Equity portfolio diversification: how many stocks are enough? Evidence from five developed markets.[☆]

Vitali Alexeev[a,*], Francis Tapon[b]

[a] *School of Economics and Finance, University of Tasmania, Hobart, Australia, Tel: +61 (3) 6226 2335*
[b] *Department of Economics, University of Guelph, Ontario, Canada, Tel: +1 (519) 824 4120 (ext. 52657)*

Abstract

Choosing the number of stocks to hold in a portfolio can significantly affect its risk. We use daily observations for traded equity returns in the US, UK, Japan, Canada and Australia from 1975 to 2011 to simulate portfolios and calculate several measures of risk, including heavy tailed. For each measure, we estimate confidence bands to assure a specific reduction in diversifiable risk. The optimal number of stocks is shown to depend on the measure of risk, level of assurance required by investors, the specific stock market, and the changing correlation structure across time.

Keywords: Portfolio diversification, international investing, heavy tailed risk, expected shortfall, time series standard deviation, terminal wealth standard deviation.

In this paper, we present new answers to a decades old question about the optimal number of stocks in a portfolio[1] required to reduce the level of diversifiable risk. We show how confidence bands around the mean number of stocks in portfolios can be used to assure a reduction in risk at a specific certainty level - the first time that the relationship between portfolio size and confidence levels has been investigated since Evans and Archer (1968). Furthermore, we calculate various measures of risk (including some that take into account black swan events) using daily data, whereas most papers use lower frequencies varying from weekly (Solnik, 1974); monthly (for example, Klemkosky and Martin, 1975; Beck, Perfect, and Peterson, 1996; Benjelloun, 2010; Kryzanowski and Singh, 2010); quarterly (Johnson and Shannon, 1974; O'Neal, 1997); semi-annual (Evans and Archer, 1968) to annual data (Fisher and Lorie, 1970; Jennings, 1971). Finally, we consider four different developed stock markets in addition to those in the United States between 1975 and 2011. When we compare the year-by-year dynamic of optimal portfolio sizes, we find that the recommended number of stocks differs depending upon whether the markets are in distress or quiescent.

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[*] Corresponding author

Email addresses: valexeev@utas.edu.au (Vitali Alexeev), ftapon@uoguelph.ca (Francis Tapon)

[1] Hereafter, referred to as portfolio size.

The number of stocks held by an average, all domestic US equity fund is 176 stocks.[2] These large portfolios are supported by some academic studies suggesting that the optimal number of stocks in a portfolio should be in excess of 50 (Copp and Cleary, 1999; Domian, Louton, and Racine, 2007; Benjelloun, 2010; Kryzanowski and Singh, 2010). However, the average fund underperforms the market index, frequently needs rebalancing and thus is expensive to manage. In contrast, other studies indicate that substantial diversification benefits can be achieved by owning as few as 6-15 stocks (Evans and Archer, 1968; Jennings, 1971; Fielitz, 1974; Johnson and Shannon, 1974; Solnik, 1974; Bird and Tippett, 1986; Tang, 2004; Brands and Gallagher, 2005).

Although it is commonly recommended that risk-averse investors hold a number of uncorrelated securities in their portfolios to achieve some degree of diversification, it is not readily obvious how many stocks are required to achieve an optimal degree of diversification. Holding too few stocks exposes the investor to unnecessary idiosyncratic risk. Holding too many stocks is costly both in terms of the cost of numerous transactions needed to build the initial portfolio and the opportunity cost of monitoring a large diversified portfolio. In addition, the larger the number of stocks in a portfolio the higher the chances of underperforming the benchmark after fees. If it is possible to eliminate most diversifiable risk with a small portfolio, the need for large portfolios typically held by equity funds is unjustified. Indeed, Campbell, Lettau, Malkiel, and Xu (2001) have shown that firm specific risk in the U.S. has grown over the past thirty years relative to the overall volatility of the stock market and that correlations between stocks have correspondingly decreased, reinforcing the advisability of smaller portfolios.

The question of the optimal number of stocks in a diversified portfolio has been extensively studied in the literature, especially for the US. Using a variety of risk measures, authors have not yet reached a definitive conclusion.[3] The pioneering paper by Evans and Archer (1968) was the first study to evaluate the reduction in portfolio risk as portfolio size increased. With standard deviation as a risk measure, they show that on average eight to ten stocks are sufficient to achieve most of the benefits of diversification. Standard deviation was commonly used as a measure of risk in earlier studies (see for example Fisher and Lorie, 1970; Wagner and Lau, 1971; Solnik, 1974; Bloomfield, Leftwich, and Jr., 1977; Bird and Tippett, 1986; Statman, 1987) and continued to be a popular choice in the past decade (e.g., Brands and Gallagher, 2005; Benjelloun, 2010). Several other measures include variance (Johnson and Shannon, 1974; Elton and Gruber, 1977; Beck, Perfect, and Peterson, 1996), mean absolute deviation (Fisher and Lorie, 1970; Fielitz, 1974), and terminal wealth standard deviation (O'Neal, 1997; Brands and Gallagher, 2005; Benjelloun, 2010). Within the linear market model framework measures of portfolio beta, R^2, residual variance have been used in assessing level of diversification (see for example Wagner and Lau, 1971; Klemkosky and Martin, 1975). Correlation between returns on a portfolio and the market index was also effectively used in Beck, Perfect, and Peterson (1996); Dbouk and

[2]On May 2012, the interquartile range for the number of holdings of 8,277 all domestic stock mutual funds registered in Morningstar database was from 128 to 583. Source: Morningstar Fund Screener.

[3]In Table 3 on page 38, we present a detailed summary of the literature.

Kryzanowski (2009); Kryzanowski and Singh (2010). In addition to the measures mentioned above, Fisher and Lorie (1970)'s study was the only one that used Gini's mean difference and coefficient of concentration as measures of portfolio diversification. Several examples of downside measures of risk have also been found in the literature: semi-variance (O'Neal, 1997; Kryzanowski and Singh, 2010) and mean shortfall (O'Neal, 1997). Although Hyung and de Vries (2005) and Ibragimov and Walden (2007) provide some theoretical justification for the use of extremely heavy-tailed risks in measuring portfolio diversification, the empirical analyses are only starting to emerge (e.g., Hyung and de Vries, 2012).

Many of the conclusions in the studies cited above are based on either the mean or the median measure of risk among a large number of randomly selected n-stock portfolios. However, as noted in Tang (2004, pp.155-156) these "...findings are based on the expected portfolio variances (or risk) for different portfolio sizes, which are different from the actual portfolio risk. There is no guarantee that the risk of one particular portfolio is the same as the expected risk with the same portfolio size (i.e., risk of the portfolio risk exists). Hence, there are additional sample risks in that your portfolio may not be the same as the population average." Evans and Archer (1968) were the first to calculate the 95% confidence limit to their central measure but do not infer the portfolio size recommendations based on this result (Evans and Archer, 1968, Figure 1, p.765). In fact, in the past 45 years (since Evans and Archer, 1968) no research has been done to investigate the relationship of portfolio size and confidence levels that assures a reduction in risk at a specific certainty level.

Finally, most of the studies on diversification and portfolio size focus on the US markets and relatively few study other markets. The exceptions are Solnik (1974) who, analyzes the US and markets in the UK, Germany, France, Switzerland, Italy, Belgium and the Netherlands; Bird and Tippett (1986) and Brands and Gallagher (2005) report on Australia; Copp and Cleary (1999) and Kryzanowski and Singh (2010) study Canada; Byrne and Lee (2000) study the UK.

In this paper, we build on the work of our predecessors regarding portfolio size. We use a simulation approach to construct random portfolios based on actual equity returns (in the US, UK, Japan, Canada and Australia markets) over the period 1975 to 2011. Using daily data, we construct equally weighted random portfolios of different sizes ranging from portfolios consisting of only one security to a broad market portfolio including all actively traded securities in the market. For each of these portfolios, we analyze several measures of risk both within a sub-period and across different stock markets. Using daily observations, we focus on the estimates of time series standard deviation (SD), expected shortfall (ES) and terminal wealth standard deviation (TWSD) for portfolios of different sizes.[4] We focus on downside risk measures, comparing five developed equity markets and trace the dynamics of diversification benefits over the past 37 years in these markets.

[4]In addition to the risk measures outlined above, we also estimate other common measures of risk, performance and diversification. We consider lower partial moments (LPMs), Value-at-Risk (VaR), mean absolute deviation (MAD) and median absolute deviation for risk; arithmetic and geometric returns, Sharpe and Treynor measures for performance; R^2 and correlation coefficients with the market portfolio for diversification. Due to article length restrictions, we provide these additional results on our website.

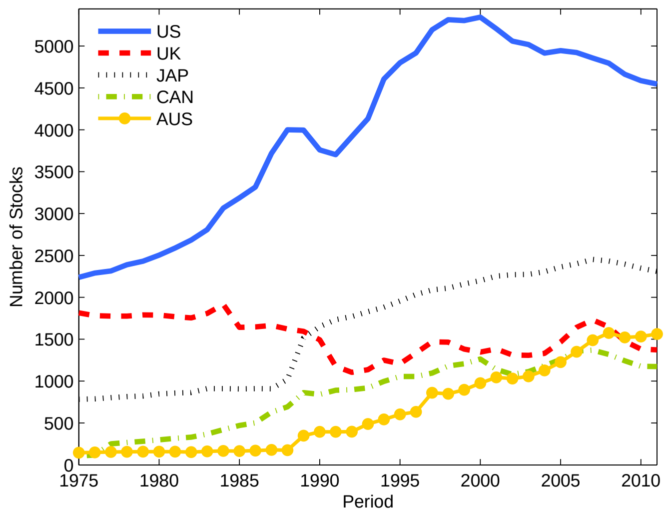


Figure 1: **Number of common stocks by country.** The lines represent the number of stocks used in our analysis after removing stocks that have traded less than 25% of the trading days in a particular year. For example, in 2011, the total number of stocks across the five markets used in the analysis is 10,964 (down from 11,456 for which data were available).

We find that investors concerned with tail risk can achieve diversification benefits with a relatively small number of stocks. For the same level of risk reduction, our results for risk measures that do not consider extreme losses find no substantial differences in portfolio sizes when compared to results using standard deviation as a risk measure. We construct confidence bands (CI) around our calculated central risk measures that give us an upper limit to the number of stocks required in a portfolio that guarantees a fixed level of risk reduction at a specified confidence level. After performing our analysis, we are able to state the following: "With y percent confidence, holding anywhere from $x + CI$ stocks in a portfolio, z percent of diversifiable risk can be eliminated". As we will show, the optimal portfolio size needed to diversify specific risk away with a given level of confidence depends on: (i) the measure of risk; (ii) required confidence level needed to achieve a 90% reduction in diversifiable risk; (iii) market locale; and (iv) the changing correlation structure between stock returns over time. However, the number of stocks required to achieve a comparable level of diversification for a specific risk measure may differ substantially from one investment locale to another.

In Section I, we discuss our data and methodology. In Section II, we present our results. We draw our conclusions in Section III.

I. Data and Methodology

We assume that (i) each portfolio contains only common stocks; (ii) purchases are financed without borrowing; (iii) investors' actions will not influence the price and/or dividend of an individual stock; (iv) taxes are not considered. These assumptions are similar to those in Jennings, 1971. We analyze each of the markets separately to avoid contaminating our results with exchange rate fluctuations. Our data are obtained from Thomson Reuters Datastream and consist of daily total return observations on common stocks listed on the NYSE-AMEX, the Nasdaq, the London, Tokyo, Toronto and Australian stock exchanges from 1975 to 2011. Figure 1 depicts the total number of stocks for each country. To avoid survivorship bias we acquire return indices for both active and delisted securities. For each of these years, we consider only securities which have traded at least 75% of the trading days in a particular year. We do not analyse continental European financial markets since our sampling period includes these countries' transition to a single currency and the subsequent increase in cross-listings of national enterprises in exchanges across the Eurozone. Such an analysis is a goal for future research.

Our methodology assumes that portfolio total risk is comprised of systematic risk and specific or unsystematic risk. As the number of securities included in a portfolio approaches the number of securities in the market, portfolio risk approaches the overall level of systematic risk, that is, market risk, suggesting a relationship which behaves as a decreasing asymptotic function. Reduction in portfolio risk can then be achieved up to the point where all unsystematic risk has been eliminated or where the incremental decrease in unsystematic risk brings insignificant benefits.[5]

We construct portfolios by randomly drawing n stocks without replacement from the entire sample for a particular stock market. These are equally weighted to give a portfolio P_i^n, where $n = 1..N$ indicates the number of stocks in the portfolio and N is the total number of actively traded stocks in the subperiod analyzed, and $i = 1..M$ represents the draw number. Given that our sample includes non-surviving stocks, a stock in the chosen portfolio that does not survive in one period, is replaced in the subsequent period with a new randomly selected stock not already in the portfolio.

We construct $M = 10,000$ n-stock portfolios for each $n = 1..N$, unless the number of combinations of n stocks out of N available is lower than M. For example, when $n = 1$, the number of unique single security portfolios equals N and when $n = N$ only one equally weighted portfolio can be constructed - we define it as the market portfolio. We find that 10,000 replications are sufficient to give a robust measure of central tendency of our risk measures. If N is the total number of stocks available in the market at any given time, the n-combination of N stocks is equal to the binomial coefficient $\left(\begin{array}{c} N \\ n \end{array}\right) = \frac{N!}{n!(N-n)!}$. To give the reader an appreciation for the computational intensity of this exercise we quote from Fisher and Lorie (1970, p.111): "..the number of possible portfolios containing eight different

[5]Several studies (Evans and Archer (1968); Latane and Young (1969); Fisher and Lorie (1970); Wagner and Lau (1971); Fielitz (1974); Klemkosky and Martin (1975); Tole (1982); Statman (1987)) have conducted statistical tests for evaluating the significance of the incremental change in reduced unsystematic risk. However, these approaches are subject to a replication sensitivity problem related to the number of replications. Increasing the number of replications to construct random portfolios, and thus reducing the estimation error, can lead to conflicting results. For large numbers of replications, these statistical tests may find significant but irrelevant differences in the risk measure. This problem is only partially mitigated in Beck, Perfect, and Peterson (1996). Consequently, we do not use this approach in this paper.

stocks that could be selected from a list of 1,000 is more than 24 quintillion [2.4×10^{19}]. At current [1969] costs for computer time, complete enumeration of all such portfolios of eight stocks would have cost approximately $150 trillion." Although the cost of modern computing facilities is no longer an issue, the estimation time to reconstruct the full set of all possible portfolio combinations for the US market alone for the period from 1975 to 2011 is approximately three years.

