A Monetary RBC Model with Nominal Rigidities

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Abstract

I want to thank Mati (2019) for creating the R package that allowed the use of MATLAB and Dynare through R. |

This chapter extends a standard real business cycle (RBC) framework with capital accumulation to incorporate New Keynesian features. I introduce consumption habits, money-in-utility, monopolistic competition with Rotemberg price rigidities, and capital adjustment costs-all specified in real terms. The resulting hybrid model provides foundations for analyzing monetary policy transmission through intertemporal substitution and nominal frictions.

Table 0.1: Key Household Real Variables

Variable	Description
c_t	Real consumption
h_t	Hours worked
m_t	Real money balances
b_t	Real bond holdings
K_t	Real capital
i_t	Real investment

1. A RBC foundation

1.1. Households

I follow the structure of Sims (2024a), but work entirely in real variables, while Sims defines the household problem in nominal terms. This shift affects the way assets and constraints are written. I

also extend Sims' setup by including capital, which is subject to adjustment costs. These are modelled after Sims (2024b), who follows the formulation in Hayashi (1982).

Households derive utility from consumption and money, and disutility from working. They form expectations over future utility and optimize subject to a standard intertemporal problem:

$$
\max_{\{c_t, h_t, m_t, b_t, K_t, i_t\}} \mathbb{E}_0 \sum_{t=1}^{\infty} \beta^{t-1} \left[\frac{\left(c_t - \eta\, c_{t-1}\right)^{1-\theta}}{1-\theta} \; - \; \chi \frac{h_t^{1+\gamma}}{1+\gamma} \; + \; \psi \ln\!\left(m_t\right) \right] \tag{1.1}
$$

Consumption habits (η) introduce inertia. The term $\psi \ln(m_t)$ captures money-in-the-utility, which matters for liquidity services. Households dislike labour h_t, with the parameter χ controlling the weight and γ the curvature of disutility.

They face a real flow constraint:

$$
c_t + i_t + b_t + m_t \le r_{t-1}^B b_{t-1} + \frac{m_{t-1}}{\pi_t} + w_t h_t + r_t^k K_{t-1} + \Pi_t^r - \tau_t \tag{1.2}
$$

Income sources include wage income $w_t h_t$, rental income from capital $r_t^k K_{t-1}$, bond returns, money holdings, and profit transfers Π_t^r. Taxes τ_t are lump-sum.

Capital evolves dynamically, with adjustment frictions:

$$
K_t \;=\; (1-\delta)\, K_{t-1} \; + \; i_t \; - \; \frac{\phi_k}{2} \left(\frac{i_t}{K_{t-1}} - \delta \right)^2 K_{t-1} \tag{1.3}
$$

The cost term penalizes investment rates that deviate from steady-state depreciation δ. The parameter ϕ_k governs the severity of this penalty. This friction dampens and spreads the effect of monetary policy over time, by making investment less elastic in the short run.

Firm profits, received by households as transfers, are given by:

$$
\Pi_t^r = Y_t - r_t^k K_{t-1} - w_t h_t - \frac{\phi_p}{2} (\pi_t - 1)^2 Y_t \tag{1.4}
$$

Households are passive recipients; they do not optimise over profits. The final term captures Rotemberg price adjustment costs.

Agents optimize with respect to *real* values. Their choices reflect a desire to smooth consumption over time and respond to shocks, particularly monetary ones, through portfolio reallocation (bonds, capital, money) and labour supply.

The first-order conditions are:

$$\lambda_t = (c_t - \eta c_{t-1})^{-\theta} - \beta \eta \, \mathbb{E}_t\big[(c_{t+1} - \eta c_t)^{-\theta}\big] \tag{1.5}$$

The marginal utility of consumption depends on current and past consumption, due to habit formation.

$$\lambda_t w_t = \chi h_t^{\gamma} \tag{1.6}$$

Agents choose labour by equating marginal disutility to the marginal benefit from wages.

$$\lambda_t = \beta \, \mathbb{E}_t\left[\lambda_{t+1} r_t^B\right] \tag{1.7}$$

Intertemporal arbitrage condition: today's marginal utility equals expected discounted future returns.

$$\frac{\psi}{m_t} = \lambda_t - \beta \, \mathbb{E}_t\left[\frac{\lambda_{t+1}}{\pi_{t+1}}\right] \tag{1.8}$$

Marginal utility of real balances trades off against expected inflation. Demand for liquidity falls with higher expected inflation.

