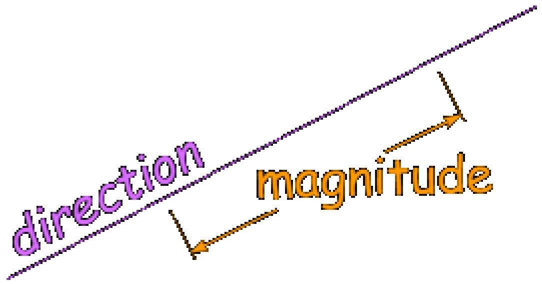
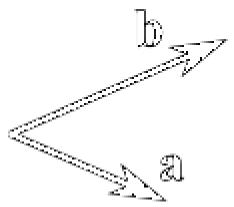
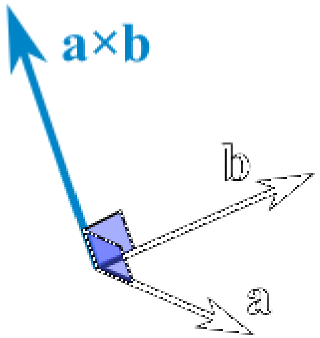
- A vector has **magnitude** (how long it is) and **direction**:



- **Two vectors** can be **multiplied** using the "**Cross Product**" *(also see Dot Product)*

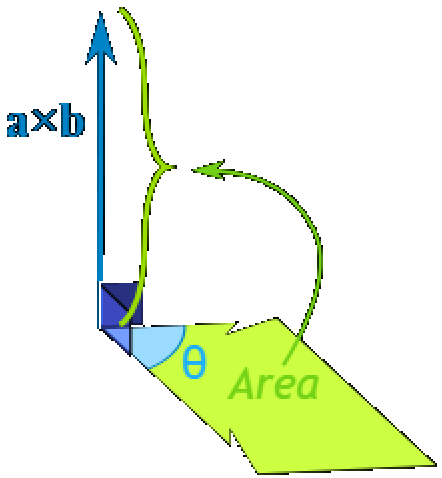


- The Cross Product **a × b** of two vectors is **another vector** that is at right angles to both:

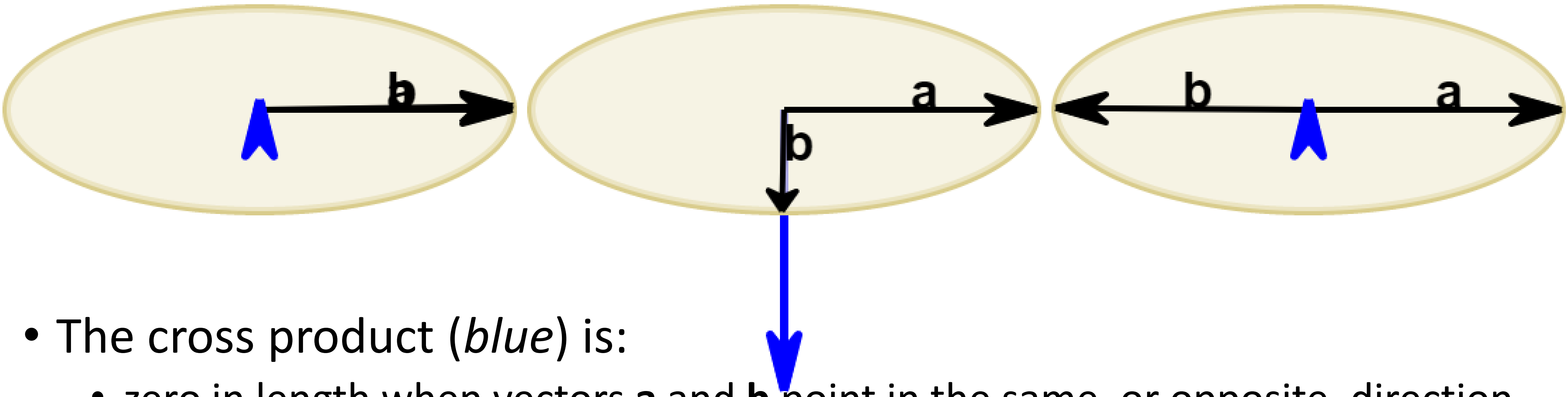


And it all happens in 3 dimensions!

