Problem Set 1.2

1. Solve to find a combination of the columns that equals b:

Triangular system
$$\begin{aligned} u - v - w &= b_1 \\ v + w &= b_2 \\ w &= b_3. \end{aligned}$$

2. Sketch these three lines and decide if the equations are solvable:

3 by 2 system
$$\begin{aligned} x + 2y &= 2 \\ x - y &= 2 \\ y &= 1. \end{aligned}$$

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

3. For the equations $x + y = 4$, $2x - 2y = 4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector $(4, 4)$ on the right side).

4. Find two points on the line of intersection of the three planes $t = 0$ and $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

5. (Recommended) Describe the intersection of the three planes $u + v + w + z = 6$ and $u + w + z = 4$ and $u + w = 2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

6. These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$\begin{aligned} ax + 2y &= 0 \\ 2x + ay &= 0 \end{aligned}$$

7. Explain why the system

$$\begin{aligned} u + v + w &= 2 \\ u + 2v + 3w &= 1 \\ v + 2w &= 0 \end{aligned}$$

is singular by finding a combination of the three equations that adds up to $0 = 1$. What value should replace the last zero on the right side to allow the equations to have solutions—and what is one of the solutions?

8. (Recommended) Under what condition on y_1, y_2, y_3 do the points $(0, y_1)$, $(1, y_2)$, $(2, y_3)$ lie on a straight line?

9. When $b = (2, 5, 7)$, find a solution (u, v, w) to equation (4) different from the solution $(1, 0, 1)$ mentioned in the text.

10. Give two more right-hand sides in addition to $b = (2, 5, 7)$ for which equation (4) can be solved. Give two more right-hand sides in addition to $b = (2, 5, 6)$ for which it cannot be solved.

11. The column picture for exercise 7 (singular system) is

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = b.$$

Show that the three columns on the left lie in the same plane by expressing the third column as a combination of the first two. What are all the solutions (u, v, w) if b is the zero vector $(0, 0, 0)$?

12. Starting with $x + 4y = 7$, find the equation for the parallel line through $x = 0$, $y = 0$. Find the equation of another line that meets the first at $x = 3$, $y = 1$.

Problems 13–15 are a review of the row and column pictures.

13. For two linear equations in three unknowns x, y, z, the row picture will show (2 or 3) (lines or planes) in (two or three)-dimensional space. The column picture is in (two or three)-dimensional space. The solutions normally lie on a ______.

14. For four linear equations in two unknowns x and y, the row picture shows four ______. The column picture is in ______-dimensional space. The equations have no solution unless the vector on the right-hand side is a combination of ______.

15. Draw the two pictures in two planes for the equations $x - 2y = 0$, $x + y = 6$.

16. Find a point with $z = 2$ on the intersection line of the planes $x + y + 3z = 6$ and $x - y + z = 4$. Find the point with $z = 0$ and a third point halfway between.

17. In Problem 22 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a "singular case" because the third column is ______. Find two combinations of the columns that give $b = (2, 3, 5)$. This is only possible for $b = (4, 6, c)$ if $c =$ ______.

18. In these equations, the third column (multiplying w) is the *same* as the right side b. The column form of the equations *immediately* gives what solution for (u, v, w)?

$$\begin{aligned} 6u + 7v + 8w &= 8 \\ 4u + 5v + 9w &= 9 \\ 2u - 2v + 7w &= 7. \end{aligned}$$

19. Move the third plane in Problem 22 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line **L**, but the third plane doesn't ______ that line.

20. When equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution?

21. If (a, b) is a multiple of (c, d) with $abcd \neq 0$, *show that* (a, c) *is a multiple of* (b, d). This is surprisingly important: call it a challenge question. You could use numbers first to see how a, b, c, and d are related. The question will lead to:

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ has dependent rows then it has dependent columns.}$$

22. The first of these equations plus the second equals the third:

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ 2x + 3y + 2z &= 5. \end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also ______. The equations have infinitely many solutions (the whole line **L**). Find three solutions.

23. Normally 4 "planes" in four-dimensional space meet at a ______. Normally 4 column vectors in four-dimensional space can combine to produce b. What combination of $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$ produces $b = (3, 3, 3, 2)$? What 4 equations for x, y, z, t are you solving?

1.3 AN EXAMPLE OF GAUSSIAN ELIMINATION

The way to understand elimination is by example. We begin in three dimensions:

$$\textbf{Original system} \qquad \begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9. \end{aligned} \qquad (1)$$

The problem is to find the unknown values of u, v, and w, and we shall apply Gaussian elimination. (Gauss is recognized as the greatest of all mathematicians, but certainly not because of this invention, which probably took him ten minutes. Ironically, constant C is so large and the coding is so awkward that the new method is largely (or entirely) of theoretical interest. The newest problem is the cost with *many processors in parallel.*

Problem Set 1.3

Problems 1–9 are about elimination on 2 by 2 systems.

1. Choose a right-hand side which gives no solution and another right-hand side which gives infinitely many solutions. What are two of those solutions?

$$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= \_\_ \end{aligned}$$

2. What multiple of equation 2 should be *subtracted* from equation 3?

$$\begin{aligned} 2x - 4y &= 6 \\ -x + 5y &= 0. \end{aligned}$$

After this elimination step, solve the triangular system. If the right-hand side changes to $(-6, 0)$, what is the new solution?

3. Choose a coefficient b that makes this system singular. Then choose a right-hand side g that makes it solvable. Find two solutions in that singular case.

$$\begin{aligned} 2x + by &= 16 \\ 4x + 8y &= g. \end{aligned}$$

4. What multiple ℓ of equation 1 should be subtracted from equation 2?

$$\begin{aligned} 2x + 3y &= 1 \\ 10x + 9y &= 11. \end{aligned}$$

After this elimination step, write down the upper triangular system and circle the two pivots. The numbers 1 and 11 have no influence on those pivots.

5. Solve the triangular system of Problem 4 by back-substitution, y before x. Verify that x times $(2, 10)$ plus y times $(3, 9)$ equals $(1, 11)$. If the right-hand side changes to $(4, 44)$, what is the new solution?

6. What multiple ℓ of equation 1 should be subtracted from equation 2?

$$\begin{aligned} ax + by &= f \\ cx + dy &= g. \end{aligned}$$

The first pivot is a (assumed nonzero). Elimination produces what formula for the second pivot? What is y? The second pivot is missing when $ad = bc$.

7. What test on b_1 and b_2 decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture.

$$\begin{aligned} 3x - 2y &= b_1 \\ 6x - 4y &= b_2. \end{aligned}$$

8. For which numbers a does elimination break down (a) permanently, and (b) temporarily?

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6. \end{aligned}$$

Solve for x and y after fixing the second breakdown by a row exchange.

9. For which three numbers k does elimination break down? Which is fixed by a row exchange? In each case, is the number of solutions 0 or 1 or ∞?

$$\begin{aligned} kx + 3y &= 6 \\ 3x + ky &= -6. \end{aligned}$$

Problems 10–19 study elimination on 3 by 3 systems (and possible failure).

10. Which number b leads later to a row exchange? Which b leads to a missing pivot? In that singular case find a nonzero solution x, y, z.

$$\begin{aligned} x + by \phantom{{}-z} &= 0 \\ x - 2y - z &= 0 \\ y + z &= 0. \end{aligned}$$

11. Which number d forces a row exchange, and what is the triangular system (not singular) for that d? Which d makes this system singular (no third pivot)?

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + dy + z &= 2 \\ y - z &= 3. \end{aligned}$$

12. Reduce this system to upper triangular form by two row operations:

$$\begin{aligned} 2x + 3y + z &= 8 \\ 4x + 7y + 5z &= 20 \\ -2y + 2z &= 0. \end{aligned}$$

Circle the pivots. Solve by back-substitution for z, y, x.

13. Apply elimination (circle the pivots) and back-substitution to solve

$$\begin{aligned} 2x - 3y \phantom{{}+3z} &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5. \end{aligned}$$

List the three row operations: Subtract ___ times row ___ from row ___.

