
Mean-Variance Versus Mean-Conditional Value-at- Risk Optimization: The Impact of Incorporating Fat Tails and Skewness into the Asset Allocation Decision

James X. Xiong, Ph.D., CFA®
Thomas Idzorek, CFA®

Feb 16, 2010

ibbotson.

a Morningstar company

Abstract

We investigate the impact of mean-conditional value-at-risk (M-CVaR) optimizations that take into account fat tails and skewness on optimal asset allocation. In a series of controlled optimizations, we compare optimal asset allocation weights obtained from the traditional mean-variance optimizations with those from M-CVaR. To arrive at efficient asset allocation weights that are different from the weights obtained from the traditional mean-variance optimization framework, the M-CVaR estimate must go beyond the first two moments, the mean and variance of the forecasted return distribution, and account for higher moments. We find that both skewness and kurtosis (fat tails) impact the M-CVaR optimization and lead to substantially different allocations than the traditional mean-variance optimizations. In particular, the combination of negative skewness coupled with a fat tail has the greatest impact on the optimal asset allocation weights. The study provides useful insights for designing optimal asset allocation mixes when investor preferences go beyond mean and variance.

James X. Xiong, Ph.D., CFA,
Senior Research Consultant
Ibbotson Associates, a Morningstar company.
22 West Washington Street,
Chicago, IL 60602
(312) 384-3764
jxiong@ibbotson.com

Thomas M. Idzorek is chief investment officer and director of research at Ibbotson Associates, a Morningstar company.

Introduction

Asset class return distributions are not normally distributed; yet, the typical Markowitz mean-variance optimization framework that has dominated the asset allocation process for over 50 years only relies on the first two moments of the return distribution. Equally important, there is considerable evidence that investor preferences go beyond mean and variance. Investors are particularly concerned with significant losses, i.e. downside risk.

Numerous alternatives to the mean-variance optimization (MVO) framework have emerged in the literature, but there is no clear leader amongst these alternative objective functions. The lack of an agreed upon alternative to MVO has slowed the development of practitioner-oriented tools. Additionally, it is well known that it is difficult to estimate the required inputs – returns, standard deviations, and correlations – for MVO; a problem that can be substantially more difficult for more advanced techniques. The future is hard to predict accurately, especially in greater detail.

This article explores one of the promising alternatives to MVO called mean-conditional value-at-risk optimization (M-CVaR). In a series of optimizations, we compare optimal asset allocation weights obtained from traditional MVO with those from M-CVaR.

Traditional MVO leads to an efficient frontier that maximizes return per unit of variance, or equivalently, minimizes variance for a given level of return. In contrast, M-CVaR maximizes return for a given level of CVaR, or equivalently, minimizes CVaR for a given level of return. Conditional value-at-risk (CVaR) measures the expected loss in the left tail *given* (i.e. conditional on) that a particular threshold has been met, such as the worst 1st or 5th percentile of outcomes in the distribution of possible future outcomes.

The M-CVaR process used in this article takes non-normal return characteristic into consideration, and in general, prefers assets with positive skewness, small kurtosis, and low variance. If the returns of the asset classes are normally distributed or if the method used to estimate the CVaR only considers the first two moments, both MVO and M-CVaR optimization lead to the same efficient frontier, and hence, the same asset allocations. In order to understand the implications of skewness and kurtosis on portfolio selection, it is critical to estimate CVaR in a manner that captures the important non-normal characteristics of the assets in the opportunity set and how those non-normal characteristics interact when combined into portfolios.

Empirically, almost all asset classes and portfolios have returns that are not normally distributed. Many assets return distribution are not symmetrical. In other words, the distribution is skewed to the left or right of the mean (expected) value. Additionally, many asset return distributions are more leptokurtic than normal or Gaussian distributions – a condition characterized by a higher peak in the center, “thinner waist” or fewer values between the center and the tails, and most importantly, “fatter tails.” The normal distribution assigns what most people would characterize as meaninglessly small probabilities to extreme events that empirically seem to occur approximately 10 times more often than the normal distribution predicts.

Some risk measurements, such as CVaR, are focused on the left tail. The normal and lognormal distribution models are considered “thin-tailed” distributions. Using thin-tailed distributions to estimate the downside risk of a portfolio can dramatically underestimate both the magnitude of the downside and the frequency with which these bad events occur. It can be shown that “fat tailed” distributions often do a better job of fitting realized returns. A better alternative model is the log-stable distribution (Mandelbrot, 1963; Fama, 1965). Martin, Rachev, and Siboulet (2003) and Kaplan (2009) show that the log-stable distribution does a good job of fitting the tails of the historical data.

Unfortunately, the tails of a log-stable distribution are so fat that the variance is infinite, limiting the usefulness of the log-stable distribution in practice as it makes the risk estimation and scenario analysis complicated or impossible. A simple solution that addresses this shortcoming is to truncate the tails of the stable distribution, which results in what is known as the Truncated Lévy Flight (TLF) distribution (see Mantegna and Stanley (1994, 1999)).

The TLF distribution is particularly well suited to financial modeling because it has finite variance, fat tails that empirically fit the data, and perhaps most importantly for financial modelers, it “scales” appropriately overtime. More specifically, it obeys Lévy stable scaling relations at small time intervals and slowly converges to a Gaussian distribution at very large time intervals, which is consistent with empirical observations. Xiong (2010) shows that the TLF model provides an excellent fit for a variety of asset classes, such as U.S. large-cap equities, U.S. long-term government bonds, and MSCI U.K. equities, in all aspects: the center, the tails as well as the minimum and maximum. Thus, in our controlled optimization in which we systematically vary the skewness and kurtosis of the asset classes we use a multivariate TLF model as the basis for generating asset returns and ultimately estimating a portfolio’s CVaR.

Conditional Value-at-Risk

Backing up temporarily, the precursor to CVaR was the standard value-at-risk (VaR) measure. VaR estimates the loss that is expected to be *exceeded* with a given level of probability over a specified time period. CVaR is closely related to VaR and is calculated by taking a probability weighted average of the possible losses *conditional* on the loss being equal to or exceeding the specified VaR. Other terms for CVaR include mean shortfall, tail VaR, and expected tail loss. CVaR is a comprehensive measure of the entire part of the tail that is being observed, and for many, the preferred measurement of downside risk. In contrast with CVaR, VaR is only a statement about one particular point on the distribution. Intuitively, CVaR is a more complete measure of risk relative to VaR and previous studies have shown that CVaR has more attractive properties (see for example, Rockafellar and Uryasev (2000) and Pflug (2000)).