$$\lambda_t = \mu_t \left(1 - \phi_k \left(\frac{i_t}{K_{t-1}} - \delta\right)\right) \tag{1.9}$$

Higher investment relative to capital raises adjustment costs, lowering the effective return.

$$\mu_t = \beta \, \mathbb{E}_t\left[\lambda_{t+1} r_{t+1}^k + \mu_{t+1}\left((1-\delta) - \frac{\phi_k}{2}\left(\delta^2 - \left(\frac{i_{t+1}}{K_t}\right)^2\right)\right)\right] \tag{1.10}$$

This defines the shadow value of capital today. It depends on the rental return next period and the adjustment cost penalty going forward.

1.2. Production

Production follows a two-tier structure: a perfectly competitive final-goods firm aggregate output from monopolistically competitive intermediate-goods producers. This allows for price-setting frictions akin to Rotemberg, following the formulation used in a working paper by the European Central Bank (2022).

1.2.1. Final Goods Producers

A final goods firm combines intermediate goods $Y_t(j)$ via a CES aggregator:

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}, \quad \epsilon > 1 \tag{1.11}$$

ϵ is the elasticity of substitution between varieties. A higher ϵ implies closer substitutes and tighter competition among intermediate firms.

Cost minimization yields the usual demand curve and aggregate price index:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t \tag{1.12}$$

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}} \tag{1.13}$$

Equation (1.12) captures firm-level demand. Equation (1.13) defines the price index that the final firm uses when aggregating differentiated goods.

1.2.2. Intermediate Goods Producers

Intermediate firms face monopolistic competition and price adjustment frictions. They produce using Cobb–Douglas technology:

$$Y_t(j) = A_t K_t(j)^\alpha h_t(j)^{1-\alpha} \tag{1.14}$$

Cost minimization yields factor demands and marginal cost:

$$r_t^k = mc_t \alpha A_t K_t(j)^{\alpha-1} h_t(j)^{1-\alpha} \tag{1.15}$$

$$w_t = mc_t(1-\alpha)A_t K_t(j)^{\alpha} h_t(j)^{-\alpha} \tag{1.16}$$

$$\mathrm{mc}_t = \frac{1}{A_t}\left(\frac{r_t^k}{\alpha}\right)^{\alpha}\left(\frac{w_t}{1-\alpha}\right)^{1-\alpha} \tag{1.17}$$

Marginal cost mc_t rises with factor prices and falls with productivity.

Firms face Rotemberg-style price frictions. The pricing problem is:

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \Lambda_{t,t+s}\left[\frac{P_{t+s}(j)}{P_{t+s}}Y_{t+s}(j) - mc_{t+s} \cdot Y_{t+s}(j) - \frac{\phi_p}{2}\left(\frac{P_{t+s}(j)}{P_{t+s-1}(j)} - 1\right)^2 Y_{t+s}\right] \tag{1.18}$$

Price setting is intertemporal: firms trade off markup gains against adjustment costs. The discount factor $\Lambda_{t,t+s} = \beta^s \frac{\lambda_{t+s}}{\lambda_t}$ reflects household valuation of future profits. The parameter ϕ_p governs the degree of price stickiness-higher ϕ_p means greater reluctance to adjust prices.

In symmetric equilibrium ($P_t(j) = P_t$), this reduces to the New Keynesian Phillips Curve:

$$(\pi_t - 1)\pi_t = \frac{\epsilon}{\phi_p}\left(mc_t - \frac{\epsilon-1}{\epsilon}\right) + \beta\mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}\frac{Y_{t+1}}{Y_t}(\pi_{t+1} - 1)\pi_{t+1}\right] \tag{1.19}$$

Equation (1.19) ties inflation to real marginal costs and future inflation expectations. A higher ϕ_p dampens inflation's response to shocks, increasing persistence-crucial for modelling monetary transmission.

1.3. Government Sector & Monetary Authority

The government sector follows the setup in Sims (2024c) with the addition of an inflation tax The government finances an exogenous stream of real spending g_t using lump-sum taxes τ_t, nominal debt issuance, and seigniorage. Its intertemporal budget constraint is:

$$g_t + r_{t-1}^B d_{t-1} = d_t + \tau_t + \frac{m_{t-1} - m_t}{\pi_t} \tag{1.20}$$

Here, d_t denotes real bond holdings by households (government debt), and r_t^B is the ex-post real gross return on bonds purchased in $t - 1$. Money balances are defined in real terms, so seigniorage revenue enters as $\frac{m_{t-1} - m_t}{\pi_t}$. Taxes are lump-sum, and the model does not account for Ricardian equivalence-government debt is assumed to be non-neutral in household decisions.

