Assignment-2

EX-2.1

1) Construct a subset of the x-y plane $\mathbb{R}^2$ that is

a) Closed under vector addition and substraction, but not scalar multiplication

b) Closed under scalar multiplication but not under vector addition

Ans a) Let $(0,0), (1,1), (-2,3)$ etc. This is closed under vector addition, since the sum or difference of two integers is always an integer. But it is not under scalar multiplication because multiplying by $1/2$ gives $(1/2, 1/2)$ which is not in set.

b) $(x_1, 0)$ and $(0, x_2)$ is closed under scalar multiplication but not under vector addition.

6)

$x+0=x$

but according to question this will give result 1. So it breaks identity operation of addition rule.

b) Show that the set of all positive real numbers, with $x+y$ and cx redifined to equal the usual xy and x^c, is a vector space: what is "zero vector"?

Ans: 1) $x \cdot y = y \cdot x$

2) $x \cdot (y \cdot z) = (y \cdot x) \cdot z$

3) Zero vector $= 1 - x = x$

4) unique "opposite" vector $x \cdot \frac{1}{x} = 1$.

5) $x^1 = x$

6) $x^{c_1 c_2} = (x^{c_1})^{c_2}$

7) $(x \cdot y)^c = x^c \cdot y^c$

8) $x^{c_1+c_2} = x_1^c \cdot x_2^c$

c) Suppose $(x_1, x_2) + (y_1, y_2)$ is defined to be (x_1+y_2, x_2+y_1). With the usual $cx = (cx_1, cx_2)$, which of the eight conditions are not satisfied?

Ans: $(x_1, x_2) + (y_1, y_2) = (x_1+y_2, x_2+y_1) \neq (y_1+x_2, y_2+x_1) = (y_1, y_2) + (x_1, x_2)$

$(c_1+c_2)x = ((c_1+c_2)x_1, (c_1+c_2)x_2)$

On the other hand

$c_1(x_1, x_2) + c_2(x_1, x_2) = (c_1x_1, c_1x_2) + (c_2x_1, c_2x_2) = (c_1x_1+c_2x_2, c_1x_2+c_2x_1)$

Which is not similar.

8) Which of the following descriptions are correct? The solution x of

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

a) a plane b) a line c) a point d) a subspace e) the nullspace of A

f) the column space of A.

Ans : $Ax = 0 \Leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\Leftrightarrow \begin{cases} x_2 = -x_1 - x_3 \\ x_1 = -2x_3 \end{cases}$$

That means that each $x \in N(A)$ has following form.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ Basis for $N(A)$ is given by: $\langle \{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \} \rangle$

b) line d) subspace e) nullspace

26) Describe the column space of these matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Ans: $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x+2y \\ 0 \\ 0 \end{bmatrix}$$

This column space is therefore equal to x-axis. That's because for any vector $\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$ that lies on the x-axis, such that $a = x + 2y$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 2y \\ 0 \end{bmatrix}$$

This column space thus equal to xy plane.

$$C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$x\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 2x \\ 0 \end{bmatrix}$$

$\therefore$ The column space is equal to line spanned by the vector $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

Q28) For which vectors do these systems have a solution.

Ans a)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Here, $b_3 = x_3$

$b_2 = x_2 + x_3$

$b_1 = x_1 + x_2 + x_3$

System will always have a solution $V = (b_1, b_2, b_3)$, $b_1, b_2, b_3 \in R$

b)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

System will have a solution only when $b_3 = 0$, so for vectors $V = (0, b_2, b_3)$, $b_2, b_3 \in \mathbb{R}$

Ex-2.2

7. Describe the set of attainable right hand sides b for finding the constraints on b that turn the third equation into $0=0$. What is the rank, and a particular solution?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Ans: Perform elimination

$$\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 2 & 3 & b_3 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 3 & b_3 - 2b_1 \end{array}\right] \xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & b_3 - 2b_1 - 3b_2 \end{array}\right]$$

Now, in order to set value of third row $0 = 0$ set of attainable RHS b is given by

$$b = \begin{bmatrix} b_1 \\ b_2 \\ 2b_1 + 3b_2 \end{bmatrix}$$

We can see one row is zero and two are LI.

$\therefore$ Rank is 2.

