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CSE - 51
2nd Year

ASSIGNMENT-2

SECTION 2·1

Q1) Construct a subset of the x-y plane $\mathbb{R}^2$ that is

a) Closed under vector addition and substraction, but not scalar multiplication

b) Closed under scalar assignment but not under vector addition.

ans) a) The set of vectors (i, j) where i and j are integers.

For example $(0,0)$, $(1,0)$, $(-2,3)$ etc.

This set is a closed under vector addition since the sum or difference of two integer is always an integer. However it is not closed under scalar multiplication, since multiplying $(1,1)$ by $1/2$ produces a result $(\frac{1}{2}, \frac{1}{2})$ not in the set.

b) The set of all points on the x axis and y axis (that is $x_1, 0)$ and $(0, x_2)$ is closed under scalar multiplication but not under vector addition.

Q6) a) Suppose addition in R^2 adds an extra 1 to each component, so that $(3,1) + (5,0)$ equals $(9,2)$ instead of $(8,1)$. With scalar multiplication, which rules are broken?

Q8) Which of the following description are correct? The solution x of

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

form

a) a plane

b) a line

c) a point

d) a subspace

e) the nullspace of A

f) The column space of A.

ans) $$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [0; 0]$$

$$\Rightarrow x_2 = -x_1 - x_3$$
$$x_1 = -2x_3$$

$$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow N(A) = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

∴ The solution is

b) A line

d) A subspace

e) The nullspace of A.

Q26) Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$$

ans) Matrix A

$$c\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c+2d \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ Column space = x axis because for any vector $\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$ that lies on x axis, there is values such that $x = c+2d$.

Matrix B

$$c\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ 2d \\ 0 \end{bmatrix}$$

$\therefore$ Column space = xy plane, because for any $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$, there is a value such that $x = c$ and $y = 2d$.

Matrix C

$$c\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2c \\ 0 \end{bmatrix}$$

$\therefore$ Column space = Line spanned by vector $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Q28) For which vectors (b_1, b_2, b_3) do these system have a solution ?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

ans) For the 1st system

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

It will always have solution.

$\therefore \quad v = (b_1, b_2, b_3),\ b_1 b_2, b_3 \in \mathbb{R}$

For the 2nd system

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

It will have solution only when $b_3 = 0$

$\therefore\ v = (0, b_2, b_3),\ b_2, b_3 \in \mathbb{R}$

SECTION 2.2

Q7) Describe the set of attainable right-hand sides b (in the column space) for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

by finding the constraints on b that turn the third equation into $0=0$. What is the rank and a particular solution ?

ans) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Performing elimination

$$\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 2 & 3 & b_3 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 3 & b_3 - 2b_1 \end{array}\right]$$

$$R_3 \to R_3 - 3R_2 \left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & b_3 - 2b_1 - 3b_2 \end{array}\right]$$

For $R_3 = 0 = 0$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ 2b_1 + 3b_2 \end{bmatrix}$$

$\therefore$ Rank = 2

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} x_p = \begin{bmatrix} b_1 \\ b_2 \\ b_3 - 2b_1 - 3b_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 - 2b_1 - 3b_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \to \text{Particular solution}$$

Q15) Find the reduced row Echelon forms R and the rank of these matrices:

(a) The 3 by 4 matrix of all 1s.

ans)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad r(R) = 1$$

(b) The 4x4 with $a_{ij} = (-1)^{ij}$

ans)

$$\begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 + R_1} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - R_2} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{Rank} = 2.$$

c) The 3 by 4 matrix with $a_{ij} = (-1)^j$

ans) $\begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$r(R) = 1$

Q32) For every C, find R and the (special solutions) for A, B, C ?

