Multi-Alpha Equity Portfolios:

An Integrated Risk Budgeting Approach for Robust Constrained Portfolios.

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Abstract

We propose a robust optimization approach to construct realistic constrained multi-strategy portfolios which starts with the identification of different sources of alpha and the risk-budgeting exercise to optimally combine them. We show how systematic alpha-capture strategies can be combined with judgmental strategies and how bottom-up based strategies for stock picking can be combined with top-down sector and country allocation strategies. The approach is shown to be fully transparent for both unconstrained and constrained portfolios with a discussion of how constraints impact the final optimal portfolio allocation. In particular we show that the constrained portfolios retain the exposures to systematic risk in the unconstrained target solution as much as possible, and that specific risk takes the toll of portfolio constraints. Through a realistic back-tested example combining different well-known alpha capture strategies we demonstrate the robustness and transparency of the approach. Finally we also discuss the advantages of this approach over the alternative process based on selecting and investing in a mix of different index-funds implementing off-the-shelf active strategies for alpha capture. We believe that our approach is particularly suited for institutional investors interested in risk budgeting the alpha in their portfolios while fully understanding the final allocation in their constrained portfolios.

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Empirical evidence that the Capital Asset Pricing Model (CAPM) does not give an accurate description of markets has been growing for years. For equities, examples include the alpha of smaller capitalization stocks over large capitalization stocks (Banz [1981]), of value stocks over growth stocks (Fama and French [1992]), of longer-term winners over losers (Jegadeesh and Titman [1993]), of short-term losers over winners (Lehmann [1990]) or of low-risk over high-risk stocks (Haugen and Heins [1972], Black et al. [1972]).

Although the focus of the literature has been mainly on alpha at bottom-up stock level, there is also empirical evidence of violations of the CAPM model at top-down sector and country level. Evidence of momentum and size anomalies at sector and industry level have been reported (Moskowitz and Grinblatt [1999], Capaul [1999]). Abnormal seasonal effects in global sector returns have been observed (Doeswijk [2008]). Empirical evidence of momentum and value anomalies at country level have been reported (Desrosiers et al. [2004]).

These alphas tend to be explained by structural mispricings due to behavioral biases of investors which are not accounted for in the simplistic CAPM. The empirical demonstration of alphas is usually done by showing how a given active systematic strategy which tilts some stocks away from their market capitalization weight would have generated abnormally higher returns that are not explained by the portfolio exposure to the market as described by its beta. The tilts are determined by the exposure of stocks to factors believed to predict investor behavior.

The active strategies designed to capture the underlying alphas may use different factors. For example, price-to-book, price-to-earnings or dividend yield are typical factors used to capture value alpha. But it is difficult to know which best predicts investor behavior. It is even possible that several factors could be at play. For instance, higher demand for risky stocks is sometimes attributed to investor preference for risk (stocks as lottery tickets) and to overconfidence in forecasting higher returns for risky stocks. But it can also be explained by the fact that most investors cannot easily leverage low-beta stocks (Baker et al. [2011]). It is therefore not surprising that both volatility of returns and beta are factors considered in strategies designed to capture the alpha from low-risk stocks.

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The empirical evidence that systematic strategies can be used to capture alpha opens the question of whether markets are really information efficient. Investors spending the extra time researching company statements and foreseeing the impact of management behavior and decisions may be able to use such information to generate alpha from their forecasts of stock returns. Although it will always be much more difficult to prove the ability to generate alpha in such a way, evidence is available (e.g. Gergaud and Ziemba [2012]).

Alpha is not equally available to investors. The larger the investor the more difficult it is to implement active strategies without leaving a market impact which will detract from performance, in particular for strategies which require high turnover. Momentum alpha, and in particular short-term momentum alpha, will be much more difficult to capture than value or low-risk alpha which require less turnover.

The optimal capture of alpha should therefore take into account investor size and constraints. The size of the investor caps turnover and limits the access to some alpha strategies like short-term momentum. Other typical constraints are i) a long-only constraint, which makes it impossible to sell-short stocks ii) liquidity constraints, in particular for larger investors who may find it difficult to invest at all in some stocks, iii) stock exclusion list constraints, where investing in some stocks is not authorized due for example to non socially-responsible behavior of companies, and finally, iv) restricted access to derivatives instruments, in particular over-the-counter. In fact, many investors are restricted to construct their portfolios with only a small number of sufficiently liquid stocks, typically avoiding those with the smaller market capitalizations and volume, and are subject to pre-defined exclusion lists.

Passively managed index funds implementing off-the-shelf active strategies exposed to some of these alphas are now available. For example, index-funds designed to capture value alpha have been shown to be fully explained by their loadings to value factors (Blitz and Swinkels [2008]). While index funds offer easy access to systematic alphas at low management fees, investors should be aware of some drawbacks. The transparency and profitability requirements in many such approaches tend to lead to over-simplification, unnecessary constraints and consequently to sub-optimal capture of the underlying alphas (Blitz [2012]). Risk management is sometimes sacrificed for simplicity with no clear definition of the target risk budget. Constraints on country and sector maximum deviations against the market capitalization index are also over-simplifications of risk controls which in many cases are not needed. The fact that index funds need to grow for profitability reasons also means that their strategies may constrain turnover to sub-optimal levels for a given investor. And some index-funds may present a higher risk of crowding as they grow. Increasing assets into sometimes relatively concentrated portfolios can lead to considerable deviation from the more liquid market capitalization allocation. Finally, the history of most active indexes available today is, for the most part, not more than a historical back-test constructed with the benefit of hindsight, free of transaction costs and market impact. The benefit of lower management fees can quite easily be offset by poorer returns resulting from such drawbacks. Non-indexed active funds clearly have more choices to adapt to since they are not slaves of an index-strategy that is difficult to change once the index-fund attracts assets.

If many index-funds are very clear about their objective to capture alpha there are also examples of increased complexity beyond what seems necessary, obscuring the real sources of alpha in the final index. Carvalho et al. [2012] have recently shown that equity risk-based strategies like Minimum Variance, Maximum Diversification or Risk Parity are examples of unnecessarily complex and sub-optimal portfolio constructions capturing a mix of low-risk, small-cap and value alphas while offering a defensive beta ($\beta < 1$).

This has important implications for the construction of a portfolio aiming at being exposed to a number of such alpha sources. The fact that index-funds are implementing alpha strategies which may be sub-optimal, that many are already exposed to more than one alpha source and that not all known alpha sources are readily available are strong constraints to a proper risk-budgeting exercise.

The alternative is to use an integrated approach to construct one single portfolio combining the different strategies to capture alpha. In our view, constructing such a portfolio starts with a risk-budgeting exercise to combine unconstrained systematic or judgmental portfolios designed for alpha capture, whether for stock picking or top-down sector or country allocation approaches. The optimal unconstrained portfolio which combines all available alpha sources is then simply a risk-weighted average of the unconstrained portfolios given by the strategies designed to capture each individual source of alpha.

A robust constrained optimal portfolio can be derived from the unconstrained optimal allocation by estimating its implied stock returns and using them as inputs in constrained mean-variance optimization. The implied returns are, by definition, the stock returns which render the underlying allocation optimal in the absence of constraints. As we shall show, constrained portfolios generated from implied returns do compare well with the starting underlying unconstrained portfolio; constraints impact essentially the specific risk of the portfolio while systematic risk exposures remain represented at a comparable level to the extent that it is possible. We show that for as long as constraints are not too binding the final resulting constrained stock allocation remains close to that in the unconstrained portfolio. The more the portfolio relies on specific risk the more constraints are likely to flatten the efficient frontier. Portfolios relying more on systematic risk will handle constraints more easily as the optimizer will be able to find similar exposures to systematic risk with small changes in the weights of other stocks.

In this paper we first show how to build an unconstrained alpha capture portfolio using both systematic and judgmental approaches. We discuss the important exercise of allocating a risk budget to each alpha source in the portfolio, which defines how alpha sources are combined to form an optimal unconstrained target portfolio allocation. In the absence of any constraints the portfolio construction would end here. But this is never the case. We show how the stock implied returns derived from this optimal unconstrained allocation can be used to generate mean-variance efficient constrained portfolios, which are robust in the sense that they retain the exposures to the alpha sources as much as constraints allow. In the second part of the paper, we present two examples to illustrate the implementation of the approach. The properties of constrained portfolios built using this approach are discussed in detail. In the annex, we discuss the differences between this approach and the well-know Black-Litterman model [Black and Litterman (1992)] and include a number of analytical insight regarding the impact of constraints.

FRAMEWORK

Below we present in detail the framework starting with i) a discussion of how to build alpha capture portfolios followed by ii) the risk budgeting exercise for the each alpha capture strategies and finally iii) the step of portfolio construction, unconstrained and with constraints.

Alpha capture portfolios

Whether a judgmental or systematic approach is used, unconstrained strategies to capture alpha can be built from a list of selected stocks which are expected to generate alpha. Systematic alpha capture strategies select stocks on the basis of their exposure to a given factor like value or momentum. Judgmental approaches will select stocks on the basis of fundamental analysis.

