Macroeconomics Assignment

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Abstract

Almost an abstract

I have used an R package from Mati (2019) and should give credit. This allowed me to code everything in R while still using Dynare.

1. Model Specification

$$\pi_t = \frac{P_t}{P_{t-1}} \tag{1.1}$$

$$c_t = \frac{C_t}{P_t}, \quad i_t = \frac{I_t}{P_t}, \quad g_t = \frac{G_t}{P_t}, \quad b_t = \frac{B_t}{P_t} \tag{1.2}$$

Above could note be a heading on itself. Just a paragraph that it is a core rbc foundations and what that means.

Note that capital letters denote nominal amounts and lowercase denote real values.

Mainly adding capital to Sims (2024a), while also introducing adjustment costs from Sims (2024b) . Rotemberg prices was used while gleeming insights from European Central Bank (2022) .

1.1. Households

Maybe I should a table for (**??**)

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$$\max_{\{C_t,\,h_t,\,M_t,\,B_t,\,K_t,\,I_t\}} \mathbb{E}_0 \sum_{t=1}^{\infty} \beta^{t-1} \left[\frac{\left(\frac{C_t}{P_t} - \eta\,\frac{C_{t-1}}{P_{t-1}}\right)^{1-\theta}}{1-\theta} \;-\; \chi\,\frac{h_t^{1+\gamma}}{1+\gamma} \;+\; \psi\,\ln\!\Big(\frac{M_t}{P_t}\Big) \right] \tag{1.3}$$

$$C_t \;+\; I_t \;+\; B_t \;+\; M_t \;\leq\; R^B_{t-1}\,B_{t-1} \;+\; M_{t-1} \;+\; W_t\,h_t \;+\; R^k_t\,K_{t-1} \;+\; \Pi_t \;-\; P_t\,\tau_t \tag{1.4}$$

$$K_t \;=\; (1-\delta)\,K_{t-1} \;+\; i_t \;-\; \frac{\phi}{2}\left(\frac{i_t}{K_{t-1}} - \delta\right)^2 K_{t-1} \tag{1.5}$$

Households maximize expected lifetime utility over consumption and labour supply, taking all after-tax income—wages, rental income, bond returns, and dividends—as given. In particular, we treat firm profits Π_t as a lump-sum transfer from firms that enters the budget constraint passively; households do not choose or time their utility over dividends, nor do they form expectations about future payout ratios.

From which we can derive the following first order conditions, found in full form in Appendix 6.1.1:

$$\lambda_t P_t = (c_t - \eta c_{t-1})^{-\theta} - \beta\eta\,\mathbb{E}_t\big[(c_{t+1} - \eta c_t)^{-\theta}\big] \tag{1.6}$$

$$\lambda_t W_t = \chi h_t^{\gamma} \tag{1.7}$$

$$\frac{\psi}{M_t} = \lambda_t - \beta\,\mathbb{E}_t[\lambda_{t+1}] \tag{1.8}$$

$$\lambda_t = \beta\,\mathbb{E}_t[\lambda_{t+1} R^B_t] \tag{1.9}$$

$$\lambda_t P_t = \mu_t\Big[1 - \phi\big(\tfrac{I_t}{P_t K_{t-1}} - \delta\big)\Big] \tag{1.10}$$

$$\mu_t = \beta\,\mathbb{E}_t\Big[\lambda_{t+1} R^k_{t+1} + \mu_{t+1}\Big((1-\delta) + \tfrac{\phi}{2}\Big(\big(\tfrac{I_{t+1}}{P_{t+1} K_t}\big)^2 - \delta^2\Big)\Big)\Big] \tag{1.11}$$

1.2. Production

This model departs from standard RBC frameworks by introducing monopolistic competition and nominal rigidities. This is achieved through two layers of firms. The first being a perfectly competitive final goods producers, who aggregate intermediate goods, and the second being a monopolistically competitive intermediate goods producers with price-setting power. This two-tiered structure captures core New Keynesian dynamics, such as price stickiness and strategic pricing, while retaining analytical tractability.

1.2.1. Final Goods Producer

Households own final good producer and gets all the profit

$$\Pi_t = P_t Y_t - R_t^k K_{t-1} - W_t h_t - \frac{\psi}{2}(\pi_t - 1)^2 P_t Y_t \tag{1.12}$$

The final goods firm combines differentiated inputs $Y_t(j)$ into final output Y_t using a CES aggregator:

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}, \quad \epsilon > 1 \tag{1.13}$$

Where ϵ captures the elasticity of substitution between varieties: the higher it is, the more easily final goods producers can substitute across inputs. Conversely, a lower ϵ implies that intermediate producers face less competition and enjoy greater market power.

Given the CES aggregator in Equation (1.13), the final goods producer chooses input varieties to minimize the cost of delivering one unit of output. The resulting demand and pricing relationships, derived in Appendix 6.2.1, are:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t \tag{1.14}$$

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}} \tag{1.15}$$

Equation (1.14) shows that more expensive varieties are purchased in smaller quantities, while Equation (1.15) reflects the minimum cost of assembling one unit of final output given prevailing input prices.

