Problem Set 2.3

① In U,

1^{st} choice of independent columns can be pivot columns,

$$\begin{bmatrix}2\\0\\0\\0\end{bmatrix}, \begin{bmatrix}3\\6\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\9\\0\end{bmatrix}$$

2^{nd} choice can be 1^{st}, 3^{rd} and 4^{th} columns as, if

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}2\\0\\0\\0\end{bmatrix} + c_2\begin{bmatrix}4\\7\\0\\0\end{bmatrix} + c_3\begin{bmatrix}1\\0\\9\\0\end{bmatrix} = \begin{bmatrix}2c_1+4c_2+c_3\\7c_2\\9c_3\\0\end{bmatrix}$$

Here, $c_2=c_3=0$ (from the second & 3^{rd} components), so the first component reduces to $0=2c_1$
$\Rightarrow \boxed{c_1=0}$

3^{rd} choice can be, 2^{nd}, ~~three~~ 3^{rd} and 4^{th} columns, if

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}3\\6\\0\\0\end{bmatrix} + c_2\begin{bmatrix}4\\7\\0\\0\end{bmatrix} + c_3\begin{bmatrix}1\\0\\9\\0\end{bmatrix} = \begin{bmatrix}3c_1+4c_2+c_3\\6c_1+7c_2\\9c_3\\0\end{bmatrix}$$

Then third component implies that $c3=0$

So, the first two component give the System of equations

$$3c_1 + 4c_2 = 0$$
$$6c_1 + 7c_2 = 0$$

$$\sim \left[\begin{array}{cc|c} 3 & 4 & 0 \\ 6 & 7 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 3 & 4 & 0 \\ 0 & -1 & 0 \end{array}\right]$$

it means c_1 & c_2 are zero too. So, these columns (2^{nd}, 3^{rd} & 4^{th}) are independent too.

★ Now since U is the echelon form of A, the choices are similar to U.

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(a) According to the question,

let $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$c_1 + 2c_2 + 3c_3 = 0$$
$$3c_1 + c_2 + 2c_3 = 0$$
$$2c_1 + 3c_2 + c_3 = 0$$

Here, $c_1 = c_2 = c_3 = 0$, Hence, the given vectors are independent.

(b) Let,

$$\begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 + 2c_2 - 3c_3 = 0$
$-3c_1 + c_2 + 2c_3 = 0$
$2c_1 - 3c_2 + c_3 = 0$

Here, $c_1 = c_2 = c_3 = 1$, Hence the given vectors are dependent.

(4) According to the question,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 + c_2 + c_3 = 0$

$c_2 + c_3 = 0$

$c_3 = 0$

So, $c_1 = c_2 = c_3 = 0$, Hence v_1, v_2 & v_3 are independent.

Now,

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 + c_2 + c_3 + 2c_4 = 0$ —— ①

$c_2 + c_3 + 3c_4 = 0$ —— ②

$c_3 + 4c_4 = 0$ —— ③

(5) According to the question,

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$, where, $c_1, c_2, c_3 \neq 0$

By substitution we get,

$c_1(w_2 - w_3) + c_2(w_1 - w_3) + c_3(w_1 - w_2) = 0$

$\Rightarrow (c_2 + c_3) w_1 + (c_1 - c_3) w_2 + (-c_1 - c_2) w_3 = 0$

So, here, $c_1 = c_3$, $c_2 = -c_3$ and $c_3 \in \mathbb{R}$, as w_1, w_2 and w_3 are LI.

Now if we take $c_3 = 1$,

$\cancel{1v_1 - 1v_2 + 1v_3 = 0}$

$1(v_1) - 1(v_2) + 1(v_3) = 0$

Hence, v_1, v_2 & v_3 are linearly dependent.

(8)

(a) $\dim(\mathbb{R}^3) = 3$

(b) v_1 is a multiple of v_2.

(c) v_1 is a multiple of $(0,0,0,0)$.

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(9) Two independent vectors on the plane $x + 2y - 3z - t = 0$ can be,

$$v_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} \text{ \& } v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

As,
$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \quad \begin{array}{l} NR_2 \to R_2 + \frac{1}{2}R_1 \\ NR_1 \to R_1 - R_4 \end{array}$$

So, $2c_1 = 0$

$c_2 = 0$

So, $\boxed{c_1 = c_2 = 0}$

So, these two are independent.

The third independent vector can be,

$$v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

As,

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \qquad \cancel{R_2 \to R_2 + \frac{1}{2} R_1}$$

$$= \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 0 & 3/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \qquad NR_2 \to R_2 + \frac{1}{2} R_1$$

$$= \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 0 & 3/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \qquad NR_1 \to R_1 - R_4$$

$$= \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \qquad NR_2 \to R_2 - \frac{3}{2} R_3$$

$NR_1 \to R_1 - R_3$

$$= \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

So, $\boxed{c_1 = c_2 = c_3 = 0}$

~~So, $c_1 = c_2 = 0$~~

So, they are independent.

We can't find 4 independent vectors as it don't span the whole $\mathbb{R}^4$,
eg.:- $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is not present.

This plane is a null space of matrix:

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 \end{bmatrix}$$

(10) According to the question,

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$\Rightarrow c_1 (w_2 + w_3) + c_2 (w_1 + w_3) + c_3 (w_1 + w_2) = 0$$

$$\Rightarrow w_1 (c_2 + c_3) + w_2 (c_1 + c_3) + w_3 (c_1 + c_2) = 0$$

As w_1, w_2 & w_3 are independent,

$$c_1 + c_2 = c_2 + c_3 = c_1 + c_3 = 0$$

So, $c_1 = c_2 = c_3 = 0$. (Ans)

(13) According to the question:-

$$c_1 + c_2 = 0$$
$$c_1 + c_4 = 0$$
$$c_2 + c_3 = 0$$
$$c_3 + c_4 = 0$$

So, $c_1 = c_2 = c_3 = c_4 = 0$, hence the vectors are LI.

Now,

$$c_1 + c_2 = 0 \quad \text{—(1)}$$

$$c_1 + c_4 = 0 \quad \text{—(2)}$$

$$c_2 + c_3 = 0 \quad \text{—(3)}$$

$$c_3 + c_4 = 1 \quad \text{—(4)}$$

Here, $c_2 = c_4$,

So, $c_3 + c_4 = 0$ ($\because c_2 = c_4$) in (3)

But $c_3 + c_4 = 1$ in (4).

So, it is not solvable, hence it doesn't span $\mathbb{R}^4$.

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(19) ~~Basis~~ Basis for the plane $x-2y+3z=0$ in $\mathbb{R}^3$ can be,

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Now, the basis that intersects with xy-plane is

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ as here, } z=0.$$

Now the basis of all vectors perpendicular to the plane is,

$$c\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$ as we are looking for vectors $\perp$er to $(1,-2,3)$.

(23) ~~basis for C(A)~~

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $C(A) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

$C(A^T) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$N(A) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

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$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$$

for $d=2$ and all values of c matrix A will have rank 2.

$B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ for $c \in \mathbb{R}$ and $d=0$ or vice versa this matrix has rank 2.

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(31)

(a) True

(b) False

(40) Here,

(c) $(1,2,2)$, $(-1,2,1)$, $(0,8,0)$ are the bases for R^3.