Artzner, Delbaen, Eber, and Heath (1999) demonstrates that one of the desirable characteristics of a “coherent measure of risk” is subadditivity, or the assertion that the risk of a combination of investments is at most as large as the sum of the individual risks. VaR is not always subadditive, which means that the VaR of a portfolio with two instruments may be greater than the sum of individual VaRs of these two instruments. In contrast, CVaR is subadditive.

In the most basic case, if one assumes that returns are normally distributed, both VaR and CVaR can be estimated using just the first two moments of the return distribution (e.g. see Rockafellar and Uryasev (2000)).

$$VaR_p = -\mu_p + 1.65 \cdot \sigma_p \quad (1)$$

$$CVaR_p = -\mu_p + 2.06 \cdot \sigma_p \quad (2)$$

Where μ_p, σ_p are the mean and standard deviation of the portfolio, respectively. The probability level for the VaR and CVaR in this paper is fixed at 5% (corresponding to a confidence level of 95%). For example, for a portfolio with $\mu_p = 10\%$, $\sigma_p = 20\%$, The VaR and CVaR of the portfolio are $VaR_p = 23.0\%$, $CVaR_p = 31.2\%$ of the portfolio's starting value, respectively.

However, introducing skewness (or asymmetry) and kurtosis to a portfolio's return distribution complicates the calculation of CVaR and moves us to Equation (3):

$$CVaR_p = -\mu_p + f(c, s, k) \cdot \sigma_p \quad (3)$$

Where $f(c, s, k)$ is a function of confidence level c , skewness s , and kurtosis k .¹ Unfortunately, the function $f(c, s, k)$ is complicated and in general has no closed form solution. However, with Monte Carlo simulations based on the TLF distribution, we can estimate equation (3) using an advanced numerical procedure that is shown later.

¹ Hallerbach (2003) provides a formula for VaR for non-normal distributions, and it is straightforward to extend it to CVaR.

MVO vs. M-CVaR

Armed with a measure of CVaR that accounts for skewness and kurtosis, we study the impact of skewness and kurtosis on asset allocation in a series of seven comparisons. We start with a simple hypothetical example and eventually move to a more applicable 14-asset-class example.

For both MVO and M-CVaR, the only constraint is no shorting. In our first scenario in which we assume normal returns, a traditional quadratic optimization routine is used to determine the optimal portfolios. In Scenario 2 through Scenario 7, which involves non-normal return distributions, we use simulation-based optimizations for both MVO and M-CVaR.² We simulate asset returns using multivariate TLF distribution models. The multivariate TLF distribution models result in return distributions that incorporate variance, skewness and kurtosis into the CVaR estimate. Appendix A provides additional simulation details.

² The M-CVaR optimization algorithm we used was developed by Rockafellar and Uryasev (2000), and it can be easily implemented with simulated stochastic returns.

Hypothetical Asset Classes

In order to isolate the impact of skewness and kurtosis on M-CVaR optimization, we assume four simple hypothetical assets – Asset A, B, C, and D. The expected returns, standard deviations, and correlation matrix are shown in Table 1 (Panels A and B). Panel C of Table 1 identifies the skewness and kurtosis for the four assets used in the first six scenarios. The normal distribution has skewness of 0 and kurtosis of 3. Kurtosis greater than 3 indicates a fatter tail than the normal distribution. The multivariate TLF model can take the inputs shown in Table 1 and generate the corresponding multivariate returns that are used for the first six scenario analyses.

Assets A, B, and C have the same ratio of return to risk (standard deviation) of 0.5. Asset D has a slightly lower return to risk ratio of 0.43. The correlation between Asset A and the other assets is “low,” while the correlations among Assets B, C, and D are “high.” One can think of Asset A as a bond index, and Asset B, C, and D as equity indices, in which Asset D has the least attractive traditional return to risk ratio.

Moving forward, we analyze the asset allocations as we vary the skewness and kurtosis of the four assets. As one can see in Panel C of Table 1, by varying the skewness and kurtosis of Asset C relative to the other assets, Asset C serves as our primary guinea pig. Since MVO ignores higher moments, the optimal allocations are nearly the same for the six scenarios based on MVO.³ In contrast, we expect the M-CVaR optimizations to lead to different allocations.

Table 1: Capital Market Assumptions with Higher Moments

Panel A. Expected Returns and Standard Deviations

	Exp. Ret.	Std. Dev.
Asset A	5%	10%
Asset B	10%	20%
Asset C	15%	30%
Asset D	13%	30%

Panel B. Correlation Matrix

Correlation	Asset A	Asset B	Asset C	Asset D
Asset A	1.00	0.34	0.32	0.32
Asset B	0.34	1.00	0.82	0.82
Asset C	0.32	0.82	1.00	0.71
Asset D	0.32	0.82	0.71	1.00

³ The slight variations in MVO results in scenarios 2-6 are the result of the simulation procedure. As the number of trials increases, the MVO results approach those from a non-simulation based quadratic programming technique.

Panel C. Skewness and Kurtosis

		Asset A	Asset B	Asset C	Asset D
Scenario 1	Skewness	0	0	0	0
	Kurtosis	3	3	3	3
Scenario 2	Skewness	0	0	0	0
	Kurtosis	6	6	6	6
Scenario 3	Skewness	0	0	0	0
	Kurtosis	3.5	3.5	6	3.5
Scenario 4	Skewness	0	0	-0.5	-0.3
	Kurtosis	3.5	3.5	3.5	3.5
Scenario 5	Skewness	0	0	-0.5	-0.3
	Kurtosis	6	6	6	6
Scenario 6	Skewness	0	0	-0.5	-0.3
	Kurtosis	3.5	3.5	6	3.5

Scenario 1: Normal Distributions (Zero Skewness and Normal Tails)

In our simplest scenario, the returns of all four hypothetical assets are assumed to be multivariate normally distributed. Table 2 shows eight optimal allocations ranging from low return (low risk) to high return (high risk). Throughout this article, we use expected return as the basis of MVO vs. M-CVaR optimizations comparisons given our two different definitions of risk. Notice that the optimal allocations for the MVO and the M-CVaR optimization are the same. Returning to Equation 2, this intuitive result flows directly from the fact that minimizing CVaR (when estimated using Equation 2) is equivalent to minimizing either σ_p or σ_p^2 for a given μ_p .