$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix}$ and $A = \begin{bmatrix} 1-C & 2 \\ 0 & 2-C \end{bmatrix}$

ans)

a) $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & c & 2 & 2 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & c-1 & 0 & 0 \end{bmatrix}$

$R_2 \longrightarrow R_3 \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

For $c = 1$

$R = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -x_2 - 2x_3 - 2x_4 \\ x_2, x_3, x_4 \in \mathbb{R} \end{cases}$ [x_2, x_3, x_4 = free]

$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ Special solution are

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

For $c \neq 1$

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{Normalisation}]{\text{2nd row}} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$R = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -2x_3 - 2x_4 \\ x_2 = 0 \\ x_3, x_4 \in \mathbb{R} \end{cases}$$

$[x_3, x_4 = \text{free}]$

$$\Rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ special solutions are: $\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

b) $c = 1$

$$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = R$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{cases} x_1 \in \mathbb{R} \\ x_2 = 0 \end{cases}$$

$\therefore$ special solution $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

C = 2

$$R = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 2x_2 \\ x_2 \in \mathbb{R} \end{cases}$$ [x_2 is a free var]

$\therefore$ special solution is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$C \notin \{1, 2\}$:

$$\begin{bmatrix} 1-C & 2 \\ 0 & 2-C \end{bmatrix} \xrightarrow[\text{normalisation}]{\text{2nd row}} \begin{bmatrix} 1-C & 2 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1-C & 0 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\text{Normalisation}]{\text{1st Row}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow x_1 = x_2 = 0$$

No free variables are there.

$\therefore$ No special solution.

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Q36) Find the complete solutions of

$$x+3y+3z=1$$
$$2x+6y+9z=5$$
$$-x-3y+3z=5$$

and $\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

a) $x+3y+3z=1$

$2x+6y+9z=5$

$-x-3y+3z=5$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

$$\Rightarrow [A \mid b] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 5 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ -1 & -3 & 3 & 5 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{array}\right]$$

$$\xrightarrow[\text{normalisation}]{\text{2nd row}} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 \to R_1 - 3R_2 \left[\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 0 \Rightarrow \begin{cases} x = -3y \\ y \in \mathbb{R} \\ z = 0 \end{cases} \Rightarrow \text{Special solution} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$Ax_p = b \Rightarrow Rx_p = C$

$y = 0$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x = -2 \\ z = 1 \end{matrix}$$

$\therefore$ Particular solution $= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

b) $$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\Downarrow$

$x = \frac{1}{2} - 3y$

$y \in \mathbb{R}$

$z = \frac{1}{2} - 2t$

$t \in R$

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - 3y \\ y \\ \frac{1}{2} - 2t \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}}_{\text{particular Sol}^n} + y \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{special Sol}^n} + t \underbrace{\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\text{special Sol}^n}$$

Q56) Reduce to $Ux = c$ and then $Rx = d$:

$$Ax = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = b$$

Find the particular solution x_p and all nullspace solution x_n.

ans) $$Ax = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = b$$

$$[A \;\; b] = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 1 & 3 & 2 & 0 & 5 \\ 2 & 0 & 4 & 9 & 10 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 3 & 0 & -3 & 3 \\ 2 & 0 & 4 & 9 & 10 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 3 & 0 & -3 & 3 \\ 0 & 0 & 0 & 3 & 6 \end{array}\right]$$

$$\xrightarrow[\text{Normalisation}]{\text{2nd row}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 & 6 \end{array}\right]$$

$$\xrightarrow[\text{Normalisation}]{\text{3rd Row}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$Ax_n = 0 \;\Rightarrow\; Rx_n = 0$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \quad x_1 = -2x_3 - 3x_4$$

$$x_2 = x_4 = 0$$

$$x_3 \in \mathbb{R}$$

$\therefore$ special sol^n = $x_n = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$Ax_p = b \Rightarrow Rx_p = d$

$$x_1 = 2 - 2x_3 - 3x_4 = -4$$

$$x_2 = 1 + x_4 = 3$$

$$x_3 = 0$$

$$x_4 = 2$$

$$\Rightarrow \quad x_p = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}$$

Q12) Under what conditions on b_1 and b_2 does $Ax = b$ have a solution?

