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Problem Category

- Statistical Pattern Recognition
- Supervised Learning
- Parametric Learning
- Bayes Decision Theory
- Univariate data
- 2-class problem
- different variances
- different priors
- Gaussian model (2 parameters)
- With conditional Risk (loss functions)

Sections

Given information:

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model: continuous univariate normal (Gaussian) density

$p(x|\omega_j) \sim N(\mu|\sigma^2)$ $p(x|\omega_j) \sim \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right]$

parameters:

$P(\omega_1) = \frac{2}{3}, \quad P(\omega_2) = \frac{1}{3}$

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$\sigma_1^2 = 0.25, \quad \sigma_2^2 = 0.04$

$\mu_1 = 2, \quad \mu_2 = 1.5$

Deriving the decision boundary

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Bayes' Rule:

$P(\omega_j|x) = \frac{p(x|\omega_j) * P(\omega_j)}{p(x)}$

Risk Functions:

$R(\alpha_1|x) = \lambda_{11} P(\omega_1|x) + \lambda_{12} P(\omega_2|x)$

$R(\alpha_2|x) = \lambda_{21} P(\omega_1|x) + \lambda_{22} P(\omega_2|x)$

Decision Rule:

Decide ω_1 if $R(\alpha_2|x) > R(\alpha_1|x)$ else decide ω_2.

$\Rightarrow \lambda_{21} P(\omega_1|x) + \lambda_{22} P(\omega_2|x) > \lambda_{11} P(\omega_1|x) + \lambda_{12} P(\omega_2|x)$

$\Rightarrow (\lambda_{21} - \lambda_{11})\ P(\omega_1|x) > (\lambda_{12} - \lambda_{22})\ P(\omega_2|x)$

$\Rightarrow \frac{P(\omega_1|x)}{P(\omega_2|x)} > \frac{(\lambda_{12}-\lambda_{22})}{(\lambda_{21}-\lambda_{11})}$

$\Rightarrow \frac{p(x|\omega_1)\ P(\omega_1)}{p(x|\omega_2)\ P(\omega_2)} > \frac{(\lambda_{12}-\lambda_{22})}{(\lambda_{21}-\lambda_{11})}$

$\Rightarrow \frac{p(x|\omega_1)}{p(x|\omega_2)} > \frac{(\lambda_{12}-\lambda_{22})\ P(\omega_2)}{(\lambda_{21}-\lambda_{11})\ P(\omega_1)}$

$\Rightarrow \frac{p(x|\omega_1)}{P(x|\omega_2)} > \frac{(1-0)\ (1/3)}{(2-0)\ (2/3)}$

$\Rightarrow \frac{p(x|\omega_1)}{P(x|\omega_2)} > \frac{1}{4}$

$$\Rightarrow \left(\frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left[-\frac{1}{2} \left(\frac{x-\mu_1}{\sigma_1} \right)^2 \right] \right) \Big/ \left(\frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left[-\frac{1}{2} \left(\frac{x-\mu_2}{\sigma_2} \right)^2 \right] \right) > \frac{1}{4} \quad \Bigg| \quad \sigma_1^2 = 0.25, \quad \sigma_2^2 = 0.04, \quad \mu_1 = 2, \quad \mu_2 = 1.5$$

$$\Rightarrow \left(\frac{1}{\sqrt{2\pi 0.25}} \exp\left[-\frac{1}{2} \left(\frac{x-2}{0.5} \right)^2 \right] \right) \Big/ \left(\frac{1}{\sqrt{2\pi 0.04}} \exp\left[-\frac{1}{2} \left(\frac{x-1.5}{0.2} \right)^2 \right] \right) > \frac{1}{4}$$

$$\Rightarrow \left(2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left[-2 \cdot (x-2)^2 \right] \right) \Big/ \left(5 \cdot \frac{1}{\sqrt{2\pi}} \exp\left[-12.5 \cdot (x-1.5)^2 \right] \right) > \frac{1}{4}$$

$$\Rightarrow \frac{2}{5} \left(\exp\left[-2 \cdot (x-2)^2 \right] \right) \Big/ \left(\exp\left[-12.5 \cdot (x-1.5)^2 \right] \right) > \frac{1}{4} \quad \Bigg| \quad ln$$

$$\Rightarrow \left(-2(x-2)^2 \right) - \left(-12.5(x-1.5)^2 \right) > ln(\frac{5}{8})$$

