# Some representation theory of finite groups

**Jake Huryn**

In this handout we attempt to thoroughly develop the machinery of representation theory necessary to prove that the set of irreducible characters of any finite group $G$ forms an orthonormal basis for the space of $\mathbb{C}$-valued class functions on $G$. Throughout the handout we refer to this result as the *ONB property of irreducible characters*. Because of the relatively narrow scope of the material—developing this one result—some important notions and tools of representation theory are omitted.

There are also exercises at the end of some of the subsections to give the reader a chance to play with the definitions and get a better intuition for representation theory. These are results which are not of central importance or things the author was too lazy to write up. Within each group the exercises are ordered roughly by difficulty, but are in general relatively easy.

The problems, proofs, and structure of the presentation of this material are often inspired by or pulled directly from the book *Representation Theory: A First Course* by William Fulton and Joe Harris.

The prerequisites for these notes are a good understanding of abstract linear algebra (in particular, dual spaces, tensor products, and inner products) and a familiarity with group theory.

## 1 Group representations

A *representation* of a group $G$ is a map $\rho\colon G \to \mathrm{GL}(V)$ where $V$ is a vector space. In other words, a representation of $G$ is an action of $G$ on a vector space by linear functions. Since even basic group theory experience shows that group actions are powerful tools for learning about $G$, and (especially finite-dimensional) linear algebra is so well understood, so one might guess that representation theory is quite useful. This is indeed the case, and representation theory is ubiquitous throughout mathematics. Unfortunately, in these notes we will not be able to accurately express this, so we hope the interested reader will pursue other sources for a deeper understanding.

For any $g \in G$, we write $g$ to denote the map $\rho(g)\colon V \to V$, and we call $V$ itself the representation of $G$. Although this notation and terminology may at first seem dangerously ambiguous, we will see that it rarely becomes problematic and in fact is very useful.

As hinted above, we are most interested in the setting when $G$ is a finite group and $V$ is a finite-dimensional complex vector space. We will make these assumptions on $G$ and $V$ throughout the rest of the handout, except when otherwise mentioned.

Any group has the *zero representation* when $V$ is zero-dimensional and the *trivial representation* when $\rho$ is the zero map, that is, when $g\colon V \to V$ is the identity map for all $g \in G$.

A *homomorphism* of representations $V$ and $W$ of $G$ is a map $\varphi\colon V \to W$ which satisfies $g \circ \varphi = \varphi \circ g$ for any $g \in G$. This can be encoded by saying that the following diagram commutes for all $g \in G$:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] & W \arrow[d, "g"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

We call a linear map *$G$-linear* if it is a homomorphism of representations.

One way to think about this definition is that $\varphi$ acts as a sort of change-of-basis transformation which turns the representation of $G$ on $V$ into the representation of $G$ on $W$. More precisely, if $\varphi$ is an isomorphism and $\{v_k\}_{k=1}^n$ is a basis for $V$, then for any $g \in G$, the matrices of $g$ with respect to the bases $\{v_k\}_{k=1}^n$ and $\{\varphi(v_k)\}_{k=1}^n$ are identical. This follows from the $G$-linearity of $\varphi$, since

$$\text{if} \quad g(v_k) = \sum_{\ell=1}^n \lambda_{\ell,k} v_\ell \quad \text{then} \quad g(\varphi(v_k)) = \varphi(g(v_k)) = \sum_{\ell=1}^n \lambda_{\ell,k} \varphi(v_\ell).$$

What this tells us is that the various linear-algebraic properties of $g$, such as trace and eigenvalues, are identical between the actions on $V$ and on $W$.

## 1.1 Maschke's Theorem

In this section we see the first nice property of the representations to which we restrict our study: they can always be broken down into irreducible pieces, somewhat similarly to prime factorizations.

