Partitions and the pentagonal number theorem

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Reading Classics

September 8, 2020

What is a partition?

A *partition* of a positive integer n is a way to write n as a sum of positive integers. For example:

$$\begin{aligned}
6 &= 1+1+1+1+1+1 \\
&= 2+1+1+1+1 \\
&= 2+2+1+1 \\
&= 2+2+2 \\
&= 3+1+1+1 \\
&= 3+2+1 \\
&= 3+3 \\
&= 4+1+1 \\
&= 4+2 \\
&= 5+1 \\
&= 6
\end{aligned}$$

Early history of partitions

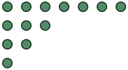
Leibniz first considered the problem of determining the number of partitions of n in a 1674 paper to J. Bernoulli.

Later, Euler was asked in 1740 by Philippe Naudé to determine the number of partitions of n. Euler's results laid the basis for the theory of partitions.

Ferrers diagrams

Partitions can be represented graphically via *Ferrers diagrams*:

$$13 = 7 + 3 + 2 + 1 \rightsquigarrow$$



These diagrams can be used to derive interesting results about partition, such as the following:

Theorem (Adams 1847, Ferrers, Sylvester 1853)

Let m and n be positive integers. Then the number of partitions of n into m parts is equal to the number of partitions of n into parts the largest of which is m.

Ferrers diagrams and a neat partition identity

Theorem

Let m and n be positive integers. Then the number of partitions of n into m parts is equal to the number of partitions of n into parts the largest of which is m.

Proof.

We associate a partition with its *conjugate partition*:

$13 = 7 + 3 + 2 + 1$ $\qquad$ $13 = 4 + 3 + 2 + 1 + 1 + 1 + 1$

$\square$

What is a generating function?

A *generating function* is a way of encoding a sequence (a_n) as a formal power series:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Generating function example

For example, if we take (a_n) to be the sequence $(0, 1, 1, 2, 3, 5, \dots)$ of Fibonacci numbers, we can compute

$$\begin{aligned} f(x) &= 0 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots \\ &= \quad x + (x^2 + \;\; x^3 + 2x^4 + 3x^5 + \cdots) \\ &\qquad\qquad + (x^3 + \;\; x^4 + 2x^5 + \cdots) \\ &= x + xf(x) + x^2 f(x), \end{aligned}$$

so

$$f(x) = \frac{x}{1 - x - x^2}.$$

Open-ended question

Why does the fact that the denominator of $f(x)$ have a zero at $x = 1/\varphi$ mean that the nth Fibonacci number is approximately φ^n?

What is the partition function?

Given a positive integer n, the *partition function* $p(n)$ is the number of partitions of n. For example:

$$\left.\begin{aligned} 5 &= 1+1+1+1+1 \\ &= 2+1+1+1 \\ &= 2+2+1 \\ &= 3+1+1 \\ &= 3+2 \\ &= 4+1 \\ &= 5 \end{aligned}\right\} p(5) = 7.$$

We take as convention that $p(0) = 1$. (Why is this the most reasonable decision?)

The generating function of the partition function

Theorem (Euler, 1751)

Let $f(x)$ denote the generating function of $p(n)$. Then

$$f(x) = \prod_{n=1}^{\infty} \frac{1}{1 - x^n}.$$

Proof.

Expand $f(x)$ using the geometric series:

$$\begin{aligned} f(x) = (1 + x^1 + x^{2\cdot 1} + x^{3\cdot 1} + \cdots) &\cdot (1 + x^2 + x^{2\cdot 2} + x^{3\cdot 2} + \cdots) \\ &\cdot (1 + x^3 + x^{2\cdot 3} + x^{3\cdot 3} + \cdots) \cdots \end{aligned}$$

Each way to get x^n when expanding this corresponds to a partition of n: the term from the first pair of parentheses tells you the number of parts of size 1; the term from the second tells you the number of parts of size 2; and so on. $\square$

Some restricted partitions functions

Let $p_d(n)$ denote the number of partitions of n into distinct parts, and let $p_o(n)$ denote the number of partitions of n into odd parts. For example:

$$\underbrace{1+1+1+1 \;=\; 2+1+1 \;=\; 2+2 \;=\; 3+1 \;=\; 4}_{p_d(4)=p_o(4)=2}$$

Exercise

Let $f_d(x)$ denote the generating function of $p_d(n)$ and let $f_o(x)$ denote the generating function of $p_o(n)$. Use the technique of the previous theorem to show that

$$f_d(x) = \prod_{n=1}^{\infty}(1+x^n) \quad \text{and} \quad f_o(x) = \prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}}.$$

Euler's identity

Theorem (Euler, 1748)

$p_d(n) = p_o(n)$ for every positive integer n.

Proof.

The following calculation shows that $f_d(x) = f_o(x)$:

$$\begin{aligned} f_d(x) &= (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots \\ &= \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \cdot \frac{1-x^8}{1-x^4} \cdots \\ &= \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots \\ &= f_o(x). \end{aligned}$$

□

Exercise

Can you find a combinatorial proof?

Generalizations of Euler's identity

Theorem (Sylvester, 1882)

The number of partitions of n into odd parts, where exactly k distinct parts appear, is equal to the number of partitions of n into distinct parts, where exactly k sequences of consecutive integers appear.

Theorem (Glaisher, 1883)

The number of partitions of n into parts not divisible by d is equal to the number of partitions $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ where $\lambda_i \geqslant \lambda_{i+1}$ and $\lambda_i > \lambda_{i+d-1}$.

Theorem (Rogers-Ramanujan, 1894)

The number of partitions of n into parts differing by at least 2 is equal to the number of partitions of n into parts which are congruent to 1 or 4 modulo 5.

The pentagonal number theorem

Theorem (Euler, 1760)

$$\prod_{n=1}^{\infty}(1-x^n) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \cdots,$$

where the exponents on the right-hand side are $k(3k-1)/2$ *for* $k = 0, 1, -1, 2, -2, 3, -3, \ldots$.

Franklin's 1881 proof of the PNT

Proof. Note that if

$$(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots = 1 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

then a_n is

the number of partitions of n into an
even number of distinct parts

minus

the number of partitions of n into an
odd number of distinct parts.

Franklin's 1881 proof of the PNT

even number of distinct parts $\Longleftrightarrow$ odd number of distinct parts

Consider the following operation on Ferrers diagrams with distinct parts:

$r > b$:

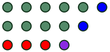
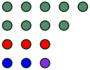
$r \leqslant b$:

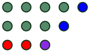
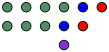
This operation switches the parity of the number of parts, preserves distinctness of the number of parts, and is reversible!

Franklin's 1881 proof of the PNT

Almost...

Consider the following exceptional cases:

$r = b + 1$:  $\rightsquigarrow$  !!?

$r = b$:  $\rightsquigarrow$  !??

These occur only when the blue and red dots overlap, and only when $r = b + 1$ or $r = b$. The sizes of these exceptional cases are $k(3k - 1)/2$ and $k(3k + 1)/2$ — exactly where the terms in the Euler function are! $\square$

Euler's pentagonal number recurrence

Theorem (Euler, 1760)

For every positive integer n,

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + \cdots .$$

Proof.

Recall that, if $f(x)$ is the generating function of $p(n)$, then

$$f(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n}.$$

Hence, by the pentagonal number theorem,

$$(1 - x - x^2 + x^5 + x^7 - \cdots) \cdot (1 + p(1)x + p(2)x^2 + p(3)x^3 + \cdots) = 1.$$

Thus the coefficient on x^n on the left side must equal zero. $\square$

References

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