be positive definite ... into

$$Ax = \lambda Mx = \lambda R^{\mathrm{T}}Rx$$

Writing C for R^{-1}, and multiplying through by $(R^{\mathrm{T}})^{-1} = C^{\mathrm{T}}$, this becomes a standard eigenvalue problem for the *single* symmetric matrix $C^{\mathrm{T}}AC$:

Equivalent problem $$C^{\mathrm{T}}ACy = \lambda y. \tag{10}$$

The eigenvalues λ_j are the same as for the original $Ax = \lambda Mx$, and the eigenvectors are related by $y_j = Rx_j$. The properties of $C^{\mathrm{T}}AC$ lead directly to the properties of $Ax = \lambda Mx$, when $A = A^{\mathrm{T}}$ and M is positive definite:

1. The eigenvalues for $Ax = \lambda Mx$ are real, because $C^{\mathrm{T}}AC$ is symmetric.
2. The λ's have the same signs as the eigenvalues of A, by the law of inertia.
3. $C^{\mathrm{T}}AC$ has orthogonal eigenvectors y_j. So the eigenvectors of $Ax = \lambda Mx$ have

"M-orthogonality" $$x_i^{\mathrm{T}}Mx_j = x_i^{\mathrm{T}}R^{\mathrm{T}}Rx_j = y_i^{\mathrm{T}}y_j = 0. \tag{11}$$

A and M are being *simultaneously diagonalized*. If S has the x_j in its columns, then $S^{\mathrm{T}}AS = \Lambda$ and $S^{\mathrm{T}}MS = I$. This is a *congruence* transformation, with S^{T} on the left, and not a similarity transformation with S^{-1}. The main point is easy to summarize: As long as M is positive definite, the generalized eigenvalue problem $Ax = \lambda Mx$ behaves exactly like $Ax = \lambda x$.

Problem Set 6.2

1. Decide for or against the positive definiteness of

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2.$$

2. For what range of numbers a and b are the matrices A and B positive definite?

$$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{bmatrix}$$

3. *If A and B are positive definite, then $A+B$ is positive definite*. Pivots and eigenvalues are not convenient for $A+B$. Much better to prove $x^{\mathrm{T}}(A+B)x > 0$.

4. Construct an *indefinite matrix* with its largest entries on the main diagonal:

$$A = \begin{bmatrix} 1 & b & -b \\ b & 1 & b \\ -b & b & 1 \end{bmatrix} \quad \text{with } |b| < 1 \text{ can have } \det A < 0.$$

5. If $A = Q\Lambda Q^{\mathrm{T}}$ is symmetric positive definite, then $R = Q\sqrt{\Lambda}Q^{\mathrm{T}}$ is its *symmetric positive definite square root*. Why does R have positive eigenvalues? Compute R and verify $R^2 = A$ for

$$A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix}.$$

6. The ellipse $u^2 + 4v^2 = 1$ corresponds to $A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$. Write the eigenvalues and eigenvectors, and sketch the ellipse.

7. Show from the eigenvalues that if A is positive definite, so is A^2 and so is A^{-1}.

8. Reduce the equation $3u^2 - 2\sqrt{2}uv + 2v^2 = 1$ to a sum of squares by finding the eigenvalues of the corresponding A, and sketch the ellipse.

9. From the pivots, eigenvalues, and eigenvectors of $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$, write A as $R^{\mathrm{T}}R$ in three ways: $(L\sqrt{D})(\sqrt{D}L^{\mathrm{T}})$, $(Q\sqrt{\Lambda})(\sqrt{\Lambda}Q^{\mathrm{T}})$, and $(Q\sqrt{\Lambda}Q^{\mathrm{T}})(Q\sqrt{\Lambda}Q^{\mathrm{T}})$.

10. Write down the five conditions for a 3 by 3 matrix to be *negative definite* ($-A$ is positive definite) with special attention to condition III: How is $\det(-A)$ related to $\det A$?

11. If $A = R^{\mathrm{T}}R$ prove the generalized Schwarz inequality $|x^{\mathrm{T}}Ay|^2 \le (x^{\mathrm{T}}Ax)(y^{\mathrm{T}}Ay)$.

12. Decide whether the following matrices are positive definite, negative definite, semidefinite, or indefinite:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}, \quad C = -B, \quad D = A^{-1}.$$

Is there a real solution to $-x^2 - 5y^2 - 9z^2 - 4xy - 6xz - 8yz = 1$?

13. If A is symmetric positive definite and C is nonsingular, prove that $B = C^{\mathrm{T}}AC$ is also symmetric positive definite.

14. In three dimensions, $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 1$ represents an ellipsoid when all $\lambda_i > 0$. Describe all the different kinds of surfaces that appear in the positive semidefinite case when one or more of the eigenvalues is zero.

