1. 方向导数

$$\frac{\partial f}{\partial l}\Big|_{P_0} = f_x' \cos\alpha + f_y' \cos\beta$$

2、梯度

$$\mathrm{grad} f = f_x'\vec{i} + f_y'\vec{j}$$

通量

$$\Phi = \oiint_{\Sigma} P\,dydz + Q\,dxdz + R\,dxdy$$

散度

$$\mathrm{div}\,\vec{A} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

环流量 $\oint P dx + Q dy$

旋度

$$\mathrm{rot}\,\vec{A} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

3.隐函数微分

$$F(x,y)=0 \longrightarrow y=f(x)$$

$$F(x,y,z)=0 \longrightarrow z=f(x,y)$$

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \longrightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$$

未知量 − 方程数 = 确定几元函数

以 $\frac{\partial u}{\partial x}$ 为例

$$\begin{cases} F(x,y,u(x,y),v(x,y))=0 \\ G(x,y,u(x,y),v(x,y))=0 \end{cases}$$

$$\begin{cases} F_x' + F_u'\frac{\partial u}{\partial x} + F_v'\frac{\partial v}{\partial x} = 0 \\ G_x' + G_u'\frac{\partial u}{\partial x} + G_v'\frac{\partial v}{\partial x} = 0 \end{cases}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -F_x' & F_v' \\ -G_x' & G_v' \end{vmatrix}}{\begin{vmatrix} F_u' & F_v' \\ G_u' & G_v' \end{vmatrix}} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}$$

4. 曲线切线与法平面

①参数方程

切向量 $\vec{S} = (x'(t_0), y'(t_0), z'(t_0))$

②一般方程

$$\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases} \Rightarrow \begin{cases} x=x \\ y=y(x) \\ z=z(x) \end{cases}$$

$$\begin{cases} F_x' + F_y'\frac{\partial y}{\partial x} + F_z'\frac{\partial z}{\partial x} = 0 \\ G_x' + G_y'\frac{\partial y}{\partial x} + G_z'\frac{\partial z}{\partial x} = 0 \end{cases}$$

$$\vec{S} = \left(1, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}\right) = \left(\frac{\partial(F,G)}{\partial(y,z)}, \frac{\partial(F,G)}{\partial(z,x)}, \frac{\partial(F,G)}{\partial(x,y)}\right)$$

5.曲面法线与切平面

① $F(x,y,z) \longrightarrow \vec{n} = (F_x', F_y', F_z')$

② $z=f(x,y) \longrightarrow \vec{n} = (f_x', f_y', -1)$

③ $\begin{cases} x=x(u,v) \\ y=y(u,v) \\ z=z(u,v) \end{cases} \longrightarrow \vec{n} = \left(\frac{\partial(y,z)}{\partial(u,v)}\Big|_{P_0}, \frac{\partial(z,x)}{\partial(u,v)}\Big|_{P_0}, \frac{\partial(x,y)}{\partial(u,v)}\Big|_{P_0}\right)$

6.极值判断

$$\underbrace{A}_{f''_{xx}}\underbrace{C}_{f''_{yy}} - \underbrace{B}_{f''_{xy}}{}^2$$

7.

$$\text{原函数连续} \longleftarrow \text{可微} \longrightarrow \text{偏导数存在}$$

$$\text{偏导数连续} \longrightarrow \text{可微}$$

证可微 → 验证 $\lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overbrace{f(x+\Delta x, y+\Delta y) - f(x,y)}^{\Delta z - (f_x\Delta x + f_y\Delta y)}}{\sqrt{\Delta x^2 + \Delta y^2}}$ 是否为 0

证偏导 → $\lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$ 极限存在

证方向导数 → $\lim\limits_{t \to 0} \frac{f(0+t\cos\theta, 0+t\sin\theta)}{t}$ 极限存在

8、二重积分（曲顶柱体体积）

极坐标 $d\sigma = r dr d\theta$

换元法 $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$ $\iint\limits_D f(x,y)d\sigma = \iint\limits_{D'} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du dv$

