$$\det P \det A = \det L \det D \det U \qquad (3)$$

Transposing $PA = LDU$ gives $A^{\mathrm{T}} P^{\mathrm{T}} = U^{\mathrm{T}} D^{\mathrm{T}} L^{\mathrm{T}}$, and again by rule 9,

$$\det A^{\mathrm{T}} \det P^{\mathrm{T}} = \det U^{\mathrm{T}} \det D^{\mathrm{T}} \det L^{\mathrm{T}}. \qquad (4)$$

This is simpler than it looks, because L, U, L^{T}, and U^{T} are triangular with unit diagonal. By rule 7, their determinants all equal 1. Also, any diagonal matrix is the same as its transpose: $D = D^{\mathrm{T}}$. We only have to show that $\det P = \det P^{\mathrm{T}}$.

Certainly $\det P$ is 1 or -1, because P comes from I by row exchanges. Observe also that $PP^{\mathrm{T}} = I$. (The 1 in the first row of P matches the 1 in the first column of P^{T}, and misses the 1s in the other columns.) Therefore $\det P \det P^{\mathrm{T}} = \det I = 1$, and P and P^{T} must have the same determinant: both 1 or both -1.

We conclude that the products (3) and (4) are the same, and $\det A = \det A^{\mathrm{T}}$. This fact practically doubles our list of properties, because every rule that applied to the rows can now be applied to the columns: *The determinant changes sign when two columns are exchanged, two equal columns (or a column of zeros) produce a zero determinant, and the determinant depends linearly on each individual column.* The proof is just to transpose the matrix and work with the rows.

I think it is time to stop and call the list complete. It only remains to find a definite formula for the determinant, and to put that formula to use.

Problem Set 4.2

1. *Row exchange*: Add row 1 of A to row 2, then subtract row 2 from row 1. Then add row 1 to row 2 and multiply row 1 by -1 to reach B. Which rules show the following?

$$\det B = \begin{vmatrix} c & d \\ a & b \end{vmatrix} \quad \text{equals} \quad -\det A = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Those rules could replace Rule 2 in the definition of the determinant.

2. If a 4 by 4 matrix has $\det A = \frac{1}{2}$, find $\det(2A)$, $\det(-A)$, $\det(A^2)$, and $\det(A^{-1})$.

3. If a 3 by 3 matrix has $\det A = -1$, find $\det\left(\frac{1}{2}A\right)$, $\det(-A)$, $\det(A^2)$, and $\det(A^{-1})$.

4. By applying row operations to produce an upper triangular U, compute

$$\det \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix} \quad \text{and} \quad \det \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}.$$

Exchange rows 3 and 4 of the second matrix and recompute the pivots and determinant.

Note Some readers will already know a formula for 3 by 3 determinants. It has six terms (equation (2) of the next section), three going parallel to the main diagonal and three others going the opposite way with minus signs. There is a similar formula for 4 by 4 determinants, ***but it contains*** $4! = 24$ ***terms*** (*not just eight*). You cannot even be sure that a minus sign goes with the reverse diagonal, as the next exercises show.

5. Find the determinants of:
   (a) a rank one matrix
$$A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}.$$
   (b) the upper triangular matrix
$$U = \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$
   (c) the lower triangular matrix U^{T}.
   (d) the inverse matrix U^{-1}.
   (e) the "reverse-triangular" matrix that results from row exchanges,
$$M = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 4 & 4 & 8 & 8 \end{bmatrix}.$$

6. Suppose you do two row operations *at once*, going from
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} a - mc & b - md \\ c - \ell a & d - \ell b \end{bmatrix}.$$
   Find the determinant of the new matrix, by rule 3 or by direct calculation.

7. Count row exchanges to find these determinants:
$$\det \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = +1 \quad \text{and} \quad \det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = -1.$$

8. If Q is an orthogonal matrix, so that $Q^{\mathrm{T}}Q = I$, prove that $\det Q$ equals $+1$ or -1. What kind of box is formed from the rows (or columns) of Q?

9. For each n, how many exchanges will put (row n, row $n-1$, ..., row 1) into the normal order (row 1, ..., row $n-1$, row n)? Find $\det P$ for the n by n permutation with 1s on the reverse diagonal. Problem 7 had $n = 4$.

10. Prove again that $\det Q = 1$ or -1 using only the product rule. If $|\det Q| > 1$ then $\det Q^n$ blows up. How do you know this can't happen to Q^n?

11. Show how rule 6 ($\det = 0$ if a row is zero) comes directly from rules 2 and 3.

12. True or false, with reason if true and counterexample if false:

(a) If A and B are identical except that $b_{11} = 2a_{11}$, then $\det B = 2 \det A$.
(b) The determinant is the product of the pivots.
(c) If A is invertible and B is singular, then $A + B$ is invertible.
(d) If A is invertible and B is singular, then AB is singular.
(e) The determinant of $AB - BA$ is zero.