7. What test on b_1 and b_2 decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture.

$$\begin{aligned} 3x - 2y &= b_1 \\ 6x - 4y &= b_2. \end{aligned}$$

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$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6. \end{aligned}$$

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13. Apply elimination (circle the pivots) and back-substitution to solve

$$\begin{aligned} 2x - 3y \phantom{{}+z} &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5. \end{aligned}$$

List the three row operations: Subtract ___ times row ___ from row ___.

14. Which number q makes this system singular and which right-hand side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t. \end{aligned}$$

15. (Recommended) It is impossible for a system of linear equations to have exactly two solutions. *Explain why*.

    (a) If (x, y, z) and (X, Y, Z) are two solutions, what is another one?
    (b) If 25 planes meet at two points, where else do they meet?

16. If rows 1 and 2 are the same, how far can you get with elimination (allowing row exchange)? If columns 1 and 2 are the same, which pivot is missing?

$$\begin{aligned} 2x - y + z &= 0 \\ 2x - y + z &= 0 \\ 4x + y + z &= 2 \end{aligned} \qquad \begin{aligned} 2x + 2y + z &= 0 \\ 4x + 4y + z &= 0 \\ 6x + 6y + z &= 2. \end{aligned}$$

17. (a) Construct a 3 by 3 system that needs two row exchanges to reach a triangular form and a solution.
    (b) Construct a 3 by 3 system that needs a row exchange to keep going, but breaks down later.

18. Three planes can fail to have an intersection point, when no two planes are parallel. The system is singular if row 3 of A is a ______ of the first two rows. Find a third equation that can't be solved if $x + y + z = 0$ and $x - 2y - z = 1$.

19. Construct a 3 by 3 example that has 9 different coefficients on the left-hand side, but rows 2 and 3 become zero in elimination. How many solutions to your system with $b = (1, 10, 100)$ and how many with $b = (0, 0, 0)$?

Problems 20–22 move up to 4 by 4 and n by n.

20. If you extend Problem 22 following the 1, 2, 1 pattern or the $-1, 2, -1$ pattern, what is the fifth pivot? What is the nth pivot?

21. Apply elimination and back-substitution to solve

$$\begin{aligned} 2u + 3v \phantom{{}+ w} &= 0 \\ 4u + 5v + w &= 3 \\ 2u - v - 3w &= 5. \end{aligned}$$

What are the pivots? List the three operations in which a multiple of one row is subtracted from another.

22. Find the pivots and the solution for these four equations:

$$\begin{aligned} 2x + y \phantom{{}+ z + t} &= 0 \\ x + 2y + z \phantom{{}+ t} &= 0 \\ y + 2z + t &= 0 \\ z + 2t &= 5. \end{aligned}$$

14. Which number q makes this system singular and which right-hand side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t. \end{aligned}$$

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    (a) If (x, y, z) and (X, Y, Z) are two solutions, what is another one?
    (b) If 25 planes meet at two points, where else do they meet?

16. If rows 1 and 2 are the same, how far can you get with elimination (allowing row exchange)? If columns 1 and 2 are the same, which pivot is missing?

$$\begin{aligned} 2x - y + z &= 0 \\ 2x - y + z &= 0 \\ 4x + y + z &= 2 \end{aligned} \qquad \begin{aligned} 2x + 2y + z &= 0 \\ 4x + 4y + z &= 0 \\ 6x + 6y + z &= 2. \end{aligned}$$

17. (a) Construct a 3 by 3 system that needs two row exchanges to reach a triangular form and a solution.
    (b) Construct a 3 by 3 system that needs a row exchange to keep going, but breaks down later.

18. Three planes can fail to have an intersection point, when no two planes are parallel. The system is singular if row 3 of A is a ________ of the first two rows. Find a third equation that can't be solved if $x + y + z = 0$ and $x - 2y - z = 1$.

19. Construct a 3 by 3 example that has 9 different coefficients on the left-hand side, but rows 2 and 3 become zero in elimination. How many solutions to your system with $b = (1, 10, 100)$ and how many with $b = (0, 0, 0)$?

Problems 20–22 move up to 4 by 4 and *n* by *n*.

20. If you extend Problem 22 following the 1, 2, 1 pattern or the $-1, 2, -1$ pattern, what is the fifth pivot? What is the nth pivot?

21. Apply elimination and back-substitution to solve

$$\begin{aligned} 2u + 3v \quad\;\; &= 0 \\ 4u + 5v + \;w &= 3 \\ 2u - \;v - 3w &= 5. \end{aligned}$$

What are the pivots? List the three operations in which a multiple of one row is subtracted from another.

22. Find the pivots and the solution for these four equations:

$$\begin{aligned} 2x + y \qquad\qquad &= 0 \\ x + 2y + z \qquad &= 0 \\ y + 2z + t &= 0 \\ z + 2t &= 5. \end{aligned}$$

23. Solve by elimination the system of two equations

$$\begin{aligned} x - y &= 0 \\ 3x + 6y &= 18. \end{aligned}$$

Draw a graph representing each equation as a straight line in the x-y plane; the line intersect at the solution. Also, add one more line—the graph of the new secon equation which arises after elimination.

24. Find three values of a for which elimination breaks down, temporarily or perma nently, in

$$\begin{aligned} au + v &= 1 \\ 4u + av &= 2. \end{aligned}$$

Breakdown at the first step can be fixed by exchanging rows—but not breakdow at the last step.

25. Solve the system and find the pivots when

$$\begin{aligned} 2u - v \phantom{{}+2w-z} &= 0 \\ -u + 2v - w \phantom{{}-z} &= 0 \\ -v + 2w - z &= 0 \\ -w + 2z &= 5. \end{aligned}$$

You may carry the right-hand side as a fifth column (and omit writing u, v, w, until the solution at the end).

26. True or false:

(a) If the third equation starts with a zero coefficient (it begins with $0u$) then n multiple of equation 1 will be subtracted from equation 3.

(b) If the third equation has zero as its second coefficient (it contains $0v$) then n multiple of equation 2 will be subtracted from equation 3.

(c) If the third equation contains $0u$ *and* $0v$, then no multiple of equation 1 o equation 2 will be subtracted from equation 3.

27. For the system

$$\begin{aligned} u + v + w &= 2 \\ u + 3v + 3w &= 0 \\ u + 3v + 5w &= 2, \end{aligned}$$

what is the triangular system after forward elimination, and what is the solution?

28. Apply elimination to the system

$$\begin{aligned} u + v + w &= -2 \\ 3u + 3v - w &= 6 \\ u - v + w &= -1. \end{aligned}$$

When a zero arises in the pivot position, exchange that equation for the one belo it and proceed. What coefficient of v in the third equation, in place of the presen -1, would make it impossible to proceed—and force elimination to break down?

29. Find experimentally the average size (absolute value) of the first and second and third pivots for MATLAB's lu(rand(3, 3)). The average of the first pivot from abs(A(1, 1)) should be 0.5.

30. For which three numbers a will elimination fail to give three pivots?

$$\begin{aligned} ax + 2y + 3z &= b_1 \\ ax + ay + 4z &= b_2 \\ ax + ay + az &= b_3. \end{aligned}$$

31. (Very optional) Normally the multiplication of two complex numbers

$$(a + ib)(c + id) = (ac - bd) + i(bc + ad)$$

involves the four separate multiplications ac, bd, bc, ad. Ignoring i, can you compute $ac - bd$ and $bc + ad$ with only three multiplications? (You may do additions, such as forming $a + b$ before multiplying, without any penalty.)

32. Use elimination to solve

$$\begin{aligned} u + v + w &= 6 \\ u + 2v + 2w &= 11 \\ 2u + 3v - 4w &= 3 \end{aligned} \quad \text{and} \quad \begin{aligned} u + v + w &= 7 \\ u + 2v + 2w &= 10 \\ 2u + 3v - 4w &= 3. \end{aligned}$$

1.4 MATRIX NOTATION AND MATRIX MULTIPLICATION

With our 3 by 3 example, we are able to write out all the equations in full. We can list the elimination steps, which subtract a multiple of one equation from another and reach a triangular matrix. For a large system, this way of keeping track of elimination would be hopeless; a much more concise record is needed.

