Is Your Covariance Matrix Still Relevant?

An asset allocation-based analysis of dynamic volatility models

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Although researchers have made many strides in modeling risk over the past two decades, practitioners continue to cling to the sample covariance matrix as a forward looking estimate of future volatility. As discussed in Colon et al. [2011], the use of the sample covariance matrix in the formulation of an asset allocation often leads to static portfolios that fail to adjust to prevailing risk environments. Furthermore, Colon et al. [2011] contend that individuals have locally stable risk tolerances, and along with Perold [2007] suggest that investment managers should tactically adjust the composition of an investor's portfolio to ensure that the risk profile of the portfolio perpetually matches the risk tolerance of the investor.

To manage the risk profile of an investor's portfolio, one must model and forecast covariance across a range of asset classes. George E.P. Box once wrote, "All models are wrong, but some are useful" [1987]. The scope of this article is to identify useful volatility models available to practitioners from a range of both popular and unknown models. Due to the large number of pairwise covariance estimates in a forecast covariance matrix and due to the latency of covariance, comparing volatility models is a rather difficult task. To simplify this comparison, we employ a new approach to determining the accuracy of the volatility model.

In our research, we generate one-month covariance forecasts for seven different volatility models at monthly time intervals between December 29, 2006 and April 29, 2011. We then use the covariance forecasts to construct risk-parity portfolios with ex-ante volatilities of 8.0 (annualized).[1] Finally, we analyze the out-of-sample volatilities of each portfolio to determine how close each process tracked the volatility target, ex-post. We contend that the most useful volatility model is the one whose corresponding portfolio best tracks the volatility target out of sample.

Our research finds that the scalar Asymmetric Dynamic Conditional Correlation model (A-DCC) of Cappiello, Engle, and Sheppard [2006] delivered superior covariance forecasts over the analysis period, followed by the Dynamic Conditional Correlation model (DCC) of Engle [2002]. Furthermore, these findings are consistent with prior research that suggests that negative co-movements in asset prices tend to be followed by increases in correlation (see Ang and Chen [2002]). Consequently, we believe that practitioners should consider adopting an A-DCC volatility model to generate the covariance forecasts required as inputs to most asset allocation models (Meucci [2005] provides a review of different asset allocation models).

This paper is organized into four sections. The first section contains a discussion of the multivariate volatility models we use in this analysis. The second section contains a discussion of the methodology used to compare the volatility models. The third section contains our findings and the fourth section offers some conclusions.

VOLATILITY MODELS

Since at least 1952, when Markowitz published his seminal work on portfolio construction, practitioners have relied upon covariance forecasts as key inputs in their asset allocation models. While researchers continue to develop new methods for combining portfolios of assets, such as those described by Michaud [1989], Black and Litterman [1992], and Qian [2005], the overwhelming majority of asset allocation models incorporate a forecasted covariance matrix.

To forecast covariance, most practitioners continue to use years of past returns to estimate an unconditional, sample covariance matrix.

$$\overline{\boldsymbol{H}} = \tfrac{1}{T-1}\boldsymbol{a}\boldsymbol{a}' \qquad\qquad \boldsymbol{a} = \boldsymbol{r} - E[\boldsymbol{r}] \tag{1}$$

where $\overline{\boldsymbol{H}}$ represents the unconditional, sample covariance matrix, **a** represents a $k \times T$ matrix of innovations, **r** represents a $k \times T$ matrix of log returns[2], and T represents the number of time periods in the sample.

Implicit in its use is that $\overline{\boldsymbol{H}}$ is stationary. Thus the sample covariance matrix is an unbiased estimator of the population covariance matrix and not dependent upon time. Furthermore, as long as all of the assets have the same amount of history, $\overline{\boldsymbol{H}}$ is guaranteed to be positive semi-definite.

If $\overline{\boldsymbol{H}}$ is stationary, the standard error of the covariance estimates will decline as the number of observations in the sample increases, leading some to incorporate an expanding estimation window to incorporate as many data points as possible.

Many have recognized, however, that covariances change through time. The simplest model to accommodate the time-conditional nature of volatility is a simple extension of the sample covariance matrix in (1). The forecasted covariance matrix is simply the sample covariance matrix of a rolling window of innovations, as discussed in Chan et al. [1999] and Gosier et al. [2005]. For example, one may only incorporate returns observed during the past five years to estimate the time-conditional covariance matrix.

$$E[\boldsymbol{H}_t|\mathcal{F}_{t-1}] = \frac{1}{T^* - 1}\boldsymbol{a}\boldsymbol{a}' \tag{2}$$

where $\boldsymbol{H_t}$ is the conditional covariance matrix, T^* is the rolling window length and $\mathcal{F}_{t-1}$ represents the sigma-algebra generated by $\{\boldsymbol{r_{t-1}}, \boldsymbol{r_{t-2}}, \dots, \boldsymbol{r_{t-T^*}}\}$.

Equation (2) is essentially equation (1), however, it implies that the covariance matrix evolves over time and is a function of the past T^* returns. As in the unconditional, sample covariance matrix, as long as all of the assets in the sample have T^* periods of data available, $\boldsymbol{H}_t$ is guaranteed to be positive semi-definite.

While the rolling covariance model may be more sensitive to recent information than a sample estimate based on all prior data, it nonetheless interprets every observation in $\mathcal{F}_{t-1}$ as equally important. For example, returns that occurred five years ago are treated with the same importance as those that occurred yesterday. To place greater importance on more recent information, some practitioners use the exponentially-weighted covariance matrix, made popular by RiskMetrics [1996].

$$E[\boldsymbol{H}_t|\mathcal{F}_{t-1}] = (1-\lambda)\boldsymbol{a}_{t-1}\boldsymbol{a}'_{t-1} + \lambda\boldsymbol{H}_{t-1} \qquad \text{where } 0 < \lambda < 1 \tag{3}$$

Equation (3) can be rewritten as follows:

$$E[\boldsymbol{H}_t|\mathcal{F}_{t-1}] = (1-\lambda)\sum_{i=1}^{\infty}\lambda^{i-1}\boldsymbol{a}_{t-i}\boldsymbol{a}'_{t-i} \tag{4}$$

From equation (3), the estimate of the covariance matrix at time t is the weighted sum of the covariance matrix at time t-1 and the innovations on t-1. From (4) we see that, as the weight assigned to λ declines, the emphasis placed on the most recent observations increases.

While this model is simple, readily available in many software packages, and yields a positive semi-definite matrix as long as $\boldsymbol{H}_1$ is positive semi-definite, it is not without several faults. Only one parameter, λ, governs the time evolution of $\boldsymbol{H}_t$. Furthermore, λ can be difficult to estimate (e.g. via maximum likelihood) for non-trivial dimensions. As the number of assets increases, λ tends toward one. As a consequence, practitioners often choose a value for λ (typically between 0.94 and 0.98) without rigorously estimating λ.

An alternative to the exponentially weighted covariance matrix is the Dynamic Conditional Correlation matrix of Engle [2002].

$$E[\boldsymbol{H}_t|\mathcal{F}_{t-1}] = \boldsymbol{\Omega} + \alpha\boldsymbol{a}_{t-1}\boldsymbol{a}'_{t-1} + \beta\boldsymbol{H}_{t-1} \tag{5}$$

where $\boldsymbol{\Omega}$ represents a $k \times k$ matrix of intercept parameters ($n(n+1)/2$ unique paramters), and α and β are the parameters associated with the innovations at time t-1 and the covariance matrix at time t-1. Although slightly more complex than the exponentially weighted covariance model, estimation of the DCC model is rather simple and takes place via quasi-maximum-likelihood estimation. In comparison with the exponentially-weighted covariance matrix, it allows a more rigorous estimation of the parameters that govern the time evolution of $\boldsymbol{H}_t$.