Table 2: Scenario 1 – Zero Skewness and Normal Tails

	Exp. Ret. = 7%		Exp. Ret. = 9%		Exp. Ret. = 11%		Exp. Ret. = 13%	
	MVO	M-CVaR	MVO	M-CVaR	MVO	M-CVaR	MVO	M-CVaR
Asset A	70.9%	70.9%	54.5%	54.5%	37.3%	37.3%	16.4%	16.4%
Asset B	18.3%	18.3%	8.4%	8.4%	0.0%	0.0%	0.0%	0.0%
Asset C	10.9%	10.9%	30.6%	30.6%	49.3%	49.3%	65.7%	65.7%
Asset D	0.0%	0.0%	6.5%	6.5%	13.4%	13.4%	17.9%	17.9%
Total	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Std. Dev.	11.2%	11.2%	14.9%	14.9%	19.4%	19.4%	24.4%	24.4%
Skewness	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Kurtosis	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0
VaR	-11.5%	-11.5%	-15.6%	-15.6%	-21.1%	-21.1%	-27.3%	-27.3%
CVaR	-16.1%	-16.1%	-21.7%	-21.7%	-29.0%	-29.0%	-37.3%	-37.3%

Of note, Asset D has lower allocations than Asset C because Asset D has lower expected return given the same volatility. Among optimal allocations with the same expected return, the CVaR is always more extreme than VaR by definition.

Scenario 2: Zero Skewness and Uniform Fat Tails

Starting in Scenario 2, we now assume that the asset class returns follow a multivariate TLF distribution to account for both skewness and fat tails. For each of the ensuing five scenarios, the higher moments of the asset classes are taken from Panel C of Table 1. In this scenario, the skewness for all four assets is approximately 0, and the kurtosis for each is approximately 6.⁴ In other words, all four assets have symmetrical return distributions, although they all have fat tails. Scenario 2 tests if uniform fat tails make a difference.

Table 3 shows that the MVO and M-CVaR optimizations lead to very similar allocations, which in turn are very similar to the Scenario 1 asset allocations presented in Table 2. In fact, the minor allocation differences in this case are due to sampling errors associated with the simulated returns. The equivalency of the asset allocations provides empirical evidence that “fat tails” alone do not invalidate MVO.⁵ The intuition behind the result shown in Table 3 can be understood by looking into the relationship between kurtosis and CVaR. In general, CVaR is approximately proportional to the kurtosis, i.e., the higher the kurtosis, the fatter the tail, so the higher the CVaR. When the kurtosis for all the assets are uniformly higher, their individual contribution to the overall portfolio’s CVaR will be higher in absolute value, but not on a relative-basis. In other words, the relative or percentage contribution to the portfolio’s CVaR for each asset remains roughly the same.

Relative to Scenario 1 (Table 2), all of the corresponding CVaRs are higher in Scenario 2 (Table 3).

Table 3 : Scenario 2 – Zero Skewness and Uniform Fat Tails

	Exp. Ret. = 7%		Exp. Ret. = 9%		Exp. Ret. = 11%		Exp. Ret. = 13%	
	MVO	M-CVaR	MVO	M-CVaR	MVO	M-CVaR	MVO	M-CVaR
Asset A	72.0%	70.3%	55.5%	53.9%	37.3%	35.8%	17.7%	16.3%
Asset B	16.8%	20.2%	8.0%	10.9%	2.6%	4.9%	0.2%	2.0%
Asset C	11.1%	9.1%	30.9%	28.7%	49.0%	46.2%	66.0%	63.0%
Asset D	0.0%	0.5%	5.5%	6.6%	11.1%	13.1%	16.0%	18.7%
Total	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Std. Dev.	11.2%	11.3%	14.9%	14.9%	19.4%	19.4%	24.4%	24.4%
Skewness	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Kurtosis	4.2	4.2	4.7	4.6	5.3	5.2	5.5	5.4
VaR	-11.5%	-11.6%	-15.0%	-15.1%	-19.7%	-19.8%	-25.4%	-25.5%
CVaR	-18.3%	-18.3%	-24.7%	-24.7%	-33.8%	-33.8%	-43.7%	-43.7%

⁴ While the parameters of the simulation are precise, we use the word “approximately” to emphasize that in any given simulation the “moment” may vary from the “target.”

⁵ Another case in which the MVO and M-CVaR optimization are equivalent is the multivariate elliptical distribution with fat tails (e.g. see Bradley and Taqqu (2003)).

Scenario 3: Zero Skewness and Mixed Tails

In Scenario 3, we continue to assume the skewness for all the four assets is approximately 0; however, we now vary the amount of kurtosis associated with each asset. In Scenario 3 Assets A, B, and D have relatively thin tails with a kurtosis of 3.5, and Asset C has a fat tail with a kurtosis of 6. Scenario 3 tests in the absence of skewness, if different levels of kurtosis impact asset allocations.

Table 4 shows the results. As one might expect, relative to the MVO allocations, at an equivalent expected return, the M-CVaR optimization have significantly lower allocations to Asset C – the only fat-tailed asset in this scenario. The M-CVaR optimization lowers the portfolio's CVaR by decreasing allocations to Asset C.

Scenario 3 also highlights one of the weakness of VaR. For an expected return of 13%, the CVaR is 0.9 percentage points less extreme for the M-CVaR optimization than it was for the MVO. In contrast, for the same two portfolios, the VaR is 1.6 percentage points more extreme for the M-CVaR than it was for the MVO. This is equivalent to saying that by trying to lower the VaR, the CVaR is accidentally increased. This unwanted result is rooted in the fact that VaR is not a coherent measure of risk (see Rockafellar and Uryasev (2000)).