We now introduce **matrix notation** to describe the original system, and **matrix multiplication** to describe the operations that make it simpler. Notice that three different types of quantities appear in our example:

Nine coefficients
Three unknowns
Three right-hand sides

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v \quad &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned} \tag{1}$$

On the right-hand side is the column vector b. On the left-hand side are the unknowns u, v, w. Also on the left-hand side are nine coefficients (one of which happens to be zero). It is natural to represent the three unknowns by a vector:

$$\text{The unknown is } x = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \text{The solution is } x = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

The nine coefficients fall into three rows and three columns, producing a **3 by 3 *matrix***:

$$\textbf{Coefficient matrix} \qquad A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}.$$

Notice that the produ...

Problem Set 1.4

1. Write down the 2 by 2 matrices A and B that have entries $a_{ij} = i + j$ and $b_{ij} =$ $(-1)^{i+j}$. Multiply them to find AB and BA.

2. Find two inner products and a matrix product:

$$\begin{bmatrix} 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} \begin{bmatrix} 3 & 5 & 1 \end{bmatrix}.$$

The first gives the length of the vector (squared).

3. If an m by n matrix A multiplies an n-dimensional vector x, how many separa multiplications are involved? What if A multiplies an n by p matrix B?

4. Give 3 by 3 examples (not just the zero matrix) of
   (a) a diagonal matrix: $a_{ij} = 0$ if $i \neq j$.
   (b) a symmetric matrix: $a_{ij} = a_{ji}$ for all i and j.
   (c) an upper triangular matrix: $a_{ij} = 0$ if $i > j$.
   (d) a skew-symmetric matrix: $a_{ij} = -a_{ji}$ for all i and j.

5. Compute the products

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For the third one, draw the column vectors (2, 1) and (0, 3). Multiplying by (1, just adds the vectors (do it graphically).

6. Multiply Ax to find a solution vector x to the system $Ax =$ *zero vector*. Can yo find more solutions to $Ax = 0$?

$$Ax = \begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

7. Working a column at a time, compute the products

$$\begin{bmatrix} 4 & 1 \\ 5 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 4 & 3 \\ 6 & 6 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}.$$

8. Do these subroutines multiply Ax by rows or columns? Start with $B(I) = 0$:

```
   DO 10 I = 1,N                     DO 10 J = 1,N
   DO 10 J = 1,N                     DO 10 I = 1,N
10 B(I) = B(I) + A(I,J) * X(J)   10  B(I) = B(I) + A(I,J) * X(J)
```

The outputs $Bx = Ax$ are the same. The second code is slightly more efficient in FORTRAN and much more efficient on a vector machine (the first changes single entries $B(I)$, the second can update whole vectors).

9. The product of two lower triangular matrices is again lower triangular (all its entries above the main diagonal are zero). Confirm this with a 3 by 3 example, and then explain how it follows from the laws of matrix multiplication.

10. Suppose A commutes with every 2 by 2 matrix ($AB = BA$), and in particular

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{commutes with} \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Show that $a = d$ and $b = c = 0$. If $AB = BA$ for all matrices B, then A is a multiple of the identity.

11. True or false? Give a specific counterexample when false.
   (a) If columns 1 and 3 of B are the same, so are columns 1 and 3 of AB.
   (b) If rows 1 and 3 of B are the same, so are rows 1 and 3 of AB.
   (c) If rows 1 and 3 of A are the same, so are rows 1 and 3 of AB.
   (d) $(AB)^2 = A^2B^2$.

12. Let x be the column vector $(1, 0, \dots, 0)$. Show that the rule $(AB)x = A(Bx)$ forces the first column of AB to equal A times the first column of B.

13. Which of the following matrices are guaranteed to equal $(A+B)^2$?

$$A^2 + 2AB + B^2, \quad A(A+B) + B(A+B), \quad (A+B)(B+A), \quad A^2 + AB + BA + B^2.$$

14. If the entries of A are a_{ij}, use subscript notation to write
   (a) the first pivot.
   (b) the multiplier ℓ_{i1} of row 1 to be subtracted from row i.
   (c) the new entry that replaces a_{ij} after that subtraction.
   (d) the second pivot.

15. By trial and error find examples of 2 by 2 matrices such that
   (a) $A^2 = -I$, A having only real entries.
   (b) $B^2 = 0$, although $B \neq 0$.
   (c) $CD = -DC$, not allowing the case $CD = 0$.
   (d) $EF = 0$, although no entries of E or F are zero.

16. The first row of AB is a linear combination of all the rows of B. What are the coefficients in this combination, and what is the first row of AB, if

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}?$$

17. Describe the rows of EA and the *columns* of AE if

$$E = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}.$$

18. *A fourth way to multiply matrices is* ***columns of A times rows of B***:

$$AB = (\text{column } 1)(\text{row } 1) + \cdots + (\text{column } n)(\text{row } n) = \text{sum of simple matric}$$

Give a 2 by 2 example of this important rule for matrix multiplication.

19. Find the powers A^2, A^3 (A^2 times A), and B^2, B^3, C^2, C^3. What are A^k, B^k, C^k?

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad C = AB = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

20. If A and B are n by n matrices with all entries equal to 1, find $(AB)_{ij}$. Summa notation turns the product AB, and the law $(AB)C = A(BC)$, into

$$(AB)_{ij} = \sum_k a_{ik} b_{kj} \qquad \sum_j \left(\sum_k a_{ik} b_{kj} \right) c_{jl} = \sum_k a_{ik} \left(\sum_j b_{kj} c_{jl} \right).$$

Compute both sides if C is also n by n, with every $c_{jl} = 2$.

21. The matrix that rotates the x-y plane by an angle θ is

$$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Verify that $A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2)$ from the identities for $\cos(\theta_1 + \theta_2)$ a $\sin(\theta_1 + \theta_2)$. What is $A(\theta)$ times $A(-\theta)$?

Problems 22–31 are about elimination matrices.

22. Suppose $a_{33} = 7$ and the third pivot is 5. If you change a_{33} to 11, the third pivo ___. If you change a_{33} to ___, there is zero in the pivot position.

23. What matrix E_{31} subtracts 7 times row 1 from row 3? To reverse that step, R_{31} shou ____ 7 times row ___ to row ___. Multiply E_{31} by R_{31}.

24. If every column of A is a multiple of (1, 1, 1), then Ax is always a multiple (1, 1, 1). Do a 3 by 3 example. How many pivots are produced by elimination?

25. In Problem 26, applying E_{21} and then E_{32} to the column $b = (1, 0, 0)$ gi $E_{32}E_{21}b =$ ___. Applying E_{32} before E_{21} gives $E_{21}E_{32}b =$ ___. When E_{32} com first, row ___ feels no effect from row ___.

26. Write down the 3 by 3 matrices that produce these elimination steps:

(a) E_{21} subtracts 5 times row 1 from row 2.
(b) E_{32} subtracts −7 times row 2 from row 3.
(c) P exchanges rows 1 and 2, then rows 2 and 3.

27. Which three matrices E_{21}, E_{31}, E_{32} put A into triangular form U?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \quad \text{and} \quad E_{32}E_{31}E_{21}A = U.$$

Multiply those E's to get one matrix M that does elimination: $MA = U$.

28. This 4 by 4 matrix needs which elimination matrices E_{21} and E_{32} and E_{43}?

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

29. Multiply these matrices:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}.$$

30. (a) What 3 by 3 matrix E_{13} will add row 3 to row 1?
(b) What matrix adds row 1 to row 3 and *at the same time* adds row 3 to row 1?
(c) What matrix adds row 1 to row 3 and *then* adds row 3 to row 1?

31. (a) E_{21} subtracts row 1 from row 2 and then P_{23} exchanges rows 2 and 3. What matrix $M = P_{23}E_{21}$ does both steps at once?
(b) P_{23} exchanges rows 2 and 3 and then E_{31} subtracts row 1 from row 3. What matrix $M = E_{31}P_{23}$ does both steps at once? Explain why the M's are the same but the E's are different.

