Macroeconomics Assignment

Liam Andrew Beattie[a]

[a] *Macroeconomics 871, Stellenbosch University, South Africa*

Abstract

Almost an abstract

I have used an R package from Mati (2019) and should give credit. This allowed me to code everything in R while still using Dynare.

1. Model Specification

Above could note be a heading on itself. Just a paragraph that it is a core rbc foundations and what that means.

Note that capital letters denote nominal amounts and lowercase denote real values.

Mainly adding capital to Sims (2024a), while also introducing adjustment costs from Sims (2024b) . Rotemberg prices was used while gleeming insights from European Central Bank (2022) .

1.1. Households

$$C_t = P_t c_t, \qquad I_t = P_t i_t \tag{1.1}$$

$$\max_{\{C_t, h_t, M_t, B_t, K_t, I_t\}} \mathbb{E}_0 \sum_{t=1}^{\infty} \beta^{t-1} \left[\frac{\left(\frac{C_t}{P_t} - \eta \frac{C_{t-1}}{P_{t-1}}\right)^{1-\theta}}{1-\theta} - \chi \frac{h_t^{1+\gamma}}{1+\gamma} + \psi \ln\left(\frac{M_t}{P_t}\right) \right] \tag{1.2}$$

Email address: `22562435@sun.ac.za` (Liam Andrew Beattie)

$$C_t + I_t + B_t + M_t \leq R^B_{t-1} B_{t-1} + M_{t-1} + W_t h_t + R^k_t K_{t-1} + \Pi_t - P_t \tau_t \tag{1.3}$$

$$K_t = (1-\delta) K_{t-1} + I_t - \frac{\phi}{2} \left(\frac{I_t}{K_{t-1}} - \delta \right)^2 K_{t-1} \tag{1.4}$$

From which we can derive the following equations, found in full form in Appendix 5.1.1

$$(c_t - \eta c_{t-1})^{-\theta} = \beta \mathbb{E}_t \Big[R^B_t \frac{P_t}{P_{t+1}} (c_{t+1} - \eta c_t)^{-\theta} \Big] \tag{1.5}$$

Equation 1.5 describes the household's optimal consumption decision over time. On the left-hand side, we have the marginal utility of consumption today, which takes into account *habit formation* - meaning that current satisfaction from consuming c_t is reduced if past consumption c_{t-1} was high. The right-hand side reflects the expected marginal benefit of postponing consumption to the next period: it combines the expected *real return* on bonds, $R^B_t \cdot \frac{P_t}{P_{t+1}}$, with the marginal utility of tomorrow's (habit-adjusted) consumption, $c_{t+1} - \eta c_t$.

Put simply, households balance the gain from consuming now against the expected value of saving and consuming later. Habit persistence, captured by η, introduces a kind of "inertia" in consumption preferences, while inflation, through the price ratio $\frac{P_t}{P_{t+1}}$, adjusts the real value of future returns.

$$\frac{W_t}{P_t} = \frac{\chi h_t^\gamma}{(c_t - \eta c_{t-1})^{-\theta} - \beta \eta \mathbb{E}_t \big[(c_{t+1} - \eta c_t)^{-\theta} \big]} \tag{1.6}$$

Equation 1.6 links the real wage $\frac{W_t}{P_t}$ to the household's labour supply decision. The right-hand side captures the trade-off between working more hours, h_t, and the (habit-adjusted) marginal utility of consumption. Stronger habit formation (η) lowers the perceived benefit of consumption, so households require a higher real wage to be willing to work the same amount - especially when the disutility of labour rises more steeply with hours (γ large).

$$M_t = \frac{\psi}{\beta \mathbb{E}_t \Big[(R^B_t - 1) \cdot \frac{(c_{t+1} - \eta c_t)^{-\theta} - \beta \eta \mathbb{E}_{t+1} \big[(c_{t+2} - \eta c_{t+1})^{-\theta} \big]}{P_{t+1}} \Big]} \tag{1.7}$$

Equation 1.7 characterises money demand as inversely related to the expected return on bonds - the higher the nominal rate $(R^B_t - 1)$, the greater the opportunity cost of holding money. The expression in the denominator reflects the *liquidity premium*, adjusted for how habits (η) and expected future consumption affect the marginal utility of spending. Inflation expectations, via P_{t+1}, also influence how attractive money is relative to interest-bearing assets.

$$
\begin{aligned}
q_t \;&\equiv\; 1 - \phi\left(\tfrac{I_t}{K_{t-1}} - \delta\right) \\
\frac{\beta\,\mathbb{E}_t[\lambda_{t+1}\,R_t^B]}{q_t} \;&=\; \beta\,\mathbb{E}_t\Big[\lambda_{t+1}\Big(R_{t+1}^k + \frac{1}{q_{t+1}}\Big(1-\delta+\tfrac{\phi}{2}\left[(I_{t+1}/K_t)^2-\delta^2\right]\Big)\Big)\Big]
\end{aligned}
\tag{1.8}
$$

Equation 1.8 defines Tobin's q_t - the value of an additional unit of capital - and describes how firms decide whether to invest. When the ratio of investment to existing capital $\frac{I_t}{K_{t-1}}$ exceeds depreciation δ, $q_t > 1$, signalling that it's profitable to expand the capital stock. The equation balances the bond-return-adjusted cost of investing (left-hand side) with the expected return on capital, including future capital gains and adjustment costs (right-hand side). The parameter ϕ governs these adjustment costs, meaning that rapid changes in investment are costly and create frictions.