Table 4: Scenario 3 – Zero Skewness and Mixed Tails

	Exp. Ret. = 7%		Exp. Ret. = 9%		Exp. Ret. = 11%		Exp. Ret. = 13%	
	MVO	M-CVaR	MVO	M-CVaR	MVO	M-CVaR	MVO	M-CVaR
Asset A	71.0%	68.2%	52.8%	47.3%	35.8%	25.7%	16.9%	4.5%
Asset B	18.8%	24.6%	13.9%	23.2%	5.1%	23.4%	0.6%	22.4%
Asset C	10.1%	7.1%	28.8%	20.6%	46.6%	33.2%	63.5%	45.9%
Asset D	0.0%	0.1%	4.5%	8.9%	12.5%	17.8%	18.9%	27.2%
Total	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Std. Dev.	11.2%	11.3%	14.8%	14.9%	19.3%	19.6%	24.2%	24.6%
Skewness	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Kurtosis	3.3	3.3	4.0	3.6	4.6	3.9	4.9	4.0
VaR	-11.4%	-11.4%	-14.9%	-15.3%	-19.9%	-20.8%	-25.4%	-27.0%
CVaR	-16.6%	-16.6%	-23.4%	-23.0%	-32.4%	-31.8%	-42.2%	-41.3%

Scenario 4: Non-Zero Skewness and Uniformly Thin Tails

In Scenario 4, Assets A and B have 0 skewness and Assets C and D have (bad) skewness of -.5 and -.3, respectively. All four assets are assumed to have relatively thin tails which are only slightly thicker than the thin-tailed normal distribution, all with a kurtosis of 3.5. Relative to normal skewness and kurtosis, Asset C is disadvantaged the most with its negative skewness of -.5. Scenario 4 tests the impact of different levels of skewness when tails are relatively thin.

The optimization results are displayed in Table 5. The allocations to Asset C in the M-CVaR optimization range from 4.6 to 10.6 percentage points lower than the corresponding MVO. This indicates that when kurtosis is controlled, the M-CVaR optimization tends to avoid the negatively skewed asset(s) in order to minimize the CVaR. The more negative the skewness, the higher the tail loss or CVaR. Based on Scenario 4, even when kurtosis is uniformly "low" across the asset classes, skewness can create significant differences between MVO and M-CVaR optimal.

Table 5: Scenario 4 – Non-Zero Skewness and Uniformly Thin Tails

	Exp. Ret. = 7%		Exp. Ret. = 9%		Exp. Ret. = 11%		Exp. Ret. = 13%	
	MVO	M-CVAR	MVO	M-CVAR	MVO	M-CVAR	MVO	M-CVAR
Asset A	70.2%	65.7%	53.5%	47.7%	37.3%	31.1%	17.3%	13.1%
Asset B	20.3%	29.2%	11.9%	22.2%	1.5%	11.4%	0.0%	4.2%
Asset C	9.5%	4.9%	29.3%	21.3%	48.2%	38.1%	64.8%	54.2%
Asset D	0.0%	0.2%	5.4%	8.9%	13.0%	19.5%	17.9%	28.5%
Total	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Std. Dev.	11.4%	11.4%	15.0%	15.1%	19.6%	19.8%	24.6%	24.8%
Skewness	-0.1	-0.1	-0.3	-0.2	-0.4	-0.3	-0.4	-0.4
Kurtosis	3.2	3.2	3.3	3.3	3.5	3.4	3.5	3.4
VaR	-12.1%	-12.0%	-16.8%	-16.9%	-23.2%	-23.3%	-30.0%	-30.2%
CVaR	-17.5%	-17.4%	-25.1%	-24.8%	-34.5%	-34.1%	-44.5%	-44.1%

Scenario 5: Non-Zero Skewness and Uniformly Fat Tails

Scenario 5 is similar to Scenario 4; skewness is the same (Assets A and B with 0 and Assets C and D with -.5 and -.3, respectively), but rather than having uniformly low levels of kurtosis of 3.5, all assets have uniformly high levels of kurtosis of 6. Scenario 5 tests the impact of different levels of skewness when tails are relatively fat.

Overall, the across-the-board increase in kurtosis did not change the Scenario 5 allocations significantly relative to the Scenario 4 allocations. The allocations to Asset C in the M-CVaR optimization range from 3.9 to 9.4 percentage points lower than that in the MVO, similar to the differences reported in Scenario 4. When the kurtosis for all four assets is uniformly higher, their individual contribution to the overall portfolio's CVaR is similar. When all assets in the universe have uniformly fat tails the net impact is from skewness.

Table 6: Scenario 5 – Non-Zero Skewness and Uniformly Fat Tails

	Exp. Ret. = 7%		Exp. Ret. = 9%		Exp. Ret. = 11%		Exp. Ret. = 13%	
	MVO	M-CVAR	MVO	M-CVAR	MVO	M-CVAR	MVO	M-CVAR
Asset A	72.1%	68.8%	56.0%	52.5%	37.7%	33.6%	17.2%	13.1%
Asset B	16.2%	22.6%	6.1%	11.9%	0.5%	6.9%	0.0%	5.1%
Asset C	11.4%	7.5%	31.0%	25.5%	48.7%	41.3%	65.2%	55.8%
Asset D	0.3%	1.2%	6.9%	10.1%	13.1%	18.2%	17.7%	26.0%
Total	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Std. Dev.	11.3%	11.3%	14.9%	15.0%	19.5%	19.5%	24.4%	24.6%
Skewness	-0.1	-0.1	-0.3	-0.3	-0.4	-0.3	-0.4	-0.3
Kurtosis	4.2	4.1	4.7	4.5	5.2	4.9	5.4	5.1
VaR	-12.1%	-12.3%	-16.4%	-16.6%	-22.2%	-22.5%	-28.8%	-29.3%
CVaR	-19.1%	-19.0%	-26.9%	-26.7%	-37.2%	-37.0%	-48.2%	-47.9%

Scenario 6: Non-Zero Skewness and Mixed Tails

Just as we did in Scenarios 4 and 5, in Scenario 6 the skewness is the same (Assets A and B with 0 and Assets C and D with -.5 and -.3, respectively); however, we now vary the kurtosis of the assets. Assets A, B, and D all have kurtosis of 3.5 while Asset C has the disadvantaging kurtosis of 6. Scenario 6 tests the impact of different levels of skewness and kurtosis on asset allocations.