Consider the likelihood function for jointly normal random variables with a zero mean:

$$L(\boldsymbol{H}_t|\boldsymbol{a}_t) = (2\pi)^{-T/2}|\boldsymbol{H}_t|\exp\left(-\frac{1}{2}\boldsymbol{a}'_t\boldsymbol{H}_t\boldsymbol{a}_t\right) \tag{6}$$

With some algebra and using the relationship $\boldsymbol{H}_t = \boldsymbol{D}_t^{1/2}\boldsymbol{R}_t\boldsymbol{D}_t^{1/2}$, where $\boldsymbol{D}_t = diag(\boldsymbol{H}_t)$ and represents the diagonal matrix of conditional volatilities, the log likelihood function becomes:

$$logL(\boldsymbol{H}_t|\boldsymbol{a}_t) = -\frac{1}{2}\sum_{t=1}^{T}\left(Tlog(2\pi) + 2\log\left|\boldsymbol{D}_t^{1/2}\right| + \boldsymbol{a}_t'\boldsymbol{D}_t\boldsymbol{a}_t\right) + \frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{\varepsilon}_t'\boldsymbol{\varepsilon}_t - log|\boldsymbol{R}_t| - \boldsymbol{\varepsilon}_t'\boldsymbol{R}_t^{-1}\boldsymbol{\varepsilon}_t) \quad (7)$$

In examining equation (7), the log likelihood function contains two distinct parts. The first summation contains only the variance parameters, while the second summation contains only the correlation parameters and the standardized innovations. While it is possible to perform full maximum likelihood estimation, separating the two summations into composite likelihoods greatly simplifies the process, and according to Engle [2009], is very close to the full maximum likelihood.

Thus, we can break the estimation process into three steps (1) derive the innovations by removing the conditional means (i.e. de-mean the returns), (2) fit conditional volatility models to each of the marginal series and standardize the innovations (i.e. de-GARCH the innovations), and (3) fit the correlation model to the standardized innovations. A summary of this process is explained in further detail below.

Step 1: De-mean the returns

If the means of the series in $\mathbf{r}$ are not statistically different from zero, we must develop a model for the means. If the returns exhibit serial cross correlations, a vector autoregressive model (VAR[p]) may be appropriate to incorporate the conditional dependence across the series.

$$\boldsymbol{a}_t = \boldsymbol{r}_t - \boldsymbol{\phi}_0 - \boldsymbol{\phi}_1\boldsymbol{r}_{t-1} - \cdots - \boldsymbol{\phi}_p\boldsymbol{r}_{t-p} \quad (8)$$

where $\boldsymbol{\phi}_i\ \forall i\ \leq p$ represents a $k \times k$ coefficient matrix. The innovations generated from the above equation have a conditional mean of zero, but time-conditional volatilities and pairwise correlations. Although not necessary, if we assume that the errors are normally distributed:

$$\boldsymbol{a}_t|\mathcal{F}_{t-1} \sim N(0, \boldsymbol{H}_t) \quad (9)$$

Step 2: De-GARCH the innovations

As previously discussed, the likelihood estimation can be broken into two parts. The first part contains the volatility parameters, and the second part contains the correlation parameters and standardized innovations. Since the calculation of the correlation parameters requires standardized innovations as necessary inputs, we must next derive the conditional volatilities of each series. The standardized innovations can then be calculated as follows:

$$\boldsymbol{\varepsilon}_t = \boldsymbol{D}_t^{-1/2}\boldsymbol{a}_t \quad (10)$$

The types of univariate volatility models for $\boldsymbol{a}_t$ do not need to be uniform across the different series. Therefore, one may wish to consider a range of univariate volatility models for each of the series in $\boldsymbol{a}_t$ and select the one that best fits the data. The following are three common GARCH-type models considered in this analysis ($where\ r_t = \mu_t + a_t$, $a_t = \sigma_t \varepsilon_t$ and ε_t is either normal or student-t distributed):

$$GARCH(1,1): \qquad \sigma_t^2 = \omega + \alpha a_{t-1}^2 + \beta \sigma_{t-1}^2 \qquad a + \beta < 1 \tag{11}$$

The GARCH(1,1) model of Bollerslev [1986] is the simplest and most common of GARCH-type models. The conditional variance at time t is a function of the squared innovations at time t-1 and the conditional variance at time t-1.

$$GJR - GARCH(1,1): \quad \sigma_t^2 = \omega + \left(\alpha + \gamma \mathbb{I}_{a_{t-1}<0}\right) a_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{12}$$

The GJR-GARCH model of Glosten, Jagannathan, and Runkle [1993] is an extension of the classic GARCH model that includes an additional parameter that delineates between positive and negative innovations. If γ is a positive coefficient, negative innovations lead to larger increases in variance.

$$EGARCH(1,1): \qquad \log(\sigma_t^2) = \omega + \alpha \frac{|a_{t-1}| + \gamma a_{t-1}}{\sigma_{t-1}} + \beta \log(\sigma_{t-1}^2) \tag{13}$$

Similar to the GJR-GARCH model, the EGARCH model developed by Nelson [1991] also takes into account asymmetric responses to positive and negative innovations.

At least as far back as Black [1976], researchers have documented the asymmetric dependence on negative innovations and have attributed its existence to: (1) the leverage effect, where negative innovations increase firms' debt-to-equity ratios, which causes greater volatility in the stock price, and/or (2) the risk premium effect, where risk averse investors sell their positions due to an expectation of higher volatility until the expected return is sufficient for the new level of expected volatility.

Once the appropriate volatility model is selected for each of the marginal series, the innovations $\boldsymbol{a}_t$ are divided by their conditional volatilities, as in (10). The resulting standardized innovations ($\boldsymbol{\varepsilon}_t$) have constant variances and zero means.

Step 3: Derive the Conditional Correlations

Now that the standardized innovations are available, we can fit the conditional correlation model. As shown in (5), we must estimate the following parameters: Ω, α, and β where Ω contains $\frac{k(k-1)}{2}$ unique parameters. As in Engle [2009], we can employ correlation targeting to estimate Ω as follows:

$$\widehat{\boldsymbol{\Omega}} = (1 - \alpha - \beta)\overline{\boldsymbol{R}} \qquad \overline{\boldsymbol{R}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t' \tag{14}$$

Finally, we maximize the second summation in (7) to find the parameter estimates α, β and $\boldsymbol{R}_t$. Combining these estimates with the conditional volatility estimates in Step 2 yields the conditional covariance matrix $\boldsymbol{H}_t$ in (5).

Cappiello, Engle, and Sheppard [2006] suggested an extension to the standard DCC model that incorporates asymmetric effects in the correlations.

$$E[\boldsymbol{H}_t|\mathcal{F}_{t-1}] = \Omega + \alpha \boldsymbol{a}_{t-1}\boldsymbol{a}'_{t-1} + \gamma \boldsymbol{\eta}_{t-1}\boldsymbol{\eta}'_{t-1} + \beta \boldsymbol{H}_{t-1} \tag{15}$$

where $\boldsymbol{\eta}_{t-1} = \min(\boldsymbol{a}_{t-1}, 0)$. By incorporating the additional parameter, we are able to allow for greater increases in correlation during negative co-movements across assets (i.e. the "correlations go to one" effect). As documented in Ang and Chen [2002], correlations across stocks tend to increase during negative environments. Thus, by incorporating an additional parameter that accounts for this phenomenon, the A-DCC model may better fit the data.