15. Suppose A is symmetric positive definite and Q is an orthogonal matrix. True or false:

(a) $Q^{\mathrm{T}}AQ$ is a diagonal matrix.
(b) $Q^{\mathrm{T}}AQ$ is symmetric positive definite.

(c) $Q^{\mathrm{T}}AQ$ has the same eigenvalues as A.
(d) e^{-A} is symmetric positive definite.

16. A positive definite matrix cannot have a zero (or even worse, a negative number) on its diagonal. Show that this matrix fails to have $x^{\mathrm{T}}Ax > 0$:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is not positive when } (x_1, x_2, x_3) = (\quad,\quad,\quad).$$

17. (Lyapunov test for stability of M) Suppose $AM + M^{\mathrm{H}}A = -I$ with positive definite A. If $Mx = \lambda x$ show that $\operatorname{Re}\lambda < 0$. (*Hint*: Multiply the first equation by x^{H} and x.)

18. If A is positive definite and a_{11} is increased, prove from cofactors that the determinant is increased. Show by example that this can fail if A is indefinite.

19. Give a quick reason why each of these statements is true:
(a) Every positive definite matrix is invertible.
(b) The only positive definite projection matrix is $P = I$.
(c) A diagonal matrix with positive diagonal entries is positive definite.
(d) A symmetric matrix with a positive determinant might not be positive definite!

20. From $A = R^{\mathrm{T}}R$, show for positive definite matrices that $\det A \le a_{11}a_{22}\cdots a_{nn}$. (The length squared of column j of R is a_{jj}. Use determinant = volume.)

21. For which s and t do A and B have all $\lambda > 0$ (and are therefore positive definite)?

$$A = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{bmatrix}.$$

22. Compute the three upper left determinants to establish positive definiteness. Verify that their ratios give the second and third pivots.

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}.$$

23. You may have seen the equation for an ellipse as $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$. What are a and b when the equation is written as $\lambda_1 x^2 + \lambda_2 y^2 = 1$? The ellipse $9x^2 + 16y^2 = 1$ has half-axes with lengths $a =$ ___ and $b =$ ___.

24. A diagonal entry a_{jj} of a symmetric matrix cannot be smaller than all the λ's. If it were, then $A - a_{jj}I$ would have ______ eigenvalues and would be positive definite. But $A - a_{jj}I$ has a ______ on the main diagonal.

25. Which 3 by 3 symmetric matrices A produce these functions $f = x^{\mathrm{T}}Ax$? Why is the first matrix positive definite but not the second one?
(a) $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$.
(b) $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3)$.

26. With positive pivots in D, the factorization $A = LDL^{\mathrm{T}}$ becomes $L\sqrt{D}\sqrt{D}L^{\mathrm{T}}$. (Square roots of the pivots give $D = \sqrt{D}\sqrt{D}$.) Then $C = L\sqrt{D}$ yields the ***Cholesky factorization*** $A = CC^{\mathrm{T}}$, which is "symmetrized LU":

$$\text{From} \quad C = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{find } A. \qquad \text{From} \quad A = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} \quad \text{find } C.$$

27. Without multiplying $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, find

(a) the determinant of A. (b) the eigenvalues of A.
(c) the eigenvectors of A. (d) a reason why A is symmetric positive definite.

28. For the semidefinite matrices

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \text{ (rank 2)} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ (rank 1)},$$

write $x^{\mathrm{T}}Ax$ as a sum of two squares and $x^{\mathrm{T}}Bx$ as one square.

29. For $C = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, confirm that $C^{\mathrm{T}}AC$ has eigenvalues of the same signs as A. Construct a chain of nonsingular matrices $C(t)$ linking C to an orthogonal Q. Why is it impossible to construct a nonsingular chain linking C to the identity matrix?

30. Draw the tilted ellipse $x^2 + xy + y^2 = 1$ and find the half-lengths of its axes from the eigenvalues of the corresponding A.

31. The symmetric factorization $A = LDL^{\mathrm{T}}$ means that $x^{\mathrm{T}}Ax = x^{\mathrm{T}}LDL^{\mathrm{T}}x$:

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & (ac-b^2)/a \end{bmatrix}\begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

The left-hand side is $ax^2 + 2bxy + cy^2$. The right-hand side is $a\left(x + \frac{b}{a}y\right)^2 +$ ______ y^2. The second pivot completes the square! Test with $a = 2$, $b = 4$, $c = 10$.

32. In the Cholesky factorization $A = CC^{\mathrm{T}}$, with $C = L\sqrt{D}$, the square roots of the pivots are on the diagonal of C. Find C (lower triangular) for

$$A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix}.$$

33. If the pivots of a matrix are all greater than 1, are the eigenvalues all greater than 1? Test on the tridiagonal $-1, 2, -1$ matrices.

