Date 14.05.24 Problem Set 3.1 ASSIGNMENT-03

(2)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2$$

For $N(A)$, $AX = 0$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow x_3 = 0$ $\quad x_1 + 2x_2 + x_3 = 0$

$\Rightarrow x_1 = -2x_2$

$$x = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix}, \quad x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ $N(A)$ is orthogonal to row space of A

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} R_2 \leftrightarrow R_3$$

for $N(A^T)$, $A^T y = 0$ $\quad y_2 + y_3 = 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0 \qquad \Rightarrow y_2 = -y_3$$

$y_1 + 2y_2 + 3y_3 = 0$ $\quad$ $y_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$\Rightarrow y_1 = -y_3$

(11) $\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix} = 0$ $\quad$ $\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 2 \\ 9 \\ 8 \\ 2 \end{bmatrix} = 0$

$\Rightarrow a + 4b + c + d = 0$ $\quad$ $\Rightarrow 2a + 9b + 8c + 2d = 0$

$$\begin{bmatrix} 1 & 4 & 4 & 1 \\ 2 & 9 & 8 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow Ax = 0$

Set of all perpendicular $= N(A)$

Basis of $N(A)$ $= \left\{ \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(10) $x + 2y - z = 0$

$\Rightarrow 1x + 2y + (-1)z = 0$

$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$\therefore \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is the vectors perpendicular to P,

Verification:- let $(1, 2, 5)$ be a pt. on P

$\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} = 1 + 4 - 5 = 0$

Since P in $\mathbb{R}^2$, then if $N(A) = P$, so A should be of order 2×3.

$x + 2y - z = 0$

$\Rightarrow z = x + 2y$

Let $x = 1$, $y = 1 \Rightarrow z = 3$

$\therefore x = 0$, $y = 1 \Rightarrow z = 2$

$\therefore A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ & $N(A) \perp P$

$C(A^T) \perp N(A)$

$\therefore B = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$

rowspace of $B = P$.

(12) Assume

$x-y \perp x+y$

$\Rightarrow (x-y)^T(x+y) = 0$

$\Rightarrow (x^T-y^T)(x+y) = 0$

$\Rightarrow x^Tx + \cancel{x^Ty} - \cancel{y^Tx} - y^Ty = 0$

$\Rightarrow \|x\|^2 - \|y\|^2 = 0$ $\left[\because x^Ty = y^Tx \quad \|x\| = x^Tx \right.$

$\Rightarrow \|x\| = \|y\|$

2nd Assume

$\Rightarrow \|x\| = \|y\|$

$\Rightarrow \|x\|^2 = \|y\|^2$

$(x-y)^T(x+y)$

$= (x^T-y^T)(x+y)$

$= x^Tx - y^Ty + \cancel{x^Ty} - \cancel{y^Tx}$

$= \|x\|^2 - \|y\|^2 = 0$ Proved

(18) Since V & W are orthogonal subspaces

$\therefore$ $u^Tw = 0$ $\forall$ $v \in V$ & $w \in W$

V & W are subspaces, so $0 \in V$ & $0 \in W$

So they interest at 0,

$V \cap W = \{0\}$

(33) (a) A is symmetric matrix $(A^T = A)$

$\therefore$ $C(A) = C(A^T) =$ rowspace of A

we know $C(A^T) \perp N(A)$

So, $C(A) \perp N(A)$

(b) $x \in N(A)$, $z \in C(A^T) = C(A)$

Problem Set - 3.2

5. (a) $a = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\therefore$ Projection Matrix $P_1 = \dfrac{aa^T}{a^Ta}$

$$\Rightarrow P_1 = \frac{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \end{bmatrix}}{\begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

$$\Rightarrow P_1 = \frac{\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}}{10}$$

$$\therefore P_1 = \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix}$$ Ans

$\Rightarrow$ A ~~line~~ line perpendicular to a = a line through $\left(b = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right)$ $\left[\because \begin{bmatrix} -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0 \right]$