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Find two vectors in the null space of A and the complete solution to $Ax = b$.

ans) $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 2 & 4 & 0 & 7 & b_2 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - 2b_1 \end{array}\right]$$

Vectors from nullspace of Matrix A satisfy

$$Ax = 0 \Rightarrow Ux = 0$$

$$\therefore \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = -2x_2 \\ x_2, x_3 \in \mathbb{R} \\ x_4 = 0 \end{array}$$

$$\Rightarrow \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \text{Null space}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \end{bmatrix} \Rightarrow \begin{array}{l} x_4 = b_2 - 2b_1 \\ x_1 + 3x_4 = b_1 \Rightarrow x_1 = 4b_1 - 3b_2 \end{array}$$

$\therefore$ Complete solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4b_1 - 3b_2 \\ 0 \\ 0 \\ b_2 - 2b_1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

SECTION 2.3

Q4) Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Solve $c_1 v_1 + \dots + c_4 v_4 = 0$ or $Ac = 0$. The v's go in the columns of A.

ans)

$$A = [\, v_1 \; v_2 \; v_3 \,] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Ac = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_2 \to R_2 - R_3 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \boxed{c_1 = 0, \; c_2 = 0, \; c_3 = 0}$$

$$\Rightarrow v_1 c_1 + v_2 c_2 + v_3 c_3 = 0$$

$$\Rightarrow v_1 \cdot 0 + v_2 \cdot 0 + v_3 \cdot 0 = 0$$

$\Rightarrow$ Linearly independent.

$$A = [V_1\ V_2\ V_3\ V_4] = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$Ac = 0 \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right]$$

$$\Rightarrow \quad c_1 = c_4$$
$$c_2 = c_4$$
$$c_3 = -4c_4$$
$$c_4 \in \mathbb{R}$$

Let $c_4 = 1$

$\Rightarrow \quad c_1 = 1, \quad c_2 = 1, \quad c_3 = -4, \quad c_4 = 1$

$\Rightarrow V_1C_1 + V_2C_2 + V_3C_3 + V_4C_4$

$= V_1 + V_2 + V_3 - 4 + V_4 = 0$

$\Rightarrow Ac = 0$ [Lineary Dependent]

$\therefore$ V_1, V_2, V_3 are independent

V_1, V_2, V_3, V_4 are dependent.

Q9) Find two independent vectors on the plane $x+2y-3z-t=0$ in R^4. Then find three independent vectors. Why not 4? This plane is the nullspace of what matrix?

ans) $x+2y-3z-t=0$

$$\Rightarrow V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$Vc = 0 \quad \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right]$$

$$R_1 \rightarrow R_1 + R_4 \left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right]$$

$$R_1 \rightarrow R_1 + 2R_2 \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] \Rightarrow c_1 = c_2 = 0$$

$\therefore$ $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ are two independent vector on the given plane.

$$V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$Vc = 0 \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_4} \left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 \to R_2 - R_3 \left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 \to R_1 - R_3 \left[\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 \to R_1 + 2R_2 \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right] \Rightarrow c_1 = c_2 = c_3 = 0$$

$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ are three independent vectors on the given plane.

We cannot find four linearly independent vectors on the given plane because this plane lies in R^4.

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ y \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ Plane $x+2y-3z-t=0$ is a nullspace of matrix $\begin{bmatrix} 1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 2 & -3 & -1 \end{bmatrix}$

Q10) If w_1, w_2, w_3 are independent vectors, show that the sums $v_1 = w_2 + w_3$, $v_2 = w_1 + w_3$ and $v_3 = w_1 + w_2$ are independent.

ans) w_1, w_2, w_3 are linearly independent

$\Rightarrow av_1 + bv_2 + cv_3 = 0$ only for $a=b=c=0$

$$a(w_2+w_3) + b(w_1+w_3) + c(w_1+w_2) = 0$$

$$w_1(b+c) + w_2(a+c) + w_3(a+b) = 0$$

The only solution for this is $a=b=c=0$.

$\therefore$ The vectors are linearly independent.