Alpha can be created either by tilting the portfolio in favor of selected stocks expected to generate positive alpha, away from those expected to generate negative alpha or both. The active portfolio representing deviations to the market capitalization portfolio can be represented by a zero-sum long-short portfolio.

The active weights allocated to each selected stock can be inversely proportional to their volatility, in which case the active allocation is mean-variance efficient if all pair-wise correlations were equal and the expected information ratio for each stock with a non-zero active weight is equal in magnitude, with the sign changing to negative for those with a negative active weight. An equally-weighted long-short portfolio representing active weights is more often used for simplicity and because it will generate less turnover. This would be the optimal solution if additional stocks had the same volatility. In general, when a sufficiently large number of stocks are retained, the difference between equally-weighting or risk weighting can be small. At each re-balancing, each alpha capture strategy will thus generate an equally-weighted long-short portfolio representing active stock weights.

Unconstrained portfolios for sectors and countries can be constructed using similar approaches.

Risk budgeting allocation to strategies

Mean-variance optimization offers a good starting point for alpha risk budgeting. Kritzman [2006] has shown that mean-variance optimization allocates sensibly when correlations are not too large, but other more robust approaches like re-sampling optimization could be considered (Scherer [2004]) when reliable data is available.

If we can forecast the expected risk-adjusted returns for each strategy (information ratio) and measure the expected level of correlation between each pair of strategies (e.g. historically), then, the mean-variance optimal risk budget is given by[1]:

$$\mathbf{RB} = \frac{1}{\eta}\mathbf{\Theta}^{-1}\mathbf{IR} \tag{1}$$

where **RB** is the vector of optimal risk budget allocated to each strategy, $\mathbf{\Theta}$ is the pair-wise correlation matrix for the strategies and **IR** is the vector with the expected information ratio for each strategy. Finally, η is a parameter which measures the overall risk aversion and can be scaled so that the ex-ante risk reaches a given target level. The total risk budget allocation is inversely proportional to the decrease in overall risk aversion.

The information ratio of the multi-strategy portfolio is the same irrespective of the level of risk aversion and is the largest possible for any combination of the strategies, i.e. the mean-variance optimization maximizes the information ratio of the unconstrained multi-strategy portfolio.

Uncorrelated strategies: if the strategies are uncorrelated then the unconstrained optimal mean-variance risk budget allocation to each strategy is simply proportional to the information ratio for each individual strategy. The equation above simplifies to:

$$\mathbf{RB} = \frac{1}{\eta}\mathbf{IR} \tag{2}$$

Equal risk-adjusted returns: if there is no reason to expect a different risk-adjusted return from each strategy (no view on the information ratio of the strategies, in which case it can makes sense to assume they will deliver the same risk-adjusted return), then equation (1) simplifies to:

$$\mathbf{RB} = \frac{IR}{\eta}\mathbf{\Theta}^{-1}\mathbf{1} \tag{3}$$

IR is the information ratio of all strategies and **1** is unit vector. Now the risk budget depends only on correlations: the optimal risk budget allocation minimizes correlation allocating higher risk budget to strategies with the lowest correlations and lower risk budget to those more correlated with others while scaling with IR/η. This strategy has been called maximum diversification in a different context (Choueifaty and Coignard [2008]).

Absence of any prior information: in the absence of any prior information about the expected future performance of the strategies and their correlation, the most obvious solution is to allocate an equal risk budget to each strategy. This is the optimal mean-variance allocation if we expect all strategies to deliver exactly the same risk-adjusted return and to be equally correlated. In this case, equation (1) simplifies to:

$$\mathbf{RB} = \frac{IR}{\eta} \tag{4}$$

Portfolio construction

This follows three steps: i) the construction of the unconstrained target allocation at stock level from the risk budget allocation to strategies, ii) the estimation of the stock implied returns for that unconstrained target allocation and iii) a mean-variance optimization step starting from the stock implied returns and applying constraints.

Aggregation of strategies into an optimal unconstrained target allocation: the optimal vector of unconstrained active stock weights $\mathbf{P}_A$ can be determined from the risk budget allocation to strategies and, from their underlying portfolios, resolved at stock level.

The vector $\mathbf{P}_A$ can be built from a matrix $\mathbf{P}_S$ with the active weight of each stock in each strategy, with strategies in rows and stocks in columns. The vector $\mathbf{P}_A$ is then given by the weighted average of the active stock weight in each strategy:

$$\mathbf{P}_A = \mathbf{P}_S \cdot \mathrm{w} \tag{5}$$

where the vector w takes into account the ex-ante volatility σ_i of the stock level allocation representing the view of strategy *i* at that point in time and the risk budget allocation to each strategy, RB_i:

$$w_i = \sigma_i^{-1} \cdot RB_i \tag{6}$$

Risk model: as usual when dealing with large stock universes, we consider a linear factor model. We assume a set of stock exposures to risk factors $\mathbf{\Phi}$ and stock specific risks $\mathbf{\Delta}$, the risk model $\mathbf{\Sigma}$ is then:

$$\mathbf{\Sigma} = \mathbf{\Phi}'\mathbf{\Lambda}\mathbf{\Phi} + \mathbf{\Delta} \tag{7}$$

with $\mathbf{\Lambda}$ the variance-covariance matrix of factor returns. The framework is independent of the risk model. Commercial risk models like Barra, Northfield, Axioma or APT can be easily employed. Alternatively, a statistical risk model based on principal component risk factor decomposition could be selected. Full covariance matrix models like Bayesian approaches can be equally considered although the results concerning separation into systematic and specific risk discussed below do not apply.

Implied active returns for each stock: once the target unconstrained active allocation $\mathbf{P}_A$ has been built and the risk model $\mathbf{\Sigma}$ chosen, we can estimate the implied excess returns $\mathbf{R}_I$ for each stock from

$$\mathbf{R}_I = \lambda \mathbf{\Sigma} \cdot \mathbf{P}_A \tag{8}$$

λ is related to the information ratio of the portfolio $\mathbf{P}_A$ by $IR_{\mathbf{P}_A} = \mathbf{P}_A{}'\mathbf{R}_I / \sigma_{\mathbf{P}_A} = \lambda\left(\mathbf{P}_A{}'\mathbf{\Sigma} \cdot \mathbf{P}_A / \sigma_{\mathbf{P}_A}\right)$. Therefore, with T = 12, 52 or 260 for monthly, weekly or daily trading data, respectively, $\lambda = IR_{\mathbf{P}_A} \cdot T \big/ \sigma_{\mathbf{P}_A}$.

The vector of implied stock excess returns is by definition the set of stock excess returns that renders the unconstrained active portfolio $y^* = (\lambda/\eta)\mathbf{P}_A$ efficient, where y^* is the solution to the unconstrained mean-variance optimization:

$$y^* = \frac{1}{\eta}\mathbf{\Sigma}^{-1}\mathbf{R}_I \tag{9}$$

for a level of risk aversion η. The implied excess returns translate at stock level the risk budgeting allocation to each alpha capture strategy and are robust with regard to the risk model. In the annex we discuss the differences between the implied returns estimated in this way and the stock returns estimated with a Black-Litterman model.

Handling constraints: the problem of portfolio constraints can now be addressed by running a constrained mean-variance optimization using the implied excess returns which efficiently represent the aggregation of views given by the different strategies.

The mean-variance optimization problem in (9) under k linear constraints, $(v_i)_{1 \le i \le k}$, can be translated into:

$$y^* = \arg\min \frac{\eta}{2} y'\Sigma y - y'\mathbf{R}_I \quad \text{u.c.} \quad v_i{}'\, y \ge u_i, \forall\, 1 \le i \le k \tag{10}$$

with the solution y^* the optimal constrained portfolio of active weights at risk aversion η. $\mathbf{R}_I$ is defined in equation (8) as the vector of implied excess return for the unconstrained portfolio of active weights $\mathbf{P}_A$.

When $k=0$, i.e. no constraints, the solution is simply $y^* = (\lambda/\eta)\mathbf{P}_A$ as seen in the previous section. Using equation (8) in equation (10) it is then relatively easy to show that the minimization is equivalent to:

$$y^* = \arg\min(y - \frac{\lambda}{\eta}\mathbf{P}_A)'\Sigma(y - \frac{\lambda}{\eta}\mathbf{P}_A) \quad \text{u.c.} \quad v_i' y \geq u_i, \forall 1 \leq i \leq k \tag{11}$$

i.e. minimizing the tracking error risk of a long-short allocation representing the active weights of the unconstrained portfolio relative to the long-short allocation representing the active weights of the constrained portfolio at the same risk aversion η.

If instead we look for the optimal constrained portfolio at a given target tracking error risk, then in equation (10) the first term is constant and we can re-write it using equation (8) for the implied returns as:

$$y^* = \arg\max y' \mathbf{\Sigma P}_A \quad \text{u.c.} \quad v_i' y \geq u_i, \forall 1 \leq i \leq k \tag{12}$$

which is the maximization of the covariance of the constrained active allocation y^* with the unconstrained active portfolio $\mathbf{P}_A$.

Mean-variance optimization can handle a number of traditional linear constraints, for example constraints on the weight of individual stocks or portfolios of stocks. In the annex we discuss analytically the impact of the most typical linear constraints on the final constrained portfolio. Turnover constraints can also be imposed but we do not include a discussion of those.