As expected, Table 7 reports that the allocations to Asset C in the M-CVaR optimization range from 5.6 to 20.6 percentage points lower than that in the MVO, an additional 2.6 to 3 percentage point reduction compared to Scenario 3 in which all of the assets had zero skewness. Asset C has the lowest allocations among the six scenarios that we covered so far. For an expected return of 13%, the portfolio's standard deviation is 0.5 percentage points higher for the M-CVaR optimization, but its CVaR or expected tail loss is reduced by 1.3 percentage points.

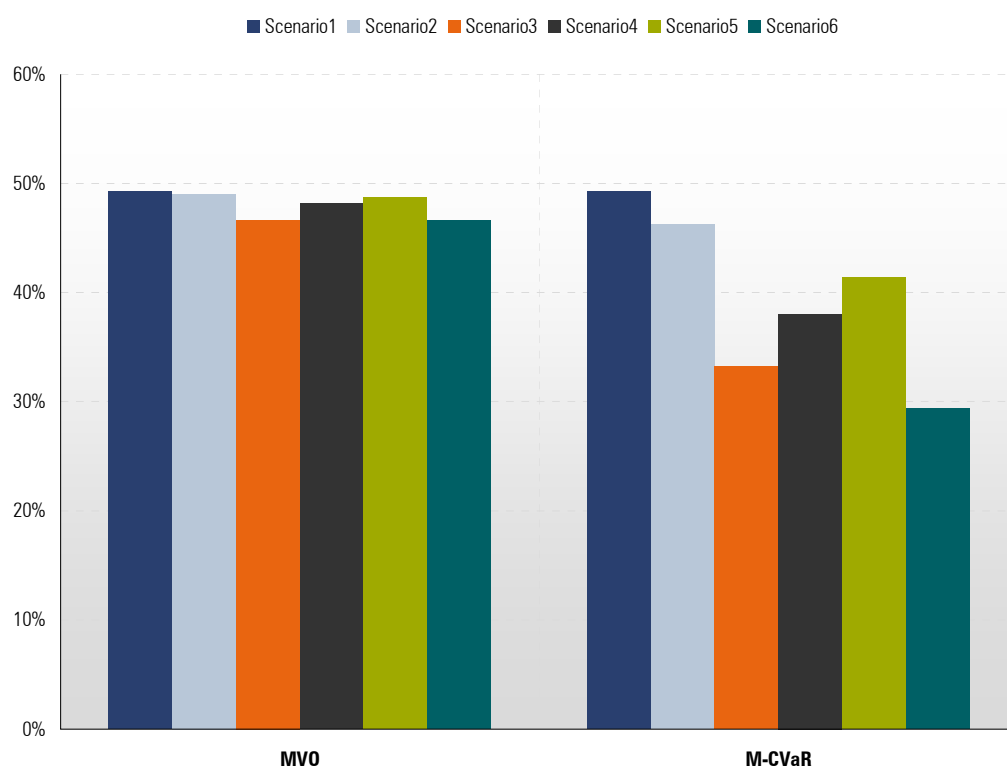
Table 7: Scenario 6 –Non-Zero Skewness and Mixed Tails

	Exp. Ret. = 7%		Exp. Ret. = 9%		Exp. Ret. = 11%		Exp. Ret. = 13%	
	MVO	M-CVAR	MVO	M-CVAR	MVO	M-CVAR	MVO	M-CVAR
Asset A	71.3%	66.0%	53.4%	44.2%	35.8%	22.9%	16.7%	2.9%
Asset B	18.1%	28.6%	11.9%	28.9%	4.4%	27.8%	0.1%	23.8%
Asset C	10.6%	5.0%	29.1%	17.3%	46.6%	29.4%	63.1%	42.4%
Asset D	0.0%	0.4%	5.6%	9.6%	13.3%	20.0%	20.1%	30.9%
Total	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Std. Dev.	11.3%	11.3%	14.8%	15.1%	19.3%	19.8%	24.3%	24.8%
Skewness	-0.1	-0.1	-0.3	-0.2	-0.4	-0.3	-0.4	-0.3
Kurtosis	3.3	3.2	4.0	3.5	4.6	3.7	4.8	3.9
VaR	-11.9%	-12.0%	-16.2%	-16.7%	-22.2%	-23.1%	-28.7%	-29.9%
CVaR	-17.6%	-17.4%	-25.7%	-25.1%	-36.0%	-34.9%	-46.8%	-45.5%

Summarizing Scenario 1 – 6

In Figure 1, we summarize the impact of skewness and kurtosis as it pertains to the asset allocation differences that result from MVO and M-CVaR optimizations as measured by the allocation to our guinea pig asset, Asset C. Across all six scenarios, MVO led to similar asset allocations at each of the corresponding expected return points. Again, the observed differences are due to sampling errors, such as a slight difference of return vector, standard deviation vector, and correlation matrix for scenarios 2-6. In contrast, M-CVaR optimization incorporated skewness and kurtosis into the asset allocation decision, which led to different optimal mixes—the allocations to Asset C varied by 20 percentage points when changes were made to skewness and kurtosis. The difference between Scenario 1 and Scenario 2 is due to sampling errors. Scenario 3 indicates kurtosis with mixed tails has significant impact even though the asset return distributions are symmetrical. Scenario 4 and 5 indicate that skewness has significant impact when the kurtosis is controlled. Scenario 6 indicates that the combination of skewness and kurtosis with mixed tails has the largest impact.

Figure 1. Allocations to Asset C in the Efficient Frontier with Portfolio Return of 11%.



These six scenarios provide useful insights. In an asset universe with mixed tails, information about skewness and kurtosis can significantly impact the optimal allocations in the M-CVaR optimization. In these cases, the CVaR or expected tail loss can be reduced by performing the M-CVaR optimization but not MVO.