13. Find the determinants of

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}, \quad A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}, \quad A - \lambda I = \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}.$$

For which values of λ is $A - \lambda I$ a singular matrix?

14. If every row of A adds to zero, prove that $\det A = 0$. If every row adds to 1, prove that $\det(A - I) = 0$. Show by example that this does not imply $\det A = 1$.

15. (a) A skew-symmetric matrix satisfies $K^{\mathrm{T}} = -K$, as in

$$K = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

In the 3 by 3 case, why is $\det(-K) = (-1)^3 \det K$? On the other hand $\det K^{\mathrm{T}} = \det K$ (always). Deduce that the determinant must be zero.

(b) Write down a 4 by 4 skew-symmetric matrix with $\det K$ *not* zero.

16. Evaluate $\det A$ by reducing the matrix to triangular form (rules 5 and 7).

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 1 & 5 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 1 & 5 & 9 \end{bmatrix}.$$

What are the determinants of B, C, AB, $A^{\mathrm{T}}A$, and C^{T}?

17. Suppose that $CD = -DC$, and find the flaw in the following argument: Taking determinants gives $(\det C)(\det D) = -(\det D)(\det C)$, so either $\det C = 0$ or $\det D = 0$. Thus $CD = -DC$ is only possible if C or D is singular.

18. Use row operations to verify that the 3 by 3 "Vandermonde determinant" is

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$$

19. Find these 4 by 4 determinants by Gaussian elimination:

$$\det \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix} \quad \text{and} \quad \det \begin{bmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{bmatrix}.$$

20. The inverse of a 2 by 2 matrix seems to have determinant = 1:

$$\det A^{-1} = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad-bc}{ad-bc} = 1.$$

What is wrong with this calculation? What is the correct $\det A^{-1}$?

21. Do these matrices have determinant 0, 1, 2, or 3?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Problems 22–28 use the rules to compute specific determinants.

22. If a_{ij} is i times j, show that $\det A = 0$. (Exception when $A = [1]$.)

23. Use row operations to simplify and compute these determinants:

$$\det \begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix} \quad \text{and} \quad \det \begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}.$$

24. If a_{ij} is $i + j$, show that $\det A = 0$. (Exception when $n = 1$ or 2.)

25. Reduce A to U and find $\det A$ = product of the pivots:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}.$$

26. Compute the determinants of these matrices by row operations:

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}.$$

27. Elimination reduces A to U. Then $A = LU$:

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = LU.$$

Find the determinants of L, U, A, $U^{-1}L^{-1}$, and $U^{-1}L^{-1}A$.

28. By applying row operations to produce an upper triangular U, compute

$$\det \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} \quad \text{and} \quad \det \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

29. What is wrong with this proof that projection matrices have $\det P = 1$?

$$P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} \quad \text{so} \quad |P| = |A|\frac{1}{|A^{\mathrm{T}}||A|}|A^{\mathrm{T}}| = 1.$$

30. (MATLAB) What is a typical determinant (experimentally) of **rand**(n) and **randn**(n) for $n = 50, 100, 200, 400$? (And what does "Inf" mean in MATLAB?)

31. Using MATLAB, find the largest determinant of a 4 by 4 matrix of 0s and 1s.

32. Suppose the 4 by 4 matrix M has four equal rows all containing a, b, c, d. We know that $\det(M) = 0$. The problem is to find $\det(I + M)$ by any method:

$$\det(I + M) = \begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}.$$

Partial credit if you find this determinant when $a = b = c = d = 1$. Sudden death if you say that $\det(I + M) = \det I + \det M$.

33. If you know that $\det A = 6$, what is the determinant of B?

$$\det A = \begin{vmatrix} \text{row } 1 \\ \text{row } 2 \\ \text{row } 3 \end{vmatrix} = 6 \qquad \det B = \begin{vmatrix} \text{row } 1 + \text{row } 2 \\ \text{row } 2 + \text{row } 3 \\ \text{row } 3 + \text{row } 1 \end{vmatrix} = \_\_$$

34. (Calculus question) Show that the partial derivatives of $\ln(\det A)$ give A^{-1}:

$$f(a, b, c, d) = \ln(ad - bc) \qquad \text{leads to} \qquad \begin{bmatrix} \partial f/\partial a & \partial f/\partial c \\ \partial f/\partial b & \partial f/\partial d \end{bmatrix} = A^{-1}.$$

35. (MATLAB) The Hilbert matrix **hilb**(n) has i, j entry equal to $1/(i + j - 1)$. Print the determinants of **hilb**(1), **hilb**(2), ..., **hilb**(10). Hilbert matrices are hard to work with! What are the pivots?