This simple fiscal structure ensures closure of the asset side of the model, anchoring bond demand from households to government issuance.

1.3.1. Monetary Authority

The central bank sets the nominal interest rate R_t according to a standard Taylor-type rule with interest smoothing:

$$R_t = \rho_R R_{t-1} + (1 - \rho_R) \left[R_* + \kappa_\pi (\pi_t - \pi_*) + \kappa_y \left(\frac{Y_t - Y_*}{Y_*} \right) \right] + \varepsilon_t^R \tag{1.21}$$

operating under discretion, re-optimising each period rather than committing to a pre-announced path. Here, ρ_R governs inertia, κ_π and κ_y the responses to inflation and output gaps, and ε_t^R is a monetary policy shock.

To link nominal and real interest rates, I include the Fisher equation:

$$R_t = r_t^B \cdot \mathbb{E}_t[\pi_{t+1}] \tag{1.22}$$

This expression defines the ex-ante real interest rate r_t^B as the nominal rate deflated by expected inflation. For now, the inflation risk premium is assumed to be zero. This identity helps bridge monetary policy to intertemporal consumption and investment decisions and will be central to the discussion in Section 4.

1.4. Exogenous Processes

To close the model, taxes, government spending, and technology are governed by exogenous AR(1) processes, as summarised in the following Table.

Table 1.1: Exogenous Processes Driving Taxes, Spending, and Technology

Variable	Description	Process
τ_t	Real lump-sum taxes	AR(1): $(1 - \rho_\tau)\bar{\tau} + \rho_\tau \tau_{t-1} + \varepsilon_t^\tau$
g_t	Real government spending	AR(1): $(1 - \rho_g)\bar{g} + \rho_g g_{t-1} + \varepsilon_t^g$
$\ln A_t$	Log total factor productivity (TFP)	AR(1): $\rho_a \ln A_{t-1} + \varepsilon_t^a$

Each shock ε_t^i (for $i \in \{\tau, g, a\}$) is mean-zero, i.i.d., and uncorrelated with others. Parameters ρ_i determine persistence, while steady-state values $\bar{\tau}$ and $\bar{g}$ anchor fiscal dynamics. Productivity is normalized to one in steady state.

1.5. Market Clearing and Equilibrium

In equilibrium, a competitive allocation consists of sequences of real allocations $\{c_t, h_t, m_t, b_t, K_t, i_t, Y_t\}_{t \geq 0}$ and prices $\{w_t, r_t^k, R_t, P_t\}_{t \geq 0}$ such that, given the exogenous processes $\{A_t, g_t, \tau_t\}$ and initial stocks (K_{-1}, b_{-1}), (i) households and firms optimise-satisfying the first-order conditions (1.5)–(1.10), (1.15)–(1.19), and the government's budget constraint (1.20); (ii) bond-market clearing holds every period, with households absorbing all government debt ($b_t = d_t$ for all t, and $b_{-1} = d_{-1}$); (iii) the goods market clears, so that output is exactly absorbed by consumption, investment, government spending, and the wedges from adjustment costs-

$$Y_t = c_t + i_t + g_t + \frac{\phi_k}{2}\left(\frac{i_t}{K_{t-1}} - \delta\right)^2 K_{t-1} + \frac{\phi_p}{2}(\pi_t - 1)^2 Y_t \tag{1.23}$$

and (iv) labour and capital markets clear under symmetry: each intermediate-firm's demand equals the household's aggregate supply ($h_t(j) = h_t$ and $K_{t-1}(j) = K_{t-1}$ for all j), yielding the aggregate Cobb–Douglas production function

$$Y_t = A_t K_{t-1}^\alpha h_t^{1-\alpha} \tag{1.24}$$

These equilibrium conditions, together with the Taylor-rule (1.21) and Fisher equation (1.22), fully characterise the law of motion for all endogenous variables in response to monetary, fiscal, and technology shocks.