$\therefore P_2 = \dfrac{bb^T}{b^Tb}$

$$\Rightarrow P_2 = \frac{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\begin{bmatrix} -3 & 1 \end{bmatrix}}{\begin{bmatrix} -3 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix}}$$

$$\Rightarrow P_2 = \frac{1}{10}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$$

$$\therefore P_2 = \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix}$$

(b) $P_1 + P_2 = \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix} + \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix}$

$\Rightarrow P_1 + P_2 = \begin{bmatrix} \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

~~The projection onto one line and then a perpendicular line gives the zero vector.~~

$\Rightarrow P_1 + P_2 = I$

The sum of the projection on to two perpendicular lines gives the vector itself.

$$P_1 P_2 = \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix}\begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix}$$

$$\Rightarrow P_1 P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The projection on to one line and then perpendicular line gives the zero vector.

(9) Given, $x+2y=0 \Rightarrow x=-2y$

Let $a_1=(2,-1)$, $a_2=(2,1)$, $a_3=(-4,2)$ are three pts on the line $x+2y=0$

Let P_1, P_2, P_3 are the projection matrix for a_1, a_2 & a_3 respectively

x	2	-2	-4
y	-1	1	2

$$\therefore P_1 = \frac{a_1 a_1^T}{a_1^T a_1} = \frac{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \end{bmatrix}}{\begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}}$$

$$= \frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

$$\therefore P_2 = \frac{a_2 a_2^T}{a_2^T a_2} = \frac{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \end{bmatrix}}{\begin{bmatrix} -2 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix}} = \frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

$$\therefore P_3 = \frac{a_3 a_3^T}{a_3^T a_3} = \frac{\begin{bmatrix} -4 \\ 2 \end{bmatrix}\begin{bmatrix} -4 & 2 \end{bmatrix}}{\begin{bmatrix} -4 & 2 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \end{bmatrix}} = \frac{1}{20}\begin{bmatrix} 16 & -8 \\ -8 & 4 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

$\therefore P_1 = P_2 = P_3$

i.e for any pt on the plane on to the line $x+2y=0$; the projection matrix is always same.

$$\therefore P = \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$$

(11) Given $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \frac{10}{3}$$

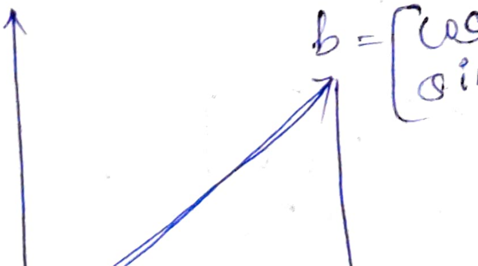
$\therefore \frac{10}{3}$ of $a = (1, 1, 1)$ i.e $(10/3, 10/3, 10/3)$ is closest to the point $\underline{b}$.

Now, closest to $\underline{a}$ on the line through $\underline{b}$

$$p' = \frac{b^T a}{b^T b} b = \frac{\begin{bmatrix} 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}}\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = \frac{10}{36}\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

$$= \frac{5}{18}\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 5/9 \\ 10/9 \\ 10/9 \end{bmatrix}$$

(17)



$b = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$p = \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}$ $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

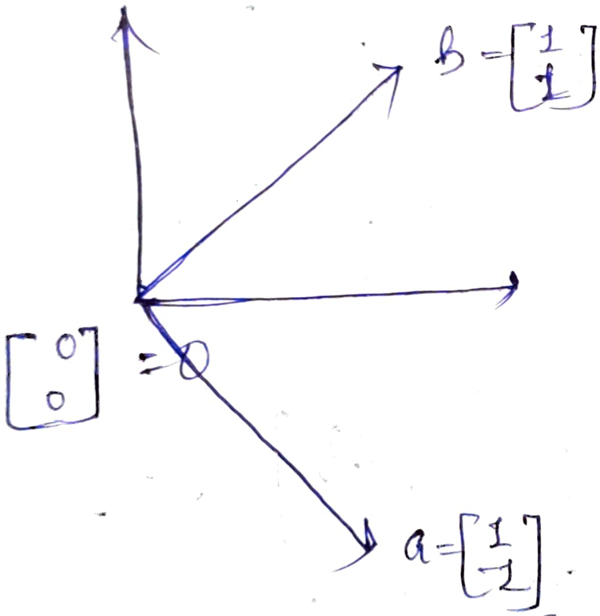
$$\hat{x} = \frac{a^T b}{a^T a} = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}}{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \frac{\cos\theta}{1} = \cos\theta$$