- The magnitude (length) of the cross product equals the area of a parallelogram with vectors **a** and **b** for sides:



- See how it changes for different angles:



- The cross product (*blue*) is:
  - zero in length when vectors **a** and **b** point in the same, or opposite, direction
  - reaches maximum length when vectors **a** and **b** are at right angles

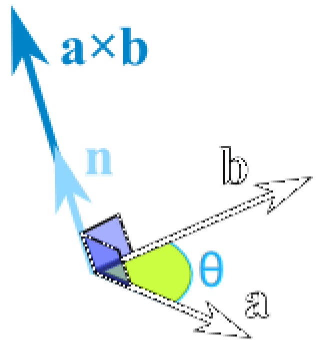
And it can point one way or the other!
So how do we calculate it?

Calculating

- We can calculate the Cross Product this way:

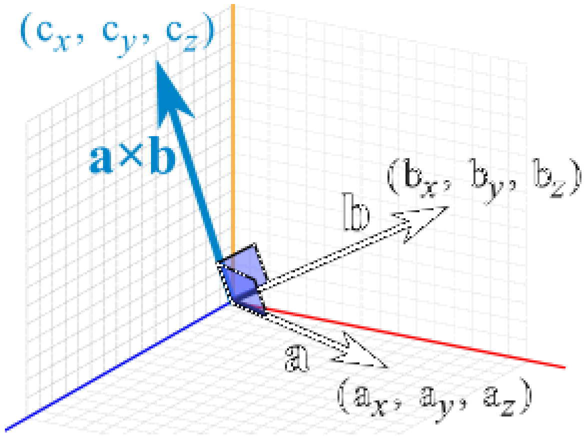
$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \sin(\theta)\ \mathbf{n}$$

- $|\mathbf{a}|$ is the magnitude (length) of vector **a**
- $|\mathbf{b}|$ is the magnitude (length) of vector **b**
- θ is the angle between **a** and **b**
- **n** is the unit vector at right angles to both **a** and **b**



- So the **length** is: the length of **a** times the length of **b** times the sine of the angle between **a** and **b**,
- Then we multiply by the vector **n** to make sure it heads in the right **direction** (at right angles to both **a** and **b**).

- OR WE CAN CALCULATE IT THIS WAY:
- When **a** and **b** start at the origin point (0,0,0), the Cross Product will end at:
  - $\mathbf{c_x = a_y b_z - a_z b_y}$
  - $\mathbf{c_y = a_z b_x - a_x b_z}$
  - $\mathbf{c_z = a_x b_y - a_y b_x}$



Example

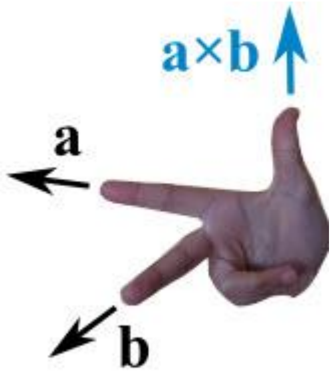
- Example: The cross product of **a** = (2,3,4) and **b** = (5,6,7)

$c_x = a_y b_z - a_z b_y = 3\times7 - 4\times6 = -3$
$c_y = a_z b_x - a_x b_z = 4\times5 - 2\times7 = 6$
$c_z = a_x b_y - a_y b_x = 2\times6 - 3\times5 = -3$

Answer: $\mathbf{a} \times \mathbf{b} = (-3,6,-3)$

Which Direction?