Scenario 7: A 14 Asset Classes Example

In our last example, Scenario 7, we move away from our four asset class examples, and apply MVO and M-CVaR to a more typical 14 asset class opportunity set. The descriptive statistics for the 14 asset classes are shown in Table 8.

In contrast to our first six scenarios in which we used the TLF distribution to estimate CVaR, in our final example we switch to a non-parametric approach based on the historical data. This enables other researchers to duplicate this portion of our analysis.⁶

Rather than simply using historical returns, we use the CAPM-based reverse optimization procedure to infer each asset class' expected future return. We then "shift" the location of the historical return distribution by either adding or subtracting a constant (the difference between the reverse optimized return and the historical average return) from each historical return. By adding or subtracting an asset class specific constant to each historical return series, standard deviation, skewness, kurtosis, and correlation matrix of the adjusted returns remain the same as those of the historical returns (from Feb 1990 to Jun 2009).

Table 8. The Descriptive Statistics for the 14 Asset Classes (from Feb 1990 to Jun 2009)

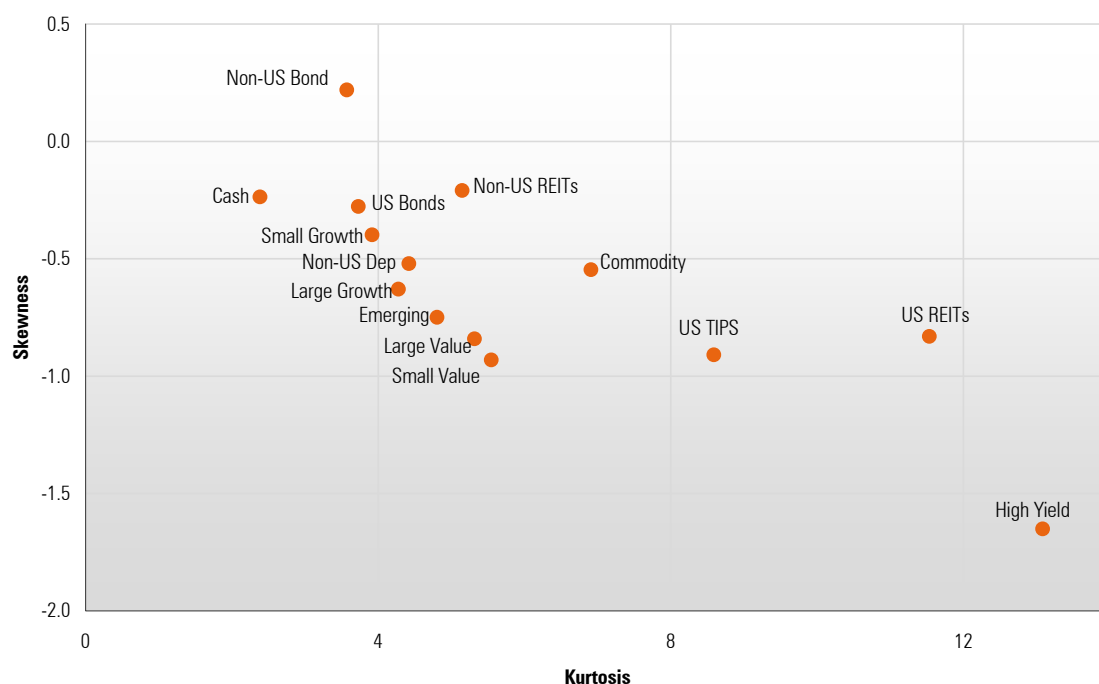
	Hist. Mean	Equilibrium Mean	Std. Dev.	Skewness	Kurtosis	Estimated Capitalization Weights
Large Value	9.47%	8.52%	14.63%	-0.84	5.32	8.59%
Large Growth	8.70%	9.20%	17.51%	-0.63	4.28	8.70%
Small Value	11.31%	8.61%	16.82%	-0.93	5.55	0.71%
Small Growth	8.51%	10.35%	23.43%	-0.40	3.92	0.67%
Non-U.S. Dev. Equity	5.58%	10.12%	17.43%	-0.52	4.42	17.66%
Emerging Market Equity	12.58%	11.43%	24.48%	-0.75	4.80	4.93%
Commodity Composite	7.50%	6.68%	15.63%	-0.55	6.91	10.82%
U.S. Nominal Bonds	7.16%	4.45%	3.86%	-0.28	3.73	21.99%
Non-U.S. Gov. Bonds	8.03%	5.39%	8.75%	0.22	3.57	15.44%
Global High Yield	10.20%	6.88%	10.62%	-1.65	13.08	1.76%
Cash	4.03%	4.01%	0.53%	-0.24	2.38	2.06%
Non-U.S. Real Estate	7.83%	10.64%	20.81%	-0.21	5.15	3.24%
U.S. Real Estates	13.15%	8.61%	20.13%	-0.83	11.54	2.68%
U.S. Inf. Linked Bonds	8.16%	4.77%	5.57%	-0.91	8.59	0.75%

Over the last 20 years from 1990 to 2009, large value, small value, global high yield, U.S. real estate, and U.S. inflation linked bonds have higher negative skewness, while only non-U.S. government bonds have positive skewness. The kurtosis for global high yield, U.S. real estate, and U.S. inflation linked bonds are higher than other asset classes. Based on the insights provided in the first six scenarios, a portfolio's downside risk could have been lowered over the last 20 years if the allocations to global high yield, U.S. real estate, and U.S. inflation linked bonds (TIPS) were lowered.

⁶ In practice, we believe it is better practice to use expected returns coupled with expected standard deviations, correlations, skewness, and kurtosis characteristics to generate the multivariate TLF returns, and then use simulation-based optimization to derive the MVO and M-CVaR efficient frontiers.

Figure 2 shows the skewness and kurtosis for the 14 asset classes. Note that TIPS, U.S. real estate, and global high yield appear in the bottom right of the Figure 2. All of these three asset classes seem to produce “relatively” stable returns during normal times, but they can suffer severe negative return during extraordinary events. For REITS, it is perhaps due to larger than normal amounts of leverage coupled with unusually smooth returns. For TIPS and global high yield, our analyses show that their correlation with U.S. equity market is asymmetrically much higher in severe downside market than a normal market or up market, which might explain their more negative skewness and higher kurtosis.⁷ If one cares about building a portfolio with less extreme downside risk, they should consider reducing allocations to assets in the bottom right of Figure 2.