1.2. Production

This model departs from standard RBC frameworks by introducing monopolistic competition and nominal rigidities. This is achieved through two layers of firms:

1. **Perfectly competitive final goods producers** who aggregate intermediate goods

2. **Monopolistically competitive intermediate goods producers** with price-setting power

This two-tiered structure captures core New Keynesian dynamics, such as price stickiness and strategic pricing, while retaining analytical tractability.

1.2.1. Final Goods Producer

(Perfectly competitive aggregator)

The final goods firm combines differentiated inputs $Y_t(j)$ into final output Y_t using a CES aggregator:

$$
Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}}\,dj\right)^{\frac{\epsilon}{\epsilon-1}}, \quad \epsilon > 1 \tag{1.9}
$$

Where ϵ aptures the elasticity of substitution between varieties: the higher it is, the more easily final goods producers can substitute across inputs. Conversely, a lower ϵ implies that intermediate producers face less competition and enjoy greater market power.

Given the CES aggregator in Equation (1.9), the final goods producer chooses input varieties to minimize the cost of delivering one unit of output. The resulting demand and pricing relationships, derived in Appendix 5.2.1, are:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t \tag{1.10}$$

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj\right)^{\frac{1}{1-\epsilon}} \tag{1.11}$$

Equation 1.10 shows that more expensive varieties are purchased in smaller quantities, while Equation 1.11 reflects the minimum cost of assembling one unit of final output given prevailing input prices.

1.2.2. Intermediate Goods Producers

(Monopolistically competitive firms, with nominal rigidities)

Each intermediate firm j produces with identical Cobb-Douglas technology:

$$Y_t(j) = A_t K_t(j)^{\alpha} h_t(j)^{1-\alpha} \tag{1.12}$$

Cost minimization yields the real marginal cost:

$$\text{MC}_t = \frac{1}{A_t}\left(\frac{R_t^k}{\alpha}\right)^{\alpha}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha} \tag{1.13}$$

Equation (5.12), which is derived in Appendix 5.2.2), captures the firm's cost of producing one unit of output, given factor prices and technology. A rise in wages or rental rates increases marginal cost, while higher productivity A_t reduces it.

Intermediate firms set prices subject to Rotemberg-style adjustment costs. The firm chooses a price path to maximize the expected discounted sum of profits, which includes a quadratic penalty for deviating from past prices. Imposing symmetry and using first-order conditions, we obtain the pricing equation:

Newer

$$\boxed{0 = \epsilon \left(mc_t - \tfrac{\epsilon-1}{\epsilon}\right) - \psi(\pi_t - 1)\pi_t + \beta \mathbb{E}_t \left[\tfrac{\lambda_{t+1}}{\lambda_t} \psi(\pi_{t+1} - 1)\pi_{t+1} \tfrac{Y_{t+1}}{Y_t}\right]} \tag{1.14}$$

Equation (5.13) is a New Keynesian Phillips Curve under Rotemberg pricing. It links inflation dynamics π_t to real marginal costs mc_t and expected future inflation. The parameter ψ governs price rigidity: higher values imply stronger penalties for price changes and thus more persistent inflation dynamics.

OLDER

$$\text{AdjCost}_t(j) = \frac{\psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)^2 Y_t$$

The real profit function is:

$$\Pi_t(j) = \underbrace{\frac{P_t(j)}{P_t} Y_t(j)}_{\text{real revenue}} - \underbrace{\frac{MC_t}{P_t} \cdot Y_t(j)}_{\text{real cost}} - \underbrace{\frac{\psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)^2 Y_t}_{\text{adjustment cost}} \tag{1.15}$$

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \Lambda_{t,t+s} \left[\frac{P_{t+s}(j)}{P_{t+s}} Y_{t+s}(j) - mc_{t+s} Y_{t+s}(j) - \frac{\psi}{2} \left(\frac{P_{t+s}(j)}{P_{t+s-1}(j)} - 1\right)^2 Y_{t+s}\right]$$

where: - $\Lambda_{t,t+s} = \beta^s \frac{\lambda_{t+s}}{\lambda_t}$ = Stochastic discount factor (from households) - $mc_t = \frac{\text{MC}_t}{P_t}$ = Real marginal cost (eq (5.12)) - $Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t$ (demand curve from eq (1.10))

Optimal Price Setting

Each intermediate firm j maximizes discounted real profits, accounting for future adjustment costs. In symmetric equilibrium ($P_t(j) = P_t$, $\pi_t = P_t/P_{t-1}$), this yields the **Rotemberg Phillips Curve**:

$$(\pi_t - 1)\pi_t = \frac{\epsilon}{\psi} \left(mc_t - \frac{\epsilon - 1}{\epsilon}\right) + \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \frac{Y_{t+1}}{Y_t} (\pi_{t+1} - 1)\pi_{t+1}\right] \tag{1.16}$$

where $mc_t = \text{MC}_t/P_t$ (real marginal cost) and λ_t is the household's marginal utility of consumption. This links inflation to:

1. Deviations of real marginal cost from its flexible-price level $\frac{\epsilon-1}{\epsilon}$
2. Expected future inflation (weighted by discounting and output growth)

3. Adjustment cost parameter ψ (higher ψ = stickier prices)

5.2.2

1.3. Government Sector

Government sector is from Sims (2024c)'s notes. Government chooses spending, (term), exogenously. It finances spending with lump-sum taxes and issues new debt.