- The cross product could point in the completely opposite direction and still be at right angles to the two other vectors, so we have the:
- "Right Hand Rule"
- With your right-hand, point your index finger along vector **a**, and point your middle finger along vector **b**: the cross product goes in the direction of your thumb.



Dot Product

- The Cross Product gives a **vector** answer, and is sometimes called the **vector product**.
- But there is also the Dot Product which gives a **scalar** (ordinary number) answer, and is sometimes called the **scalar product**.

Question

Your Turn

- Vector **a** has magnitude 3, vector **b** has magnitude 4, the angle between **a** and **b** is 30° and **n** is the unit vector at right angles to both **a** and **b**
  What is **a × b** ?
- A 3**n**
- B 6**n**
- C 9**n**
- D 10.39**n**

- Vector **a** has magnitude $3\sqrt{2}$, vector **b** has magnitude 5.
  The angle between **a** and **b** is 135° and **n** is the unit vector at right angles to both **a** and **b**.

  What is the value of $\mathbf{a} \times \mathbf{b}$?
- A
- $-15\sqrt{2}\mathbf{n}$
- B
- $-15\mathbf{n}$
- C
- $15\mathbf{n}$
- D
- $15\sqrt{2}\mathbf{n}$

- Vector **a** has magnitude $1/\sqrt{3}$, vector **b** has magnitude 4, the angle between **a** and **b** is 60° and **n** is the unit vector at right angles to both **a** and **b**

  What is the value of $\mathbf{a} \times \mathbf{b}$?
- A
- $2\mathbf{n}$
- B
- $2\sqrt{3}\mathbf{n}$
- C
- $(2/\sqrt{3})\mathbf{n}$
- D
- $4\mathbf{n}$

- What is the cross product of **a** = (1, 2, 3) and **b** = (4, 5, 6)?
- A
- (4, 10, 18)
- B
- (-3, 18, -3)
- C
- (3, 6, -3)
- D
- (-3, 6, -3)

- What is the cross product of **a** = (-2, 3, 5) and **b** = (-4, 1, -6)?
- A
- (-23, -32, 10)
- B
- (-23, -8, 10)
- C
- (-23, -32, -14)
- D
- (8, 3, -30)

- What is the cross product of **a** = (2, -5, 1) and **b** = (3, -2, -4)?
- A
- (18, 11, 11)
- B
- (22, 11, 11)
- C
- (22, -5, 11)
- D
- (22, 11, -19)

- If **a** = (-2, 1, 1), **b** = (2, 1, 1) and **c** = **a** × **b**, what is the magnitude of **c**?
- A
- 2√2
- B
- 4
- C
- 4√2
- D
- 8

- If **a** = (2, 0, 1), **b** = (0, 1, ½) and **c** = **a** × **b**, what is the magnitude of **c**?
- A
- √6
- B
- 3
- C
- 2√3
- D
- 2√6

- If **a** = (2, -4, 4), **b** = (4, 0, 3) and **c** = **a** × **b**, what is the magnitude of **c**?
- A
- $4\sqrt{13}$
- B
- 20
- C
- $10\sqrt{5}$
- D
- 30

- **a**, **b** and **c** are three vectors such that **c** is perpendicular to both **a** and **b** What is the value of **a** × **b** × **c**?
- A
- (1, 1, 1)
- B
- (0, 0, 0)
- C
- (1, 1, 0)
- D
- (0, 0, 1)

- Vector **a** has magnitude 3, vector **b** has magnitude 4, the angle between **a** and **b** is 30° and **n** is the unit vector at right angles to both **a** and **b** What is **a** × **b** ?
- A
- 3**n**
- B
- 6**n**
- C
- 9**n**
- D
- 10.39**n**

- Vector **a** has magnitude $3\sqrt{2}$, vector **b** has magnitude 5.
  The angle between **a** and **b** is 135° and **n** is the unit vector at right angles to both **a** and **b**.

