The vectors b are in the column space and the vectors x are in the nullspace. We shall compute the dimensions of those subspaces and a convenient set of vectors to generate them. We hope to end up by understanding all *four* of the subspaces that are intimately related to each other and to A—the column space of A, the nullspace of A, and their two perpendicular spaces.

Problem Set 2.1

1. Construct a subset of the x-y plane $\mathbf{R}^2$ that is

   (a) closed under vector addition and subtraction, but not scalar multiplication.
   (b) closed under scalar multiplication but not under vector addition.

   Hint: Starting with u and v, add and subtract for (a). Try cu and cv for (b).

2. Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

   (a) The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0$.

(b) The plane of vectors b with $b_1 = 1$.
(c) The vectors b with $b_2 b_3 = 0$ (this is the union of two subspaces, the pl[ane]
$b_2 = 0$ and the plane $b_3 = 0$).
(d) All combinations of two given vectors $(1, 1, 0)$ and $(2, 0, 1)$.
(e) The plane of vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0$.

3. What is the smallest subspace of 3 by 3 matrices that contains all symmetric matr[ices] *and* all lower triangular matrices? What is the largest subspace that is containe[d in] both of those subspaces?

4. Which of the following are subspaces of $\mathbf{R}^\infty$?
(a) All sequences like $(1, 0, 1, 0, \ldots)$ that include infinitely many zeros.
(b) All sequences $(x_1, x_2, \ldots)$ with $x_j = 0$ from some point onward.
(c) All decreasing sequences: $x_{j+1} \le x_j$ for each j.
(d) All convergent sequences: the x_j have a limit as $j \to \infty$.
(e) All arithmetic progressions: $x_{j+1} - x_j$ is the same for all j.
(f) All geometric progressions $(x_1, kx_1, k^2x_1, \ldots)$ allowing all k and x_1.

5. Describe the column space and the nullspace of the matrices

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

6. Addition and scalar multiplication are required to satisfy these eight rules:

1. $x + y = y + x$.
2. $x + (y + z) = (x + y) + z$.
3. There is a unique "zero vector" such that $x + 0 = x$ for all x.
4. For each x there is a unique vector $-x$ such that $x + (-x) = 0$.
5. $1x = x$.
6. $(c_1 c_2)x = c_1(c_2 x)$.
7. $c(x + y) = cx + cy$.
8. $(c_1 + c_2)x = c_1 x + c_2 x$.

(a) Suppose addition in $\mathbf{R}^2$ adds an extra 1 to each component, so that $(3, 1) + ($[5, 0]$)$ equals $(9, 2)$ instead of $(8, 1)$. With scalar multiplication unchanged, w[hich] rules are broken?
(b) Show that the set of all positive real numbers, with $x + y$ and cx redefin[ed to] equal the usual xy and x^c, is a vector space. What is the "zero vector"?
(c) Suppose $(x_1, x_2) + (y_1, y_2)$ is defined to be $(x_1 + y_2, x_2 + y_1)$. With the u[sual] $cx = (cx_1, cx_2)$, which of the eight conditions are not satisfied?

7. Let $\mathbf{P}$ be the plane in 3-space with equation $x + 2y + z = 6$. What is the equ[ation] of the plane $\mathbf{P}_0$ through the origin parallel to $\mathbf{P}$? Are $\mathbf{P}$ and $\mathbf{P}_0$ subspaces of $\mathbf{R}^3$?

8. Which of the following descriptions are correct? The solutions x of

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

form

(a) a plane.
(b) a line.
(c) a point.
(d) a subspace.
(e) the nullspace of A.
(f) the column space of A.

9. (a) Describe a subspace of **M** that contains $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but not $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.
(b) If a subspace of **M** contains A and B, must it contain I?
(c) Describe a subspace of **M** that contains no nonzero diagonal matrices.

10. Show that the set of nonsingular 2 by 2 matrices is not a vector space. Show also that the set of *singular* 2 by 2 matrices is not a vector space.

11. Describe the smallest subspace of the 2 by 2 matrix space **M** that contains

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. (d) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

12. $\mathbf{P}_0$ is the plane through (0, 0, 0) parallel to the plane **P** in Problem 17. What is the equation for $\mathbf{P}_0$? Find two vectors in $\mathbf{P}_0$ and check that their sum is in $\mathbf{P}_0$.

13. The functions $f(x) = x^2$ and $g(x) = 5x$ are "vectors" in the vector space **F** of all real functions. The combination $3f(x) - 4g(x)$ is the function $h(x) =$ ______. Which rule is broken if multiplying $f(x)$ by c gives the function $f(cx)$?

14. The four types of subspaces of $\mathbf{R}^3$ are planes, lines, $\mathbf{R}^3$ itself, or **Z** containing only (0, 0, 0).
(a) Describe the three types of subspaces of $\mathbf{R}^2$.
(b) Describe the five types of subspaces of $\mathbf{R}^4$.

15. (a) The intersection of two planes through (0, 0, 0) is probably a ______ but it could be a ______. It can't be the zero vector **Z**!
(b) The intersection of a plane through (0, 0, 0) with a line through (0, 0, 0) is probably a ______ but it could be a ______.
(c) If **S** and **T** are subspaces of $\mathbf{R}^5$, their intersection $\mathbf{S} \cap \mathbf{T}$ (vectors in both subspaces) is a subspace of $\mathbf{R}^5$. *Check the requirements on $x + y$ and cx.*

16. If the sum of the "vectors" $f(x)$ and $g(x)$ in **F** is defined to be $f(g(x))$, then the "zero vector" is $g(x) = x$. Keep the usual scalar multiplication $cf(x)$, and find two rules that are broken.

17. Let **P** be the plane in $\mathbf{R}^3$ with equation $x + y - 2z = 4$. The origin (0, 0, 0) is not in **P**! Find two vectors in **P** and check that their sum is not in **P**.