Gov budget constraint (nominal)

$$G_t + R^B_{t-1} D_{t-1} \leq D_t + P_t \tau_t \tag{1.17}$$

With $R^B t$ being the interest rate on bonds for period t.

To ensure internal consistency, I introduce a simple government sector. The government issues one-period nominal bonds purchased by households, uses tax revenues to finance an exogenous stream of government spending, and services its debt obligations. The government budget constraint equates the sum of nominal spending and interest payments to the sum of new debt issuance and tax revenues. We assume lump-sum taxation and do not model Ricardian equivalence effects explicitly. Bonds held by households are thus assumed to be government-issued, closing the financial side of the model.

2. Policy, Equilibrium, and Aggregation

A competitive equilibrium is a set of prices () and allocations () such that (i) household and rm optimality conditions all hold, (ii) the rm hires all the labour and capital supplied by the household, (iii) the household and rm budget constraints hold with equality, and (iv) household bond-holdings equal government debt issuance in all periods (i.e. Bt+1 = Dt+1, and we require that Bt = Dt initially), given values and stochastic processes of Gt and At, as well as initial values of government debt and household bond-holdings, which must be equal (e.g. Bt =Dt).

$$B_t \;=\; D_t \tag{Bond market clear}$$

2.1. Exogenous Processes

Government spending, taxes, and technology evolve according to exogenous AR(1) processes.

Real taxes follow Equation (2.1),

$$\tau_t = (1 - \rho_\tau)\bar{\tau} + \rho_\tau \tau_{t-1} + \varepsilon^\tau_t \tag{2.1}$$

where $\bar{\tau}$ is the steady-state level, ρ_τ controls persistence, and ε_t^τ is a fiscal shock.

Government spending is governed by Equation (2.2),

$$g_t = (1 - \rho_g)\bar{g} + \rho_g g_{t-1} + \varepsilon_t^g \tag{2.2}$$

with similar dynamics: mean reversion around $\bar{g}$ and shocks ε_t^g.

Technology evolves log-linearly as in Equation (2.3),

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_t^a \tag{2.3}$$

ensuring a unit steady-state level and allowing for persistent TFP shocks.

3. old

$$T_t \;=\; (1 - \rho_T)\,\bar{T} \;+\; \rho_T\,T_{t-1} \;+\; \varepsilon_t^T, \qquad 0 \leq \rho_T < 1, \tag{3.1}$$

where

- $\bar{T}$ is the (constant) long-run level of nominal taxes,
- ρ_T governs the persistence of tax shocks, and
- ε_t^T is an i.i.d. fiscal-shock process.

Because $\tau_t = T_t/P_t$, this fully pins down the lump-sum tax rate each period (once you know the price level P_t). In your code you would simply treat T_t as an exogenous state whose only "choice" is driven by the AR(1) shock ε_t^T.

Unsure on the following before i look up if it at all breaks my model - TFP shock: $\ln A_t = (1 - \rho_A)\ln A_{ss} + \rho_A \ln A_{t-1} + \varepsilon_t^A$

- Monetary policy shocks ($\varepsilon_t^r, \varepsilon_t^m$
- Output Gap: $\hat{Y}_t = Y_t - Y_t^n$ (natural rate output)

- Market clearing conditions
- Determinacy Requirements: Blanchard-Kahn conditions for policy rules

Market Clearing Aggregate production:

$$Y_t = A_t K_{t-1}^{\alpha} h_t^{1-\alpha} \tag{3.2}$$

Resource constraint (adjustment costs reduce output):

$$Y_t = C_t + I_t + \underbrace{\frac{\psi}{2}(\Pi_t - 1)^2 Y_t}_{\text{price adjustment cost}} \tag{3.3}$$

Factor markets clear:

$$\int_0^1 h_t(j)dj = h_t \tag{3.4}$$
$$\int_0^1 K_t(j)dj = K_{t-1} \tag{3.5}$$

3.1. Full Set of Conditions

$$K_t \;=\; (1-\delta)\,K_{t-1} \;+\; I_t \;-\; \frac{\phi}{2}\left(\frac{I_t}{K_{t-1}} - \delta\right)^2 K_{t-1} \tag{1.4}$$

$$(c_t - \eta\, c_{t-1})^{-\theta} \;=\; \beta\,\mathbb{E}_t\!\left[R_t^B\, \frac{P_t}{P_{t+1}}\, (c_{t+1} - \eta\, c_t)^{-\theta}\right] \tag{1.5}$$

$$\frac{W_t}{P_t} \;=\; \frac{\chi\, h_t^{\gamma}}{(c_t - \eta\, c_{t-1})^{-\theta} \;-\; \beta\,\eta\,\mathbb{E}_t\!\left[(c_{t+1} - \eta\, c_t)^{-\theta}\right]} \tag{1.6}$$

$$M_t \;=\; \frac{\psi}{\beta\,\mathbb{E}_t\!\left[(R_t^B - 1)\;\cdot\;\dfrac{(c_{t+1} - \eta\, c_t)^{-\theta} \;-\; \beta\,\eta\,\mathbb{E}_{t+1}\!\left[(c_{t+2} - \eta\, c_{t+1})^{-\theta}\right]}{P_{t+1}}\right]} \tag{1.7}$$

$$q_t \equiv 1 - \phi\left(\tfrac{I_t}{K_{t-1}} - \delta\right)$$

$$\frac{\beta\,\mathbb{E}_t[\lambda_{t+1}\,R_t^B]}{q_t} = \beta\,\mathbb{E}_t\Big[\lambda_{t+1}\Big(R_{t+1}^k + \frac{1}{q_{t+1}}\Big(1 - \delta + \tfrac{\phi}{2}\big[(I_{t+1}/K_t)^2 - \delta^2\big]\Big)\Big)\Big] \tag{1.8}$$