1.2.2. Intermediate Goods Producers

Each intermediate firm j produces with identical Cobb-Douglas technology:

$$Y_t(j) = A_t K_t(j)^{\alpha} h_t(j)^{1-\alpha} \tag{1.16}$$

Cost minimization yields capital and labour demand conditions as well as the marginal cost:

$$R_t^k = MC_t \alpha A_t K_t(j)^{\alpha-1} h_t(j)^{1-\alpha} \tag{1.17}$$

$$W_t = MC_t(1-\alpha) A_t K_t(j)^{\alpha} h_t(j)^{-\alpha} \tag{1.18}$$

$$\mathrm{MC}_t = \frac{1}{A_t}\left(\frac{R_t^k}{\alpha}\right)^{\alpha}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha} \tag{1.19}$$

Equation (1.19), which is derived in Appendix 6.2.2, captures the firm's cost of producing one unit of output, given factor prices and technology. A rise in wages or rental rates increases marginal cost, while higher productivity A_t reduces it.

Intermediate firms set prices subject to Rotemberg-style adjustment costs. The firm chooses a price path to maximize the expected discounted sum of profits, which includes a quadratic penalty for deviating from past prices.

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \Lambda_{t,t+s} \left[\underbrace{\frac{P_{t+s}(j)}{P_{t+s}} Y_{t+s}(j)}_{\text{real revenue}} - \underbrace{\frac{\mathrm{MC}_{t+s}}{P_{t+s}} \cdot Y_{t+s}(j)}_{\text{real cost}} - \underbrace{\frac{\psi}{2}\left(\frac{P_{t+s}(j)}{P_{t+s-1}(j)} - 1\right)^2 Y_{t+s}}_{\text{price adjustment cost}} \right] \tag{1.20}$$

In equation (1.20), $\Lambda_{t,t+s} = \beta^s \frac{\lambda_{t+s}}{\lambda_t}$ is the stochastic discount factor, capturing how households value future profits relative to today, where λ_t is the marginal utility of consumption. The parameter $\psi > 0$ is the magnitude of price adjustment costs - a higher ψ implies greater penalties for changing prices rapidly. Finally, $Y_{t+s}(j)$ is the firm-specific demand, which depends on the relative price $\frac{P_{t+s}(j)}{P_{t+s}}$ and aggregate output Y_{t+s}, as derived from the CES demand curve in Equation (1.14).

Each intermediate firm j chooses its price path to maximize the expected discounted sum of real profits, taking into account Rotemberg-style quadratic adjustment costs. In symmetric equilibrium—where $P_t(j) = P_t$ for all j and $\pi_t \equiv P_t/P_{t-1}$—the first-order condition collapses into the familiar Rotemberg Phillips Curve (see Appendix 6.2.2 for the full derivation):

$$(\pi_t - 1)\pi_t = \frac{\epsilon}{\psi}\left(mc_t - \frac{\epsilon - 1}{\epsilon}\right) + \beta \mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}\frac{Y_{t+1}}{Y_t}(\pi_{t+1} - 1)\pi_{t+1}\right] \tag{1.21}$$

$$mc_t = \frac{\mathrm{MC}_t}{P_t} \tag{1.22}$$

In this expression, $mc_t = \frac{\mathrm{MC}_t}{P_t}$ denotes the real marginal cost, and $\frac{\epsilon-1}{\epsilon}$ is its flexible-price benchmark. The left-hand side captures current inflation's departure from unity, while the first term on the right relates that departure to real marginal cost gaps scaled by the adjustment-cost parameter ψ. A larger ψ makes prices stickier, damping the response of inflation to cost pressures. The second term brings in expected future inflation, discounted by β, adjusted for households' intertemporal marginal-utility ratio λ_{t+1}/λ_t and relative output growth Y_{t+1}/Y_t. Together, these components link today's inflation dynamics to both real-economic conditions and expectations of tomorrow's price adjustments.

1.3. Government Sector

Government sector is from Sims (2024c)'s notes. Government chooses spending, (term), exogenously. It finances spending with lump-sum taxes and issues new debt.

Gov budget constraint (nominal)

$$G_t + R^B_{t-1} D_{t-1} \leq D_t + P_t \tau_t \tag{1.23}$$

With $R^B t$ being the interest rate on bonds for period t.

To ensure internal consistency, I introduce a simple government sector. The government issues one-period nominal bonds purchased by households, uses tax revenues to finance an exogenous stream of government spending, and services its debt obligations. The government budget constraint equates the sum of nominal spending and interest payments to the sum of new debt issuance and tax revenues. We assume lump-sum taxation and do not model Ricardian equivalence effects explicitly. Bonds held by households are thus assumed to be government-issued, closing the financial side of the model.