4.3 FORMULAS FOR THE DETERMINANT

The first formula has already appeared. Row operations produce the pivots in D:

4A If A is invertible, then $PA = LDU$ and $\det P = \pm 1$. The product rule gives

$$\det A = \pm \det L \det D \det U = \pm(\textbf{product of the pivots}). \tag{1}$$

The sign ± 1 depends on whether the number of row exchanges is even or odd. The triangular factors have $\det L = \det U = 1$ and $\det D = d_1 \cdots d_n$.

In the 2 by 2 case, the standard LDU factorization is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \ldots \end{bmatrix} \begin{bmatrix} a & \ldots \end{bmatrix}$$

This gives the determinant of increasingly bigger matrices. At every step the determinant of A_n is $n+1$, from the previous determinants n and $n-1$:

−1, 2, −1 matrix $\qquad \det A_n = 2(n) - (n-1) = n+1\,.$

The answer $n+1$ agrees with the product of pivots at the start of this section.

Problem Set 4.3

1. *True or false?*
   (a) The determinant of $S^{-1}AS$ equals the determinant of A.
   (b) If $\det A = 0$ then at least one of the cofactors must be zero.
   (c) A matrix whose entries are 0s and 1s has determinant 1, 0, or −1.

2. Let F_n be the determinant of the 1, 1, −1 tridiagonal matrix (n by n):

$$F_n = \det\begin{bmatrix} 1 & -1 & & & \\ 1 & 1 & -1 & & \\ & 1 & 1 & -1 & \\ & & \cdot & \cdot & \cdot \\ & & & 1 & 1 \end{bmatrix}.$$

By expanding in cofactors along row 1, show that $F_n = F_{n-1} + F_{n-2}$. This yields the *Fibonacci sequence* 1, 2, 3, 5, 8, 13, ... for the determinants.

3. For these matrices, find the only nonzero term in the big formula (6):

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

There is only one way of choosing four nonzero entries from different rows and different columns. By deciding even or odd, compute $\det A$ and $\det B$.

4. Expand those determinants in cofactors of the first row. Find the cofactors (they include the signs $(-1)^{i+j}$) and the determinants of A and B.

5. Suppose A_n is the n by n tridiagonal matrix with 1s on the three diagonals:

$$A_1 = [1], \qquad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \ldots$$

Let D_n be the determinant of A_n; we want to find it.
   (a) Expand in cofactors along the first row to show that $D_n = D_{n-1} - D_{n-2}$.
   (b) Starting from $D_1 = 1$ and $D_2 = 0$, find $D_3, D_4, \ldots, D_8$. By noticing how these numbers cycle around (with what period?) find D_{1000}.

6. (a) Find the LU factorization, the pivots, and the determinant of the 4 by 4 matrix whose entries are a_{ij} = smaller of i and j. (Write out the matrix.)
   (b) Find the determinant if a_{ij} = smaller of n_i and n_j, where $n_1 = 2$, $n_2 = 6$, $n_3 = 8$, $n_4 = 10$. Can you give a general rule for any $n_1 \le n_2 \le n_3 \le n_4$?

7. In a 5 by 5 matrix, does a $+$ sign or $-$ sign go with $a_{15}a_{24}a_{33}a_{42}a_{51}$ down the reverse diagonal? In other words, is $P = (5, 4, 3, 2, 1)$ even or odd? The checkerboard pattern of $\pm$ signs for cofactors does *not* give $\det P$.

8. How many multiplications to find an n by n determinant from
   (a) the big formula (6)?
   (b) the cofactor formula (10), building from the count for $n-1$?
   (c) the product of pivots formula (including the elimination steps)?

9. Suppose the matrix A is fixed, except that a_{11} varies from $-\infty$ to $+\infty$. Give examples in which $\det A$ is always zero or never zero. Then show from the cofactor expansion (8) that otherwise $\det A = 0$ for exactly *one value* of a_{11}.

10. Compute the determinants of A_2, A_3, A_4. Can you predict A_n?

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Use row operations to produce zeros, or use cofactors of row 1.

11. (a) Evaluate this determinant by cofactors of row 1:

$$\begin{vmatrix} 4 & 4 & 4 & 4 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \end{vmatrix}$$

(b) Check by subtracting column 1 from the other columns and recomputing.

12. If A is m by n and B is n by m, explain why

$$\det \begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} = \det AB. \qquad \left(\textit{Hint}\text{: Postmultiply by } \begin{bmatrix} I & 0 \\ B & I \end{bmatrix}. \right)$$

Do an example with $m < n$ and an example with $m > n$. Why does your second example automatically have $\det AB = 0$?