Each firm j produces with identical Cobb-Douglas technology:

$$Y_t(j) = A_t K_t(j)^{\alpha} h_t(j)^{1-\alpha} \tag{1.12}$$

$$(\pi_t - 1)\pi_t = \frac{\epsilon}{\psi}\left(mc_t - \frac{\epsilon - 1}{\epsilon}\right) + \beta\mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}\frac{Y_{t+1}}{Y_t}(\pi_{t+1} - 1)\pi_{t+1}\right] \tag{1.16}$$

4. Steady State

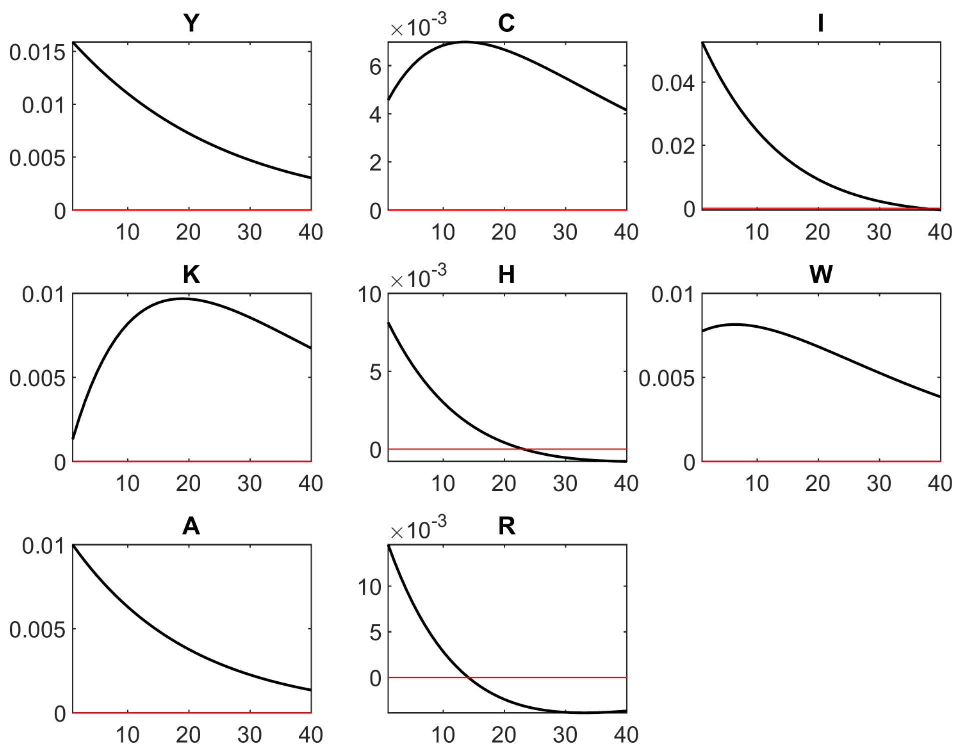

Figure 4.1: image

5. Appendix

5.1. Households

Define Lagrangian

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=1}^{\infty} \beta^{t-1} \Big\{ \underbrace{\frac{\left(\frac{C_t}{P_t} - \eta \frac{C_{t-1}}{P_{t-1}}\right)^{1-\theta}}{1-\theta}}_{\text{Consumption utility}} - \underbrace{\chi \frac{h_t^{1+\gamma}}{1+\gamma}}_{\text{Labor disutility}} + \underbrace{\psi \ln\left(\frac{M_t}{P_t}\right)}_{\text{Money utility}}$$

$$+ \underbrace{\lambda_t \left[R_{t-1}^{B} B_{t-1} + M_{t-1} + W_t h_t + R_t^{k} K_{t-1} + \Pi_t - P_t \tau_t - C_t - I_t - B_t - M_t \right]}_{\text{Nominal flow constraint}} + \underbrace{\mu_t \left[(1-\delta) K_{t-1} + I_t - \frac{\phi}{2}\left(\frac{I_t}{K_{t-1}}\right.\right.}_{\text{Capital accumulatio}}$$

5.1.1. First Order Conditions

FOC w.r.t. Consumption :

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0$$

$$\left[(c_t - \eta c_{t-1})^{-\theta} / P_t - \lambda_t \right] - \beta \, \mathbb{E}_t \left[\eta \, (c_{t+1} - \eta c_t)^{-\theta} / P_t \right] = 0$$

Combine terms over $1/P_t$

$$\frac{1}{P_t} \left[(c_t - \eta c_{t-1})^{-\theta} - \beta \eta \, \mathbb{E}_t [(c_{t+1} - \eta c_t)^{-\theta}] \right] - \lambda_t = 0$$