Figure 2. Skewness and Kurtosis for the 14 Asset Classes



As before, we identified efficient asset allocations from the MVO and M-CVaR optimization approaches based on expected return. More specifically, in Table 11 we identify allocations with expected returns of 5.5%, 6.8%, 8.1%, and 9.4%, respectively.⁸

Two asset classes, non-U.S. government bonds and U.S. large value, have the largest allocation differences. Compared to the MVO, the M-CVaR optimization under-weights the U.S. large value, Emerging Market Equity, and U.S. REITs because of their relatively lower skewness and higher kurtosis, and over-weights the non-U.S. government bonds, non-U.S. Developed Equity, and non-U.S. REITs because of their relatively higher skewness and lower kurtosis.

⁷ We use measures presented in Longin and Solnik (2001) and Ang and Chen (2002) called “exceedence correlations” to determine the asymmetric correlation between equity market and TIPS or global high yield.

⁸ The MVO and M-CVaR optimizations are performed on monthly expected returns, and the reported expected returns are annualized.

Global high yield has the lowest skewness and highest kurtosis, and it receives zero allocations in M-CVaR and small allocations in MVO. To test the impact of skewness and kurtosis, we increase its expected return by 1% (from 6.88% to 7.88%) so that it becomes more attractive. We observed significant allocation differences to the global high yield asset class between MVO and M-CVaR optimization. The M-CVaR optimization has significantly lower allocations to the global high yield class, and the allocation difference between M-CVaR and MVO ranges from 2.5% to 30% across the efficient frontier.

Overall, these observations are consistent with our previous discussions that M-CVaR optimization tends to pick positively skewed and thin-tailed assets, while the MVO ignores the information from skewness and kurtosis. At the portfolio level, the skewness is higher (less negative), the kurtosis is lower, and CVaR is lower with the M-CVaR optimization. As shown in the bottom of the Table 9, for the portfolio with expected return of 9.4%, the expected volatility increased by only 0.3 percentage points, but the CVaR is lowered by 1.6 percentage points with the M-CVaR optimization.

Table 9: Scenario 7 – Optimal Allocations and Statistics

	Exp. Ret. = 5.5%		Exp. Ret. = 6.8%		Exp. Ret. = 8.1%		Exp. Ret. = 9.4%	
	MVO	M-CVAR	MVO	M-CVAR	MVO	M-CVAR	MVO	M-CVAR
Large Value	3.62%	0.00%	6.52%	0.00%	9.8%	0.0%	9.4%	0.0%
Large Growth	2.68%	3.02%	6.04%	7.47%	11.0%	10.1%	13.3%	12.0%
Small Value	0.47%	0.00%	1.13%	0.00%	0.1%	0.0%	0.0%	0.0%
Small Growth	0.49%	0.00%	0.76%	0.00%	0.3%	0.8%	0.5%	0.0%
Non-U.S. Dev. Equity	7.27%	11.41%	13.73%	17.99%	20.2%	27.0%	30.9%	44.2%
Emerging Market Equity	2.28%	0.00%	4.36%	0.00%	5.1%	0.0%	6.7%	0.0%
Commodity Composite	4.32%	2.93%	7.63%	5.80%	12.2%	9.2%	13.3%	13.9%
U.S. Nominal Bonds	9.68%	8.61%	14.61%	13.08%	7.0%	6.4%	0.0%	0.0%
Non-U.S. Gov. Bonds	6.50%	12.00%	11.56%	22.46%	17.0%	34.2%	9.5%	13.1%
Global High Yield	0.97%	0.00%	0.49%	0.00%	2.2%	0.0%	0.0%	0.0%
Cash	59.18%	58.15%	24.33%	24.43%	1.0%	0.0%	0.0%	0.0%
Non-U.S. Real Estate	1.61%	3.88%	2.47%	8.78%	4.2%	12.4%	10.9%	16.8%
U.S. Real Estates	0.93%	0.00%	1.76%	0.00%	3.7%	0.0%	5.6%	0.0%
U.S. Inf. Linked Bonds	0.00%	0.00%	4.62%	0.00%	6.5%	0.0%	0.0%	0.0%
Total	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Std. Dev.	3.6%	3.8%	6.8%	7.0%	9.9%	10.2%	13.1%	13.4%
Skewness	-1.2	-0.6	-1.2	-0.6	-1.2	-0.6	-1.0	-0.7
Kurtosis	7.8	5.4	7.8	5.3	7.9	5.2	6.9	5.7
VaR	-4.5%	-4.1%	-9.0%	-8.7%	-14.3%	-13.7%	-20.6%	-18.7%
CVaR	-7.8%	-7.1%	-15.2%	-14.0%	-23.0%	-21.1%	-30.9%	-29.3%

Our CVaR estimate has been based on a confidence level of 95%. In order to test the impact of the confidence level, we performed the same M-CVaR optimization analyses shown in Table 9 but with a confidence level of 99%. As we expected, the assets with higher skewness and lower kurtosis are allocated even more, because the M-CVaR optimization puts more penalty weights on the extreme left tail which has greater impact on skewness and kurtosis. The average of absolute allocation difference between MVO and M-CVaR optimization is about 3% less per asset class due to this confidence level change.

Finally, to test the impact of the financial crisis started in Sep 2008, we repeated the same experiment except that the data from Sep 2008 to June 2009 were removed. The average of absolute allocation difference between MVO and M-CVaR optimization is about 0.5% less per asset class when this block of data was removed. The main reason is that the financial crisis has put more weights to the left tail of majority equity asset classes by lowering their skewness and increasing their kurtosis, so that their impact on the M-CVaR optimization is higher.

Conclusions

Practitioners are well aware that asset returns are not normally distributed and that investor preferences often go beyond mean and variance; however, the implications for portfolio choice are not well known.

In a series of controlled traditional MVOs and M-CVaR optimizations we gain insights into the ramification of skewness and kurtosis on optimal asset allocations. In our first six scenarios, prior to running the optimizations, we use the multivariate TLF distribution model to simulate a large number of returns with appropriate variance, skewness and kurtosis, this in turn enables us to more accurately measure the downside risk of a portfolio using CVaR.