$$\therefore p = \hat{x} a = \cos\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}$$

$$\therefore e = b - p = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} - \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \sin\theta \end{bmatrix}$$

$$\therefore P = \frac{a a^T}{a^T a} = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}}{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

b)



$$\hat{x} = \frac{a^T b}{a^T a} = \frac{\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \frac{0}{2} = 0$$

$$\therefore p = \hat{x} a = 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore e = b - p = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore P = \frac{a a^T}{a^T a} = \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}}{\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

19. (a) Given $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \frac{5}{3}$$

$$\therefore p = \hat{x} a = \frac{5}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix}$$

$$\therefore e = b - p = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$e^T a = \begin{bmatrix} -2/3 & 1/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

$e \perp a$ (checked)

(b) Given, $b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ & $a = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{\begin{bmatrix} -1 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 & -3 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}} = \frac{-11}{11} = -1$$

$$\therefore p = \hat{x} a = -1\begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\therefore e = b - p = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore e^T a = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix} = 0 \Rightarrow e \perp a \text{ (checked)}$$

Problem Set 3.2

(2) Given $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Normal Equations :- $A^T A \hat{x} = A^T b$ —— (1)

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

from (1), $\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A^T A} \underbrace{\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}}_{\hat{x}} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{A^T b}$

$$\Rightarrow \hat{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \hat{x} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \hat{x} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \hat{x} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

$$\Rightarrow P = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$e = b - p = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$e^T(\text{Col}1) = \begin{bmatrix} 2/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$ $\therefore$ Error $(b-p)$ is $\perp$ to the

$e^T(\text{col}2) = \begin{bmatrix} 2/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$ columns of A verified.

③ Given, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $x = \begin{bmatrix} u \\ v \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

$$Ax - b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} u \\ v \\ u+v \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} u-1 \\ v-3 \\ u+v-4 \end{bmatrix}$$

$$E^2 = \|Ax - b\|^2$$

$$\Rightarrow E^2 = (u-1)^2 + (v-3)^2 + (u+v-4)^2$$

To minimize E^2, $\dfrac{\partial E^2}{\partial u} = 0$

$$\Rightarrow 2(u-1) + 2(u+v-4) = 0$$

$$\Rightarrow 4u + 2v = 10 \quad \text{—— ①}$$

& $\dfrac{\partial E^2}{\partial v} = 0$

$$\Rightarrow 2(v-3) + 2(u+v-4) = 0$$

$$\Rightarrow 2u + 4v = 14 \quad \text{—— ②}$$

From ① & ②, we get

$$4\hat{u} + 2\hat{v} = 10 \Rightarrow 2\hat{u} + \hat{v} = 5$$

$$2\hat{u} + 4\hat{v} = 14 \Rightarrow \hat{u} + 2\hat{v} = 7$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \quad \text{—— ③}$$

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$A^T A \hat{x} = A^T b$

$\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \hat{x} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ ——— (4)

(3) & (4) ~~are~~ same

$\therefore \hat{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

$\Rightarrow \hat{x} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

$\Rightarrow \hat{x} = \frac{1}{3} \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

$\Rightarrow \hat{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$P = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\Rightarrow P = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = b$

Here $p = b$ because $b \in C(A)$

$\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(6) Given, $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

Projection of b onto the $C(A)$:

$P = A\bar{x} = A(A^T A)^{-1} A^T b$

$A^T A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix}$

$\therefore (A^T A)^{-1} = \frac{1}{44} \begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$

$$A^Tb = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} -11 \\ 27 \end{bmatrix}$$

$$A(A^TA)^{-1} = \frac{1}{22} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

$$= \frac{1}{22} \begin{bmatrix} 13 & 7 \\ 5 & 1 \\ -2 & 4 \end{bmatrix}$$

$$p = \frac{1}{22} \begin{bmatrix} 13 & 7 \\ 5 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -11 \\ 27 \end{bmatrix}$$

$$= \frac{1}{22} \begin{bmatrix} 46 \\ -28 \\ 130 \end{bmatrix}$$

$$p = \frac{1}{11} \begin{bmatrix} 23 \\ -14 \\ 65 \end{bmatrix} = \begin{bmatrix} 23/11 \\ -14/11 \\ 65/11 \end{bmatrix}$$

We know that

$e = b - p$

i.e $e = q = b - p = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} - \begin{bmatrix} 23/11 \\ -14/11 \\ 65/11 \end{bmatrix} = \begin{bmatrix} -12/11 \\ 36/11 \\ 12/11 \end{bmatrix}$

i.e $b = p + q$

q is perpendicular to the $C(A)$

i.e $q^TA = \begin{bmatrix} -12/11 & 36/11 & 12/11 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} = 0$

$\therefore q \in N(A^T)$ (because $N(A^T) \perp C(A)$)

(9) The best straight line fit is g $b = C + Dt$, C & D are the components of least square.

Given, $b = 4$ at $t = -2$ $\Rightarrow C - 2D = 4$

$b = 3$ at $t = -1 \Rightarrow C - D = 3$

$b = 1$ at $t = 0 \Rightarrow C = 1$

$b = 0$ at $t = 2 \Rightarrow C + 2D = 0$

$\therefore$ Minimize $E^2 = \|Ax - b\|^2$

$= (C - 2D - 4)^2 + (C - D - 3)^2 + (C - 1)^2 + (C + 2D)^2$

$$\frac{\partial E^2}{\partial C} = 2(C - 2D - 4) + 2(C - D - 3) + 2(C - 1) + 2(C + 2D) = 0$$

$\Rightarrow 8C - 2D = 16$

$\Rightarrow 4C - D = 8$ —— (1)

$$\frac{\partial E^2}{\partial D} = 2(C - 2D - 4)(-2) + 2(C - D - 3)(-1) + 2(C + 2D) \cdot 2 = 0$$

$\Rightarrow -2C + 18D = -22$

$\Rightarrow C - \cancel{9}D = 11$ —— (2)

From (1) and (2)

$$\begin{bmatrix} 4 & -1 \\ 1 & -9 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$

$\Rightarrow A^T A \hat{x} = A^T b$

$\Rightarrow \hat{x} = (A^T A)^{-1} A^T b$

$$\Rightarrow \hat{x} = \frac{1}{-35} \begin{bmatrix} -9 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$

$$\Rightarrow \hat{x} = \frac{-1}{35} \begin{bmatrix} -61 \\ 36 \end{bmatrix}$$

$\therefore C = \frac{61}{35}, \; D = -36/35$

$\therefore$ Best line: $61/35 - 36/35 \, t$

$P = A\hat{x}$

$$P = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 61/35 \\ -36/35 \end{bmatrix} = \begin{bmatrix} 133/35 \\ 97/35 \\ 61/35 \\ -11/35 \end{bmatrix}$$

Or From (1)

$t = -2 \Rightarrow P_1 = \frac{61}{35} + \frac{72}{35} = \frac{133}{35}$

$t = -1 \Rightarrow P_2 = \frac{61}{35} + \frac{36}{35} = \frac{97}{35}$

$t = 0 \Rightarrow P_3 = \frac{61}{35}$

$t = 2 \Rightarrow P_4 = \frac{61}{35} - \frac{72}{35} = \frac{-11}{35}$

(12) Given, $a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$

Projection Matrix :

$P = A(A^TA)^{-1}A^T$

$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$\therefore (A^TA)^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} =$

$A(A^TA)^{-1} = \frac{1}{6}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 & 2 \\ 0 & 2 \\ 3 & -2 \end{bmatrix}$