Problems 32–44 are about creating and multiplying matrices.

32. A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. *All entries are* 1. Which of these matrix operations are allowed, and what are the results?

$$BA \qquad AB \qquad ABD \qquad DBA \qquad A(B+C).$$

33. If E adds row 1 to row 2 and F adds row 2 to row 1, does EF equal FE?

34. The first component of Ax is $\sum a_{1j}x_j = a_{11}x_1 + \cdots + a_{1n}x_n$. Write formulas for the third component of Ax and the $(1, 1)$ entry of A^2.

35. The parabola $y = a + bx + cx^2$ goes through the points $(x, y) = (1, 4)$ and $(2, 8)$ and $(3, 14)$. Find and solve a matrix equation for the unknowns (a, b, c).

36. Multiply these matrices in the orders EF and FE and E^2:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}.$$

37. Write these ancient problems in a 2 by 2 matrix form $Ax = b$ and solve them:
(a) X is twice as old as Y and their ages add to 39.
(b) $(x, y) = (2, 5)$ and $(3, 7)$ lie on the line $y = mx + c$. Find m and c.

38. (a) Suppose all columns of B are the same. Then all columns of EB are the same, because each one is E times _______.
(b) Suppose all rows of B are $[1\ \ 2\ \ 4]$. Show by example that all rows of EB are *not* $[1\ \ 2\ \ 4]$. It is true that those rows are _______.

39. If $AB = I$ and $BC = I$, use the associative law to prove $A = C$.

40. True or false?

(a) If A^2 is defined then A is necessarily square.
(b) If AB and BA are defined then A and B are square.
(c) If AB and BA are defined then AB and BA are square.
(d) If $AB = B$ then $A = I$.

41. If A is m by n, how many separate multiplications are involved when

(a) A multiplies a vector x with n components?
(b) A multiplies an n by p matrix B? Then AB is m by p.
(c) A multiplies itself to produce A^2? Here $m = n$.

42. To prove that $(AB)C = A(BC)$, use the column vectors $b_1, \dots, b_n$ of B. suppose that C has only one column c with entries $c_1, \dots, c_n$: AB has columns $Ab_1, \dots, Ab_n$ and Bc has one column $c_1b_1 + \cdots + c_nb_n$. Then $(AB)c = c_1Ab_1 + \cdots + c_nAb_n$ equals $A(c_1b_1 + \cdots + c_nb_n) = A(Bc)$. *Linearity* gives equality of those two sums, and $(AB)c = A(Bc)$. The same is for all other ___ of C. Therefore $(AB)C = A(BC)$.

43. What rows or columns or matrices do you multiply to find

(a) the third column of AB?
(b) the first row of AB?
(c) the entry in row 3, column 4 of AB?
(d) the entry in row 1, column 1 of CDE?

44. (3 by 3 matrices) Choose the only B so that for every matrix A,

(a) $BA = 4A$.
(b) $BA = 4B$.
(c) BA has rows 1 and 3 of A reversed and row 2 unchanged.
(d) All rows of BA are the same as row 1 of A.

The next problems use column–row multiplication and block multiplication.

45. ***Block multiplication*** separates matrices into blocks (submatrices). If their sh make block multiplication possible, then it is allowed. Replace these x's by num and confirm that block multiplication succeeds.

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} AC + BD \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{cc|c} x & x & x \\ x & x & x \\ \hline x & x & x \end{array}\right] \left[\begin{array}{cc|c} x & x & x \\ x & x & x \\ \hline x & x & x \end{array}\right]$$

46. Multiply AB using columns times rows:

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \underline{\qquad} = \underline{\qquad}.$$

47. Draw the cuts in A and B and AB to show how each of the four multiplication is really a block multiplication to find AB:

(a) Matrix A times columns of B.
(b) Rows of A times matrix B.

(c) Rows of A times columns of B.
(d) Columns of A times rows of B.

48. If you multiply a *northwest matrix* A and a *southeast matrix* B, what type of matrices are AB and BA? "Northwest" and "southeast" mean zeros below and above the antidiagonal going from $(1, n)$ to $(n, 1)$.

49. If the three solutions in Question 58 are $x_1 = (1, 1, 1)$ and $x_2 = (0, 1, 1)$ and $x_3 = (0, 0, 1)$, solve $Ax = b$ when $b = (3, 5, 8)$. Challenge problem: What is A?

50. Write $2x + 3y + z + 5t = 8$ as a matrix A (how many rows?) multiplying the column vector (x, y, z, t) to produce b. The solutions fill a plane in four-dimensional space. *The plane is three-dimensional with no 4D volume.*

51. Write the inner product of $(1, 4, 5)$ and (x, y, z) as a matrix multiplication Ax. A has one row. The solutions to $Ax = 0$ lie on a ______ perpendicular to the vector ______. The columns of A are only in ______-dimensional space.

52. *Elimination for a 2 by 2 block matrix*: When $A^{-1}A = I$, multiply the first block row by CA^{-1} and subtract from the second row, to find the "*Schur complement*" S:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix}.$$

53. What 2 by 2 matrix P_1 projects the vector (x, y) onto the x axis to produce $(x, 0)$? What matrix P_2 projects onto the y axis to produce $(0, y)$? If you multiply $(5, 7)$ by P_1 and then multiply by P_2, you get (______) and (______).

54. In MATLAB notation, write the commands that define the matrix A and the column vectors x and b. What command would test whether or not $Ax = b$?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad x = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

55. Block multiplication says that elimination on column 1 produces

$$EA = \begin{bmatrix} 1 & \mathbf{0} \\ -\boldsymbol{c}/a & I \end{bmatrix} \begin{bmatrix} a & \boldsymbol{b} \\ \boldsymbol{c} & D \end{bmatrix} = \begin{bmatrix} a & \boldsymbol{b} \\ \mathbf{0} & \underline{\quad} \end{bmatrix}.$$

~~56. Find~~ all matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{that satisfy} \quad A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A.$$

57. With $i^2 = -1$, the product $(A + iB)(x + iy)$ is $Ax + iBx + iAy - By$. Use blocks to separate the real part from the imaginary part that multiplies i:

$$\begin{bmatrix} A & -B \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax - By \\ ? \end{bmatrix} \begin{matrix} \text{real part} \\ \text{imaginary part} \end{matrix}$$

58. Suppose you solve $Ax = b$ for three special right-hand sides b:

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If the solutions x_1, x_2, x_3 are the columns of a matrix X, what is AX?

(c) Rows of A times columns of B.
(d) Columns of A times rows of B.

48. If you multiply a *northwest matrix* A and a *southeast matrix* B, what type of matrices are AB and BA? "Northwest" and "southeast" mean zeros below and above the antidiagonal going from $(1, n)$ to $(n, 1)$.

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$$EA = \begin{bmatrix} 1 & \mathbf{0} \\ -c/a & I \end{bmatrix} \begin{bmatrix} a & b \\ c & D \end{bmatrix} = \begin{bmatrix} a & b \\ \mathbf{0} & \_\_ \end{bmatrix}.$$

56. Find all matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{that satisfy} \quad A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A.$$

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If the solutions x_1, x_2, x_3 are the columns of a matrix X, what is AX?

59. Invent a 3 by 3 **magic matrix** M with entries $1, 2, \ldots, 9$. All rows and columns an diagonals add to 15. The first row could be 8, 3, 4. What is M times $(1, 1, 1)$? Wha is the row vector $[1 \quad 1 \quad 1]$ times M?

60. The MATLAB commands A = eye(3) and v = [3:5]' produce the 3 by 3 identit matrix and the column vector (3, 4, 5). What are the outputs from A * v and v' * v (Computer not needed!) If you ask for v * A, what happens?

61. If you multiply the 4 by 4 all-ones matrix A = ones(4,4) and the column v = ones(4,1), what is A * v? (Computer not needed.) If you multiply B = eye(4) + ones(4,4) times w = zeros(4,1) + 2 * ones(4,1), what is B * w?