18. The matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is a "vector" in the space **M** of all 2 by 2 matrices. Write the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$. What matrices are in the smallest subspace containing A?

19. True or false for $\mathbf{M}$ = all 3 by 3 matrices (check addition using an example)?
(a) The skew-symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} = -A$) form a subspace.
(b) The unsymmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} \neq A$) form a subspace.
(c) The matrices that have $(1, 1, 1)$ in their nullspace form a subspace.

20. Suppose $\mathbf{P}$ is a plane through $(0, 0, 0)$ and $\mathbf{L}$ is a line through $(0, 0, 0)$. The smal vector space containing both $\mathbf{P}$ and $\mathbf{L}$ is either ______ or ______.

Problems 21–31 are about column spaces $C(A)$ and the equation $Ax = b$.

21. Adding row 1 of A to row 2 produces B. Adding column 1 to column 2 produ C. A combination of the columns of ______ is also a combination of the colum of A. Which two matrices have the same column ______?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

22. (Recommended) If we add an extra column b to a matrix A, then the column sp gets larger unless ______. Give an example in which the column space gets lar and an example in which it doesn't. Why is $Ax = b$ solvable exactly when the colu space *doesn't* get larger by including b?

23. If A is any 8 by 8 invertible matrix, then its column space is ______. Why?

24. For which right-hand sides (find a condition on b_1, b_2, b_3) are these syste solvable?

$$\text{(a)} \quad \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \qquad \text{(b)} \quad \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

25. The columns of AB are combinations of the columns of A. This means: *The colu space of AB is contained in* (possibly equal to) *the column space of A.* Give example where the column spaces of A and AB are not equal.

26. Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

27. True or false (with a counterexample if false)?
(a) The vectors b that are not in the column space $C(A)$ form a subspace.
(b) If $C(A)$ contains only the zero vector, then A is the zero matrix.
(c) The column space of $2A$ equals the column space of A.
(d) The column space of $A - I$ equals the column space of A.

28. For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

29. Why isn't $\mathbf{R}^2$ a subspace of $\mathbf{R}^3$?

30. Construct a 3 by 3 matrix whose column space contains $(1, 1, 0)$ and $(1, 0, 1)$ but not $(1, 1, 1)$. Construct a 3 by 3 matrix whose column space is only a line.

31. If the 9 by 12 system $Ax = b$ is solvable for every b, then $C(A) =$ ______.

2.2 SOLVING $Ax=0$ AND $Ax=b$

Chapter 1 concentrated on square invertible matrices. There was one solution to $Ax = b$, and it was $x = A^{-1}b$. That solution was found by elimination (not by computing A^{-1}). A rectangular matrix brings new possibilities—U may not have a full set of pivots. This section goes onward from U to a reduced form R—**the simplest matrix that elimination can give.** R reveals all solutions immediately.

For an invertible matrix, the nullspace contains only $x = 0$ (multiply $Ax = 0$ by A^{-1}). The column space is the whole space ($Ax = b$ has a solution for every b). The new questions appear when the nullspace contains *more than the zero vector* and/or the column space contains *less than all vectors*:

1. Any vector x_n in the nullspace can be added to a particular solution x_p. The solutions to all linear equations have this form, $x = x_p + x_n$:

$$\textbf{Complete solution} \qquad Ax_p = b \ \ \textbf{and} \ \ Ax_n = 0 \ \ \textbf{produce} \ \ A(x_p + x_n) = b.$$

2. When the column space doesn't contain every b in $\mathbf{R}^m$, we need the conditions on b that make $Ax = b$ solvable.

A 3 by 4 example will be a good size. We will write down all solutions to $Ax = 0$. We will find the conditions for b to lie in the column space (so that $Ax = b$ is solvable). The 1 by 1 system $0x = b$, one equation and one unknown, shows two possibilities:

$0x = b$ has *no solution* unless $b = 0$. The column space of the 1 by 1 zero matrix contains only $b = 0$.

$0x = 0$ has *infinitely many solutions*. The nullspace contains *all* x. A particular solution is $x_p = 0$, and the complete solution is $x = x_p + x_n = 0 +$ (any x).

Simple, I admit. If you move up to 2 by 2, it's more interesting. The matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not invertible: $y + z = b_1$ and $2y + 2z = b_2$ usually have no solution.

There is ***no solution*** unless $b_2 = 2b_1$. The column space of A contains only those b's, the multiples of $(1, 2)$.

When $b_2 = 2b_1$ there are ***infinitely many solutions***. A particular solution to $y + z = 2$ and $2y + 2z = 4$ is $x_p = (1, 1)$. The nullspace of A in Figure 2.2 contains $(-1, 1)$ and all its multiples $x_n = (-c, c)$:

$$\textbf{Complete solution} \quad \begin{aligned} y + z &= 2 \\ 2y + 2z &= 4 \end{aligned} \quad \text{is solved by} \quad x_p + x_n = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - c \\ 1 + c \end{bmatrix}.$$

That final matrix $[R \ \ d]$ is $\text{rref}([A \ \ b]) = \text{rref}([U \ \ c])$. The numbers 2 and 0 and 2 and 1 in the free columns of R have opposite sign in the special solutions (the nullspace matrix N). Everything is revealed by $Rx = d$.

Problem Set 2.2

1. Find the value of c that makes it possible to solve $Ax = b$, and solve it:

$$\begin{aligned} u + v + 2w &= 2 \\ 2u + 3v - w &= 5 \\ 3u + 4v + w &= c. \end{aligned}$$

2. Find the echelon form U, the free variables, and the special solutions:

$$A = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

$Ax = b$ is consistent (has a solution) when b satisfies $b_2 =$ ______. Find the complete solution in the same form as equation (4).

3. Construct a system with more unknowns than equations, but no solution. Change the right-hand side to zero and find all solutions x_n.