1.4. Monetary Authority

The central bank sets the nominal interest rate R_t according to:

$$R_t = \rho_R R_{t-1} + (1-\rho_R)\left[R_* + \phi_\pi(\pi_t - \pi_*) + \phi_y\left(\frac{Y_t - Y_*}{Y_*}\right)\right] + \varepsilon_t^R \tag{1.24}$$

The policy rate (R_t) sets the nominal return on government bonds (R_t^B) implying with no risk premium.

2. Market Clearing and Equilibrium

2.1. Resource Constraint

(Goods Market Clearing)

$$Y_t = c_t + i_t + g_t + \frac{\phi}{2}\left(\frac{i_t}{K_{t-1}} - \delta\right)^2 K_{t-1} + \frac{\psi}{2}(\pi_t - 1)^2 Y_t \tag{2.1}$$

$$R_t^B = R_t \tag{2.2}$$

2.2. Factor Market Clearing

2.3. Definition of Equilibrium

A competitive equilibrium is a set of prices () and allocations () such that (i) household and rm optimality conditions all hold, (ii) the rm hires all the labour and capital supplied by the household, (iii) the household and rm budget constraints hold with equality, and (iv) household bond-holdings equal government debt issuance in all periods (i.e. Bt+1 = Dt+1, and we require that Bt = Dt initially), given values and stochastic processes of Gt and At, as well as initial values of government debt and household bond-holdings, which must be equal (e.g. Bt =Dt).

$$B_t = D_t \tag{2.3}$$

2.3.1. Natural (flexible-price) equilibrium

The equilibrium when prices are fully flexible. Therefore set ψ to 0 and solve. This should set prices opitimally each period without adjustment costs. Inflation dynamics vanish, simplifying the philips curve. Output and the natrual real rate are purely determined by real economic forces (technology, preferences, and capital accumulation)

Natural (Flexible-Price) Equilibrium Set $\psi = 0$ in all equations. The equation firms are simplifies to:

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \Lambda_{t,t+s} \left[\underbrace{\frac{P_{t+s}(j)}{P_{t+s}} Y_{t+s}(j)}_{\text{real revenue}} - \underbrace{\frac{\mathrm{MC}_{t+s}}{P_{t+s}} \cdot Y_{t+s}(j)}_{\text{real cost}} \right] \tag{2.4}$$

Thus we get

$$mc_t = \frac{\epsilon - 1}{\epsilon} \tag{2.5}$$

Real marginal cost mc_t is constant at $\frac{\epsilon-1}{\epsilon}$ (inverse markup). The natural level of output Y_t^* is the output level consistent with this mc_t in the flexible-price equilibrium. Using the production structure and resource constraints from the provided equations. Derived in Appendix.

3. Exogenous Processes

Government spending, taxes, and technology evolve according to exogenous AR(1) processes.

Real taxes follow Equation (3.1),

$$\tau_t = (1 - \rho_\tau)\bar{\tau} + \rho_\tau \tau_{t-1} + \varepsilon_t^\tau \tag{3.1}$$

where $\bar{\tau}$ is the steady-state level, ρ_τ controls persistence, and ε_t^τ is a fiscal shock.

Government spending is governed by Equation (3.2),

$$g_t = (1 - \rho_g)\bar{g} + \rho_g g_{t-1} + \varepsilon_t^g \tag{3.2}$$

with similar dynamics: mean reversion around $\bar{g}$ and shocks ε_t^g.

Technology evolves log-linearly as in Equation (3.3),

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_t^a \tag{3.3}$$

ensuring a unit steady-state level and allowing for persistent TFP shocks.

4. Full Set of Conditions

$$\pi_t = \frac{P_t}{P_{t-1}} \tag{1.1}$$

$$c_t = \frac{C_t}{P_t}, \quad i_t = \frac{I_t}{P_t}, \quad g_t = \frac{G_t}{P_t}, \quad b_t = \frac{B_t}{P_t} \tag{1.2}$$

$$C_t \;+\; I_t \;+\; B_t \;+\; M_t \;\leq\; R^B_{t-1}\, B_{t-1} \;+\; M_{t-1} \;+\; W_t\, h_t \;+\; R^k_t\, K_{t-1} \;+\; \Pi_t \;-\; P_t\, \tau_t \tag{1.4}$$

$$K_t \;=\; (1-\delta)\, K_{t-1} \;+\; i_t \;-\; \frac{\phi}{2}\left(\frac{i_t}{K_{t-1}} - \delta\right)^2 K_{t-1} \tag{1.5}$$

$$\lambda_t P_t = (c_t - \eta c_{t-1})^{-\theta} - \beta\eta\, \mathbb{E}_t\big[(c_{t+1} - \eta c_t)^{-\theta}\big] \tag{1.6}$$

