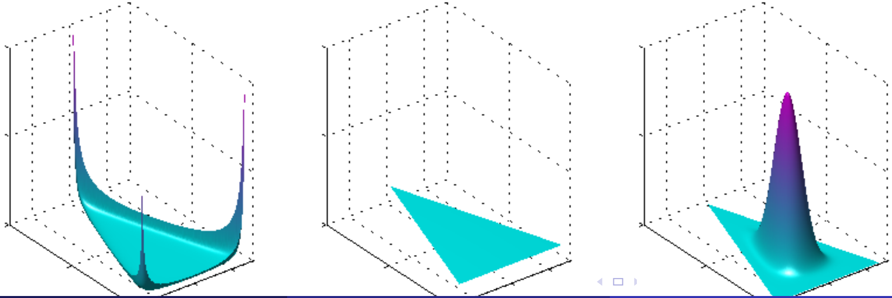



$\Leftarrow$ The domain of the Dirichlet distribution with $K = 3$

Plots of the Dirichlet distribution ($K = 3$)

$\Downarrow$

$\alpha_k = 0.1$ $\alpha_k = 1$ $\alpha_k = 10$



# Single Variable Gaussian

## For a single variable $x$

$$\mathbf{N}(x|\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

where $\mu$ is the mean and $\sigma^2$ is the variance

# Multivariable Gaussian

For a $d$-dimensional vector $\mathbf{x}$

$$\mathbf{N}(x|\mu,\Sigma) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mu)^\top \Sigma^{-1}(\mathbf{x}-\mu)\right\}$$

where $\mu$ is a $d$-dimensional mean vector and $\Sigma$ is a $d \times d$ covariance matrix, and $|\Sigma|$ denotes the determinant of $\Sigma$

# Geometrical Form

## Mahalanobis distance

The functional dependence of the Gaussian on $\mathbf{x}$ is through the quadratic form

$$\Delta^2 = (\mathbf{x} - \boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})$$

The quantity $\Delta$ is called the Mahalanobis distance from $\boldsymbol{\mu}$ to $\mathbf{x}$ and

# Geometrical Form

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The quantity $\Delta$ is called the Mahalanobis distance from $\boldsymbol{\mu}$ to **x** and reduces to the Euclidean distance when $\Sigma$ is the identity matrix

Consider the eigenvector equation for the covariance matrix

$$\Sigma \boldsymbol{\mu}_i = \lambda_i \boldsymbol{\mu}_i$$

Since $\Sigma$ is a real, symmetric matrix, its eigenvalues will be real, and its eigenvectors can be chosen to form an orthonormal set, so that,

$$\boldsymbol{\mu}_i^\top \boldsymbol{\mu}_j = \mathrm{I}_{ij}$$

# Geometrical Form (cont'd)

The covariance matrix can be expressed as an expansion in terms of its eigenvectors in the form

$$\Sigma = \sum_{i=1}^{d} \lambda_i \boldsymbol{\mu}_i \boldsymbol{\mu}_j^\top$$

and similarly the inverse covariance matrix $\Sigma^{-1}$ can be expressed as

# Geometrical Form (cont'd)

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and similarly the inverse covariance matrix $\Sigma^{-1}$ can be expressed as

$$\Sigma^{-1} = \sum_{i=1}^{d} \frac{1}{\lambda_i} \boldsymbol{\mu}_i \boldsymbol{\mu}_j^\top$$

# Geometrical Form (cont'd)

$$\Sigma^{-1} = \sum_{i=1}^{d} \frac{1}{\lambda_i} \boldsymbol{\mu}_i \boldsymbol{\mu}_j^\top \text{ and } \Delta^2 = (\mathbf{x} - \boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}),$$