Fortunately, incorporating the additional parameter does not impact the likelihood function of (7). However, two changes are required in the estimation process. As noted in Capiello, Engle, and Sheppard [2006], the estimator of $\boldsymbol{\Omega}$ is revised to incorporate the additional information in the asymmetric parameter:

$$\widehat{\boldsymbol{\Omega}} = (1 - \alpha - \beta)\overline{\boldsymbol{R}} - \gamma \overline{\boldsymbol{N}} \qquad \overline{\boldsymbol{N}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{\eta}_t \boldsymbol{\eta}'_t \tag{16}$$

Furthermore, to ensure that $\boldsymbol{H}_t$ is positive semi-definite, we enforce the following linear constraint during the maximization of the log-likelihood function:

$$1 - \alpha - \beta - \gamma g > 0 \tag{17}$$

where g is the maximum eigenvalue of $\left(\overline{\boldsymbol{R}}^{-1/2}\overline{\boldsymbol{N}}\overline{\boldsymbol{R}}^{-1/2}\right)$, which can be evaluated prior to the maximum likelihood step.

The final volatility model analyzed in this paper relies on the following simple relationship:

$$Cov[\boldsymbol{r}_1, \boldsymbol{r}_2] = \frac{Var[\boldsymbol{r}_1 + \boldsymbol{r}_2] - Var[\boldsymbol{r}_1 - \boldsymbol{r}_2]}{4} \tag{18}$$

Thus pairwise covariances can be calculated by fitting variance models to the series $\boldsymbol{r}_{1+2} = \boldsymbol{r}_1 + \boldsymbol{r}_2$ and $\boldsymbol{r}_{1-2} = \boldsymbol{r}_1 + \boldsymbol{r}_2$ for each pair.

The problem with this technique is that the resulting covariance matrix is not guaranteed to be positive semi-definite. Higham [2002] proposes a process to find the nearest positive semi-definite correlation matrix from a non-positive definite correlation matrix. The process employs alternating projections with respect to a weighted Frobenius norm to find a positive semi-definite correlation matrix determined to be "nearest" to the original matrix.

Herein referred to as the GARCH-Implied Covariance with Higham PSD Correction, we will fit univariate GARCH models to each of the pairwise series in (18), solve for the pairwise covariance in (18), and then employ Higham's nearest correlation matrix approach to find the nearest positive semi-definite covariance matrix.

METHODOLOGY

Data

To compare the out-of-sample performance of the various volatility models, we acquired daily prices for assets representing seven different asset classes from Bloomberg and transformed them into log returns:

$$r_t = \log\left(\frac{P_t}{P_{t-1}}\right) \tag{19}$$

where P_t represents the price of a security at time t.

While index data was available for all seven of the asset classes, we chose to use exchange traded fund (ETF) prices for six of the seven asset classes. The indices representing two of the asset classes, MSCI EAFE and MSCI Emerging Markets, trade at different times than the other indices (7:00 PM to 1:30 PM EST). To avoid the asynchronous trading time problems (see Burns et al. [1998]), as well as differing holiday schedules across markets, we chose to use ETFs where available. Due to the limited history of commodity ETFs, we elected to use the index returns to represent the asset class.

The following table summarizes our proxy choices for each asset class:

Exhibit 1: Translation of Asset Class to Investment Vehicle

Asset Class	Index	Security
US Equity	S&P 500 Index	SPDR S&P 500 (SPY)
International Developed Equity	MSCI EAFE Index	iShares MSCI EAFE (EFA)
Emerging Market Equity	MSCI Emerging Market Index	iShares MSCI Emerging Markets (EEM)
US Public Real Estate	Dow Jones US Real Estate Index	iShares Dow Jones US Real Est (IYR)
Long Duration Treasuries	Barclays Capital 7-10 Yr Treasury Index	iShares Barclays 7-10 Yr Treasury (IEF)
Credit	Barclays Capital US Aggregate Index	iShares Barclays Agg Bond (AGG)
Commodities	Dow Jones UBS Commodity Index	-

Upon inspection of the data, we observed that not all securities had prices available on all of the NYSE trading dates between December 29, 2003 and April 29, 2011. We observed six dates on which at least one vehicle did not report a price and deleted those observations from the sample (casewise deletion).[3]

The following are the descriptive statistics of the log returns in our sample:

Exhibit 2: Descriptive statistics of the daily log returns from January 4, 2004 – April 29, 2011

	Geometric Mean*	Standard Deviation*	Skewness	Kurtosis
SPDR S&P 500 (SPY)	4.93	21.49	0.37	19.48
iShares MSCI EAFE (EFA)	7.15	25.71	0.39	16.15
iShares MSCI Emerging Markets (EEM)	15.70	37.26	1.03	20.82
iShares Dow Jones US Real Est (IYR)	8.02	38.82	0.02	13.51
iShares Barclays 7-10 Yr Treasury (IEF)	5.56	7.21	0.18	5.79
iShares Barclays Agg Bond (AGG)	4.90	5.89	-2.63	76.50
Dow Jones UBS Commodity Index	5.69	19.95	-0.16	4.86

*Annualized based on 252 trading days in a year.

The descriptive statistics reveal the imperfect nature of proxies. A sample kurtosis of 76.50 for the iShares Barclays Aggregate Bond ETF is in stark contrast to the sample kurtosis of 1.92 for the corresponding index. This is due to the idiosyncratic risk associated with the ETF, which makes ETFs imperfect proxies to the indices they attempt to replicate.

A prime example occurred in October of 2008, when markets were reeling from the onset of the subprime crisis. The Barclays Aggregate Bond Index returned -0.47% on October 10th, while the ETF returned -6.84%.[4] Since the Barclays Aggregate Index includes over 7,000 bonds, many of which do not trade throughout the day, it is unclear which return best characterizes the true return on that day.

Analysis Process

The following is an outline of the methodology we used to assess the performance of the different volatility models:

Step 1: At each month end, refit each of the volatility models to generate a forecast covariance matrix.
Step 2: Use each model's forecast covariance matrix to construct a risk-parity portfolio with a target volatility of 8.0 (annualized).
Step 3: Repeat this process throughout the out-of-sample period.
Step 4: Derive the realized volatilities of each portfolio by fitting a univariate GARCH model (selected by AIC) to each of the series.
Step 5: Analyze how well each portfolio was able to meet the ex-ante volatility target of 8.0 (annualized).

To compare the performance of the different volatility models, we generated volatility forecasts using each model discussed in the preceding section starting on December 29, 2006 and each subsequent month through March 31, 2011. The following is a summary of the volatility models we compared:

Exhibit 3: Summary of Multivariate Volatility Models

Symbol	Name
$\bar{H}$	Unconditional Sample Covariance[5]
H_t^{RS}	3-Year Rolling Sample Covariance[6]
H_t^{EW}	Exponentially-Weighted Moving Average ($\lambda = 0.96$)[7]
H_t^{DCC}	DCC with GARCH(1,1) marginal volatility models and normal innovations[8]
H_t^{DCC2}	DCC with AIC selected marginal volatility models and normal or student-t innovations[9]
H_t^{A-DCC}	Asymmetric DCC (A-DCC) with AIC selected marginal volatility models and normal or student-t innovations[10]
H_t^{GI}	GARCH Implied Covariance with Higham PSD correction[11]

With the volatility forecasts at each month-end we constructed a risk-parity portfolio with a targeted volatility of 8.0 (annualized).

Risk parity is a portfolio construction technique that distributes risk evenly across all assets in the portfolio (see Maillard et al. [2010] for a review). In our case, we define risk as variance in returns. The ex-ante marginal contribution to risk is found by taking the first-order partial derivative of portfolio variance with respect to portfolio weights.

First, the ex-ante variance of a portfolio is estimated as follows:

$$\sigma_{P,t}^2 = \boldsymbol{w}'\boldsymbol{H}_t\boldsymbol{w} \tag{20}$$

where $\sigma_{P,t}^2$ represents the ex-ante variance of the portfolio at time t, and **w** represents a $k \times 1$ vector of portfolio weights.

The marginal risk contributions of each asset can be calculated as follows:

$$\boldsymbol{C}_t = \frac{\partial \sigma_{P,t}^2}{\partial \boldsymbol{w}} = 2\boldsymbol{H}_t\boldsymbol{w} \tag{21}$$

Where **C** is a $k \times 1$ vector of marginal risk contributions representing the increase in portfolio variance due to a small increase in the weight of the respective asset.

