Problem Set 3.1

1. Which pairs are orthogonal among the vectors v_1, v_2, v_3, v_4?

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

2. Find a vector x orthogonal to the row space of A, and a vector y orthogonal to the column space, and a vector z orthogonal to the nullspace:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}.$$

3. Find all vectors in $\mathbf{R}^3$ that are orthogonal to $(1, 1, 1)$ and $(1, -1, 0)$. Produce an orthonormal basis from these vectors (mutually orthogonal unit vectors).

4. Two lines in the plane are perpendicular when the product of their slopes is -1. Apply this to the vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$, whose slopes are x_2/x_1 and y_2/y_1, to derive again the orthogonality condition $x^{\mathrm{T}}y = 0$.

5. Give an example in $\mathbf{R}^2$ of linearly independent vectors that are not orthogonal. Also, give an example of orthogonal vectors that are not independent.

6. How do we know that the ith row of an invertible matrix B is orthogonal to the jth column of B^{-1}, if $i \neq j$?

7. Find the lengths and the inner product of $x = (1, 4, 0, 2)$ and $y = (2, -2, 1, 3)$.

8. *Why are these statements false?*
   (a) If **V** is orthogonal to **W**, then $\mathbf{V}^\perp$ is orthogonal to $\mathbf{W}^\perp$.
   (b) **V** orthogonal to **W** and **W** orthogonal to **Z** makes **V** orthogonal to **Z**.

9. Find a basis for the orthogonal complement of the row space of A:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

Split $x = (3, 3, 3)$ into a row space component x_r and a nullspace component x_n.

10. Let **P** be the plane in $\mathbf{R}^2$ with equation $x + 2y - z = 0$. Find a vector perpendicular to **P**. What matrix has the plane **P** as its nullspace, and what matrix has **P** as its row space?

11. Find all vectors that are perpendicular to $(1, 4, 4, 1)$ and $(2, 9, 8, 2)$.

12. Show that $x - y$ is orthogonal to $x + y$ if and only if $\|x\| = \|y\|$.

13. Let **S** be the subspace of $\mathbf{R}^4$ containing all vectors with $x_1 + x_2 + x_3 + x_4 = 0$. Find a basis for the space $\mathbf{S}^\perp$, containing all vectors orthogonal to **S**.

14. Find the orthogonal complement of the plane spanned by the vectors $(1, 1, 2)$ and $(1, 2, 3)$, by taking these to be the rows of A and solving $Ax = 0$. Remember that the complement is a whole line.

15. Let **S** be a subspace of $\mathbf{R}^n$. Explain what $(\mathbf{S}^\perp)^\perp = \mathbf{S}$ means and why it is true.

16. Illustrate the action of A^{T} by a picture corresponding to Figure 3.4, sending $\boldsymbol{C}(A)$ back to the row space and the left nullspace to zero.

17. If $\mathbf{S} = \{0\}$ is the subspace of $\mathbf{R}^4$ containing only the zero vector, what is $\mathbf{S}^\perp$? If **S** is spanned by $(0, 0, 0, 1)$, what is $\mathbf{S}^\perp$? What is $(\mathbf{S}^\perp)^\perp$?

18. If **V** and **W** are orthogonal subspaces, show that the only vector they have in common is the zero vector: $\mathbf{V} \cap \mathbf{W} = \{0\}$.

19. The fundamental theorem is often stated in the form of *Fredholm's alternative*: For any A and b, one and only one of the following systems has a solution:
   (i) $Ax = b$.
   (ii) $A^{\mathrm{T}}y = 0$, $y^{\mathrm{T}}b \neq 0$.

   Either b is in the column space $\boldsymbol{C}(A)$ or there is a y in $\boldsymbol{N}(A^{\mathrm{T}})$ such that $y^{\mathrm{T}}b \neq 0$. Show that it is contradictory for (i) and (ii) both to have solutions.

20. If **V** is the orthogonal complement of **W** in $\mathbf{R}^n$, is there a matrix with row space **V** and nullspace **W**? Starting with a basis for **V**, construct such a matrix.

21. Find a matrix whose row space contains $(1, 2, 1)$ and whose nullspace contains $(1, -2, 1)$, or prove that there is no such matrix.

22. Construct a homogeneous equation in three unknowns whose solutions are the linear combinations of the vectors $(1, 1, 2)$ and $(1, 2, 3)$. This is the reverse of the previous exercise, but the two problems are really the same.

23. If $AB = 0$ then the columns of B are in the ______ of A. The rows of A are in the ______ of B. Why can't A and B be 3 by 3 matrices of rank 2?

24. In Figure 3.4, how do we know that Ax_r is equal to Ax? How do we know that this vector is in the column space? If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, what is x_r?

25. (a) If $Ax = b$ has a solution and $A^{\mathrm{T}}y = 0$, then y is perpendicular to ______.
(b) If $A^{\mathrm{T}}y = c$ has a solution and $Ax = 0$, then x is perpendicular to ______.

26. If Ax is in the nullspace of A^{T} then $Ax = 0$. Reason: Ax is also in the ______ of A and the spaces are ______. *Conclusion: $A^{\mathrm{T}}A$ has the same nullspace as A.*

27. (Recommended) Draw Figure 3.4 to show each subspace for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

28. Redraw Figure 3.4 for a 3 by 2 matrix of rank $r = 2$. Which subspace is $\mathbf{Z}$ (zero vector only)? The nullspace part of any vector x in $\mathbf{R}^2$ is $x_n =$ ______.

29. This is a system of equations $Ax = b$ with *no solution*:

$$\begin{aligned} x + 2y + 2z &= 5 \\ 2x + 2y + 3z &= 5 \\ 3x + 4y + 5z &= 9. \end{aligned}$$

Find numbers y_1, y_2, y_3 to multiply the equations so they add to $0 = 1$. You have found a vector y in which subspace? The inner product $y^{\mathrm{T}}b$ is 1.