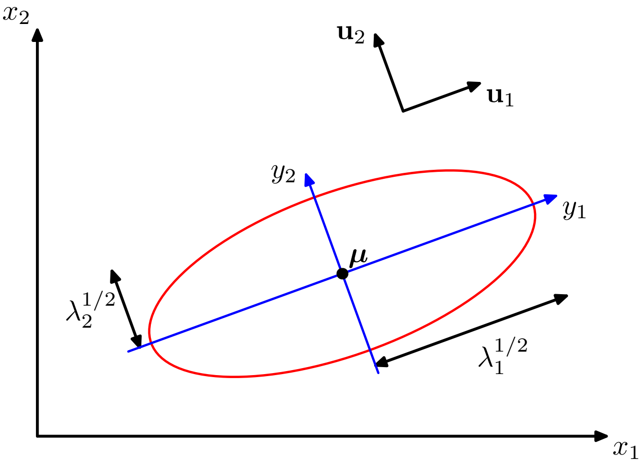
therefore we have,

$$\Delta^2 = \sum_{i=1}^{d} \frac{y_i^2}{\lambda_i}, \; y_i = \boldsymbol{\mu}_i^\top (\mathbf{x} - \boldsymbol{\mu})$$

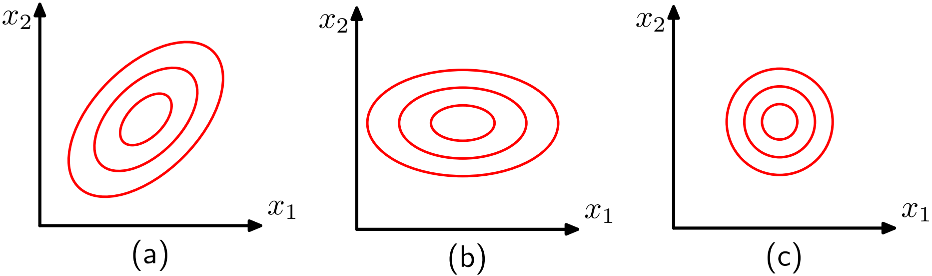
- We can interpret $\{y_i\}$ as a new coordinate system defined by the orthonormal vectors $\boldsymbol{\mu}_i$ that are shifted and rotated with respect to the original $x_i$ coordinates
- Forming the vector $\mathbf{y} = (y_1, \cdots, y_d)^\top$, we have

$$\mathbf{y} = \mathbf{U}(\mathbf{x} - \boldsymbol{\mu})$$

# Geometrical Form (cont'd)



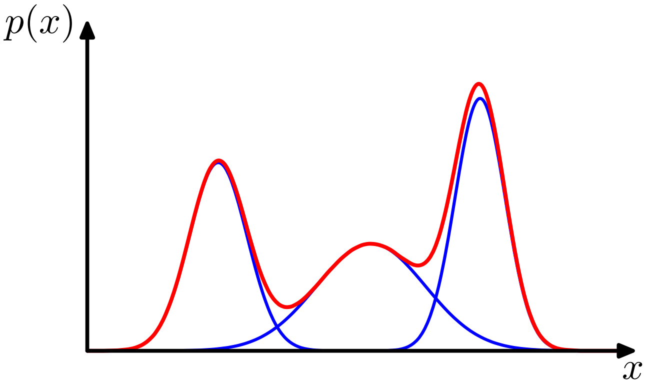
# One of Limitations of Gaussian Distribution




| $\Sigma$ | $\Sigma = \mathrm{diag}(\sigma_i^2)$ | $\Sigma = \sigma^2 \mathbf{I}$ |
|---|---|---|
| $D(D+3)/2$ | $2D$ | $D+1$ |

# Mixture of Gaussians

Another limitations of Gaussian distribution is that it is uni-modal
The superpositions, formed by taking linear combinations of more basic distributions, can be formulated as probabilistic models known as mixture distribution