1.5 TRIANGULAR FACTORS AND ROW EXCHANGES

form. $Ax = b$ reduces to two triangular systems. This is the practical equivalent of the calculation we do next—*to find the inverse matrix* A^{-1} *and the solution* $x = A^{-1}b$.

Problem Set 1.5

1. Multiply the matrix $L = E^{-1}F^{-1}G^{-1}$ in equation (6) by GFE in equation (3):

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{times} \quad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

Multiply also in the opposite order. *Why are the answers what they are?*

2. When is an upper triangular matrix nonsingular (a full set of pivots)?

3. What multiple ℓ_{32} of row 2 of A will elimination subtract from row 3 of A? Use the factored form

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix}.$$

What will be the pivots? Will a row exchange be required?

4. Find the products FGH and HGF if (with upper triangular zeros omitted)

$$F = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

5. (a) Under what conditions is the following product nonsingular?

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Solve the system $Ax = b$ starting with $Lc = b$:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = b.$$

6. (*Second proof of* $A = LU$) The third row of U comes from the third row of A by subtracting multiples of rows 1 and 2 (*of* U!):

$$\text{row 3 of } U = \text{row 3 of } A - \ell_{31}(\text{row 1 of } U) - \ell_{32}\ (\text{row 2 of } U).$$

(a) Why are rows of U subtracted off and not rows of A? Answer: Because by the time a pivot row is used, ___.

(b) The equation above is the same as

$$\text{row 3 of } A = \ell_{31}(\text{row 1 of } U) + \ell_{32}(\text{row 2 of } U) + 1\ (\text{row 3 of } U).$$

Which rule for matrix multiplication makes this row 3 of L times U?

The other rows of LU agree similarly with the rows of A.

7. Factor A into LU, and write down the upper triangular system $Ux = c$ which appears after elimination, for

$$Ax = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}.$$

8. (a) Why does it take approximately $n^2/2$ multiplication–subtraction steps to solve each of $Lc = b$ and $Ux = c$?

(b) How many steps does elimination use in solving 10 systems with the same 60 by 60 coefficient matrix A?

9. Apply elimination to produce the factors L and U for

$$A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}.$$

10. Find E^2 and E^8 and E^{-1} if

$$E = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}.$$

11. Decide whether the following systems are singular or nonsingular, and whether they have no solution, one solution, or infinitely many solutions:

$$\begin{aligned} v - w &= 2 \\ u - v \quad &= 2 \\ u \quad - w &= 2 \end{aligned} \quad \text{and} \quad \begin{aligned} v - w &= 0 \\ u - v \quad &= 0 \\ u \quad - w &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} v + w &= 1 \\ u + v \quad &= 1 \\ u \quad + w &= 1. \end{aligned}$$

12. Write down all six of the 3 by 3 permutation matrices, including $P = I$. Identify their inverses, which are also permutation matrices. The inverses satisfy $PP^{-1} = I$ and are on the same list.

13. Find a 4 by 4 permutation matrix that requires three row exchanges to reach the end of elimination (which is $U = I$).

14. The less familiar form $A = LPU$ exchanges rows only at the end:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \to L^{-1}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} = PU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}.$$

What is L is this case? Comparing with $PA = LU$ in Box 1J, the multipliers now stay in place (ℓ_{21} is 1 and ℓ_{31} is 2 when $A = LPU$).

15. How could you factor A into a product UL, upper triangular times lower triangular? Would they be the same factors as in $A = LU$?

16. Which numbers a, b, c lead to row exchanges? Which make the matrix singular?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ a & 8 & 3 \\ 0 & b & 5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} c & 2 \\ 6 & 4 \end{bmatrix}.$$

17. Solve by elimination, exchanging rows when necessary:

$$\begin{aligned} u + 4v + 2w &= -2 \\ -2u - 8v + 3w &= 32 \\ v + w &= 1 \end{aligned} \quad \text{and} \quad \begin{aligned} v + w &= 0 \\ u + v \quad &= 0 \\ u + v + w &= 1. \end{aligned}$$

Which permutation matrices are required?

18. Solve as two triangular systems, without multiplying LU to find A:

$$LUx = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

19. Find the $PA = LDU$ factorizations (and check them) for

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Problems 20–31 compute the factorization $A = LU$ (and also $A = LDU$).

20. What are the 3 by 3 triangular systems $Lc = b$ and $Ux = c$ from Problem 2[...] Check that $c = (5, 2, 2)$ solves the first one. Which x solves the second one?

21. What three elimination matrices E_{21}, E_{31}, E_{32} put A into upper triangular fo[rm] $E_{32}E_{31}E_{21}A = U$? Multiply by E_{32}^{-1}, E_{31}^{-1} and E_{21}^{-1} to factor A into LU whe[re] $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$. Find L and U:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}.$$

22. When zero appears in a pivot position, $A = LU$ *is not possible*! (We need nonze[ro] pivots d, f, i in U.) Show directly why these are both impossible:

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix}\begin{bmatrix} d & e \\ 0 & f \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ \ell & 1 & \\ m & n & 1 \end{bmatrix}\begin{bmatrix} d & e & g \\ & f & h \\ & & i \end{bmatrix}$$

23. Forward elimination changes $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]x = b$ to a triangular $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]x = c$:

$$\begin{aligned} x + y &= 5 \\ x + 2y &= 7 \end{aligned} \quad \longrightarrow \quad \begin{aligned} x + y &= 5 \\ y &= 2 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

That step subtracted $\ell_{21} =$ ______ times row 1 from row 2. The reverse step ad[ds] ℓ_{21} times row 1 to row 2. The matrix for that reverse step is $L =$ ______. Multip[ly] this L times the triangular system $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]x = \left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right]$ to get ______ = ______. [...] letters, L multiplies $Ux = c$ to give ______.

24. (Move to 3 by 3) Forward elimination changes $Ax = b$ to a triangular $Ux = c$:

$$\begin{aligned} x + y + z &= 5 \\ x + 2y + 3z &= 7 \\ x + 3y + 6z &= 11 \end{aligned} \qquad \begin{aligned} x + y + z &= 5 \\ y + 2z &= 2 \\ 2y + 5z &= 6 \end{aligned} \qquad \begin{aligned} x + y + z &= 5 \\ y + 2z &= 2 \\ z &= 2. \end{aligned}$$

The equation $z = 2$ in $Ux = c$ comes from the original $x + 3y + 6z = 11$ i[n] $Ax = b$ by subtracting $\ell_{31} =$ ___ times equation 1 and $\ell_{32} =$ ___ times the fin[al] equation 2. Reverse that to recover $[1\ \ 3\ \ 6\ \ 11]$ in $[A\ \ b]$ from the final $[1\ \ 1\ \ 1\ \ 5]$ and $[0\ \ 1\ \ 2\ \ 2]$ and $[0\ \ 0\ \ 1\ \ 2]$ in $[U\ \ c]$:

$$\text{Row 3 of } \begin{bmatrix} A & b \end{bmatrix} = (\ell_{31} \text{ Row } 1 + \ell_{32} \text{ Row } 2 + 1 \text{ Row } 3) \text{ of } \begin{bmatrix} U & c \end{bmatrix}.$$

In matrix notation this is multiplication by L. So $A = LU$ and $b = Lc$.

25. What two elimination matrices E_{21} and E_{32} put A into upper triangular form $E_{32}E_{21}A = U$? Multiply by E_{32}^{-1} and E_{21}^{-1} to factor A into $LU = E_{21}^{-1}E_{32}^{-1}U$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{bmatrix}.$$

26. Which number c leads to zero in the second pivot position? A row exchange i[s] needed and $A = LU$ is not possible. Which c produces zero in the third pivo[t] position? Then a row exchange can't help and elimination fails:

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

27. (Recommended) Compute L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

28. *Tridiagonal matrices* have zero entries except on the main diagonal and the two adjacent diagonals. Factor these into $A = LU$ and $A = LDV$:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix}.$$

29. Solve $Lc = b$ to find c. Then solve $Ux = c$ to find x. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

30. A and B are symmetric across the diagonal (because $4 = 4$). Find their triple factorizations LDU and say how U is related to L for these symmetric matrices:

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}.$$

31. Solve the triangular system $Lc = b$ to find c. Then solve $Ux = c$ to find x:

$$L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 11 \end{bmatrix}.$$

For safety find $A = LU$ and solve $Ax = b$ as usual. Circle c when you see it.

