Problem Set 2.2

(1)

$$u + v + 2w = 2$$
$$2u + 3v - w = 5$$
$$3u + 4v + w = c$$

Here,

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & 3 & -1 & 5 \\ 3 & 4 & 1 & c \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 1 & -5 & c-6 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & c-7 \end{array}\right] \quad R_3 \to R_3 - R_2$$

So, for $Ax = b$ to have a solution,

$$c - 7 = 0$$

$$\Rightarrow \boxed{c = 7}$$

Date
Page

4

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$NR_2 \leftarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Now make $v=0$,

So, $\boxed{w = 2}$

$u + 2v + 2w = 1$

$\Rightarrow u + 0 + 4 = 1$

$\Rightarrow \boxed{u = -3}$

So, $x_p = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$

Now,

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 \to R_1 - 2R_2$$

$$\sim \begin{array}{c} \quad I \qquad F \\ \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 0 \end{array}\right] \end{array}$$ By column exchange

So, $x_n = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

So, $x_c = x_p + x_n$

$$= \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + v \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$ (Ans)

(5)

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 - 2R_2$$

$\hookrightarrow$ Row reduced Echelon form

Rank $(A) = 2$

let the variables be $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$, free variables are z & w.

Date
Page

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 4R_1 \\ R_3 \leftarrow R_3 - 7R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \leftarrow R_3 - 2R_2$$

So, as x_p isn't possible, x_c isn't possible.

Rank$(B) = 2$

Free variable $= z$ (if $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ are variable)

7)
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

So,
$$\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 2 & 3 & b_3 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 3 & b_3 - 2b_1 \end{array}\right] \quad R_3 \rightarrow R_3 - 2R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & b_3-2b_1-3b_2 \end{array}\right] \quad R_3 \to R_3 - 3R_2$$

So, $b_3 - 2b_1 - 3b_2 = 0$

$\Rightarrow b_3 = 2b_1 + 3b_2$

So, $b = \begin{bmatrix} b_1 \\ b_2 \\ 2b_1+3b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

Rank = 2

$x_p = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

(12) $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 2 & 4 & 0 & 7 & b_2 \end{array}\right]$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 1 & b_2-2b_1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & b_1-3b_2+6b_1 \\ 0 & 0 & 0 & 1 & b_2-2b_1 \end{array}\right]$$

, take $y=0$ & $z=0$,

$x_p = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 7b_1-3b_2 \\ 0 \\ 0 \\ b_2-2b_1 \end{bmatrix}$

Date
Page

Now,
$$\begin{array}{cccc} I & & F & \\ \end{array}$$
$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

So, $x_n = \begin{bmatrix} -F \\ I \end{bmatrix}$

$$= \begin{bmatrix} 0 & -2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

So, $x_c = x_p + x_n$

$$= \begin{bmatrix} 7b_1 - 3b_2 \\ 0 \\ 0 \\ b_2 - 2b_1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(13)

(a) $Ux = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So, $U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_1 \to R_1 - 3R_2$

$\hookrightarrow R$

Pivot colums

$x_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\cancel{2x_4 = 0}$ $x_3 = 0$

$\cancel{\Rightarrow x_4 = 0}$

~~Also,~~ $\cancel{2x_2 = 0}$ Also, $x_1 = 0$

$\cancel{\Rightarrow x_2 = 0}$

Now,

$$R = \begin{array}{c} \begin{array}{cccc} \quad I & & \quad F & \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

So, $x_n = \begin{bmatrix} -F \\ I \end{bmatrix}$

$$= \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

So, $x_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

(b) If $(0,0,0)$ is $(a,b,0)$

So,
$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & a \\ 0 & 0 & 1 & 2 & b \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & a-3b \\ 0 & 0 & 1 & 2 & b \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a-3b \\ 0 \\ b \\ 0 \end{bmatrix}$$

$$x_3 = b$$

Also, $x_1 = a-3b$

Now, $x_n = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

So, $x_c = \begin{bmatrix} a-3b \\ 0 \\ b \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

15

(a)
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Row reduced echelon form}$$

Rank = 1

(b)
$$\begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 + R_1 \\ R_3 \rightarrow R_3 - R_1 \\ R_4 \rightarrow R_4 + R_1 \end{array}$$

$$\sim \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \rightarrow R_4 - R_2$$

$$\sim \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \text{Row reduced echelon Form}$$

Rank = 2

32

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & c-1 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & c-1 & 0 & 0 \end{bmatrix} \quad \cancel{R_1 \to R_1 - c}$$

for $c=1$, there are 3 free columns.

~~and 3 null space.~~

else if $c \neq 1$, there are 2 free colums

~~and 2 null space.~~

for

$$A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

for $c=1$ or $c=2$, 1 free column exist and 1 null space exist. Otherwise x_n doesn't exist as there is no free column available.

34

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

So,

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2-2b_1 \\ 0 & 1 & b_3-2b_1 \\ 0 & 3 & b_4-3b_1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - 3R_1$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 0 & b_4 - 3b_1 - 3b_3 + 6b_1 \end{array}\right] \quad R_4 \to R_4 - 3R_3$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 0 & b_4 - 3b_3 + 3b_1 \end{array}\right]$$

$\sim$

So, for $b_4 - 3b_3 + 3b_1 = 0$

$\Rightarrow b_4 = 3b_3 - 3b_1$, it is solvable

So, for $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 3b_3 - 3b_1 \end{bmatrix}$, it is solvable.

Date
Page

36

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

So,

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So, Here x is the only pivot variable.

So, for $y=0, z=0, t=0,$

$$x_p = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{x=1}$$

Now,

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, $x_n = \begin{bmatrix} -F \\ I \end{bmatrix}$

$$= \begin{bmatrix} -3 & -2 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

So, $x_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(44)

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}$$

$$\sim \begin{bmatrix} 6 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & q-3 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + \frac{1}{2} R_1 \\ R_3 \to R_3 - 9/6\, R_1 \end{array}$$

~~So, there values of q can be anything~~

~~as~~ for $q = 3$,

Rank = 1

for $q \neq 3$,

Rank = 2

(59)

(a) false,

counter example,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$$R \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ ~~Total~~ free variables are 2.

free columns

(b) ~~False~~ True, as in an invertible matrix, ~~the~~ ~~bottom~~ every column has a pivot, so there are no free variables.

(c) True, as the pivot ~~elements~~ variables can't exceed the no. of columns. because each column has only one pivot element.

(d) False,

e.g:- $\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 5 \end{bmatrix}_{4 \times 2}$

$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}_{4 \times 2}$

Here, $m=4$ but no. of pivot variables $= 2 = n$.

Date ____
Page ____

56

$$Ax = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = b$$

So,

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 1 & 3 & 2 & 0 & 5 \\ 2 & 0 & 4 & 9 & 10 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 & 6 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -4 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 & 6 \end{array}\right] \quad R_1 \to R_1 - R_3$$

free variable x_3

So, $x_3 = 0$,

Here, $\boxed{x_4 = 6/3 = 2}$

$\boxed{x_2 = 1}$

$\boxed{x_1 = -4}$

So,

$$x_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

So,

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ - making pivot entries as 1.

So, $x_n = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ ($\because$ free variable is x_3)

So, $x_c = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

59) Here,

$x - 3y - z = 0$,

So, $x = 3y + z$

So, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y+z \\ y \\ z \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

So, Special solutions are (3,1,0) and (1,0,1).

Here, $A = \begin{bmatrix} 1 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The complete solution will be,

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore x_p = (12, 0, 0)$ as per question.