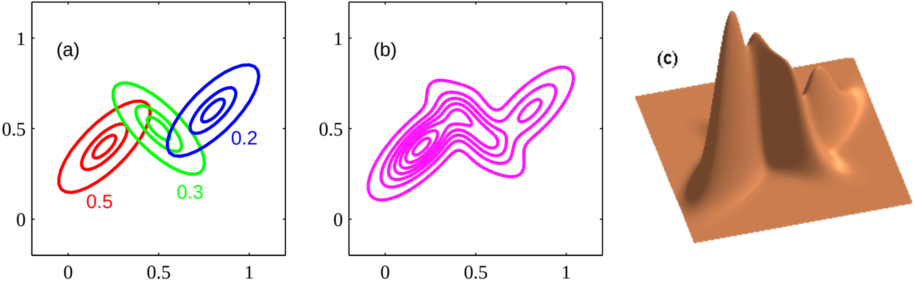


# Mixture of Gaussians (cont'd)

Consider a superposition of $K$ Gaussian densities of the form

$$p(x) = \sum_{k=1}^{K} \pi_k \mathbf{N}(\mathbf{x}|\boldsymbol{\mu}_k, \Sigma_k)$$

which is called a mixture of Gaussians



# General Form

The exponential family of distributions over $\mathbf{x}$, given parameters $\boldsymbol{\eta}$, is defined to be the set of distributions of the form

$$p(\mathbf{x}|\boldsymbol{\eta}) = h(\mathbf{x})g(\boldsymbol{\eta})\exp\left\{\boldsymbol{\eta}^\top \mathbf{u}(\mathbf{x})\right\}$$

- $\boldsymbol{\eta}$ are called the natural parameters of the distribution, and $\mathbf{u}(\mathbf{x})$ is some function of $\mathbf{x}$
- the function $g(\boldsymbol{\eta})$ can be interpreted as the coefficient that the distribution is normalized and therefore satisfies

$$g(\boldsymbol{\eta})\int h(\mathbf{x})\exp\left\{\boldsymbol{\eta}^\top \mathbf{u}(\mathbf{x})\right\} d\mathbf{x} = 1$$

# Bernoulli Distribution

$$p(x|\mu) = \mathrm{Bern}(x|\mu) = \mu^x(1-\mu)^{1-x}$$

Expressing the right-hand side as the exponential of the logarithm, we have

$$p(x|\mu) \quad = \exp\left\{x \ln \mu + (1-x)\ln(1-\mu)\right\}$$

[1] this is called the logistic sigmoid function

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$$\begin{cases} \eta = \ln\left(\frac{\mu}{1-\mu}\right) \\ \sigma(\boldsymbol{\eta}) = \frac{1}{1+\exp(-\eta)} \end{cases}$$ [1]

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# Bernoulli Distribution (cont'd)

$$p(x|\mu) = \sigma(-\eta)\exp(\eta x)$$

# Bernoulli Distribution (cont'd)

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$$\begin{aligned} u(x) &= x \\ h(x) &= 1 \\ g(\eta) &= \sigma(-\eta) \end{aligned}$$

# Multinomial Distribution

$$p(\mathbf{x}|\boldsymbol{\mu}) = \prod_{k=1}^{M}(\mu_k^{x_k}) = \exp\left\{\sum_{k=1}^{M} x_k \ln \mu_k\right\}$$

where $\mathbf{x} = (x_1, \cdots, x_N)^\top$

We can write this in the standard representation so that

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# Multinomial Distribution (cont'd)

$\mu_k - \eta_k$?

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- By expressing it in terms of the remaining $\{\mu_k, k = 1, \cdots, M-1\}$, there remaining parameters are still s.t. the constraints

$$0 \leq \mu_k \leq 1, \quad \sum_{k=1}^{M-1} \mu_k \leq 1$$

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$$\exp\left\{\sum_{k=1}^{M} x_k \ln \mu_k\right\}$$

# Multinomial Distribution (cont'd)

$$
\begin{aligned}
& \exp\left\{\sum_{k=1}^{M} x_k \ln \mu_k\right\} \\
&= \exp\left\{\sum_{k=1}^{M-1} x_k \ln \mu_k + \left(1 - \sum_{k=1}^{M-1} x_k\right) \ln\left(1 - \sum_{k=1}^{M-1} \mu_k\right)\right\} \\
&= \exp\left\{\sum_{k=1}^{M-1} x_k \ln\left(\frac{\mu_k}{1 - \sum_{j=1}^{M-1} \mu_j}\right) + \ln\left(1 - \sum_{k=1}^{M-1} \mu_k\right)\right\}
\end{aligned}
$$