For each stock j at time t we define returns as

$$r_{j,t} = log\left(RI_{j,t}\right) - log\left(RI_{j,t-1}\right), \tag{1}$$

where $RI_{j,t}$ is the total return index inclusive of dividends. For $n = 1..N$ the return of the n-stock equally weighted random portfolio i at time t is defined as

$$P_{i,t}^{n} = \sum_{j=1}^{n} \frac{\{r_{j,t}\}_i}{n} \tag{2}$$

and the average time series return over time $t = 1..T$ of portfolio i can be expressed as

$$\bar{P}_i^n = \sum_{t=1}^{T} \frac{P_{i,t}^n}{T}. \tag{3}$$

Let Ω_i^n represent a risk measure of an n-stock portfolio i. We define the average risk metric of M portfolios, each of size n, as follows:

$$\overline{\Omega^n} = \sum_{i=1}^{M} \frac{\Omega_i^n}{M}. \tag{4}$$

Most studies define the location of a sample center as in (4) above. A few studies use median as a central tendency for the risk metric.

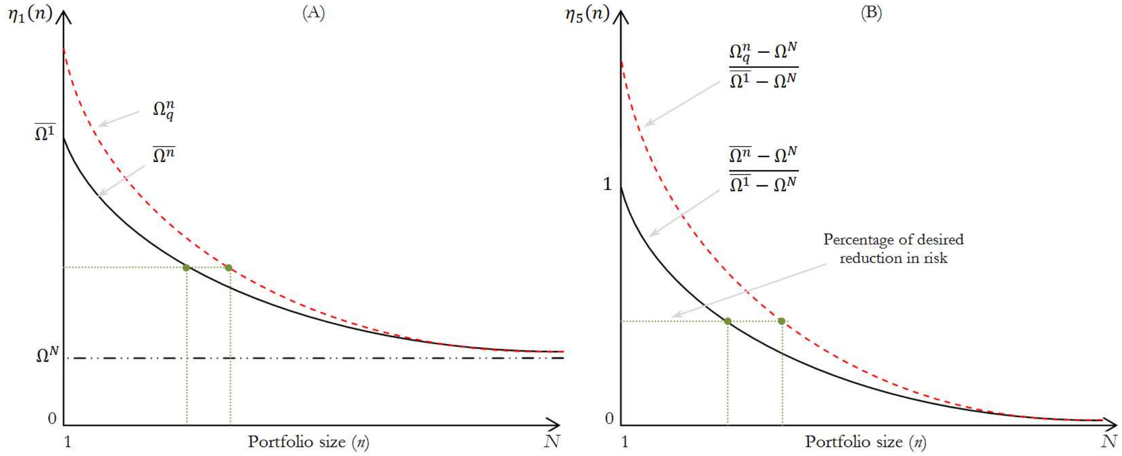
When equally weighted, the market portfolio consisting of all available securities is a unique portfolio, and $\overline{\Omega^N} = \Omega^N$. If $\overline{\Omega^1}$ and Ω^N are risk metrics for the average single-stock and market portfolios, we can define several measures of diversification for an n-stock portfolio. In fact, the first measure is an unscaled and unstandardized risk measure. It is simply defined as the average risk metric and is consistent with those in Evans and Archer (1968); Fielitz (1974); Johnson and Shannon (1974); Beck, Perfect, and Peterson (1996); Brands and Gallagher (2005). We define this measure as follows:

$$\eta_1\left(n\right) = \overline{\Omega^n} \tag{5}$$

Using the measure defined above, as $n \to N$, the asymptote approaches market risk (see Figure 2 panel A).

The second measure, showing only unsystematic risk, is consistent with definitions in Klemkosky and Martin (1975); Kryzanowski and Singh (2010). It is adjusted for market risk and can be represented, in general form, as:

Figure 2: Risk as a function of portfolio size.



$$\eta_2(n) = \overline{\Omega^n} - \Omega^N. \tag{6}$$

In this case the asymptote approaches 0 as the number of stocks in the portfolio increases.

The third measure is standardized by $\overline{\Omega^1}$ and is consistent with those in Solnik (1974); O'Neal (1997); Tang (2004):

$$\eta_3(n) = \frac{\overline{\Omega^n}}{\overline{\Omega^1}}. \tag{7}$$

The measure expressed above is bounded by 1 from above: when $n = 1$, $\overline{\Omega^1}$ is at its highest. The asymptote in this case represents the percentage of the average security risk that cannot be diversified away.

Under certain circumstances, it is convenient to represent an alternative standardization such as the one below:

$$\eta_4(n) = \frac{\overline{\Omega^n}}{\Omega^N}. \tag{8}$$

When $n = 1$, the measure shows how many times the average security risk is higher than market risk. As the number of stocks in the portfolio increases, the measure approaches 1 from above. Since the market risk level changes from year to year, we pick this particular measure to facilitate our visual analysis . We depict this measure in panels (A) and (B) of Figures 3, 6, 9, 12 and 15.

Since we are using several measures of risk across many subperiods, we require a normalized measure that adjusts for the average security risk and for the level of market risk. To obtain the required number of securities for portfolios with a given level of diversifiable risk, we find it useful to define a measure exclusively focused on diversifiable risk that is bounded from 0 to 1 as follows:

$$\eta_5(n) = \frac{\overline{\Omega^n} - \Omega^N}{\overline{\Omega^1} - \Omega^N}. \tag{9}$$

The graph in Figure 2 panel (B) illustrates this measure.

In addition, for a series of random draws of n-stock portfolios, let Ω_q^n be a qth percentile of a risk measure Ω^n. Similar to (9) we define:

$$\eta_5(n, q) = \frac{\Omega_q^n - \Omega^N}{\overline{\Omega^1} - \Omega^N}. \tag{10}$$

We depict (10) in Figure 2 panel (B). The advantage of the risk measures in (9) and (10) is that they can be used to compare diversification benefits across time periods and against other risk measures.

When the market model holds and the variance of portfolio returns is used as the risk measure, equation (9) is the level of idiosyncratic risk of an n-stock portfolio relative to the average level of idiosyncratic risk of a single security.

Because in the finance literature measuring risk is more contentious than measuring return, we consider several types of risk measures. Our first measure is time series standard deviation (SD), a well accepted measurement of risk of a financial asset or portfolio. This measure allows us to compare our results with those in previous studies and with the results obtained with other risk measures. Other important risk measures are downside (or tail) risk measures. Downside risk measures account for deviations below a certain threshold, unlike standard deviation, where positive and negative deviations from the expected level are penalized equally. One advantage of downside risk measures is that they account, to some extent, for the asymmetries in returns during bull and bear markets. Specifically, we use the expected shortfall (ES) measure. An additional conceptually different measure, reported in this study, is terminal wealth standard deviation (TWSD).

A. Time series standard deviation

When the risk metric is the standard deviation of a portfolio, we define it as follows:

$$\Omega_i^n \equiv \sigma_i^n = \sqrt{\sum_{t=1}^{T} \frac{\left(P_{i,t}^n - \bar{P}_i^n\right)^2}{T-1}} \tag{11}$$

and the average standard deviation of M random portfolios, each of size n is

$$\overline{\sigma^n} = \sum_{i=1}^{M} \frac{\sigma_i^n}{M}. \tag{12}$$

Despite its drawbacks, standard deviation is the most used measure of risk in the finance literature.[6]

[6]We have also estimated our results using equation (9) with mean absolute deviation (MAD), median absolute deviation, and variance as measures of risk. These results are similar to those obtained with SD.

B. Expected Shortfall

Currently, the most used choice in finance for measuring downside risk is Value-at-Risk (VaR), which replaced the previous favorite, semi variance. However, VaR is not a coherent risk measure and fails to accommodate one important property of the risk measure, namely, sub-additivity (see Artzner, Delbaen, Eber, and Heath, 1997; Annaert, Osselaer, and Verstraete, 2009; for an excellent summary of risk measures see also Szego, 2002). Sub-additivity insures that the risk measure of a portfolio containing two securities is smaller or equal to the two combined risk measures for these two securities. We believe that this property in a risk measure is paramount when analyzing diversification. Expected shortfall (ES) satisfies the sub-additivity condition and provides a natural extension to VaR.[7]

Given a confidence level $\alpha \in (0,1)$, the portfolio's Value-at-Risk at the confidence level α, VaR_α, is defined as a threshold value such that the probability that the loss on the portfolio exceeds this value is no larger than α.[8] Then, expected shortfall at level α, ES_α, can be defined as the expected value of the losses exceeding VaR_α.[9]

We estimate expected shortfall from the portfolio returns. Let $P^n_{i,t,\alpha}$ be the empirical αth quantile. We define:

$$\Omega_i^n \equiv ES^n_{\alpha,i} = -\frac{1}{T_\alpha}\sum_{t=1}^{T} P^n_{i,t}\mathbf{1}\left(P^n_{i,t} \leq P^n_{i,t,\alpha}\right) \tag{13}$$

where $\mathbf{1}\,(.) = 1$ if $P^n_{i,t} \leq P^n_{i,t,\alpha}$ and 0 otherwise, and T_α denotes the number of $P^n_{i,t}$ no greater than $P^n_{i,t,\alpha}$.

We choose expected shortfall at 1% level, $ES_{1\%}$, as our measure of downside risk.[10] When we estimate extreme risk measures, we are faced with a dilemma. We realize that market conditions change from one year to the next and so do the estimates of extreme measures. Taking one year as a sub-period provides us on average with 250 daily observations sufficient to get a sensible historical estimate for $ES_{10\%}$ or $ES_{5\%}$. However, this sample size is not nearly enough to get a good historical estimate for $ES_{1\%}$. Even if we take a rolling 3 year period and increase our sample to approximately 750 daily observations, the estimate of the $ES_{1\%}$ is not satisfactory. Instead, we use a model based simulation approach to increase the size of the sample for the particular subperiod in order to get a more reliable estimate. We perform GARCH filtration by using the GJR(1,1) model (Glosten, Jagannathan, and

[7]In the case of a continuous random variable, the definition of ES coincides with that of Conditional VaR (CVaR).

[8]The exact definition of the VaR_α of a random variable X is based on the α-quantile, taken with a negative sign of the distribution function F_X as $VaR_\alpha(X) = -F_X^{-1}(\alpha)$. VaR does not measure losses exceeding VaR, the tail of the distribution of returns may contain many local extremes leading to unstable VaR rankings. Most importantly, the non-sub-additivity implies that portfolio diversification may lead to an increase in risk.

[9]If the underlying distribution for portfolio returns, X, is a continuous distribution then the expected shortfall is defined by $ES_\alpha(X) = E\left[-X \,|X \leq -VaR_\alpha(X))\right]$.

[10]We have also estimated our results using equation (9) with $ES_{10\%}$, $ES_{5\%}$ and lower partial moments (e.g., semi-variance). These results, again, are similar to those obtained with SD. However, focusing on extreme (black swan) events, $ES_{1\%}$ yields considerably divergent results.

Runkle, 1993) as our parametric benchmark for each of our simulated portfolios. By adopting this model we account for some stylized facts commonly attributed to financial time series. Namely, we control for autoregressive and conditional heteroskedasticity and the asymmetry in the conditional variances. In addition, by estimating the GJR model with t-distributed disturbances, we compensate for the presence of fat tails in equity portfolio returns. Once we estimate the model parameters, we standardize the residuals by the estimated conditional variance to represent a zero mean and unit variance disturbances. We use a semi-parametric approach to estimate the distribution of these standardized residuals: the 10% lower and upper tails are estimated parametrically by fitting a Generalized Pareto Distribution using the peak over threshold (POT) approach (Pickands, 1975; Davison and Smith, 1990). We estimate the middle, 10%-90%, of the distribution non parametrically using normal kernel smoothing. We sample the residuals from the resulting estimated distributions which we then use in the estimated GJR(1,1) models to obtain simulated observations. This allows us to simulate samples of lengths greater than the 250 trading days in a typical year. We find that 10,000 observations give us a reliable estimate of $ES_{1\%}$.[11]

C. Terminal Wealth Standard Deviation

Similar to O'Neal (1997), Brands and Gallagher (2005) and Benjelloun (2010) we use terminal wealth standard deviation (TWSD) as one of the risk measures. The advantage of using this risk measure is that while it is independent of the data frequency, it does rely on the length of the investment horizon.

The terminal wealth of the n-stock equally weighted random portfolio i at time t is defined as

$$TW_i^n = \prod_{t=1}^{T} \left(1 + P_{i,t}^n\right) \tag{14}$$

and the terminal wealth standard deviation can be expressed as

$$TWSD^n = \sqrt{\sum_{i=1}^{M} \frac{\left(TW_i^n - \overline{TW^n}\right)^2}{M-1}} \tag{15}$$

where $\overline{TW^n} = \sum_{i=1}^{M} \frac{TW_i^n}{M}$ is the average terminal wealth over M portfolios.[12] Thus, the TWSD risk measure accounts for the volatility of terminal wealth.

II. Results

Table 2 and Figures 3 to 5 summarize our findings for the US market. The vertical axis in Figure 3.A shows the average level of standard deviation normalized by the market standard deviation as defined by equation (8) above. For each year from 1975 to 2011, we construct an asymptote and show the average n-stock portfolio standard deviation for $n = 1..20$ relative to the market standard deviation for each year. We observe in Figure 3.A that the average security risk, measured by standard deviation, was at its highest from 1994 to 1996, amounting to five times greater than the market standard deviation. In contrast, in 2008 (see Figure 4.A), the average security standard deviation was only two times the standard deviation of the market, due to the fact that the overall market risk was one of the highest. Similarly, Figure 4.B depicts the average security $ES_{1\%}$ relative to $ES_{1\%}$ of the overall market. For the average security, $ES_{1\%}$ was closer to the market's during the 1987 crash, but not as close to the market's $ES_{1\%}$ during the 2008-2009 Global Financial Crisis (GFC). This difference may be due to the rise of globalization since 1987 when many more international firms were listed on US markets, and the sectoral composition of US markets was increasingly diversified.