2. Full set of Conditions

Below is a compact summary of every equilibrium condition in my NK DSGE model; Section 2.1 then presents their log-linearised counterparts for easier interpretation.

$$c_t + i_t + b_t + m_t \leq r_{t-1}^B b_{t-1} + \frac{m_{t-1}}{\pi_t} + w_t h_t + r_t^k K_{t-1} + \Pi_t^r - \tau_t \tag{1.2}$$

$$K_t = (1-\delta)K_{t-1} + i_t - \frac{\phi_k}{2}\left(\frac{i_t}{K_{t-1}} - \delta\right)^2 K_{t-1} \tag{1.3}$$

$$\Pi_t^r = Y_t - r_t^k K_{t-1} - w_t h_t - \frac{\phi_p}{2}(\pi_t - 1)^2 Y_t \tag{1.4}$$

$$\lambda_t = (c_t - \eta c_{t-1})^{-\theta} - \beta, \eta, \mathbb{E}t\left[(ct + 1 - \eta c_t)^{-\theta}\right] \tag{1.5}$$

$$h_t^\gamma = \frac{\lambda_t w_t}{\chi} \tag{1.6}$$

$$1 = \beta, \mathbb{E}t\left[\frac{\lambda t + 1}{\lambda_t} r_t^B\right] \tag{1.7}$$

$$\lambda_t = \frac{\psi}{m_t} + \beta, \mathbb{E}t\left[\frac{\lambda t + 1}{\pi_{t+1}}\right] \tag{1.8}$$

$$\mu_t = \lambda_t\left(1 - \phi_k\left(\frac{i_t}{K_{t-1}} - \delta\right)\right)^{-1} \tag{1.9}$$

$$1 = \beta, \mathbb{E}t\left[\frac{\lambda t + 1}{\mu_t} r_{t+1}^k + \frac{\mu_{t+1}}{\mu_t}\left((1-\delta) - \frac{\phi_k}{2}\left(\delta^2 - \left(\frac{i_{t+1}}{K_t}\right)^2\right)\right)\right] \tag{1.10}$$

$$Y_t = A_t K_{t-1}^\alpha h_t^{1-\alpha} \tag{1.24}$$

$$(\pi_t - 1)\pi_t = \frac{\epsilon}{\phi_p}\left(mc_t - \frac{\epsilon - 1}{\epsilon}\right) + \beta\mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}\frac{Y_{t+1}}{Y_t}(\pi_{t+1} - 1)\pi_{t+1}\right] \tag{1.19}$$

$$mc_t = \frac{1}{A_t}\left(\frac{r_t^k}{\alpha}\right)^\alpha \left(\frac{w_t}{1-\alpha}\right)^{1-\alpha} \tag{1.17}$$

$$g_t + r_{t-1}^B d_{t-1} \le d_t + \tau_t + \frac{m_{t-1} - m_t}{\pi_t} \tag{1.20}$$

$$Y_t = c_t + i_t + g_t + \frac{\phi_k}{2}\left(\frac{i_t}{K_{t-1}} - \delta\right)^2 K_{t-1} + \frac{\phi_p}{2}(\pi_t - 1)^2 Y_t \tag{1.23}$$

$$b_t = d_t \tag{2.1}$$

$$R_t = r_t^B \cdot \mathbb{E}_t[\pi_{t+1}] \tag{1.22}$$

$$R_t = \rho_R R_{t-1} + (1 - \rho_R)\left[R_* + \kappa_\pi(\pi_t - \pi_*) + \kappa_y\left(\frac{Y_t - Y_*}{Y_*}\right)\right] + \varepsilon_t^R \tag{1.21}$$

$$\tau_t = (1 - \rho_\tau)\bar\tau + \rho_\tau \tau_{t-1} + \varepsilon_t^\tau \tag{2.2}$$

$$g_t = (1 - \rho_g)\bar g + \rho_g g_{t-1} + \varepsilon_t^g \tag{2.3}$$

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_t^a \tag{2.4}$$

2.1. Log-Linearised

$$\begin{aligned}
\bar C \hat c_t + \bar I \hat i_t + \bar B \hat b_t + \bar M \hat m_t =& \\
& \bar r^B \bar B (\hat r_{t-1}^B + \hat b_{t-1}) + \bar M(\hat m_{t-1} - \hat\pi_t) \\
& + \bar w \bar H(\hat w_t + \hat h_t) + \bar r^k \bar K(\hat r_t^k + \hat K_{t-1}) \\
& + \overline\Pi^r \widehat\Pi_t^r - \bar T \hat\tau_t
\end{aligned} \tag{2.5}$$