Using equations (20) and (21), we can decompose a portfolio's variance into the sum of its component risk contributions:

$$RC_{i,t} = w_i \frac{C_{i,t}}{2} \tag{22}$$

where $RC_{i,t}$ represents the variance contribution of asset i to the portfolio's total variance at time t. Thus, a portfolio's variance can be written as a linear combination of the variance contributions:

$$\sigma_{P,t}^2 = \sum_{i=1}^{k} w_i RC_{i,t} \tag{23}$$

While a closed-form solution for finding the risk-parity portfolio is not available, we can use numerical optimization techniques to find the portfolio:

$$\boldsymbol{w}^* = \arg\min f(\boldsymbol{w})$$

$$f(\boldsymbol{w}) = \mathbf{1}'\left(\boldsymbol{w} \odot \frac{\boldsymbol{H}_t \boldsymbol{w}}{\boldsymbol{w}' \boldsymbol{H}_t \boldsymbol{w}} - \mathbf{1}\frac{1}{k}\right)^2 \tag{24a}$$

where **1** represents a $k \times 1$ ones vector and $\odot$ represents the Hadamard Product.

We impose a long-only constraint to ensure that the portfolios generated are reasonably consistent with what an actual investor may follow:

$$\boldsymbol{A}\boldsymbol{w} \geq \boldsymbol{b}$$

Where **A** is a $k \times k$ identity matrix and **b** is a $k \times 1$ zero vector. We chose not to impose a general budget constraint, allowing the sum of the weights to exceed zero (implying leverage) or fall below zero (implying cash is a holding).

Since we are testing how well the portfolios constructed with different volatility models track a constant volatility target, we impose the following constraint:

$$\boldsymbol{w}' \boldsymbol{H}_t \boldsymbol{w} = \sigma_T^2$$

where σ_T represents the constant volatility target.

Thus (24a) simplifies to the following:

$$f(\boldsymbol{w}) = \mathbf{1}'\left(\boldsymbol{w} \odot \frac{\boldsymbol{H}_t \boldsymbol{w}}{\sigma_T^2} - \mathbf{1}\frac{1}{k}\right)^2 \tag{24b}$$

In practice, one must guard against local minimums, as the space is not convex.

We chose to construct the month-end portfolios using a risk-parity approach for three primary reasons:

1) ***Risk parity does not depend on mean estimation.*** Mean-variance optimization, the most common approach to portfolio selection, requires forecasts of means. Furthermore, as shown in Chopra and Ziemba [1993], mean-variance optimization is highly sensitive to estimation error in means (relative to estimation error in variances and covariances). Introducing an additional set of

parameters that require estimation would increase noise in the results, as portfolio selection would be due to both covariances and means.

2) ***Risk parity distributes estimation error in the covariances evenly.*** An alternative option is to simply allocate 1/k percent of the portfolio to each asset, and lever the portfolio until it hits its risk target ex-ante. This approach, however, would be most sensitive to estimation error in those assets with the highest covariance with the portfolio. Volatility models that were accurate across the assets, with the exception of one asset, would be severely penalized if that asset had a high covariance with the portfolio. Instead, risk parity allows the impact of estimation error to be spread evenly across the assets analyzed, leading to more robust results.

FINDINGS

Ex-post Volatility Analysis

Having generated covariance forecasts using each of the seven methods and constructed risk-parity portfolios with the corresponding covariance matrices, we now have seven daily return series representing the out-of-sample performance for each covariance method. The following table summarizes the annualized out-of-sample performance of each of the seven approaches:

Exhibit 4: Summary Statistics on the Out-of-Sample Returns from January 3, 2007 through April 29, 2011

	$\overline{H}$	H_t^{RS}	H_t^{EW}	H_t^{DCC}	H_t^{DCC2}	H_t^{A-DCC}	H_t^{GI}
Geometric Mean	7.21	4.82	9.89	9.47	9.46	10.13	16.16
Std. Deviation	12.5	10.1	9.2	8.5	8.6	8.9	11.5
Skew	0.02	-0.11	-0.59	-0.49	-0.52	-0.45	-0.30
Kurt	8.29	12.18	3.50	2.95	2.98	2.17	6.29

*Returns annualized based on 252 trading days per year. The return of cash was assumed to be 0% per day.
**See the Appendix for the DCC parameter values and area graphs of the portfolio holdings for each method.

From above, we observe that, on average, the portfolio constructed using the DCC-GARCH(1,1) covariance model came closest to meeting the volatility target of 8.0. The two other DCC-type models also achieved out-of-sample volatility that, on average, was within 1% of the target.

However, as we previously noted, many practitioners use the sample covariance matrix to generate their volatility forecasts. From Exhibit 4, we observe that the static portfolio constructed with the unconditional, sample covariance matrix at the beginning of the out-of-sample period performed poorly relative to the volatility target. While the target was 8.0, this portfolio generated an ex-post volatility of 12.5. The portfolio constructed with the rolling 3-year sample covariance matrix performed better, although over two percent higher than the volatility target.

Although the above table provides a glimpse at the performance of the different covariance models, we are able to examine the realized volatilities more extensively by looking at the daily volatilities. To perform this analysis, we fit each of the seven series with univariate GARCH models (GARCH(1,1), GJR-GARCH(1,1), and EGARCH(1,1)) with both normal and t-distributed errors, and based on the resulting AIC values, we selected the univariate GARCH model that best fit each series. The following table summarizes the resulting daily volatility series:

Exhibit 5: Summary Statistics on the Realized Volatilities* from January 3, 2007 through April 29, 2011

	$\bar{H}$	H_t^{RS}	H_t^{EW}	H_t^{DCC}	H_t^{DCC2}	H_t^{A-DCC}	H_t^{GI}
Mean	11.23	8.67	8.81	8.24	8.36	8.75	10.71
Std. Deviation	4.75	5.19	2.26	1.67	1.73	1.48	3.79
Skewness	3.76	3.17	4.05	3.91	3.67	2.48	2.88
Kurtosis	20.55	15.86	24.94	25.04	22.33	12.19	13.17
Max	45.58	41.57	28.10	23.04	23.26	18.89	37.29
Min	6.95	3.88	6.83	6.71	6.67	6.98	7.25

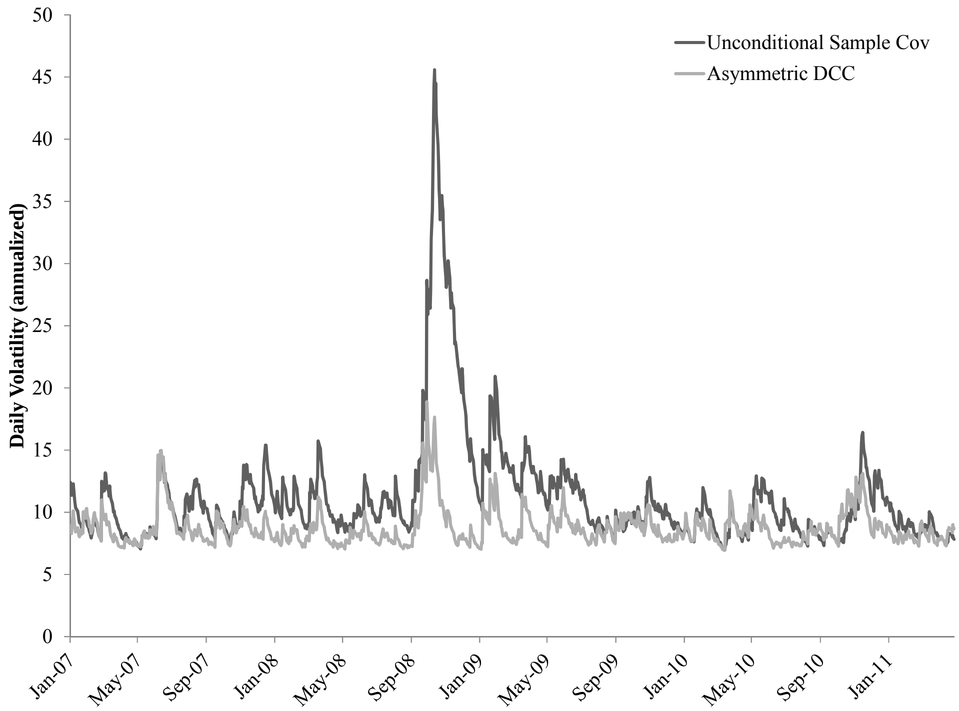
*Realized volatilities have been annualized based on 252 trading days per year