$$\lambda_t W_t = \chi h_t^{\gamma} \tag{1.7}$$

$$\frac{\psi}{M_t} = \lambda_t - \beta\, \mathbb{E}_t[\lambda_{t+1}] \tag{1.8}$$

$$\lambda_t = \beta\, \mathbb{E}_t[\lambda_{t+1} R^B_t] \tag{1.9}$$

$$\lambda_t P_t = \mu_t\Big[1 - \phi\big(\tfrac{I_t}{P_t K_{t-1}} - \delta\big)\Big] \tag{1.10}$$

$$\mu_t = \beta\, \mathbb{E}_t\Big[\lambda_{t+1} R^k_{t+1} + \mu_{t+1}\Big((1-\delta) + \tfrac{\phi}{2}\Big(\big(\tfrac{I_{t+1}}{P_{t+1} K_t}\big)^2 - \delta^2\Big)\Big)\Big] \tag{1.11}$$

$$Y_t(j) = A_t K_t(j)^{\alpha} h_t(j)^{1-\alpha} \tag{1.16}$$

$$Y_t = A_t K_{t-1}^{\alpha} h_t^{1-\alpha} \tag{5.3}$$

$$\Pi_t = P_t Y_t - R_t^k K_{t-1} - W_t h_t - \frac{\psi}{2}(\pi_t - 1)^2 P_t Y_t \tag{1.12}$$

$$R_t^k = MC_t \alpha A_t K_t(j)^{\alpha-1} h_t(j)^{1-\alpha} \tag{1.17}$$

$$W_t = MC_t(1-\alpha) A_t K_t(j)^{\alpha} h_t(j)^{-\alpha} \tag{1.18}$$

$$\mathrm{MC}_t = \frac{1}{A_t}\left(\frac{R_t^k}{\alpha}\right)^{\alpha}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha} \tag{1.19}$$

$$mc_t = \frac{\mathrm{MC}_t}{P_t} \tag{1.22}$$

$$(\pi_t - 1)\pi_t = \frac{\epsilon}{\psi}\left(mc_t - \frac{\epsilon-1}{\epsilon}\right) + \beta \mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}\frac{Y_{t+1}}{Y_t}(\pi_{t+1}-1)\pi_{t+1}\right] \tag{1.21}$$

$$G_t + R_{t-1}^B D_{t-1} \leq D_t + P_t \tau_t \tag{1.23}$$

$$B_t = D_t \tag{2.3}$$

$$R_t^B = R_t \tag{2.2}$$

$$R_t = \rho_R R_{t-1} + (1-\rho_R)\left[R_* + \phi_\pi(\pi_t - \pi_*) + \phi_y\left(\frac{Y_t - Y_*}{Y_*}\right)\right] + \varepsilon_t^R \tag{1.24}$$

$$Y_t = c_t + i_t + g_t + \frac{\phi}{2}\left(\frac{i_t}{K_{t-1}} - \delta\right)^2 K_{t-1} + \frac{\psi}{2}(\pi_t - 1)^2 Y_t \tag{2.1}$$

$$\tau_t = (1-\rho_\tau)\bar{\tau} + \rho_\tau \tau_{t-1} + \varepsilon_t^\tau \tag{3.1}$$

$$g_t = (1-\rho_g)\bar{g} + \rho_g g_{t-1} + \varepsilon_t^g \tag{3.2}$$

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_t^a \tag{3.3}$$

$$R_* = \frac{\pi_*}{\beta} \tag{5.1}$$

$$r_*^k = \frac{1}{\beta} - (1-\delta) \tag{5.2}$$

5. Steady State

Equations (5.1) and (5.2) are derived in the appendix under 6.3

$$R_* = \frac{\pi_*}{\beta} \tag{5.1}$$

$$r_*^k = \frac{1}{\beta} - (1-\delta) \tag{5.2}$$

$$Y_t = A_t K_{t-1}^{\alpha} h_t^{1-\alpha} \tag{5.3}$$

Equation (5.3) is added because I know dynare drops the j subscript.