$\therefore P = A(A^TA)^{-1}A^T = \frac{1}{6}\begin{bmatrix} 3 & 2 \\ 0 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix}$

$\therefore P = \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix}$

Problem Set - 4.2

(2) Given, $A \to 4\times4$ matrix

$\det(A) = \frac{1}{2}$

$\det(2A) = 2^4 \det(A) = 16 \times \frac{1}{2} = 8$ [$\because \det(A) = t^n \det A$, $n \to$ order(A)]

$\det(-A) = (-1)^4 \det A = \frac{1}{2}$

$\det(A^2) = \det(A\cdot A) = (\det A)^2 = \frac{1}{4}$

$\det(A^{-1}) = \frac{1}{\det A} = \frac{1}{1/2} = 2$

(5) (a) $A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 8 & -4 & 8 \\ 4 & -2 & 4 \end{bmatrix}$

$\det(A) = 0$ [$\because$ A is singular]

(b) $U = \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ $\therefore \det U = 4\cdot1\cdot2\cdot2 = 16$

(c) $\det U^T = \det(U) = 16$

(d) $\det(U^{-1}) = \frac{1}{\det(U)} = \frac{1}{16} =$

(e) $M = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 4 & 4 & 8 & 8 \end{bmatrix}$

$\therefore \det M = (-1)^2 \det U = (-1)^2\, 16 = 16$
($\because$ 2 row exchange happened)

(12) (a) $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} = 12 - 2 = 10$

(b) $A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$ ~~$= \frac{1}{10}[12+2]$~~ $=$

$\det A^{-1} = \frac{1}{\det A} = \frac{1}{10}$

$A - \lambda I = \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (4-\lambda)(3-\lambda) - 2$

$= \lambda^2 - 7\lambda + 12 - 2$

$= \lambda^2 - 7\lambda + 10$

To make $(A - \lambda I)$ singular, $\det(A - \lambda I) = 0$

$\Rightarrow \lambda^2 - 7\lambda + 10 = 0$

$\Rightarrow (\lambda - 2)(\lambda - 5) = 0$

$\Rightarrow \lambda = 2 \mid \lambda = 5$

(b) L.H.S $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \xrightarrow[R_3 = R_3 - R_2]{R_2 = R_2 - R_1} \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-b & c^2-b^2 \end{vmatrix}$

Taking common ~~from~~ $(b-a)$ from row 2 and $(c-b)$ from row 3.

$= (b-a)(c-b) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+b \end{vmatrix}$

$= (b-a)(c-b) \left[1 \begin{vmatrix} 1 & b+a \\ 1 & c+b \end{vmatrix} \right]$

$= (b-a)(c-b)\,(\cancel{c+b-b+a})\,((c+b) - (b+a))$

$= (b-a)(c-b)(c + \cancel{b} - \cancel{b} - a) = (b-a)(c-b)(c-a)$

Prove

(21) $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{R_1 \leftrightarrow R_2} (-1)^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\det(A) = 1$ (identity matric)

$B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$\det(B) = -1(1-0) + 1(1-0) = -1+1 = 0$

$C = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$ $\det(C) = 0$

(14) Let $A = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \vdots & \\ a_{n1} & a_{n2} \cdots a_{nn} \end{bmatrix}$

Clearly, from the given condition,

$$\sum_{k=1}^{n} a_{ik} = 1 \text{ for each } i.$$

$\Rightarrow A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\Rightarrow$ The equ $Ax = 0$ has a non zero sol^n $x = [1\ 1 \cdots 1]^T$. Therefore, A is a singular matrix, Thus, $\det(A) = 0$

(18) $K^T = -K$, K skew-symmetric

$\Rightarrow \det(K^T) = \det(-K)$

$\Rightarrow \det(K) = (-1)^n \det(K)$,

where n = no of rows in K.