Problems 13–23 use the big formula with $n!$ terms: $|A| = \sum \pm a_{1\alpha}a_{2\beta}\cdots a_{n\nu}$.

13. Compute the determinants of A, B, C. Are their columns independent?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

14. Compute the determinants of A, B, C from six terms. Independent rows?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 6 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

15. Place the smallest number of zeros in a 4 by 4 matrix that will guarantee det $A = 0$. Place as many zeros as possible while still allowing det $A \neq 0$.

16. How many 5 by 5 permutation matrices have det $P = +1$? Those are even permutations. Find one that needs four exchanges to reach the identity matrix.

17. This problem shows in two ways that det $A = 0$ (the x's are any numbers):

$$A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}. \qquad \begin{array}{l} \text{5 by 5 matrix} \\ \text{3 by 3 zero matrix} \\ \text{Always singular} \end{array}$$

(a) How do you know that the rows are linearly dependent?
(b) Explain why all 120 terms are zero in the big formula for det A.

18. Prove that 4 is the largest determinant for a 3 by 3 matrix of 1s and -1s.

19. If det $A \neq 0$, at least one of the $n!$ terms in the big formula (6) is not zero. Deduce that some ordering of the rows of A leaves no zeros on the diagonal. (Don't use P from elimination; that PA can have zeros on the diagonal.)

20. Show that det $A = 0$, regardless of the five nonzeros marked by x's:

$$A = \begin{bmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}. \qquad \text{(What is the rank of } A\text{?)}$$

21. (a) If $a_{11} = a_{22} = a_{33} = 0$, how many of the six terms in det A will be zero?
(b) If $a_{11} = a_{22} = a_{33} = a_{44} = 0$, how many of the 24 products $a_{1j}a_{2k}a_{3\ell}a_{4m}$ are sure to be zero?

22. How many permutations of $(1, 2, 3, 4)$ are even and what are they? Extra credit: What are all the possible 4 by 4 determinants of $I + P_{\text{even}}$?

23. Find two ways to choose nonzeros from four different rows and columns:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}. \quad (B \text{ has the same zeros as } A.)$$

Is det A equal to $1 + 1$ or $1 - 1$ or $-1 - 1$? What is det B?

Problems 24–33 use cofactors $C_{ij} = (-1)^{i+j} \det M_{ij}$. Delete row i, column j.

24. Find cofactors and then transpose. Multiply C_A^{T} and C_B^{T} by A and B!

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}.$$

25. Explain why this Vandermonde determinant contains x^3 but not x^4 or x^5:

$$V_4 = \det\begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & x & x^2 & x^3 \end{bmatrix}.$$

The determinant is zero at $x =$ ___, ___, and ___. The cofactor of x^3 is $V_3 = (b-a)(c-a)(c-b)$. Then $V_4 = (x-a)(x-b)(x-c)V_3$.

26. B_n is still the same as A_n except for $b_{11} = 1$. So use linearity in the first row, where $[1\ \ -1\ \ 0]$ equals $[2\ \ -1\ \ 0]$ minus $[1\ \ 0\ \ 0]$:

$$|B_n| = \begin{vmatrix} 1 & -1 & & 0 \\ -1 & & & \\ & & A_{n-1} & \\ 0 & & & \end{vmatrix} = \begin{vmatrix} 2 & -1 & & 0 \\ -1 & & & \\ & & A_{n-1} & \\ 0 & & & \end{vmatrix} - \begin{vmatrix} 1 & 0 & & 0 \\ -1 & & & \\ & & A_{n-1} & \\ 0 & & & \end{vmatrix}.$$

Linearity in row 1 gives $|B_n| = |A_n| - |A_{n-1}| =$ ___.

27. Find the cofactor matrix C and compare AC^{T} with A^{-1}:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

28. Problem 30 has 1s just above and below the main diagonal. Going down the matrix, which order of columns (if any) gives all 1s? Explain why that permutation is *even* for $n = 4, 8, 12, \ldots$ and *odd* for $n = 2, 6, 10, \ldots$

$$C_n = 0 \text{ (odd } n) \qquad C_n = 1\ (n = 4, 8, \ldots) \qquad C_n = -1\ (n = 2, 6, \ldots).$$

29. The matrix B_n is the $-1, 2, -1$ matrix A_n except that $b_{11} = 1$ instead of $a_{11} = 2$. Using cofactors of the *last* row of B_4, show that $|B_4| = 2|B_3| - |B_2| = 1$:

$$B_4 = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \qquad B_3 = \begin{bmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix}.$$

The recursion $|B_n| = 2|B_{n-1}| - |B_{n-2}|$ is the same as for the A's. The difference is in the starting values 1, 1, 1 for $n = 1, 2, 3$. *What are the pivots?*

30. The n by n determinant C_n has 1s above and below the main diagonal:

$$C_1 = |0| \quad C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \quad C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

(a) What are the determinants of C_1, C_2, C_3, C_4?