From the above table we again see that, on average, the DCC-GARCH(1,1) came closest to meeting the volatility target of 8.0. However, the A-DCC model outperformed the DCC-GARCH(1,1) model based on all other metrics. The returns generated from the A-DCC portfolio had a lower "volatility of volatility," meaning that the realized daily volatilities were closer to the average daily volatility than the DCC-GARCH(1,1) portfolio returns.

Furthermore, we see that the kurtosis of the portfolio generated from the A-DCC model was less than half of the kurtosis of the DCC-GARCH(1,1) model. This difference is evident in the range. At the end of September 2008, when the markets were suffering significant losses from the Subprime Crisis, the daily volatility of the A-DCC constructed portfolio rose to 18.89, while the volatility of the DCC-GARCH(1,1) portfolio rose to 23.04.

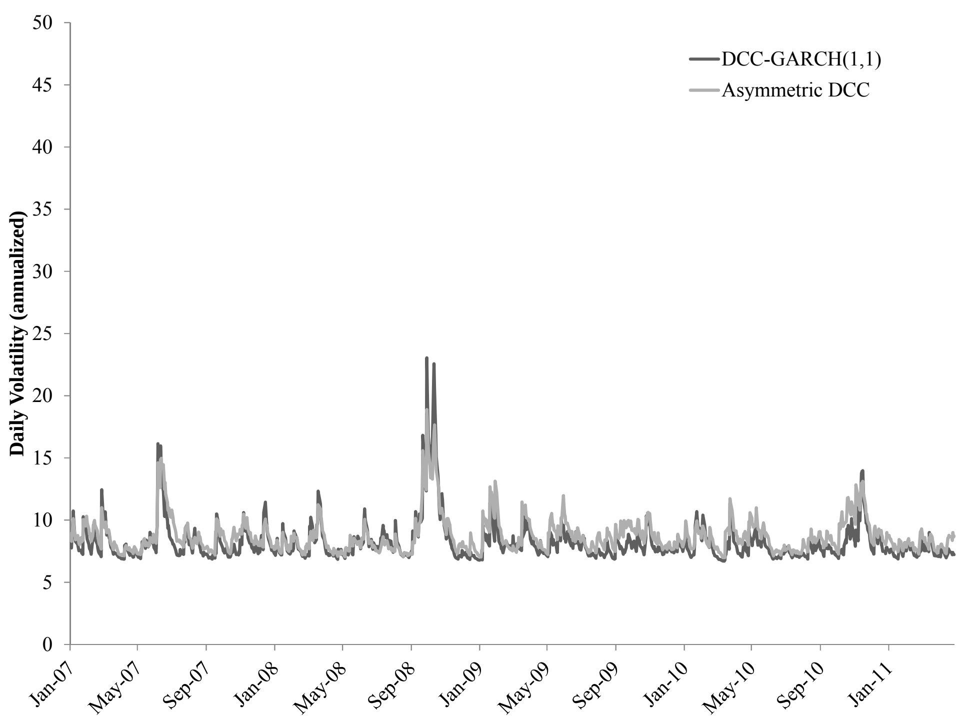
The following graph compares the out-of-sample volatilities of the static portfolio, which was constructed using the unconditional sample covariance matrix, and the portfolio constructed using the A-DCC covariance model.

Exhibit 6: Annualized Volatility of Unconditional Sample Covariance and Asymmetric DCC Portfolios



From Exhibit 6, we clearly see that the A-DCC model consistently keeps realized volatility much closer to the volatility target of 8.0. During October of 2008, the annualized volatility of the A-DCC portfolio rose to 18.89 while the volatility of the unconditional sample covariance portfolio rose to 45.58.

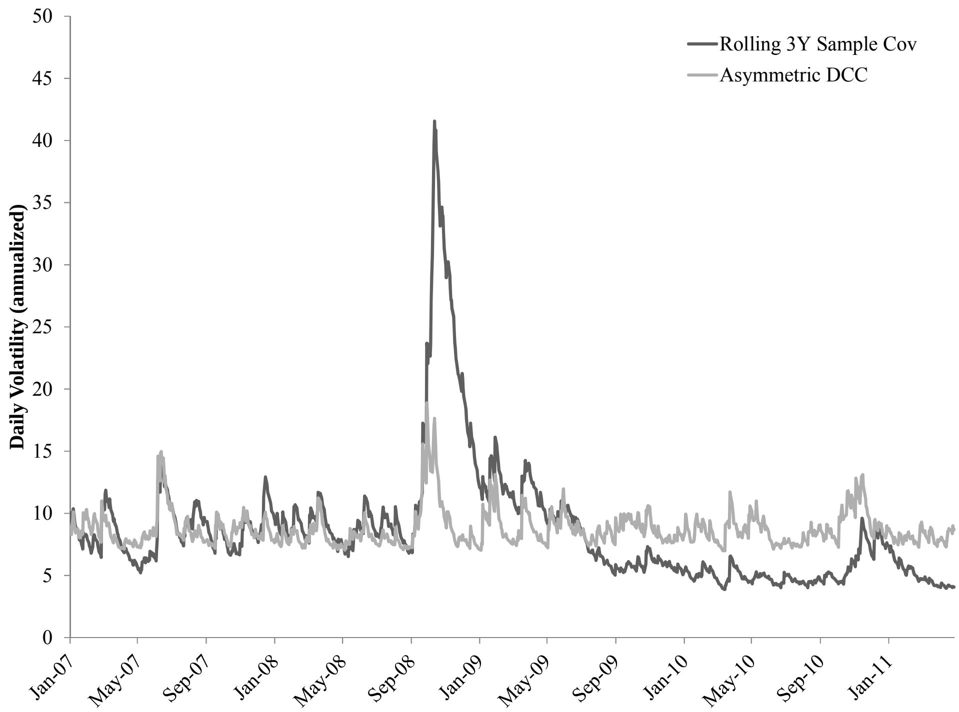
Exhibit 7: Annualized Volatility of GARCH(1,1) DCC and Asymmetric DCC Portfolios



When comparing the GARCH(1,1) DCC model, we see that the performance is quite similar to that of the Asymmetric DCC model. During the heart of the Subprime Crisis, the annualized volatility of the GARCH(1,1) DCC portfolio reached 23.04, while the volatility of the A-DCC portfolio reached 18.89.

Finally, it is of interest to examine the ex-post volatility of the portfolio constructed using the rolling 3-year sample covariance matrix, since the use of a rolling covariance matrix is quite common among practitioners.

Exhibit 8: Annualized Volatility of Rolling 3Y Sample Covariance and Asymmetric DCC Portfolios



While the portfolio constructed using the rolling 3-year sample covariance matrix as a forecast did not reach the ex-post volatility levels of the unconditional sample covariance portfolio, it did come close. On October 14, 2008, the annualized volatility of this portfolio reached 41.57, while the contemporaneous volatility of the A-DCC constructed portfolio was 17.65.

Analysis of Magnitude in Dependence Changes

So far, the data suggest that the more dynamic covariance models, such as the exponentially-weighted moving average and the DCC-type models are better able to adapt to changing economic conditions and generate short-term forecasts that best represent the prevailing volatilities and correlations.