$\Rightarrow \det(K) = -\det(K)$ ($\because$ if n is odd)

$\Rightarrow \det(K) = 0$

(26) $A = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix} \Rightarrow \det(A) = abc$

$B = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\substack{R_3 \leftrightarrow R_2 \\ R_2 \leftrightarrow R_1}]{R_4 \leftrightarrow R_3} \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{bmatrix} (-1)^3$

$= (-1)\, d \cdot a \cdot b \cdot c = -(abcd)$

$C = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}$

$= a(bc - b^2) - a(ac - ab) + a(\cancel{ab - ab})^{0}$

$= abc - \cancel{ab^2} - a^2c + a^2b$

$= abc - a^2c$

Problem Set 4.4

7(a) $ax + by = 1$
$cx + dy = 0$

$$x = \frac{\det B_1}{\det A} = \frac{\begin{vmatrix} 1 & b \\ 0 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$= \frac{d}{ad - bc}$$

$$y = \frac{\det B_2}{\det A} = \frac{\begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-c}{ad-bc}$$

Ans

(b)

$x + 4y - z = 1$
$x + y + z = 0$
$2x + 3z = 0$

$$\Rightarrow \begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x = \frac{\det B_1}{\det A} = \frac{3}{1} = 3$$

$$y = \frac{\det B_2}{\det A} = \frac{-1}{1} = -1$$

$$z = \frac{\det B_3}{\det A} = \frac{-2}{1} = -2$$

$$\det A = \begin{vmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix}$$

$$= 1(3-0) - 1(12-0) + 2(4+1)$$

$$= 3 - 12 + 10$$

$$= 1$$

$$\det B_1 = \begin{vmatrix} 1 & 4 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 3$$

$$\det B_2 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 0 & 3 \end{vmatrix} =$$

$= 1 \times 0 - 1\,(3-0) + 2(1-0)$

$= -3 + 2 = -1$

$\det B_3 = \begin{bmatrix} 1 & 4 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} = -1(0-2) + 2(0-1)$

$= -2$

(13) Let $A = [a_1 \ a_2 \ a_3]_{3\times3}$ & b is the right-hand-side vector.

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the sol^n of $Ax = b$

Therefore, $b = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a_1x_1 + a_2x_2 + a_3x_3$

$\Rightarrow |b \ a_2 \ a_3| = |a_1x_1 + a_2x_2 + a_3x_3 \ \ a_2 \ a_3|$

$= x_1|a_1 \ a_2 \ a_3| + x_2|a_2 \ a_2 \ a_3|$
$+ x_3 \ |a_3 \ a_2 \ a_3|$

$= x_1 |A| +$

$\Rightarrow x_1 = \dfrac{|b \ a_2 \ a_3|}{|A|} = \dfrac{|B_1|}{|A|}$

(14) a) $2x_1 + 5x_2 = 1$

$x_1 + 4x_2 = 2$

$\det A = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} = 8-5$
$= 3$

$\Rightarrow \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\det B_1 = \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = 4 - 10 = -6$$

$$\det B_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$$x_1 = \frac{\det B_1}{\det A} = \frac{-6}{3} = -2$$

$$x_2 = \frac{\det B_2}{\det A} = \frac{3}{3} = 1$$

$\therefore$ sol^n : $x_1 = -2$, $x_2 = 1$

b)

$$2x_1 + x_2 = 1$$
$$x_1 + 2x_2 + x_3 = 0$$
$$x_2 + 2x_3 = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2(4-1) - 1(2-0) + 0(1-0)$$
$$= 2 \times 3 - 1 \times 2 + 0$$
$$= 6 - 2 = 4$$

$$|B_1| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 1(4-1) = 3$$

$$|B_2| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2(0-0) - 1(2-0)$$
$$= -2$$

$$|B_3| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 2(0-0) - 1(0-1)$$
$$= 1$$

$$x_1 = \frac{|B_1|}{|A|} = \frac{3}{4}$$

$$x_2 = \frac{|B_2|}{|A|} = \frac{-2}{4} = -\frac{1}{2}$$

$$x_3 = \frac{|B_3|}{|A|} = \frac{1}{4}$$

Solⁿ: $x_1 = 3/4$, $x_2 = -1/2$, $x_3 = 1/4$