Now,

$$\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix} x_p = \begin{bmatrix}b_1\\b_2\\b_3-2b_1-3b_2\end{bmatrix} \Rightarrow \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3-2b_1-3b_2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$$ (Particular Solution)

12) under what condition on b_1 and b_2 does $Ax=b$ have a solution?

$$A = \begin{bmatrix}1&2&0&3\\2&4&0&7\end{bmatrix}, \quad b = \begin{bmatrix}b_1\\b_2\end{bmatrix}$$

Find two vectors in the null space of A, and the complete solution to $Ax=b$.

Ans: $$\left[\begin{array}{cccc|c}1&2&0&3&b_1\\2&4&0&7&b_2\end{array}\right] \xrightarrow{R_2\to R_2-2R_1} \left[\begin{array}{cccc|c}1&2&0&3&b_1\\0&0&0&1&b_2-2b_1\end{array}\right]$$

$Ax=b$ into $Ux=c$

Vectors of ~~the~~ nullspace of matrix A satisfy

$$Ax=0 \iff Ux=0$$

Hence,

$$\begin{bmatrix}1&2&0&3\\2&4&0&7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \iff \begin{cases}x_1=-2x\\x_2, x_3 \in \mathbb{R}\\x_4=0\end{cases}$$

which means null space is generated by vectors

$$\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\}$$

To complete solution it only remains to find particular solution which satisfy $Ax_p=b \iff Ux_p=c$

$$\begin{bmatrix}1&2&0&3\\0&0&0&1\end{bmatrix}\begin{bmatrix}x_1\\0\\0\\x_4\end{bmatrix} = \begin{bmatrix}b_1\\b_2-2b_1\end{bmatrix} \Rightarrow \begin{cases}x_4=b_2-2b_1\\x_1+3x_4=b_1 \Rightarrow x_1=4b_1-3b_2\end{cases}$$

complete solution is given by

$$x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}4b_1-3b_2\\0\\0\\b_2-2b_1\end{bmatrix} + x_2\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$ Ans

15) Find the row reduced echelon form R and the rank of these matrices

a) The 3 by 4 matrix of all 1s.

b) The 4 by 4 matrix with $a_{ij} = (-1)^{ij}$

c) The 3 by 4 matrix with $a_{ij} = (-1)^j$

Ans a) 3x4 matrix of all 1s:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r(R) = 1$

b) $$\begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_4 \to R_4 + R_1} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - R_2} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 \to 1/2 R_2]{R_1 \to R_1 - 1/2 R_2} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r(R) = 2$

c) $$\begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r(R) = 1$

32) $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix}$ and $A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$

For every c, find R and the special solutions to $Ax = 0$

Ans: $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
$R_2 \Leftrightarrow R_3$

Here, y & z are free variable. If c=1 v will be a free variable

For c=1

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$u + v + 2w + 2y = 0 \Rightarrow u - v - 2w - 2y$

$$x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Special solutions are

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

For $c \neq 1$

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$u + v + w + y = 0$$

$$\Rightarrow (c-1) \cdot v = 0 \Rightarrow v = 0$$

$$u = -w - y$$

$$x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Special solutions: $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

b) $A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

For $c = 1$ u is a free variable, for $c = 2$ v is a free variables in any other case there is no free variables

For $c = 1$

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$0u + 2v = 0$$

$$0u + 1v = 0$$

$$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For $c = 2$

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-u + 2v = 0$$

$$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Special solution $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

36) $x + 3y + 3z = 1$
$2x + 6y + 9z = 5$
$-x - 3y + 3z = 5$

and $\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

Ans

$$\begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

By Elimination, $\left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 5 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ -1 & -3 & 3 & 5 \end{array}\right]$.

$\xrightarrow{R_3 \to R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\xrightarrow{R_2 \to 1/3 R_2} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - 3R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow [R|C]$

To find special solution

$Rx = 0$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x = -3y \\ y \in \mathbb{R} \\ z = 0 \end{array}$$

$\therefore y$ is free variable and special solution is given by

$$\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

Now, find particular solution to

$Ax_p = b \Leftrightarrow Rx_p = C$

with free variables $y = 0$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x = -2 \\ z = 1 \end{cases}$$

Particular solution is $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

complete solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{2} R_2} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1/2 \\ 0 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x + 3y + 2t = 1/2 \Rightarrow x = \frac{1}{2} - 3y$

$y \in \mathbb{R}$

$z = \frac{1}{2} - 2t$

$t \in \mathbb{R}$

Here, y & t are free variables and complete solution is given by

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1/2 - 3y \\ y \\ 1/2 - 2t \\ t \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

56) $Ux = C$ and then $Rx = d$

$$Ax = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = b$$

Find particular solution and null space solutions x_n.