  What is the value of **a × b** ?
- A
- $-15\sqrt{2}$**n**
- B
- -15**n**
- C
- 15**n**
- D
- $15\sqrt{2}$**n**

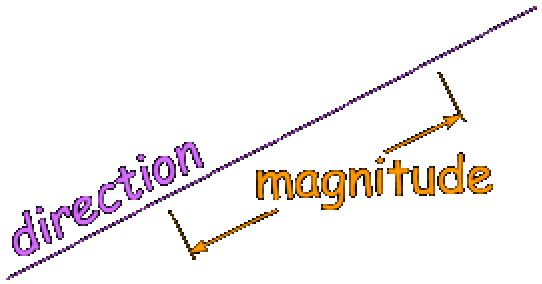
- Vector **a** has magnitude 1/√3, vector **b** has magnitude 4, the angle between **a** and **b** is 60° and **n** is the unit vector at right angles to both **a** and **b**

  What is the value of **a × b** ?
- A
- 2**n**
- B
- 2√3**n**
- C
- (2/√3)**n**
- D
- 4**n**

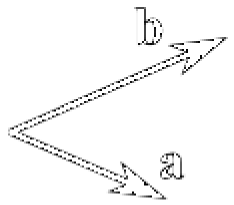
- What is the cross product of **a** = (1, 2, 3) and **b** = (4, 5, 6)?
- A
- (4, 10, 18)
- B
- (-3, 18, -3)
- C
- (3, 6, -3)
- D
- (-3, 6, -3)

Dot Product

- A vector has **magnitude** (how long it is) and **direction**:



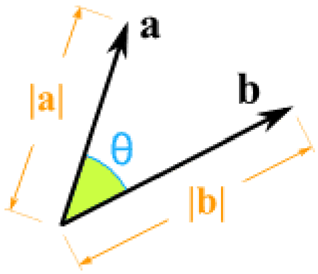
- They can be **multiplied** using the "**Dot Product**" (also see Cross Product).



Calculating

- The Dot Product gives a **number** as an answer (a "scalar", not a vector).
- The Dot Product is written using a central dot:

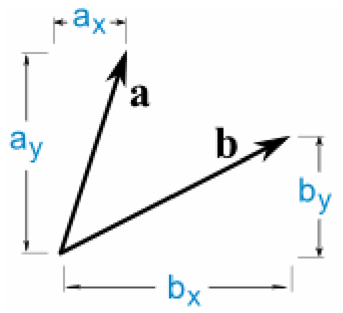
a · b
This means the Dot Product of **a** and **b**



- We can calculate the Dot Product of two vectors this way:
- **a · b** = |**a**| × |**b**| × cos(θ)
- Where:
  |**a**| is the magnitude (length) of vector **a**
  |**b**| is the magnitude (length) of vector **b**
  θ is the angle between **a** and **b**

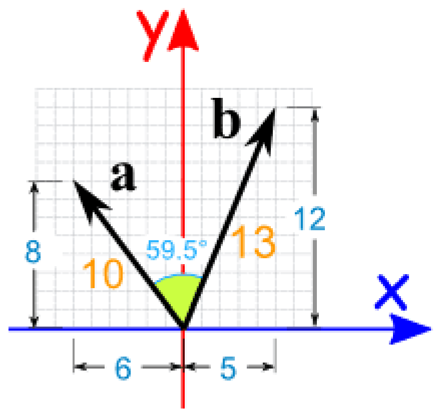
- OR we can calculate it this way:

- $\mathbf{a} \cdot \mathbf{b} = a_x \times b_x + a_y \times b_y$
- So we multiply the x's, multiply the y's, then add.