(b) By cofactors find the relation between C_n and C_{n-1} and C_{n-2}. Find C_{10}.

31. Change 3 to 2 in the upper left corner of the matrices in Problem 33. Why does that subtract S_{n-1} from the determinant S_n? Show that the determinants become the Fibonacci numbers 2, 5, 13 (always F_{2n+1}).

32. Compute the determinants S_1, S_2, S_3 of these 1, 3, 1 tridiagonal matrices:

$$S_1 = |3| \qquad S_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \qquad S_3 = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix}.$$

Make a Fibonacci guess for S_4 and verify that you are right.

33. Cofactors of those 1, 3, 1 matrices give $S_n = 3S_{n-1} - S_{n-2}$. Challenge: *Show that S_n is the Fibonacci number F_{2n+2} by proving $F_{2n+2} = 3F_{2n} - F_{2n-2}$.* Keep using Fibonacci's rule $F_k = F_{k-1} + F_{k-2}$.

Problems 34–36 are about block matrices and block determinants.

34. Block elimination subtracts CA^{-1} times the first row $[A \ \ B]$ from the second row $[C \ \ D]$. This leaves the *Schur complement* $D - CA^{-1}B$ in the corner:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}.$$

Take determinants of these matrices to prove correct rules for square blocks:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \underset{\text{if } A^{-1} \text{ exists}}{|A|\,|D - CA^{-1}B|} = \underset{\text{if } AC = CA}{|AD - CB|}.$$

35. With 2 by 2 blocks, you cannot always use block determinants!

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A|\,|D| \quad \text{but} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A|\,|D| - |C|\,|B|.$$

(a) Why is the first statement true? Somehow B doesn't enter.
(b) Show by example that equality fails (as shown) when C enters.
(c) Show by example that the answer $\det(AD - CB)$ is also wrong.

36. With block multiplication, $A = LU$ has $A_k = L_kU_k$ in the upper left corner:

$$A = \begin{bmatrix} A_k & * \\ * & * \end{bmatrix} = \begin{bmatrix} L_k & 0 \\ * & * \end{bmatrix} \begin{bmatrix} U_k & * \\ 0 & * \end{bmatrix}.$$

(a) Suppose the first three pivots of A are $2, 3, -1$. What are the determinants of L_1, L_2, L_3 (with diagonal 1s), U_1, U_2, U_3, and A_1, A_2, A_3?
(b) If A_1, A_2, A_3 have determinants 5, 6, 7, find the three pivots.

37. A 3 by 3 determinant has three products "down to the right" and three "down to the left" with minus signs. Compute the six terms in the figure to find D. Then explain without determinants why this matrix is or is not invertible:

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \begin{matrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{matrix}$$

$$- \quad - \quad - \qquad + \quad + \quad +$$

38. $A = 2 * \text{eye}(n) - \text{diag}(\text{ones}(n-1, 1), 1) - \text{diag}(\text{ones}(n-1, 1), -1)$ is the $-1, 2, -1$ matrix. Change $A(1, 1)$ to 1 so $\det A = 1$. Predict the entries of A^{-1} based on $n = 3$ and test the prediction for $n = 4$.

39. For A_4 in Problem 5, five of the $4! = 24$ terms in the big formula (6) are nonzero. Find those five terms to show that $D_4 = -1$.

40. For the 4 by 4 tridiagonal matrix (entries $-1, 2, -1$), find the five terms in the big formula that give $\det A = 16 - 4 - 4 - 4 + 1$.

41. All **Pascal matrices** have determinant 1. If I subtract 1 from the n, n entry, why does the determinant become zero? (Use rule 3 or a cofactor.)

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & \mathbf{20} \end{bmatrix} = 1 \text{ (known)} \qquad \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & \mathbf{19} \end{bmatrix} = \mathbf{0} \text{ (explain)}.$$

42. Find the determinant of this cyclic P by cofactors of row 1. How many exchanges reorder 4, 1, 2, 3 into 1, 2, 3, 4? Is $|P^2| = +1$ or -1?

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

43. (MATLAB) The $-1, 2, -1$ matrices have determinant $n + 1$. Compute $(n + 1)A^{-1}$ for $n = 3$ and 4, and verify your guess for $n = 5$. (Inverses of tridiagonal matrices have the rank-1 form uv^{T} above the diagonal.)

4.4 APPLICATIONS OF DETERMINANTS

This section follows through on four major applications: *inverse of A, solving $Ax = b$, volumes of boxes*, and *pivots*. They are among the key computations in linear algebra (done by elimination). Determinants give formulas for the answers.