When we compare Figure 4.C (dashed line) with Figure 5.A in Campbell, Lettau, Malkiel, and Xu (2001, p.24), we note that the downward trend in average correlations among stocks reported by Campbell, Lettau, Malkiel, and Xu between 1960 and 2000, reversed itself between 2000 and 2011, including the period 2002 to 2007 when the economy was not in crisis. After 2000, increased correlations among stocks made it harder to achieve portfolio diversification with a small number of stocks, as can be seen in Table 2 (columns 1 to 3).

Figure 5 and Table 2 (columns 1 to 3) show the average number of stocks required to achieve 90% reduction in diversifiable risk as defined by equation (9). Using standard deviation as the measure of risk, an average investor (solid darker line) must hold from 16 to 31 stocks (see Table 2 column 1). This result is stable across all years with only a slight increase in the number of stocks required in the period 1990 to 2007. Looking at the dashed darker line in Figure 5, we note that an investor who wants to achieve a 90% diversifiable risk reduction with 90% certainty instead of on average, should hold a portfolio comprised of 39-73 stocks (see Table 2 column 1a). The number of stocks to achieve this level of certainty, i.e., to reduce 90% of diversifiable risk 90% of the time, increases during periods when markets are in financial distress (for example, in Table 2 column 1a, 49-51 stocks following the market crash in 1987, 50-60 stocks from 2000 to 2002 right after the burst of the Dot Com Bubble, and 56-73 stocks in 2008-2011 during the GFC). During such periods, events which normally would not threaten a firm's survival might lead to bankruptcy or

[11] We performed simulations to check the convergence of $ES_{1\%}$ for sample lengths from 250 to 50,000 in 250 increments. Results are available upon request.

[12] Adopting the idea behind the Sortino measure that deviations above the market terminal wealth should not be penalized and following O'Neal (1997), we calculate the terminal wealth standard semi-deviation (TWSSD). We find that the number of stocks required in an optimal portfolio is similar to our results using TWSD as a measure of risk.

takeover. Thus, investors need a larger number of stocks to guard against diversifiable risk with this level of certainty.

From 1975 to 2011 (Table 2 column 2 and Figure 5), using $ES_{1\%}$ as a measure of risk, the solid lighter line shows that average investors need to hold 10 (in 1987-1988) to 24 (in 2004-2005) securities in their portfolios to reduce the $ES_{1\%}$ diversifiable risk by 90%. However, Table 2 column 2a and the dashed lighter line in Figure 5 indicate that to achieve this reduction with 90% certainty for most years a portfolio of 33 to 55 holdings is required, with exceptions in 1977-1981 (50-64 stocks), 1987-1989 (109-113 stocks) and 2008-2011 (63-72 stocks). The solid line with circles (in Figure 5) represents the number of portfolio holding requirements using terminal wealth standard deviation as a measure of risk which is always higher than the number based on the standard deviation and $ES_{1\%}$ measures. It is relatively constant throughout the period requiring on average holdings of 83-99 stocks (Table 2 column 3) to reduce the level of diversifiable risk by 90%.

Turning our attention to UK market results in Table 2, columns 4 to 6 and Figures 6 to 8, we observe that 1978-1986 was the period with the highest number of holdings in portfolios of an average investor concerned with standard deviation as a measure of risk (25-36 stocks in Table 2 column 4). After 1986, a relatively low number of stock holdings is required to achieve 90% reduction in diversifiable risk (14-22 stocks). For UK investors concerned with standard deviation as a measure of risk (also shown by the darker dashed line in Figure 8), we recommend holding 29-66 stocks to eliminate 90% of diversifiable risk with 90% assurance (Table 2 column 4a). With the exception of the market crash in 1987, the results based on $ES_{1\%}$ (Table 2 column 5) require, on average, holding 13-14 fewer securities to achieve 90% diversification risk reduction. Using TWSD as a risk measure, an average investor should have held 85-95 stocks to eliminate 90% of diversifiable risk from 1975 to 1987; 63-66 stocks, a substantially lower requirement, from 1990 to 1998; 66-86 stocks, a steadily increasing requirement from 1999 and 2011 (Table 2 column 6 and Figure 8, solid line with circles).

Results for the Japanese financial market in Table 2, columns 7-9 and Figure 11 present an interesting feature: prior to 1989, the dispersion of risk measures of randomly chosen portfolios resulted in a very narrow spread between the average and the 90th percentile. This situation changed after 1989, when it became necessary to hold more and more securities to guarantee the 90% certainty of 90% reduction in diversifiable risk measured by both SD and $ES_{1\%}$ measures (see Table 1 columns 7a and 8a). Between 1990 and 2011 using standard deviation as a risk measure, 14-20 stocks were needed, but to assure 90% certainty of the 90% risk reduction between 30-53 stocks were required. We observe similar patterns when risk is measured as $ES_{1\%}$. However, when risk is measured as TWSD (Table 2 column 9 or Figure 11) the number of stocks required is stable across the 1975 to 2011 period ranging from 82 to 97 stocks.

For the Canadian market, in Table 2, columns 10-12 and Figure 14, we observe that between 1992 and 2007 the average investor needed to hold a higher number of stocks (26-39 stocks) as did the investor wishing to reduce diversifiable risk with 90% certainty (34-52 stocks). The general trend is consistent with the US markets; however, the average US investor in the period from 1992 to 2006 required a portfolio consisting of 24 to 31 stocks to achieve a 90% reduction in diversifiable risk as measured by standard deviation, while the average Canadian investor required 26-39 stocks. Given the closeness of the US and Canadian economies, we are not surprised to find similarities between required portfolio holdings in the US and Canada compared to different requirements in the UK, Japan or Australia.

For Australian investors concerned with standard deviation as a measure of risk, portfolios on average require 14-30 stocks (Table 2, column 13 and Figure 17, solid darker line). But from 1991 to 2007, investors had to hold a slightly larger portfolio (22-30 stocks) to achieve 90% in diversifiable risk on average. Between 1975 and 1987, to assure the 90% reduction 90% of the time, Australian investors needed to hold 31-39 stocks. Between 1988 to 2011 portfolio size increased to 34-52 stocks (Table 2, column 13a and Figure 17, dashed darker line). Consistent with the results from analyzing other countries, the portfolio size recommendations for an average Australian investor based on $ES_{1\%}$ are generally lower by 5-7 stocks over the entire period when compared with SD. To achieve 90% reduction in risk with 90% certainty, a similar pattern emerges except when financial markets are in distress. During periods of extreme market volatility or market crashes (black swan events), it is not surprising that when we use the measures of extreme risk such as $ES_{1\%}$, the portfolio size requirements to achieve the desired level of diversification with 90% certainty increase dramatically, even doubling for the US and the UK (refer to Table 2, 'Average' row vs. columns 2a, 5a, 8a, 11a, and 14a). Using TWSD as a risk measure (Table 2 column 15 and Figure 17, solid line with circles), an average Australian investor would have held 52-57 stocks (compared to 85-95 in the UK) to eliminate 90% of diversifiable risk from 1975 to 1988, with a substantially lower requirement (43-49 stocks) in 1989-1990 and incrementally and steadily increasing portfolio sizes from 1991 (55 stocks) to 2011 (81 stocks).

Our results for other risk measures for 10% and 5% expected shortfall, lower partial moments (e.g., semi-variance) are very similar to the results when the risk measure used is standard deviation but are included in this paper for brevity's sake.[13]

III. Conclusion

In this study of the optimal size of portfolios we use daily traded equity returns for five developed countries (the US, the UK, Japan, Canada and Australia) between 1975 to 2011. We examine the optimal number of stocks investor should hold in their portfolios in order to achieve a particular level of reduction in diversifiable risk with a specific degree of confidence. We calculate several widely accepted (including heavy tailed) measures of risk. For each measure, in addition to providing portfolio size recommendations for an average investor, we estimated confidence bands to assure a specific reduction in diversifiable risk. The period from 1975 to 2011 allowed us to account for some significant events in financial market history (see Table 1). We conclude that the optimal portfolio size is influenced by market conditions, which in turn determine the level of the particular risk measure. We identify two types of crises, general drops in the market (1987 and 2008-2011) and industry

[13]The results and figures are available from the authors upon request.

specific meltdowns (2000-2002).[14] Such crises are characterized by a few common features. In case of market crashes, we noted increased market risk measured by SD and especially $ES_{1\%}$ coupled with greatly increased average correlation among securities as well as the correlation of the average security with the market portfolio (see Figures 4, 7, 10, 13, 16). During market crashes in the US, Figure 5 (solid darker line) shows that the number of stocks required for an average investor to obtain a 90% reduction in diversifiable risk using standard deviation as a measure decreases. At the same time, the number of stocks required to achieve the same level of diversification but with 90% confidence increases. For the average US investor, Tables 1 and 2 confirm that during major stock market crashes (1987,1989, and 2008) the number of stocks required to eliminate 90% of diversifiable risk was the lowest (16-17 stocks when risk is measured by SD and 9-10 stocks when risk is measured by $ES_{1\%}$). The number of stocks required to achieve the same level of diversification but with 90% confidence was the highest across all periods when risk is measured as $ES_{1\%}$ (71-113 stocks). This result generally holds across all five markets.

With the exception of the UK's financial market, we note that the downward trend in average correlations among stocks reported by Campbell, Lettau, Malkiel, and Xu for the US between 1960 and 2000, reversed itself between 2000 and 2011, including the period 2002 to 2007 when the economy was not in crisis. Increased correlations among stocks after 2000, made portfolio diversification harder to achieve with a small number of stocks.

In the case of industry specific meltdowns, such as the bursting of the Dot Com bubble, the market experienced high volatility. The average correlation among securities within the market and with the market portfolio, however, were among the lowest (Figures 4, 7, 10, 13, 16). As shown for the US in Figure 5, fewer stocks are needed to get the desired level of diversification with 90% certainty: 35 stocks in 2000 using expected shortfall at 1% level as a measure. One interpretation of the findings in this paper is that the recommended number of stocks in a buy-and-hold portfolio to attain on average (or with a particular degree of certainty) to reduce diversifiable risk should not be based on results in periods when markets are in distress. Rather, as shown in Figure 5, during 2008-2010, US long term investors should instead have relied on results obtained during normal financial market periods.

While the average all domestic US equity fund holds 176 stocks, we find that this number is considerably larger than the number of stocks required to assure a 90% reduction in diversifiable risk 90% of the time for either of the measures used in our analysis under most conditions since 1975. We recommend that professional portfolio managers who use standard deviation as a measure of risk, and who seek to reduce 90% of diversifiable risk 90% percent of the time should hold between 40 and 70 stocks for the US; between 30 and 65 for the UK; between 30 and 50 for Japan; between 20 and 50 for Canada; and between 30 and 50 for Australia.

[14]The OECD dated recessions (OECD, 2012) during the period of our study do not appear to have had an impact on the number of stocks needed for diversifying specific risk, unless these recessions coincide with financial market meltdowns.

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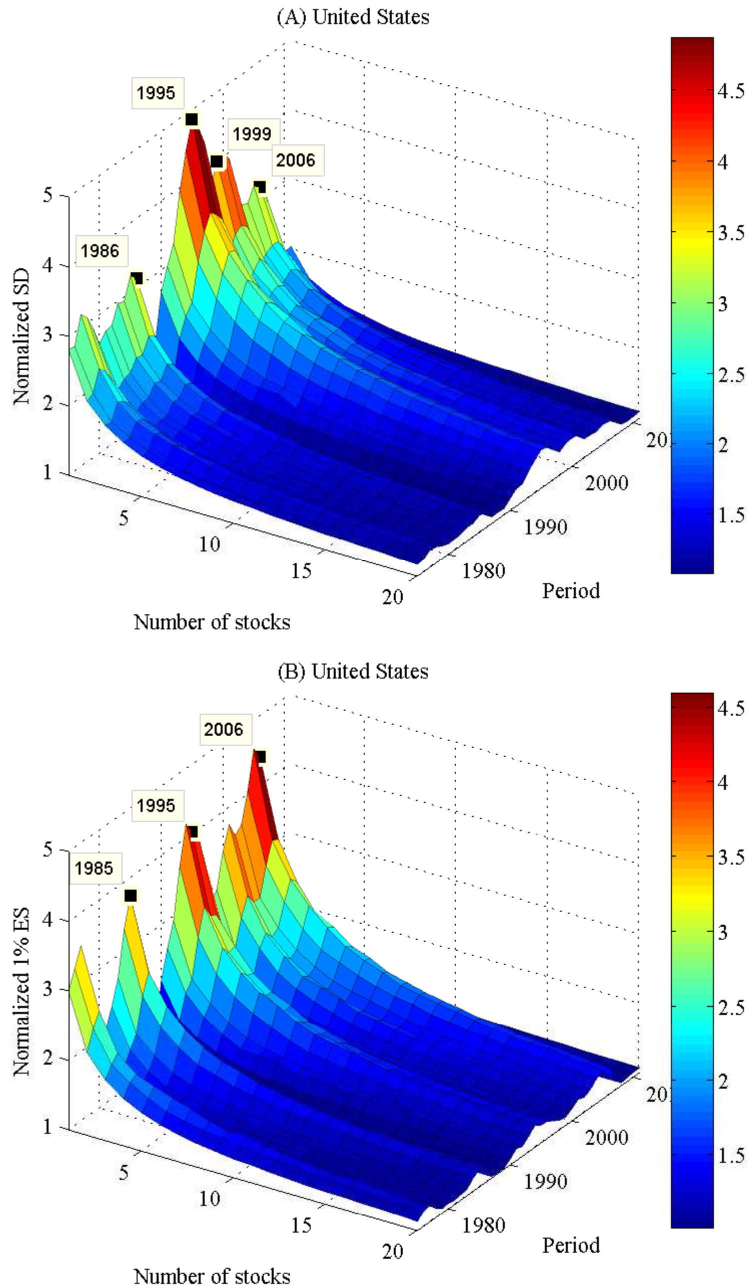


Figure 3: **United States**. Panel (A) shows the average level of an n-stock portfolio standard deviation normalized by the market standard deviation as defined by (8) above. For each year from 1975 to 2011 we construct an asymptote and show the average n-stock portfolio standard deviation for $n = 1..20$ relative to the market standard deviation for each year (e.g. in 1995 the average security standard deviation was almost five times higher than the market standard deviation). Similarly, Panel (B) shows the average level of $ES_{1\%}$ for an n-stock portfolio normalized by the market $ES_{1\%}$ (e.g. in 1995 the average security $ES_{1\%}$ was four times higher than that of the market portfolio). In both panels, when $n \to N$, the normalized measure approaches 1. In both cases, slower convergence requires a larger portfolio size to reduce diversifiable risk.