9、三重积分（物体质量）

$dV = r\, d\theta\, dr\, dz$ 柱坐标

$dV = \rho^2 \sin\theta\, d\theta\, d\varphi\, d\rho$ 球坐标

10、对弧长曲线积分（弧段质量）

$ds = \sqrt{x'^2(t) + y'^2(t)}\, dt$

对坐标曲线积分（变力做功）

$dx + dy \rightarrow x'(t)dt + y'(t)dt$

对面积曲面积分（求曲面质量）

$dS = \sqrt{1 + z_x'^2 + z_y'^2}\, dxdy$

$dS = R^2 \sin\varphi\, d\theta\, d\rho$（球坐标系）

对坐标曲面积分

合一投影 $\pm$ $(-z_x')$ $(-z_y')$ (1) $dxdy$

分面投影

11、4等价

① 对任闭曲线 $\oint_L Pdx + Qdy = 0$

② $\int_{AB} Pdx + Qdy$ 仅与起、终点有关

③ D内存在可微函数 $du = Pdx + Qdy$

④ $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ 在D内各点成立

12、格林公式（P,Q要有连续偏导数）

$$\oint_{\partial D} Pdx + Qdy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

13、高斯公式（要是闭区域）

$$\oiint_{\partial V} Pdydz + Qdxdz + Rdxdy = \iiint\limits_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz$$

14、斯托克斯公式（P.Q.R要有连续偏导数）（高斯升级）

$$\oint_{\partial S} Pdx + Qdy + Rdz = \iint\limits_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dxdz + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

15、傅里叶（周期 $2l$）

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}$$

$$\begin{cases} a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\, dx \\ b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\, dx \end{cases}$$

16、常用级数

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad x \in (-\infty, +\infty)$

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\quad x \in (-\infty, +\infty)$

$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $\quad x \in (-\infty, +\infty)$

$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} x^{n+1}$ $\quad x \in (-1, 1]$

$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$ $\quad x \in (-1, 1)$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\quad x \in (-1, 1)$

$\sqrt{1+x} = 1 + \frac{1}{2}x + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(2n-3)!!}{(2n)!!} x^n$ $\quad x \in [-1, 1]$

$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} x^n$ $\quad x \in [-1, 1]$

$(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n$ $\quad x \in (-1, 1)$ 端点看情况

$(n=0 \quad \alpha \to 0! = 1)$

微分方程（一阶）

• $y' + P(x)y = Q(x)$

通解：$y = e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx + C\right)$

• $y' + P(x)y = Q(x)y^{\alpha}$

令 $z = y^{1-\alpha}$ 原式 $\Rightarrow z' + (1-\alpha)P(x)z = (1-\alpha)Q(x)$

18. 微分方程（二阶）

• $y'' + py' + qy = 0 \rightarrow r^2 + pr + q = 0$
（通解）

- $\lambda_1, \lambda_2 \rightarrow y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$
- 重根 $\rightarrow y = C_1e^{\lambda x} + C_2xe^{\lambda x}$
- $\alpha \pm \beta i \rightarrow y = C_1e^{\alpha x}\sin\beta x + C_2e^{\alpha x}\cos\beta x$

• $y'' + py' + qy = f(x)$
（特解）

- $f(x) = P_m(x)e^{\lambda x}$
  - 不为根 $y^* = e^{\lambda x}Q_m(x)$
  - 单根 $y^* = xe^{\lambda x}Q_m(x)$
  - 重根 $y^* = x^2e^{\lambda x}Q_m(x)$
- $f(x) = e^{\lambda x}[P_l(x)\cos wx + P_n(x)\sin wx]$
  - $\lambda \pm wi$ 不为根 $y^* = e^{\lambda x}[P_m(x)\cos wx + Q_m(x)\sin wx]$
  - 为根 $y^* = xe^{\lambda x}[P_m(x)\cos wx + Q_m(x)\sin wx]$