高数上知识点总结

1. $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\frac{x}{a}+C$

$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$

$\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a}+C$

$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left|x+\sqrt{x^2-a^2}\right|+C$

$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)+C$

$\int \sqrt{a^2+x^2}dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|+C$

$\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C$

$\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C$

$\int \sec x dx = \ln|\sec x+\tan x|+C$

$\int \csc x dx = \ln|\csc x-\cot x|+C$

2. $\sin(wx+\varphi)^{(n)} = w^n\sin\left(wx+p+\frac{n}{2}\pi\right)$

$\cos(wx+\varphi)^{(n)} = w^n\cos\left(wx+p+\frac{n}{2}\pi\right)$

$\left(\frac{1}{ax+b}\right)^{(n)} = (-1)^n\frac{a^n n!}{(ax+b)^{n+1}}$

$\left(\frac{1}{a-x}\right)^{(n)} = \frac{n!}{(a-x)^{n+1}}$

$\left(e^{ax}\right)^{(n)} = a^n e^{ax}$

$(uv)^{(n)} = C_n^0 u^{(n)}v + C_n^1 u^{(n-1)}v' + \cdots + C_n^{n-1}u'v^{(n-1)} + C_n^n uv^{(n)}$

3. $\sec^2 x = 1+\tan^2 x$

$\csc^2 x = 1+\cot^2 x$

令 $t=\tan\frac{x}{2}$ — $\sin x = \frac{2t}{1+t^2}$

$x = 2\arctan t$ $\cos x = \frac{1-t^2}{1+t^2}$

$dx = \frac{2}{1+t^2}dt$

4. $\sin x \sim x$

$\arcsin x \sim x$

$\tan x \sim x$

$\arctan x \sim x$

$1-\cos x \sim \frac{1}{2}x^2$

$\log_a(x+1) \sim \frac{x}{\ln a}$

$\ln(x+1) \sim x$

$a^x - 1 \sim x\ln a$

$e^x - 1 \sim x$

$(1+x)^\alpha - 1 \sim \alpha x$

$(\tan x)' = \sec^2 x$

$(\sec x)' = \tan x\sec x$

$(\cot x)' = -\csc^2 x$

$(\csc x)' = -\cot x\csc x$

$(a^x)' = a^x\ln a$

$(\log_a x)' = \frac{1}{x\ln a}$

5. 弧微分

$ds = \sqrt{1+y'^2}dx$

$= \sqrt{\varphi'^2(t)+\psi'^2(t)}dt$

$= \sqrt{\rho^2(\theta)+\rho'^2(\theta)}d\theta$

曲率 $k = \frac{d\alpha}{ds} = \frac{|y''|}{(1+y'^2)^{\frac{3}{2}}}$

$k_{圆} = \frac{1}{R}$

6. $\frac{d^2y}{dx^2} = \frac{x_t'y_t'' - x_t''y_t'}{(x_t')^3}$

7. $\frac{A}{(x-a)^k}$, $\frac{Mx+N}{(x^2+px+q)^k}$

8. 泰勒

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

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$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$$

$$e^x=1+x+\frac{x^2}{2!}+o(x^2)$$

$$\ln(1+x)=x-\frac{x^2}{2!}+o(x^2)$$

$$\sin x=x-\frac{x^3}{3!}+o(x^4)$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+o(x^5)$$

$$\tan x=x+\frac{1}{3}x^3+o(x^4)$$

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+o(x^2)$$

$$\frac{1}{1-x}=1+x+x^2+o(x^2)$$

$$\frac{1}{1+x}=1-x+x^2+o(x^2)$$

9. $y'+P(x)y=Q(x)$

$\Rightarrow y=e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx+C\right)$ 通解

$y'+P(x)y=Q(x)y^\alpha$

$\Rightarrow z=y^{1-\alpha}$

$z'+(1-\alpha)P(x)z=(1-\alpha)Q(x)$

其他 $\begin{cases}\text{分离} \\ u=\frac{y}{x}, & u=x+a \\ & v=y+b\end{cases}$ 化可分离

降阶 $\begin{cases}y''\times y' \rightarrow P(x)=y' & y''=P' \\ y''\ y y' \rightarrow P(y)=y' & y''=P\frac{dP}{dy}\end{cases}$

$$\frac{dP}{dx}=\frac{dP}{dy}\cdot\frac{dy}{dx}=P\frac{dP}{dy}$$

10.

$y''+py'+qy=0 \to r^2+pr+q=0$ 的根：

- $\lambda_1, \lambda_2 \to y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$
- 重根 $\to y = C_1e^{\lambda x} + C_2 x e^{\lambda x}$
- $\alpha \pm \beta i \to y = C_1 e^{\alpha x}\sin\beta x + C_2 e^{\alpha x}\cos\beta x$

$y''+py'+qy=f(x)$

- $f(x)=P_m(x)e^{\lambda x}$
  - 不为根 $y=e^{\lambda x}Q_m(x)$
  - 单根 $y=xe^{\lambda x}Q_m(x)$
  - 重根 $y=x^2e^{\lambda x}Q_m(x)$
- $e^{\lambda x}[P_l(x)\cos\omega x + P_n(x)\sin\omega x]$
  - $\lambda \pm \omega i$ 不为根 $y^*=e^{\lambda x}[P_m(x)\cos\omega x + Q_m(x)\sin\omega x]$
  - 为根 $y^*=xe^{\lambda x}[P_m(x)\cos\omega x + Q_m(x)\sin\omega x]$

11. 欧拉方程

$$x^n y^{(n)} + P_1 x^{n-1} y^{(n-1)} + \cdots + P_{n-1} x y' + P_n y = f(x)$$

令 $x=e^t$，$t=\ln x \Rightarrow x^k y^{(k)} = D(D-1)\cdots(D-k+1)y$

例 $xy' = Dy = \frac{dy}{dt}$　　$x^2y'' = \frac{d^2y}{dt^2} - \frac{dy}{dt}$　　$x^3y^{(3)} = D(D-1)(D-2)$

$= D^3 - 2D^2 + 2D$

$= \frac{d^3y}{dt^3} - \frac{2d^2y}{dt^2} + \frac{2dy}{dt}$

12. 常数项 $\int_a^{+\infty}|f(x)|dx$ 收敛 $\xrightarrow[\text{不一定}]{\text{一定}}$ $\int_a^{+\infty} f(x)dx$ 收敛

小的发散 $\Rightarrow$ 大的发散

大的收敛 $\Rightarrow$ 小的收敛

$x \to +\infty$ $\lim\limits_{x\to+\infty} \frac{f(x)}{\frac{1}{x^p}} = l$

- $0<l<+\infty$ 和 $\int_a^{+\infty}\frac{1}{x^p}dx$ 敛散性相同
- $l=0$，$p>1$，$f(x)$ 收敛
- $l=+\infty$，$p\le 1$，$f(x)$ 发散

$x \to a^+$ $\lim\limits_{x\to a^+} \frac{f(x)}{\frac{1}{(x-a)^p}} = l$

- $0<l<+\infty$，和 $\int_a^{b}\frac{1}{(x-a)^p}dx$ 敛散性相同
- $l=0$，$0<p<1$，$f(x)$ 收敛
- $l=+\infty$，$p\ge 1$，$f(x)$ 发散