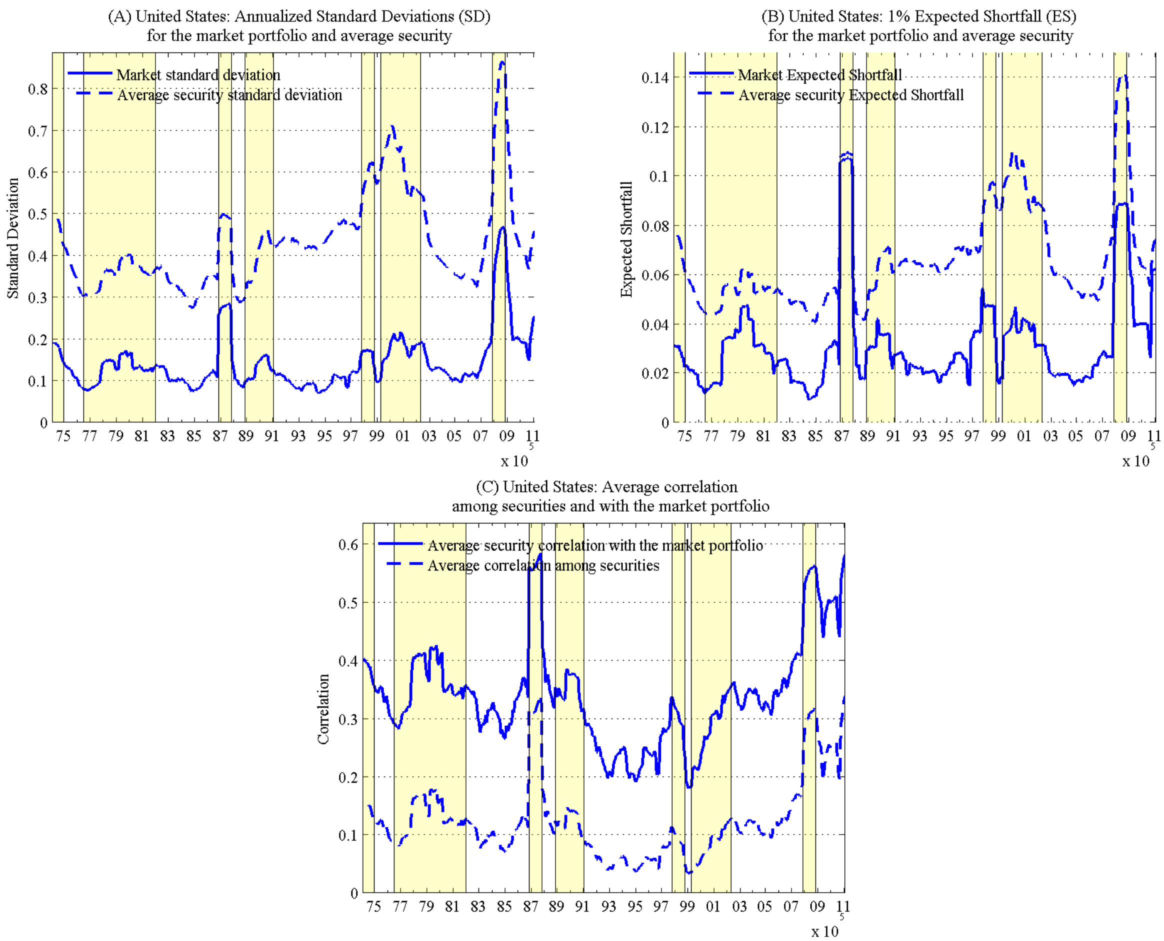


Figure 4: **United States**. In panel (A) the solid line shows the annualized standard deviation within each month of daily market returns based on the past 12 months' returns. The dashed line represents the average security standard deviation. Panels (B) depicts $ES_{1\%}$ of the market portfolio (solid line) and the average security $ES_{1\%}$ (dashed line). Panel (C) shows the average security correlation with the market portfolio (solid line) and the average correlation among securities (dashed line). Shaded regions in the figure represent periods of crises and correspond to events presented in Table 1.

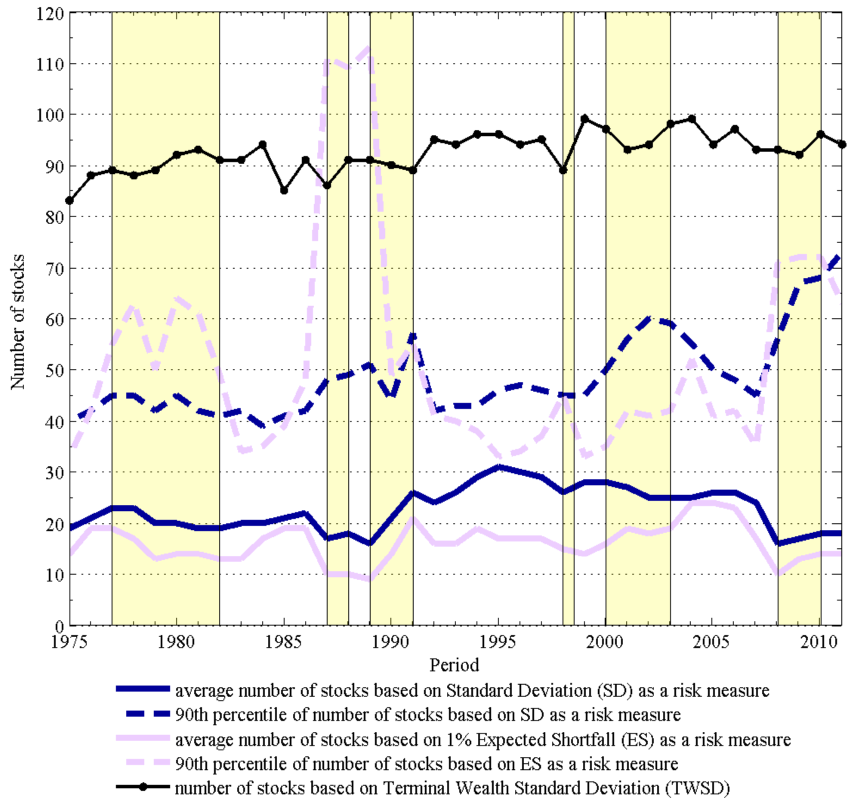
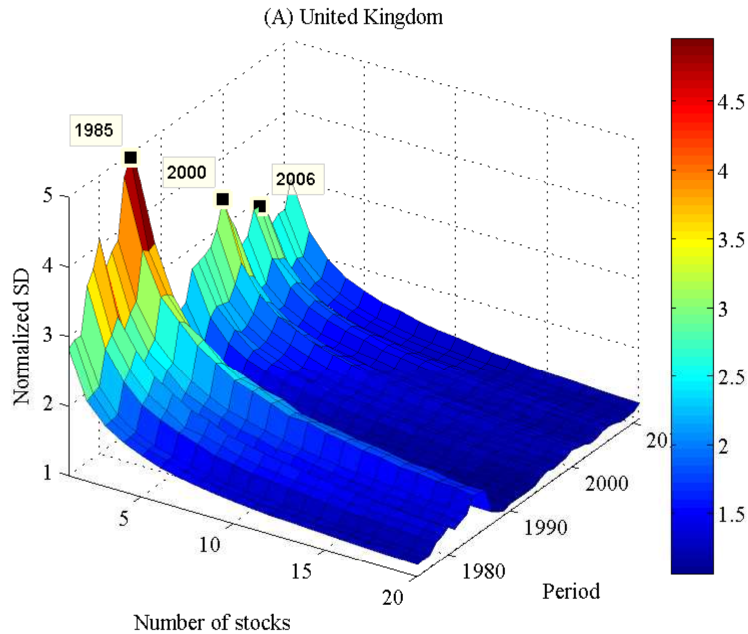


Figure 5: **United States. Recommended portfolio size to achieve 90% reduction in diversifiable risk.** The solid dark line represents the number of stocks recommended for an average investor to achieve 90% reduction in diversifiable risk when standard deviation is used as a risk measure. To achieve this reduction 90% of the time, portfolio size is depicted by the dashed dark line. Similarly, for investors concerned with extreme risk and using $ES_{1\%}$ as the risk measure, the portfolio size for an average investor is depicted by the solid light line and the size of the portfolio that assures this reduction 90% of the time is shown by the dashed light line. For investors concerned with terminal wealth standard deviation, our recommended portfolio size is shown by the dark solid line with circles. Shaded regions in the figure represent periods of crises and correspond to events presented in Table 1.



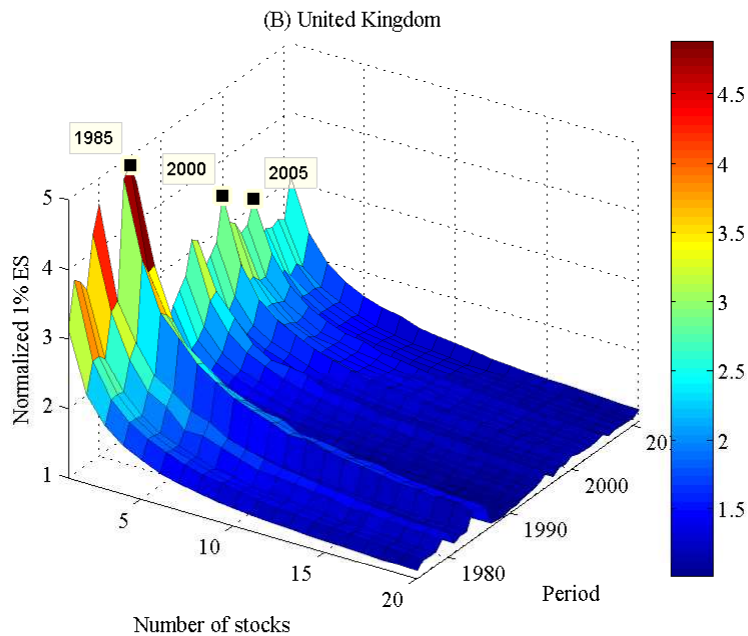


Figure 6: **United Kingdom**. Panel (A) shows the average level of an n-stock portfolio standard deviation normalized by the market standard deviation as defined by (8) above. For each year from 1975 to 2011 we construct an asymptote and show the average n-stock portfolio standard deviation for $n = 1..20$ relative to the market standard deviation for each year (e.g. in 1985 the average security standard deviation was almost five times higher than the market standard deviation). Similarly, Panel (B) shows the average level of $ES_{1\%}$ for an n-stock portfolio normalized by the market $ES_{1\%}$ (e.g. in 1985 the average security $ES_{1\%}$ was five times higher than that of the market portfolio). In both panels, when $n \to N$, the normalized measure approaches 1. In both cases, slower convergence requires a larger portfolio size to reduce diversifiable risk.

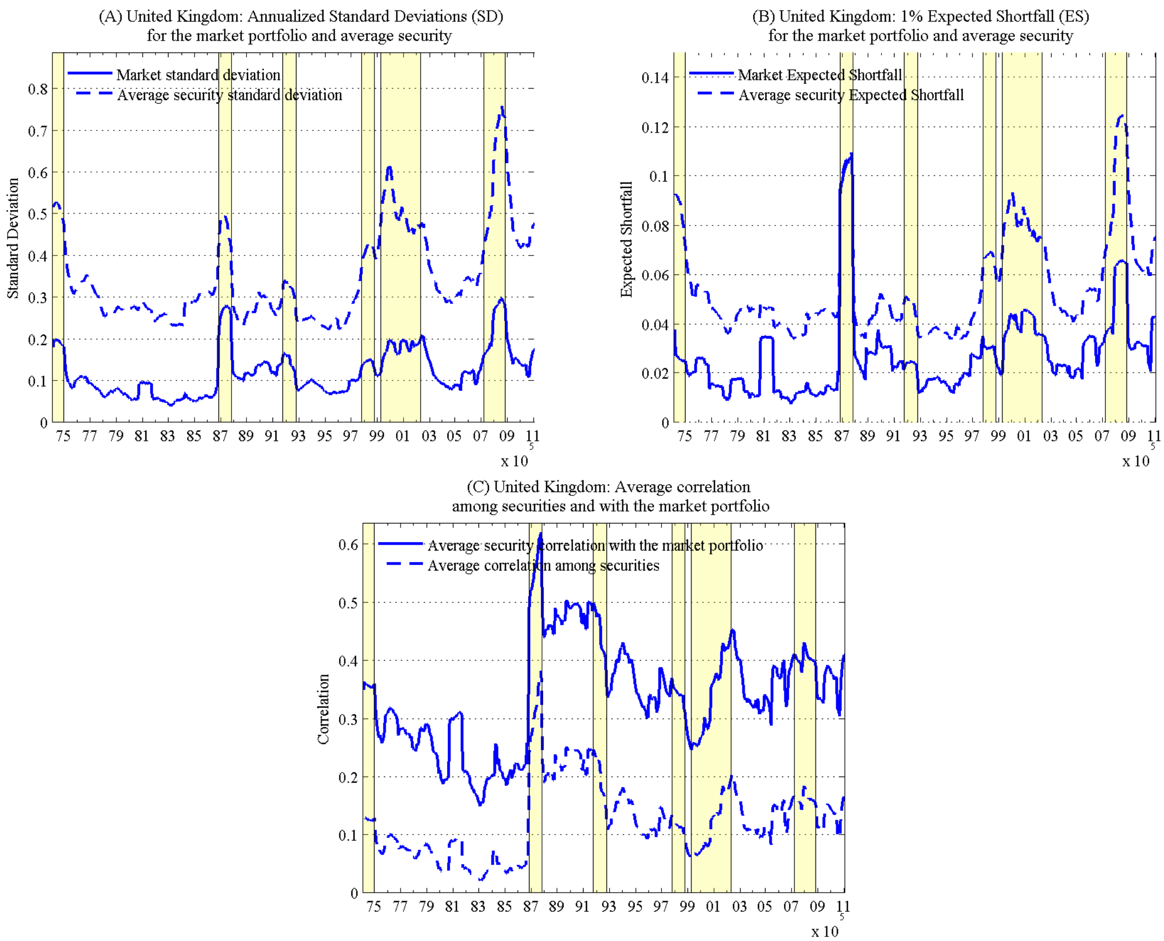


Figure 7: **United Kingdom**. In panel (A) the solid line shows the annualized standard deviation within each month of daily market returns based on the past 12 months' returns. The dashed line represents the average security standard deviation. Panels (B) depicts $ES_{1\%}$ of the market portfolio (solid line) and the average security $ES_{1\%}$ (dashed line). Panel (C) shows the average security correlation with the market portfolio (solid line) and the average correlation among securities (dashed line). Shaded regions in the figure represent periods of crises and correspond to events presented in Table 1.