1. Computation of A^{-1}. The 2 by 2 case shows how cofactors go into A^{-1}:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}.$$

We are dividing by the determinant, and A is invertible exactly when $\det A$ is nonzero. The number $C_{11} = d$ is the cofactor of a. The number $C_{12} = -c$ is the cofactor of b

That does it for determinants, except for an optional remark on property 2—the sign reversal on row exchanges. The ***determinant of a permutation matrix P*** was the only questionable point in the big formula. Independent of the particular row exchanges linking P to I, is the number of exchanges always even or always odd? If so, its determinant is well defined by rule 2 as either $+1$ or -1.

Starting from $(3, 2, 1)$, a single exchange of 3 and 1 would achieve the natural order $(1, 2, 3)$. So would an exchange of 3 and 2, then 3 and 1, and then 2 and 1. In both sequences, the number of exchanges is odd. The assertion is that *an even number of exchanges can never produce the natural order, beginning with* $(3, 2, 1)$.

Here is a proof. Look at each pair of numbers in the permutation, and let N count the pairs in which the larger number comes first. Certainly $N = 0$ for the natural order $(1, 2, 3)$. The order $(3, 2, 1)$ has $N = 3$ since all pairs $(3, 2)$, $(3, 1)$, and $(2, 1)$ are wrong. We will show that *every exchange alters N by an odd number*. Then to arrive at $N = 0$ (the natural order) takes a number of exchanges having the same evenness or oddness as N.

When neighbors are exchanged, N changes by $+1$ or -1. *Any exchange can be achieved by an odd number of exchanges of neighbors*. This will complete the proof; an odd number of odd numbers is odd. To exchange the first and fourth entries below, which happen to be 2 and 3, we use five exchanges (an odd number) of neighbors:

$$(\mathbf{2}, 1, 4, \mathbf{3}) \to (1, \mathbf{2}, 4, \mathbf{3}) \to (1, 4, \mathbf{2}, \mathbf{3}) \to (1, 4, \mathbf{3}, 2) \to (1, \mathbf{3}, 4, 2) \to (\mathbf{3}, 1, 4, 2).$$

We need $\ell - k$ exchanges of neighbors to move the entry in place k to place ℓ. Then $\ell - k - 1$ exchanges move the one originally in place ℓ (and now found in place $\ell - 1$) back down to place k. Since $(\ell - k) + (\ell - k - 1)$ is odd, the proof is complete. The determinant not only has all the properties found earlier, it even exists.

Problem Set 4.4

1. Find the determinant and all nine cofactors C_{ij} of this triangular matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Form C^{T} and verify that $AC^{\mathrm{T}} = (\det A)I$. What is A^{-1}?

2. (a) Draw the triangle with vertices $A = (2, 2)$, $B = (-1, 3)$, and $C = (0, 0)$. By regarding it as half of a parallelogram, explain why its area equals

$$\text{area}(ABC) = \frac{1}{2}\det\begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix}.$$

(b) Move the third vertex to $C = (1, -4)$ and justify the formula

$$\text{area}(ABC) = \frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = \frac{1}{2}\det\begin{bmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & -4 & 1 \end{bmatrix}.$$

Hint: Subtracting the last row from each of the others leaves

$$\det\begin{bmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & -4 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 6 & 0 \\ -2 & 7 & 0 \\ 1 & -4 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 6 \\ -2 & 7 \end{bmatrix}.$$

Sketch $A' = (1, 6)$, $B' = (-2, 7)$, $C' = (0, 0)$ and their relation to A, B, C.

3. Predict in advance, and confirm by elimination, the pivot entries of

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 5 & 0 \\ 2 & 7 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 5 & 3 \\ 2 & 7 & 0 \end{bmatrix}.$$

4. Find all the odd permutations of the numbers $\{1, 2, 3, 4\}$. They come from an odd number of exchanges and lead to $\det P = -1$.

5. Use the cofactor matrix C to invert these symmetric matrices:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

6. Suppose the permutation P takes $(1, 2, 3, 4, 5)$ to $(5, 4, 1, 2, 3)$.

(a) What does P^2 do to $(1, 2, 3, 4, 5)$?

(b) What does P^{-1} do to $(1, 2, 3, 4, 5)$?

7. Find x, y, and z by Cramer's Rule in equation (4):

$$\begin{aligned} ax + by &= 1 \\ cx + dy &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} x + 4y - z &= 1 \\ x + y + z &= 0 \\ 2x \qquad + 3z &= 0. \end{aligned}$$

8. Explain in terms of volumes why $\det 3A = 3^n \det A$ for an n by n matrix A.

9. (a) Find the determinant when a vector x replaces column j of the identity (consider $x_j = 0$ as a separate case):

$$\text{if } M = \begin{bmatrix} 1 & & x_1 & & \\ & 1 & \cdot & & \\ & & x_j & & \\ & & \cdot & 1 & \\ & & x_n & & 1 \end{bmatrix} \quad \text{then } \det M = \_\_\_.$$

(b) If $Ax = b$, show that AM is the matrix B_j in equation (4), with b in column j.