Multiply by P_t

$$(c_t - \eta c_{t-1})^{-\theta} - \beta \eta \, \mathbb{E}_t [(c_{t+1} - \eta c_t)^{-\theta}] - \lambda_t P_t = 0$$

$$\boxed{\lambda_t P_t = (c_t - \eta c_{t-1})^{-\theta} - \beta \eta \, \mathbb{E}_t \left[(c_{t+1} - \eta c_t)^{-\theta} \right]} \tag{5.1}$$

FOC w.r.t. Labour :

$$\frac{\partial \mathcal{L}}{\partial h_t} = 0$$

$$- \chi h_t^{\gamma} + \lambda_t W_t = 0$$

Rearrange

$$\lambda_t W_t = \chi h_t^{\gamma}$$

$$\boxed{\lambda_t W_t = \chi h_t^\gamma} \tag{5.2}$$

FOC w.r.t. Real Money Balances:

$$\frac{\partial \mathcal{L}}{\partial M_t} = 0$$

$$\beta^{t-1}[\psi/M_t - \lambda_t] + \beta^t \mathbb{E}_t[\lambda_{t+1}] = 0$$

Divide by β^{t-1} and rearrange

$$\psi/M_t - \lambda_t + \beta\, \mathbb{E}_t[\lambda_{t+1}] = 0$$

$$\boxed{\frac{\psi}{M_t} = \lambda_t - \beta\, \mathbb{E}_t[\lambda_{t+1}]} \tag{5.3}$$

FOC w.r.t. Bonds (5.4)):

$$\frac{\partial \mathcal{L}}{\partial B_t} = 0$$

$$-\beta^{t-1}\lambda_t + \beta^t \mathbb{E}_t[\lambda_{t+1} R_t^B] = 0$$

Divide by β^{t-1} and simplify

$$-\lambda_t + \beta\, \mathbb{E}_t[\lambda_{t+1} R_t^B] = 0$$

$$\boxed{\lambda_t = \beta\, \mathbb{E}_t[\lambda_{t+1} R_t^B]} \tag{5.4}$$

FOC w.r.t. Capital :

$$\frac{\partial \mathcal{L}}{\partial K_t} = 0$$

$$-\beta^{t-1}\mu_t + \beta^t \mathbb{E}_t\big[\lambda_{t+1} R_{t+1}^k + \mu_{t+1}(1 - \delta + \tfrac{\phi}{2}((I_{t+1}/K_t)^2 - \delta^2))\big] = 0$$

Divide by β^{t-1} and solve

$$\mu_t = \beta\, \mathbb{E}_t\big[\lambda_{t+1} R_{t+1}^k + \mu_{t+1}(1 - \delta + \tfrac{\phi}{2}((I_{t+1}/K_t)^2 - \delta^2))\big]$$

$$\boxed{\mu_t = \beta\, \mathbb{E}_t\big[\lambda_{t+1} R_{t+1}^k + \mu_{t+1}(1 - \delta + \tfrac{\phi}{2}((I_{t+1}/K_t)^2 - \delta^2))\big]} \tag{5.5}$$

FOC w.r.t. Investment :

$$\frac{\partial \mathcal{L}}{\partial I_t} = 0$$
$$\beta^{t-1}\left[-\lambda_t + \mu_t(1 - \phi(\tfrac{I_t}{K_{t-1}} - \delta))\right] = 0$$

Divide by β^{t-1} and isolate

$$\lambda_t = \mu_t\big(1 - \phi(\tfrac{I_t}{K_{t-1}} - \delta)\big)$$

$$\boxed{\lambda_t = \mu_t\big(1 - \phi(\tfrac{I_t}{K_{t-1}} - \delta)\big)} \tag{5.6}$$

5.1.2. Household Final Equations

Consumption Euler Equation

Combines consumption–habit dynamics with bond returns (from (5.4) and (5.1)) :

Start with FOC for Bonds

$$\lambda_t = \beta\,\mathbb{E}_t[\lambda_{t+1} R_t^B] \quad \text{(Equation 5.4)}$$

Substitute λ_t and λ_{t+1} from FOC for Consumption

$$\lambda_t = \frac{(c_t - \eta\, c_{t-1})^{-\theta}}{P_t} \quad \text{(from Equation 5.1 rearranged)}$$
$$\lambda_{t+1} = \frac{(c_{t+1} - \eta\, c_t)^{-\theta}}{P_{t+1}} \quad \text{(time-shifted)}$$

Combine results

$$\frac{(c_t - \eta\, c_{t-1})^{-\theta}}{P_t} = \beta\,\mathbb{E}_t\left[R_t^B \cdot \frac{(c_{t+1} - \eta\, c_t)^{-\theta}}{P_{t+1}}\right]$$

Clear denominator

$$(c_t - \eta\, c_{t-1})^{-\theta} = \beta\,\mathbb{E}_t\left[R_t^B \cdot \frac{P_t}{P_{t+1}} \cdot (c_{t+1} - \eta\, c_t)^{-\theta}\right]$$

$$\boxed{(c_t - \eta\, c_{t-1})^{-\theta} \;=\; \beta\, \mathbb{E}_t\Big[R_t^B\, \frac{P_t}{P_{t+1}}\, (c_{t+1} - \eta\, c_t)^{-\theta}\Big]} \tag{5.7}$$