From a practical point of view, the impact of constraints in the final stock allocation can be monitored by comparing not only the resulting constrained active weights with the initial target unconstrained active weights but also the factor risk exposures in the initial unconstrained target allocation to risk exposures in the constrained allocation. As demonstrated in the annex, the optimal mean-variance unconstrained and constrained portfolios exhibit similar exposures to the risk factors in the risk model unless constraints become too binding. The impact of constraints will be felt essentially in the exposure to stock specific risks.

We can also monitor how much return is detracted by the constraints when comparing the expected return of the constrained portfolio with that of the unconstrained target portfolio. The percentage of return lost to constraints is a useful indicator to find the best working range of ex-ante risk. If risk is too high, there is the danger that the higher risk is no longer compensated by higher returns when constraints flatten the efficient frontier too much. Constraints can also have a major impact at lower risk levels and, as we shall see later, taking too little active risk can lead to a less optimal representation of views too.

EXAMPLES

Portfolio allocation at a given date for European stocks

In this first example we want to illustrate step-by-step how the methodology can be implemented and discuss in detail the impact of different constraints on the final portfolio.

Risk model: we used a principal components analysis (PCA) risk model based on two years of weekly returns following the methodology proposed by Plerou et al. [2002] that considers results from random matrix theory showing that the eigenvalues λ of a $T \times N$ random matrix with variance σ^2 are capped asymptotically at $\lambda_{max} = \sigma^2 \ (1 + N / T + 2 \ (N / T)^{1/2})$ with T the number of periods and N the number of stocks. Thus we discard all eigenvalues smaller than λ_{max}, considered to be statistical noise.

Benchmark index: for the sake of illustration we chose a simplified example starting from a market capitalization index with a relatively small number of stocks. We built an active portfolio based on the 50 European stocks in the Eurostoxx 50 index overlaying active strategies on this market capitalization index on 16th August 2012. Three systematic sources of alpha at stock level were taken into account: small-cap, value and momentum. We also considered a systematic momentum approach at sector level as an additional systematic source of alpha. Finally we include a typified example of judgmental views.

Alpha capture portfolios: for the sake of transparency, the alpha capture strategies here considered have been deliberately kept simple and by no means do we pretend that they represent the most efficient approach to capture the underlying alpha. In exhibit one we show an unconstrained active portfolio for each alpha capture strategy on this date. The stocks have been screened by market capitalization for small-cap, book-to-price for value and the average 11 month returns ending the 18th July 2012 (excluding the last four weeks) for momentum. The portfolios are long (short) the smaller (larger) capitalization stocks, those with the larger (lowest) book-to-price and with the strongest (weakest) momentum. For simplification we chose to equally-weight stocks in each long-short portfolio. For sector momentum, the strategy portfolio is long sectors with the strongest momentum measured by the last 12 month return and short sectors with the weakest momentum. Finally, we assumed that a judgmental process had selected a number of stocks based on their higher expected returns. As it is not our purpose to build investment cases for companies but just to show how these could have been taken into account, we arbitrarily selected all companies starting with the letter “A”. The judgmental strategy can also be represented using an equally-weighted long-short portfolio, long all stocks selected and selling-short all other stocks in the index.

Stocks	Sectors	Active Weights (%) Small	Value	Momentum	Judgmental	Sector Momentum		Unconstrained	Implied Excess Returns (%) Unconstrained Portfolio	Portfolio Weights (%) Eurostoxx Index	Unconstrained	Long-only 1	Long-only + Liq Constraints	Long-only 2
Ex-ante tracking error risk (%)		5.0	5.0	5.0	5.0	5.0		5.0		0.0	5.0	4.0	3.2	5.0
Daimler AG		-2.62	0.00	0.00	-1.79		0.00	-3.84	-4.081	2.50	-1.34	0.00	0.00	0.00
LVMH Moet Hennessy Louis Vuitton		-2.62	-1.20	0.00	-1.79		0.00	-4.88	-2.176	2.44	-2.44	0.00	0.00	0.00
Volkswagen AG (Pfd Non-Vtg)	Consumer Discretionary	0.00	0.00	1.35	-1.79	0.00	0.00	-0.39	-3.375	1.43	1.05	0.34	1.66	0.00
Bayerische Motorenwerke AG BMW		0.00	0.00	0.00	-1.79		0.00	-1.56	-3.197	1.33	-0.23	0.00	0.48	0.00
Industria de Diseno Textil S.A.		0.00	-1.20	1.35	-1.79		0.00	-1.43	-1.649	1.26	-0.17	0.00	0.00	0.00
Anheuser-Busch InBev		-2.62	-1.20	1.35	11.02		3.12	10.13	3.354	3.36	13.49	10.00	10.00	10.00
Unilever N.V.		-2.62	-1.20	1.35	-1.79		2.77	-1.30	-0.253	2.99	1.68	1.95	1.72	0.51
Danone S.A.	Consumer Staples	0.00	-1.20	0.00	-1.79	9.96	1.95	-0.91	-0.877	2.10	1.18	1.06	1.66	0.00
L'Oreal S.A.		0.00	-1.20	1.35	-1.79		1.53	-0.10	-0.601	1.65	1.56	1.55	1.43	0.30
Carrefour S.A.		2.62	-1.20	-1.35	-1.79		0.59	-0.99	-1.318	0.64	-0.35	0.00	0.00	0.00
Total S.A.		-2.62	0.00	0.00	-1.79		0.00	-3.84	-1.315	6.15	2.31	1.86	4.22	0.00
ENI S.p.A.	Energy	-2.62	0.00	1.35	-1.79	0.00	0.00	-2.66	-0.450	3.26	0.59	0.14	5.21	0.00
Repsol S.A.		2.62	1.20	-1.35	-1.79		0.00	0.59	1.253	0.83	1.42	1.26	0.00	1.72
Banco Santander S.A.		-2.62	1.20	-1.35	-1.79		-1.62	-5.37	-0.582	3.61	-1.75	0.00	0.00	0.00
Allianz SE		-2.62	0.00	0.00	11.02		-1.21	6.24	-0.266	2.71	8.95	7.31	10.00	7.97
BNP Paribas S.A.		-2.62	1.20	-1.35	-1.79		-1.06	-4.88	-1.303	2.37	-2.51	0.00	0.00	0.00
Banco Bilbao Vizcaya Argentaria S.A.		0.00	1.20	-1.35	-1.79		-0.94	-2.50	-0.615	2.11	-0.40	0.00	1.02	0.00
Deutsche Bank AG		0.00	1.20	0.00	-1.79		-0.74	-1.15	-1.545	1.65	0.50	0.00	1.15	0.00
ING Groep N.V.		0.00	1.20	-1.35	-1.79		-0.69	-2.29	-3.032	1.55	-0.74	0.00	0.40	0.00
AXA S.A.	Financials	0.00	1.20	0.00	11.02	-9.96	-0.67	10.04	2.916	1.49	11.53	10.00	10.00	10.00
Muenchener Rueckversicherungs-Gesellschaft AG		0.00	0.00	1.35	-1.79		-0.59	-0.90	-1.461	1.32	0.42	0.00	0.23	0.00
UniCredit S.p.A.		2.62	1.20	-1.35	-1.79		-0.52	0.14	5.263	1.16	1.30	1.49	0.67	3.78
Societe Generale S.A. (France)		2.62	1.20	-1.35	-1.79		-0.50	0.16	0.002	1.12	1.28	0.44	2.10	0.00
Intesa Sanpaolo S.p.A.		2.62	1.20	-1.35	-1.79		-0.50	0.16	1.214	1.12	1.28	0.58	1.97	0.00
Unibail-Rodamco SE		2.62	0.00	0.00	-1.79		-0.46	0.32	0.410	1.03	1.35	1.19	0.00	0.67
Assicurazioni Generali S.p.A.		2.62	0.00	-1.35	11.02		-0.45	10.28	4.789	1.01	11.29	10.00	0.00	10.00
Sanofi S.A.		-2.62	0.00	1.35	-1.79		5.48	2.10	0.065	5.64	7.74	7.65	8.15	9.13
Bayer AG	Health Care	-2.62	-1.20	1.35	-1.79	9.96	3.49	-0.67	-2.082	3.60	2.92	2.55	2.62	0.00
Essilor International S.A.		2.62	-1.20	1.35	-1.79		0.98	1.70	1.250	1.01	2.71	3.12	0.00	7.01
Siemens AG		-2.62	-1.20	0.00	-1.79		0.00	-4.88	-2.562	4.50	-0.38	0.00	1.38	0.00
Schneider Electric S.A.		0.00	-1.20	0.00	-1.79		0.00	-2.60	-3.116	1.90	-0.70	0.00	0.35	0.00
Vinci S.A.	Industrials	0.00	0.00	0.00	-1.79	0.00	0.00	-1.56	-0.791	1.28	-0.28	0.00	2.19	0.00
Koninklijke Philips Electronics N.V.		2.62	0.00	1.35	-1.79		0.00	1.89	0.160	1.18	3.08	2.78	3.67	3.91
Compagnie de Saint-Gobain S.A.		2.62	1.20	-1.35	-1.79		0.00	0.59	-1.124	0.82	1.41	0.67	0.00	0.00
SAP AG		-2.62	-1.20	1.35	-1.79		0.00	-3.71	-1.978	3.30	-0.41	0.00	0.19	0.00
ASML Holding N.V.	Information Technology	0.00	-1.20	1.35	11.02	0.00	0.00	9.70	2.300	1.34	11.04	10.00	10.00	10.00
Nokia Corp.		2.62	1.20	-1.35	-1.79		0.00	0.59	3.615	0.57	1.16	0.79	0.00	2.15
BASF SE		-2.62	-1.20	0.00	-1.79		0.00	-4.88	-3.948	3.89	-0.99	0.00	0.00	0.00
Air Liquide S.A.		0.00	-1.20	0.00	11.02		0.00	8.53	0.625	2.07	10.59	10.00	10.00	10.00
ArcelorMittal SA	Materials	2.62	1.20	-1.35	11.02	0.00	0.00	11.72	3.156	0.81	12.53	10.00	0.00	10.00
CRH PLC		2.62	0.00	1.35	-1.79		0.00	1.89	0.110	0.71	2.60	2.20	0.00	2.84
Telefonica S.A.		-2.62	-1.20	-1.35	-1.79		0.00	-6.05	-0.845	2.86	-3.19	0.00	0.98	0.00
Deutsche Telekom AG		0.00	0.00	0.00	-1.79		0.00	-1.56	-1.405	1.94	0.38	0.20	0.00	0.00
France Telecom	Telecom Svs	0.00	0.00	0.00	-1.79	0.00	0.00	-1.56	0.180	1.51	-0.05	0.00	3.95	0.00
Vivendi		0.00	0.00	0.00	-1.79		0.00	-1.56	-0.006	1.40	-0.16	0.00	1.36	0.00
E.ON AG		-2.62	0.00	1.35	-1.79		-3.35	-5.57	-4.778	2.52	-3.05	0.00	0.00	0.00
GDF Suez S.A.		0.00	1.20	0.00	-1.79		-2.50	-2.68	-2.555	1.88	-0.80	0.00	0.00	0.00
RWE AG	Utilities	2.62	0.00	1.35	-1.79	-9.96	-1.49	0.60	-3.373	1.12	1.72	0.89	0.38	0.00
Enel S.p.A.		2.62	1.20	-1.35	-1.79		-1.48	-0.69	-0.741	1.11	0.42	0.00	0.85	0.00
Iberdrola S.A.		2.62	1.20	-1.35	-1.79		-1.14	-0.40	-0.873	0.86	0.46	0.00	0.00	0.00