Q13) Decide whether or not the following vectors are linearly independent, by solving $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

ans) $Vc = 0 \Rightarrow [v_1 \;\; v_2 \;\; v_3 \;\; v_4]\, c = 0$

$\Rightarrow c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$

$$Vc = 0 \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xRightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xRightarrow{R_1 \to R_3 - R_1} \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$R_4 \to R_4 - R_3 \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} c_1 = -c_4 \\ c_2 = c_4 \\ c_3 = -c_4 \\ c_4 \in \mathbb{R} \end{cases}$$

$\Rightarrow -c_4 \cdot (v_1 + v_3) + c_4 \cdot (v_2 + v_4) = 0, \quad c_4 \in \mathbb{R}$

$\Rightarrow$ vectors v_1, v_2, v_3, v_4 are linearly dependent.

since they are not independent, they cannot span $\mathbb{R}^4$.

$$Vc = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$\xRightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$R_4 \to R_4 - R_3 \Rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Here the last row has $1 = 0$ which can't be true

$\therefore$ The given vector don't span $\mathbb{R}^4$

Q32) Find a counter example to the following statement: If V_1, V_2, V_3, V_4 is a basis

Q31) For which numbers c and d do these matrices have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$$

ans) By performing row elimination

$$\begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \xrightarrow[d \neq 0]{R_2 \to R_2 - \frac{2}{d} R_3} \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 0 & 2 - 2/d \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$$

The only way possible for this matrix to have rank 2 is to have 0 row as 2nd row.

$\therefore \quad c = 0, \; d = 2$

For the very same c and d matrix B has rank 2.

Q32) True or False?

a) If the columns of A are linearly independent, then $Ax = b$ has exactly one solution for every b.

ans) True:

Let us assume it to false.

$\Rightarrow Ax = b, \; Ay = b$

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$\Rightarrow$ $Ax = b$ $\Rightarrow Ax = Ay$
$Ay = b$

$\Rightarrow A(x-y) = 0 \xRightarrow{A \neq 0} x = y$

Last conclusion about A not being null matrix was a consequence of A having independent columns. If any of those columns was a null column they would be dependent.

$\therefore$ The claim is True.

b) A 5×7 matrix never has linearly independent columns.

ans) True.

This is true because in order for all columns to be independent matrix need to be square.

Q40) Review: Which of the following bases are for $\mathbb{R}^3$?

a) (1,2,0) and (0,1,-1)

ans) Not a bases of $\mathbb{R}^3$

In order to have bases for $\mathbb{R}^3$ we need to have three linearly independent vectors, and here we have two.

b) (1,1,-1), (2,3,4), (4,1,-1), (0,1,-1)

ans) This is Not bases of $\mathbb{R}^3$

Let us denote the vectors as :-

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, V_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, V_3 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}, V_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Now to check if they are linearly independent or not :-

$Vc = 0 \rightarrow [V_1 \; V_2 \; V_3]\, C = 0 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 1 & 3 & 1 & 0 \\ -1 & 4 & -1 & 0 \end{array}\right]$

$R_2 \rightarrow R_2 - R_1 \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ -1 & 4 & -1 & 0 \end{array}\right] \xrightarrow{R_3 \rightarrow R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 6 & 3 & 0 \end{array}\right]$

$R_3 \rightarrow R_3 - 6R_2 \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 21 & 0 \end{array}\right] \xrightarrow[\text{Normalisation}]{\text{3rd row}} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\Rightarrow c_1 = c_2 = c_3 = 0 \Rightarrow V_1, V_2, V_3$ are independent.

$V_4 = \frac{4}{3} V_1 - \frac{1}{3} V_3$

$\therefore$ They do not form basis for $\mathbb{R}^3$.

c) $(1,2,2), \; (-1,2,1), \; (0,8,0)$

ans) This is a bases of $\mathbb{R}^3$.

$\Rightarrow$ Let $V_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, V_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, V_3 = \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix}$

$Vc = 0 \Rightarrow [V_1 \; V_2 \; V_3]\, C = 0 \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & 2 & 8 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right]$

$R_2 \rightarrow R_2 - 2R_1 \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 4 & 8 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \rightarrow R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right]$

$R_2 \rightarrow R_2 - \frac{4}{3} R_3 \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] \xrightarrow[\text{Normalisation}]{\text{3rd and 2nd row}} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$

which means that V_1, V_2, V_3 are linearly independent vectors and since we have 3 of them and $\dim(\mathbb{R}^3) = 3$, they form bases.