Labour Supply

Real wage equals the marginal rate of substitution between leisure and consumption (from (5.1) and (5.2)) :

Start with FOC for Hours Worked

$\lambda_t W_t = \chi\, h_t^{\gamma} \quad$ (Equation 5.2)

Solve for λ_t

$$\lambda_t = \frac{\chi\, h_t^{\gamma}}{W_t}$$

Equate to FOC of Consumption expression

$$\frac{\chi\, h_t^{\gamma}}{W_t} = \frac{(c_t - \eta\, c_{t-1})^{-\theta} - \beta\, \eta\, \mathbb{E}_t[(c_{t+1} - \eta\, c_t)^{-\theta}]}{P_t}$$

Solve for real wage (W_t/P_t)

$$\frac{W_t}{P_t} = \frac{\chi\, h_t^{\gamma}}{(c_t - \eta\, c_{t-1})^{-\theta} - \beta\, \eta\, \mathbb{E}_t[(c_{t+1} - \eta\, c_t)^{-\theta}]}$$

$$\boxed{\frac{W_t}{P_t} \;=\; \frac{\chi\, h_t^{\gamma}}{(c_t - \eta\, c_{t-1})^{-\theta} \;-\; \beta\, \eta\, \mathbb{E}_t\big[(c_{t+1} - \eta\, c_t)^{-\theta}\big]}} \tag{5.8}$$

Money Demand

Opportunity cost of holding money vs. bonds (from (5.3), (5.4) and (5.1)) :

Combine FOC for Money and Bonds

$$\frac{\psi}{M_t} = \lambda_t - \beta\,\mathbb{E}_t[\lambda_{t+1}] \quad \text{(Equation 5.3)}$$

$$\lambda_t = \beta\,\mathbb{E}_t[\lambda_{t+1} R_t^B] \quad \text{(Equation 5.4)}$$

Substitute λ_t into FOC of money

$$\frac{\psi}{M_t} = \beta\,\mathbb{E}_t[\lambda_{t+1} R_t^B] - \beta\,\mathbb{E}_t[\lambda_{t+1}]$$

$$\frac{\psi}{M_t} = \beta\,\mathbb{E}_t\left[\lambda_{t+1}(R_t^B - 1)\right]$$

Substitute λ_{t+1} from FOC of Consumption

$$\lambda_{t+1} = \frac{(c_{t+1} - \eta\,c_t)^{-\theta} - \beta\,\eta\,\mathbb{E}_{t+1}[(c_{t+2} - \eta\,c_{t+1})^{-\theta}]}{P_{t+1}}$$

Solve for M_t

$$M_t = \frac{\psi}{\beta\,\mathbb{E}_t\left[(R_t^B - 1) \cdot \dfrac{(c_{t+1} - \eta\,c_t)^{-\theta} - \beta\,\eta\,\mathbb{E}_{t+1}[(c_{t+2} - \eta\,c_{t+1})^{-\theta}]}{P_{t+1}}\right]}$$

$$\boxed{M_t \;=\; \frac{\psi}{\beta\,\mathbb{E}_t\Big[(R_t^B - 1) \;\cdot\; \dfrac{(c_{t+1} - \eta\,c_t)^{-\theta} \;-\; \beta\,\eta\,\mathbb{E}_{t+1}\big[(c_{t+2} - \eta\,c_{t+1})^{-\theta}\big]}{P_{t+1}}\Big]}} \tag{5.9}$$

Capital Euler Equation Defines Tobin's q and links required returns on capital to bond returns (from (5.6), (5.5) and (5.4)) :

Define Tobin's q from FOC for Investment

$\lambda_t = \mu_t q_t \quad \text{where} \quad q_t \equiv 1 - \phi\left(\frac{I_t}{K_{t-1}} - \delta\right)$

Rearrange FOC for Capital

$\mu_t = \beta\,\mathbb{E}_t\Big[\lambda_{t+1} R^k_{t+1} + \mu_{t+1}\Big(1 - \delta + \frac{\phi}{2}\left[(I_{t+1}/K_t)^2 - \delta^2\right]\Big)\Big]$

Substitute $\mu_t = \lambda_t/q_t$ and $\mu_{t+1} = \lambda_{t+1}/q_{t+1}$

$$\frac{\lambda_t}{q_t} = \beta\,\mathbb{E}_t\left[\lambda_{t+1} R^k_{t+1} + \frac{\lambda_{t+1}}{q_{t+1}}\left(1 - \delta + \frac{\phi}{2}\left[(I_{t+1}/K_t)^2 - \delta^2\right]\right)\right]$$

Factor λ_{t+1}

$$\frac{\lambda_t}{q_t} = \beta\,\mathbb{E}_t\left[\lambda_{t+1}\left(R^k_{t+1} + \frac{1}{q_{t+1}}\left(1 - \delta + \frac{\phi}{2}\left[(I_{t+1}/K_t)^2 - \delta^2\right]\right)\right)\right]$$