32. What are L and D for this matrix A? What is U in $A = LU$ and what is the new U in $A = LDU$?

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}.$$

33. Find L and U for the nonsymmetric matrix

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}.$$

Find the four conditions on a, b, c, d, r, s, t to get $A = LU$ with four pivots.

34. (Review) For which numbers c is $A = LU$ impossible—with three pivots?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

35. Estimate the time difference for each new right-hand side b when $n = 800$. Cre A = rand(800) and b = rand(800,1) and B = rand(800,9). Compare the times fr tic; $A \backslash b$; toc and tic; $A \backslash B$; toc (which solves for 9 right sides).

36. Use chol(pascal(5)) to find the triangular factors of MATLAB's pascal(5). R exchanges in $[L,\ U]$ = lu(pascal(5)) spoil Pascal's pattern!

37. If A and B have nonzeros in the positions marked by x, which zeros are still ze in their factors L and U?

$$A = \begin{bmatrix} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} x & x & x & 0 \\ x & x & 0 & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix}.$$

38. Starting from a 3 by 3 matrix A with pivots 2, 7, 6, add a fourth row and column produce M. What are the first three pivots for M, and why? What fourth row a column are sure to produce 9 as the fourth pivot?

39. (Important) If A has pivots 2, 7, 6 with no row exchanges, what are the pivots f the upper left 2 by 2 submatrix B (without row 3 and column 3)? Explain why.

Problems 40–48 are about permutation matrices.

40. (Try this question.) Which permutation makes PA upper triangular? Which perm tations make P_1AP_2 lower triangular? ***Multiplying A on the right by P_2 exchang the ______ of A.***

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}.$$

41. If P_1 and P_2 are permutation matrices, so is P_1P_2. This still has the rows of I some order. Give examples with $P_1P_2 \neq P_2P_1$ and $P_3P_4 = P_4P_3$.

42. Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$). Find a 4 by permutation $\widehat{P}$ with $\widehat{P}^4 \neq I$.

43. There are 12 "*even*" permutations of (1, 2, 3, 4), with an *even number of exchange* Two of them are (1, 2, 3, 4) with no exchanges and (4, 3, 2, 1) with two exchange List the other ten. Instead of writing each 4 by 4 matrix, use the numbers 4, 3, 2, to give the position of the 1 in each row.

44. How many exchanges will permute (5, 4, 3, 2, 1) back to (1, 2, 3, 4, 5)? How man exchanges to change (6, 5, 4, 3, 2, 1) to (1, 2, 3, 4, 5, 6)? One is even and the othe is odd. For $(n, \ldots, 1)$ to $(1, \ldots, n)$, show that $n = 100$ and 101 are even, $n = 10$ and 103 are odd.

34. (Review) For which numbers c is $A = LU$ impossible—with three pivots?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

35. Estimate the time difference for each new right-hand side b when $n = 800$. Cre A=rand(800) and b=rand(800,1) and B=rand(800,9). Compare the times fr tic; A\ b; toc and tic; A\ B; toc (which solves for 9 right sides).

36. Use chol(pascal(5)) to find the triangular factors of MATLAB's pascal(5). R exchanges in [L, U] = lu(pascal(5)) spoil Pascal's pattern!

37. If A and B have nonzeros in the positions marked by x, which zeros are still ze in their factors L and U?

$$A = \begin{bmatrix} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} x & x & x & 0 \\ x & x & 0 & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix}.$$

38. Starting from a 3 by 3 matrix A with pivots 2, 7, 6, add a fourth row and column produce M. What are the first three pivots for M, and why? What fourth row a column are sure to produce 9 as the fourth pivot?

39. (Important) If A has pivots 2, 7, 6 with no row exchanges, what are the pivots f the upper left 2 by 2 submatrix B (without row 3 and column 3)? Explain why.

Problems 40–48 are about permutation matrices.

40. (Try this question.) Which permutation makes PA upper triangular? Which perm tations make P_1AP_2 lower triangular? ***Multiplying A on the right by P_2 exchang the _______ of A.***

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}.$$

41. If P_1 and P_2 are permutation matrices, so is P_1P_2. This still has the rows of I some order. Give examples with $P_1P_2 \neq P_2P_1$ and $P_3P_4 = P_4P_3$.

42. Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$). Find a 4 by permutation $\widehat{P}$ with $\widehat{P}^4 \neq I$.

43. There are 12 "*even*" permutations of (1, 2, 3, 4), with an *even number of exchange* Two of them are (1, 2, 3, 4) with no exchanges and (4, 3, 2, 1) with two exchange List the other ten. Instead of writing each 4 by 4 matrix, use the numbers 4, 3, 2, to give the position of the 1 in each row.

44. How many exchanges will permute (5, 4, 3, 2, 1) back to (1, 2, 3, 4, 5)? How man exchanges to change (6, 5, 4, 3, 2, 1) to (1, 2, 3, 4, 5, 6)? One is even and the othe is odd. For $(n, \ldots, 1)$ to $(1, \ldots, n)$, show that $n = 100$ and 101 are even, $n = 10$ and 103 are odd.

45. The matrix P that multiplies (x, y, z) to give (z, x, y) is also a rotation matrix. Find P and P^3. The rotation axis $a = (1, 1, 1)$ doesn't move, it equals Pa. What is the angle of rotation from $v = (2, 3, -5)$ to $Pv = (-5, 2, 3)$?

46. If P is any permutation matrix, find a nonzero vector x so that $(I - P)x = 0$. (This will mean that $I - P$ has no inverse, and has determinant zero.)

47. If P has 1s on the antidiagonal from $(1, n)$ to $(n, 1)$, describe PAP.

48. If you take powers of a permutation, why is some P^k eventually equal to I? Find a 5 by 5 permutation P so that the smallest power to equal I is P^6. (This is a challenge question. Combine a 2 by 2 block with a 3 by 3 block.)

1.6 INVERSES AND TRANSPOSES

The inverse of an n by n matrix is another n by n matrix. The inverse of A is written A^{-1} (and pronounced "A inverse"). The fundamental property is simple: *If you multiply by A and then multiply by A^{-1}, you are back where you started*:

$$\textbf{Inverse matrix} \qquad \text{If} \quad b = Ax \quad \text{then} \quad A^{-1}b = x.$$

Thus $A^{-1}Ax = x$. The matrix A^{-1} times A is the identity matrix. ***Not all matrices have inverses. An inverse is impossible when Ax is zero and x is nonzero.*** Then A^{-1} would have to get back from $Ax = 0$ to x. No matrix can multiply that zero vector Ax and produce a nonzero vector x.

Our goals are to define the inverse matrix and compute it and use it, when A^{-1} exists—and then to understand which matrices don't have inverses.

1K The **inverse** of A is a matrix B such that $BA = I$ and $AB = I$. There is at most one such B, and it is denoted by A^{-1}:

$$A^{-1}A = I \qquad \text{and} \qquad AA^{-1} = I. \tag{1}$$

Note 1 ***The inverse exists if and only if elimination produces n pivots*** (row exchanges allowed). Elimination solves $Ax = b$ without explicitly finding A^{-1}.

Note 2 The matrix A cannot have two different inverses. Suppose $BA = I$ and also $AC = I$. Then $B = C$, according to this "proof by parentheses":

$$B(AC) = (BA)C \quad \text{gives} \quad BI = IC \quad \text{which is} \quad B = C. \tag{2}$$

This shows that a *left-inverse* B (multiplying from the left) and a *right-inverse* C (multiplying A from the right to give $AC = I$) must be the *same matrix*.