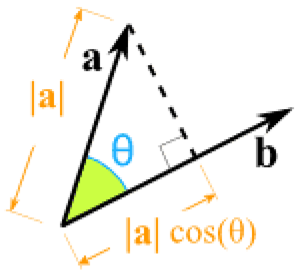
Example

- Example: Calculate the dot product of vectors **a** and **b**:
- $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta)$
  - $\mathbf{a} \cdot \mathbf{b} = 10 \times 13 \times \cos(59.5°)$
  - $\mathbf{a} \cdot \mathbf{b} = 10 \times 13 \times 0.5075...$
  - $\mathbf{a} \cdot \mathbf{b} = 65.98... = 66$ (rounded)
- or we can calculate it this way:
- $\mathbf{a} \cdot \mathbf{b} = a_x \times b_x + a_y \times b_y$
  - $\mathbf{a} \cdot \mathbf{b} = -6 \times 5 + 8 \times 12$
  - $\mathbf{a} \cdot \mathbf{b} = -30 + 96$
  - $\mathbf{a} \cdot \mathbf{b} = 66$
- Both methods came up with the same result (after rounding)
- Also note that we used **minus 6** for a_x (it is heading in the negative x-direction)

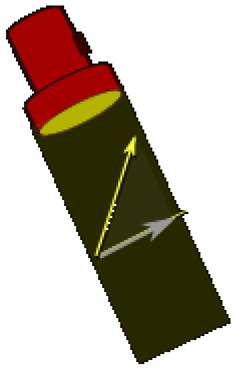


Why cos(θ) ?

- OK, to multiply two vectors it makes sense to multiply their lengths together ***but only when they point in the same direction*.**
- So we make one "point in the same direction" as the other by multiplying by cos(θ):



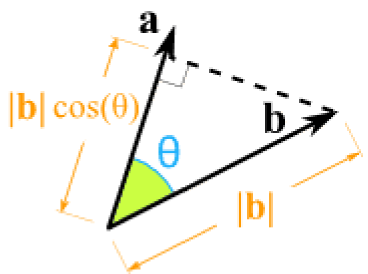
We take the component of **a** that lies alongside **b**



Like shining a light to see where the shadow lies

- THEN we multiply !
- It works exactly the same if we "projected" **b** alongside **a** then multiplied:
- Because it doesn't matter which order we do the multiplication:

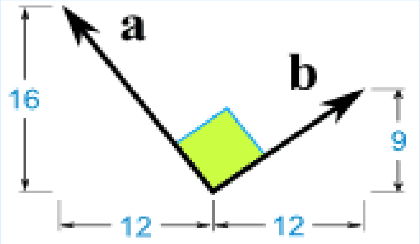
$$|\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta) = |\mathbf{a}| \times \cos(\theta) \times |\mathbf{b}|$$



Right Angles

- When two vectors are at right angles to each other the dot product is **zero**.

Example: calculate the Dot Product for:



$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta)$

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(90°)$

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times 0$

$\mathbf{a} \cdot \mathbf{b} = 0$

or we can calculate it this way:

$\mathbf{a} \cdot \mathbf{b} = a_x \times b_x + a_y \times b_y$

$\mathbf{a} \cdot \mathbf{b} = -12 \times 12 + 16 \times 9$

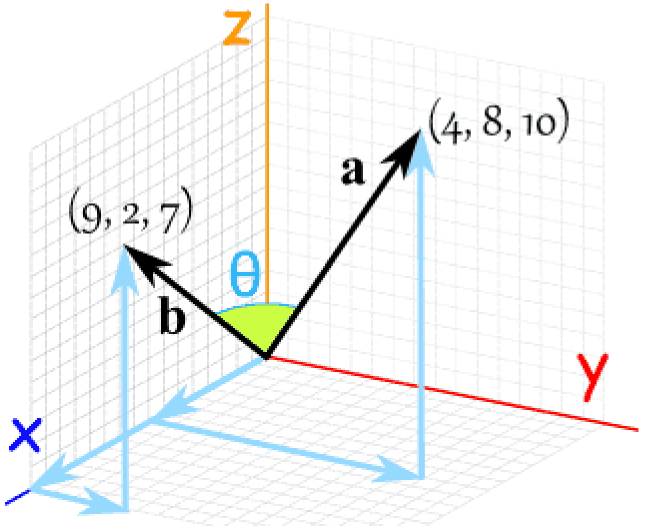
$\mathbf{a} \cdot \mathbf{b} = -144 + 144$

$\mathbf{a} \cdot \mathbf{b} = 0$

Three or More Dimensions

- This all works fine in 3 (or more) dimensions, too.
- And can actually be very useful!