Substitute FOC for Bonds ($\lambda_t = \beta\,\mathbb{E}_t[\lambda_{t+1} R^B_t]$)

$$\frac{\beta\,\mathbb{E}_t[\lambda_{t+1} R^B_t]}{q_t} = \beta\,\mathbb{E}_t\left[\lambda_{t+1}\left(R^k_{t+1} + \frac{1}{q_{t+1}}\Gamma_{t+1}\right)\right]$$

where $\Gamma_{t+1} \equiv 1 - \delta + \frac{\phi}{2}\left[(I_{t+1}/K_t)^2 - \delta^2\right]$

$$\boxed{\begin{aligned} q_t \;&\equiv\; 1 - \phi\left(\tfrac{I_t}{K_{t-1}} - \delta\right) \\ \frac{\beta\,\mathbb{E}_t[\lambda_{t+1}\, R^B_t]}{q_t} \;&=\; \beta\,\mathbb{E}_t\Big[\lambda_{t+1}\Big(R^k_{t+1} + \frac{1}{q_{t+1}}\Big(1 - \delta + \tfrac{\phi}{2}\left[(I_{t+1}/K_t)^2 - \delta^2\right]\Big)\Big)\Big] \end{aligned}} \tag{5.10}$$

5.2. Production

5.2.1. Final Good Producer

Derivation of Intermediate Goods Demand and Aggregate Price Index

Final goods producer's profit:

$$\Pi_t = P_t Y_t - \int_0^1 P_t(j) Y_t(j) dj$$

subject to $Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$

Substitute production function into profit:

$$\Pi_t = P_t \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 P_t(j) Y_t(j) dj$$

First-order condition for $Y_t(j)$:

$$\frac{\partial \Pi_t}{\partial Y_t(j)} = P_t \cdot \frac{\epsilon}{\epsilon - 1} \left(\int_0^1 Y_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{1}{\epsilon-1}} \cdot \frac{\epsilon - 1}{\epsilon} Y_t(j)^{-\frac{1}{\epsilon}} - P_t(j) = 0$$

$$\Rightarrow P_t \cdot Y_t^{\frac{1}{\epsilon}} Y_t(j)^{-\frac{1}{\epsilon}} = P_t(j)$$

Rearrange to obtain demand curve:

$$Y_t(j) = \left(\frac{P_t}{P_t(j)} \right)^{\epsilon} Y_t$$

Substitute demand into production function:

$$Y_t = \left(\int_0^1 \left[\left(\frac{P_t}{P_t(j)} \right)^{\epsilon} Y_t \right]^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

$$= Y_t \left(\int_0^1 \left(\frac{P_t}{P_t(j)} \right)^{\epsilon-1} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

Simplify to obtain price index:

$$1 = \left(\int_0^1 \left(\frac{P_t}{P_t(j)}\right)^{\epsilon-1} dj\right)^{\frac{\epsilon}{\epsilon-1}}$$

$$\Rightarrow P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon} dj$$

$$\Rightarrow P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj\right)^{\frac{1}{1-\epsilon}}$$

$$\boxed{\begin{array}{c} Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t \\ \\ P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj\right)^{\frac{1}{1-\epsilon}} \end{array}} \tag{5.11}$$

5.2.2. Intermediate Goods Producers

Cost minimization for intermediate firm j :

$$\min_{K_t(j),h_t(j)} \left\{ R_t^k K_t(j) + W_t h_t(j) \right\}$$

subject to $Y_t(j) = A_t K_t(j)^\alpha h_t(j)^{1-\alpha}$

Lagrangian:

$$\mathcal{L} = R_t^k K_t(j) + W_t h_t(j) + \lambda_t \left[A_t K_t(j)^\alpha h_t(j)^{1-\alpha} - Y_t(j) \right]$$

First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial K_t(j)} = 0 : \quad R_t^k = \lambda_t \alpha A_t K_t(j)^{\alpha-1} h_t(j)^{1-\alpha}$$

$$\frac{\partial \mathcal{L}}{\partial h_t(j)} = 0 : \quad W_t = \lambda_t (1-\alpha) A_t K_t(j)^\alpha h_t(j)^{-\alpha}$$

Rearrange FOCs:

$$\lambda_t = \frac{R_t^k}{\alpha} \left(\frac{K_t(j)}{h_t(j)} \right)^{1-\alpha} \frac{1}{A_t}, \quad \lambda_t = \frac{W_t}{1-\alpha} \left(\frac{K_t(j)}{h_t(j)} \right)^{\alpha} \frac{1}{A_t}$$

Equate expressions:

$$\frac{R_t^k}{\alpha} \left(\frac{K_t(j)}{h_t(j)} \right)^{-\alpha} = \frac{W_t}{1-\alpha} \left(\frac{K_t(j)}{h_t(j)} \right)^{1-\alpha}$$

$$\Rightarrow \frac{K_t(j)}{h_t(j)} = \frac{\alpha}{1-\alpha} \frac{W_t}{R_t^k}$$

Substitute into capital FOC:

$$\lambda_t = \frac{R_t^k}{\alpha A_t} \left(\frac{\alpha}{1-\alpha} \frac{W_t}{R_t^k} \right)^{\alpha-1}$$

$$= \frac{1}{A_t} \left(\frac{R_t^k}{\alpha} \right)^\alpha \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha}$$

$$\boxed{MC_t = \frac{1}{A_t} \left(\frac{R_t^k}{\alpha} \right)^\alpha \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha}} \tag{5.12}$$