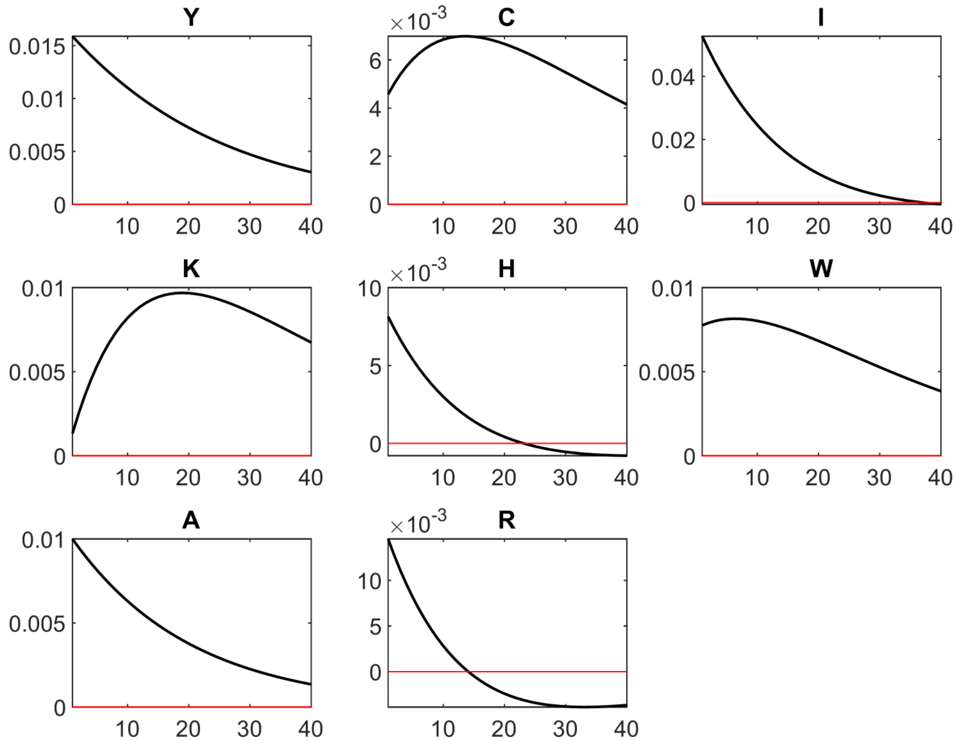

Figure 5.1: image

6. Appendix

6.1. Households

Define Lagrangian

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=1}^{\infty} \beta^{t-1} \Big\{ \underbrace{\frac{\big(\frac{C_t}{P_t} - \eta \frac{C_{t-1}}{P_{t-1}}\big)^{1-\theta}}{1-\theta}}_{\text{Consumption utility}} - \underbrace{\chi \frac{h_t^{1+\gamma}}{1+\gamma}}_{\text{Labor disutility}} + \underbrace{\psi \ln\big(\tfrac{M_t}{P_t}\big)}_{\text{Money utility}}$$

$$+ \underbrace{\lambda_t \big[R_{t-1}^B B_{t-1} + M_{t-1} + W_t h_t + R_t^k K_{t-1} + \Pi_t - P_t \tau_t - C_t - I_t - B_t - M_t \big]}_{\text{Nominal flow constraint}} + \underbrace{\mu_t \big[(1-\delta) K_{t-1} + \tfrac{I_t}{P_t} - \tfrac{\phi}{2} \big(\tfrac{I_t/P_t}{K_{t-1}}}_{\text{Capital accumulati}}$$

6.1.1. First Order Conditions

FOC w.r.t. Consumption :

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0$$

$$\big[(c_t - \eta c_{t-1})^{-\theta} / P_t - \lambda_t \big] - \beta \, \mathbb{E}_t \big[\eta \, (c_{t+1} - \eta c_t)^{-\theta} / P_t \big] = 0$$

Combine terms over $1/P_t$

$$\frac{1}{P_t} \big[(c_t - \eta c_{t-1})^{-\theta} - \beta \eta \, \mathbb{E}_t [(c_{t+1} - \eta c_t)^{-\theta}] \big] - \lambda_t = 0$$

Multiply by P_t

$$(c_t - \eta c_{t-1})^{-\theta} - \beta \eta \, \mathbb{E}_t [(c_{t+1} - \eta c_t)^{-\theta}] - \lambda_t P_t = 0$$

$$\boxed{\lambda_t P_t = (c_t - \eta c_{t-1})^{-\theta} - \beta \eta \, \mathbb{E}_t \big[(c_{t+1} - \eta c_t)^{-\theta} \big]} \tag{6.1}$$

FOC w.r.t. Labour :

$$\frac{\partial \mathcal{L}}{\partial h_t} = 0$$

$$- \chi h_t^{\gamma} + \lambda_t W_t = 0$$

Rearrange

$$\lambda_t W_t = \chi h_t^{\gamma}$$

$$\boxed{\lambda_t W_t = \chi h_t^{\gamma}} \tag{6.2}$$

FOC w.r.t. Real Money Balances:

$$\frac{\partial \mathcal{L}}{\partial M_t} = 0$$

$$\beta^{t-1}[\psi/M_t - \lambda_t] + \beta^t \mathbb{E}_t[\lambda_{t+1}] = 0$$

Divide by β^{t-1} and rearrange

$$\psi/M_t - \lambda_t + \beta\,\mathbb{E}_t[\lambda_{t+1}] = 0$$

$$\boxed{\frac{\psi}{M_t} = \lambda_t - \beta\,\mathbb{E}_t[\lambda_{t+1}]} \tag{6.3}$$

FOC w.r.t. Bonds :

$$\frac{\partial \mathcal{L}}{\partial B_t} = 0$$

$$-\beta^{t-1}\lambda_t + \beta^t \mathbb{E}_t[\lambda_{t+1} R_t^B] = 0$$

Divide by β^{t-1} and simplify

$$-\lambda_t + \beta\,\mathbb{E}_t[\lambda_{t+1} R_t^B] = 0$$

$$\boxed{\lambda_t = \beta\,\mathbb{E}_t[\lambda_{t+1} R_t^B]} \tag{6.4}$$

FOC w.r.t. Investment:

$$\frac{\partial \mathcal{L}}{\partial I_t} = 0$$

$$\beta^{t-1}\Big[-\lambda_t + \mu_t \cdot \frac{1}{P_t}\Big(1 - \phi\big(\tfrac{I_t/P_t}{K_{t-1}} - \delta\big)\Big)\Big] = 0$$