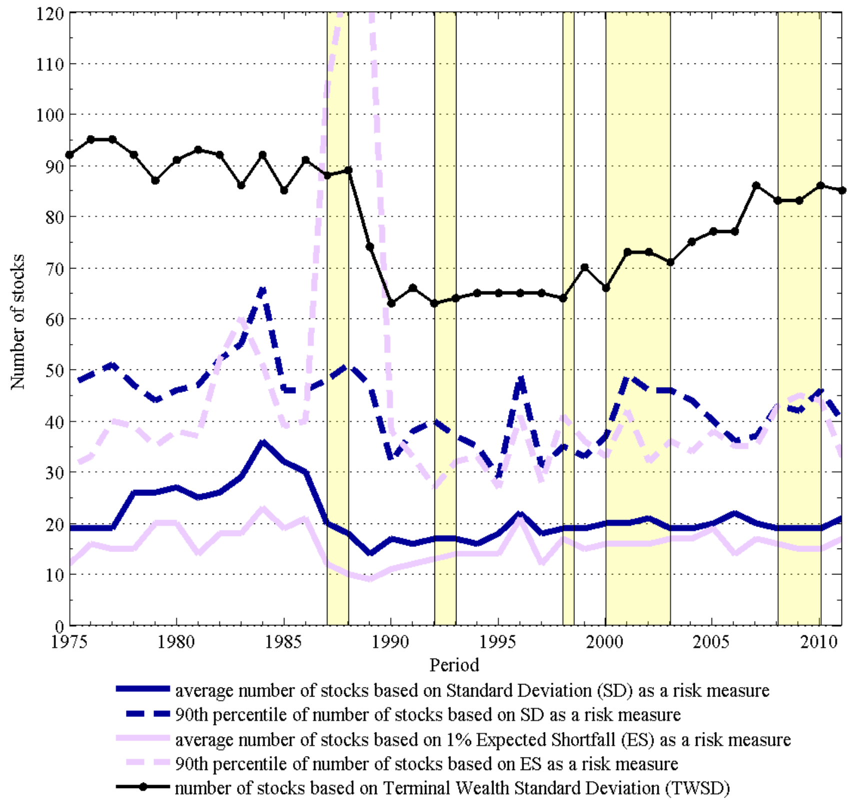


Figure 8: **United Kingdom. Recommended portfolio size to achieve 90% reduction in diversifiable risk.** The solid dark line represents the number of stocks recommended for an average investor to achieve 90% reduction in diversifiable risk when standard deviation is used as a risk measure. To achieve this reduction 90% of the time, portfolio size is depicted by the dashed dark line. Similarly, for investors concerned with extreme risk and using $ES_{1\%}$ as the risk measure, the portfolio size for an average investor is depicted by the solid light line and the size of the portfolio that assures this reduction 90% of the time is shown by the dashed light line. For investors concerned with terminal wealth standard deviation, our recommended portfolio size is shown by the dark solid line with circles. Shaded regions in the figure represent periods of crises and correspond to events presented in Table 1.

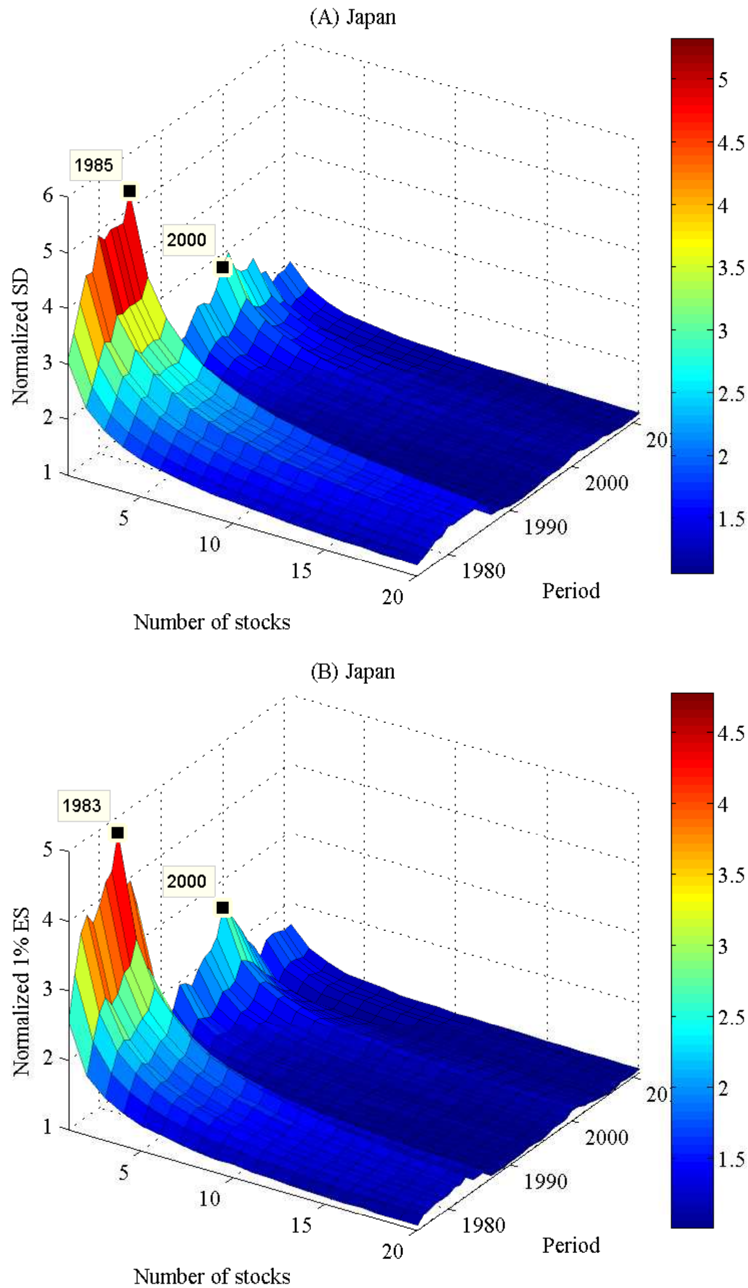


Figure 9: **Japan**. Panel (A) shows the average level of an n-stock portfolio standard deviation normalized by the market standard deviation as defined by (8) above. For each year from 1975 to 2011 we construct an asymptote and show the average n-stock portfolio standard deviation for $n = 1..20$ relative to the market standard deviation for each year (e.g. in 1985 the average security standard deviation was more than five times higher than the market standard deviation). Similarly, Panel (B) shows the average level of $ES_{1\%}$ for an n-stock portfolio normalized by the market $ES_{1\%}$ (e.g. in 1983 the average security $ES_{1\%}$ was five times higher than that of the market portfolio). In both panels, when $n \to N$, the normalized measure approaches 1. In both cases, slower convergence requires a larger portfolio size to reduce diversifiable risk.

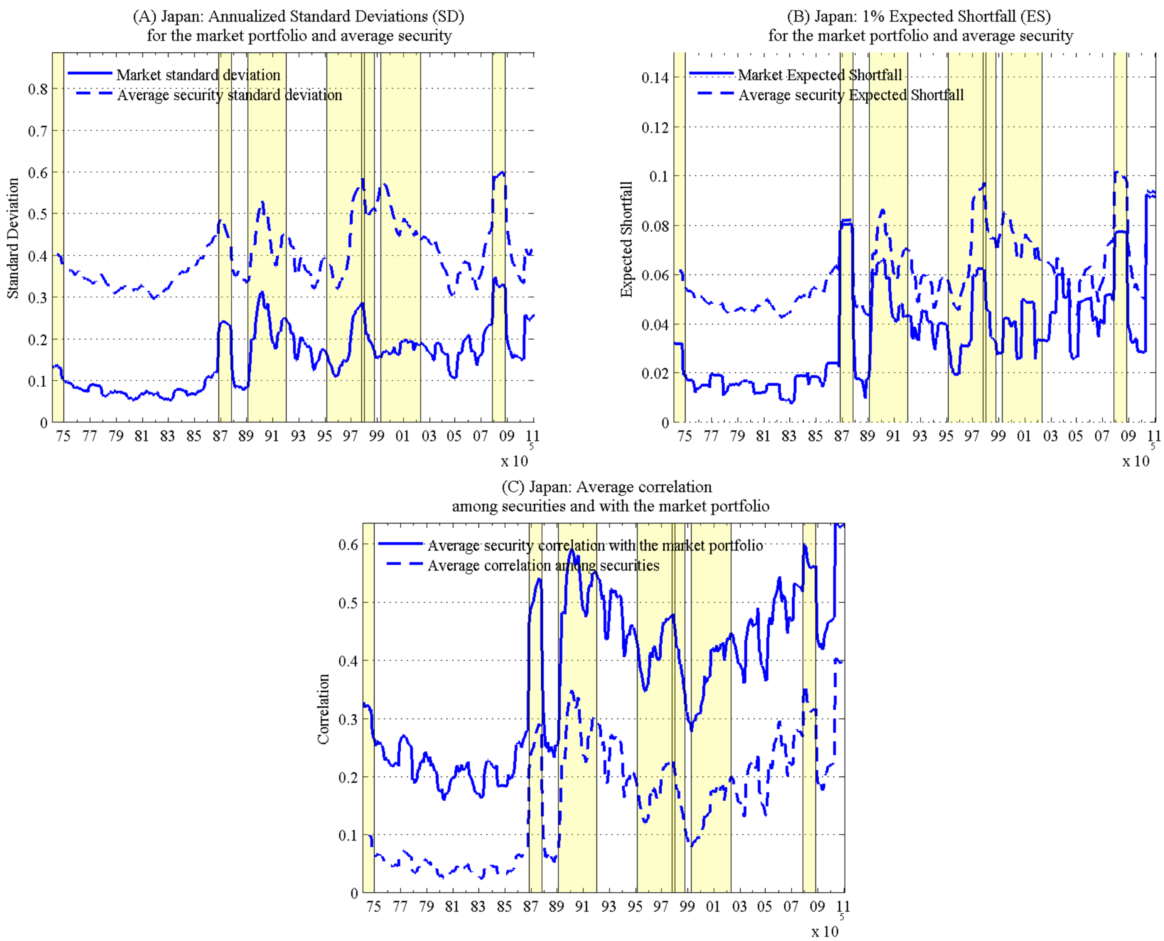


Figure 10: **Japan**. In panel (A) the solid line shows the annualized standard deviation within each month of daily market returns based on the past 12 months' returns. The dashed line represents the average security standard deviation. Panels (B) depicts $ES_{1\%}$ of the market portfolio (solid line) and the average security $ES_{1\%}$ (dashed line). Panel (C) shows the average security correlation with the market portfolio (solid line) and the average correlation among securities (dashed line). Shaded regions in the figure represent periods of crises and correspond to events presented in Table 1.

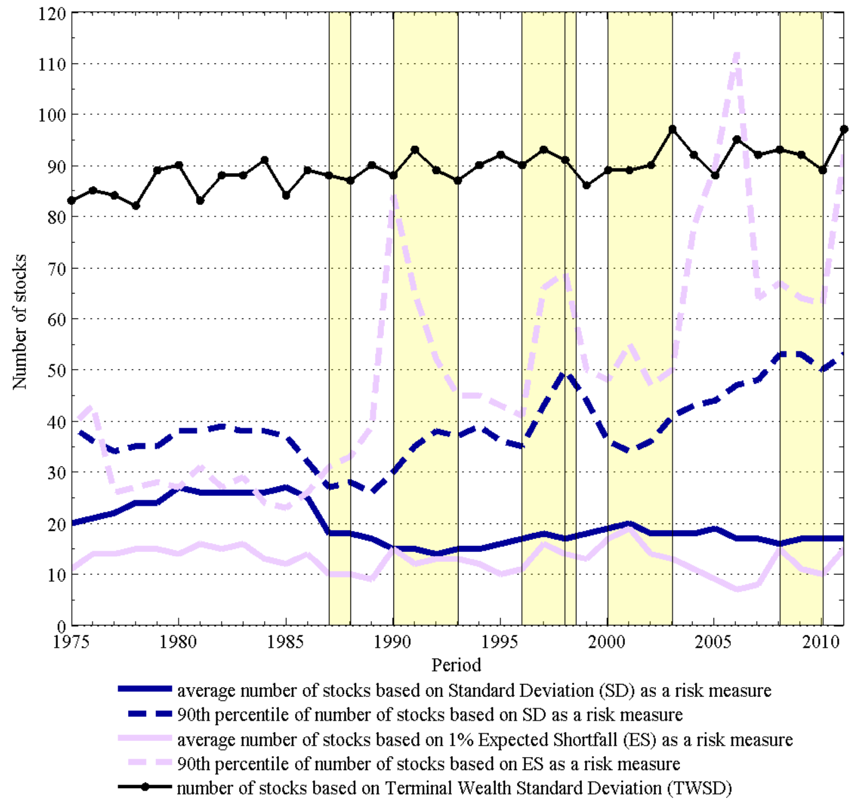


Figure 11: **Japan. Recommended portfolio size to achieve 90% reduction in diversifiable risk.** The solid dark line represents the number of stocks recommended for an average investor to achieve 90% reduction in diversifiable risk when standard deviation is used as a risk measure. To achieve this reduction 90% of the time, portfolio size is depicted by the dashed dark line. Similarly, for investors concerned with extreme risk and using $ES_{1\%}$ as the risk measure, the portfolio size for an average investor is depicted by the solid light line and the size of the portfolio that assures this reduction 90% of the time is shown by the dashed light line. For investors concerned with terminal wealth standard deviation, our recommended portfolio size is shown by the dark solid line with circles. Shaded regions in the figure represent periods of crises and correspond to dates presented in Table 1.

