11.10 - Taylor and Maclaurin Series

Table of Maclaurin Series

$R = 1$	$R = \infty$
$\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$	$e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$	$\sin x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
$\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$\cos x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

Example 1: Given $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, evaluate $\int x^4 e^{x^2}\, dx$ as a power series.

$$\int x^4 e^{x^2}\, dx = \int x^4 \left(\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}\right) dx = \int x^4 \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\right) dx = \int \sum_{n=0}^{\infty} \frac{x^{2n+4}}{n!}\, dx = C + \sum_{n=0}^{\infty} \frac{x^{2n+5}}{(2n+5)\cdot n!}$$

Example 2: Given $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, evaluate $\int x^5 \cos(\sqrt{x})\, dx$ as a power series.

$$\int x^5 \cos(\sqrt{x})\, dx = \int x^5 \sum_{n=0}^{\infty} \frac{(-1)^n(\sqrt{x})^{2n}}{(2n)!}\, dx = \int x^5 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}\, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+5}}{(2n)!}\, dx$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+6}}{(n+6)(2n)!}$$

Recall the following regarding alternating series:

Alternating Series Estimation

If $\sum (-1)^{n-1} b_n$ satisfies

- $b_{n+1} \le b_n$ for all n
- $\lim_{n\to\infty} b_n = 0$

and $\sum (-1)^{n-1} b_n = s$, then

$|R_n| = |s - s_n| \le b_{n+1}$

In other words, the absolute value of the remainder is less than or equal to the absolute value of the first term in the remainder. Therefore, in order to approximate a sum with a given level of accuracy, we test each term in the series starting from the first term. As soon as we find a term less than the given error threshold, we discard it and add the preceding terms.

Example 3: Given $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, approximate $\int_0^{1/2} x^3 \tan^{-1} x \, dx$ with $|$ error $| < 0.00005$.

$$\int_0^{1/2} x^3 \tan^{-1} x \, dx = \int_0^{1/2} x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \, dx = \int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{2n+1} \, dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+5}}{(2n+5)(2n+1)} \right]_0^{1/2}$$

$$= \left(\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2})^{2n+5}}{(2n+5)(2n+1)} \right) - \left(\sum_{n=0}^{\infty} \frac{(-1)^n 0}{(2n+5)(2n+1)} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+5}(2n+5)(2n+1)}$$

$$\approx .00625 - .000372 + .0000434$$

$$\approx .00625 - .000372 \approx 0.00588$$

Example 4: Given $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, approximate $\int_0^1 \sin(x^4) \, dx$ with $|\text{error}| < 0.00005$.

$$\int_0^1 \sin(x^4) \, dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n (x^4)^{2n+1}}{(2n+1)!} \, dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+4}}{(2n+1)!} \, dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+5}}{(8n+5)\cdot(2n+1)!} \right]_0^1$$

$$= \left(\sum_{n=0}^{\infty} \frac{(-1)^n 1^{8n+5}}{(8n+5)\cdot(2n+1)!} \right) - \left(\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 0}{(8n+5)\cdot(2n+1)!} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(8n+5)\cdot(2n+1)!}$$

$$\approx 0.2 - 0.012821 + 0.000397 - 0.0000068$$

$$\approx 0.2 - 0.012821 + 0.000397 \approx 0.187576 \approx 0.18758$$

The following formula is an extremely useful tool for working with complex numbers, and is used in future courses. We provide the proof here using Maclaurin series.

Euler's Formula
Let x be a real number, and $i = \sqrt{-1}$. Then

$$e^{ix} = \cos x + i \sin x$$

One particular example is to plug in $x = \pi$, then add 1 to both sides of the equation to get

$$e^{i\pi} + 1 = 0$$

Proof: Recall that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and $i^5 = i$. Also, recall that if a series is absolutely convergent, then we can rearrange the order of the addition within the series without changing its sum. Then:

$$
\begin{aligned}
e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \cdots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \left(ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} + \cdots\right) \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\
&= \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right) + i\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) \\
&= \cos x + i \sin x
\end{aligned}
$$