If $V$ is a representation of $G$ and $W \subseteq V$ is a $G$-invariant subspace, that is, $g(W) = W$, then $W$ is called a *subrepresentation* of $V$. A nonzero representation with no nonzero proper subrepresentations is called *irreducible*. We see that if $V$ is a trivial representation, then it is irreducible if and only if $V$ is one-dimensional. On the other hand, we call $V$ *indecomposable* if, whenever $V = W_1 \oplus W_2$ with $W_1$ and $W_2$ two $G$-invariant subspaces, then either $W_1$ is trivial or $W_2$ is trivial. Clearly, irreducibility implies indecomposability. A priori, indecomposability and irreducibility are different notions, and in fact there are representations of infinite groups which are reducible but indecomposable (Exercise 2). However, in the case of finite, groups, these notions coincide.

In order to prove the following theorem, we will use the fact that for any representation $V$ of $G$ there is an inner product $\langle\cdot,\cdot\rangle_G$ on $V$ which is $G$-invariant, meaning that $\langle g(v), g(w)\rangle_G = \langle v, w\rangle_G$ for any $v, w \in V$. Indeed, if $\langle\cdot,\cdot\rangle$ is any inner product on $V$, we can define

$$\langle v, w \rangle_G \coloneqq \sum_{g \in G} \langle g(v), g(w) \rangle.$$

This inner product will also be useful when we begin the study of characters.

**Theorem 1** (Maschke's theorem)**.** *If $V$ is a representation of $G$ then there is a decomposition $V = \bigoplus V_n$ in which each $V_n$ is irreducible.*

*Proof.* We induct on the dimension of $V$: the zero representation is the empty direct sum of irreducible representations. If $V$ is irreducible we are done, so suppose $W \subseteq V$ is a nonzero proper subrepresentation. Then, with respect to $\langle \cdot, \cdot \rangle_G$, the orthogonal complement $W^\perp$ is also a subrepresentation and $V = W \oplus W^\perp$. By induction we may decompose each of $W$ and $W^\perp$ into irreducible representations, which gives the decomposition of $V$. □

It will be deduced (Corollary 10) as an immediate consequence the ONB property of irreducible characters that this decomposition is unique up to isomorphism and reordering of its factors, strengthening our feelings that these representations are fantastically well-behaved.

**Exercise 2.** Find a representation of $\mathbb{Z}$ which is irreducible but not decomposable.

## 1.2 Some constructions

In this section, we describe some ways to construct new representations from old representations. These constructions will be crucial in proving the ONB property of irreducible characters.

We can build representations using tensor products. If $V$ and $W$ are two representations of $G$, then we define $V \otimes W$ to be a representation of $G$ by $g(v \otimes w) \coloneqq g(v) \otimes g(w)$.

We can also define the *dual* of a representation $V$ by $g(v^*) \coloneqq v^* \circ g^{-1}$ for any $g \in G$ and $v^* \in V^*$. Under this definition the *natural pairing* of $V$ and $V^*$ is preserved by $G$: for any $v \in V$ and $v^* \in V^*$, $g(v^*)(g(v)) = v^*(v)$. It is not hard to see that $V$ is irreducible if and only if $V^*$ is irreducible.

We make $\mathrm{Hom}(V, W)$ into a representation of $G$ by $g(T) \coloneqq g \circ T \circ g^{-1}$. This definition is chosen for the following reason:

**Lemma 3.** *The natural isomorphism $\varphi \colon V^* \otimes W \to \mathrm{Hom}(V, W)$ is $G$-linear.*

*Proof.* Recall that the natural isomorphism is defined by $\varphi(v^* \otimes w) \coloneqq v^* \cdot w$ (that is, $\varphi(v^* \otimes w)$ is defined to be the linear map from $V$ to $W$ sending $v$ to $v^*(v) \cdot w$). We then compute

$$\begin{aligned} g \circ \varphi \circ g^{-1}(v^* \otimes w) &= g \circ \varphi(v^* \circ g \otimes g^{-1}(w)) \\ &= g((v^* \circ g) \cdot g^{-1}(w)) \\ &= (v^* \circ g \circ g^{-1}) \cdot g(g^{-1}(w)) \qquad (1) \\ &= v^* \cdot w, \end{aligned}$$

(1) coming from the definition of the action of $G$ on $\mathrm{Hom}(V, W)$ (note that $v^* \circ g \circ g^{-1}$ is just a scalar). That is, we have shown that $g \circ \varphi \circ g^{-1} = \varphi$, verifying the $G$-linearity of $\varphi$. □

Finally, $G$ acts by permutation on some set $X$, then we get a *permutation representation* by letting $V_X$ be the space with basis $\{v_x\}_{x \in X}$ and defining $g(v_x) = v_{g(x)}$. The *regular representation* $R_G$ is the permutation representation obtained from $G$ acting on itself by left-mutiplication.