30. Construct an unsymmetric 2 by 2 matrix of rank 1. Copy Figure 3.4 and put one vector in each subspace. Which vectors are orthogonal?

31. Find the pieces x_r and x_n, and draw Figure 3.4 properly, if

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

32. Construct a matrix with the required property or say why that is impossible.

(a) Column space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b) Row space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(c) $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution and $A^{\mathrm{T}}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(d) Every row is orthogonal to every column (A is not the zero matrix).

(e) The columns add up to a column of 0s, the rows add to a row of 1s.

33. Suppose A is a symmetric matrix ($A^{\mathrm{T}} = A$).

(a) Why is its column space perpendicular to its nullspace?

(b) If $Ax = 0$ and $Az = 5z$, which subspaces contain these "eigenvectors" x and z? **Symmetric matrices have perpendicular eigenvectors** (see Section 5.5).

Problems 34–44 are about orthogonal subspaces.

34. The floor and the wall are not orthogonal subspaces because they share a nonzero vector (along the line where they meet). Two planes in $\mathbf{R}^3$ cannot be orthogonal! Find a vector in both column spaces $C(A)$ and $C(B)$:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{bmatrix}.$$

This will be a vector Ax and also $B\widehat{x}$. Think 3 by 4 with the matrix $[A \ \ B]$.

35. Suppose $\mathbf{S}$ is spanned by the vectors $(1, 2, 2, 3)$ and $(1, 3, 3, 2)$. Find two vectors that span $\mathbf{S}^\perp$. This is the same as solving $Ax = 0$ for which A?

36. Suppose $\mathbf{S}$ only contains $(1, 5, 1)$ and $(2, 2, 2)$ (not a subspace). Then $\mathbf{S}^\perp$ is the nullspace of the matrix $A =$ ______. $\mathbf{S}^\perp$ is a subspace even if $\mathbf{S}$ is not.

37. Extend Problem 34 to a p-dimensional subspace $\mathbf{V}$ and a q-dimensional subspace $\mathbf{W}$ of $\mathbf{R}^n$. What inequality on $p + q$ guarantees that $\mathbf{V}$ intersects $\mathbf{W}$ in a nonzero vector? These subspaces cannot be orthogonal.

38. If a subspace $\mathbf{S}$ is contained in a subspace $\mathbf{V}$, prove that $\mathbf{S}^\perp$ contains $\mathbf{V}^\perp$.

39. If $\mathbf{P}$ is the plane of vectors in $\mathbf{R}^4$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$, write a basis for $\mathbf{P}^\perp$. Construct a matrix that has $\mathbf{P}$ as its nullspace.

40. Suppose $\mathbf{V}$ is the whole space $\mathbf{R}^4$. Then $\mathbf{V}^\perp$ contains only the vector ______. Then $(\mathbf{V}^\perp)^\perp$ is ______. So $(\mathbf{V}^\perp)^\perp$ is the same as ______.

41. If $\mathbf{S}$ is the subspace of $\mathbf{R}^3$ containing only the zero vector, what is $\mathbf{S}^\perp$? If $\mathbf{S}$ is spanned by $(1, 1, 1)$, what is $\mathbf{S}^\perp$? If $\mathbf{S}$ is spanned by $(2, 0, 0)$ and $(0, 0, 3)$, what is $\mathbf{S}^\perp$?

42. Put bases for the orthogonal subspaces $\mathbf{V}$ and $\mathbf{W}$ into the columns of matrices V and W. *Why does* $V^{\mathrm{T}}W =$ *zero matrix*? This matches $v^{\mathrm{T}}w = 0$ for vectors.

43. Prove that every y in $N(A^{\mathrm{T}})$ is perpendicular to every Ax in the column space, using the matrix shorthand of equation (8). Start from $A^{\mathrm{T}}y = 0$.

44. Suppose $\mathbf{L}$ is a one-dimensional subspace (a line) in $\mathbf{R}^3$. Its orthogonal complement $\mathbf{L}^\perp$ is the ______ perpendicular to $\mathbf{L}$. Then $(\mathbf{L}^\perp)^\perp$ is a ______ perpendicular to $\mathbf{L}^\perp$. In fact $(\mathbf{L}^\perp)^\perp$ is the same as ______.

Problems 45–50 are about perpendicular columns and rows.

45. Find $A^{\mathrm{T}}A$ if the columns of A are unit vectors, all mutually perpendicular.

46. Construct a 3 by 3 matrix A with no zero entries whose columns are mutually perpendicular. Compute $A^{\mathrm{T}}A$. Why is it a diagonal matrix?

47. Suppose an n by n matrix is invertible: $AA^{-1} = I$. Then the first column of A^{-1} is orthogonal to the space spanned by which rows of A?

48. The lines $3x + y = b_1$ and $6x + 2y = b_2$ are ______. They are the same line if ______. In that case (b_1, b_2) is perpendicular to the vector ______. The nullspace of the matrix is the line $3x + y =$ ______. One particular vector in that nullspace is ______.

49. The command $N = \mathsf{null}(A)$ will produce a basis for the nullspace of A. Then the command $B = \mathsf{null}(N')$ will produce a basis for the ______ of A.

50. Why is each of these statements false?
    (a) $(1, 1, 1)$ is perpendicular to $(1, 1, -2)$, so the planes $x + y + z = 0$ and $x + y - 2z = 0$ are orthogonal subspaces.
    (b) The subspace spanned by $(1, 1, 0, 0, 0)$ and $(0, 0, 0, 1, 1)$ is the orthogonal complement of the subspace spanned by $(1, -1, 0, 0, 0)$ and $(2, -2, 3, 4, -4)$.
    (c) Two subspaces that meet only in the zero vector are orthogonal.