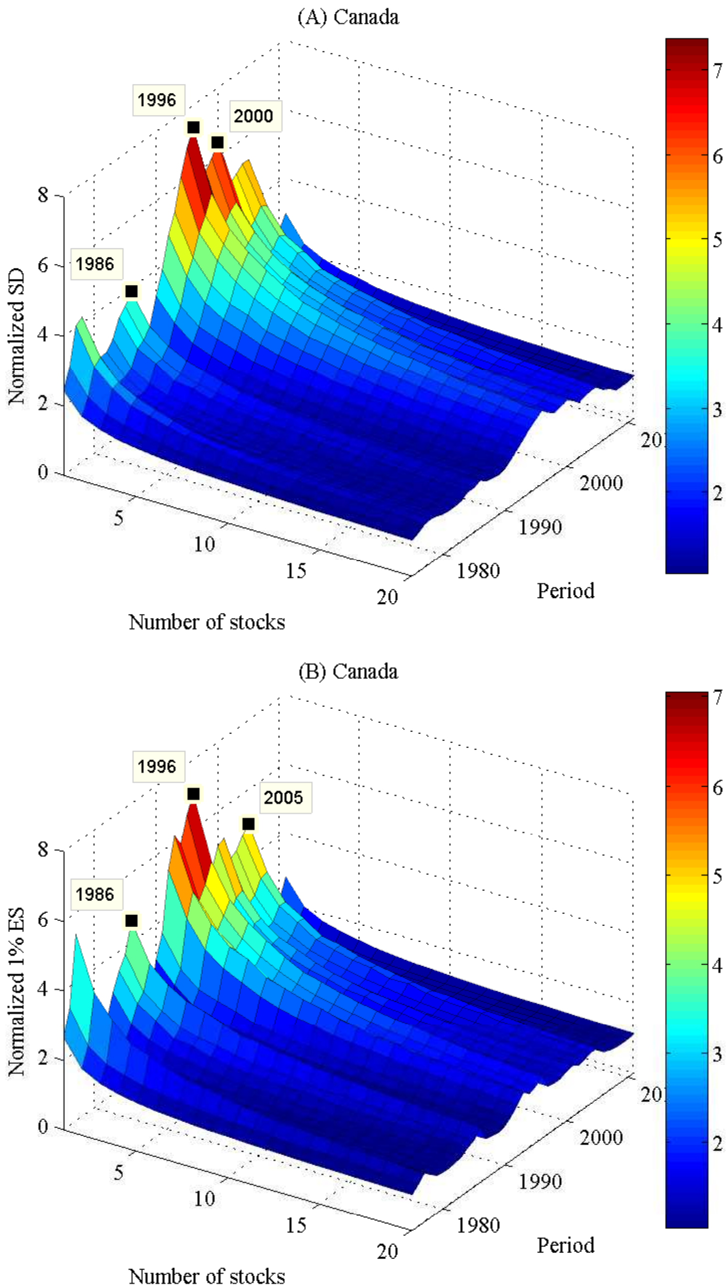


Figure 12: **Canada**. Panel (A) shows the average level of an n-stock portfolio standard deviation normalized by the market standard deviation as defined by (8) above. For each year from 1975 to 2011 we construct an asymptote and show the average n-stock portfolio standard deviation for $n = 1..20$ relative to the market standard deviation for each year (e.g. in 1996 the average security standard deviation was almost seven times higher than the market standard deviation). Similarly, Panel (B) shows the average level of $ES_{1\%}$ for an n-stock portfolio normalized by the market $ES_{1\%}$ (e.g. in 1996 the average security $ES_{1\%}$ was seven times higher than that of the market portfolio). In both panels, when $n \to N$, the normalized measure approaches 1. In both cases, slower convergence requires a larger portfolio size to reduce diversifiable risk.

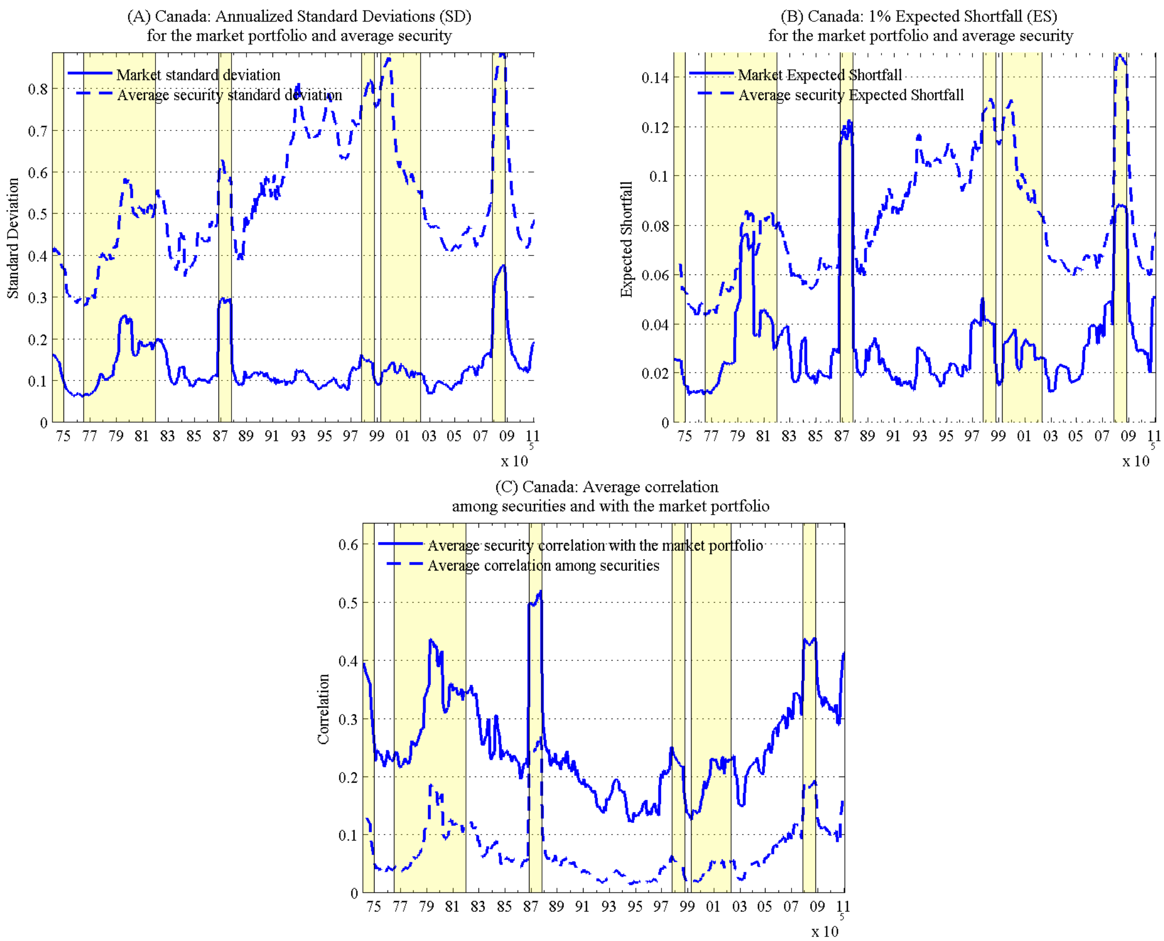


Figure 13: **Canada**. In panel (A) the solid line shows the annualized standard deviation within each month of daily market returns based on the past 12 months' returns. The dashed line represents the average security standard deviation. Panels (B) depicts $ES_{1\%}$ of the market portfolio (solid line) and the average security $ES_{1\%}$ (dashed line). Panel (C) shows the average security correlation with the market portfolio (solid line) and the average correlation among securities (dashed line). Shaded regions in the figure represent periods of crises and correspond to events presented in Table 1.

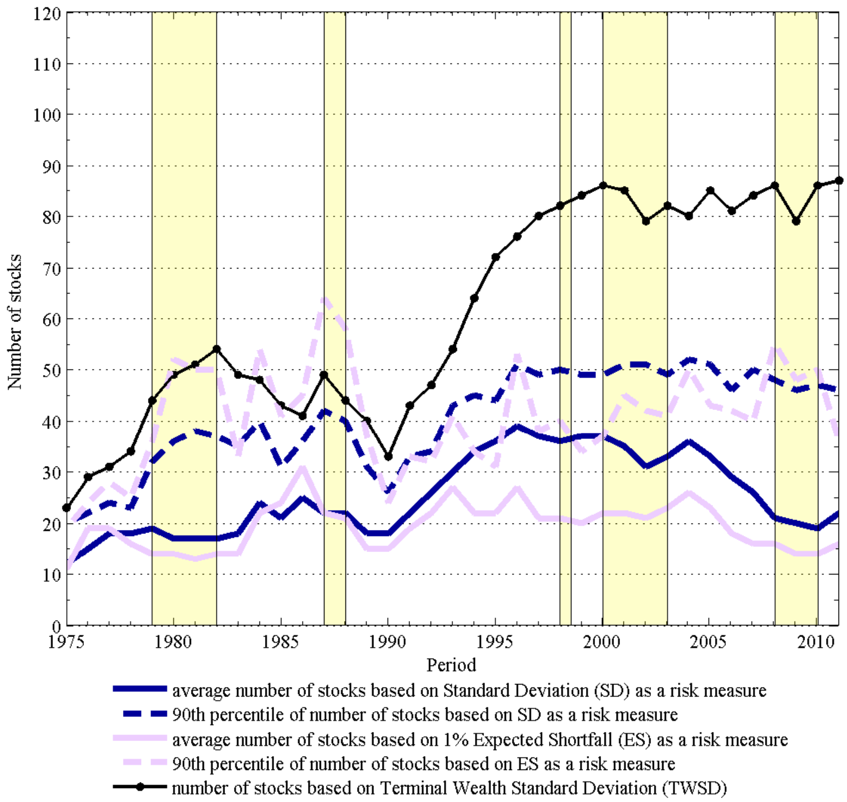


Figure 14: **Canada. Recommended portfolio size to achieve 90% reduction in diversifiable risk.** The solid dark line represents the number of stocks recommended for an average investor to achieve 90% reduction in diversifiable risk when standard deviation is used as a risk measure. To achieve this reduction 90% of the time, portfolio size is depicted by the dashed dark line. Similarly, for investors concerned with extreme risk and using $ES_{1\%}$ as the risk measure, the portfolio size for an average investor is depicted by the solid light line and the size of the portfolio that assures this reduction 90% of the time is shown by the dashed light line. For investors concerned with terminal wealth standard deviation, our recommended portfolio size is shown by the dark solid line with circles. Shaded regions in the figure represent periods of crises and correspond to events presented in Table 1.

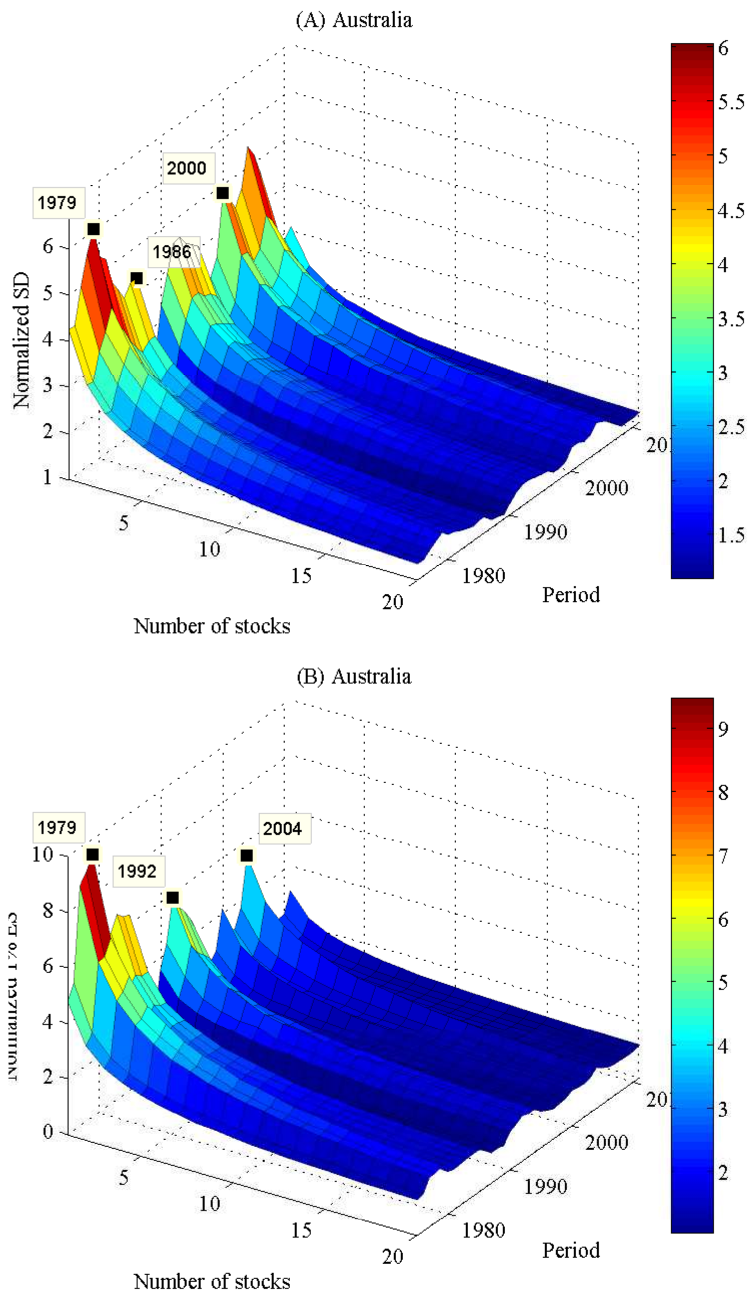


Figure 15: **Australia**. Panel (A) shows the average level of an n-stock portfolio standard deviation normalized by the market standard deviation as defined by (8) above. For each year from 1975 to 2011 we construct an asymptote and show the average n-stock portfolio standard deviation for $n = 1..20$ relative to the market standard deviation for each year (e.g. in 1979 the average security standard deviation was six times higher than the market standard deviation). Similarly, Panel (B) shows the average level of $ES_{1\%}$ for an n-stock portfolio normalized by the market $ES_{1\%}$ (e.g. in 1979 the average security $ES_{1\%}$ was nine times higher than that of the market portfolio). In both panels, when $n \rightarrow N$, the normalized measure approaches 1. In both cases, slower convergence requires a larger portfolio size to reduce diversifiable risk.