**Exercise 4.** Verify that the natural pairing is preserved by $G$.

**Exercise 5.** Show that $V$ is irreducible if and only if $V^*$ is irreducible.

# 2 Characters

Suppose we have an orthonormal basis $\{v_k\}_{k=1}^n$ of $V$ with respect to $\langle \cdot, \cdot \rangle_G$ (recall this inner product from Subsection 1.1). Then $\{gv_k\}_{k=1}^n$ will also be an orthonormal basis for any $g \in G$, ensuring that each $g \in G$ is a unitary transformation with respect to $\langle \cdot, \cdot \rangle_G$. It follows that each $g \in G$ has an eigenbasis with eigenvalues all being $n$th roots of unity, where $n = \mathrm{ord}(g)$.

Although this result is interesting on its own, it perhaps motivates us to study the eigenvalues of the group elements under the representation. We have two objects which are incredibly useful and can be defined in terms of eigenvalues: the determinant (the product of the eigenvalues) and the trace (the sum of the eigenvalues). Although the determinant might be the more obvious choice, we note two reasons why it is the wrong one. First, by the multiplicativity of the determinant, $\det g$ is restricted to being an $n$th root of unity, where $n = \mathrm{ord}(g)$, for all $g \in G$. Second, again by the multiplicativity of the determinant, $\det g^n$ is immediately known from $\det g$ and thus gives no more information about the eigenvalues of $g$.

We are thus led to the following definition. If $V$ is a representation of $G$, the *character* of $V$ is the function $\chi_V : G \to \mathbb{C}$ defined by $\chi_V(g) := \mathrm{tr}(g)$. We call $\chi_V$ *irreducible* if $V$ is irreducible. It follows immediately from the well-known identity $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ that $\chi_V$ is a *class function*, that is, $\chi_V(g) = \chi_V(h)$ for any $g, h \in G$ which are conjugate.

**Lemma 6.** *Let $V$ and $W$ be representations of $G$. Then*

(a) $\chi_{V \oplus W} = \chi_V + \chi_W$,

(b) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$, *and*

(c) $\chi_{V^*} = \overline{\chi_V}$.

*Proof.* Fix $g \in G$ and let $\mathcal{B}_V := \{v_k\}_{k=1}^m$ and $\mathcal{B}_W := \{w_k\}_{k=1}^n$ be eigenbases for $V$ and $W$, respectively, with respect to $g$, and write $g(v_k) = \lambda_k v_k$ and $g(w_k) = \mu_k w_k$ with $\lambda_k, \mu_k \in \mathbb{C}$ for all $k$. We prove each stated formula for $\chi_U$ by finding an eigenbasis for $U$ and summing the eigenvalues.

($a$) The space $V \oplus W$ has basis $\mathcal{B}_V \cup \mathcal{B}_W$.

($b$) The space $V \otimes W$ has basis $\{v \otimes w : v \in \mathcal{B}_V, w \in \mathcal{B}_W\}$, and $g(v_k \otimes w_\ell) = (\lambda_k \mu_\ell)(v_k \otimes w_\ell)$.

($c$) Defining $v_k^*$ to be the indicator function of $\{v_k\}$, $V^*$ has basis $\{v_k^*\}_{k=1}^m$, and $g(v_k^*) = v_k^* \circ g^{-1} = \lambda_k^{-1} v_k^*$. The claim then follows from the fact that the eigenvalues are roots of unity. □

**Exercise 7.** Given $G$ acting by permutation on a set $X$ with associated permutation representation $V_X$, show that $\chi_{V_X}(g)$ is the number of elements of $X$ fixed by $g$.