(c) Derive *Cramer's rule* by taking determinants in $AM = B_j$.

10. Prove that if you keep multiplying A by the same permutation matrix P, the first row eventually comes back to its original place.

11. If A is a 5 by 5 matrix with all $|a_{ij}| \le 1$, then $\det A \le$ ___. Volumes or the big formula or pivots should give some upper bound on the determinant.

12. If P is an odd permutation, explain why P^2 is even but P^{-1} is odd.

Problems 13–17 are about Cramer's Rule for $x = A^{-1}b$.

13. *Quick proof of Cramer's rule.* The determinant is a linear function of column 1. It is zero if two columns are equal. When $b = Ax = x_1 a_1 + x_2 a_2 + x_3 a_3$ goes into column 1 to produce B_1, the determinant is

$$\begin{vmatrix} b & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} x_1 a_1 + x_2 a_2 + x_3 a_3 & a_2 & a_3 \end{vmatrix} = x_1 \begin{vmatrix} a_1 & a_2 & a_3 \end{vmatrix} = x_1 \det A.$$

(a) What formula for x_1 comes from left side = right side?
(b) What steps lead to the middle equation?

14. Solve these linear equations by Cramer's Rule $x_j = \det B_j / \det A$:

$$\text{(a)}\quad \begin{aligned} 2x_1 + 5x_2 &= 1 \\ x_1 + 4x_2 &= 2. \end{aligned} \qquad \text{(b)}\quad \begin{aligned} 2x_1 + x_2 \phantom{{}+x_3} &= 1 \\ x_1 + 2x_2 + x_3 &= 0 \\ x_2 + 2x_3 &= 0. \end{aligned}$$

15. If the right side b is the *last column* of A, solve the 3 by 3 system $Ax = b$. Explain how each determinant in Cramer's Rule leads to your solution x.

16. Cramer's Rule breaks down when $\det A = 0$. Example (a) has no solution, whereas (b) has infinitely many. What are the ratios $x_j = \det B_j / \det A$?

$$\text{(a)}\quad \begin{aligned} 2x_1 + 3x_2 &= 1 \\ 4x_1 + 6x_2 &= 1. \end{aligned} \quad \text{(parallel lines)} \qquad \text{(b)}\quad \begin{aligned} 2x_1 + 3x_2 &= 1 \\ 4x_1 + 6x_2 &= 2. \end{aligned} \quad \text{(same line)}$$

17. Use Cramer's Rule to solve for y (only). Call the 3 by 3 determinant D:

$$\text{(a)}\quad \begin{aligned} ax + by &= 1 \\ cx + dy &= 0. \end{aligned} \qquad \text{(b)}\quad \begin{aligned} ax + by + cz &= 1 \\ dx + ey + fz &= 0 \\ gx + hy + iz &= 0. \end{aligned}$$

Problems 18–26 are about $A^{-1} = C^{\mathrm{T}} / \det A$. Remember to transpose C.

18. If all the cofactors are zero, how do you know that A has no inverse? If none of the cofactors are zero, is A sure to be invertible?

19. Find the cofactors of A and multiply AC^{T} to find $\det A$:

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}, \qquad C = \begin{bmatrix} 6 & -3 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}, \qquad \text{and} \quad AC^{\mathrm{T}} = \underline{\qquad}.$$

If you change that corner entry from 4 to 100, why is $\det A$ unchanged?

20. Find A^{-1} from the cofactor formula $C^{\mathrm{T}} / \det A$. Use symmetry in part (b):

$$\text{(a)}\ A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}. \qquad \text{(b)}\ A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

21. From the formula $AC^T = (\det A)I$ show that $\det C = (\det A)^{n-1}$.

22. If all entries of A are integers, and $\det A = 1$ or -1, prove that all entries of A^{-1} are integers. Give a 2 by 2 example.

23. Suppose $\det A = 1$ and you know all the cofactors. How can you find A?

24. For $n = 5$ the matrix C contains ___ cofactors and each 4 by 4 cofactor contains ___ terms and each term needs ___ multiplications. Compare with $5^3 = 125$ for the Gauss–Jordan computation of A^{-1}.

25. (For professors only) If you know all 16 cofactors of a 4 by 4 invertible matrix A, how would you find A?

26. L is lower triangular and S is symmetric. Assume they are invertible:

$$L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \qquad S = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}.$$

(a) Which three cofactors of L are zero? Then L^{-1} is lower triangular.
(b) Which three pairs of cofactors of S are equal? Then S^{-1} is symmetric.

