for $\mathbb{R}^3$.

Problem Set 2.4

(2)

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{R}$$

Basis for $C(A) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$$\dim(C(A)) = 2$$

Now,

$$\begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So, bases of $N(A) = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\dim(N(A)) = n - r = 4 - 2 = 2$

basis for $C(A^T) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

$\dim(C(A^T)) = 2$

Now,

$$\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|cccc} 1 & -2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \to R_1 - 2R_2$$

So, basis of $N(A^T) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\dim(N(A^T)) = m - r$

$= 3 - 2$

$= 1$

3

$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

~~$\sim \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ by column exchange~~

Basis for $C(A) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\dim(C(A)) = 1$

Basis for $N(A) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\dim(N(A)) = n - r$

$= 4 - 1 = 3$

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(i) $A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

Rank (A) = 1

Now, $A = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \end{bmatrix}$

where, $u = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ & $v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}$

(ii)

$A = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix}$

$\sim \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}$ $R_2 \to R_2 - 3R_1$

Rank (A) = 1

Now, $A = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & -2 \end{bmatrix}$ $\left(\because u = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ \& } v = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \right)$

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(13)

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, basis of $C(A) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$

$\dim(C(A)) = 2$

basis of $N(A) = \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\dim(N(A)) = n - r = 5 - 2 = 3$

basis of $C(A^T) = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 6 \end{bmatrix}$

basis of $N(A^T) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

~~then Rank(A) = 3~~

(17) According to the question,

If $m < n$,

$\text{Rank}(A) = m$.

If $m > n$,

$\text{Rank}(A) = n$

If $m = n$,

$\text{Rank}(A) = n$

~~so~~

The columns of A are linearly Dependent.

(18) The linear equation is given by,

$x_1 + 2x_2 + 4x_3 = 0$

The augmented matrix is,

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, $x_1 + 2x_2 + 4x_3 = 0$

$\Rightarrow x_1 = -2x_2 - 4x_3$

Then,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 4x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

Hence, a 1×3 matrix of ~~same~~ null space is

$$\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}.$$

And 3×3 matrix consisting the null space is

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(24)

(a) lets consider,

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \quad R_2 \to R_2 + 2R_3$$

$$\sim \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 2 & 5 \end{bmatrix} \quad R_3 \to R_3 + 2R_2$$

Here, the resultant matrix ~~has~~ has column space $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and row space $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

(b) Constructing such a matrix isn't possible as we know, $A \in \mathbb{R}^{m \times n}$.

column space is given to be 1.
so, $Rank(A) = 1$

Null space is given to be 1. So, n would have to be 2.

This is not possible as null space basis is a 3-dimentional vector.

(c) It can be the matrix A as follows:-

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Here, pivot variable is 1.

So, Rank (A) = 1

So, $\dim(N(A)) = n - r$

$= 2 - 1$

$= 1$

$\dim(N(A^T)) = m - r$

$= 1 - 1$

$= 0$

So, $\dim(N(A)) = 1 + \dim(N(A^T))$, is satisfied.

(d) The matrix can be,

$$A = \begin{bmatrix} 3 & 1 \\ -1 & -1/3 \end{bmatrix}$$

So, $A^T x = \begin{bmatrix} 3 & -1 \\ 1 & -1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

Here, $x_2 = 3x_1$

So, $x = \begin{bmatrix} x_1 \\ 3x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Here, ~~null~~ left null space is $(1,3)$ and row space is $(3,1)$.

(e) Such a matrix is not possible, as we require matrix A to have the same column and row space, that means matrix A is symmetric. So, we must have,

null space = left null space.

(32)

(a) True, because $\dim(C(A)) = \dim(C(A^T))$.

(b) False, as, $\dim(N(A)) = n-r$ whereas $\dim(N(A^T)) = m-r$

(c) False,

Ex: $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ basis for $C(A) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

basis for $C(A^T) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

but $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

So, $A \neq A^T$.

(d) True,