Exhibit 1: Portfolios combining five alpha capture strategies with the same risk budget: Small, Value, Momentum, Judgmental and Sector Momentum. On the left, the active weights for each alpha strategy represented by long-short portfolios. The sizing of the active weights has been set so that their ex-ante volatility is 5%. The average of these portfolios, Unconstrained, has been scaled so that its ex-ante volatility is also 5%. The portfolio weights of the unconstrained and constrained portfolios are shown on the right. Long-only and liquidity constraints were applied.

Risk budget: to keep the exercise simple we assumed that all alpha capture strategies should deliver the same information ratio of 0.3 and that they are fully uncorrelated. Hence, an equal risk budgeting approach applies and information ratio of the combination of strategies is $IR_{\mathbf{P}_A} = 0.3 \times \sqrt{5}$. Since all alpha capture portfolios have the same ex-ante risk, 5%, in equation (6) σ_i is the same for all strategies. We target $\sigma_{\mathbf{P}_A} = 5\%$ ex-ante tracking error for the unconstrained portfolio. Then, in equation (8) λ = 697.7, obtained from $\lambda = IR_{\mathbf{P}_A} \cdot T / \sigma_{\mathbf{P}_A}$ with T=52 weeks, as defined before.

Unconstrained portfolio: the final unconstrained active allocation is then simply the weighted average of the stock weights in each alpha capture portfolio. In exhibit one we also show the implied excess returns calculated from the product of the variance-covariance matrix with these active weights. The fully invested active portfolio can be obtained by adding the unconstrained active allocation to the benchmark index.

Constrained portfolio: the implied excess returns were used as inputs in the optimizer to maximize excess returns against tracking error risk. A budget constraint stating that the sum of stock active weights must be zero applies. In exhibit one we show two portfolios optimized under long-only constraints and limiting the absolute weight of each stock to 10%. The latter can be expressed in terms of stock active weights as Δw_i < 10% - w_i^{MC} , with w_i^{MC} the stock weight in the market capitalization index, whereas the long-only constraint can be expressed as Δw_i > -w_i^{MC}. The first of these two portfolios, Long-only 1, comes at the same risk aversion as the unconstrained portfolio, $\eta = \lambda$, where as the second, Long-only 2, comes at the same ex-ante tracking error risk of 5% which required $\eta^* = \lambda / 2.39$. We also include a portfolio optimized at the same risk aversion, $\eta = \lambda$, with an additional constraint for liquidity not allowing it to invest in stocks with the lowest market capitalizations, i.e. $\Delta w_i = -w_i^{MC}$, for any of the 10 stocks in the index with the lowest market capitalization.

Analysis of portfolios: the portfolio weights of the Long-only 1 portfolio remain close to those in the unconstrained portfolio but no short positions or stock weights above 10% are found. As seen in exhibit two, the tracking error risk is lower for the same risk aversion as a result of the application of constraints. Exposure to systematic risk is not too different but exposure to specific risk was reduced as expected (see annex).

Portfolios:	Unconstrained	Long-only 1	Long-only + Liq Constraints	Long-only 2
Ex-ante tracking error risk (%)	5.0	4.0	3.2	5.0
Systematic tracking error risk (%)	1.5	1.4	1.3	2.9
Specific tracking error risk (%)	4.8	3.8	3.0	4.1
Expected excess return (%)	3.4	2.7	1.5	3.0
Expected information ratio	0.67	0.66	0.47	0.60

Exhibit 2: The tracking error risk, expected excess return and information ratio for the unconstrained and constrained portfolios shown in exhibit 1. The decomposition of tracking error risk into systematic and specific components is also shown.

Even after applying the liquidity constraint, the portfolio still remains relatively close to the starting initial unconstrained portfolio. Nevertheless, the liquidity constraints impact two stocks which were before assigned large positive weights. The portfolio will not invest in those stocks and reallocates their weight, increasing the difference with the originally unconstrained portfolio.

In exhibit two, we can see that the contribution from systematic risk to tracking error risk is slightly lower but still remains comparable to that in the unconstrained portfolio. The contribution from specific risk is now even smaller, and so is the ex-ante tracking error.

If we target the same tracking error risk as in the unconstrained portfolio, then (see annex) the contribution of the systematic risk to tracking error risk increases in the constrained portfolio. Thus the portfolio distances itself more from the unconstrained solution in terms of stock weights. It would be better compared to an unconstrained portfolio of higher tracking error risk obtained at the same level of risk aversion, $\lambda^* = \eta^*$.

In exhibit two we also show that the expected information ratio for the unconstrained portfolio is higher than that for the constrained portfolios. The more we constrain the portfolio the lower the expected information ratio.

The different portfolios obtained with the same risk aversion $\eta = \lambda$ can be more easily compared in exhibit three where we plot the stock weights of the unconstrained portfolio on the horizontal axis against those of the constrained portfolios on the vertical axis. It is clear that constrained allocations remain relative close to those in the unconstrained portfolio. The more constrained the portfolio is, the more it deviates from the unconstrained allocation. Negative views are more difficult to express than positive views since short-selling is not possible. The constraint limiting the maximum stock weight at 10% also has some impact. Adding liquidity constraints also impacts stocks for which the unconstrained portfolio has a positive weight and which are excluded from the constrained portfolio.