In our first example, when returns are normally distributed, MVO and M-CVaR lead to the same results. Next, when returns are symmetrical but with uniform kurtosis, this too leads to very similar results. When there are varying levels of skewness and kurtosis among the opportunity set of assets, MVO and M-CVaR lead to significantly different asset allocations. More specifically, M-CVaR prefers assets with higher skewness, lower kurtosis, and lower variance.

Over the last 20 years, value stocks, global high yield, U.S. REITs, and U.S. inflation linked bonds (since inception) have had significant negative skewness, while non-U.S. government bonds have had positive skewness. The kurtosis for global high yield, U.S. REITs, and U.S. TIPS are higher than other asset classes. In a 14 asset class analysis, relative to traditional MVO, M-CVaR leads to higher allocations to non-U.S. government bonds, non-U.S. developed equity and non-U.S. REITs, and lower allocations to U.S. large value, emerging market equity, TIPS and U.S. REITs.

While we are beginning to understand the impact of higher moments on asset allocation policy and further study is needed, these optimizations drive home a critical implication of modern portfolio theory: what matters is the overall impact on the portfolio's characteristics.

Appendix A. Multivariate TLF Distribution

We generate multivariate Lévy stable distributed returns through a numerical software package written by John Nolan (for details, see <http://academic2.american.edu/~jpnolan/stable/stable.html>). Next, we apply a truncation method on these returns with infinite variance so that each return series follows a Truncated Lévy flight model and preserves the desired pair-wise correlations.

The multivariate Lévy stable returns are generated by the software's multivariate stable functions with discrete spectral measure. The inputs to the function are a matrix specifying the location of the point masses as columns, a lamda vector, a beta vector, a location vector, and α . Conceptually, this matrix specification serves a role similar to that of the covariance matrix that drives a traditional multivariate normal simulation. The lamda vector is proportional to the volatility vector. The beta vector specifies the skewness for each asset. The location vector corresponds to an expected return vector. The α indicates the heaviness of the tails.

Note that the disadvantage of the multivariate stable, and hence the TLF distribution, is that it requires the same tail index (α) for each marginal or individual distribution. In a typical portfolio, stocks and bonds tend to have slightly different tail indexes. The tail index (α) is closely related to kurtosis and is the parameter used to control the kurtosis. The lower the tail index (α), the higher the kurtosis, and vice versa.

In Scenario 3 and Scenario 6 we assume two different kurtosis levels (kurtosis of 6 for Asset C, and 3.5 for Assets A, B, and D). To implement these more complicated scenarios, we call the multivariate TLF function twice for each of the two kurtosis. We then replace the Asset C returns in the first set of multivariate TLF returns (kurtosis = 3.5) with the Asset C returns in the second set of multivariate TLF returns (kurtosis = 6). The correlation matrix will be slightly distorted by this replacement, but this is not important since our interest is to compare the MVO with the M-CVaR optimization given the same multivariate returns.

References

- Artzner, P., Delbaen F., Eber, J. M. and D. Heath, (1999). "Coherent Measures of Risk." *Mathematical Finance*, 9, 203-228.
- Bradley, B., and M.S. Taqqu. 2003. "Financial Risk and Heavy Tails." In *Handbook of Heavy-Tailed Distributions in Finance*, ed. S.T. Rachev, 35-103. Amsterdam: Elsevir.
- Fama, E. F., 1965. "The Behavior of Stock-Market Prices." *The Journal of Business*, Vol. 38, No. 1, pp. 34-105.
- Hallerbach, Winfried G., 2003, "Decomposing Portfolio Value-at-Risk: A General Analysis." *The Journal of Risk*, 5, Feb. 2003.
- Kaplan, P.D., (2009). "Déjà Vu All Over Again", *Morningstar Advisor*, Fed/March, 29-33.
- Lévy, P. 1925. *Calcul des probabilités* (Gauthier-Villars, Paris).
- Longin, F. and B. Solnik, (2001), "Extreme Correlation of International Equity Markets", *Journal of Finance*, Vol.56, No. 2 (April 2001), page 649 – 676.
- Ang, A., and J. Chen. (2002). "Asymmetric Correlations of Equity Portfolios." *Journal of Financial Economics* 63, 443--494.
- Mandelbrot, B. (1963). "The variation of certain speculative prices." *Journal of Business* 36, 392–417.
- Mantegna, R.N., H.E. Stanley, (1994). "Stochastic Process with Ultraslow Convergence to a Gaussian: The Truncated Levy Flight." *Physical Review Letters*, 73, 2946-2949.
- Mantegna, R.N., H.E. Stanley, (1999). "An Introduction to Econophysics: Correlations and Complexity in Finance." Cambridge University Press, Cambridge.
- Martin, R.D., S. Rachev, and F. Siboulet, (2003). "Phi-Alpha Optimal Portfolios & Extreme Risk Management." *Wilmott Magazine of Finance*, November: 70-83.
- Nolan, J., *Software Stable 5.1 for Matlab*, 2009.
- Pflug, G.Ch. (2000). Some Remarks on the Value-at-Risk and the Conditional Value-at-Risk. In "Probabilistic Constrained Optimization: Methodology and Applications", Ed. S. Uryasev, Kluwer Academic Publishers, 2000.
- Rachev S.T., C. Menn, and F.J. Fabozzi, (2005). "Fat-Tailed and Skewed Asset Return Distributions. Implications for Risk Management, Portfolio Selection, and Option Pricing." *Wiley Finance*.
- Rockafellar, R.T. and S.Uryasev. (2000). "Optimization of Conditional Value-At-Risk." *The Journal of Risk*, Vol. 2, No. 3, 2000, 21-41.
- Uryasev, S. (2000). "Conditional Value-at-Risk: Optimization Algorithms and Applications." *Financial Engineering News*, 14, February, 2000.
- Xiong, J.X. (2010). "Using Truncated Lévy Flight to Estimate Downside Risk." *Journal of Risk Management in Financial Institutions*, forthcoming in June, 2010.

Visit www.ibbotson.com to access all of Ibbotson's research documents