Note 3 If A is invertible, the one and only solution to $Ax = b$ is $x = A^{-1}b$:

$$\textbf{\textit{Multiply}} \quad Ax = b \quad \textbf{\textit{by}} \quad A^{-1}. \quad \textbf{\textit{Then}} \quad x = A^{-1}Ax = A^{-1}b.$$

Note 4 (Important) ***Suppose there is a nonzero vector x such that $Ax = 0$. Then A cannot have an inverse.*** To repeat: No matrix can bring 0 back to x.

If A is invertible, then $Ax = 0$ can only have the zero solution $x = 0$.

Problem Set 1.6

1. Suppose elimination fails because there is no pivot in column 3:

$$\textbf{Missing pivot} \qquad A = \begin{bmatrix} 2 & 1 & 4 & 6 \\ 0 & 3 & 8 & 5 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 9 \end{bmatrix}.$$

Show that A cannot be invertible. The third row of A^{-1}, multiplying A, should g the third row $[0 \;\; 0 \;\; 1 \;\; 0]$ of $A^{-1}A = I$. Why is this impossible?

2. If the inverse of A^2 is B, show that the inverse of A is AB. (Thus A is inverti whenever A^2 is invertible.)

3. Find three 2 by 2 matrices, other than $A = I$ and $A = -I$, that are their o inverses: $A^2 = I$.

4. Find the inverses (in any legal way) of

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

5. Show that $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ has no inverse by solving $Ax = 0$, and by failing to solv

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

6. (a) If A is invertible and $AB = AC$, prove quickly that $B = C$.
   (b) If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, find an example with $AB = AC$ but $B \neq C$.

7. (a) Find the inverses of the permutation matrices

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

   (b) Explain for permutations why P^{-1} *is always the same as* P^{T}. Show that the 1 are in the right places to give $PP^{\mathrm{T}} = I$.

8. From $AB = C$ find a formula for A^{-1}. Also find A^{-1} from $PA = LU$.

9. Find the inverses (no special system required) of

$$A_1 = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

10. Use the Gauss–Jordan method to invert

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

11. If B is square, show that $A = B + B^{\mathrm{T}}$ is always symmetric and $K = B - B^{\mathrm{T}}$ is always *skew-symmetric*—which means that $K^{\mathrm{T}} = -K$. Find these matrices A and K when $B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$, and write B as the sum of a symmetric matrix and a skew-symmetric matrix.

12. Compute the symmetric LDL^{T} factorization of

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.$$

13. (a) If $A = LDU$, with 1s on the diagonals of L and U, what is the corresponding factorization of A^{T}? Note that A and A^{T} (square matrices with no row exchanges) share the same pivots.
(b) What triangular systems will give the solution to $A^{\mathrm{T}}y = b$?

14. Find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$

15. Under what conditions on their entries are A and B invertible?

$$A = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}.$$

16. If $A = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, compute $A^{\mathrm{T}}B$, $B^{\mathrm{T}}A$, AB^{T}, and BA^{T}.

17. Give examples of A and B such that
(a) $A + B$ is not invertible although A and B are invertible.
(b) $A + B$ is invertible although A and B are not invertible.
(c) all of A, B, and $A + B$ are invertible.

In the last case use $A^{-1}(A + B)B^{-1} = B^{-1} + A^{-1}$ to show that $C = B^{-1} + A^{-1}$ is also invertible—and find a formula for C^{-1}.

18. (a) How many entries can be chosen independently in a symmetric matrix of order n?
(b) How many entries can be chosen independently in a skew-symmetric matrix ($K^{\mathrm{T}} = -K$) of order n? The diagonal of K is zero!

19. If A is invertible, which properties of A remain true for A^{-1}?
(a) A is triangular. (b) A is symmetric. (c) A is tridiagonal. (d) All entries are whole numbers. (e) All entries are fractions (including numbers like $\frac{3}{1}$).

20. If $A = L_1D_1U_1$ and $A = L_2D_2U_2$, prove that $L_1 = L_2$, $D_1 = D_2$, and $U_1 = U_2$. If A is invertible, the factorization is unique.
(a) Derive the equation $L_1^{-1}L_2D_2 = D_1U_1U_2^{-1}$, and explain why one side is lower triangular and the other side is upper triangular.
(b) Compare the main diagonals and then compare the off-diagonals.

21. (Important) If A has row 1 + row 2 = row 3, show that A is not invertible:

(a) Explain why $Ax = (1, 0, 0)$ cannot have a solution.
(b) Which right-hand sides (b_1, b_2, b_3) might allow a solution to $Ax = b$?
(c) What happens to row 3 in elimination?

22. Suppose A is invertible and you exchange its first two rows to reach B. Is the new matrix B invertible? How would you find B^{-1} from A^{-1}?

23. (a) What matrix E has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then subtract row 2 from row 3.
(b) What single matrix L has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.

24. (Remarkable) If A and B are square matrices, show that $I - BA$ is invertible if $I - AB$ is invertible. Start from $B(I - AB) = (I - BA)B$.

25. If the product $M = ABC$ of three square matrices is invertible, then A, B, C are invertible. Find a formula for B^{-1} that involves M^{-1} and A and C.

26. Find the numbers a and b that give the inverse of 5 * eye(4) − ones(4,4):

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}.$$

What are a and b in the inverse of 6 * eye(5) − ones(5,5)?

27. Show that $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has no inverse by trying to solve for the column (x, y):

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x & t \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{must include} \quad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

28. Multiply $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ times $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. What is the inverse of each matrix if $ad \neq bc$?

29. Solve for the columns of $A^{-1} = \begin{bmatrix} x & t \\ y & z \end{bmatrix}$:

$$\begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

30. If A has column 1 + column 2 = column 3, show that A is not invertible:

(a) Find a nonzero solution x to $Ax = 0$. The matrix is 3 by 3.
(b) Elimination keeps column 1 + column 2 = column 3. Explain why there is no third pivot.

31. Find the inverses (directly or from the 2 by 2 formula) of A, B, C:

$$A = \begin{bmatrix} 0 & 3 \\ 4 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}.$$

32. Prove that a matrix with a column of zeros cannot have an inverse.

33. There are sixteen 2 by 2 matrices whose entries are 1s and 0s. How many of them are invertible?

34. Show that A = 4 * eye(4) − ones(4,4) is *not* invertible: Multiply A * ones(4,1).

Problems 35–39 are about the Gauss–Jordan method for calculating A^{-1}.

35. Use Gauss–Jordan elimination on $[A \quad I]$ to solve $AA^{-1} = I$:

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

36. Exchange rows and continue with Gauss–Jordan to find A^{-1}:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}.$$

37. Follow the 3 by 3 text example but with plus signs in A. Eliminate above and below the pivots to reduce $[A \quad I]$ to $[I \quad A^{-1}]$:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{0} & 1 & 0 & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & 0 & 1 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & 0 & 0 & 1 \end{bmatrix}.$$

38. Change I into A^{-1} as you reduce A to I (by row operations):

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix}.$$

39. Invert these matrices A by the Gauss–Jordan method starting with $[A \quad I]$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

40. Prove that A is invertible if $a \neq 0$ and $a \neq b$ (find the pivots and A^{-1}):

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}.$$

41. True or false (with a counterexample if false and a reason if true):
    (a) A 4 by 4 matrix with a row of zeros is not invertible.
    (b) A matrix with 1s down the main diagonal is invertible.
    (c) If A is invertible then A^{-1} is invertible.
    (d) If A^{T} is invertible then A is invertible.

42. For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}.$$

43. Use inv(S) to invert MATLAB's 4 by 4 symmetric matrix S = pascal(4). C
Pascal's lower triangular A = abs(pascal(4,1)) and test inv(S) = inv(A')*inv

44. This matrix has a remarkable inverse. Find A^{-1} by elimination on $[A \quad I]$. Ex
to a 5 by 5 "alternating matrix" and guess its inverse:

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

45. M^{-1} shows the change in A^{-1} (useful to know) when a matrix is subtracted fro
Check part 3 by carefully multiplying MM^{-1} to get I:

1. $M = I - uv^{\mathrm{T}}$ and $M^{-1} = I + uv^{\mathrm{T}}/(1 - v^{\mathrm{T}}u)$.
2. $M = A - uv^{\mathrm{T}}$ and $M^{-1} = A^{-1} + A^{-1}uv^{\mathrm{T}}A^{-1}/(1 - v^{\mathrm{T}}A^{-1}u)$.
3. $M = I - UV$ and $M^{-1} = I_n + U(I_m - VU)^{-1}V$.
4. $M = A - UW^{-1}V$ and $M^{-1} = A^{-1} + A^{-1}U(W - VA^{-1}U)^{-1}VA^{-1}$.