Ans $[A \;\; b] = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 1 & 3 & 2 & 0 & 5 \\ 2 & 0 & 4 & 9 & 10 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 3 & 0 & -3 & 3 \\ 2 & 0 & 4 & 9 & 10 \end{array}\right] = [U \; c]$

$$\xrightarrow{R_2 \to \frac{1}{3} R_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 & 6 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{3} R_3} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] = [R \; d]$$

To find special solutions $Rx_n = 0$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = -2x_3 - 3x_4 = -2x_3 \\ x_2 = x_4 = 0 \\ x_3 \in \mathbb{R} \end{array}$$

$$x_n = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Particular Solution

$Rx_n = d$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{l} x_1 = 2-2x_3-3x_4 = -4 \\ x_2 = 1+x_4 = 3 \\ x_3 = 0 \\ x_4 = 2 \end{array}\right.$$

$$x_p = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}$$

Ex-2.3

4) Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Solve $c_1v_1 + \cdots + c_4v_4 = 0$ or $Ac = 0$ The v's go in the column of A.

Ans $A = [v_1, v_2\ v_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Now, Solve system $AC=0$

$$AC = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_2 \to R_2 - R_3]{R_1 \Leftrightarrow R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$c_1 = 0, c_2 = 0, c_3 = 0$

$\therefore$ Linear independent

Now, we have to show v_1, v_2, v_3 & v_4 are linear dependent

$$A = [v_1, v_2, v_3, v_4] = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$AC = 0 \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right]$$

$$c_1 = \begin{cases} c_1 = c_4 \\ c_2 = c_4 \\ c_3 = -4c_4 \end{cases}$$

$c_4 \in \mathbb{R}$

$c_1 = 1, c_2 = 1, c_3 = -4, c_4 = 1$

$\therefore$ Linear dependent

10) If w_1, w_2, w_3 are independent vectors, show that sums $v_1 = w_2 + w_3$, $v_2 = w_1 + w_3$ and $v_3 = w_1 + w_2$ are independent.

Ans:

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \vec{0}$$

are $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

Let's consider an arbitary linear combination of the vectors v_1, v_2 and v_3

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \vec{0}$$

Substituting values of vectors v_i and arranging the vectors, we obtain

$$\alpha_1 (w_2 + w_3) + \alpha_2 (w_1 + w_3) + \alpha_3 (w_1 + w_2) = \vec{0};$$

$$(\alpha_2 + \alpha_3) w_1 + (\alpha_1 + \alpha_3) w_2 + (\alpha_1 + \alpha_2) w_3 = \vec{0}.$$

Since, the vectors w_i, are linearly independent we obtain the following linear system of equation α_i, $i = 1, 2, 3 \ldots$

$$\alpha_2 + \alpha_3 = 0$$
$$\alpha_1 + \alpha_3 = 0$$
$$\alpha_1 + \alpha_2 = 0,$$

$$|A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = (0 + 1 + 1) - (0 + 0 + 0) = 2 \neq 0.$$

$$\alpha_1 = \alpha_2 = \alpha_3 = 0$$

13) Decide whether it is linear independent or not by solving $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$Vc = 0 \Rightarrow [v_1, v_2\ v_3\ v_4]\, c = 0$$

$$\Rightarrow c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$$

$$Vc = 0 \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1}$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_4 \to R_4 - R_3} \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} c_1 = -c_4 \\ c_2 = c_4 \\ c_3 = -c_4 \\ c_4 \in \mathbb{R} \end{cases}$$

$\therefore$ $v_1\ v_2\ v_3\ v_4$ are linearly dependant

In the second part it is not linearly independent

31) For which numbers c and d these matrices have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$$

Rank = no. of linear independent columns.

$$\begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2/d R_3} \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 0 & 2-2/d \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$$

$\therefore \boxed{c = 0 \text{ and } d = 2}$

32) True or false?

a) If the columns of A are linearly independent, then $Ax = b$ has exactly one solution for every b.

b) A 5 by 7 matrix never has linear independent columns.