Divide by β^{t-1} and rearrange

$$\lambda_t = \frac{\mu_t}{P_t}\Big[1 - \phi\big(\tfrac{I_t}{P_t K_{t-1}} - \delta\big)\Big]$$

Multiply both sides by P_t

$$\lambda_t P_t = \mu_t\Big[1 - \phi\big(\tfrac{I_t}{P_t K_{t-1}} - \delta\big)\Big]$$

$$\boxed{\lambda_t P_t = \mu_t \Big[1 - \phi\big(\tfrac{I_t}{P_t K_{t-1}} - \delta\big)\Big]} \tag{6.5}$$

FOC w.r.t. Capital:

$$\frac{\partial \mathcal{L}}{\partial K_t} = 0$$

$$-\beta^{t-1}\mu_t + \beta^t \mathbb{E}_t \Big[\lambda_{t+1} R^k_{t+1} + \mu_{t+1}\Big((1-\delta) + \tfrac{\phi}{2}\Big(\big(\tfrac{I_{t+1}}{P_{t+1}K_t}\big)^2 - \delta^2\Big)\Big)\Big] = 0$$

Divide by β^{t-1} and solve

$$\mu_t = \beta\, \mathbb{E}_t \Big[\lambda_{t+1} R^k_{t+1} + \mu_{t+1}\Big((1-\delta) + \tfrac{\phi}{2}\Big(\big(\tfrac{I_{t+1}}{P_{t+1}K_t}\big)^2 - \delta^2\Big)\Big)\Big]$$

$$\boxed{\mu_t = \beta\, \mathbb{E}_t \Big[\lambda_{t+1} R^k_{t+1} + \mu_{t+1}\Big((1-\delta) + \tfrac{\phi}{2}\Big(\big(\tfrac{I_{t+1}}{P_{t+1}K_t}\big)^2 - \delta^2\Big)\Big)\Big]} \tag{6.6}$$

6.2. Production

6.2.1. Final Good Producer

Derivation of Intermediate Goods Demand and Aggregate Price Index

Final goods producer's profit:

$$\Pi_t = P_t Y_t - \int_0^1 P_t(j) Y_t(j) dj$$

subject to $Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$

Substitute production function into profit:

$$\Pi_t = P_t \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 P_t(j) Y_t(j) dj$$

First-order condition for $Y_t(j)$:

$$\frac{\partial \Pi_t}{\partial Y_t(j)} = P_t \cdot \frac{\epsilon}{\epsilon - 1} \left(\int_0^1 Y_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{1}{\epsilon-1}} \cdot \frac{\epsilon - 1}{\epsilon} Y_t(j)^{-\frac{1}{\epsilon}} - P_t(j) = 0$$

$$\Rightarrow P_t \cdot Y_t^{\frac{1}{\epsilon}} Y_t(j)^{-\frac{1}{\epsilon}} = P_t(j)$$

Rearrange to obtain demand curve:

$$Y_t(j) = \left(\frac{P_t}{P_t(j)} \right)^{\epsilon} Y_t$$

Substitute demand into production function:

$$Y_t = \left(\int_0^1 \left[\left(\frac{P_t}{P_t(j)} \right)^{\epsilon} Y_t \right]^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

$$= Y_t \left(\int_0^1 \left(\frac{P_t}{P_t(j)} \right)^{\epsilon-1} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

Simplify to obtain price index:

$$1 = \left(\int_0^1 \left(\frac{P_t}{P_t(j)} \right)^{\epsilon - 1} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

$$\Rightarrow P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon} dj$$

$$\Rightarrow P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}}$$

$$\boxed{\begin{gathered} Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t \\ \\ P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}} \end{gathered}} \tag{6.7}$$

6.2.2. Intermediate Goods Producers

Cost minimization for intermediate firm j :

$$\min_{K_t(j),h_t(j)} \left\{ R_t^k K_t(j) + W_t h_t(j) \right\}$$

subject to $Y_t(j) = A_t K_t(j)^\alpha h_t(j)^{1-\alpha}$

Lagrangian:

$$\mathcal{L} = R_t^k K_t(j) + W_t h_t(j) + \lambda_t \left[A_t K_t(j)^\alpha h_t(j)^{1-\alpha} - Y_t(j) \right]$$

First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial K_t(j)} = 0 : \quad R_t^k = \lambda_t \alpha A_t K_t(j)^{\alpha-1} h_t(j)^{1-\alpha}$$

$$\frac{\partial \mathcal{L}}{\partial h_t(j)} = 0 : \quad W_t = \lambda_t (1-\alpha) A_t K_t(j)^\alpha h_t(j)^{-\alpha}$$