- Example: Sam has measured the end-points of two poles, and wants to know **the angle between them**:



- We have 3 dimensions, so don't forget the z-components:
- $\mathbf{a} \cdot \mathbf{b} = a_x \times b_x + a_y \times b_y + a_z \times b_z$
  - $\mathbf{a} \cdot \mathbf{b} = 9 \times 4 + 2 \times 8 + 7 \times 10$
  - $\mathbf{a} \cdot \mathbf{b} = 36 + 16 + 70$
  - $\mathbf{a} \cdot \mathbf{b} = 122$

- Now for the other formula:
- $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta)$
- But what is $|\mathbf{a}|$? It is the magnitude, or length, of the vector **a**. We can use Pythagoras:
  - $|\mathbf{a}| = \sqrt{}(4^2 + 8^2 + 10^2)$
  - $|\mathbf{a}| = \sqrt{}(16 + 64 + 100)$
  - $|\mathbf{a}| = \sqrt{}180$

- Likewise for $|\mathbf{b}|$:
  - $|\mathbf{b}| = \sqrt{(9^2 + 2^2 + 7^2)}$
  - $|\mathbf{b}| = \sqrt{(81 + 4 + 49)}$
  - $|\mathbf{b}| = \sqrt{134}$
- And we know from the calculation above that $\mathbf{a} \cdot \mathbf{b} = 122$, so:
- $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta)$

  $122 = \sqrt{180} \times \sqrt{134} \times \cos(\theta)$

  $\cos(\theta) = 122 / (\sqrt{180} \times \sqrt{134})$

  $\cos(\theta) = 0.7855...$

  $\theta = \cos^{-1}(0.7855...) = 38.2...°$
- Done!

Cross Product

- The Dot Product gives a **scalar** (ordinary number) answer, and is sometimes called the **scalar product**.
- But there is also the Cross Product which gives a **vector** as an answer, and is sometimes called the **vector product**.

Source

- https://www.mathsisfun.com/algebra/vectors-dot-product.html
- https://www.mathsisfun.com/algebra/vectors-cross-product.html

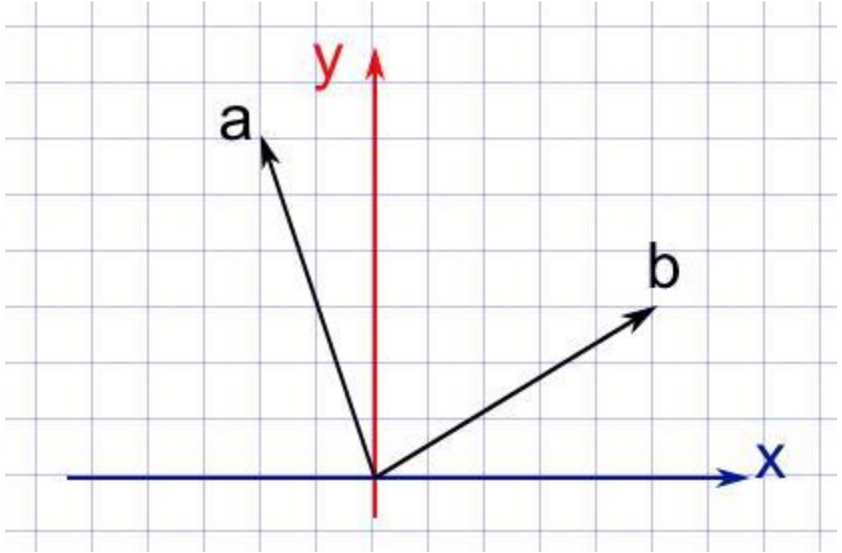
Question

- Vector **a** has magnitude 3, vector **b** has magnitude 4 and the angle between **a** and **b** is 60°
  What is the value of **a.b** ?
- A
- 3
- B
- 5
- C
- 6
- D
- 10.39

- Vector **a** has magnitude $3\sqrt{2}$, vector **b** has magnitude 5 and the angle between **a** and **b** is 135°.