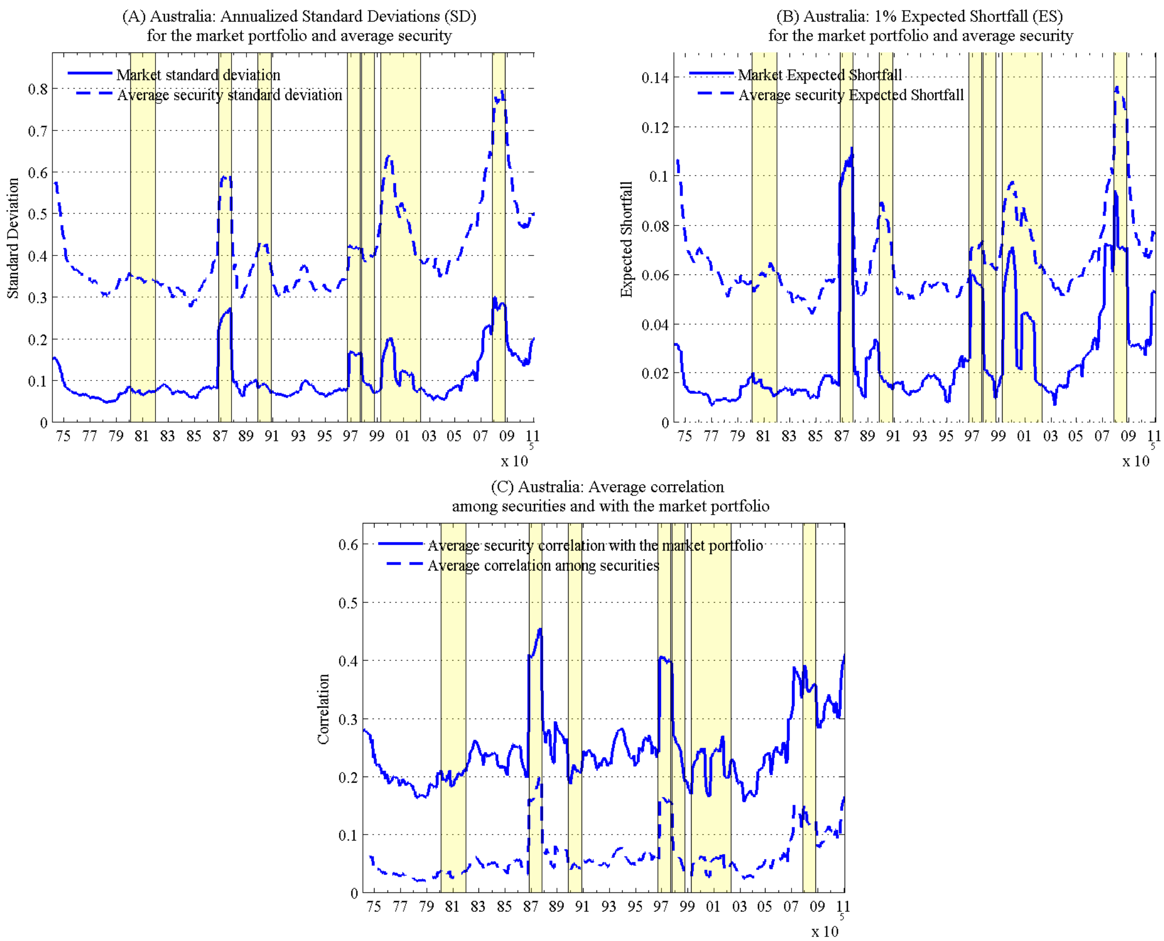


Figure 16: **Australia**. In panel (A) the solid line shows the annualized standard deviation within each month of daily market returns based on the past 12 months' returns. The dashed line represents the average security standard deviation. Panels (B) depicts $ES_{1\%}$ of the market portfolio (solid line) and the average security $ES_{1\%}$ (dashed line). Panel (C) shows the average security correlation with the market portfolio (solid line) and the average correlation among securities (dashed line). Shaded regions in the figure represent periods of crises and correspond to events presented in Table 1.

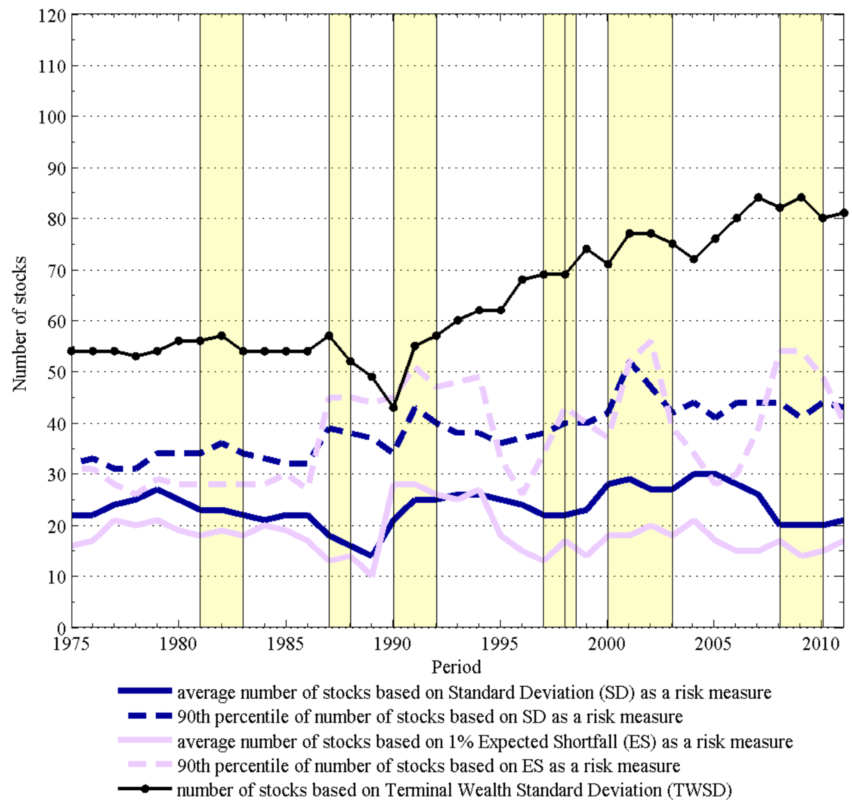


Figure 17: **Australia. Recommended portfolio size to achieve 90% reduction in diversifiable risk.** The solid dark line represents the number of stocks recommended for an average investor to achieve 90% reduction in diversifiable risk when standard deviation is used as a risk measure. To achieve this reduction 90% of the time, portfolio size is depicted by the dashed dark line. Similarly, for investors concerned with extreme risk and using $ES_{1\%}$ as the risk measure, the portfolio size for an average investor is depicted by the solid light line and the size of the portfolio that assures this reduction 90% of the time is shown by the dashed light line. For investors concerned with terminal wealth standard deviation, our recommended portfolio size is shown by the dark solid line with circles. Shaded regions in the figure represent periods of crises and correspond to events presented in Table 1.

Event	**US**		**UK**		**Japan**		**Canada**		**Australia**	
	SD	$ES_{1\%}$	**SD**	$ES_{1\%}$	**SD**	$ES_{1\%}$	**SD**	$ES_{1\%}$	**SD**	$ES_{1\%}$
The 1973 oil crisis (1973-1974)	✓		✓		✓		✓		+	
Bear market of 1977-1978	✓ 23 (45)	17-19 (55-63)								
The 1979 (or second) oil crisis (1979-1982)	✓ 19-20 (41-45)	13-14 (49-64)					+ 17-19 (32-38)	13-14 (36-52)		
Bear market of 1981-1982									✓ 23 (34-36)	18-19 (28)
Black Monday (1987)	+ 17 (48)	10 (111)	+ 20 (48)	12 (105)	+ 18 (27)	10 (31)	+ 22 (42)	22 (64)	+ 18 (39)	13 (45)
Friday the 13th mini-crash (1989)	✓ 16 (51)	9 (113)								
First Gulf War (1990-1991)	✓ 21-26 (44-57)	14-21 (49-55)			✓ 15 (30-35)	12-15 (65-84)			✓ 21-25 (34-33)	28 (45-51)
Japanese Asset Price bubble (1990-1992)					✓ 14-15 (30-38)	12-15 (52-84)				
Black Wednesday (1992)			+ 17 (40)	13 (27)						
Bear market of 1996-1998					✓ 17-18 (35-50)	11-16 (41-69)				
Asian Financial Crisis (1997)					✓ 18 (43)	16 (66)			+ 22 (38)	13 (34)
Collapse of LTCM (1998)	+ 26 (45)	15 (45)	+ 19 (35)	17 (31)	+ 17 (50)	14 (69)	+ 36 (50)	21 (40)	+ 22 (40)	17 (43)
Dot-com bubble (2000-2002)	✓ 25-28 (50-60)	16-19 (35-42)	✓ 20-21 (37-49)	16 (32-42)	✓ 18-20 (34-36)	14-19 (47-55)	✓ 31-37 (49-51)	21-22 (42-45)	+ 27-29 (42-52)	18-20 (37-56)
Global Financial Crisis (2008)	✓ 16 (56)	10 (71)	✓ 19 (43)	16 (43)	✓ 16 (53)	15 (67)	✓ 21 (48)	16 (55)	✓ 20 (44)	17 (54)

Table 1: **Dates for crises leading to stock market crashes and their aftermath**. Source: Symbol ✓refers to events identified in Reinhart and Rogoff (2009) and *Dates for Banking Crises, Currency Crashes, Sovereign Domestic or External Default, Inflation Crises, and Stock Market Crashes* webpage on the authors website: *http://www.reinhartandrogoff.com/data/*. The symbol + refers to events we include in addition to those identified in Reinhart and Rogoff (2009): for all five countries in our sample - Black Monday (1987), and Long Term Capital Management (LTCM) in 1998; for the UK - Black Wednesday (1992) ; for Australia - the 1973 oil crisis (1973-1974), the Asian Financial Crisis (1997) and the Dot-com bubble (2000-2002); for Canada - the 1979 oil crisis (1979-1982). The numbers in the SD and $ES_{1\%}$ columns represent the range of the number of stocks required to reduce diversifiable risk by 90% (with 90% certainty).

	United States			United Kingdom			Japan			Canada			Australia		
	1 (1a)	2 (2a)	3	4 (4a)	5 (5a)	6	7 (7a)	8 (8a)	9	10 (10a)	11 (11a)	12	13 (13a)	14 (14a)	15
Year	**SD**	**ES**	**TWSD**	**SD**	**ES**	**TWSD**	**SD**	**ES**	**TWSD**	**SD**	**ES**	**TWSD**	**SD**	**ES**	**TWSD**
1975	19 (40)	14 (33)	83	19 (47)	12 (31)	92	20 (39)	11 (39)	83	12 (20)	11 (19)	23	22 (32)	16 (31)	54
1976	21 (42)	19 (42)	88	19 (49)	16 (33)	95	21 (36)	14 (43)	85	15 (22)	19 (24)	29	22 (33)	17 (31)	54
1977	23 (45)	19 (55)	89	19 (51)	15 (40)	95	22 (34)	14 (26)	84	18 (24)	19 (28)	31	24 (31)	21 (28)	54
1978	23 (45)	17 (63)	88	26 (47)	15 (39)	92	24 (35)	15 (27)	82	18 (23)	16 (25)	34	25 (31)	20 (26)	53
1979	20 (42)	13 (50)	89	26 (44)	20 (35)	87	24 (35)	15 (28)	89	19 (32)	14 (36)	44	27 (34)	21 (29)	54
1980	20 (45)	14 (64)	92	27 (46)	20 (38)	91	27 (38)	14 (27)	90	17 (36)	14 (52)	49	25 (34)	19 (28)	56
1981	19 (42)	14 (61)	93	25 (47)	14 (37)	93	26 (38)	16 (31)	83	17 (38)	13 (50)	51	23 (34)	18 (28)	56
1982	19 (41)	13 (49)	91	26 (52)	18 (52)	92	26 (39)	15 (27)	88	17 (37)	14 (50)	54	23 (36)	19 (28)	57
1983	20 (42)	13 (34)	91	29 (55)	18 (60)	86	26 (38)	16 (29)	88	18 (35)	14 (33)	49	22 (34)	18 (28)	54
1984	20 (39)	17 (35)	94	36 (66)	23 (51)	92	26 (38)	13 (24)	91	24 (40)	22 (54)	48	21 (33)	20 (28)	54
1985	21 (41)	19 (39)	85	32 (46)	19 (39)	85	27 (37)	12 (23)	84	21 (31)	24 (41)	43	22 (32)	19 (30)	54
1986	22 (42)	19 (48)	91	30 (46)	21 (40)	91	25 (32)	14 (26)	89	25 (36)	31 (45)	41	22 (32)	17 (27)	54
1987	17 (48)	10 (111)	86	20 (48)	12 (105)	88	18 (27)	10 (31)	88	22 (42)	22 (64)	49	18 (39)	13 (45)	57
1988	18 (49)	10 (109)	91	18 (51)	10 (129)	89	18 (28)	10 (33)	87	22 (40)	21 (58)	44	16 (38)	14 (45)	52
1989	16 (51)	9 (113)	91	14 (47)	9 (127)	74	17 (26)	9 (39)	90	18 (31)	15 (37)	40	14 (37)	10 (44)	49
1990	21 (44)	14 (49)	90	17 (32)	11 (38)	63	15 (30)	15 (84)	88	18 (26)	15 (24)	33	21 (34)	28 (45)	43
1991	26 (57)	21 (55)	89	16 (38)	12 (33)	66	15 (35)	12 (65)	93	22 (33)	19 (33)	43	25 (43)	28 (51)	55
1992	24 (42)	16 (41)	95	17 (40)	13 (27)	63	14 (38)	13 (52)	89	26 (34)	22 (32)	47	25 (40)	26 (47)	57
1993	26 (43)	16 (40)	94	17 (37)	14 (32)	64	15 (37)	13 (45)	87	30 (43)	27 (41)	54	26 (38)	25 (48)	60
1994	29 (43)	19 (38)	96	16 (35)	14 (33)	65	15 (39)	12 (45)	90	34 (45)	22 (34)	64	26 (38)	27 (49)	62
1995	31 (46)	17 (33)	96	18 (29)	14 (27)	65	16 (36)	10 (43)	92	36 (44)	22 (31)	72	25 (36)	18 (33)	62
1996	30 (47)	17 (34)	94	22 (49)	21 (41)	65	17 (35)	11 (41)	90	39 (51)	27 (53)	76	24 (37)	15 (26)	68
1997	29 (46)	17 (37)	95	18 (31)	12 (28)	65	18 (43)	16 (66)	93	37 (49)	21 (38)	80	22 (38)	13 (34)	69
1998	26 (45)	15 (45)	89	19 (35)	17 (41)	64	17 (50)	14 (69)	91	36 (50)	21 (40)	82	22 (40)	17 (43)	69
1999	28 (45)	14 (33)	99	19 (33)	15 (36)	70	18 (44)	13 (50)	86	37 (49)	20 (34)	84	23 (40)	14 (40)	74
2000	28 (50)	16 (35)	97	20 (37)	16 (33)	66	19 (36)	17 (48)	89	37 (49)	22 (37)	86	28 (42)	18 (37)	71
2001	27 (56)	19 (42)	93	20 (49)	16 (42)	73	20 (34)	19 (55)	89	35 (51)	22 (45)	85	29 (52)	18 (52)	77
2002	25 (60)	18 (41)	94	21 (46)	16 (32)	73	18 (36)	14 (47)	90	31 (51)	21 (42)	79	27 (47)	20 (56)	77
2003	25 (59)	19 (42)	98	19 (46)	17 (36)	71	18 (41)	13 (50)	97	33 (49)	23 (41)	82	27 (42)	18 (39)	75
2004	25 (55)	24 (52)	99	19 (44)	17 (34)	75	18 (43)	11 (78)	92	36 (52)	26 (50)	80	30 (44)	21 (34)	72
2005	26 (50)	24 (41)	94	20 (40)	19 (38)	77	19 (44)	9 (90)	88	33 (51)	23 (43)	85	30 (41)	17 (28)	76
2006	26 (48)	23 (42)	97	22 (36)	14 (35)	77	17 (47)	7 (112)	95	29 (46)	18 (42)	81	28 (44)	15 (30)	80
2007	24 (45)	17 (35)	93	20 (37)	17 (35)	86	17 (48)	8 (64)	92	26 (50)	16 (40)	84	26 (44)	15 (39)	84
2008	16 (56)	10 (71)	93	19 (43)	16 (43)	83	16 (53)	15 (67)	93	21 (48)	16 (55)	86	20 (44)	17 (54)	82
2009	17 (67)	13 (72)	92	19 (42)	15 (45)	83	17 (53)	11 (64)	92	20 (46)	14 (48)	79	20 (41)	14 (54)	84
2010	18 (68)	14 (72)	96	19 (46)	15 (44)	86	17 (50)	10 (63)	89	19 (47)	14 (50)	86	20 (44)	15 (49)	80
2011	18 (73)	14 (63)	94	21 (40)	17 (33)	85	17 (53)	15 (92)	97	22 (46)	16 (36)	87	21 (43)	17 (40)	81
Average	23 (49)	16 (52)	92	21 (43)	16 (44)	79	19 (39)	13 (50)	89	25 (40)	19 (41)	61	24 (38)	18 (38)	64