51. Find a matrix with $v = (1, 2, 3)$ in the row space and column space. Find another matrix with v in the nullspace and column space. Which pairs of subspaces can v *not* be in?

52. Suppose A is 3 by 4, B is 4 by 5, and $AB = 0$. Prove $\operatorname{rank}(A) + \operatorname{rank}(B) \leq 4$.

3.2 COSINES AND PROJECTIONS ONTO LINES

Vectors with $x^{\mathrm{T}}y = 0$ are orthogonal. Now we allow inner products that are not zero

Problem Set 3.2

1. (a) Given any two positive numbers x and y, choose the vector b equal to $(\sqrt{x}, \sqrt{y})$, and choose $a = (\sqrt{y}, \sqrt{x})$. Apply the Schwarz inequality to compare the arithmetic mean $\frac{1}{2}(x + y)$ with the geometric mean $\sqrt{xy}$.

   (b) Suppose we start with a vector from the origin to the point x, and then add a vector of length $\|y\|$ connecting x to $x + y$. The third side of the triangle goes from the origin to $x + y$. *The triangle inequality asserts that this distance* *cannot be greater than the sum of the first two:*

$$\|x+y\| \le \|x\| + \|y\|.$$

After squaring both sides, and expanding $(x+y)^{\mathrm{T}}(x+y)$, reduce this to the Schwarz inequality.

2. Square the matrix $P = aa^{\mathrm{T}}/a^{\mathrm{T}}a$, which projects onto a line, and show that $P^2 = P$. (Note the number $a^{\mathrm{T}}a$ in the middle of the matrix $aa^{\mathrm{T}}aa^{\mathrm{T}}$!)

3. By choosing the correct vector b in the Schwarz inequality, prove that

$$(a_1 + \cdots + a_n)^2 \le n\left(a_1^2 + \cdots + a_n^2\right).$$

When does equality hold?

4. Verify that the length of the projection in Figure 3.7 is $\|p\| = \|b\| \cos\theta$, using formula (5).

5. (a) Find the projection matrix P_1 onto the line through $a = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and also the matrix P_2 that projects onto the line perpendicular to a.
(b) Compute $P_1 + P_2$ and $P_1 P_2$ and explain.

6. The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$—note that all six edges have length $\sqrt{2}$, so the tetrahedron is regular—what is the cosine of the angle between the rays going from the center $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ to the vertices? (The bond angle itself is about 109.5°, an old friend of chemists.)

7. Explain why the Schwarz inequality becomes an equality in the case that a and b lie on the same line through the origin, and only in that case. What if they lie on opposite sides of the origin?

8. Prove that the *trace* of $P = aa^{\mathrm{T}}/a^{\mathrm{T}}a$—which is the sum of its diagonal entries—always equals 1.

9. Find the matrix that projects every point in the plane onto the line $x + 2y = 0$.

10. In n dimensions, what angle does the vector $(1, 1, \ldots, 1)$ make with the coordinate axes? What is the projection matrix P onto that vector?

11. What multiple of $a = (1, 1, 1)$ is closest to the point $b = (2, 4, 4)$? Find also the point closest to a on the line through b.

12. Is the projection matrix P invertible? Why or why not?

13. The Schwarz inequality has a one-line proof if a and b are normalized ahead of time to be unit vectors:

$$|a^{\mathrm{T}}b| = \left|\sum a_j b_j\right| \le \sum |a_j||b_j| \le \sum \frac{|a_j|^2 + |b_j|^2}{2} = \frac{1}{2} + \frac{1}{2} = \|a\|\,\|b\|.$$

Which previous problem justifies the middle step?

14. Show that the length of Ax equals the length of $A^{\mathrm{T}}x$ if $AA^{\mathrm{T}} = A^{\mathrm{T}}A$.

15. Suppose P is the projection matrix onto the line through a.

(a) Why is the inner product of x with Py equal to the inner product of Px with y?

(b) Are the two angles the same? Find their cosines if $a = (1, 1, -1)$, $x = (2, 0, 1)$, $y = (2, 1, 2)$.

(c) Why is the inner product of Px with Py again the same? What is the angle between those two?

16. What matrix P projects every point in $\mathbf{R}^3$ onto the line of intersection of the planes $x + y + t = 0$ and $x - t = 0$?

Problems 17–26 ask for projections onto lines. Also errors $e = b - p$ and matrices P.

17. *Draw* the projection of b onto a and also compute it from $p = \widehat{x}a$:

(a) $b = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ and $a = \begin{bmatrix}1\\0\end{bmatrix}$. (b) $b = \begin{bmatrix}1\\1\end{bmatrix}$ and $a = \begin{bmatrix}1\\-1\end{bmatrix}$.

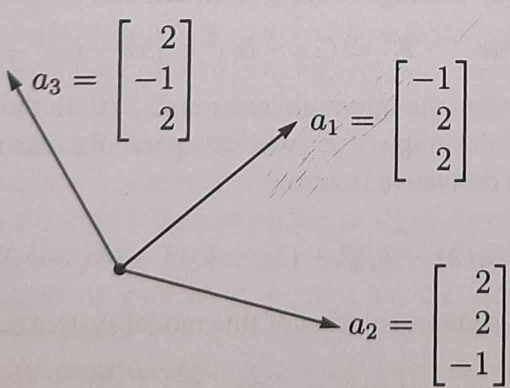
18. Construct the projection matrices P_1 and P_2 onto the lines through the a's in Problem 17. Is it true that $(P_1 + P_2)^2 = P_1 + P_2$? This *would* be true if $P_1P_2 = 0$.