  What is the value of **a.b** ?
- A
- -15
- B
- $-15\sqrt{2}$
- C
- $15\sqrt{2}$
- D
- 15

What is the value of **a.b**



A

3

B

8

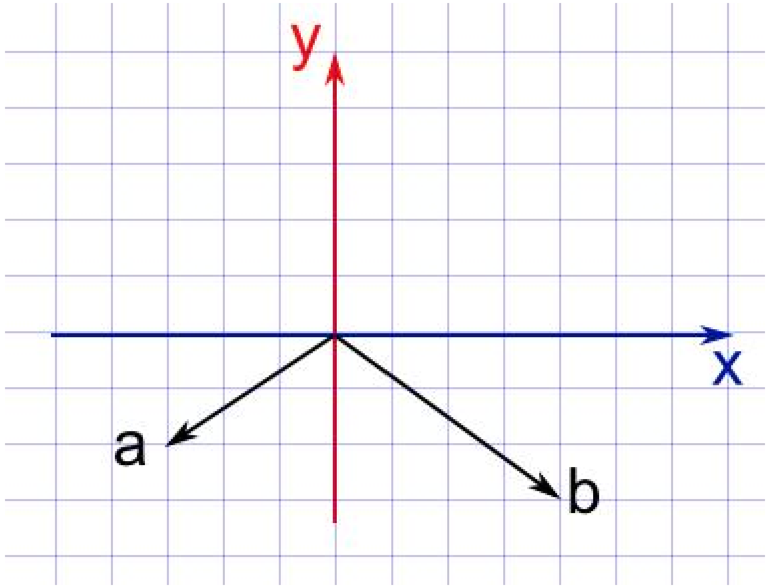
C

24

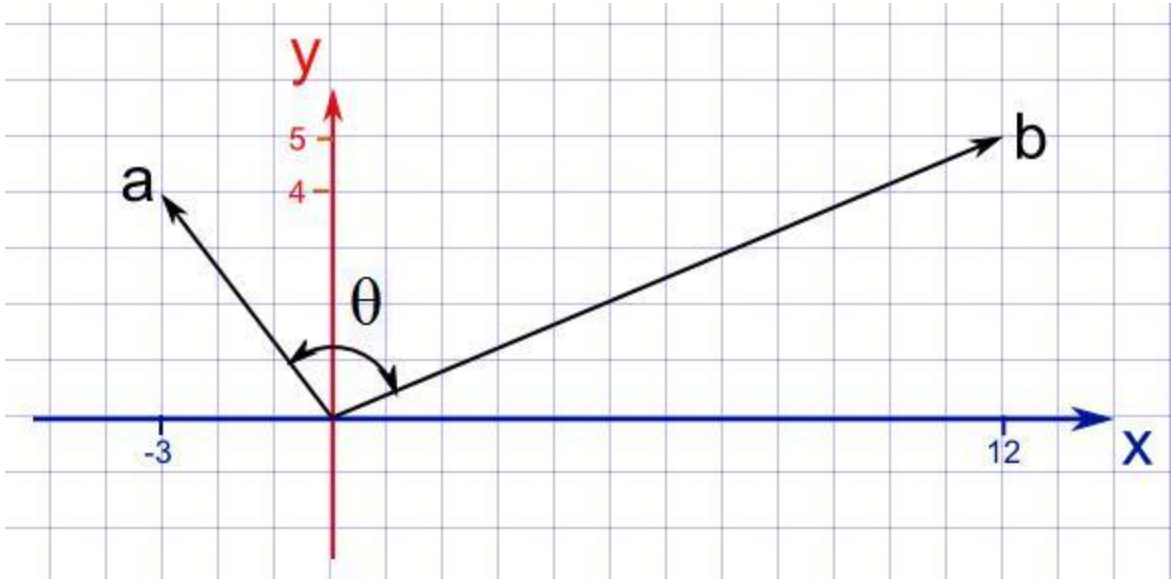
D

28

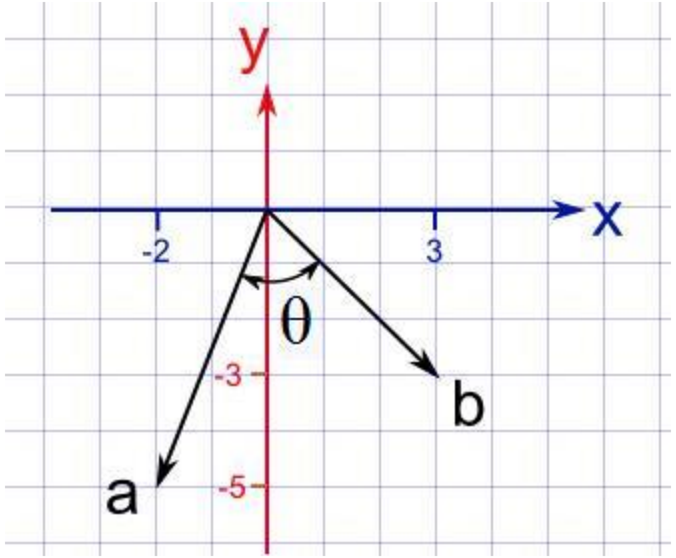
- What is the value of **a.b** ?
- A
- -6
- B
- 1
- C
- 6
- D
- 18



- Use the dot product to find the size of angle θ?
- A
- 75.7°
- B
- 90°
- C
- 104.3°
- D
- 110°



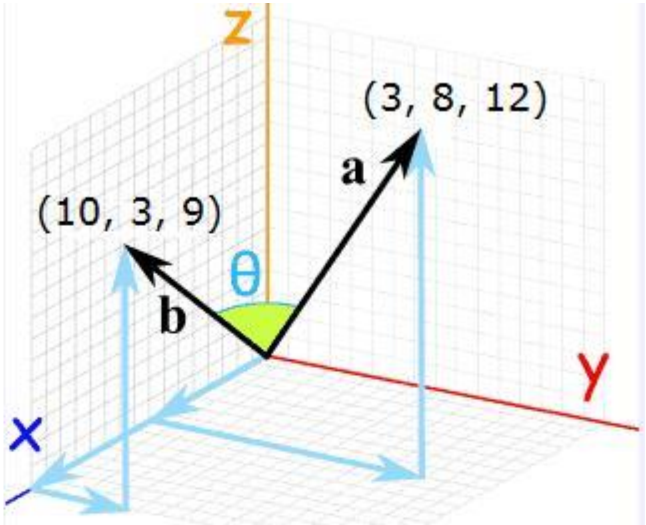
- Use the dot product to find the size of angle θ?
- A
- 113.2°
- B
- 73.2°
- C
- 70°
- D
- 66.8°



- **a** = (4, 3), **b** = (4, -3), **c** = (-3, 4) and **d** = (-3, -4)

  Which of the following pairs of vectors is perpendicular?
- A
- **a** and **b**
- B
- **a** and **c**
- C
- **b** and **c**
- D
- **c** and **d**

- Sally has measured the end-points of two poles, and wants to know the angle between them:
- Use the dot product to calculate the size of angle θ.
- A
- 142.9°
- B
- 90°
- C
- 52.9°
- D
- 37.1°



- What is the size of the angle between the vectors **a** = (2, 5, -1) and **b** = (-3, 2, 6)?
- A
- 3.0°
- B
- 87.0°
- C
- 93.0°
- D
- 177.0°

END