Ans: a) Let, this is false

$Ax = b$ and $Ay = b$

Then we would have

$$\begin{cases} Ax = b \\ Ay = b \end{cases} \Rightarrow Ax = Ay$$

$$\Rightarrow A(x-y) = 0 \xrightarrow{A \neq 0} x = y$$

True

b) True

This is true because in order for all columns to be independent matrix needs to be square.

40) Which are bases for $\mathbb{R}^3$?

a) $(1,2,0)$ and $(0,1,-1)$

b) $(1,1,-1)$, $(2,3,4)$, $(4,1,-1)$, $(0,1,-1)$

c) $(1,2,2)$, $(-1,2,1)$, $(0,8,0)$

d) $(1,2,2)$, $(-1,2,1)$, $(0,8,6)$

Ans: a) This is NOT BASIS FOR $\mathbb{R}^3$

In order to have basis for $\mathbb{R}^3$ we need to have three linearly independent Vectors and here we have only two.

b) Not basis for $\mathbb{R}^3$.

Here, $C_1 = C_2 = C_3 = 0 \Rightarrow V_1, V_2, V_3$ are independent.

c) Basis for $\mathbb{R}^3$

$\boxed{C_1 = C_2 = C_3 = 0}$

d) Not basis for $\mathbb{R}^3$

$C_1 = -2C_3$, $C_2 = -2C_3$, $C_3 \in \mathbb{R}$

V_1, V_2, V_3 are linearly dependent vectors and therefore don't make basis for $\mathbb{R}^3$.

Ex-2.4.5

3) Find dimension and construct a basis for the four subspaces associated with each matrix

$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Ans: Row space

U has rank =1, we know non-zero rows from basis for the row spaces $C(A^T)$ & $C(U^T)$.

$\therefore \dim C(A^T) = \dim C(U^T) = 1$ and $\left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$ is basis for both row

Null space

$Ux = 0$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \quad \Rightarrow x_2 + 4x_3 = 0 \Rightarrow x_2 = -4x_3$$

$$x = \begin{bmatrix} x_1 \\ -4x_3 \\ x_3 \\ x_4 \end{bmatrix} \Rightarrow x = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad x_1, x_3, x_4 \in \mathbb{R}$$

$\dim N(A) = \dim N(U) = 3$ and $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis for null spaces

Column Space

Only the pivot columns of matrix U form basis for column space
As there is only one pivot column, and that is second column

$$\dim C(U)=1 \text{ and } \left\{\begin{bmatrix}1\\0\end{bmatrix}\right\} \text{ is basis for } C(U).$$

For matrix A

$$\dim C(A)=1 \text{ and } \left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} \text{ is basis for } C(A).$$

Left null space

$U^T x=0$

$$\Rightarrow \begin{bmatrix}0&0\\1&0\\4&0\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=0 \Rightarrow x_1=0 \Rightarrow x=\begin{bmatrix}0\\x_2\end{bmatrix} \Rightarrow x=x_2\begin{bmatrix}0\\1\end{bmatrix}, x\in\mathbb{R}$$

$$\dim N(U^T)=1 \text{ and } \left\{\begin{bmatrix}0\\1\end{bmatrix}\right\} \text{ is basis for } N(U^T)$$

Matrix: A

$$\begin{bmatrix}0&0\\1&2\\4&8\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=0 \quad \Rightarrow x_1=-2x_2$$

$$\Rightarrow x=\begin{bmatrix}-2x_2\\x_2\end{bmatrix} \Rightarrow x_2\begin{bmatrix}-2\\1\end{bmatrix}, x_2\in\mathbb{R}$$

$$\dim N(A^T)=1 \text{ and } \left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\} \text{ is basis for } N(A^T).$$

6> Find rank A and write the matrix as $A=uv^T$

$$A=\begin{bmatrix}1&0&0&3\\0&0&0&0\\2&0&0&6\end{bmatrix} \text{ and } A=\begin{bmatrix}2&-2\\6&-6\end{bmatrix}$$

Ans: $A=\begin{bmatrix}1&0&0&3\\0&0&0&0\\2&0&0&6\end{bmatrix} \xrightarrow{R_3\to R_3-2R_1} \begin{bmatrix}1&0&0&3\\0&0&0&0\\0&0&0&0\end{bmatrix}$

rank $(A)=1$

$$A=uv^T=\begin{bmatrix}1\\0\\2\end{bmatrix}\begin{bmatrix}1&0&0&3\end{bmatrix} \text{ Ans}$$