4. Write the complete solutions $x = x_p + x_n$ to these systems, as in equation (4):

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

5. Reduce A and B to echelon form, to find their ranks. Which variables are free?

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Find the special solutions to $Ax = 0$ and $Bx = 0$. Find all solutions.

6. Carry out the same steps as in the previous problem to find the complete solution of $Mx = b$:

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

7. Describe the set of attainable right-hand sides b (in the column space) for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

by finding the constraints on b that turn the third equation into $0 = 0$ (after elimination). What is the rank, and a particular solution?

8. Find R for each of these (block) matrices, and the special solutions:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} A & A \end{bmatrix} \qquad C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}.$$

9. Find a 2 by 3 system $Ax = b$ whose complete solution is

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

Find a 3 by 3 system with these solutions exactly when $b_1 + b_2 = b_3$.

10. Which of these rules give a correct definition of the *rank* of A?
    (a) The number of nonzero rows in R.
    (b) The number of columns minus the total number of rows.
    (c) The number of columns minus the number of free columns.
    (d) The number of 1s in R.

11. If the r pivot variables come first, the reduced R must look like

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \qquad \begin{matrix} I \text{ is } r \text{ by } r \\ F \text{ is } r \text{ by } n - r \end{matrix}$$

What is the nullspace matrix N containing the special solutions?

12. Under what conditions on b_1 and b_2 (if any) does $Ax = b$ have a solution?

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Find two vectors in the nullspace of A, and the complete solution to $Ax = b$

13. (a) Find the special solutions to $Ux = 0$. Reduce U to R and repeat:

$$Ux = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

    (b) If the right-hand side is changed from $(0, 0, 0)$ to $(a, b, 0)$, what are the solutions?

14. Write a 2 by 2 system $Ax = b$ with many solutions x_n but no solution x_p. (The the system has no solution.) Which b's allow an x_p?

15. Find the reduced row echelon forms R and the rank of these matrices:
    (a) The 3 by 4 matrix of all 1s.
    (b) The 4 by 4 matrix with $a_{ij} = (-1)^{ij}$.
    (c) The 3 by 4 matrix with $a_{ij} = (-1)^j$.

16. If A is 2 by 3 and C is 3 by 2, show from its rank that $CA \neq I$. Give an example in which $AC = I$. For $m < n$, a right inverse is not a left inverse.

17. Find the ranks of AB and AM (rank 1 matrix times rank 1 matrix):

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1.5 & 6 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & b \\ c & bc \end{bmatrix}.$$

18. *If A has rank r, then it has an r by r submatrix S that is invertible.* Find that submatrix S from the pivot rows and pivot columns of each A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

19. If A has r pivot columns, then A^{T} has r pivot columns. Give a 3 by 3 example for which the column numbers are different for A and A^{T}.

20. Multiplying the rank 1 matrices $A = uv^{\mathrm{T}}$ and $B = wz^{\mathrm{T}}$ gives uz^{T} times the number ____. AB has rank 1 unless ____ $= 0$.

21. (Important) Suppose A and B are n by n matrices, and $AB = I$. Prove from $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$ that the rank of A is n. So A is invertible and B must be its two-sided inverse. Therefore $BA = I$ (*which is not so obvious!*).

22. Suppose A and B have the *same* reduced-row echelon form R. Explain how to change A to B by elementary row operations. So B equals an ________ matrix times A.

23. Suppose all r pivot variables come *last*. Describe the four blocks in the m by n reduced echelon form (the block B should be r by r):

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

What is the nullspace matrix N of special solutions? What is its shape?

24. Explain why the pivot rows and pivot columns of A (not R) always give an r by r invertible submatrix of A.

25. (Silly problem) Describe all 2 by 3 matrices A_1 and A_2 with row echelon forms R_1 and R_2, such that $R_1 + R_2$ is the row echelon form of $A_1 + A_2$. Is it true that $R_1 = A_1$ and $R_2 = A_2$ in this case?

26. What are the special solutions to $Rx = 0$ and $R^{\mathrm{T}}y = 0$ for these R?

$$R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

27. Every column of AB is a combination of the columns of A. Then the dimensions of the column spaces give $\mathbf{rank}(AB) \leq \mathbf{rank}(A)$. Problem: Prove also that $\mathbf{rank}(AB) \leq \mathbf{rank}(B)$.

28. Every m by n matrix of rank r reduces to (m by r) times (r by n):

$$A = (\text{pivot columns of } A)(\text{first } r \text{ rows of } R) = (\mathbf{COL})(\mathbf{ROW}).$$

16. If A is 2 by 3 and C is 3 by 2, show from its rank that $CA \neq I$. Give an example in which $AC = I$. For $m < n$, a right inverse is not a left inverse.

17. Find the ranks of AB and AM (rank 1 matrix times rank 1 matrix):

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1.5 & 6 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & b \\ c & bc \end{bmatrix}.$$

18. *If A has rank r, then it has an r by r submatrix S that is invertible.* Find that submatrix S from the pivot rows and pivot columns of each A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

19. If A has r pivot columns, then A^{T} has r pivot columns. Give a 3 by 3 example for which the column numbers are different for A and A^{T}.

20. Multiplying the rank 1 matrices $A = uv^{\mathrm{T}}$ and $B = wz^{\mathrm{T}}$ gives uz^{T} times the number ___. AB has rank 1 unless ___ $= 0$.

21. (Important) Suppose A and B are n by n matrices, and $AB = I$. Prove from $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$ that the rank of A is n. So A is invertible and B must be its two-sided inverse. Therefore $BA = I$ *(which is not so obvious!)*.

22. Suppose A and B have the *same* reduced-row echelon form R. Explain how to change A to B by elementary row operations. So B equals an ______ matrix times A.

23. Suppose all r pivot variables come *last*. Describe the four blocks in the m by n reduced echelon form (the block B should be r by r):

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

What is the nullspace matrix N of special solutions? What is its shape?