34. Apply any three tests to each of the matrices

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix},$$

to decide whether they are positive definite, positive semidefinite, or indefinite.

35. An algebraic proof of the *law of inertia* starts with the orthonormal eigenvectors $x_1, \ldots, x_p$ of A corresponding to eigenvalues $\lambda_i > 0$, and the orthonormal eigenvectors $y_1, \ldots, y_q$ of $C^{\mathrm{T}}AC$ corresponding to eigenvalues $\mu_i < 0$.

(a) To prove that the $p + q$ vectors $x_1, \ldots, x_p, Cy_1, \ldots, Cy_q$ are independent, assume that some combination gives zero:

$$a_1x_1 + \cdots + a_px_p = b_1Cy_1 + \cdots + b_qCy_q \ (= z, \text{ say}).$$

Show that $z^{\mathrm{T}}Az = \lambda_1a_1^2 + \cdots + \lambda_pa_p^2 \geq 0$ and $z^{\mathrm{T}}Az = \mu_1b_1^2 + \cdots + \mu_qb_q^2 \leq 0$.

(b) Deduce that the a's and b's are zero (proving linear independence). From that deduce $p + q \leq n$.

(c) The same argument for the $n - p$ negative λ's and the $n - q$ positive μ's gives $n - p + n - q \leq n$. (We again assume no zero eigenvalues—which are handled separately). Show that $p + q = n$, so the number p of positive λ's equals the number $n - q$ of positive μ's—which is the law of inertia.

36. If C is nonsingular, show that A and $C^{\mathrm{T}}AC$ have the same rank. Thus they have the same number of zero eigenvalues.

37. In equation (9) with $m_1 = 1$ and $m_2 = 2$, verify that the normal modes are M-orthogonal: $x_1^{\mathrm{T}}Mx_2 = 0$.

38. A *group* of nonsingular matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these sets are groups? *Positive definite symmetric matrices A, orthogonal matrices Q, all exponentials e^{tA} of a fixed matrix A, matrices P with positive eigenvalues, matrices D with determinant* 1. Invent a group containing only positive definite matrices.

39. Use the pivots of $A - \frac{1}{2}I$ to decide whether A has an eigenvalue smaller than $\frac{1}{2}$:

$$A - \frac{1}{2}I = \begin{bmatrix} 2.5 & 3 & 0 \\ 3 & 9.5 & 7 \\ 0 & 7 & 7.5 \end{bmatrix}.$$

40. Do A and $C^{\mathrm{T}}AC$ always satisfy the law of inertia when C is not square?

41. If the symmetric matrices A and M are indefinite, $Ax = \lambda Mx$ might not have real eigenvalues. Construct a 2 by 2 example.

42. Find by experiment the number of positive, negative, and zero eigenvalues ot

$$A = \begin{bmatrix} I & B \\ B^{\mathrm{T}} & 0 \end{bmatrix}$$

when the block B (of order $\frac{1}{2}n$) is nonsingular.

43. Find the eigenvalues and eigenvectors of $Ax = \lambda Mx$:

$$\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} x = \frac{\lambda}{18} \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} x.$$

$$\|Ax - b\| = \|U\Sigma V^{\mathrm{T}}x - b\| = \|\Sigma V^{\mathrm{T}}x - U^{\mathrm{T}}b\|.$$

Introduce the new unknown $y = V^{\mathrm{T}}x = V^{-1}x$, which has the same length as x. Then, minimizing $\|Ax - b\|$ is the same as minimizing $\|\Sigma y - U^{\mathrm{T}}b\|$. Now Σ is diagonal and we know the best y^+. It is $y^+ = \Sigma^+U^{\mathrm{T}}b$, so the best x^+ is Vy^+:

Shortest solution $\quad x^+ = Vy^+ = V\Sigma^+U^{\mathrm{T}}b = A^+b.$

Vy^+ is in the row space, and $A^{\mathrm{T}}Ax^+ = A^{\mathrm{T}}b$ from the **SVD**. ■

Problem Set 6.3

Problems 1–2 compute the SVD of a square singular matrix A.

1. (a) Compute AA^{T} and its eigenvalues $\sigma_1^2, 0$ and unit eigenvectors u_1, u_2.
   (b) Choose signs so that $Av_1 = \sigma_1 u_1$ and verify the SVD:
   $$\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{\mathrm{T}}.$$
   (c) Which four vectors give orthonormal bases for $\boldsymbol{C}(A), \boldsymbol{N}(A), \boldsymbol{C}(A^{\mathrm{T}}), \boldsymbol{N}(A^{\mathrm{T}})$?