The four identities come from the 1, 1 block when inverting these matrices:

$$\begin{bmatrix} I & u \\ v^{\mathrm{T}} & 1 \end{bmatrix} \qquad \begin{bmatrix} A & u \\ v^{\mathrm{T}} & 1 \end{bmatrix} \qquad \begin{bmatrix} I_n & U \\ V & I_m \end{bmatrix} \qquad \begin{bmatrix} A & U \\ V & W \end{bmatrix}.$$

46. If $A =$ ones(4,4) and $b =$ rand(4,1), how does MATLAB tell you that $Ax = b$
no solution? If $b =$ ones(4,1), which solution to $Ax = b$ is found by $A\backslash b$?

47. If B has the columns of A in reverse order, solve $(A - B)x = 0$ to show that A
is not invertible. An example will lead you to x.

48. Find and check the inverses (assuming they exist) of these block matrices:

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}.$$

Problems 49–55 are about the rules for transpose matrices.

49. (a) The matrix $((AB)^{-1})^{\mathrm{T}}$ comes from $(A^{-1})^{\mathrm{T}}$ and $(B^{-1})^{\mathrm{T}}$. *In what order?*
(b) If U is upper triangular then $(U^{-1})^{\mathrm{T}}$ is ________ triangular.

50. Find A^{T} and A^{-1} and $(A^{-1})^{\mathrm{T}}$ and $(A^{\mathrm{T}})^{-1}$ for

$$A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix} \quad \text{and also} \quad A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}.$$

51. Show that $A^2 = 0$ is possible but $A^{\mathrm{T}}A = 0$ is not possible (unless $A =$ zero matr

52. Verify that $(AB)^{\mathrm{T}}$ equals $B^{\mathrm{T}}A^{\mathrm{T}}$ but those are different from $A^{\mathrm{T}}B^{\mathrm{T}}$:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \qquad AB = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

In case $AB = BA$ (not generally true!), how do you prove that $B^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$?

53. Explain why the inner product of x and y equals the inner product of Px and Py. Then $(Px)^{\mathrm{T}}(Py) = x^{\mathrm{T}}y$ says that $P^{\mathrm{T}}P = I$ for any permutation. With $x = (1, 2, 3)$ and $y = (1, 4, 2)$, choose P to show that $(Px)^{\mathrm{T}}y$ is not always equal to $x^{\mathrm{T}}(P^{\mathrm{T}}y)$.

54. (a) The row vector x^{T} times A times the column y produces what number?

$$x^{\mathrm{T}}Ay = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \underline{\quad}.$$

(b) This is the row $x^{\mathrm{T}}A = \underline{\qquad}$ times the column $y = (0, 1, 0)$.
(c) This is the row $x^{\mathrm{T}} = [0\ \ 1]$ times the column $Ay = \underline{\qquad}$.

55. When you transpose a block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ the result is $M^{\mathrm{T}} = \underline{\qquad}$. Test it. Under what conditions on A, B, C, D is the block matrix symmetric?

Problems 56–60 are about symmetric matrices and their factorizations.

56. Factor these symmetric matrices into $A = LDL^{\mathrm{T}}$. The matrix D is diagonal:

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

57. (a) How many entries of A can be chosen independently, if $A = A^{\mathrm{T}}$ is 5 by 5?
(b) How do L and D (5 by 5) give the same number of choices in LDL^{T}?

58. If $A = A^{\mathrm{T}}$ and $B = B^{\mathrm{T}}$, which of these matrices are certainly symmetric?
(a) $A^2 - B^2$ (b) $(A + B)(A - B)$ (c) ABA (d) $ABAB$.

59. If $A = A^{\mathrm{T}}$ needs a row exchange, then it also needs a column exchange to stay symmetric. In matrix language, PA loses the symmetry of A but __ recovers the symmetry.

60. Suppose R is rectangular (m by n) and A is symmetric (m by m).
(a) Transpose $R^{\mathrm{T}}AR$ to show its symmetry. What shape is this matrix?
(b) Show why $R^{\mathrm{T}}R$ has no negative numbers on its diagonal.

The next three problems are about applications of $(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y)$.

61. Producing x_1 trucks and x_2 planes requires $x_1 + 50x_2$ tons of steel, $40x_1 + 1000x_2$ pounds of rubber, and $2x_1 + 50x_2$ months of labor. If the unit costs y_1, y_2, y_3 are \$700 per ton, \$3 per pound, and \$3000 per month, what are the values of one truck and one plane? Those are the components of $A^{\mathrm{T}}y$.

62. Wires go between Boston, Chicago, and Seattle. Those cities are at voltages x_B, x_C, x_S. With unit resistances between cities, the three currents are in y:

$$y = Ax \quad \text{is} \quad \begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_B \\ x_C \\ x_S \end{bmatrix}.$$

(a) Find the total currents $A^{\mathrm{T}}y$ out of the three cities.
(b) Verify that $(Ax)^{\mathrm{T}}y$ agrees with $x^{\mathrm{T}}(A^{\mathrm{T}}y)$—six terms in both.

63. Prove that no reordering of rows and reordering of columns can transpose a ty matrix.

64. Compare tic; inv(A); toc for A = rand(500) and A = rand(1000). The n^3 says that computing time (measured by tic; toc) should multiply by 8 when doubled. Do you expect these random A to be invertible?

65. Show that L^{-1} has entries j/i for $i \le j$ (the $-1, 2, -1$ matrix has this L):

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix} \quad \text{and} \quad L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \end{bmatrix}.$$

Test this pattern for L = eye(5) − diag(1:5)\diag(1:4,−1) and inv(L).

66. A *group* of matrices includes AB and A^{-1} if it includes A and B. "Products inverses stay in the group." Which of these sets are groups? Lower triangular mat L with 1s on the diagonal, symmetric matrices S, positive matrices M, diag invertible matrices D, permutation matrices P. Invent two more matrix group

67. I = eye(1000); A = rand(1000); B = triu(A); produces a random trian matrix B. Compare the times for inv(B) and $B\backslash I$. Backslash is engineered t the zeros in B, while inv uses the zeros in I when reducing $[B \;\; I]$ by Gauss–Jo (Compare also with inv(A) and $A\backslash I$ for the full matrix A.)

68. Here is a new factorization of A into *triangular times symmetric*:

Start from $A = LDU$. Then A equals $L(U^{\mathrm{T}})^{-1}$ times $U^{\mathrm{T}}DU$.

Why is $L(U^{\mathrm{T}})^{-1}$ triangular? Its diagonal is all 1s. Why is $U^{\mathrm{T}}DU$ symmetric?

69. A square ***northwest matrix*** B is zero in the southeast corner, below the antidiag that connects $(1, n)$ to $(n, 1)$. Will B^{T} and B^2 be northwest matrices? Will B^{T} northwest or southeast? What is the shape of BC = ***northwest times south*** You are allowed to combine permutations with the usual L and U (southwest northeast).

70. Ax gives the amounts of steel, rubber, and labor to produce x in Problem 61. Fin Then $(Ax)^{\mathrm{T}}y$ is the ______ of inputs while $x^{\mathrm{T}}(A^{\mathrm{T}}y)$ is the value of ______.

71. If every row of a 4 by 4 matrix contains the numbers 0, 1, 2, 3 in some order the matrix be symmetric? Can it be invertible?

1.7 SPECIAL MATRICES AND APPLICATIONS

This section has two goals. The first is to explain one way in which large linear syst $Ax = b$ can arise in practice. The truth is that a large and completely realistic proble engineering or economics would lead us far afield. But there is one natural and impo application that does not require a lot of preparation.