Table 2: **Recommended portfolio size to achieve 90% reduction in diversifiable risk**. The table presents the average number of stocks and the 90th percentile of number of stocks (in parenthesis) based on SD, ES at the 1% level and TWSD as risk measures.

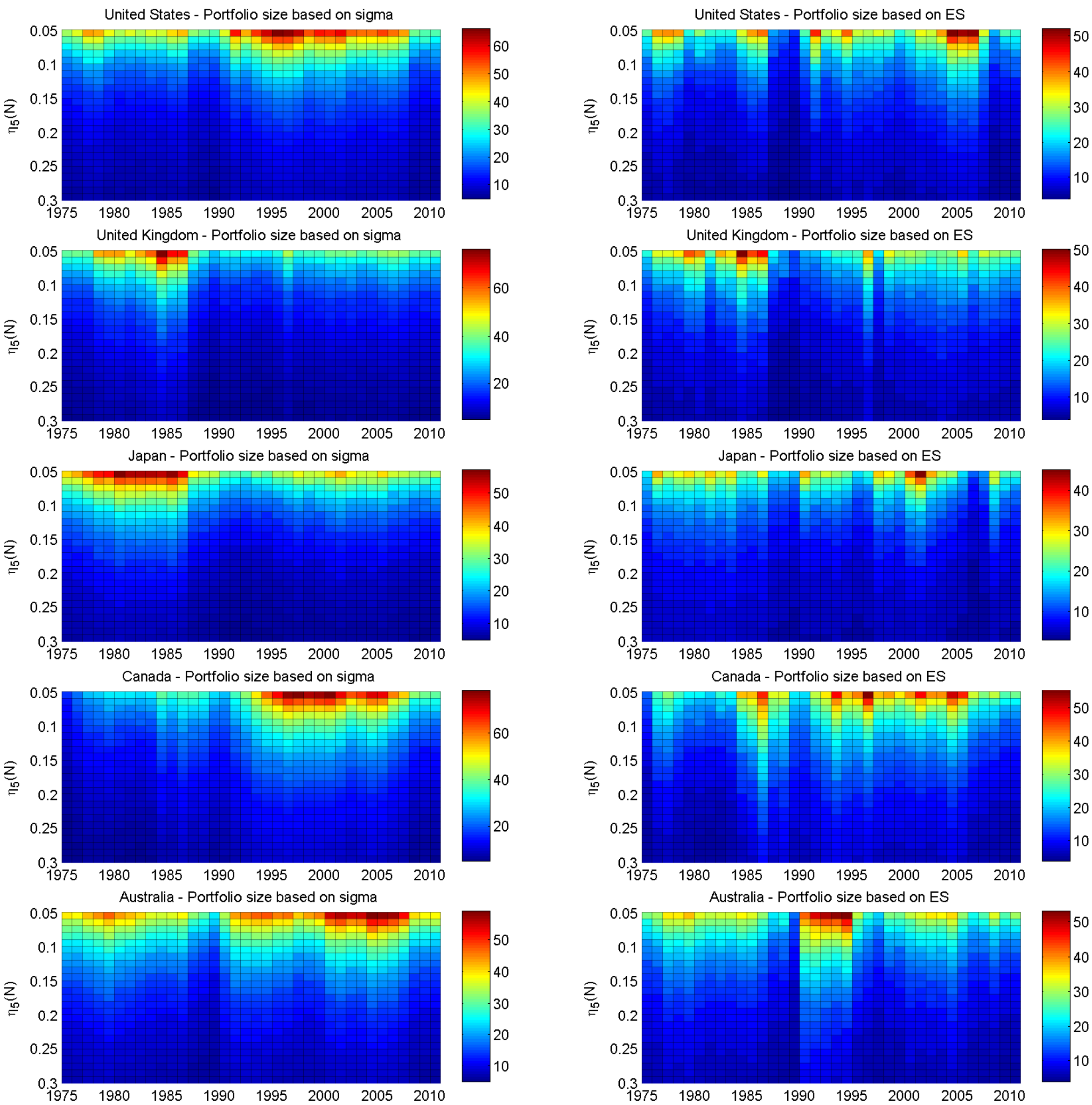


Figure 18: **Portfolio size (color bar) by the level of diversification (vertical axis) across time (horizontal axis)**. Based on our collected historical data we estimate the number of stocks an investor should have held on average (higher number of stocks are represented in red, small number of stocks are in blue) to achieve a desired level of diversification (x-axis) throughout the last 39 years. Left panels represent our recommendations based on variance as a measure of risk. Right panels outline recommendations based on 1% Expected Shortfall (ES) if taken as a measure of risk. For all the countries and across most years the number of stocks required to reduce the level of the risk measure concerned is lower if the investor is concerned with reduction of extreme losses. When $\eta_5(n)$ is equal to 0.05 or 0.1, for example, the level of diversification is equivalent to 95% or 90% reduction in diversifiable risk respectively.

Literature Summary

The summary of studies listed chronologically in the first column outlines the **period of the study** (column 2), the **data frequency** used (column 3) and the **market analyzed** (column 4). Column 5 lists **measures used** for (i) risk, (ii) performance, and (iii) diversification. Stock sample figures (in column 6) represent the **sample size** of (or the universe of investable) common stocks in the period considered, unless stated otherwise. Column 7 represents the **maximum size of a portfolio** (in terms of the number of stocks used to construct random portfolio). For most of the studies, the market portfolio proxy is constructed based on equal **weighting** of all securities available in the market, unless specifically stated otherwise in column 8. We also note down, in column 9, the **number of simulations**, M, used to derive the central tendency of a risk measure of each n-stock portfolio. Column 10 provides the **recommendation on the portfolio size** required to achieve a specified level/percentage of diversification in parentheses. Column 11 indicates the **sampling procedure** used to construct random portfolios.

Study	**Period**	**Freq.**	**Market**	**Measures**	**Sample**	**Max size**	**Market proxy**	M	**No. of stocks (diversif.%)**	**Sampling**
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Evans and Archer (1968)	1958-1967	semi-annual	US	SD	470	40		60	8-10	uniform
Fisher and Lorie (1970)	1926-1965	annual	US	wealth ratios: SD, MAD, mean difference, Gini's mean difference, coefficient of concentration		128			8-16 (85%), 16-32 (90%), 128 (99%)	stratified sampling (industry)
Jennings (1971)	1955-1965	annual	US	probability of loss, terminal wealth	all listed NYSE stocks (≈1715)		full sample, equally weighted		15	uniform
Wagner and Lau (1971)	1960-1970	monthly	US	SD, β, R^2	all listed NYSE stocks	200	full sample, equally weighted	10	20	uniform, classified (low risk vs. high risk)
Fielitz (1974)	1964-1968	quarterly	US	MAD, Sharpe, Treynor	200	20	full sample, equally weighted	40	8	uniform

Study	**Period**	**Freq.**	**Market**	**Measures**	**Sample**	**Max size**	**Market proxy**	M	**No. of stocks (diversif.%)**	**Sampling**
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Johnson and Shannon (1974)	1965-1972	quarterly	US	variance	50	17		20	3-7	quadratic optimiza-tion
Solnik (1974)	1966-1971	weekly	US, UK, DE, FR, CH, IT, BE, NL	SD	all listed on NYSE, AMEX +300 European stocks	300		60	10-15	uniform
Klemkosky and Martin (1975)	1963-1973	monthly	US	residual variance, residual variance of low vs. high beta portfolios	350	25	S&P500	350/(2..25)	3-17	uniform
Elton and Gruber (1977)	analytical	-	-	variance					20	uniform
Bloomfield et al (1977)	1953-1970	monthly	US	SD	823-893, 3 subperiods	50	?	20	?	uniform, quadratic optimiza-tion
Bird and Tippett (1986)	1958-1973	monthly	AUS	SD	188	25	full sample, equally weighted	40	10-15	uniform
Statman (1987)	1926-1984	?	US	SD	US		S&P500		30-40	uniform
Beck et al. (1996)	1982-1991	monthly	US	variances, correlations, variance ratios	all listed on NYSE, AMEX (1221)	70	full sample, equally weighted	50, 100, 200, 500, 1000, 2000	14-20	uniform
O'Neal (1997)	1976-1994	quarterly	US	TWSD and SD, mean shortfall, semi-variance	103 growth and 65 growth and income funds	30		1000	16-18 FoF	uniform

Study (1)	**Period** (2)	**Freq.** (3)	**Market** (4)	**Measures** (5)	**Sample** (6)	**Max size** (7)	**Market proxy** (8)	M (9)	**No. of stocks (diversif.%)** (10)	**Sampling** (11)
Copp and Cleary (1999)	1985-1997	monthly	CAN	variance	222 and 236	200	full sample, equally weighted	5000	10 (68%), 20 (78%), 30 (84%), 50 (90%), 90 (99.6%)	uniform
Byrne and Lee (2000)			UK						real estate	
Tang (2004)	analytical	-	-	variance					10 (90%), 20 (95%), 90 (99%)	uniform
Brands and Gallagher (2005)	1989-1999	monthly	AUS	SD, TWSD, Sharpe, Skew, Kurtosis	134 open-end equity funds	30	full sample, equally weighted	10000	6	uniform
Domian et al. (2007)	1985-2004	daily (indi-rectly)	US	TW, safety first criterion	1000 largest common stock series from NYSE, NASDAQ, AMEX with preference for large cap	200	full sample, equally weighted		164	uniform, industry
Goetzmann and Kumar (2008)	1991-1996	monthly		NV, sum of squared stocks weights in a portfolio, num. of stocks			S&P500 and full sample equally weighted			
Dbouk and Kryzanowski (2009)	1985-1997	monthly	US	correlation of bond returns, excess standard and mean derived deviation, Sortino, skew, kurtosis	bonds from Lehman Brothers Fixed Income Database (total 27,497 bonds)				bond portfolios of 25-40 bonds	
Benjelloun (2010)	1980-2000	monthly	US	TWSD, SD	all listed on CRSP	100	estimated asymptote from regression	10000	40-50	uniform and market weights

Study	**Period**	**Freq.**	**Market**	**Measures**	**Sample**	**Max size**	**Market proxy**	M	**No. of stocks (diversif.%)**	**Sampling**
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Kryzanowski and Singh (2010)	1975-2003	monthly	CAN	correlation, cross-sectional and time-series (semi) variation, MDD, MRD, NV	listed on TSX, cross-listed on TSX/US, small vs. Big, IT firms	100	several samples, all equally weighted	5000	20-25 (90%) corr., 40-45 (90%) MDD, 95 (90%) for small firms	uniform
This study	1973-2011	daily	US, UK, JAP, CAN, AUS	(i) SD, TWSD, MAD, ES, LPM (ii) Sharpe, Sortino, Omega, skew, kurtosis (iii)R^2	all listed and delisted stocks on NYSE, NASDAQ, LSE, TSE, TSX, ASX, 37 subperiods	1000	full sample, equally weighted	10000		uniform

Abbreviations: SD - time series standard deviation of portfolio returns, TWSD - portfolio terminal wealth standard deviation, MAD - mean absolute deviation, MDD - mean derived dispersion, MRD - mean realized dispersion, NV - normalized variance

Countries: US - United States, UK - United Kingdom, FR - France, DE - Germany, CH - Switzerland, BE - Belgium, NL - Netherlands, IT - Italy, AUS - Australia, CAN - Canada

Exchanges: NYSE - New York Stock Exchange, NASDAQ, LSE - London Stock Exchange, TSE - Tokyo Stock Exchange, TSX - Toronto Stock Exchange, ASX - Australian Stock Exchange.