24. Explain why the pivot rows and pivot columns of A (not R) always give an r by r invertible submatrix of A.

25. (Silly problem) Describe all 2 by 3 matrices A_1 and A_2 with row echelon forms R_1 and R_2, such that $R_1 + R_2$ is the row echelon form of $A_1 + A_2$. Is it true that $R_1 = A_1$ and $R_2 = A_2$ in this case?

26. What are the special solutions to $Rx = 0$ and $R^{\mathrm{T}}y = 0$ for these R?

$$R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

27. Every column of AB is a combination of the columns of A. Then the dimensions of the column spaces give $\mathbf{rank}(AB) \leq \mathbf{rank}(A)$. Problem: Prove also that $\mathbf{rank}(AB) \leq \mathbf{rank}(B)$.

28. Every m by n matrix of rank r reduces to (m by r) times (r by n):

$$A = (\text{pivot columns of } A)(\text{first } r \text{ rows of } R) = (\mathbf{COL})(\mathbf{ROW}).$$

Write the 3 by 4 matrix A at the start of this section as the product of the 3 by [cut off] matrix from the pivot columns and the 2 by 4 matrix from R:

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

29. (Recommended) Execute the six steps following equation (6) to find the column space and nullspace of A and the solution to $Ax = b$:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}.$$

30. What is the nullspace matrix N (of special solutions) for A, B, C?

$$A = \begin{bmatrix} I & I \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} I & I & I \end{bmatrix}.$$

31. Suppose A is an m by n matrix of rank r. Its reduced echelon form is R. Describe exactly the *reduced row echelon form of* R^{T} (not A^{T}).

32. For every c, find R and the special solutions to $Ax = 0$:

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}.$$

Problems 33–36 are about the solution of $Ax = b$. Follow the steps in the text to x_p and x_n. Reduce the augmented matrix $[A \;\; b]$.

33. Which vectors (b_1, b_2, b_3) are in the column space of A? Which combinations of the rows of A give zero?

$$\text{(a) } A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix} \qquad \text{(b) } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}.$$

34. What conditions on b_1, b_2, b_3, b_4 make each system solvable? *Solve for* x:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

35. Under what condition on b_1, b_2, b_3 is the following system solvable? Include b as a fourth column in $[A \;\; b]$. Find all solutions when that condition holds:

$$\begin{aligned} x + 2y - 2z &= b_1 \\ 2x + 5y - 4z &= b_2 \\ 4x + 9y - 8z &= b_3. \end{aligned}$$

36. Find the complete solutions of

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5 \end{aligned} \quad \text{and} \quad \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

37. If you know x_p (free variables $= 0$) and all special solutions for $Ax = b$, find x_p and all special solutions for these systems:

$$Ax = 2b \qquad \begin{bmatrix} A & A \end{bmatrix} \begin{bmatrix} x \\ X \end{bmatrix} = b \qquad \begin{bmatrix} A \\ A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix}.$$

38. If $Ax = b$ has infinitely many solutions, why is it impossible for $Ax = B$ (new right-hand side) to have only one solution? Could $Ax = B$ have no solution?

39. Why can't a 1 by 3 system have $x_p = (2, 4, 0)$ and $x_n =$ any multiple of $(1, 1, 1)$?

40. (a) If $Ax = b$ has two solutions x_1 and x_2, find two solutions to $Ax = 0$.
(b) Then find another solution to $Ax = b$.

41. Explain why all these statements are false:

(a) The complete solution is any linear combination of x_p and x_n.
(b) A system $Ax = b$ has at most one particular solution.
(c) The solution x_p with all free variables zero is the shortest solution (minimum length $\|x\|$). (Find a 2 by 2 counterexample.)
(d) If A is invertible there is no solution x_n in the nullspace.

42. Give examples of matrices A for which the number of solutions to $Ax = b$ is

(a) 0 or 1, depending on b.
(b) ∞, regardless of b.
(c) 0 or ∞, depending on b.
(d) 1, regardless of b.

43. Write all known relations between r and m and n if $Ax = b$ has

(a) no solution for some b.
(b) infinitely many solutions for every b.
(c) exactly one solution for some b, no solution for other b.
(d) exactly one solution for every b.

44. Choose the number q so that (if possible) the ranks are (a) 1, (b) 2, (c) 3:

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}.$$

45. Apply Gauss–Jordan elimination (right-hand side becomes extra column) to $Ux = 0$ and $Ux = c$. Reach $Rx = 0$ and $Rx = d$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix}.$$

Solve $Rx = 0$ to find x_n (its free variable is $x_2 = 1$). Solve $Rx = d$ to find x_p (its free variable is $x_2 = 0$).

46. Suppose column 5 of U has no pivot. Then x_5 is a _______ variable. The zero vector (is) (is not) the only solution to $Ax = 0$. If $Ax = b$ has a solution, then it has _______ solutions.

47. Find A and B with the given property or explain why you can't.

(a) The only solution to $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) The only solution to $Bx = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

48. Is there a 3 by 3 matrix with no zero entries for which $U = R = I$?

49. Reduce these matrices A and B to their ordinary echelon forms U:

(a) $A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$.

Find a special solution for each free variable and describe every solution to $Ax =$ and $Bx = 0$. Reduce the echelon forms U to R, and draw a box around the iden matrix in the pivot rows and pivot columns.

50. Apply elimination with the extra column to reach $Rx = 0$ and $Rx = d$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & \mathbf{0} \\ 0 & 0 & 2 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & \mathbf{9} \\ 0 & 0 & 2 & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}.$$

Solve $Rx = 0$ (free variable = 1). What are the solutions to $Rx = d$?

51. Suppose column 4 of a 3 by 5 matrix is all 0s. Then x_4 is certainly a ______ variab The special solution for this variable is the vector $x =$ ______.