## 2.1 The main result

We are almost ready to prove the main result of this handout, the ONB property of irreducible characters. To do so we require an easy but powerful characterization of homomorphisms between irreducible representations.

**Lemma 8** (Schur's lemma)**.** *Let $\varphi\colon V \to W$ be a homomorphism of irreducible representations of $G$. Then either $\varphi = 0$ or $\varphi$ is an isomorphism, and if $W = V$, then $\varphi = \lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$.*

*Proof.* Since $\ker\varphi$ and $\operatorname{im}\varphi$ are subrepresentations of $V$ and $W$ respectively, we either have $\ker\varphi = V$ and $\operatorname{im}\varphi = 0$ or $\ker\varphi = 0$ and $\operatorname{im}\varphi = W$. If $V = W$ then $\varphi$ has some eigenvalue $\lambda$, and $\varphi - \lambda \cdot \mathrm{id}_V$ is an endomorphism of $V$. But this map cannot be an isomorphism since its kernel is nontrivial, so $\varphi = \lambda \cdot \mathrm{id}_V$. □

In this proof again we see the good behavior of our restricted setting, in the form of the algebraic closure of $\mathbb{C}$ to guarantee the existence of some eigenvalue.

Now define an inner product on the space $\mathbb{C}^{\mathrm{Cl}(G)}$ of $\mathbb{C}$-valued class functions on $G$ by

$$\langle \alpha, \beta \rangle := \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g).$$

It is this inner product with respect to which the irreducible characters form an orthonormal basis. Here again we see the niceness of our setting, since this inner product only exists if the characteristic of the field in which we are working does not divide $|G|$.

**Theorem 9** (The ONB property of irreducible characters)**.** *The irreducible characters of $G$ form an orthonormal basis of $\mathbb{C}^{\mathrm{Cl}(G)}$ with respect to $\langle \cdot, \cdot \rangle$.*

*Proof.* We first will determine the multiplicity of the trivial representation in a given representation $V$. We write $V_G := \{v \in V : g(v) = v \text{ for all } g \in G\}$. Define $\varphi\colon V \to V$ by

$$\varphi(v) := \frac{1}{|G|} \sum_{g \in G} g(v).$$

This is easily seen to be $G$-linear and moreover is a projection onto $V_G$: that $\operatorname{im}\varphi \subseteq V_G$ is clear, and if $v \in V_G$ then $\varphi(v) = v$. Thus

$$\dim V_G = \operatorname{tr}\varphi = \frac{1}{|G|}\sum_{g\in G}\chi_V(g).$$

Now let $V$ and $W$ be any representations of $G$. Then $\operatorname{Hom}(V,W)_G$ is simply the space of $G$-linear maps from $V$ to $W$, and, in particular, Schur's lemma tells us that if $V$ and $W$ are irreducible then $\dim\operatorname{Hom}(V,W)_G$ is either 1 if $V \cong W$ or 0 otherwise. But $\operatorname{Hom}(V,W) \cong V^* \otimes W$ as representations, so

$$\dim\operatorname{Hom}(V,W)_G = \frac{1}{|G|}\sum_{g\in G}\chi_{\operatorname{Hom}(V,W)}(g) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}\chi_W(g). \tag{2}$$

This shows that the irreducible characters form an orthonormal system.

To show that this is a basis, suppose $\alpha\colon G \to \mathbb{C}$ is a class function; we will show that if $\langle\alpha,\chi_V\rangle = 0$ for all irreducible representations $V$ of $G$ then $\alpha = 0$. First, for any representation $V$, define $\varphi_V\colon V \to V$ by

$$\varphi_V(v) := \sum_{g\in G}\alpha(g)\cdot g(v). \tag{3}$$

This map is $G$-linear, since

$$\varphi_V \circ h = \sum_{g\in G}\alpha(g)\cdot gh = \sum_{g\in G}\alpha(hgh^{-1})\cdot hg = \sum_{g\in G}\alpha(g)\cdot hg = h\circ\varphi_V.$$