Intermediate-goods producer's problem:

$$\max_{P_t(j)} \; \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s} \Big[\big(\tfrac{P_{t+s}(j)}{P_{t+s}}\big)^{1-\epsilon} Y_{t+s} - mc_{t+s} \, \big(\tfrac{P_{t+s}(j)}{P_{t+s}}\big)^{-\epsilon} Y_{t+s} - \tfrac{\psi}{2} \big(\tfrac{P_{t+s}(j)}{P_{t+s-1}(j)} - 1\big)^2 Y_{t+s} \Big]$$

subject to $\quad Y_t(j) = \big(\tfrac{P_t(j)}{P_t}\big)^{-\epsilon} Y_t$

First-Order Condition w.r.t. $P_t(j)$:

$$\mathbb{E}_t \Big[\frac{\partial \Pi_t(j)}{\partial P_t(j)} + \beta \, \Lambda_{t,t+1} \frac{\partial \Pi_{t+1}(j)}{\partial P_t(j)} \Big] = 0$$

$$\frac{\partial \Pi_t}{\partial P_t(j)} = (1-\epsilon) \big(\tfrac{P_t(j)}{P_t}\big)^{-\epsilon} \tfrac{Y_t}{P_t} + \epsilon \, mc_t \big(\tfrac{P_t(j)}{P_t}\big)^{-\epsilon-1} \tfrac{Y_t}{P_t} - \psi \big(\tfrac{P_t(j)}{P_{t-1}(j)} - 1\big) \tfrac{Y_t}{P_{t-1}(j)}$$

$$\frac{\partial \Pi_{t+1}}{\partial P_t(j)} = \psi \big(\tfrac{P_{t+1}(j)}{P_t(j)} - 1\big) \, \tfrac{P_{t+1}(j)}{P_t(j)^2} \, Y_{t+1}$$

Impose symmetry: $P_t(j) = P_t, \; Y_t(j) = Y_t, \; \pi_t = \frac{P_t}{P_{t-1}}$.

$(1-\epsilon) + \epsilon \, mc_t = \epsilon \big(mc_t - \tfrac{\epsilon-1}{\epsilon}\big), \quad \tfrac{P_t(j)}{P_{t-1}(j)} = \pi_t, \; \tfrac{P_{t+1}(j)}{P_t(j)} = \pi_{t+1}$

$$0 = \epsilon \big(mc_t - \tfrac{\epsilon-1}{\epsilon}\big) - \psi \, (\pi_t - 1) \, \pi_t + \beta \, \mathbb{E}_t \Big[\Lambda_{t,t+1} \, \psi \, (\pi_{t+1} - 1) \, \pi_{t+1} \, \tfrac{Y_{t+1} P_t}{Y_t P_{t+1}} \Big]$$

Noting $\Lambda_{t,t+1} = \beta \, \frac{\lambda_{t+1}}{\lambda_t}$ and $P_{t+1}/P_t = \pi_{t+1}$, the bracket simplifies to $\beta \, \frac{\lambda_{t+1}}{\lambda_t} \, \frac{Y_{t+1}}{Y_t}$.

$$\boxed{0 = \epsilon \Big(mc_t - \tfrac{\epsilon-1}{\epsilon}\Big) - \psi \, (\pi_t - 1) \, \pi_t + \beta \, \mathbb{E}_t \Big[\tfrac{\lambda_{t+1}}{\lambda_t} \, \psi \, (\pi_{t+1} - 1) \, \pi_{t+1} \, \tfrac{Y_{t+1}}{Y_t} \Big]} \tag{5.13}$$

6. Old Stuff

Components breakdown:

Expenditures:

Consumption: $P_t c_t$

Investment: $P_t i_t$

Bonds: B_t

Money holdings: M_t

Income sources:

Bond returns: $(1 + i_{t-1})B_{t-1}$

Money carryover: M_{t-1}

Labor income: $W_t h_t$

Capital returns: $R_t^k K_{t-1}$ (Key addition missing in Sims)

Firm profits: Π_t

Net transfers: $-P_t \tau_t$

References

European Central Bank. 2022. *Economic and monetary developments no. 770.* (ECB Working Paper 770). European Central Bank. [Online], Available: https://www.ecb.europa.eu/pub/pdf/scpwps/ecbwp770.pdf.

Mati, S. 2019. DynareR: Bringing the power of Dynare to R, R Markdown, and Quarto. *CRAN*. [Online], Available: https://CRAN.R-project.org/package=DynareR.

Sims, E. 2024a. *Graduate macro theory II: The basic new keynesian model.* (Lecture Notes ECON 60202 – Spring 2024). Notre Dame, IN: University of Notre Dame. [Online], Available: https://sites.nd.edu/esims/files/2024/03/notes_new_keynesian_2024-1.pdf.

Sims, E. 2024b. *Graduate macro theory II: Extensions of basic RBC framework.* (Lecture Notes ECON 60202 – Spring 2024). Notre Dame, IN: University of Notre Dame. [Online], Available: https://sites.nd.edu/esims/files/2024/02/notes_rbc_extensions_2024.pdf.

Sims, E. 2024c. *Fiscal policy in the RBC model.* (Lecture Notes ECON 60202 – Spring 2024). Notre Dame, IN: University of Notre Dame. [Online], Available: https://sites.nd.edu/esims/files/2024/02/notes_fiscal_policy_2024.pdf.