52. Put as many 1s as possible in a 4 by 7 echelon matrix U and in a *reduced* form whose pivot columns are 2, 4, 5.

53. The nullspace of a 3 by 4 matrix A is the line through $(2, 3, 1, 0)$.

(a) What is the *rank* of A and the complete solution to $Ax = 0$?
(b) What is the exact row reduced echelon form R of A?

54. True or False? (Give reason if true, or counterexample to show it is false.)

(a) A square matrix has no free variables.
(b) An invertible matrix has no free variables.
(c) An m by n matrix has no more than n pivot variables.
(d) An m by n matrix has no more than m pivot variables.

55. The complete solution to $Ax = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find A.

56. Reduce to $Ux = c$ (Gaussian elimination) and then $Rx = d$:

$$Ax = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = b.$$

Find a particular solution x_p and all nullspace solutions x_n.

57. Suppose column 1 + column 3 + column 5 = 0 in a 4 by 5 matrix with four pivots. Which column is sure to have no pivot (and which variable is free)? What is the special solution? What is the nullspace?

58. Suppose the first and last columns of a 3 by 5 matrix are the same (nonzero). Then ______ is a free variable. Find the special solution for this variable.

59. The equation $x - 3y - z = 0$ determines a plane in $\mathbf{R}^3$. What is the matrix A in this equation? Which are the free variables? The special solutions are $(3, 1, 0)$ and ______. The parallel plane $x - 3y - z = 12$ contains the particular point $(12, 0, 0)$. All points on this plane have the following form (fill in the first components):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \ \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \ \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \ \\ 0 \\ 1 \end{bmatrix}.$$

The next problems ask for matrices (if possible) with specific properties.

60. Construct a 2 by 2 matrix whose nullspace equals its column space.

61. Explain why A and $-A$ always have the same reduced echelon form R.

62. The reduced form R of a 3 by 3 matrix with randomly chosen entries is almost sure to be ______. What R is virtually certain if the random A is 4 by 3?

63. If the special solutions to $Rx = 0$ are in the columns of these N, go backward to find the nonzero rows of the reduced matrices R:

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \ \end{bmatrix} \quad \text{(empty 3 by 1)}.$$

64. Show by example that these three statements are generally *false*:

(a) A and A^{T} have the same nullspace.
(b) A and A^{T} have the same free variables.
(c) If R is the reduced form $\mathrm{rref}(A)$ then R^{T} is $\mathrm{rref}(A^{\mathrm{T}})$.

65. Construct a matrix whose nullspace consists of all combinations of $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$.

66. Construct a matrix whose column space contains $(1, 1, 1)$ and whose nullspace is the line of multiples of $(1, 1, 1, 1)$.

67. Construct a matrix whose nullspace consists of all multiples of $(4, 3, 2, 1)$.

68. Why does no 3 by 3 matrix have a nullspace that equals its column space?

69. Construct a matrix whose column space contains $(1, 1, 0)$ and $(0, 1, 1)$ and whose nullspace contains $(1, 0, 1)$ and $(0, 0, 1)$.

70. Construct a matrix whose column space contains $(1, 1, 5)$ and $(0, 3, 1)$ and whose nullspace contains $(1, 1, 2)$.

four-dimensional ***subspace***; an example is the set of vectors in $\mathbf{R}^6$ whose first a last components are zero. The members of this four-dimensional subspace are dimensional vectors like $(0, 5, 1, 3, 4, 0)$.

One final note about the language of linear algebra. We never use the terms "basi a matrix" or "rank of a space" or "dimension of a basis." These phrases have no mean It is *the dimension of the column space* that equals *the rank of the matrix*, as we pro in the coming section.

Problem Set 2.3

Problems 1–10 are about linear independence and linear dependence.

1. Choose three independent columns of U. Then make two other choices. Do the sa for A. You have found bases for which spaces?

$$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}.$$

2. Prove that if $a = 0$, $d = 0$, or $f = 0$ (3 cases), the columns of U are dependent:

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

3. Decide the dependence or independence of
   (a) the vectors $(1, 3, 2)$, $(2, 1, 3)$, and $(3, 2, 1)$.
   (b) the vectors $(1, -3, 2)$, $(2, 1, -3)$, and $(-3, 2, 1)$.

4. Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve $c_1v_1 + \cdots + c_4v_4 = 0$ or $Ac = 0$. The v's go in the columns of A.

5. If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w$ $v_2 = w_1 - w_3$, and $v_3 = w_1 - w_2$ are *dependent*. Find a combination of the v's th gives zero.

6. If a, d, f in Problem 2 are all nonzero, show that the only solution to $Ux = 0$ $x = 0$. Then U has independent columns.

7. Find the largest possible number of independent vectors among

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad v_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

This number is the ______ of the space spanned by the v's.

8. Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbf{R}^3$.

(a) These four vectors are dependent because _______.

(b) The two vectors v_1 and v_2 will be dependent if _______.

(c) The vectors v_1 and $(0, 0, 0)$ are dependent because _______.

9. Find two independent vectors on the plane $x+2y-3z-t=0$ in $\mathbf{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

10. If w_1, w_2, w_3 are independent vectors, show that the sums $v_1 = w_2 + w_3$, $v_2 = w_1 + w_3$, and $v_3 = w_1 + w_2$ are *independent*. (Write $c_1v_1 + c_2v_2 + c_3v_3 = 0$ in terms of the w's. Find and solve equations for the c's.)

Problems 11–18 are about the space *spanned* by a set of vectors. Take all linear combinations of the vectors.

11. The vector b is in the subspace spanned by the columns of A when there is a solution to _______. The vector c is in the row space of A when there is a solution to _______. *True or false*: If the zero vector is in the row space, the rows are dependent.

12. $v+w$ and $v-w$ are combinations of v and w. Write v and w as combinations of $v+w$ and $v-w$. The two pairs of vectors _______ the same space. When are they a basis for the same space?