Rearrange FOCs:

$$\lambda_t = \frac{R_t^k}{\alpha} \left(\frac{K_t(j)}{h_t(j)} \right)^{1-\alpha} \frac{1}{A_t}, \quad \lambda_t = \frac{W_t}{1-\alpha} \left(\frac{K_t(j)}{h_t(j)} \right)^{\alpha} \frac{1}{A_t}$$

Equate expressions:

$$\frac{R_t^k}{\alpha} \left(\frac{K_t(j)}{h_t(j)} \right)^{-\alpha} = \frac{W_t}{1-\alpha} \left(\frac{K_t(j)}{h_t(j)} \right)^{1-\alpha}$$

$$\Rightarrow \frac{K_t(j)}{h_t(j)} = \frac{\alpha}{1-\alpha} \frac{W_t}{R_t^k}$$

Substitute into capital FOC:

$$\lambda_t = \frac{R_t^k}{\alpha A_t} \left(\frac{\alpha}{1-\alpha} \frac{W_t}{R_t^k} \right)^{\alpha-1}$$

$$= \frac{1}{A_t} \left(\frac{R_t^k}{\alpha} \right)^{\alpha} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha}$$

$$\boxed{MC_t = \frac{1}{A_t} \left(\frac{R_t^k}{\alpha} \right)^{\alpha} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha}} \tag{6.8}$$

Intermediate-goods producer's problem:

$$\max_{P_t(j)} \; \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\Big[\big(\tfrac{P_{t+s}(j)}{P_{t+s}}\big)^{1-\epsilon} Y_{t+s} - mc_{t+s}\,\big(\tfrac{P_{t+s}(j)}{P_{t+s}}\big)^{-\epsilon} Y_{t+s} - \tfrac{\psi}{2}\big(\tfrac{P_{t+s}(j)}{P_{t+s-1}(j)} - 1\big)^2 Y_{t+s}\Big]$$

subject to $\quad Y_t(j) = \big(\frac{P_t(j)}{P_t}\big)^{-\epsilon} Y_t$

First-Order Condition w.r.t. $P_t(j)$:

$$\mathbb{E}_t\Big[\frac{\partial \Pi_t(j)}{\partial P_t(j)} + \beta\,\Lambda_{t,t+1}\frac{\partial \Pi_{t+1}(j)}{\partial P_t(j)}\Big] = 0$$

$$\frac{\partial \Pi_t}{\partial P_t(j)} = (1-\epsilon)\big(\tfrac{P_t(j)}{P_t}\big)^{-\epsilon}\tfrac{Y_t}{P_t} + \epsilon\, mc_t\big(\tfrac{P_t(j)}{P_t}\big)^{-\epsilon-1}\tfrac{Y_t}{P_t} - \psi\big(\tfrac{P_t(j)}{P_{t-1}(j)} - 1\big)\tfrac{Y_t}{P_{t-1}(j)}$$

$$\frac{\partial \Pi_{t+1}}{\partial P_t(j)} = \psi\big(\tfrac{P_{t+1}(j)}{P_t(j)} - 1\big)\,\tfrac{P_{t+1}(j)}{P_t(j)^2}\,Y_{t+1}$$

Impose symmetry: $P_t(j) = P_t,\ Y_t(j) = Y_t,\ \pi_t = \frac{P_t}{P_{t-1}}$.

$$(1-\epsilon) + \epsilon\, mc_t = \epsilon\big(mc_t - \tfrac{\epsilon-1}{\epsilon}\big), \quad \tfrac{P_t(j)}{P_{t-1}(j)} = \pi_t,\ \tfrac{P_{t+1}(j)}{P_t(j)} = \pi_{t+1}$$

$$0 = \epsilon\big(mc_t - \tfrac{\epsilon-1}{\epsilon}\big) - \psi\,(\pi_t - 1)\,\pi_t + \beta\,\mathbb{E}_t\Big[\Lambda_{t,t+1}\,\psi\,(\pi_{t+1}-1)\,\pi_{t+1}\,\tfrac{Y_{t+1}P_t}{Y_t P_{t+1}}\Big]$$

Noting $\Lambda_{t,t+1} = \beta\,\frac{\lambda_{t+1}}{\lambda_t}$ and $P_{t+1}/P_t = \pi_{t+1}$, the bracket simplifies to $\beta\,\frac{\lambda_{t+1}}{\lambda_t}\,\frac{Y_{t+1}}{Y_t}$.

$$\boxed{0 = \epsilon\Big(mc_t - \tfrac{\epsilon-1}{\epsilon}\Big) - \psi\,(\pi_t - 1)\,\pi_t + \beta\,\mathbb{E}_t\Big[\tfrac{\lambda_{t+1}}{\lambda_t}\,\psi\,(\pi_{t+1}-1)\,\pi_{t+1}\,\tfrac{Y_{t+1}}{Y_t}\Big]} \tag{6.9}$$

