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Problem Category

- Statistical Pattern Recognition
- Supervised Learning
- Parametric Learning
- Bayes Decision Theory
- Univariate data
- 2-class problem
- different variances
- Gaussian model (2 parameters)
- No Risk function

Sections

Given information:

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model: continuous univariate normal (Gaussian) density

$p(x|\omega_j) \sim N(\mu|\sigma^2)$

$p(x|\omega_j) \sim \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right]$

parameters:

$P(\omega_1) = P(\omega_2) = 0.5$

$\sigma_1^2 = 4, \quad \sigma_2^2 = 1$

$\mu_1 = 4, \quad \mu_2 = 10$

Deriving the decision boundary

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Bayes' Rule:

$P(\omega_j|x) = \frac{p(x|\omega_j) * P(\omega_j)}{p(x)}$

Bayes' Decision Rule:

Decide ω_1 if $P(\omega_1|x) > P(\omega_2|x)$ else decide ω_2.

$$\Rightarrow \frac{p(x|\omega_1) * P(\omega_1)}{p(x)} > \frac{p(x|\omega_2) * P(\omega_2)}{p(x)}$$

Bayes' Decision Rule:

Decide ω_1 if $P(\omega_1|x) > P(\omega_2|x)$ else decide ω_2.

$\Rightarrow \frac{p(x|\omega_1)*P(\omega_1)}{p(x)} > \frac{p(x|\omega_2)*P(\omega_2)}{p(x)}$

We can drop $p(x)$ since it is just a scale factor.

$\Rightarrow P(x|\omega_1) * P(\omega_1) > p(x|\omega_2) * P(\omega_2)$

$\Rightarrow \frac{p(x|\omega_1)}{p(x|\omega_2)} > \frac{P(\omega_2)}{P(\omega_1)}$

$\Rightarrow \frac{p(x|\omega_1)}{p(x|\omega_2)} > \frac{0.5}{0.5}$

$\Rightarrow \frac{p(x|\omega_1)}{p(x|\omega_2)} > 1$

$\Rightarrow \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right] > \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2\right] \quad \Big| \quad ln$

$\Rightarrow ln(1) - ln\left(\sqrt{2\pi\sigma_1^2}\right) - \frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2 > ln(1) - ln\left(\sqrt{2\pi\sigma_2^2}\right) - \frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2 \quad \Big| \quad \sigma_1^2 = 4, \quad \sigma_2^2 = 1, \quad \mu_1 = 4, \quad \mu_2 = 10$

$\Rightarrow -ln(\sqrt{2\pi 4}) - \frac{1}{2}\left(\frac{x-4}{2}\right)^2 > -ln(\sqrt{2\pi}) - \frac{1}{2}(x-10)^2$

$\Rightarrow -\frac{1}{2}ln(2\pi) - ln(2) - \frac{1}{8}(x-4)^2 > -\frac{1}{2}ln(2\pi) - \frac{1}{2}(x-10)^2 \quad \Big| \times 2$

$\Rightarrow -ln(2\pi) - 2ln(2) - \frac{1}{4}(x-4)^2 > -ln(2\pi) - (x-10)^2 \quad \Big| + ln(2\pi)$

$\Rightarrow -4ln(4) - (x-4)^2 > -4(x-10)^2$

$\Rightarrow -ln(4) - \frac{1}{4}(x-4)^2 > -(x-10)^2 \quad \Big| \times 4$

$\Rightarrow -8ln(2) - x^2 + 8x - 16 > -4x^2 + 80x - 400$

$\Rightarrow 3x^2 - 72x + 384 - 8ln(2) > 0$

$\Rightarrow x < 7.775 \quad and \quad x > 16.225$

Plotting the class conditional densities, posterior probabilities, and decision boundary

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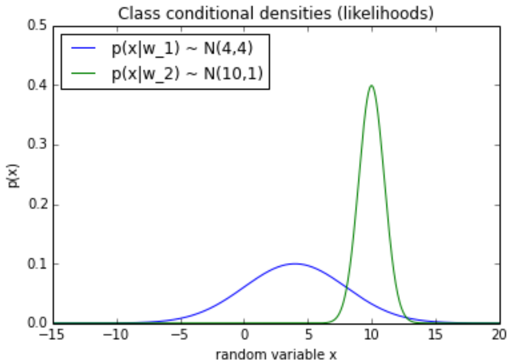
In [3]:
```
%pylab inline

import numpy as np
from matplotlib import pyplot as plt

def pdf(x, mu, sigma):
    """
    Calculates the normal distribution's probability density
    function (PDF).

    """
    term1 = 1.0 / ( math.sqrt(2*np.pi) * sigma )
    term2 = np.exp( -0.5 * ( (x-mu)/sigma )**2 )
    return term1 * term2

# generating some sample data
x = np.arange(-100, 100, 0.05)

# probability density functions
pdf1 = pdf(x, mu=4, sigma=4)
pdf2 = pdf(x, mu=10, sigma=1)

# Class conditional densities (likelihoods)
plt.plot(x, pdf1)
plt.plot(x, pdf2)
plt.title('Class conditional densities (likelihoods)')
```

