

21Lectures

Bitcoin Masterclass
Modular Arithmetic
Day 1

Objective

Understand from first principles how bitcoins are locked to keypairs

Digital signatures: ECDSA & Schnorr

modular arithmetic, finite fields, elliptic curves,
discrete log-problem, public key cryptography

Trigger warning: Math

Signatures in Bitcoin: Basic requirements

When Alice's coins are spent, we require some proof w that:

- Can be produced by Alice
- Can't be produced (efficiently) by anybody else
- Can be verified by everybody
- Commits to the details of the transaction

Roadmap

Algebra

- Groups
- Modular arithmetic
- (Finite) fields
- Discrete log

Elliptic curves

- EC math

ECs over finite fields

- Schnorr
- ECDSA

Algebraic groups

A group is a set G together with an operation $*$ such that:

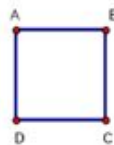
- G is "closed" under $*$: $a*b$ is in G
- $*$ is associative: $(a*b)*c = a*(b*c)$
- There exists an identity element e : $e*a = a*e = a$ (Blackboard: prove uniqueness)
- Every element a has an inverse b : $a*b = b*a = e$

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

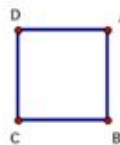
+

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{N}; b \neq 0 \right\}, *, *$$

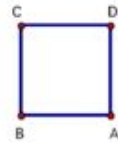
E = identity
(do nothing)



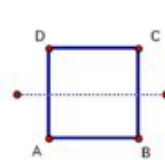
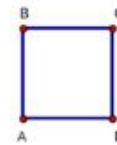
rotate 90
degrees



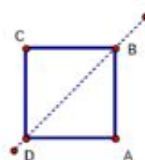
rotate 180
degrees



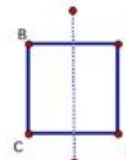
rotate 270
degrees



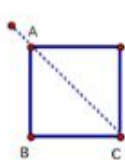
M1 reflection



M2 reflection

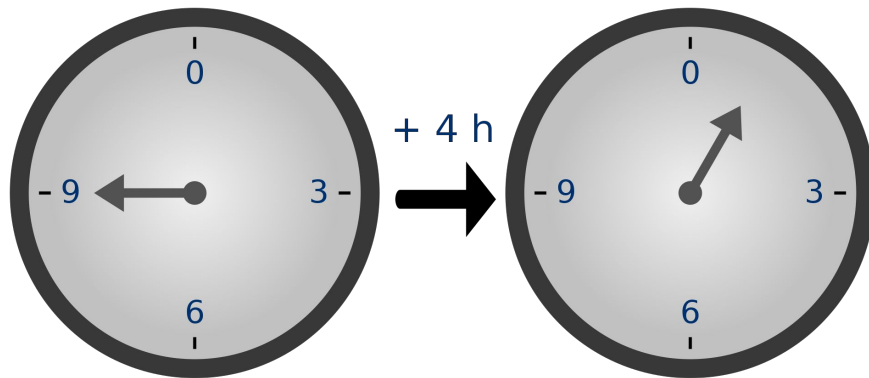


M3 reflection



M4 reflection

Constructing finite groups (why?): Modular arithmetic



$$13 \equiv 1 \pmod{12}$$

$$a \equiv b \pmod{n} \Rightarrow \exists k : a = k * n + b$$

Blackboard: some examples and basic properties

Two basic examples of finite modular groups

$\mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, \dots, n-1\}$, with addition

$(\mathbb{Z}/n\mathbb{Z})^\times = \{1, 2, \dots, n-1 \mid \text{coprime to } n\}$, with multiplication

Why coprime?
Multiplicative inverse
a.k.a. “modular division”

Multiplication Mod 6

*	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

Two more salient features of finite prime groups

- They are cyclic, i.e. generator a generates the group: $\{a, a^2, a^3, \dots\} = G$
 - Blackboard: Example
- Computing the multiplicative inverse is easy (important for later!):
 - Fermat's Little Theorem (blackboard examples):

$$a^{p-1} \equiv 1 \pmod{p} \Rightarrow a \cdot a^{p-2} \equiv 1 \pmod{p}$$

Fields: Introducing a second group operation

- Addition, Multiplication
 - Associative, commutative, have identities
 - Distributive
- Every element has inverses, except the multiplicative inverse for the additive identity element
- Fields induce two groups
- Blackboard example: rational numbers

“Galois Field”:

$$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \text{GF}(p) = \{0, 1, \dots, p-1\}$$

In a well-defined sense, this is the *only* field with p elements (“isomorphism”)

The discrete logarithm problem for finite cyclic groups

- In cyclic groups every element can be obtained by applying the group operation on the generator a certain number of times
 - Additive groups: $b = g * n \pmod{p}$
 - Multiplicative groups: $b = g^n \pmod{p}$
- Discrete log problem: Invert
 - $n = b / g \pmod{p}$
 - $n = \log(b) \text{ (base } g \text{) (naming!)}$
- Not all discrete log problems are hard, depends on the group and operation
 - Judge the above over $GF(p)$

Up next: An interesting candidate group to exploit this hardness

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Elliptic curves

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Elliptic curves

- Not ellipses
 - connected to finding arc lengths on ellipses

- Cubic equations

- With some requirements on A, B

$$y^2 = x^3 + Ax + B$$

- Group structure

- Line between A and B will intersect a third time