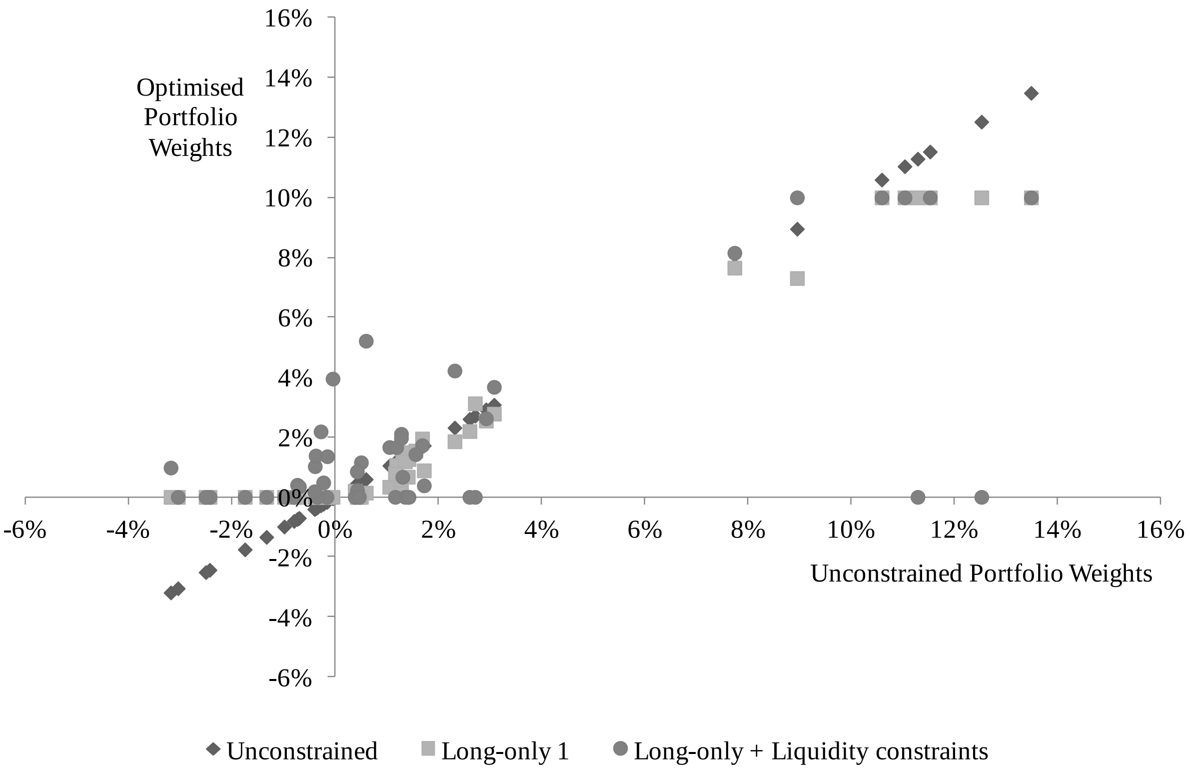

Exhibit 3: The stocks weights of constrained portfolios plotted against the stock weights of the unconstrained portfolio as shown in Exhibit 1. All portfolios have been optimized with the same level of risk aversion.

In exhibit four we see how constraints impact exposures to the risk factors in the risk model. There is no significant exposure in the unconstrained portfolio to the first factor (which typically corresponds to the market factor). This is also observed in the constrained portfolios. Exposures to other factors are smaller in the constrained portfolios obtained with the same level of risk aversion but always in line with those in the unconstrained portfolio. The Long-only 2 constrained portfolio which comes with the same tracking error risk as the unconstrained portfolio shows larger exposures to the systematic factors, sensibly 2.39 times larger than those found in Long-only 1 (see annex).

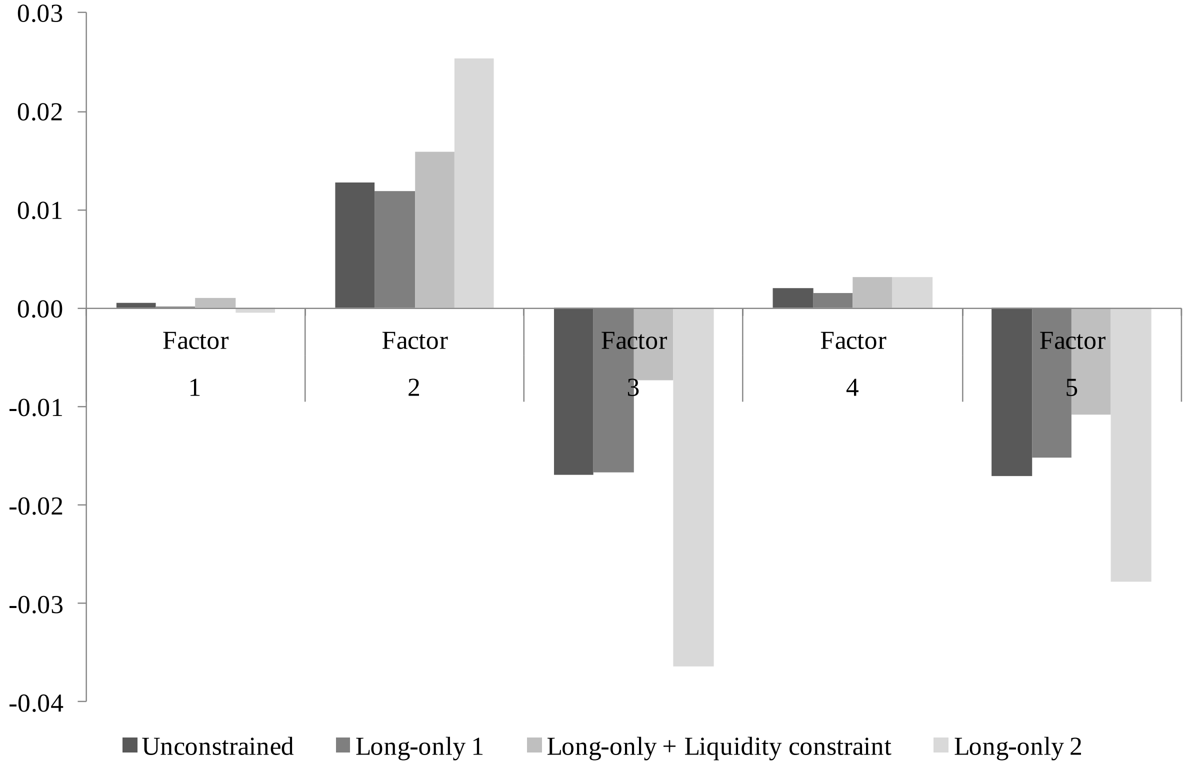

Exhibit 4: Exposures to the first five risk factors in the PCA risk model for unconstrained portfolio, the two long-only portfolios and for the portfolio with long-only and liquidity constraints. Exposures are given in terms of contribution to tracking error risk.

In exhibit five, we show the impact of adding constraints on the efficient frontier which in the absence of any constraints is simply a straight line (solid line) with excess return increasing proportionally to tracking error risk. The line goes to infinity. The dashed line represents the efficient frontier for portfolios optimized with long-only constraints and maximum 10% for each stock weight. The efficient frontier overlaps the unconstrained one until the first constraint is hit (portfolio C). Beyond that point the efficient frontier starts flattening and the expected information ratio decreases. The more constraints are binding, the more the efficient frontier flattens. All portfolios above the inflection point C have lower ex-ante information ratios than the unconstrained portfolio. The frontier will be capped at the portfolio with the maximum expected excess return. When liquidity constraints are added the efficient frontier (dotted line) no longer starts at the origin. The starting point is now portfolio A with the lowest ex-ante tracking error risk, a portfolio which is independent of expected excess returns that optimally mimics the market capitalization index when some stocks are removed from the index. The optimal portfolio with the largest information ratio can be found at the tangent point B. The lower the tracking error risk, the higher the danger of simply replicating the index while excluding illiquid stocks, hardly exploiting the alpha capture active strategies. A too large tracking error risk is also not compensating risk with higher excess returns, as the constraints flatten the efficient frontier and the information ratio falls beyond B. Even if the optimizer still finds optimal solutions beyond B, from a practical point there is little sense in chasing marginally higher returns at an increasingly large incremental tracking error risk.

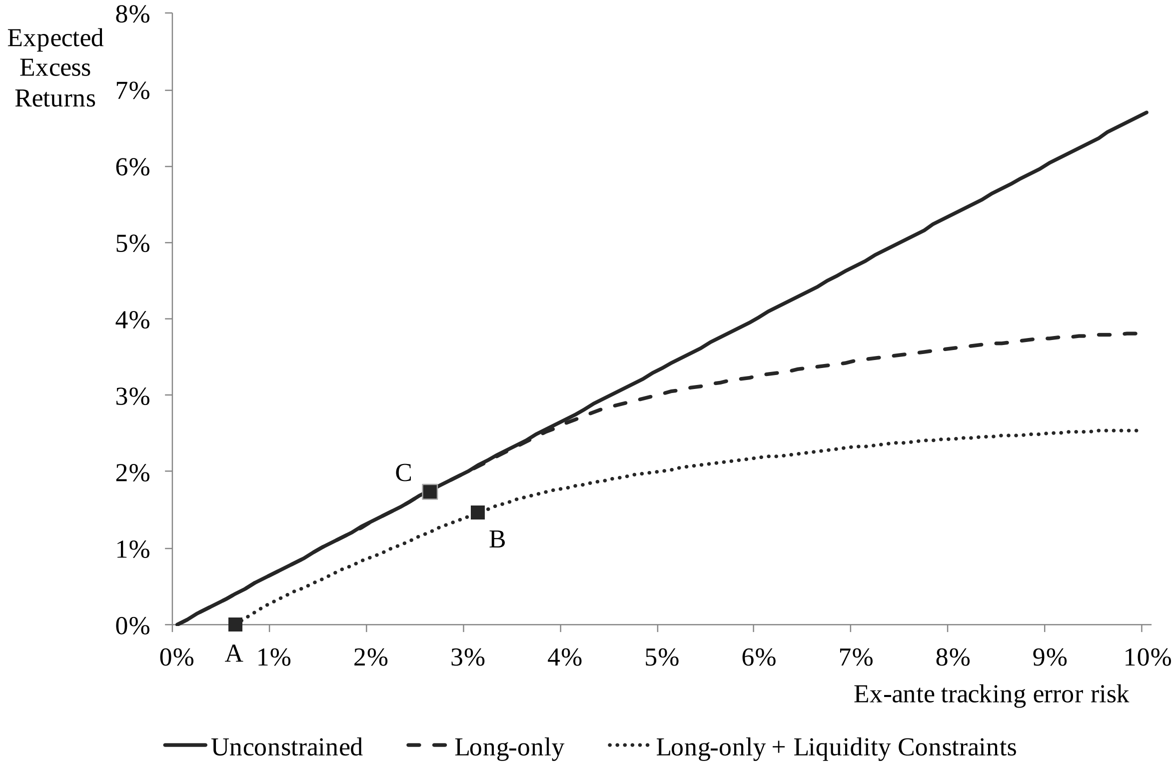

Exhibit 5: Efficient frontiers with expected excess returns against ex-ante tracking error risk for the unconstrained and constrained portfolios. Portfolio B is the long-only portfolio with liquidity constraints with the largest information ratio and point A is the minimum tracking error risk portfolio

for the same set of constraints. Portfolio C is the riskier long-only portfolio with the largest information ratio but all efficient long-only portfolios with lower tracking error risk have the same information ratio.