d) $(1,2,2)$, $(-1,2,1)$, $(0,8,6)$

ans) Not a bases for $\mathbb{R}^3$

Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 8 \\ 6 \end{bmatrix}$

$Vc = 0 \Rightarrow [v_1 \ v_2 \ v_3]\, c = 0 \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & 2 & 8 & 0 \\ 2 & 1 & 6 & 0 \end{array}\right]$

$\xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 4 & 8 & 0 \\ 2 & 1 & 6 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right]$

$\xrightarrow{R_2 \to R_2 - \frac{4}{3}R_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right] \xrightarrow{R_3 \to \frac{R_3}{3}} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$

$\xrightarrow{R_1 \to R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \Rightarrow \begin{cases} c_1 = -2c_3 \\ c_2 = -2c_3 \\ c_3 \in \mathbb{R} \end{cases}$

which means that v_1, v_2 and v_3 are linearly dependent vectors and therefore do not make bases for $\mathbb{R}^3$.

SECTION 2.4

Q3) Find the dimensions and a basis for the four fundamental subspaces for associated with each of the matrices

$$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ans) U = Echelon Matrix

$$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

ROW SPACE

$r = 1$ pivot

$\Rightarrow \dim C(A^T) = \dim C(U^T) = 1$ [Non zero rows form basis for the row spaces $C(A^T)$ and $C(U^T)$]

$= \left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$ is basis for both row spaces.

NULL SPACE

$$Ux = 0$$

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} x = 0 \quad \Rightarrow \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$\Rightarrow x_2 + 4x_3 = 0$$

$$\Rightarrow x = \begin{bmatrix} x_1 \\ -4x_3 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\Rightarrow x = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad [x_1, x_2, x_3, x_4 \in \mathbb{R}]$$

$\dim N(A) = \dim N(U) = 3$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis for both null spaces.

COLUMN SPACE

- Only the pivot columns of Matrix U form basis for column space

- Here there is only one pivot column and that is the second column of matrix U.

$\therefore \dim C(U) = 1$ and $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is basis for $C(U)$

Matrix A

We will use same pivot colums from U but ~~for~~ from A.

$\Rightarrow \dim C(A) = 1$ and $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is basis for $C(A)$.

LEFT NULLSPACE

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 4 & 0 \\ 0 & 0 \end{bmatrix} x = 0 \Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow x_1 = 0$

$\Rightarrow x = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \Rightarrow x = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}, x_2 \in \mathbb{R}$

$\Rightarrow \dim N(U^T) = 1$ and $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is basic for $N(U^T)$

Matrix A

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 4 & 0 \\ 0 & 0 \end{bmatrix} x = 0 \Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 4 & 8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow x_1 = -2x_2$$

$$\Rightarrow x = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} \Rightarrow x = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}, x_2 \in \mathbb{R}$$

dim $N(A^T) = 1$ and $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ is basis for $N(A^T)$.

Q6) Find the rank of A and write the matrix as $A = uv^T$:

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix}$$

ans) $A_1 = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix}$

Here there is only 1 linearly independent column in Matrix A_1.

$$\Rightarrow r = 1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \end{bmatrix}$$

In Matrix A_2

$C_2 =$ L.C of 1st A_1

$$\Rightarrow r = 1$$

$$\Rightarrow A_2 = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Q11) Find a matrix A that has V as its row space, and a Matrix B that has a V as its null space, if V is the subspace spanned by.

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$$

ans) $$A = \begin{bmatrix} \alpha \cdot 1 & \alpha \cdot 1 & \alpha \cdot 1 \\ \beta \cdot 1 & \beta \cdot 2 & \beta \cdot 5 \\ \gamma \cdot 1 & \gamma \cdot 0 & \gamma \cdot 0 \end{bmatrix}$$ where $\alpha, \beta, \gamma \neq 0$

If $\alpha = \beta = \gamma = 1$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 5 & 0 \end{bmatrix}$$

Matrix $B \in \mathbb{R}^{3 \times 3}$ whose null space contains subspace V satisfies following

