

# LECTURES

Bitcoin Advanced Course  
Monday morning session  
22.11.2021

# Objective

Understand from first principles how bitcoins are locked to keypairs

## **Digital signatures: ECDSA & Schnorr**

modular arithmetic, finite fields, elliptic curves,  
discrete log-problem, public key cryptography

Trigger warning: Math

# Signatures in Bitcoin: Basic requirements

When Alice's coins are spent, we require some proof  $w$  that:

- Can be produced by Alice
- Can't be produced (efficiently) by anybody else
- Can be verified by everybody
- Commits to the details of the transaction

# Roadmap

## Algebra

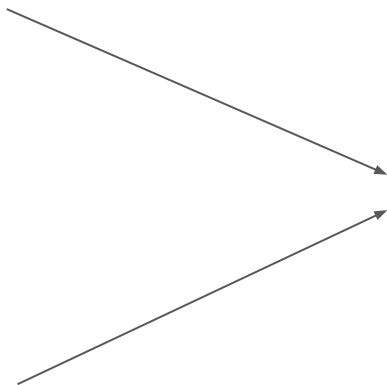
- Groups
- Modular arithmetic
- (Finite) fields
- Discrete log

## Elliptic curves

- EC math

## ECs over finite fields

- Schnorr
- ECDSA



# Algebraic groups

A group is a set  $G$  together with an operation  $*$  such that:

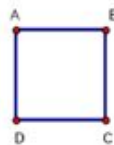
- $G$  is "closed" under  $*$ :  $a*b$  is in  $G$
- $*$  is associative:  $(a*b)*c = a*(b*c)$
- There exists an identity element  $e$ :  $e*a = a*e = a$  (Blackboard: prove uniqueness)
- Every element  $a$  has an inverse  $b$ :  $a*b = b*a = e$

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

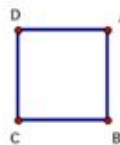
+

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{N}; b \neq 0 \right\}, *, *$$

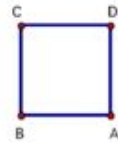
E = identity  
(do nothing)



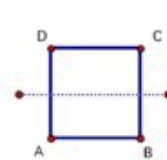
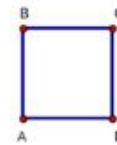
rotate 90  
degrees



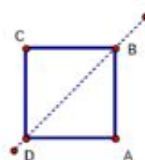
rotate 180  
degrees



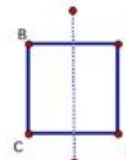
rotate 270  
degrees



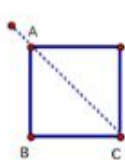
M1 reflection



M2 reflection

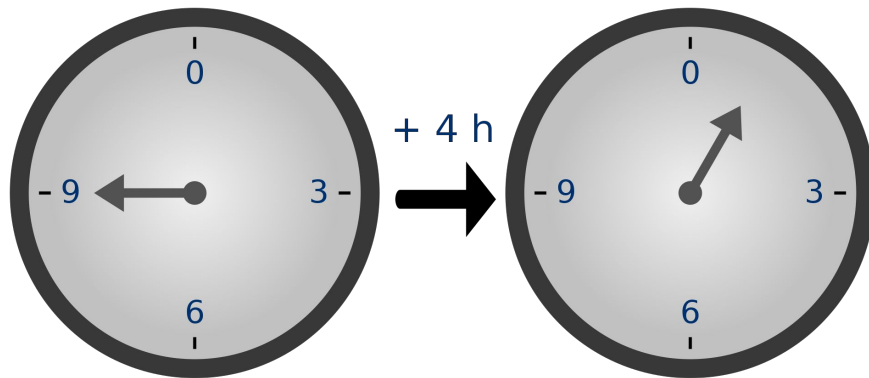


M3 reflection



M4 reflection

# Constructing finite groups (why?): Modular arithmetic



$$13 \equiv 1 \pmod{12}$$

$$a \equiv b \pmod{n} \Rightarrow \exists k : a = k * n + b$$

Blackboard: some examples and basic properties

## Two basic examples of finite modular groups

$\mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, \dots, n-1\}$ , with addition

$(\mathbb{Z}/n\mathbb{Z})^\times = \{1, 2, \dots, n-1 \mid \text{coprime to } n\}$ , with multiplication

Why coprime?  
Multiplicative inverse  
a.k.a. “modular division”

Multiplication Mod 6

*	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

# Two more salient features of finite prime groups

- They are cyclic, i.e. generator  $a$  generates the group:  $\{a, a^2, a^3, \dots\} = G$ 
  - Every non-identity member is a generator
  - Blackboard: Example
- Computing the multiplicative inverse is easy: Fermat's Little Theorem:
  - Blackboard: Examples and proof

$$a^{p-1} \equiv 1 \pmod{p} \Rightarrow a \cdot a^{p-2} \equiv 1 \pmod{p}$$



# Fields: Introducing a second group operation

- Addition, Multiplication
    - Associative, commutative, have identities
    - Distributive
  - Every element has inverses, except the multiplicative inverse for the additive identity element
  - Fields induce two groups
- 
- Example: rational numbers

$$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \text{GF}(p) = \{0, 1, \dots, p-1\}$$

In a very well-defined sense, this is the *only* field with  $p$  elements

# The discrete logarithm problem for finite cyclic groups

- In cyclic groups every element can be obtained by applying the group operation on the generator a certain number of times
  - Additive groups:  $b = g * n \pmod{p}$
  - Multiplicative groups:  $b = g^n \pmod{p}$
- Discrete log problem: Invert
  - $n = b / g \pmod{p}$
  - $n = \log(b) \text{ (base } g \text{) (naming!)}$
- Not all discrete log problems are hard, depends on the group and operation
  - Judge the above over  $\text{GF}(p)$
- “Problem”: Still too easy

Up next: An interesting candidate group

# Roadmap

## Algebra

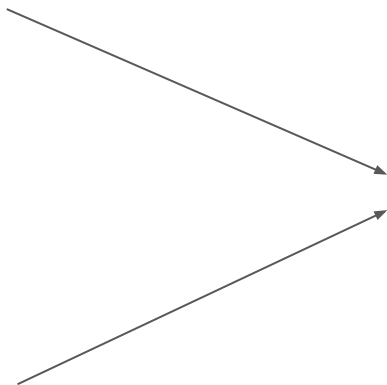
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# Elliptic curves

- Not ellipses
  - connected to finding arc lengths on ellipses
- Cubic equations
  - With some requirements on A, B

$$y^2 = x^3 + Ax + B$$

- Group structure
  - A general line intersects a cubic at three points