The impact of constraints can be measured by comparing the ex-ante information ratios of the constrained portfolios with that of an unconstrained portfolio at the same ex-ante tracking error risk. When the difference is too large then constraints are eating too much performance and the constrained portfolio reflects poorly the combination of alpha capture strategies.

Back-test of an example for global equities

We now turn to an example designed to show how the approach would have behaved over a given period of time.

Benchmark index: we chose to build portfolios benchmarked against the MSCI World Index[2].

Alpha capture portfolios: we considered four alpha capture strategies which again have been deliberately kept simple and are not necessarily the most efficient to capture the underlying alpha. The first invests in small-cap stocks, the second in value stocks using the book-to-price as definition of value, the third in winners defined by the momentum of monthly returns estimated over 11 months ending one month prior to estimation date, and the fourth invests in low-risk stocks with the lowest two year volatility of weekly returns. All these alpha capture strategies are represented by long-short portfolios rebalanced at the start of each month, investing in the top ranked 300 stocks and selling short the MSCI World Index[2].

In exhibit six we show how these individual strategies would have performed in the absence of constraints and their respective pair-wise correlations. The information ratio in the period Jan '95 through Dec '11 was positive. The leverage of each alpha capture portfolio was chosen so that the ex-ante risk is equal to 5% for all when they were re-balanced at the start of each month. Ex-ante risk is estimated from a statistical risk model based on a principal component approach, as described in the previous section, using two years of rolling historical weekly data.

	Information Ratio	Correlation (%)		
		Value	Momentum	Low Volatility
Small	0.43	75	0	-4
Value	0.25		-26	4
Momentum	0.79			-7
Low Volatility	0.47			

Exhibit 6: Information ratios for the four alpha capture strategies and their respective pair-wise correlations in the period Jan-95 through Dec-11. The MSCI World index defines the universe of stocks and simulations were performed in USD.

Risk budget: we considered three approaches to risk budgeting. For the first, mean-variance, we applied equation (1) and used the results in exhibit six, for the second, maximum diversification, we applied equation (3), and finally, for equal risk budgeting we used equation (4). The risk aversion was set so that the ex-ante risk budget of their combination is 5% at each monthly re-balancing. The risk budget allocation to each alpha factor strategy can be found in exhibit seven. Naturally, mean-variance tilts in favor of momentum with the highest information ratio and being the most uncorrelated of them all. Small-cap and value are quite correlated. Despite the lower information ratio value gets a larger risk budget in mean-variance because it is negatively correlated with momentum. In maximum diversification momentum gets the largest risk budget because it is the least correlated and small-cap gets an even smaller risk budget than in mean-variance because this approach assumes all strategies have the same expected information ratio.

	Target Risk Budget (%)		
	Mean Variance	Maximum Diversification	Equal Risk Budget
Small	0.19	0.05	0.45
Value	0.30	0.64	0.45
Momentum	0.85	0.72	0.45
Low Volatility	0.49	0.53	0.45

	Factor exposures from regression		
Unconstrained			
Intercept	0.00	0.00	0.00
Small	0.21	0.08	0.43
Value	0.22	0.52	0.39
Momentum	0.70	0.57	0.36
Low Volatility	0.40	0.43	0.38
R-square	96%	96%	98%
Long-only + Liquidity constraints			
Intercept	0.00	0.00	0.00
Small	0.14	-0.02	0.28
Value	0.21	0.50	0.38
Momentum	0.73	0.59	0.39
Low Volatility	0.45	0.49	0.45
R-square	91%	90%	89%
Long-only + Liquidity + # stocks constraints			
Intercept	0.00	0.00	0.00
Small	-0.05	-0.11	0.04
Value	0.18	0.41	0.37
Momentum	0.78	0.63	0.45
Low Volatility	0.48	0.42	0.55
R-square	73%	66%	70%

Exhibit 7: The risk budget for each alpha capture strategy estimated from three different approaches. The ex-ante tracking error risk is 5%. Below the factor exposures obtained from regression of the excess returns of each aggregate strategy, unconstrained and constrained, against the factor returns. The MSCI World index defines the universe of stocks and simulations were performed in USD from Jan-95 through Dec-11.

The portfolios combining the different alpha capture strategies for each set of risk budgets were re-balanced each month applying two sets of constraints. The risk aversion parameter was set in order that the tracking error risk of the unconstrained allocation was 5% each month at the time of re-balancing. The risk aversion used for the optimal constrained allocations was set so that all reach 5% ex-ante tracking error risk against the MSCI World index, and is therefore lower than that in the respective unconstrained portfolios on the same date.

Analysis of portfolios: in exhibit seven we also show the ex-post observed exposures to the different alpha capture strategies obtained from regression of the excess return against the MSCI World index for each combined portfolio strategy. Portfolio drift during the month and uncertainty in risk and correlation forecasts, which are not constant, explain the differences between ex-post and ex-ante target risk budgets. Nevertheless, in light of that, the differences are remarkably small: the unconstrained portfolio strategies remain very close to the target risk budgets for each alpha strategy and capture more than 96% of the expected variation of the combination of alphas.

Two sets of constraints were considered. The first set, “Long-only + Liquidity constraints”, includes the long-only constraint and a cap on the weight of each stock at the lowest of either between either 5% or 20 times its market capitalization weight. The second set, “Long-only + Liquidity + # stocks constraints”, includes an additional constraint to cap the number of stocks[3] in the final portfolio at 250.

As shown in exhibit seven, the exposure to the different alpha strategies in the constrained portfolios remains close to target risk budgets but exposure to the alpha from small capitalization stocks in particular, tends to be more difficult to retain. When the number of stocks is capped, the exposure to small-cap alpha disappears. Nevertheless, the excess returns of the most constrained optimal portfolios still show r-squares of about 70% when regressed against the excess returns of the unconstrained alpha capture strategies. If the constraint on the number of stocks is removed, then the r-squares jump to 90%. The long-only and liquidity constraints are much less binding in this example.

In exhibit eight, we show the simulation results for these strategies. The unconstrained portfolios are long-portfolios with a short-extension. It turns out that the number of stocks in that short-extension is slightly higher than that in the long-leg. If the number of stocks in the portfolio is not constrained, the final optimal allocation still invests in a large number of stocks, nearly 1 000 on average. The universe average in the period was nearly 1 700.

	MSCI World Index	Mean-Variance Risk Budget: Unconstrained	Long-only + Liquidty constraints	Long-only + Liquidty + # of stocks constraints
Average return (%)	6.1	11.4	11.7	12.0
Volatility (%)	16.0	15.2	15.1	14.7
Sharpe ratio	0.15	0.51	0.53	0.56
Excess return (%)		5.3	5.6	5.9
Tracking error risk (%)		4.7	5.1	5.6
Information ratio		1.13	1.09	1.04
Average number of stocks (long / short)		839 / 883	1008 / 0	216 / 0
		Maximum Diversification Risk Budget		
		Unconstrained	Long-only + Liquidty constraints	Long-only + Liquidty + # of stocks constraints
Average return (%)		10.9	11.1	11.6
Volatility (%)		15.4	15.2	15.2
Sharpe ratio		0.47	0.49	0.52
Excess return (%)		4.8	5.0	5.5
Tracking error risk (%)		4.9	5.0	5.2
Information ratio		0.98	1.00	1.07
Average number of stocks (long / short)		837 / 885	954 / 0	226 / 0
		Equal-Risk Budget		
		Unconstrained	Long-only + Liquidty constraints	Long-only + Liquidty + # of stocks constraints
Average return (%)		10.6	10.9	11.0
Volatility (%)		16.0	15.6	14.9
Sharpe ratio		0.43	0.46	0.49
Excess return (%)		4.5	4.8	4.8
Tracking error risk (%)		5.1	5.1	5.0
Information ratio		0.87	0.95	0.97
Average number of stocks (long / short)		825 / 897	1083 / 0	220 / 0

Exhibit 8: Simulation results for the different strategies, unconstrained and constrained. The MSCI World index defines the universe of stocks and simulations were performed in USD from Jan-95 through Dec-11.