2. Compute $A^{\mathrm{T}}A$ and its eigenvalues $\sigma_1^2, 0$ and unit eigenvectors v_1, v_2:
   $$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}.$$

Problems 3–5 ask for the SVD of matrices of rank 2.

3. Compute $A^{\mathrm{T}}A$ and AA^{T}, and their eigenvalues and unit eigenvectors, for
   $$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$
   Multiply the three matrices $U\Sigma V^{\mathrm{T}}$ to recover A.

4. Find the SVD from the eigenvectors v_1, v_2 of $A^{\mathrm{T}}A$ and $Av_i = \sigma_i u_i$:
   $$\textbf{Fibonacci matrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

5. Use the SVD part of the MATLAB demo **eigshow** (or Java on the course page `web.mit.edu/18.06`) to find the same vectors v_1 and v_2 graphically.

Problems 6–13 bring out the underlying ideas of the SVD.

6. Find $U\Sigma V^{\mathrm{T}}$ if A has orthogonal columns $w_1, \ldots, w_n$ of lengths $\sigma_1, \ldots, \sigma_n$.

7. Explain how $U\Sigma V^{\mathrm{T}}$ expresses A as a sum of r rank-1 matrices in equation (3):
   $$A = \sigma_1 u_1 v_1^{\mathrm{T}} + \cdots + \sigma_r u_r v_r^{\mathrm{T}}.$$

8. Construct the matrix with rank 1 that has $Av = 12u$ for $v = \frac{1}{2}(1, 1, 1, 1)$ and $u = \frac{1}{3}(2, 2, 1)$. Its only singular value is $\sigma_1 =$ ___.

9. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are orthonormal bases for $\mathbf{R}^n$. Construct the matrix A that transforms each v_j into u_j to give $Av_1 = u_1, \ldots, Av_n = u_n$.

10. (a) If A changes to $4A$, what is the change in the SVD?
(b) What is the SVD for A^{T} and for A^{-1}?

11. Suppose A is a 2 by 2 symmetric matrix with unit eigenvectors u_1 and u_2. If its eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$, what are U, Σ, and V^{T}?

12. Why doesn't the SVD for $A + I$ just use $\Sigma + I$?

13. Suppose A is invertible (with $\sigma_1 > \sigma_2 > 0$). Change A by as small a matrix as possible to produce a *singular* matrix A_0. Hint: U and V do not change:

$$\text{Find } A_0 \text{ from } \quad A = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{\mathrm{T}}.$$

14. Find the SVD and the pseudoinverse $V\Sigma^{+}U^{\mathrm{T}}$ of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

15. Is $(AB)^{+} = B^{+}A^{+}$ always true for pseudoinverses? I believe not.

16. (a) If A has independent columns, its left-inverse $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is A^{+}.
(b) If A has independent rows, its right-inverse $A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$ is A^{+}.
In both cases, verify that $x^{+} = A^{+}b$ is in the row space, and $A^{\mathrm{T}}Ax^{+} = A^{\mathrm{T}}b$.

17. Find the SVD and the pseudoinverse 0^{+} of the m by n *zero matrix*.

18. If an m by n matrix Q has orthonormal columns, what is Q^{+}?

19. Explain why AA^{+} and $A^{+}A$ are projection matrices (and therefore symmetric). What fundamental subspaces do they project onto?

20. Diagonalize $A^{\mathrm{T}}A$ to find its positive definite square root $S = V\Sigma^{1/2}V^{\mathrm{T}}$ and its polar decomposition $A = QS$:

$$A = \frac{1}{\sqrt{10}} \begin{bmatrix} 10 & 6 \\ 0 & 8 \end{bmatrix}.$$

21. Removing zero rows of U leaves $A = \underline{L}\,\underline{U}$, where the r columns of $\underline{L}$ span the column space of A and the r rows of $\underline{U}$ span the row space. Then A^{+} has the explicit formula $\underline{U}^{\mathrm{T}}(\underline{U}\,\underline{U}^{\mathrm{T}})^{-1}(\underline{L}^{\mathrm{T}}\underline{L})^{-1}\underline{L}^{\mathrm{T}}$.
Why is $A^{+}b$ in the row space with $\underline{U}^{\mathrm{T}}$ at the front? Why does $A^{\mathrm{T}}AA^{+}b = A^{\mathrm{T}}b$, so that $x^{+} = A^{+}b$ satisfies the normal equation as it should?

22. Split $A = U\Sigma V^{\mathrm{T}}$ into its reverse polar decomposition QS'.

23. What is the minimum-length least-squares solution $x^{+} = A^{+}b$ to the following?

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

You can compute A^{+}, or find the general solution to $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$ and choose the solution that is in the row space of A. This problem fits the best plane $C + Dt + Ez$ to $b = 0$ *and also* $b = 2$ at $t = z = 0$ (and $b = 2$ at $t = z = 1$).