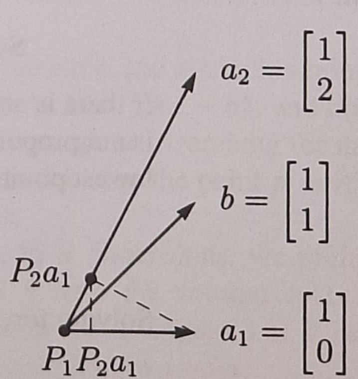
19. Project the vector b onto the line through a. Check that e is perpendicular to a:

(a) $b = \begin{bmatrix}1\\2\\2\end{bmatrix}$ and $a = \begin{bmatrix}1\\1\\1\end{bmatrix}$. (b) $b = \begin{bmatrix}1\\3\\1\end{bmatrix}$ and $a = \begin{bmatrix}-1\\-3\\-1\end{bmatrix}$.

20. In Problem 19, find the projection matrix $P = aa^{\mathrm{T}}/a^{\mathrm{T}}a$ onto the line through each vector a. Verify in both cases that $P^2 = P$. Multiply Pb in each case to compute the projection p.

For Problems 21–26, consult the accompanying figures.





21. Compute the projection matrices $aa^{\mathrm{T}}/a^{\mathrm{T}}a$ onto the lines through $a_1 = (-1, 2, 2)$ and $a_2 = (2, 2, -1)$. Multiply those projection matrices and explain why their product P_1P_2 is what it is.

22. Project the vector $b = (1, 1)$ onto the lines through $a_1 = (1, 0)$ and $a_2 = (1, 2)$. Draw the projections p_1 and p_2 and add $p_1 + p_2$. The projections do not add to b because the a's are not orthogonal.

23. In Problem 22, the projection of b onto the *plane* of a_1 and a_2 will equal b. Find $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ for $A = \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

24. Project $b = (1, 0, 0)$ onto the lines through a_1 and a_2 in Problem 21 and also onto $a_3 = (2, -1, 2)$. Add the three projections $p_1 + p_2 + p_3$.

25. Project $a_1 = (1, 0)$ onto $a_2 = (1, 2)$. Then project the result back onto a_1. Draw these projections and multiply the projection matrices $P_1 P_2$: Is this a projection?

26. Continuing Problems 21, 24 find the projection matrix P_3 onto $a_3 = (2, -1, 2)$. Verify that $P_1 + P_2 + P_3 = I$. The basis a_1, a_2, a_3 is orthogonal!

3.3 PROJECTIONS AND LEAST SQUARES

Up to this point, $Ax = b$ either has a solution or not. If b is not in the column space

Problem Set 3.3

1. Solve $Ax = b$ by least squares, and find $p = A\widehat{x}$ if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Verify that the error $b - p$ is perpendicular to the columns of A.

2. Write out $E^2 = \|Ax - b\|^2$ and set to zero its derivatives with respect to u and v, if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad x = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

Compare the resulting equations with $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$, confirming that calculus as well as geometry gives the normal equations. Find the solution $\widehat{x}$ and the projection $p = A\widehat{x}$. Why is $p = b$?

3. Suppose the values $b_1 = 1$ and $b_2 = 7$ at times $t_1 = 1$ and $t_2 = 2$ are fitted by a line $b = Dt$ *through the origin*. Solve $D = 1$ and $2D = 7$ by least squares, and sketch the best line.

4. The following system has no solution:

$$Ax = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix} = b.$$

Sketch and solve a straight-line fit that leads to the minimization of the quadratic $(C - D - 4)^2 + (C - 5)^2 + (C + D - 9)^2$. What is the projection of b onto the column space of A?

5. Find the best least-squares solution $\widehat{x}$ to $3x = 10$, $4x = 5$. What error E^2 is minimized? Check that the error vector $(10 - 3\widehat{x}, 5 - 4\widehat{x})$ is perpendicular to the column $(3, 4)$.

6. Find the projection of b onto the column space of A:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$

Split b into $p+q$, with p in the column space and q perpendicular to that space. Which of the four subspaces contains q?

7. If the vectors a_1, a_2, and b are orthogonal, what are $A^{\mathrm{T}}A$ and $A^{\mathrm{T}}b$? What is the projection of b onto the plane of a_1 and a_2?

8. If P is the projection matrix onto a line in the x-y plane, draw a figure to describe the effect of the "reflection matrix" $H = I - 2P$. Explain both geometrically and algebraically why $H^2 = I$.

9. Find the best straight-line fit (least squares) to the measurements

$$\begin{aligned} b &= 4 \quad \text{at} \quad t = -2, & b &= 3 \quad \text{at} \quad t = -1, \\ b &= 1 \quad \text{at} \quad t = 0, & b &= 0 \quad \text{at} \quad t = 2. \end{aligned}$$

Then find the projection of $b = (4, 3, 1, 0)$ onto the column space of

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

10. (a) If $P = P^{\mathrm{T}}P$, show that P is a projection matrix.
(b) What subspace does the matrix $P = 0$ project onto?

11. The vectors $a_1 = (1, 1, 0)$ and $a_2 = (1, 1, 1)$ span a plane in $\mathbf{R}^3$. Find the projection matrix P onto the plane, and find a nonzero vector b that is projected to zero.

12. Find the projection matrix P onto the space spanned by $a_1 = (1, 0, 1)$ and $a_2 = (1, 1, -1)$.

13. What 2 by 2 matrix projects the x-y plane onto the $-45°$ line $x + y = 0$?

14. Show that if u has unit length, then the rank-1 matrix $P = uu^{\mathrm{T}}$ is a projection matrix: It has properties (i) and (ii) in 3N. By choosing $u = a/\|a\|$, P becomes the projection onto the line through a, and Pb is the point $p = \widehat{x}a$. Rank-1 projections correspond exactly to least-squares problems in one unknown.