Of particular interest is the magnitude of month-to-month changes in the forecasted correlation matrix. To evaluate the monthly changes in the correlation matrix for each forecasting method, we calculated the Frobenius Norm of the change in the correlation matrix at each month end in the out-of-sample period.[12]

Using prior notation, the correlation matrix can be recovered from the covariance matrix as follows:

$$\boldsymbol{R}_t = \boldsymbol{D}_t^{-1/2} \boldsymbol{H}_t \boldsymbol{D}_t^{-1/2} \tag{25}$$

Thus, we define the change matrix of the correlations as:

$$\boldsymbol{A}_t = \boldsymbol{R}_t - \boldsymbol{R}_{t-1} \tag{26}$$

The Frobenius Norm of the change matrix can then be calculated as:

$$\|\boldsymbol{A}_t\|_F = \left(\sum_{i=1}^{k} \sum_{j=1}^{k} \left| a_{ij}, t \right|^2 \right)^{1/2} \tag{27}$$

When evaluating the resulting Frobenius Norms of the change matrices, higher values indicate greater changes in the total correlation matrix. The following graphs display the time series of norms for each of the covariance models considered in this analysis:

Exhibit 9: Frobenius Norms of Correlation Matrix Differences

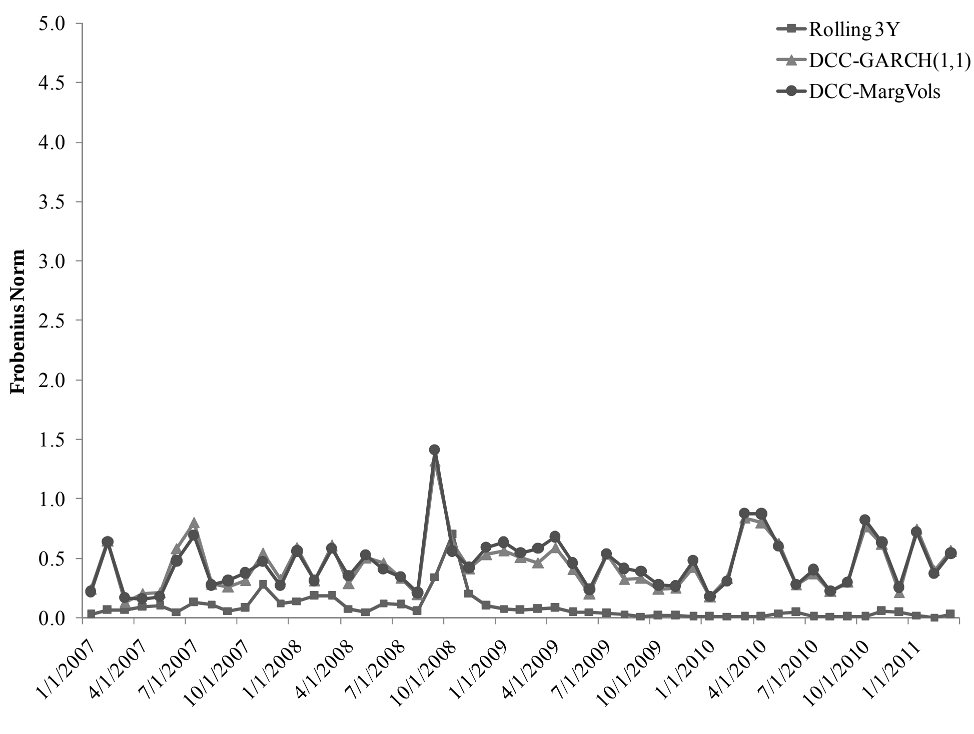
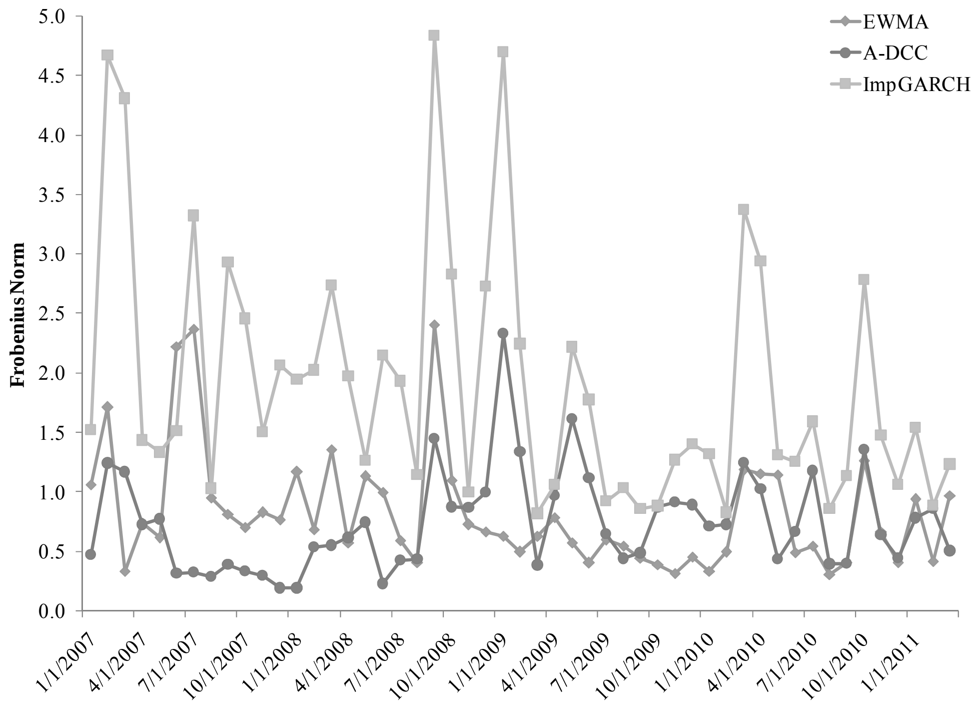


Exhibit 10: Frobenius Norms of Correlation Matrix Differences



In the first group, which contains the difference norms of the rolling 3-year sample covariance, DCC-GARCH(1,1) covariance, and DCC with AIC selected marginals covariance matrices, we see that the magnitudes of forecast correlation changes were rather small month to month. In fact, the rolling 3-year sample correlations experienced very small incremental changes over the course of the out-of-sample period. It is also of interest to note that the magnitude of changes in correlation were extremely close between the DCC-GARCH(1,1) and DCC with AIC selected marginals methods. This similarity suggests that de-GARCHing the innovations with different univariate volatility models had little impact on the correlation dynamics.

The second group, which contains the difference norms of the exponentially-weighted, asymmetric DCC, and GARCH implied covariance matrices, experienced much greater changes in forecast correlations over the out-of-sample period. During the fourth quarter of 2008, all three of these models experienced significant changes in forecasted correlations, as evidenced by the large spikes in their Frobenius Norms.

An additional interesting observation is that the magnitudes of the A-DCC changes were substantially larger than the magnitudes of changes in the DCC models without the asymmetric parameter. This difference implies that the incorporation of an asymmetric dependence parameter in modeling correlations may lead to a forecast covariance matrix that is more responsive to the prevailing environment.

CONCLUSIONS

As discussed in Colon et al. [2011], individuals (and likely institutions and other entities) have locally stable risk tolerances. Those practitioners that continue to use the unconditional sample covariance matrix as a forecasting model will inevitably construct portfolios that are inappropriate for the end investor's risk tolerance, as the forecasted covariance matrix is simply an average of past years' volatilities and is not reflective of the current environment. Although marginally better, rolling covariance estimates also do a poor job at capturing the prevailing covariances across markets.