The most striking conclusion from the analysis of exhibit eight is that all portfolios deliver quite comparable performances and generate levels of risk not too different from one another, even when strongly constrained. The ex-post level of tracking error risk also comes out sensibly in line with ex-ante target, demonstrating that the risk model performed relatively well. This proves the robustness of the methodology considered here, as an efficient approach to build constrained portfolios. Even if we were helped in the example by the fact that the alpha from small capitalization stocks, the most difficult to pass on to the constrained portfolio, was quite correlated with the alpha from value, the approach provides the means to detect which alphas are being eroded by constraints as demonstrated in exhibit seven.

It is also interesting to see that both maximum diversification and equal-risk budgeting approaches achieve comparable results to those from mean-variance. That the latter comes with the highest Sharpe ratio and information ratio should come as no surprise: the risk budgets are optimized *in-sample*, with the benefit of hindsight. But it is comforting to realize that simplified approaches requiring less inputs, reach only marginally worse results.

What we also show is that the constraint to reduce the number of stocks in the portfolio can be quite damaging. This is a constraint could have been justified on an operational basis in the past but makes little sense today since the cost of implementing portfolios with large numbers of stocks has fallen substantially in recent years. Putting more stocks in the portfolio allows for better risk control (some stocks may be needed just for tracking error risk control) and for a better representation of the originally intended unconstrained allocation. It is important to constrain portfolios in sensible ways, either to comply with regulations, to better control risks or to ease operational burdens, but not more than that. The idea that concentrated portfolios carry stronger convictions and may deliver more alpha is not true.

It is also interesting that the long-only portfolio delivers results comparable to those for the unconstrained portfolio. From an operational point of view it is clearly much simpler, cheaper and less risky since there are no short positions, no leverage and it requires no counter-party risk. That even raises the question of whether the so called 130/30 portfolios, or portfolios with short-extensions, are really needed when the sources of alpha are systematic factor strategies like those used here. In the same spirit, alpha-beta separation achieved through an investment in the market capitalization index-fund complemented with highly leveraged portable alpha strategies, provided through long-short equity hedged funds or absolute returns funds, can be questioned when their excess returns are derived from similar systematic alpha capture strategies.

CONCLUSIONS

In a world where markets are not information efficient there are opportunities to generate alpha by investing away from the market capitalization index. Several systematic active strategies are known to generate abnormal returns not explained by CAPM. The alpha generated by these strategies is believed to arise from mis-pricings due to investor behavior that is not taken into account by CAPM. The idea that diligent investors dedicating time to analyze information not necessarily easy to cast in a systematic fashion may generate alpha from their information advantage can therefore not be discarded. The question then is how to build a portfolio deviating from the market capitalization index while trying to efficiently exploit different independent alphas.

We suggest that alphas should be captured using constraint-free strategies and that portfolio constraints should be dealt with at the level of portfolio construction. The portfolio construction starts with a risk budgeting exercise to allocate risk to different sources of alpha at an aggregate level. Risk budgeting should take into account any available information about the expected risk-adjusted performance of each alpha capture strategy and their interactions. Alpha strategies are not restricted to bottom-up but can also include top-down sector and country decisions, and can be designed to systematically handle information or be based on judgmental analysis of information that is not easy to cast in a systematic fashion.

We show that constrained portfolios built from the stock implied returns of the unconstrained allocation retain much the same exposures to systematic risk factors than the unconstrained portfolio. We also show that the constrained portfolio remains very close to the unconstrained target portfolio when constraints are not too binding. When constraints are too binding, the risk-adjusted returns of the constrained portfolio can be too low to justify taking active risk. It is also shown that when constraints force stocks out of the portfolio, taking a too low tracking error risk may be sub-optimal with the efficient portfolio simply mimicking the market capitalization index. It is suggested that constrained portfolios are considered at the levels of tracking error which maximize the information ratio. At very high tracking error levels, the remuneration of each unit of tracking error risk will be smaller and therefore sub-optimal; there is no point in seeking levels of tracking error risk for which the information ratio of the constrained portfolio is far below that of the underlying unconstrained portfolio as constraints eat too much alpha.

Finally, we show with back-tested examples how robust the approach proposed here is when applied to the example of combining four well known alpha capture strategies applied to global stocks and working with realistic sets of constraints. The results suggest that in particular when the alpha capture portfolios are exposed to systematic risk the final constrained portfolios manage to deliver risk-adjusted performances comparable to those from the unconstrained portfolio and that their excess returns over the market capitalization index are largely correlated.

Institutional investors seems to recognize more and more that there are indeed some systematic mis-pricings in the market which justify tilting away from the market-capitalization portfolios and taking active positions (Chambers *et al.* [2012]). We propose a robust framework to budget the risk allocated to each alpha and to build sensibly constrained portfolios in an extremely transparent fashion. The portfolios retain as much as possible the alphas derived from systematic or judgmental approaches. We believe that this type of framework is particularly suited to those institutional mandates where investors will be able to decide on their own risk budget allocation as a function of their conviction on the sources of alpha, as represented by their expected information ratios for the different alpha capture strategies. The framework can also be easily applied to multi-asset and fixed income portfolios.

ANNEX

The Black-Litterman approach

The unconstrained solution to the Black-Litterman (BL) model can also be written in the form of a market capitalization portfolio plus a weighted sum of portfolios with active weights (He and Litterman [1999]). Using our notation, the optimal BL active weights are:

$$y^* = \mathbf{P}_S \mathrm{w} \quad \text{where} \quad \mathrm{w} = (\boldsymbol{\Omega}/\tau + \mathbf{P}_S{}'\boldsymbol{\Sigma}\mathbf{P}_S)^{-1}(\mathbf{R}/\delta - \mathbf{P}_S{}'\Pi) \qquad \text{(A1)}$$

with $\boldsymbol{\Omega}$ the variance-covariance matrix of strategy returns, $\Pi = \boldsymbol{\Sigma}\mathrm{w}_{MC}$ is proportional to the market capitalization portfolio implied stock returns, δ is the risk aversion in terms of absolute risk of the portfolio and τ a level of confidence in the strategies. The BL portfolios are obtained from an optimization in a space of returns over the risk free rate and volatility rather than excess returns and tracking error risk against the market capitalization portfolio. For a given δ, a smaller τ increases the tracking error risk allocation in BL. In our framework the choice of tracking error risk is left to the investor. Rather than use a parameter like τ to decide on a relative allocation between CAPM and strategies simply focus on the problem of budgeting the strategies and leave the choice of tracking error risk to the investor who can decide in a much simpler manner what to use e.g. pick the maximum information ratio portfolio.

Equation (A1) is more complex than our framework, where the weight of the strategies[1] is just $w = 1/\eta\left(\mathbf{\Omega}^{-1}\mathbf{R}\right)$. It shows that the risk budget of a strategy in BL will be positive only if its expected return given in **R**, adjusted for the risk aversion δ, is larger than the strategy expected return as estimated from today's underlying allocation in $\mathbf{P}_S$ and the market implied returns Π for each stock (CAPM). The strategy weights depend not just on $\mathbf{\Omega}^{-1}$ but on the inverse of the average of $\mathbf{\Omega}$ weighted by τ with today's strategy variances determined by their underlying stock allocation $\mathbf{P}_S$ and the risk model $\mathbf{\Sigma}$. The final result is therefore more complex than in our framework and requires an approach to estimate τ for which there is no established procedure. The BL approach needs to consider the CAPM implied returns in the final risk budgeting of strategies with a weight determined by this parameter τ because it was conceived for an optimization in the space of absolute performance and risk. The problem with the BL model is that it assumes zero correlation between the returns to the market capitalization portfolio and the alpha capture strategies (or views), which is often not the case in practical applications. The residual correlation with the market capitalization portfolio returns of some well-known simple alpha capture strategies like those here discussed, e.g. small capitalization stock alpha or low-volatility alpha, is therefore not properly accounted for. A recent extension of the BL model, called the non-Orthogonal Black-Litterman model, has been proposed by Ogliaro et al. [2012] to take those correlations (coupling) into account showing that they are the key to determine τ. Nevertheless, the framework here presented still has the advantage of being simpler than the non-Orthogonal BL approach.

Impact of long-only and cash neutral constraints

We shall analyze the impact of two common constraints: i) that the portfolio must be long-only and ii) fully invested in equities. We shall first look at the impact of constraints on the tracking error and on the exposure to systematic risk for a given level of risk aversion. We shall also compare the unconstrained portfolio with the constrained one when a similar tracking error exists.