15. If $\mathbf{V}$ is the subspace spanned by $(1, 1, 0, 1)$ and $(0, 0, 1, 0)$, find
(a) a basis for the orthogonal complement $\mathbf{V}^{\perp}$.
(b) the projection matrix P onto $\mathbf{V}$.
(c) the vector in $\mathbf{V}$ closest to the vector $b = (0, 1, 0, -1)$ in $\mathbf{V}^{\perp}$.

16. If P is the projection matrix onto a k-dimensional subspace $\mathbf{S}$ of the whole space $\mathbf{R}^n$, what is the column space of P and what is its rank?

17. Suppose P is the projection matrix onto the subspace $\mathbf{S}$ and Q is the projection onto the orthogonal complement $\mathbf{S}^{\perp}$. What are $P+Q$ and PQ? Show that $P-Q$ is its own inverse.

18. We want to fit a plane $y = C + Dt + Ez$ to the four points

$$\begin{aligned} y &= 3 \quad \text{at} \quad t = 1,\ z = 1 & y &= 6 \quad \text{at} \quad t = 0,\ z = 3 \\ y &= 5 \quad \text{at} \quad t = 2,\ z = 1 & y &= 0 \quad \text{at} \quad t = 0,\ z = 0. \end{aligned}$$

(a) Find 4 equations in 3 unknowns to pass a plane through the points (if there is such a plane).
(b) Find 3 equations in 3 unknowns for the best least-squares solution.

19. Suppose L_1 is the line through the origin in the direction of a_1 and L_2 is the line through b in the direction of a_2. To find the closest points x_1a_1 and $b + x_2a_2$ on the two lines, write the two equations for the x_1 and x_2 that minimize $\|x_1a_1 - x_2a_2 - b\|$. Solve for x if $a_1 = (1, 1, 0)$, $a_2 = (0, 1, 0)$, $b = (2, 1, 4)$.

20. Show that the best least-squares fit to a set of measurements $y_1, \ldots, y_m$ by a *horizontal line* (a constant function $y = C$) is their average

$$C = \frac{y_1 + \cdots + y_m}{m}.$$

21. Suppose that instead of a straight line, we fit the data in Problem 23 by a parabola: $y = C + Dt + Et^2$. In the inconsistent system $Ax = b$ that comes from the four measurements, what are the coefficient matrix A, the unknown vector x, and the data vector b? You need not compute $\widehat{x}$.

22. If P is the projection onto the column space of A, what is the projection onto the left nullspace?

23. Find the best straight-line fit to the following measurements, and sketch your solution:

$$\begin{aligned} y &= 2 \quad \text{at} \quad t = -1, \qquad & y &= 0 \quad \text{at} \quad t = 0, \\ y &= -3 \quad \text{at} \quad t = 1, \qquad & y &= -5 \quad \text{at} \quad t = 2. \end{aligned}$$

24. A Middle-Aged man was stretched on a rack to lengths $L = 5$, 6, and 7 feet under applied forces of $F = 1$, 2, and 4 tons. Assuming Hooke's law $L = a + bF$, find his normal length a by least squares.

25. If $P_C = A(A^TA)^{-1}A^T$ is the projection onto the column space of A, what is the projection P_R onto the row space? (It is not P_C^T!)

26. Find the best line $C + Dt$ to fit $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

Problems 27–31 introduce basic ideas of statistics—the foundation for least squares.

27. Second assumption behind least squares: The m errors e_i are independent with variance σ^2, so the average of $(b - Ax)(b - Ax)^T$ is $\sigma^2 I$. Multiply on the left by $(A^TA)^{-1}A^T$ and on the right by $A(A^TA)^{-1}$ to show that the average of $(\widehat{x}-x)(\widehat{x}-x)^T$ is $\sigma^2(A^TA)^{-1}$. This is the all-important ***covariance matrix*** for the error in $\widehat{x}$.

28. (Recommended) This problem projects $b = (b_1, \ldots, b_m)$ onto the line through $a = (1, \ldots, 1)$. We solve m equations $ax = b$ in 1 unknown (by least squares).
(a) Solve $a^Ta\widehat{x} = a^Tb$ to show that $\widehat{x}$ is the *mean* (the average) of the b's.
(b) Find $e = b - a\widehat{x}$, the *variance* $\|e\|^2$, and the *standard deviation* $\|e\|$.
(c) The horizontal line $\widehat{b} = 3$ is closest to $b = (1, 2, 6)$. Check that $p = (3, 3, 3)$ is perpendicular to e and find the projection matrix P.

29. First assumption behind least squares: Each measurement error has ***mean zero***. Multiply the 8 error vectors $b - Ax = (\pm 1, \pm 1, \pm 1)$ by $(A^TA)^{-1}A^T$ to show that the 8 vectors $\widehat{x} - x$ also average to zero. The estimate $\widehat{x}$ is *unbiased*.

30. If you know the average $\widehat{x}_9$ of 9 numbers $b_1, \ldots, b_9$, how can you quickly find the average $\widehat{x}_{10}$ with one more number b_{10}? The idea of *recursive* least squares is to avoid adding 10 numbers. What coefficient of $\widehat{x}_9$ correctly gives $\widehat{x}_{10}$?

$$\widehat{x}_{10} = \tfrac{1}{10} b_{10} + \underline{\quad}\, \widehat{x}_9 = \tfrac{1}{10}(b_1 + \cdots + b_{10}).$$

31. A doctor takes four readings of your heart rate. The best solution to $x = b_1, \ldots, x = b_4$ is the average $\widehat{x}$ of $b_1, \ldots, b_4$. The matrix A is a column of 1s. Problem 27 gives the expected error $(\widehat{x} - x)^2$ as $\sigma^2(A^{\mathrm{T}}A)^{-1} = \underline{\qquad}$. By averaging, the variance drops from σ^2 to $\sigma^2/4$.