13. Decide whether or not the following vectors are linearly independent, by solving $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Decide also if they span $\mathbf{R}^4$, by trying to solve $c_1v_1 + \cdots + c_4v_4 = (0, 0, 0, 1)$.

14. Suppose the vectors to be tested for independence are placed into the rows instead of the columns of A. How does the elimination process from A to U decide for or against independence?

15. Find the dimensions of (a) the column space of A, (b) the column space of U, (c) the row space of A, (d) the row space of U. Which two of the spaces are the same?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

16. Describe the subspace of $\mathbf{R}^3$ (is it a line or a plane or $\mathbf{R}^3$?) spanned by

(a) the two vectors $(1, 1, -1)$ and $(-1, -1, 1)$.

(b) the three vectors $(0, 1, 1)$ and $(1, 1, 0)$ and $(0, 0, 0)$.

(c) the columns of a 3 by 5 echelon matrix with 2 pivots.

(d) all vectors with positive components.

17. Choose $x = (x_1, x_2, x_3, x_4)$ in $\mathbf{R}^4$. It has 24 rearrangements like (x_2, x_1, x_3, x_4) and (x_4, x_3, x_1, x_2). Those 24 vectors, including x itself, span a subspace $\mathbf{S}$. Find specific vectors x so that the dimension of $\mathbf{S}$ is: (a) 0, (b) 1, (c) 3, (d) 4.

18. To decide whether b is in the subspace spanned by $w_1, \ldots, w_n$, let the vectors w b[e] the columns of A and try to solve $Ax = b$. What is the result for

(a) $w_1 = (1, 1, 0)$, $w_2 = (2, 2, 1)$, $w_3 = (0, 0, 2)$, $b = (3, 4, 5)$?
(b) $w_1 = (1, 2, 0)$, $w_2 = (2, 5, 0)$, $w_3 = (0, 0, 2)$, $w_4 = (0, 0, 0)$, and any b?

Problems 19–37 are about the requirements for a basis.

19. Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbf{R}^3$. Then find a basis for the interse[c]tion of that plane with the xy-plane. Then find a basis for all vectors perpendicula[r] to the plane.

20. If $v_1, \ldots, v_n$ are linearly independent, the space they span has dimension ______. These vectors are a ______ for that space. If the vectors are the columns of an m b[y] n matrix, then m is ______ than n.

21. Suppose $\mathbf{S}$ is a five-dimensional subspace of $\mathbf{R}^6$. True or false?

(a) Every basis for $\mathbf{S}$ can be extended to a basis for $\mathbf{R}^6$ by adding one more vecto[r].
(b) Every basis for $\mathbf{R}^6$ can be reduced to a basis for $\mathbf{S}$ by removing one vector.

22. The columns of A are n vectors from $\mathbf{R}^m$. If they are linearly independent, what is the rank of A? If they span $\mathbf{R}^m$, what is the rank? If they are a basis for $\mathbf{R}^m$, what the[n]?

23. U comes from A by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspaces.

24. Find three different bases for the column space of U above. Then find two differe[nt] bases for the row space of U.

25. Find a basis for each of these subspaces of $\mathbf{R}^4$:

(a) All vectors whose components are equal.
(b) All vectors whose components add to zero.
(c) All vectors that are perpendicular to $(1, 1, 0, 0)$ and $(1, 0, 1, 1)$.
(d) The column space (in $\mathbf{R}^2$) and nullspace (in $\mathbf{R}^5$) of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

26. Suppose the columns of a 5 by 5 matrix A are a basis for $\mathbf{R}^5$.

(a) The equation $Ax = 0$ has only the solution $x = 0$ because ______.
(b) If b is in $\mathbf{R}^5$ then $Ax = b$ is solvable because ______.

Conclusion: A is invertible. Its rank is 5.

27. Suppose $v_1, v_2, \ldots, v_6$ are six vectors in $\mathbf{R}^4$.

(a) Those vectors (do)(do not)(might not) span $\mathbf{R}^4$.
(b) Those vectors (are)(are not)(might be) linearly independent.
(c) Any four of those vectors (are)(are not)(might be) a basis for $\mathbf{R}^4$.
(d) If those vectors are the columns of A, then $Ax = b$ (has) (does not have) (mig[ht] not have) a solution.

28. Find a counterexample to the following statement: If v_1, v_2, v_3, v_4 is a basis for the vector space $\mathbf{R}^4$, and if $\mathbf{W}$ is a subspace, then some subset of the v's is a basis for $\mathbf{W}$.

29. If A is a 64 by 17 matrix of rank 11, how many independent vectors satisfy $Ax = 0$? How many independent vectors satisfy $A^{\mathrm{T}}y = 0$?

30. Suppose $\mathbf{V}$ is known to have dimension k. Prove that

(a) any k independent vectors in $\mathbf{V}$ form a basis;
(b) any k vectors that span $\mathbf{V}$ form a basis.

In other words, if the number of vectors is known to be correct, either of the two properties of a basis implies the other.

31. For which numbers c and d do these matrices have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}.$$

32. *True or false*?

(a) If the columns of A are linearly independent, then $Ax = b$ has exactly one solution for every b.
(b) A 5 by 7 matrix never has linearly independent columns.

33. Find a basis for each of these subspaces of 3 by 3 matrices:

(a) All diagonal matrices.
(b) All symmetric matrices ($A^{\mathrm{T}} = A$).
(c) All skew-symmetric matrices ($A^{\mathrm{T}} = -A$).

34. By locating the pivots, find a basis for the column space of

$$U = \begin{bmatrix} 0 & 5 & 4 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Express each column that is not in the basis as a combination of the basic columns. Find also a matrix A with this echelon form U, but a different column space.

35. Prove that if $\mathbf{V}$ and $\mathbf{W}$ are three-dimensional subspaces of $\mathbf{R}^5$, then $\mathbf{V}$ and $\mathbf{W}$ must have a nonzero vector in common. *Hint*: Start with bases for the two subspaces, making six vectors in all.