---

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# Multinomial Distribution (cont'd)

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We can identify

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\ln\left(\frac{\mu_k}{1 - \sum_{j=1}^{M-1} \mu_j}\right) = \eta_k
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&= \exp\left\{\sum_{k=1}^{M-1} x_k \ln\left(\frac{\mu_k}{1 - \sum_{j=1}^{M-1} \mu_j}\right) + \ln\left(1 - \sum_{k=1}^{M-1} \mu_k\right)\right\}
\end{aligned}
$$

We can identify

$$
\ln\left(\frac{\mu_k}{1 - \sum_{j=1}^{M-1} \mu_j}\right) = \eta_k \Rightarrow \mu_k = \frac{\exp(\eta_k)}{1 - \sum_j \exp(\eta_j)} \quad [2]
$$

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# Multinomial Distribution (cont'd)

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$$p(\mathbf{x}|\boldsymbol{\eta}) =$$

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$$p(\mathbf{x}|\boldsymbol{\eta}) = \left(1 + \sum_{k=1}^{M-1} \exp(\eta_k)\right)^{-1} \exp(\boldsymbol{\eta}^\top \mathbf{x})$$

This is the standard form of the exponential family, with parameter vector $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_{M-1})^\top$ in which

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This is the standard form of the exponential family, with parameter vector $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_{M-1})^\top$ in which

$$\begin{aligned}
&\mathbf{u}(\mathbf{x}) = \mathbf{x} \\
&h(\mathbf{x}) = 1 \\
&g(\boldsymbol{\eta}) = \left(1 + \textstyle\sum_{k=1}^{M-1} \exp(\eta_k)\right)^{-1}
\end{aligned}$$

# Gaussian Distribution

## For the univariate Gaussian

$$p(x|\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

$$=$$

# Gaussian Distribution

## For the univariate Gaussian

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\begin{aligned}
p(x|\mu,\sigma^2) &= \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\} \\
&= \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x - \frac{1}{2\sigma^2}\mu^2\right\}
\end{aligned}
$$

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$$

$$
\boldsymbol{\eta} = \begin{pmatrix} \mu/\sigma^2 \\ -1/2\sigma^2 \end{pmatrix}
$$

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&= \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x - \frac{1}{2\sigma^2}\mu^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\eta} &= \begin{pmatrix} \mu/\sigma^2 \\ -1/2\sigma^2 \end{pmatrix} \\
\mathbf{u}(\mathbf{x}) &= \begin{pmatrix} x \\ x^2 \end{pmatrix} \\
h(\mathbf{x}) &= (2\pi)^{-1/2} \\
g(\boldsymbol{\eta}) &= (-2\eta_2)^{1/2} \exp\left(\frac{\eta_1^2}{4\eta_2}\right)
\end{aligned}
$$

# Maximum Likelihood

- estimating the parameter vector $\boldsymbol{\eta}$ in the general exponential family distribution
- if using the ML technique, we can take the gradient of $g(\boldsymbol{\eta}) \int h(\mathbf{x}) \exp\left\{\boldsymbol{\eta}^\top \mathbf{u}(\mathbf{x})\right\} d\mathbf{x} = 1$,