11> Find a matrix A that has V as its row space and a matrix B that has V as its null space, if V is the subspace spanned by $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\5\\0\end{bmatrix}$

Ans: $A=\begin{bmatrix}\alpha\cdot 1 & \alpha\cdot 1 & \alpha\,0\\ \beta\cdot 1 & \beta\cdot 2 & \beta\cdot 5\\ \gamma\cdot 1 & \gamma\cdot 5 & \gamma\cdot 0\end{bmatrix}$

Ans: let, $\alpha = \beta = \gamma = 1$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 5 \\ 1 & 5 & 0 \end{bmatrix}$$

Matrix $B \in \mathbb{R}^{3\times3}$ whose null space contains subspace V satisfies following:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0 \Rightarrow \begin{cases} a+b=0 \\ d+e=0 \\ g+h=0 \end{cases}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0 \Rightarrow \begin{cases} a+2b=0 \\ d+2e=0 \\ g+2h=0 \end{cases}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} = 0 \Rightarrow \begin{cases} a+5b=0 \\ d+5c=0 \\ g+5h=0 \end{cases}$$

we get $a=b=d=e=g=h=0$

$$B = \begin{bmatrix} 0 & 0 & c \\ 0 & 0 & f \\ 0 & 0 & i \end{bmatrix}, \text{ where } c, f, i \in \mathbb{R}$$

17) Suppose the only solution to $Ax=0$ is $x=0$. What is the rank and why?

Ans: If $x=0$, the A has full rank, $r=n$

because $Ax=0 \Rightarrow [a_1, a_2, \dots a_n]\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0$

$\Rightarrow a_1.0 + a_2.0 + \dots + a_n.0 = 0$

$\therefore$ columns of A are linearly independent.

28) Without computing A, find basis for four fundamental subspaces.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Ans: Here LU factorization is given of matrix A.

rank $(A) = 3$

$\dim C(U) = \dim C(A) = \dim C(U^T) = \dim C(A^T) = 3$

$\Rightarrow C(A) = \mathbb{R}^3$ and $C(A^T) = \mathbb{R}^4$

$\dim N(U) = \dim N(A) = \dim N(U^T) = \dim N(A^T) = 3-3 = 0$

$\Rightarrow N(A) = N(A^T) = \{0\}$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is basis for row space of matrix A

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis for column space of matrix A

$\{0\}$ is basis for null space and left null space

31) Reduce A to echelon form and look at zero rows.

a) $\begin{bmatrix} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2 - 3b_1 \\ 4 & 6 & b_3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 4R_1} \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2 - 3b_1 \\ 0 & -2 & b_3 - 4b_1 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2 - 3b_1 \\ 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix}$

It has 2 independent rows.

$\dim N(A^T) = m - r = 3 - 2 = 1$

$b_3 - b_1 - b_2 = 0 \Rightarrow \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is basis for $N(A^T)$

b) $\begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1,\ R_4 \to R_4 - 2R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 - 2b_1 \\ 0 & 1 & b_4 - 2b_1 \end{bmatrix} \xrightarrow{R_4 \to R_4 + R_2} \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2 - 2R_1 \\ 0 & 0 & b_3 - 2b_1 \\ 0 & 0 & b_4 - 4b_1 + b_2 \end{bmatrix}$

$\dim N(B^T) = m - r = 4 - 2 = 2$

$b_3 - 2b_1 = 0$ and $b_4 - 4b_1 + b_2 = 0$

$\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis for $N(B^T)$.

32) True or false

a) A and A^T have same no. of pivots

b) A and A^T have same left nullspace.

c) If the row space equals the column space then $A^T = A$.

d) If $A^T = -A$ then row space of A equals column space.

Ans : a) we know row space and column space have same rank.

$\therefore r(A) = r(A^T)$

As they have same rank, they have same no. of pivots.

True

b) Let, $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

$A^T \cdot x = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 + 2x_2 = 0$

Base is $(-2, 1)$

$(A^T)A^T \cdot x = 0$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow x_1 = 0, \; x_2 = 0$

Base of left null space $(0,1)$

False

c) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

Row space: $\{(1,2), (1,3)\}$

Column space: $\{(1,2), (1,3)\}$

$\therefore$ False

d) when A^T would be the same as A claim will be true. As we are multiplying the whole matrix with factor -1 claim stays true. Vectors that span row space of A are same as vectors that span row space of A.