36. True or false (give a good reason)?

(a) If the columns of a matrix are dependent, so are the rows.
(b) The column space of a 2 by 2 matrix is the same as its row space.
(c) The column space of a 2 by 2 matrix has the same dimension as its row space.
(d) The columns of a matrix are a basis for the column space.

37. Find the dimensions of these vector spaces:

(a) The space of all vectors in $\mathbf{R}^4$ whose components add to zero.
(b) The nullspace of the 4 by 4 identity matrix.
(c) The space of all 4 by 4 matrices.

The next problems are about spaces in which the "vectors" are functions.

38. The cosine space $\mathbf{F}_3$ contains all combinations $y(x) = A\cos x + B\cos 2x + C\cos 3x$. Find a basis for the subspace that has $y(0) = 0$.

39. Write the 3 by 3 identity matrix as a combination of the other five permutation matrices! Then show that those five matrices are linearly independent. (Assume a combination gives zero, and check entries to prove each term is zero.) The five permutations are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

40. *Review*: Which of the following are bases for $\mathbf{R}^3$?

(a) $(1, 2, 0)$ and $(0, 1, -1)$.
(b) $(1, 1, -1)$, $(2, 3, 4)$, $(4, 1, -1)$, $(0, 1, -1)$.
(c) $(1, 2, 2)$, $(-1, 2, 1)$, $(0, 8, 0)$.
(d) $(1, 2, 2)$, $(-1, 2, 1)$, $(0, 8, 6)$.

41. Find a basis for the space of polynomials $p(x)$ of degree ≤ 3. Find a basis for the subspace with $p(1) = 0$.

42. (a) Find all functions that satisfy $\frac{dy}{dx} = 0$.
(b) Choose a particular function that satisfies $\frac{dy}{dx} = 3$.
(c) Find all functions that satisfy $\frac{dy}{dx} = 3$.

43. Suppose $y_1(x)$, $y_2(x)$, $y_3(x)$ are three different functions of x. The vector space they span could have dimension 1, 2, or 3. Give an example of y_1, y_2, y_3 to show each possibility.

44. *Review*: Suppose A is 5 by 4 with rank 4. Show that $Ax = b$ has no solution when the 5 by 5 matrix $[A \ \ b]$ is invertible. Show that $Ax = b$ is solvable when $[A \ \ b]$ is singular.

45. Find a basis for the space of functions that satisfy

(a) $$\frac{dy}{dx} - 2y = 0.$$

(b) $$\frac{dy}{dx} - \frac{y}{x} = 0.$$

2.4 THE FOUR FUNDAMENTAL SUBSPACES

The previous section dealt with definitions rather than constructions. We know what a basis is, but not how to find one. Now, starting from an explicit description of a su
we would like to compute an explicit basis.

Subspaces can be described in two ways. First, we may be giv
that span the space. (*Example*: The columns span the column spac
be told which conditions the vectors in the space must satisfy. (*Exa*
consists of all vectors that satisfy $Ax = 0$.)

The first description may include useless vectors (dependent c
description may include repeated conditions (dependent rows).
by inspection and a systematic

rank 1. At the same time, the columns are all multiples of the same vector; the column space shares the dimension $r = 1$ and reduces to a line.

> ***Every matrix of rank* 1 *has the simple form* $A = uv^{\mathrm{T}}$ = *column times row.***

The rows are all multiples of the same vector v^{T}, and the columns are all multiples of u. The row space and column space are lines—the easiest case.

Problem Set 2.4

1. Describe the four subspaces in three-dimensional space associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

2. Find the dimension and a basis for the four fundamental subspaces for

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. Find the dimension and construct a basis for the four subspaces associated with each of the matrices

$$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

4. If the product AB is the zero matrix, $AB = 0$, show that the column space of B is contained in the nullspace of A. (Also the row space of A is in the left nullspace of B, since each row of A multiplies B to give a zero row.)

5. True or false: If $m = n$, then the row space of A equals the column space. If $m < n$, then the nullspace has a larger dimension than ______.

6. Find the rank of A and write the matrix as $A = uv^{\mathrm{T}}$:

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix}.$$

7. If the columns of A are linearly independent (A is m by n), then the rank is _______, the nullspace is _______, the row space is _______, and there exists a _______-inverse.

8. If $Ax = b$ always has at least one solution, show that the only solution to $A^{\mathrm{T}}y = 0$ is $y = 0$. *Hint*: What is the rank?

9. Find a left-inverse and/or a right-inverse (when they exist) for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}.$$

10. Suppose A is an m by n matrix of rank r. Under what conditions on those numbers does

(a) A have a two-sided inverse: $AA^{-1} = A^{-1}A = I$?
(b) $Ax = b$ have *infinitely many* solutions for *every* b?

11. Find a matrix A that has $\mathbf{V}$ as its row space, and a matrix B that has $\mathbf{V}$ as its nullspace, if $\mathbf{V}$ is the subspace spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}.$$

12. Why is there no matrix whose row space and nullspace both contain (1, 1, 1)?

13. Find a basis for each of the four subspaces of

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

14. If a, b, c are given with $a \neq 0$, choose d so that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = uv^{\mathrm{T}}$$

has rank 1. What are the pivots?

15. (*A paradox*) Suppose A has a right-inverse B. Then $AB = I$ leads to $A^{\mathrm{T}}AB = A^{\mathrm{T}}$ or $B = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. But that satisfies $BA = I$; it is a *left*-inverse. Which step is not justified?

16. If $Ax = 0$ has a nonzero solution, show that $A^{\mathrm{T}}y = f$ fails to be solvable for some right-hand sides f. Construct an example of A and f.

17. Suppose the only solution to $Ax = 0$ (m equations in n unknowns) is $x = 0$. What is the rank and why? The columns of A are linearly _______.