Derivation of Firm's First-Order Condition with $\psi = 0$

Starting from the firm's optimization problem under Rotemberg pricing, set $\psi = 0$ to eliminate price-adjustment costs:

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s} \Big[\big(\tfrac{P_{t+s}(j)}{P_{t+s}}\big)^{1-\epsilon} Y_{t+s} - mc_{t+s} \big(\tfrac{P_{t+s}(j)}{P_{t+s}}\big)^{-\epsilon} Y_{t+s} \Big]$$
$$\text{subject to} \quad Y_t(j) = \big(\tfrac{P_t(j)}{P_t}\big)^{-\epsilon} Y_t$$

With $\psi = 0$, the problem becomes **static** (no intertemporal links). For any period t, maximize per-period profits:

$$\Pi_t(j) = \underbrace{\big(\tfrac{P_t(j)}{P_t}\big)^{1-\epsilon} Y_t}_{\text{Revenue}} - \underbrace{mc_t \big(\tfrac{P_t(j)}{P_t}\big)^{-\epsilon} Y_t}_{\text{Production cost}}$$

First-order condition with respect to $P_t(j)$:

$$\frac{\partial \Pi_t(j)}{\partial P_t(j)} = (1-\epsilon)\big(\tfrac{P_t(j)}{P_t}\big)^{-\epsilon} \tfrac{1}{P_t} Y_t + \epsilon mc_t \big(\tfrac{P_t(j)}{P_t}\big)^{-\epsilon-1} \tfrac{1}{P_t} Y_t = 0$$

Factor out $\frac{1}{P_t}\big(\frac{P_t(j)}{P_t}\big)^{-\epsilon} Y_t$:

$$\big(\tfrac{P_t(j)}{P_t}\big)^{-\epsilon} \tfrac{Y_t}{P_t} \Big[(1-\epsilon) + \epsilon mc_t \big(\tfrac{P_t(j)}{P_t}\big)^{-1} \Big] = 0$$

Since $\big(\frac{P_t(j)}{P_t}\big)^{-\epsilon} \frac{Y_t}{P_t} \neq 0$, solve:

$$\begin{aligned}
(1-\epsilon) + \epsilon mc_t \big(\tfrac{P_t(j)}{P_t}\big)^{-1} &= 0 \\
\epsilon mc_t \big(\tfrac{P_t(j)}{P_t}\big)^{-1} &= \epsilon - 1 \\
mc_t \big(\tfrac{P_t(j)}{P_t}\big)^{-1} &= \frac{\epsilon - 1}{\epsilon} \\
\big(\tfrac{P_t(j)}{P_t}\big) &= \frac{\epsilon}{\epsilon - 1} mc_t
\end{aligned}$$

In symmetric equilibrium, all firms set $P_t(j) = P_t$. Substituting:

$$1 = \frac{\epsilon}{\epsilon - 1} mc_t$$

$$\boxed{mc_t = \frac{\epsilon - 1}{\epsilon}} \tag{6.10}$$

6.3. Steady State

Steady-State Nominal Interest Rate

$$\begin{aligned}
&\text{From the bond FOC:} && \lambda_t = \beta \mathbb{E}_t[\lambda_{t+1} R_t^B] \\
&\text{In steady state:} && \lambda = \beta \lambda R_* \\
&\text{Solving for real rate:} && R_* = \frac{1}{\beta} \\
&\text{Adjusting for inflation:} && R_*^{\text{nominal}} = R_* \cdot \pi_* = \frac{\pi_*}{\beta}
\end{aligned}$$

$$\boxed{R_* = \frac{\pi_*}{\beta}} \tag{6.11}$$

Steady-State Return on Capital

$$\begin{aligned}
&\text{From the capital FOC:} && \mu_t = \beta \mathbb{E}_t\left[\lambda_{t+1} R_{t+1}^k + \mu_{t+1}(1-\delta)\right] \\
&\text{Investment adjustment cost zero:} && \frac{I}{PK} = \delta \\
&\text{Thus:} && \mu = \beta\left[\lambda R_*^k + \mu(1-\delta)\right] \\
&\text{From investment FOC:} && \lambda P = \mu \\
&\text{Substitute } \mu = \lambda P: && \lambda P = \beta\left[\lambda R_*^k + \lambda P(1-\delta)\right] \\
&\text{Divide by } \lambda: && P = \beta R_*^k + \beta P(1-\delta) \\
&\text{Rearrange:} && \beta R_*^k = P\left[1 - \beta(1-\delta)\right] \\
&\text{Divide by } \beta: && R_*^k = \frac{P}{\beta}\left[1 - \beta(1-\delta)\right] \\
&\text{Real return:} && r_*^k = \frac{R_*^k}{P} = \frac{1}{\beta} - (1-\delta)
\end{aligned}$$

$$\boxed{r^k_* = \frac{1}{\beta} - (1 - \delta)} \tag{6.12}$$

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