Q10) a) If $x = re^{i\theta}$ what are x^2, x^{-1} and $\bar{x}$ in polar coordinates? Where are the complex numbers that have $x^{-1} = \bar{x}$?

b) At $t = 0$, the complex number $(e^{(-1+i)t})$ equals one. Sketch its path in the complex plane as t increases from 0 to 2π.

Ans)

$$x = re^{i\theta}$$

$$\Rightarrow x^2 = (re^{i\theta})^2 = r^2 e^{i2\theta}$$

$$x^{-1} = r^{-1} e^{-i\theta} = \frac{1}{r} e^{-i\theta}$$

$$\bar{x} = (\overline{r e^{i\theta}}) = re^{-i\theta}$$

In polar coordinates

$$x \equiv (r, \theta)$$

$$x^2 \equiv (r^2, 2\theta)$$

$$x^{-1} \equiv \left(\frac{1}{r}, -\theta\right)$$

$$\bar{x} = (r, -\theta)$$

Now

$$x^{-1} = \bar{x}$$

$$\Rightarrow \left(\frac{1}{r}, -\theta\right) = (r, -\theta)$$

$$\Rightarrow \frac{1}{r} = r$$

$$\Rightarrow r^2 - 1 = 0$$

$$\Rightarrow r = \pm 1, \text{ but } r \geq 0$$

$$\Rightarrow r = 1$$

So for $x = e^{i\theta}$ are the complex number such that $x^{-1} = \bar{x}$, where $0 \leq \theta \leq 2\pi$.

Q3) a) With the preceding A, use elimination to solve $Ax = 0$.

b) Show that the null space you just computed is orthogonal to $C(A^H)$ and not to the usual row space.

ans) a) $A = \begin{pmatrix} 1 & i & 0 \\ i & 0 & 1 \end{pmatrix}_{2\times 3}$

$Ax = 0 \Rightarrow \begin{pmatrix} 1 & i & 0 \\ i & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + x_2 i = 0 \;\&\; ix_1 + x_3 = 0$

Let $x_1 = a \Rightarrow x_2 = a i$ and $x_3 = -ai$

$\therefore$ Null space is $\{(1, i, -i)\}$

b) $C(A^T) = \{(1, i, 0), (i, 0, 1)\}$.

For orthogonal $u \cdot v = 0$

$(1, i - i) \cdot (1 \cdot i \cdot 0) = 1^2 + i^2 + 0 = 0$

$(1, i, - i) \cdot (i, 0, 1) = i - i = 0$

Hence null space is orthogonal to $C(A^T)$

But $C(A^*) = \{(1, -i, 0), (-i, 0, 1)\}$

Now

$(1, i, -i)(1, -i, 0) = 1^2 - i^2 + 0 = 2 \neq 0$

$(1, i, -i)(-i, 0, 1) = -i + 0 - i = -2i \neq 0$

So null space is not orthogonal to $C(A^*)$.

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SECTION 5.5

Q22) Prove that $A^H A$ is always a Hermitian matrix. Compute $A^H A$ and AA^H.

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}$$

ans) $$A^T = \begin{bmatrix} -i & 1 \\ 1 & -i \\ i & i \end{bmatrix}$$

$$A^T A = \begin{bmatrix} -i & 1 \\ 1 & -i \\ i & 1 \end{bmatrix} \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2i \\ 0 & 2 & 2i \\ 2i & 2i & 4 \end{bmatrix}$$

$$(A^{\cancel{H}T}A)^{\cancel{H}T} = \begin{bmatrix} 2 & 0 & -2i \\ 0 & 2 & -2i \\ -2i & -2i & 4 \end{bmatrix}$$

Here $\cancel{(A^H A)^H = A^H A}$. $A^T A = AA^T$

$\therefore$ $A^H A$ is hermitian.

$$AA^H = \begin{bmatrix} i & 1 & i \\ 1 & i & 1 \end{bmatrix} \begin{bmatrix} -i & 1 \\ 1 & -i \\ i & i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^H A = \begin{bmatrix} -i & 1 \\ 1 & -1 \\ i & i \end{bmatrix} \begin{bmatrix} i & 1 & i \\ 1 & i & 1 \end{bmatrix} = \begin{bmatrix} 0 & -i & -i \\ -i & 0 & 2 \\ -i & 2 & 2 \end{bmatrix}$$

Q43) A matrix with orthonormal eigenvectors has the form $A = U\Lambda U^{-1} = U\Lambda U^{H}$. Prove that $AA^{H} = A^{H}A$.

ans) $A = U\Lambda U^{-1} = U\Lambda U^{H}$

$\Rightarrow AA^{H} = (U\Lambda U^{H})(U\Lambda U^{H}) = (U\Lambda U^{H})(U\Lambda U^{H})^{H}$

$= U\Lambda U^{H} U \Lambda^{*} U^{H}$

$= U(\Lambda\Lambda^{*})U^{H}$

$\Rightarrow A^{H}A = (U\Lambda U^{H})^{T}(U\Lambda U^{H})$

$= (U\Lambda U^{H})^{H}(U\Lambda U^{H})$

$= U\Lambda^{*}\Lambda U^{H}$

Since Λ is a diagonal matrix $\Lambda\Lambda^{*} = (\Lambda\Lambda^{*})^{H}$ which implies $\Lambda\Lambda^{*}$ is Hermitian.