Instead, we believe practitioners should work to develop alternative methods for forecasting covariance, which are likely to provide superior inputs to their asset allocation models. As shown in this article, the Asymmetric DCC model of Cappiello, Engle, and Sheppard [2006] yields far superior results when forecasting covariance over one-month horizons. We have shown that those who incorporate DCC covariance forecasts are better able to manage short-term volatility by constructing portfolios that perpetually meet the risk tolerance of the investor.

APPENDIX

Exhibit 11: DCC Parameters of H_t^{DCC2} Model over time

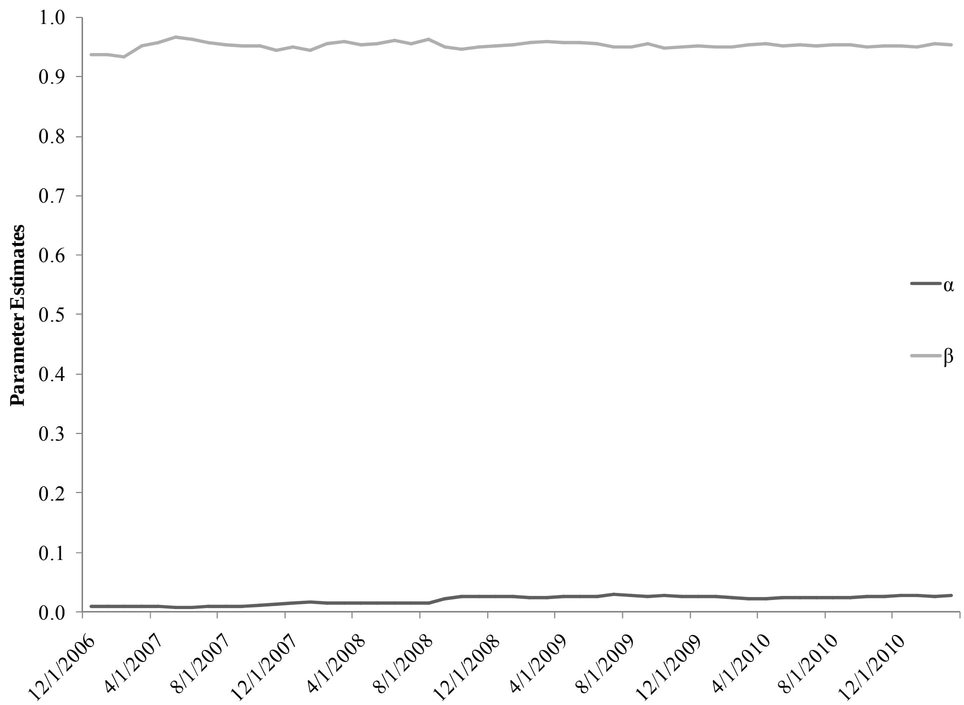


Exhibit 12: A-DCC Parameters of H_t^{A-DCC} Model over time

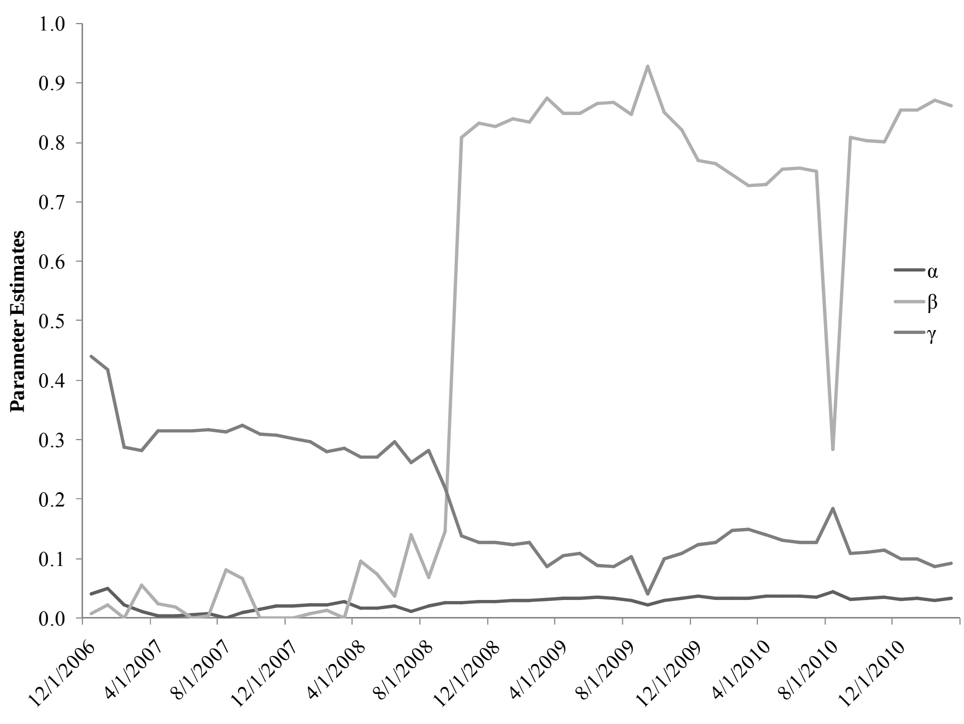
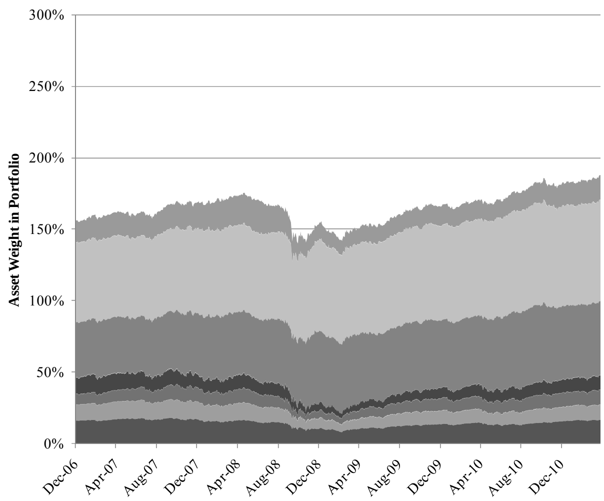
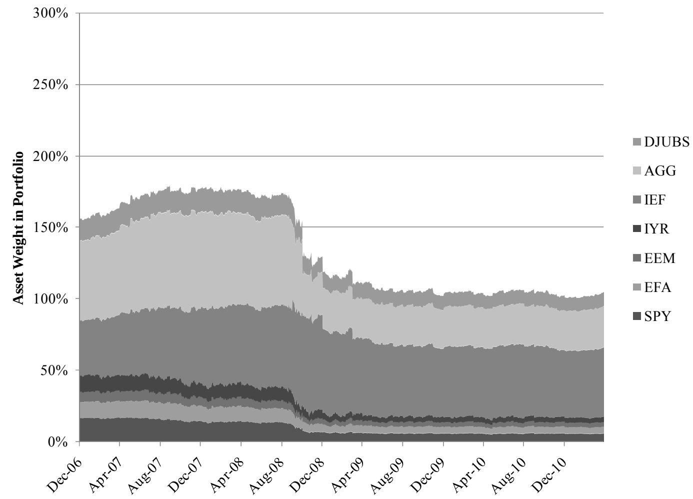


Exhibit 13: Asset allocation for each portfolio (excluding cash)

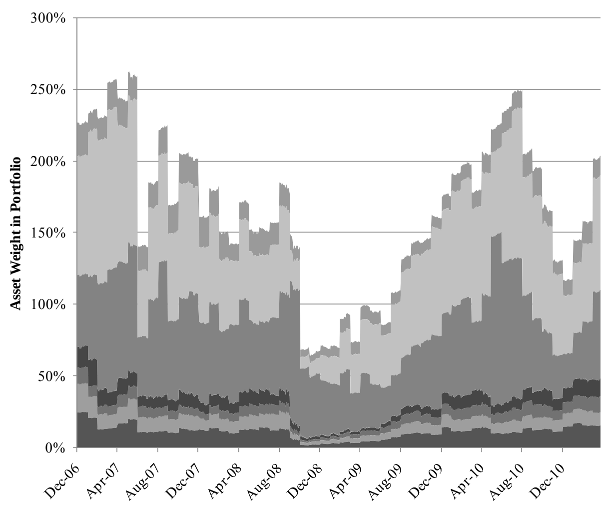
a. Unconditional Sample Cov Portfolio Wts