18. Find a 1 by 3 matrix whose nullspace consists of all vectors in $\mathbf{R}^3$ such that $x_1 + 2x_2 + 4x_3 = 0$. Find a 3 by 3 matrix with that same nullspace.

19. Construct a matrix with $(1, 0, 1)$ and $(1, 2, 0)$ as a basis for its row space and its column space. Why can't this be a basis for the row space and nullspace?

20. Suppose the 3 by 3 matrix A is invertible. Write bases for the four subspaces for A and also for the 3 by 6 matrix $B = [A \;\; A]$.

21. If A has the same four fundamental subspaces as B, does $A = cB$?

22. If the entries of a 3 by 3 matrix are chosen randomly between 0 and 1, what are the most likely dimensions of the four subspaces? What if the matrix is 3 by 5?

23. (Important) A is an m by n matrix of rank r. Suppose there are right-hand sides for which $Ax = b$ has *no solution*.
   (a) What inequalities ($<$ or $\leq$) must be true between m, n, and r?
   (b) How do you know that $A^{\mathrm{T}}y = 0$ has a nonzero solution?

24. Construct a matrix with the required property, or explain why you can't.
   (a) Column space contains $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$, row space contains $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}2\\5\end{bmatrix}$.
   (b) Column space has basis $\begin{bmatrix}1\\2\\3\end{bmatrix}$, nullspace has basis $\begin{bmatrix}3\\2\\1\end{bmatrix}$.
   (c) Dimension of nullspace $= 1 +$ dimension of left nullspace.
   (d) Left nullspace contains $\begin{bmatrix}1\\3\end{bmatrix}$, row space contains $\begin{bmatrix}3\\1\end{bmatrix}$.
   (e) Row space $=$ column space, nullspace $\neq$ left nullspace.

25. Which subspaces are the same for these matrices of different sizes?
   (a) $[A]$ and $\begin{bmatrix}A\\A\end{bmatrix}$. (b) $\begin{bmatrix}A\\A\end{bmatrix}$ and $\begin{bmatrix}A & A\\A & A\end{bmatrix}$.
   Prove that all three matrices have the same rank r.

26. What are the dimensions of the four subspaces for A, B, and C, if I is the 3 by 3 identity matrix and 0 is the 3 by 2 zero matrix?
$$A = \begin{bmatrix}I & 0\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}I & I\\0^{\mathrm{T}} & 0^{\mathrm{T}}\end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix}0\end{bmatrix}.$$

27. (a) If a 7 by 9 matrix has rank 5, what are the dimensions of the four subspaces? What is the sum of all four dimensions?
   (b) If a 3 by 4 matrix has rank 3, what are its column space and left nullspace?

28. Without computing A, find bases for the four fundamental subspaces:
$$A = \begin{bmatrix}1 & 0 & 0\\6 & 1 & 0\\9 & 8 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 2 & 3\\0 & 0 & 1 & 2\end{bmatrix}.$$

29. Without elimination, find dimensions and bases for the four subspaces for
$$A = \begin{bmatrix}0 & 3 & 3 & 3\\0 & 0 & 0 & 0\\0 & 1 & 0 & 1\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}1 & 1\\4 & 4\\5 & 5\end{bmatrix}.$$

30. (Left nullspace) Add the extra column b and reduce A to echelon form:

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}.$$

A combination of the rows of A has produced the zero row. What combination is it? (Look at $b_3 - 2b_2 + b_1$ on the right-hand side.) Which vectors are in the nullspace of A^{T} and which are in the nullspace of A?

31. Following the method of Problem 30, reduce A to echelon form and look at zero rows. The b column tells which combinations you have taken of the rows:

(a) $\begin{bmatrix} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix}$.

From the b column after elimination, read off $m-r$ basis vectors in the left nullspace of A (combinations of rows that give zero).

32. True or false (with a reason or a counterexample)?

(a) A and A^{T} have the same number of pivots.
(b) A and A^{T} have the same left nullspace.
(c) If the row space equals the column space then $A^{\mathrm{T}} = A$.
(d) If $A^{\mathrm{T}} = -A$ then the row space of A equals the column space.

33. If you exchange the first two rows of a matrix A, which of the four subspaces stay the same? If $y = (1, 2, 3, 4)$ is in the left nullspace of A, write down a vector in the left nullspace of the new matrix.

34. Without multiplying matrices, find bases for the row and column spaces of A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

How do you know from these shapes that A is not invertible?

35. Explain why $v = (1, 0, -1)$ cannot be a row of A and also be in the nullspace.

36. Describe the four subspaces of $\mathbf{R}^3$ associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

37. Suppose A is the sum of two matrices of rank one: $A = uv^{\mathrm{T}} + wz^{\mathrm{T}}$.

(a) Which vectors span the column space of A?
(b) Which vectors span the row space of A?
(c) The rank is less than 2 if ______ or if ______.
(d) Compute A and its rank if $u = z = (1, 0, 0)$ and $v = w = (0, 0, 1)$.

38. Redraw Figure 2.5 for a 3 by 2 matrix of rank $r = 2$. Which subspace is $\mathbf{Z}$ (zero vector only)? The nullspace part of any vector x in $\mathbf{R}^2$ is $x_n =$ ______.

39. Construct any 2 by 3 matrix of rank 1. Copy Figure 2.5 and put one vector in each subspace (two in the nullspace). Which vectors are orthogonal?

40. If $AB = 0$, the columns of B are in the nullspace of A. If those vectors are in $\mathbf{R}^n$, prove that $\text{rank}(A) + \text{rank}(B) \leq n$.

41. Can tic-tac-toe be completed (5 ones and 4 zeros in A) so that $\text{rank}(A) = 2$ but neither side passed up a winning move?

2.5 GRAPHS AND NETWORKS

I am not entirely happy with the 3 by 4 matrix in the previous section. From a theoretical point of view it was very satisfactory; the four subspaces were computable and their dimensions $r, n-r, r, m-r$ were nonzero. But the example was not produced by