Problems 33, 34, 38–40 use four points $b = (0, 8, 8, 20)$ to bring out more ideas.

32. From m independent measurements $b_1, \ldots, b_m$ of your pulse rate, weighted by $w_1, \ldots, w_m$, what is the weighted average that replaces equation (9)? It is the best estimate when the statistical variances are $\sigma_i^2 = 1/w_i^2$.

33. For the closest cubic $b = C + Dt + Et^2 + Ft^3$ to the same four points, write the four equations $Ax = b$. Solve them by elimination. This cubic now goes exactly through the points. What are p and e?

34. (Line $C + Dt$ does go through p's) With $b = 0, 8, 8, 20$ at times $t = 0, 1, 3, 4$, write the four equations $Ax = b$ (unsolvable). Change the measurements to $p = 1, 5, 13, 17$ and find an exact solution to $A\widehat{x} = p$.

35. If $W = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, find the W-inner product of $x = (2, 3)$ and $y = (1, 1)$, and the W-length of x. What line of vectors is W-perpendicular to y?

36. What happens to the weighted average $\widehat{x}_W = (w_1^2 b_1 + w_2^2 b_2)/(w_1^2 + w_2^2)$ if the first weight w_1 approaches zero? The measurement b_1 is totally unreliable.

37. Check that $e = b - p = (-1, 3, -5, 3)$ is perpendicular to both columns of A. What is the shortest distance $\|e\|$ from b to the column space of A?

38. For the closest parabola $b = C + Dt + Et^2$ to the same four points, write the unsolvable equations $Ax = b$ in three unknowns $x = (C, D, E)$. Set up the three normal equations $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$ (solution not required). You are now fitting a parabola to four points—what is happening in Figure 3.9b?

39. With $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$, set up and solve the normal equations $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$. For the best straight line as in Figure 3.9a, find its four heights p_i and four errors e_i. What is the minimum value $E^2 = e_1^2 + e_2^2 + e_3^2 + e_4^2$?

40. The average of the four times is $\widehat{t} = \frac{1}{4}(0 + 1 + 3 + 4) = 2$. The average of the four b's is $\widehat{b} = \frac{1}{4}(0 + 8 + 8 + 20) = 9$.

(a) Verify that the best line goes *through the center point* $(\widehat{t}, \widehat{b}) = (2, 9)$.
(b) Explain why $C + D\widehat{t} = \widehat{b}$ comes from the first equation in $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$.

41. Find the weighted least-squares solution $\widehat{x}_W$ to $Ax = b$:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad W = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Check that the projection $A\widehat{x}_W$ is still perpendicular (in the W-inner product!) to the error $b - A\widehat{x}_W$.

42. (a) Suppose you guess your professor's age, making errors $e = -2, -1, 5$ with probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$. Check that the expected error $E(e)$ is zero and find the variance $E(e^2)$.

(b) If the professor guesses too (or tries to remember), making errors $-1, 0, 1$ with probabilities $\frac{1}{8}, \frac{6}{8}, \frac{1}{8}$, what weights w_1 and w_2 give the reliability of your guess and the professor's guess?

Problem Set 3.4

1. Project $b = (0, 3, 0)$ onto each of the orthonormal vectors $a_1 = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$ and $a_2 = \left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$, and then find its projection p onto the plane of a_1 and a_2.

2. (a) Write the four equations for fitting $y = C + Dt$ to the data

$$\begin{array}{llllll} y = -4 & \text{at} & t = -2, & y = -3 & \text{at} & t = -1 \\ y = -1 & \text{at} & t = 1, & y = 0 & \text{at} & t = 2. \end{array}$$

Show that the columns are orthogonal.

(b) Find the optimal straight line, draw its graph, and write E^2.

(c) Interpret the zero error in terms of the original system of four equations in two unknowns: The right-hand side $(-4, -3, -1, 0)$ is in the ______ space.

3. If u is a unit vector, show that $Q = I - 2uu^{\mathrm{T}}$ is a symmetric orthogonal matrix. (It is a reflection, also known as a Householder transformation.) Compute Q when $u^{\mathrm{T}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

4. Project the vector $b = (1, 2)$ onto two vectors that are not orthogonal, $a_1 = (1, 0)$ and $a_2 = (1, 1)$. Show that, unlike the orthogonal case, the sum of the two one-dimensional projections does not equal b.

5. Find also the projection of $b = (0, 3, 0)$ onto $a_3 = (\frac{2}{3}, -\frac{1}{3}, \frac{2}{3})$, and add the three projections. Why is $P = a_1a_1^{\mathrm{T}} + a_2a_2^{\mathrm{T}} + a_3a_3^{\mathrm{T}}$ equal to I?

6. From the nonorthogonal a, b, c, find orthonormal vectors q_1, q_2, q_3:

$$a = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

7. Show that an orthogonal matrix that is upper triangular must be diagonal.

8. If q_1 and q_2 are the outputs from Gram-Schmidt, what were the possible input vectors a and b?

9. Apply the Gram-Schmidt process to

$$a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and write the result in the form $A = QR$.

10. Find a third column so that the matrix

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & \\ 1/\sqrt{3} & 2/\sqrt{14} & \\ 1/\sqrt{3} & -3/\sqrt{14} & \end{bmatrix}$$

is orthogonal. It must be a unit vector that is orthogonal to the other columns; how much freedom does this leave? Verify that the rows automatically become orthonormal at the same time.