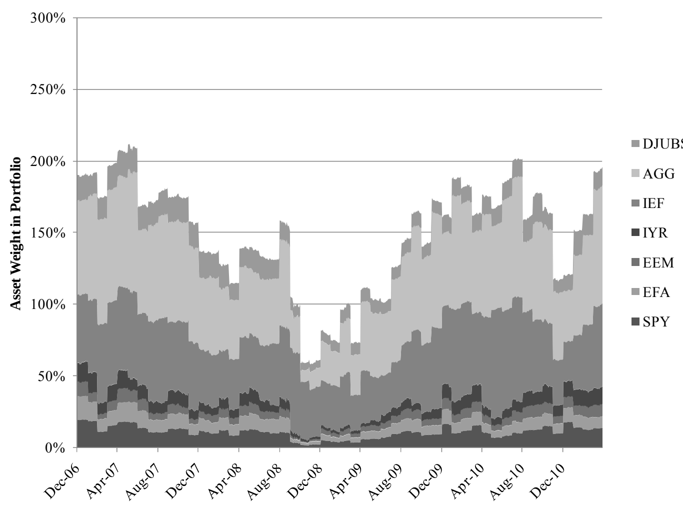
b. Rolling 3Y Sample Cov Portfolio Wts



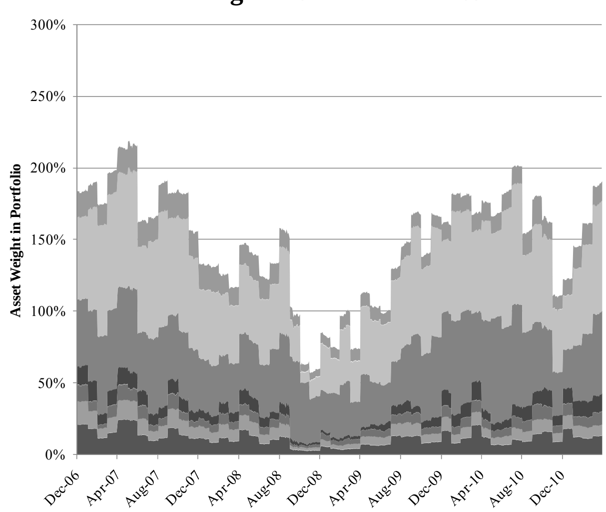
c. Expo Weighted Cov Portfolio Wts



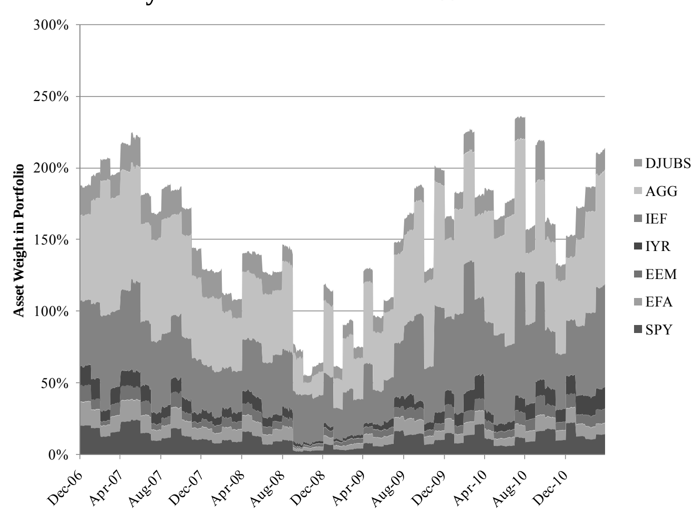
d. DCC-GARCH(1,1) Portfolio Wts



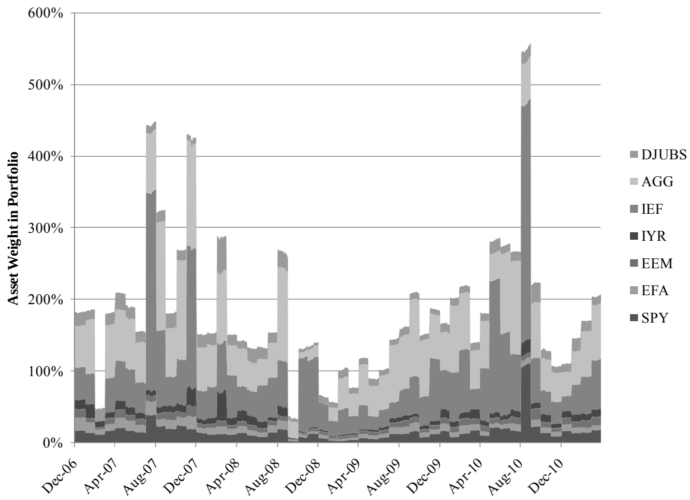
e. DCC-Marginal Vols Portfolio Wts



f. Asymmetric DCC Portfolio Wts



g. GARCH Implied Cov Portfolio Wts



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ENDNOTES

[1] We chose 8.0 since the annualized volatility of a 50% S&P 500, 50% Barclays Aggregate portfolio was 8.06 over the past 10 years (May 2001 – April 2011). Further research should consider the impact from alternative volatility targets.

[2]Log denotes the natural logarithm, unless otherwise specified. If the periodicity is monthly or greater, many subtract the risk-free return.

[3]This is due principally to the DJ UBS Commodities Index, which did not price on five of the NYSE trading dates in the sample.

[4]These are log returns.

[5]The unconditional covariance matrix $\bar{H}$ was estimated as of December 29, 2006, and was not reestimated during the out-of-sample period. The portfolio was rebalanced to the risk-parity portfolio determine on December 29, 2006 at each subsequent month end.

[6]The rolling sample covariance matrix H_t^{RS} was reestimated at each month end from December 29, 2006 through March 31, 2011 using the prior three years of daily log returns. A new risk-parity portfolio was constructed and implemented at each month end using the updated H_t^{RS}.

[7]The exponentially-weighted covariance matrix H_t^{EW} was reestimated at each month end using $\lambda = 0.96$ and the prior three years of daily log returns. A new risk-parity portfolio was constructed and implemented at each month-end using the updated H_t^{EW}.

[8]For the DCC with GARCH(1,1) marginals H_t^{DCC}, the conditional means were remodeled at each month end using a VAR(p) model, with the order determined by the Akaike Information Criterion (AIC). The univariate GARCH(1,1) models were refit at each month end as were the DCC parameters using the last three years of daily log returns. The updated covariance matrix H_t^{DCC} was then used to construct a new risk-parity portfolio, which was implemented at each month end.

[9]The DCC with AIC selected marginal volatility models H_t^{DCC2} was reestimated at each month end using the same process outlined for the DCC with GARCH(1,1) marginals, except that the marginal volatility models were selected from GARCH(1,1), GJR-GARCH(1,1), and EGARCH(1,1) models with both normal and student-t distributed errors. Selection was based on AIC.

[10]The A-DCC with AIC selected marginal volatility models H_t^{A-DCC} followed the process outlined for H_t^{DCC2} except that an extra parameter is estimated at each month end to account for asymmetric dependence.

[11]The GARCH Implied Covariance with Higham PSD Correction model H_t^{GI} was re-estimated at each month end, using AIC to select the marginal volatility models.

[12] The unconditional sample covariance matrix $\bar{H}$ was only calculated at the beginning of the out-of-sample period. Therefore, the Frobenius Norms of the change matrix at each month end were zero.