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0 \Rightarrow \begin{cases} a+b = 0 \\ d+e = 0 \\ g+h = 0 \end{cases}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0 \Rightarrow \begin{cases} a+2b = 0 \\ d+2e = 0 \\ g+2h = 0 \end{cases}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} = 0 \Rightarrow \begin{cases} a+5b = 0 \\ d+5e = 0 \\ g+5h = 0 \end{cases}$$

Q17) Suppose the only solution to $Ax=0$ (m equations in n unknowns) is $x=0$. What is the rank and why? The columns of A are linearly ______.

ans) Let Matrix $A \in \mathbb{R}^{m \times n}$ and only 1 sol^n exist

$$Ax = 0 \quad [x=0]$$

$\Rightarrow$ Rank $\Rightarrow r = n$

Because only trivial combination of column of Matrix A yields zero.

The columns of A are linearly <u>independent</u>.

Q28) Without computing A, find bases for the 4 fundamental subspaces:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

ans) Rank $= 3$

$\Rightarrow \dim C(U) = \dim C(A) = \dim C(U^T) = \dim C(A^T) = 3$

$\Rightarrow C(A) = \mathbb{R}^3$ and $C(A^T) = \mathbb{R}^4$

$\Rightarrow \dim N(U) = \dim N(A) = \dim N(U^T) = \dim N(A^T) = 3-3=0$

$\Rightarrow N(A) = N(A^T) = \{0\}$

$\therefore \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is basis for row space of Matrix A

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis for column space of Matrix A

$\{0\}$ is basis for null space and left null space.

Q31) Following the method of Problem 33, reduce A to Echelon form and look at zero rows. The b column tells which combinations you have taken of the rows:

(a) $\begin{bmatrix} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix}$

ans) a) $A = \begin{bmatrix} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{bmatrix}$ $R_2 \to R_2 - 3R_1$ $\begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2-3b_1 \\ 4 & 6 & b_3 \end{bmatrix}$

$R_3 \to R_3 - 4R_1$ $\begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2-3b_1 \\ 0 & -2 & b_3-4b_1 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - R_2}$ $\begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2-3b_1 \\ 0 & 0 & b_3-b_1-b_2 \end{bmatrix}$

$\Rightarrow \dim N(A^T) = m - r = 3 - 2 = 1$

$b_3 - b_1 - b_2 = 0 \Rightarrow \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is the basis for $N(A^T)$

b) $B = \begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix}$ $\xrightarrow{R_2 \to R_2 - 2R_1}$ $\begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2-2b_1 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - 2R_1}$ $\begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2-2b_1 \\ 0 & 0 & b_3-2b_1 \\ 2 & 5 & b_4 \end{bmatrix}$

$$\xrightarrow{R_4 \to R_4 - 2R_1} \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 - 2b_1 \\ 0 & 1 & b_4 - 2b_1 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 + R_2} \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 - 2b_1 \\ 0 & 0 & b_4 - 4b_1 + b_2 \end{bmatrix}$$

$\Rightarrow \dim N(B^T) = m - r = 4 - 2 = 2$

$b_3 - 2b_1 = 0$ and $b_4 - 4b_1 + b_2 = 0$

$\Rightarrow \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis for $N(B^T)$.

Q32) True or False (with a reason or a counter example)

a) A and A^T have the same number of pivots.

ans) True

Row space and Row column space have the same rank

$r(A) = r(A^T)$

Since they have same rank, they have same no. of pivots.

b) A and A^T have the same left subspace.

ans) False:

Eg:

$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

$A^T \cdot x = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 \cdot 1 + x_2 \cdot 2 = 0$

The base of the left nullspace is (-2,1).

$$A^T = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$(A^T)^T \cdot x = 0$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 = 0$$

$$2 \cdot x_1 = 0$$

The base of left ~~subspace~~ nullspace is (0,1).

∴ The spaces are not the same.

c) If the row space equals the column space then $A^T = A$.

ans) False

Eg: $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

- Row space : {(1,2), (1,3)}
- Column space : {(1,2), (1,3)}

d) If $A^T = -A$, then row space of A equals the column space.

ans) True

When A^T would be same as A claim is obvious. Since we are multiplying the whole matrix with factor -1 claim stays true. Vectors that span row spaces of A are the same as vectors that span row space of -A.