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$$
\begin{aligned}
&\nabla g(\boldsymbol{\eta}) \int h(\mathbf{x}) \exp\left\{\boldsymbol{\eta}^\top \mathbf{u}(\mathbf{x})\right\} d\mathbf{x} \\
&+ g(\boldsymbol{\eta}) \int h(\mathbf{x}) \exp\left\{\boldsymbol{\eta}^\top \mathbf{u}(\mathbf{x})\right\} \mathbf{u}(\mathbf{x}) d\mathbf{x} = 0 \\
&\Rightarrow -\frac{1}{g(\boldsymbol{\eta})} \nabla g(\boldsymbol{\eta}) = g(\boldsymbol{\eta}) \int h(\mathbf{x}) \exp\left\{\boldsymbol{\eta}^\top \mathbf{u}(\mathbf{x})\right\} \mathbf{u}(\mathbf{x}) d\mathbf{x}
\end{aligned}
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$$\begin{aligned}
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&+ g(\boldsymbol{\eta}) \int h(\mathbf{x}) \exp\left\{\boldsymbol{\eta}^\top \mathbf{u}(\mathbf{x})\right\} \mathbf{u}(\mathbf{x}) d\mathbf{x} = 0 \\
&\Rightarrow -\frac{1}{g(\boldsymbol{\eta})} \triangledown g(\boldsymbol{\eta}) = g(\boldsymbol{\eta}) \int h(\mathbf{x}) \exp\left\{\boldsymbol{\eta}^\top \mathbf{u}(\mathbf{x})\right\} \mathbf{u}(\mathbf{x}) d\mathbf{x} \\
&= \mathcal{E}[\mathbf{u}(\mathbf{x})]
\end{aligned}$$

# Sufficient Statistics

Considering a set of i.i.d. data denoted by $\mathbf{X} = (\mathbf{x}_n)_{n=1}^N$, for which the likelihood function is given by

$$p(\mathbf{X}|\eta) = \left(\prod_{n=1}^{N} h(\mathbf{x}_n)\right) g(\eta)^N \exp\left\{\eta^\top \sum_{n=1}^{N} \mathbf{u}(\mathbf{x}_n)\right\}$$

$$\Rightarrow -\bigtriangledown \ln g(\eta_{\text{ML}}) = \frac{1}{N}\sum_{n=1}^{N} \mathbf{u}(\mathbf{x}_n)$$

- The solution for the MLE dependends on the data only through $\sum_n \mathbf{u}(\mathbf{x}_n)$
- this is called the sufficient statistic of the distribution $h(\mathbf{x})g(\eta)\exp\left\{\eta^\top \mathbf{u}(\mathbf{x})\right\}$
- we donot need to store the entire dataset itself but the value of the sufficient statistic

# Sufficient Statistics (cont'd)

- for Bernoulli, Multinomial distribution, $\mathbf{u}(\mathbf{x}) = \mathbf{x}$, and so we only keep $\sum_n \mathbf{x}_n$
- for Gaussian, $\mathbf{u}(\mathbf{x}) = (\mathbf{x}, \mathbf{x}^2)^\top$, we should keep only $\sum_n \mathbf{x}_n$ and $\sum_n \mathbf{x}_n^2$

[3]

[3] Common distributions and the corresponding sufficient statistics are listed in PP. 108-109 of Pattern Classification

# Conjugate Priors

- In general, for a given probability distribution $p(\mathbf{x}|\boldsymbol{\eta})$, we can seek a prior $p(\boldsymbol{\eta})$ that is conjugate to the likelihood function
- so the posterior distribution has the same functional form as the prior
- for any member of the exponential family, there exists a conjugate prior in the form

$$p(\boldsymbol{\eta}|\boldsymbol{\chi}, \nu) = f(\boldsymbol{\chi}, \nu)g(\boldsymbol{\eta})^{\nu} \exp\{\nu\boldsymbol{\eta}^{\top}\boldsymbol{\chi}\}$$

where $f(\boldsymbol{\chi}, \nu)$ is a normalization coefficient, and $g(\boldsymbol{\eta})$ is the same function in the exponential family

$$p(\boldsymbol{\eta}|\mathbf{X}, \boldsymbol{\chi}, \nu) \propto g(\boldsymbol{\eta})^{\nu+N} \exp\left\{\boldsymbol{\eta}^{\top}\left(\sum_{n=1}^{N}\mathbf{u}(\mathbf{x}_n) + \nu\boldsymbol{\chi}\right)\right\}$$