11. What multiple of $a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ should be subtracted from $a_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ to make the result orthogonal to a_1? Factor $\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}$ into QR with orthonormal vectors in Q.

12. Show, by forming $b^{\mathrm{T}}b$ directly, that Pythagoras's law holds for any combination $b = x_1q_1 + \cdots + x_nq_n$ of orthonormal vectors: $\|b\|^2 = x_1^2 + \cdots + x_n^2$. In matrix terms, $b = Qx$, so this again proves that lengths are preserved: $\|Qx\|^2 = \|x\|^2$.

13. If the vectors q_1, q_2, q_3 are orthonormal, what combination of q_1 and q_2 is closest to q_3?

14. If Q_1 and Q_2 are orthogonal matrices, so that $Q^{\mathrm{T}}Q = I$, show that Q_1Q_2 is also orthogonal. If Q_1 is rotation through θ, and Q_2 is rotation through ϕ, what is Q_1Q_2? Can you find the trigonometric identities for $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$ in the matrix multiplication Q_1Q_2?

15. If $A = QR$, find a simple formula for the projection matrix P onto the column space of A.

16. Find an orthonormal set q_1, q_2, q_3 for which q_1, q_2 span the column space of

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

Which fundamental subspace contains q_3? What is the least-squares solution of $Ax = b$ if $b = \begin{bmatrix} 1 & 2 & 7 \end{bmatrix}^{\mathrm{T}}$?

17. What is the closest function $a \cos x + b \sin x$ to the function $f(x) = \sin 2x$ on the interval from $-\pi$ to π? What is the closest straight line $c + dx$?

18. What is the closest straight line to the parabola $y = x^2$ over $-1 \le x \le 1$?

19. By setting the derivative to zero, find the value of b_1 that minimizes

$$\|b_1 \sin x - \cos x\|^2 = \int_0^{2\pi} (b_1 \sin x - \cos x)^2 \, dx.$$

Compare with the Fourier coefficient b_1.

20. Express the Gram-Schmidt orthogonalization of a_1, a_2 as $A = QR$:

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \qquad a_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

Given n vectors a_i with m components, what are the shapes of A, Q, and R?

21. Find the fourth Legendre polynomial. It is a cubic $x^3 + ax^2 + bx + c$ that is orthogonal to 1, x, and $x^2 - \frac{1}{3}$ over the interval $-1 \le x \le 1$.

22. In Hilbert space, find the length of the vector $v = (1/\sqrt{2}, 1/\sqrt{4}, 1/\sqrt{8}, \ldots)$ and the length of the function $f(x) = e^x$ (over the interval $0 \le x \le 1$). What is the inner product over this interval of e^x and e^{-x}?

23. With the same matrix A as in Problem 16, and with $b = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}$, use $A = QR$ to solve the least-squares problem $Ax = b$.

24. In the Gram-Schmidt formula (10), verify that C is orthogonal to q_1 and q_2.

25. Show that these *modified Gram-Schmidt* steps produce the same C as in equation (10):

$$C^* = c - (q_1^{\mathrm{T}} c) q_1 \qquad \text{and} \qquad C = C^* - (q_2^{\mathrm{T}} C^*)\, q_2.$$

This is much more stable, to subtract the projections one at a time.

26. Find the Fourier coefficients a_0, a_1, b_1 of the step function $y(x)$, which equals 1 on the interval $0 \le x \le \pi$ and 0 on the remaining interval $\pi < x < 2\pi$:

$$a_0 = \frac{(y, 1)}{(1, 1)} \qquad a_1 = \frac{(y, \cos x)}{(\cos x, \cos x)} \qquad b_1 = \frac{(y, \sin x)}{(\sin x, \sin x)}.$$

27. If $A = QR$ then $A^{\mathrm{T}}A = R^{\mathrm{T}}R =$ _____ triangular times _____ triangular. *Gram-Schmidt on A corresponds to elimination on $A^{\mathrm{T}}A$.* Compare

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{with} \quad A^{\mathrm{T}}A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

For $A^{\mathrm{T}}A$, the pivots are $2, \frac{3}{2}, \frac{4}{3}$ and the multipliers are $-\frac{1}{2}$ and $-\frac{2}{3}$.

(a) Using those multipliers in A, show that column 1 of A and $B =$ column 2 $-\frac{1}{2}$(column 1) and $C =$ column 3 $-\frac{2}{3}$ (column 2) are orthogonal.

(b) Check that $\|\text{column } 1\|^2 = 2$, $\|B\|^2 = \frac{3}{2}$, and $\|C\|^2 = \frac{4}{3}$, using the pivots.

28. True or false (give an example in either case):

(a) Q^{-1} is an orthogonal matrix when Q is an orthogonal matrix.

(b) If Q (3 by 2) has orthonormal columns then $\|Qx\|$ always equals $\|x\|$.

29. Apply Gram-Schmidt to $(1, -1, 0)$, $(0, 1, -1)$, and $(1, 0, -1)$, to find an orthonormal basis on the plane $x_1 + x_2 + x_3 = 0$. What is the dimension of this subspace, and how many nonzero vectors come out of Gram-Schmidt?

30. Find an orthonormal basis for the subspace spanned by $a_1 = (1, -1, 0, 0)$, $a_2 = (0, 1, -1, 0)$, $a_3 = (0, 0, 1, -1)$.

31. (a) Find a basis for the subspace $\mathbf{S}$ in $\mathbf{R}^4$ spanned by all solutions of

$$x_1 + x_2 + x_3 - x_4 = 0.$$

(b) Find a basis for the orthogonal complement $\mathbf{S}^{\perp}$.