Problems 27–36 are about area and volume by determinants.

27. The parallelogram with sides (2, 1) and (2, 3) has the same area as the parallelogram with sides (2, 2) and (1, 3). Find those areas from 2 by 2 determinants and say why they must be equal. (I can't see why from a picture. Please write to me if you do.)

28. If the columns of a 4 by 4 matrix have lengths L_1, L_2, L_3, L_4, what is the largest possible value for the determinant (based on volume)? If all entries are 1 or -1, what are those lengths and the maximum determinant?

29. A box has edges from $(0, 0, 0)$ to $(3, 1, 1)$, $(1, 3, 1)$, and $(1, 1, 3)$. Find its volume and also find the area of each parallelogram face.

30. When the edge vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are perpendicular, the volume of the box is $\|\boldsymbol{a}\|$ times $\|\boldsymbol{b}\|$ times $\|\boldsymbol{c}\|$. The matrix A^TA is ______. Find $\det A^TA$ and $\det A$.

31. Show by a picture how a rectangle with area x_1y_2 minus a rectangle with area x_2y_1 produces the area $x_1y_2 - x_2y_1$ of a parallelogram.

32. (a) The corners of a triangle are $(2, 1)$, $(3, 4)$, and $(0, 5)$. What is the area?
(b) A new corner at $(-1, 0)$ makes it lopsided (four sides). Find the area.

33. (a) Find the area of the parallelogram with edges $v = (3, 2)$ and $w = (1, 4)$.
(b) Find the area of the triangle with sides v, w, and $v + w$. Draw it.
(c) Find the area of the triangle with sides v, w, and $w - v$. Draw it.

34. The Hadamard matrix H has orthogonal rows. The box is a hypercube!

$$\text{What is } \det H = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} = \text{volume of a hypercube in } \mathbf{R}^4?$$

35. The triangle with corners (0, 0), (1, 0), (0, 1) has area $\frac{1}{2}$. The pyramid with four corners (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1) has volume ______. The pyramid in $\mathbf{R}^4$ with five corners at (0, 0, 0, 0) and the rows of I has what volume?

36. An n-dimensional cube has how many corners? How many edges? How many $(n-1)$-dimensional faces? The n-cube whose edges are the rows of $2I$ has volume ______. A hypercube computer has parallel processors at the corners with connections along the edges.

Problems 37–40 are about areas dA and volumes dV in calculus.

37. The matrix that connects r, θ to x, y is in Problem 40. Invert that matrix:

$$J^{-1} = \begin{vmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{vmatrix} = \begin{vmatrix} \cos\theta & ? \\ ? & ? \end{vmatrix} = ?$$

It is surprising that $\partial r/\partial x = \partial x/\partial r$. The product $JJ^{-1} = I$ gives the chain rule

$$\frac{\partial x}{\partial x} = \frac{\partial x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial x}{\partial\theta}\frac{\partial\theta}{\partial x} = 1.$$

38. The triangle with corners (0, 0), (6, 0), and (1, 4) has area ______. When you rotate it by $\theta = 60°$ the area is ______. The rotation matrix has

$$\text{determinant} = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & ? \\ ? & ? \end{vmatrix} = ?$$

39. Spherical coordinates ρ, ϕ, θ give $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$. Find the Jacobian matrix of 9 partial derivatives: $\partial x/\partial\rho$, $\partial x/\partial\phi$, $\partial x/\partial\theta$ are in row 1. Simplify its determinant to $J = \rho^2\sin\phi$. Then $dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$.

40. Polar coordinates satisfy $x = r\cos\theta$ and $y = r\sin\theta$. Polar area $J\,dr\,d\theta$ includes J:

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}.$$

The two columns are orthogonal. Their lengths are ______. Thus $J =$ ______.

41. Suppose (x, y, z), (1, 1, 0), and (1, 2, 1) lie on a plane through the origin. What determinant is zero? What equation does this give for the plane?

42. (VISA to AVIS) This takes an odd number of exchanges (IVSA, AVSI, AVIS). Count the pairs of letters in VISA and AVIS that are reversed from alphabetical order. The difference should be odd.

43. Let $P = (1, 0, -1)$, $Q = (1, 1, 1)$, and $R = (2, 2, 1)$. Choose S so that $PQRS$ is a parallelogram, and compute its area. Choose T, U, V so that $OPQRSTUV$ is a tilted box, and compute its volume.

44. Suppose (x, y, z) is a linear combination of (2, 3, 1) and (1, 2, 3). What determinant is zero? What equation does this give for the plane of all combinations?

45. If $Ax = (1, 0, \ldots, 0)$ show how Cramer's Rule gives $x =$ *first column of* A^{-1}.