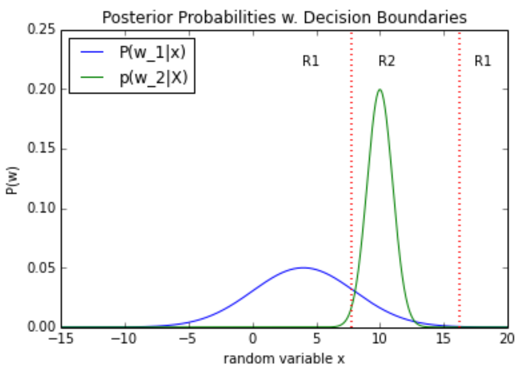
```
plt.ylabel('p(x)')
plt.xlabel('random variable x')
plt.legend(['p(x|w_1) ~ N(4,4)', 'p(x|w_2) ~ N(10,1)'], loc='upper left')
plt.ylim([0,0.5])
plt.xlim([-15,20])
plt.show()
```

```
Populating the interactive namespace from numpy and matplotlib
```



In [4]:

```
def posterior(likelihood, prior):
    """
    Calculates the posterior probability (after Bayes Rule) without
    the scale factor p(x) (=evidence).

    """
    return likelihood * prior

# probability density functions
posterior1 = posterior(pdf(x, mu=4, sigma=4), 0.5)
posterior2 = posterior(pdf(x, mu=10, sigma=1), 0.5)

# Class conditional densities (likelihoods)
plt.plot(x, posterior1)
plt.plot(x, posterior2)
plt.title('Posterior Probabilities w. Decision Boundaries')
plt.ylabel('P(w)')
plt.xlabel('random variable x')
plt.legend(['P(w_1|x)', 'p(w_2|X)'], loc='upper left')
plt.ylim([0,0.25])
plt.xlim([-15,20])
plt.axvline(7.775, color='r', alpha=0.8, linestyle=':', linewidth=2)
plt.axvline(16.225, color='r', alpha=0.8, linestyle=':', linewidth=2)
plt.annotate('R2', xy=(10, 0.2), xytext=(10, 0.22))
plt.annotate('R1', xy=(4, 0.2), xytext=(4, 0.22))
plt.annotate('R1', xy=(17, 0.2), xytext=(17.5, 0.22))
plt.show()
```



Classifying some random example data

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In [24]:

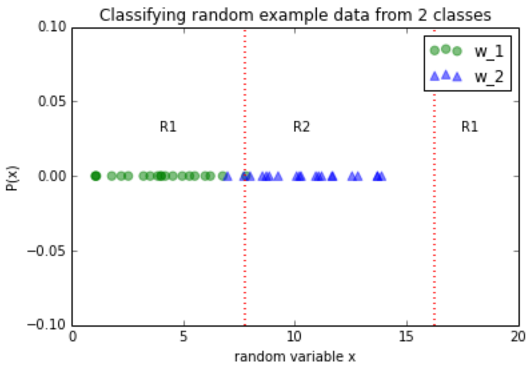
```
# Parameters
mu_1 = 4
mu_2 = 10
sigma_1_sqr = 4
sigma_2_sqr = 1

# Generating 10 random samples drawn from a Normal Distribution for class 1 & 2
```

```
x1_samples = sigma_1_sqr**0.5 * np.random.randn(20) + mu_1
x2_samples = sigma_1_sqr**0.5 * np.random.randn(20) + mu_2
y = [0 for i in range(20)]

# Plotting sample data with a decision boundary

plt.scatter(x1_samples, y, marker='o', color='green', s=40, alpha=0.5)
plt.scatter(x2_samples, y, marker='^', color='blue', s=40, alpha=0.5)
plt.title('Classifying random example data from 2 classes')
plt.ylabel('P(x)')
plt.xlabel('random variable x')
plt.legend(['w_1', 'w_2'], loc='upper right')
plt.ylim([-0.1,0.1])
plt.xlim([0,20])
plt.axvline(7.775, color='r', alpha=0.8, linestyle=':', linewidth=2)
plt.axvline(16.225, color='r', alpha=0.8, linestyle=':', linewidth=2)
plt.annotate('R2', xy=(10, 0.03), xytext=(10, 0.03))
plt.annotate('R1', xy=(4, 0.03), xytext=(4, 0.03))
plt.annotate('R1', xy=(17, 0.03), xytext=(17.5, 0.03))
plt.show()
```



Calculating the empirical error rate

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In [27]:
```
w1_as_w2, w2_as_w1 = 0, 0
for x1,x2 in zip(x1_samples, x2_samples):
    if x1 > 7.775 and x1 < 16.225:
        w1_as_w2 += 1
    if x2 <= 7.775 and x2 >= 16.225:
        w2_as_w1 += 1

emp_err =  (w1_as_w2 + w2_as_w1) / float(len(x1_samples) + len(x2_samples))

print('Empirical Error: {}%'.format(emp_err * 100))
```

```
Empirical Error: 2.5%
```