(c) Find b_1 in $\mathbf{S}$ and b_2 in $\mathbf{S}^{\perp}$ so that $b_1 + b_2 = b = (1, 1, 1, 1)$.

32. (Recommended) Find orthogonal vectors A, B, C by Gram-Schmidt from a, b, c:

$$a = (1, -1, 0, 0) \qquad b = (0, 1, -1, 0) \qquad c = (0, 0, 1, -1).$$

A, B, C and a, b, c are bases for the vectors perpendicular to $d = (1, 1, 1, 1)$.

Problem Set 3.5

1. Mark all the sixth roots of 1 in the complex plane. What is the primitive root w_6? (Find its real and imaginary part.) Which power of w_6 is equal to $1/w_6$? What is $1 + w + w^2 + w^3 + w^4 + w^5$?

2. Multiply the three matrices in equation (16) and compare with F. In which six entries do you need to know that $i^2 = -1$?

3. Solve the 4 by 4 system (6) if the right-hand sides are $y_0 = 2$, $y_1 = 0$, $y_2 = 2$, $y_3 = 0$. In other words, solve $F_4c = y$.

4. Compute $y = F_4c$ by the three steps of the Fast Fourier Transform if $c = (1, 0, 1, 0)$.

5. (a) If $y = (1, 1, 1, 1)$, show that $c = (1, 0, 0, 0)$ satisfies $F_4c = y$.
   (b) Now suppose $y = (1, 0, 0, 0)$, and find c.

6. F is symmetric. So transpose equation (14) to find a new Fast Fourier Transform!

7. Compute $y = F_8c$ by the three steps of the Fast Fourier Transform if $c = (1, 0, 1, 0, 1, 0, 1, 0)$. Repeat the computation with $c = (0, 1, 0, 1, 0, 1, 0, 1)$.

8. If you form a 3 by 3 submatrix of the 6 by 6 matrix F_6, keeping only the entries in its first, third, and fifth rows and columns, what is that submatrix?

9. For the 4 by 4 matrix, write out the formulas for c_0, c_1, c_2, c_3 and verify that *if f is odd then c is odd*. The vector f is odd if $f_{n-j} = -f_j$; for $n = 4$ that means $f_0 = 0$, $f_3 = -f_1$, $f_2 = 0$ as in $\sin 0$, $\sin \pi/2$, $\sin \pi$, $\sin 3\pi/2$. This is copied by c and it leads to a fast sine transform.

10. What are F^2 and F^4 for the 4 by 4 Fourier matrix F?

11. What are the square and the square root of w_{128}, the primitive 128th root of 1?

12. Invert the three factors in equation (14) to find a fast factorization of F^{-1}.

13. Find all solutions to the equation $e^{ix} = -1$, and all solutions to $e^{i\theta} = i$.

14. Find a permutation P of the columns of F that produces $FP = \overline{F}$ (n by n). Combine with $F\overline{F} = nI$ to find F^2 and F^4 for the n by n Fourier matrix.

15. For $n = 2$, write y_0 from the first line of equation (13) and y_1 from the second line. For $n = 4$, use the first line to find y_0 and y_1, and the second to find y_2 and y_3, all in terms of y' and y''.

16. Solve system in Problem 3 with $y = (2, 0, -2, 0)$ by knowing F_4^{-1} and computing $c = F_4^{-1}y$. Verify that $c_0 + c_1e^{ix} + c_2e^{2ix} + c_3e^{3ix}$ takes the values $2, 0, -2, 0$ at the points $x = 0, \pi/2, \pi, 3\pi/2$.

17. Find the eigenvalues of the "periodic" $-1, 2, -1$ matrix C. The -1s in the corners of C make it periodic (**a circulant matrix**):

$$C = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad \text{has } c_0 = 2,\ c_1 = -1,\ c_2 = 0,\ c_3 = -1.$$

Problems 18–20 introduce the idea of an eigenvector and eigenvalue, when a matrix times a vector is a multiple of that vector. This is the theme of Chapter 5.

18. To multiply C times x, when $C = FEF^{-1}$, we can multiply $F(E(F^{-1}x))$ instead. The direct Cx uses n^2 separate multiplications. Knowing E and F, the second way uses only $n \log_2 n + n$ multiplications. How many of those come from E, how many from F, and how many from F^{-1}?

19. All entries in the factorization of F_6 involve powers of w = sixth root of 1:

$$F_6 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_3 & \\ & F_3 \end{bmatrix} \begin{bmatrix} & P & \end{bmatrix}.$$

Write these factors with $1, w, w^2$ in D and $1, w^2, w^4$ in F_3. Multiply!

20. The columns of the Fourier matrix F are the *eigenvectors* of the cyclic permutation P. Multiply PF to find the eigenvalues λ_0 to λ_3:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \begin{bmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3 \end{bmatrix}.$$

This is $PF = F\Lambda$ or $P = F\Lambda F^{-1}$.

21. How could you quickly compute these four components of Fc starting from $c_0 + c_2, c_0 - c_2, c_1 + c_3, c_1 - c_3$? You are finding the Fast Fourier Transform!

$$Fc = \begin{bmatrix} c_0 + c_1 + c_2 + c_3 \\ c_0 + ic_1 + i^2c_2 + i^3c_3 \\ c_0 + i^2c_1 + i^4c_2 + i^6c_3 \\ c_0 + i^3c_1 + i^6c_2 + i^9c_3 \end{bmatrix}.$$

22. Two eigenvectors of this circulant matrix C are $(1, 1, 1, 1)$ and $(1, i, i^2, i^3)$. What are the eigenvalues e_0 and e_1?

$$\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = e_0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad C \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix} = e_1 \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix}.$$