In a universe of n stocks the cash neutral and long-only constraints translate into $\sum_i^n y_i = 0$ and $y_i \geq -w_i^{MC}$, with w_i^{MC} the weight of stock *i* in the market capitalization index. Applying the Kuhn-Tucker conditions to (10):

$$y^* = \mathbf{\Sigma}^{-1}(\Lambda + \mu \cdot \mathrm{I}) + \frac{\lambda}{\eta}\mathbf{P}_A \quad \text{where} \begin{cases} \Lambda_i \cdot (y_i^* + w_i^{MC}) = 0 \\ \Lambda_i \geq 0 \end{cases} \tag{A2}$$

Multiplying this equation by $y^*{}'\mathbf{\Sigma}$, using the fact that y^* is a zero sum portfolio and finally applying $\Lambda_i \cdot (y_i^* + w_i^{MC}) = 0$, leads to:

$$\begin{aligned} & y^*{}'\mathbf{\Sigma}y^* - y^*{}'\mathbf{\Sigma}(\frac{\lambda}{\eta}\mathbf{P}_A) = y^*{}'\Lambda + \mu \cdot y^*{}'\mathrm{I} = y^*{}'\Lambda = -\mathrm{w}'_{MC}\,\Lambda \leq 0 \\ & y^*{}'\mathbf{\Sigma}y^* - \sqrt{y^*{}'\mathbf{\Sigma}y^*} \cdot \sqrt{\frac{\lambda}{\eta}\mathbf{P}_A{}'\mathbf{\Sigma}\frac{\lambda}{\eta}\mathbf{P}_A} \leq y^*{}'\mathbf{\Sigma}y^* - y^*{}'\mathbf{\Sigma}(\frac{\lambda}{\eta}\mathbf{P}_A) \leq 0 \\ & y^*{}'\mathbf{\Sigma}y^* \leq \frac{\lambda}{\eta}\mathbf{P}_A{}'\mathbf{\Sigma}\frac{\lambda}{\eta}\mathbf{P}_A \end{aligned} \tag{A3}$$

showing that for the same level of risk aversion η the cash neutral and long-only constrained portfolio y^* has a lower tracking error risk than the unconstrained portfolio $(\lambda/\eta)\mathbf{P}_A$.

Now we look at the impact of the cash neutral and long-only constraints on the systematic risk exposure of the constrained solution at same risk aversion. The risk model $\mathbf{\Sigma}$ can be separated into its systematic and specific risk terms, $\mathbf{\Sigma} = \mathbf{\Psi} + \mathbf{\Delta}$. The systematic term is defined from the *m* eigenvectors $\phi_{1 \leq i \leq m}$ and the eigenvalues $\Lambda_1 \geq \Lambda_2 \geq ... \geq \Lambda_m$ of the $n \times n$ matrix $\mathbf{\Psi} = \mathbf{\Phi\Lambda\Phi}'$. The specific term $\mathbf{\Delta}$ is simply a $n \times n$ diagonal matrix of residual variances $\theta^2{}_{1 \leq i \leq n}$.

If we complete the ortho-normal vector base $(v_i)_{1 \leq i \leq m} \Rightarrow (v_i)_{1 \leq i \leq n}$ we can then write the difference between the constrained and unconstrained solution as:

$$y^* - (\lambda/\eta)\mathbf{P}_A = \sum_{\mathrm{i}=1}^{\mathrm{n}} x_i \cdot v_i \tag{A4}$$

$(x_i)_{1 \leq i \leq m}$ are the differences in exposure to systematic factor *i* between the constrained portfolio and the unconstrained portfolio. If we use this result in (11) then

$$
\begin{aligned}
(y-\frac{\lambda}{\eta}\mathbf{P}_A)'\Sigma(y-\frac{\lambda}{\eta}\mathbf{P}_A) &= \sum_{i=1}^{m}\Lambda_i\cdot x_i^2+\sum_{i=1}^{n}\theta_i^2\cdot(y-\frac{\lambda}{\eta}\mathbf{P}_A)_i^2 \\
&\approx \sum_{i=1}^{m}\Lambda_i\cdot x_i^2+\overline{\theta^2}\cdot\sum_{i=1}^{n}(y-\frac{\lambda}{\eta}\mathbf{P}_A)_i^2 \quad \text{where } \overline{\theta^2}=\frac{1}{n}\sum_{i=1}^{n}\theta_i^2 \qquad \text{(A5)} \\
&= \sum_{i=1}^{m}\Lambda_i\cdot x_i^2+\overline{\theta^2}\cdot\sum_{i=1}^{n}x_i^2
\end{aligned}
$$

while in the first step we used the decomposition of risk into systematic and specific terms, in the second step we approximated with the average over θ_i^2 and in the last step we used the result in equation (A4).

In CAPM, $m=1$ with $v_1 \propto \boldsymbol{\beta}$ and also $\bar{\boldsymbol{\beta}} \approx 1 \Rightarrow \|\boldsymbol{\beta}\| \geq \sqrt{n}$. We can then show that $\Lambda_1+\overline{\theta^2} \geq n\cdot\sigma_{market}^2+\overline{\theta^2} >> \overline{\theta^2}$. Consequently, in CAPM the optimizer will essentially minimize the difference between the constrained portfolio's active exposure to the market factor and that of the unconstrained portfolio.

More generally, under such constrains, the optimizer tends to preserve the active exposures to all systematic factors when compared to those in the unconstrained portfolio.

We now consider the constrained portfolio with the same tracking error risk of an unconstrained portfolio $(\lambda/\eta)\mathbf{P}_A$. When the former exists then there is a risk aversion η^* such that its tracking error risk is equal to that of $(\lambda/\eta)\mathbf{P}_A$. (A4) tells us that $\eta/\eta^*>1$. As shown in (A3), the systematic risk exposures after the optimization are then most likely η/η^* larger than that of the unconstrained active allocation $(\lambda/\eta)\mathbf{P}_A$ and consequently the exposure to systematic risk is larger in the constrained solution y^* of same tracking error risk.

Cash neutral and beta neutral constraints

In a universe of n stocks, with V the $n\times k$ constraint matrix $(v_i)_{1\leq i\leq k}$ and u the $k\times 1$ vector $(u_i)_{1\leq i\leq k}$ we can show by using Lagrange multipliers that when all constraints are based on equalities, $v_i{}'y=u_i$, then the solution to (10) is:

$$
y^*=\frac{\lambda}{\eta}\mathbf{P}_A+\boldsymbol{\Sigma}^{-1}V(V'\boldsymbol{\Sigma}^{-1}V)^{-1}(u-V'\frac{\lambda}{\eta}\mathbf{P}_A) \qquad \text{(A6)}
$$

There are two special cases of particular interest. The first is imposing that the solution y^* is cash neutral. In this case $k = 1$ and $u = 0$ and every single coefficient v_i in V is 1, noted as a matrix I. Then (A5) can be simplified to:

$$y^* = \frac{\lambda}{\eta}\mathbf{P}_A - \frac{\lambda}{\eta}\mathrm{I}'\mathbf{P}_A\frac{\mathbf{\Sigma}^{-1}\mathrm{I}}{\mathrm{I}'\mathbf{\Sigma}^{-1}\mathrm{I}} \tag{A6}$$

The final solution is equal to the unconstrained solution $(\lambda/\eta)\mathbf{P}_A$ minus the product of the cash exposure in $\mathbf{P}_A$, which is $\mathrm{I}'\mathbf{P}_A$, with the unconstrained minimum variance portfolio $(\mathbf{\Sigma}^{-1}\mathrm{I})/(\mathrm{I}'\mathbf{\Sigma}^{-1}\mathrm{I})$, the equity portfolio with the smallest possible ex-ante risk. The solution will sell the minimum variance portfolio if the cash exposure in the unconstrained portfolio is positive and buy otherwise. This makes sense intuitively as the minimum variance portfolio is the closest you can get to cash while investing in equities.

The second case of interest is the neutralization of the exposure to the market as measured by β. In this case $k = 1$ and $u = 0$ and the constraint vector V is equal to the matrix $\boldsymbol{\beta}$:

$$y^* = \frac{\lambda}{\eta}\mathbf{P}_A - \frac{\lambda}{\eta}\boldsymbol{\beta}'\mathbf{P}_A\frac{\mathbf{\Sigma}^{-1}\boldsymbol{\beta}}{\boldsymbol{\beta}'\mathbf{\Sigma}^{-1}\boldsymbol{\beta}} \tag{A7}$$

From the definition of $\beta_i = \mathrm{cov}(r_i, r_{MC})/\mathrm{var}(r_{MC}, r_{MC})$ and with w_{MC} the vector with market capitalization weights, $\boldsymbol{\beta} = \mathbf{\Sigma}\mathrm{w}_{MC}/(\mathrm{w}'_{MC}\mathbf{\Sigma}\mathrm{w}_{MC}) \Rightarrow \mathrm{w}_{MC} = \mathbf{\Sigma}^{-1}\boldsymbol{\beta}/(\boldsymbol{\beta}'\mathbf{\Sigma}^{-1}\boldsymbol{\beta})$. Thus, the beta neutral constrained solution is exactly equal to the unconstrained solution $(\lambda/\eta)\mathbf{P}_A$ minus the beta exposure of this unconstraint solution times the market capitalization portfolio. The solution will sell the market capitalization portfolio when the beta is positive and buy otherwise.

NOTES

[1]This can be derived from $\mathrm{w} = 1/\eta(\mathbf{\Omega}^{-1}\mathbf{R})$, with w the vector of weights allocated to each strategy and **R** the vector of expected returns for each strategy, by decomposing the variance-covariance matrix of strategy returns, $\mathbf{\Omega}$, into correlations $\mathbf{\Theta}$ and variances $\boldsymbol{\sigma}^2$.

[2]Due to licensing constraints we have used the constituents of our proprietary replication of the MSCI World index prior to 2005. The tracking error risk of the replication prior to 2005 is very small.

[3]We use an iterative procedure to find the optimal portfolio which respects the constraint of maximum number of stocks.

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