By Schur's lemma, if $V$ is irreducible we have $\varphi_V = \lambda\cdot\operatorname{id}_V$, and

$$\lambda = \frac{1}{\dim V}\operatorname{tr}\varphi_V = \frac{1}{\dim V}\sum_{g\in G}\alpha(g)\chi_V(g) = \frac{1}{\dim V}\overline{\langle\alpha,\chi_{V^*}\rangle} = 0,$$

since $V^*$ is irreducible (by either Exercise 5 or equation (2); for the latter, see Corollary 10). It follows that $\varphi_V = 0$ for any representation of $V$ by Maschke's theorem. In particular, we have $\varphi_{R_G} = 0$; but in $R_G$ the elements $g \in G$, thought of as linear transformations of $R_G$, are linearly independent, since, for example, $\{g(v_1) = v_g : g \in G\} \subseteq R_G$ is linearly independent. Thus $\alpha = 0$. □

It follows from Theorem 9 that the irreducible representations of $G$ are in bijection with the conjugacy classes of $G$.

Using this theorem we can provide the promised uniqueness component of Maschke's theorem, and we see that the character completely determines a representation.

**Corollary 10.** *Suppose $V := \bigoplus V_n^{\oplus a_n}$ is a representation of $G$, where the $V_n$ are distinct irreducible representations of $G$. Then $a_n = \langle\chi_V,\chi_{V_n}\rangle$. Also, $\langle\chi_V,\chi_V\rangle = \sum a_n^2$, and, in particular, $V$ is irreducible if and only if $\langle\chi_V,\chi_V\rangle = 1$.*

*Proof.* These follow immediately from the facts that $\chi_V = \sum a_n \chi_{V_n}$ and that the irreducible characters $\chi_{V_n}$ are orthogonal with respect to $\langle \cdot, \cdot \rangle$. $\square$

We can also derive orthogonality relations for the columns of the matrix whose rows are the irreducible characters.

**Corollary 11.** *Let $\chi_1, \ldots, \chi_n$ be the irreducible characters for $G$. For any $g \in G$,*

$$\sum_{k=1}^{n} |\chi_k(g)|^2 = \frac{|G|}{c(g)},$$

*where $c(g)$ is the number of elements of the conjugacy class of $G$. If $h \in G$ is not conjugate to $g$, then*

$$\sum_{k=1}^{n} \overline{\chi_k(g)} \chi_k(h) = 0.$$

*Proof.* Let $T$ be the matrix whose $k, \ell$ entry is $\chi_k(g_\ell)$, where $g_\ell$ ranges over the conjugacy classes of $G$, and let $D$ be the diagonal matrix whose $k, k$ entry is $|G|/c(g_k)$. Then $TDT^* = I$ by Theorem 9, so $T^*T = D^{-1}$. $\square$

**Exercise 12.** Let $V$ and $W$ be irreducible representations of $G$ and let $L_0 \colon V \to W$ be any linear map. Show that the map $L \colon V \to W$ defined by

$$L(v) = \frac{1}{|G|} \sum_{g \in G} (g^{-1} \circ L_0 \circ g)(v)$$

is the zero map if $V \ncong W$ as representations, and, if $V = W$, is multiplication by $(\operatorname{tr} L_0)/(\dim V)$.

**Exercise 13.** Given $\alpha \colon G \to \mathbb{C}$, prove that if $\alpha$ is not a class function then the map $\varphi_{R_G}$ defined as in equation (3) is not $G$-linear (remember that $R_G$ is the regular representation—see section 1.2).

**Exercise 14.** Show that, if $V$ is irreducible, then $\langle \cdot, \cdot \rangle_G$ is, up to a scalar, the unique $G$-invariant inner product on $V$, as follows. Let $\langle \cdot, \cdot \rangle'_G$ be another $G$-invariant inner product on $V$. Define $H_1, H_2 \colon V \to V^*$ by $H_1(v) = \langle v, \cdot \rangle_G$ and $H_2(v) = \langle v, \cdot \rangle'_G$. Deduce from the $G$-invariance of $\langle \cdot, \cdot \rangle_G$ and $\langle \cdot, \cdot \rangle'_G$ that $H_2^{-1} \circ H_1$ is a $G$-linear map, and apply Schur's lemma.