$$\Rightarrow -2(x-2)^2 + 12.5(x-1.5)^2 - 0.47 > 0$$

$$\Rightarrow -2x^2 + 8x - 8 + 12.5x^2 - 37.5x + 27.655 > 0$$

$$\Rightarrow 10.5x^2 - 29.5x + 19.655 > 0$$

$$\Rightarrow x = 1.08625 \quad and \quad x = 1.72328$$

Plotting the class posterior probabilities and decision boundary

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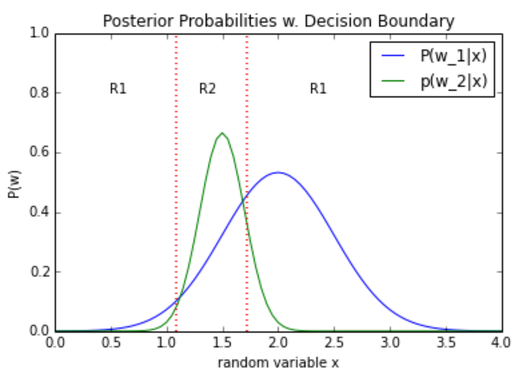
In [1]:
```
%pylab inline

import numpy as np
from matplotlib import pyplot as plt

def pdf(x, mu, sigma_sqr):
    """
    Calculates the normal distribution's probability density
    function (PDF).

    """
    term1 = 1.0 / ( math.sqrt(2*np.pi * sigma_sqr))
    term2 = np.exp( (-1/2.0) * ( ((x-mu)**2 / sigma_sqr) ))
```

```
    return term1 * term2

# generating some sample data
x = np.arange(0, 50, 0.05)

def posterior(likelihood, prior):
    """
    Calculates the posterior probability (after Bayes Rule) without
    the scale factor p(x) (=evidence).

    """
    return likelihood * prior

# probability density functions
posterior1 = posterior(pdf(x, mu=2, sigma_sqr=0.25), prior=2/3.0)
posterior2 = posterior(pdf(x, mu=1.5, sigma_sqr=0.04), prior=1/3.0)

# Class conditional densities (likelihoods)
plt.plot(x, posterior1)
plt.plot(x, posterior2)
plt.title('Posterior Probabilities w. Decision Boundary')
plt.ylabel('P(w)')
plt.xlabel('random variable x')
plt.legend(['P(w_1|x)', 'p(w_2|x)'], loc='upper right')
plt.ylim([0,1])
plt.xlim([0,4])
plt.axvline(1.08625, color='r', alpha=0.8, linestyle=':', linewidth=2)
plt.axvline(1.72328, color='r', alpha=0.8, linestyle=':', linewidth=2)
plt.annotate('R1', xy=(0.5, 0.8), xytext=(0.5, 0.8))
plt.annotate('R2', xy=(1.3, 0.8), xytext=(1.3, 0.8))
plt.annotate('R1', xy=(2.3, 0.8), xytext=(2.3, 0.8))
plt.show()
```

```
Populating the interactive namespace from numpy and matplotlib
```



Classifying some random example data

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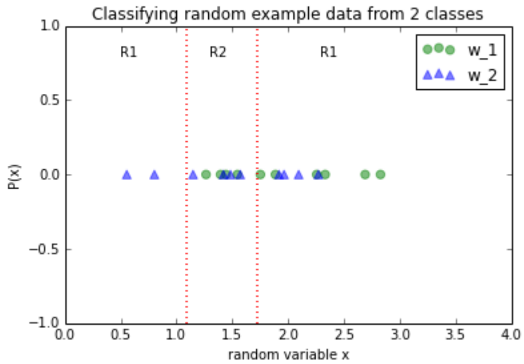
In [6]:
```
# Parameters
mu_1 = 2
mu_2 = 1.5
sigma_1_sqr = 0.25
sigma_2_sqr = 0.04

# Generating 10 random samples drawn from a Normal Distribution for class 1 & 2
x1_samples = sigma_1_sqr**0.5 * np.random.randn(10) + mu_1
x2_samples = sigma_1_sqr**0.5 * np.random.randn(10) + mu_2
y = [0 for i in range(10)]

# Plotting sample data with a decision boundary

plt.scatter(x1_samples, y, marker='o', color='green', s=40, alpha=0.5)
plt.scatter(x2_samples, y, marker='^', color='blue', s=40, alpha=0.5)
plt.title('Classifying random example data from 2 classes')
plt.ylabel('P(x)')
plt.xlabel('random variable x')
plt.legend(['w_1', 'w_2'], loc='upper right')
plt.ylim([-1,1])
plt.xlim([0,4])
plt.axvline(1.08625, color='r', alpha=0.8, linestyle=':', linewidth=2)
plt.axvline(1.72328, color='r', alpha=0.8, linestyle=':', linewidth=2)
plt.annotate('R1', xy=(0.5, 0.8), xytext=(0.5, 0.8))
plt.annotate('R2', xy=(1.3, 0.8), xytext=(1.3, 0.8))
plt.annotate('R1', xy=(2.3, 0.8), xytext=(2.3, 0.8))
```

```
plt.show()
```



Calculating the empirical error rate

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In [7]:
```
w1_as_w2, w2_as_w1 = 0, 0
for x1,x2 in zip(x1_samples, x2_samples):
    if x1 > 1.08625 and x1 < 1.72328:
        w1_as_w2 += 1
    if x2 <= 1.08625 and x2 >= 1.72328:
        w2_as_w1 += 1

emp_err =  (w1_as_w2 + w2_as_w1) / float(len(x1_samples) + len(x2_samples))

print('Empirical Error: {}%'.format(emp_err * 100))
```

```
Empirical Error: 20.0%
```