$$\hat K_t = (1-\delta)\hat K_{t-1} + \delta\hat i_t \tag{2.6}$$

$$\widehat\Pi_t^r = \frac{\bar Y}{\overline\Pi^r}\hat Y_t - \frac{\bar r^k \bar K}{\overline\Pi^r}(\hat r_t^k + \hat K_{t-1}) - \frac{\bar w \bar H}{\overline\Pi^r}(\hat w_t + \hat h_t) \tag{2.7}$$

$$\hat{\lambda}_t = -\frac{\theta}{(1-\eta)} \left(\hat{c}_t - \eta \hat{c}_{t-1} - \beta \eta \mathbb{E}_t \left[\hat{c}_{t+1} - \eta \hat{c}_t \right] \right) \tag{2.8}$$

$$\hat{\lambda}_t + \hat{w}_t = \gamma \hat{h}_t \tag{2.9}$$

$$\hat{r}_t^B = \hat{\lambda}_t - \mathbb{E}_t \hat{\lambda}_{t+1} \tag{2.10}$$

$$\hat{\lambda}_t = -(1-\beta)\hat{m}_t + \beta \mathbb{E}_t \left[\hat{\lambda}_{t+1} - \hat{\pi}_{t+1} \right] \tag{2.11}$$

$$\hat{\mu}_t = \hat{\lambda}_t + \phi_k \delta \left(\hat{i}_t - \hat{K}_{t-1} \right) \tag{2.12}$$

$$\mathbb{E}_t \left[\bar{r}^k (\hat{\lambda}_{t+1} + \hat{r}_{t+1}^k) + (1-\delta)\hat{\mu}_{t+1} + \phi_k \delta^2 (\hat{i}_{t+1} - \hat{K}_t) \right] = (\bar{r}^k + 1 - \delta)\hat{\mu}_t \tag{2.13}$$

$$\hat{Y}_t = \hat{A}_t + \alpha \hat{K}_{t-1} + (1-\alpha)\hat{h}_t \tag{2.14}$$

$$\widehat{mc}_t = -\hat{A}_t + \alpha \hat{r}_t^k + (1-\alpha)\hat{w}_t \tag{2.15}$$

$$\hat{\pi}_t = \frac{\epsilon - 1}{\phi_p} \widehat{mc}_t + \beta \mathbb{E}_t \hat{\pi}_{t+1} \tag{2.16}$$

$$\bar{G}\hat{g}_t + \bar{r}^B \bar{D}(\hat{r}_{t-1}^B + \hat{d}_{t-1}) = \bar{D}\hat{d}_t + \bar{T}\hat{\tau}_t + \bar{M}\hat{m}_{t-1} - \bar{M}\hat{m}_t \tag{2.17}$$

$$\hat{Y}_t = \frac{\bar{C}}{\bar{Y}}\hat{c}_t + \frac{\bar{I}}{\bar{Y}}\hat{i}_t + \frac{\bar{G}}{\bar{Y}}\hat{g}_t \tag{2.18}$$

$$\hat{b}_t = \hat{d}_t \tag{2.19}$$

$$\hat{R}_t = \hat{r}_t^B + \mathbb{E}_t \hat{\pi}_{t+1} \tag{2.20}$$

$$\hat{R}_t = \rho_R \hat{R}_{t-1} + \frac{(1 - \rho_R)}{\bar{R}} \left(\kappa_\pi \hat{\pi}_t + \kappa_y \bar{Y} \hat{Y}_t \right) + \frac{\varepsilon_t^R}{\bar{R}} \tag{2.21}$$

$$\hat{\tau}_t = \rho_\tau \hat{\tau}_{t-1} + \varepsilon_t^\tau \tag{2.22}$$

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \varepsilon_t^g \tag{2.23}$$

$$\hat{A}_t = \rho_a \hat{A}_{t-1} + \varepsilon_t^a \tag{2.24}$$

3. Steady State

In steady state we assume all frictions vanish and stochastic shocks are at their mean, so firms optimally reset prices each period. From the flexible-price first-order condition it follows that real marginal cost is constant at the inverse-markup

$$mc^* = \frac{\epsilon - 1}{\epsilon} \tag{3.1}$$

which pins down the natural equilibrium around which monetary shocks will later operate.

Given this constant markup and the condition for the steady real return on capital, $r^{k,*} = 1/\beta - (1 - \delta)$, the capital–labour ratio has a closed-form solution

$$k = \left(\frac{mc^* \alpha A}{r^{k,*}} \right)^{1/(1-\alpha)}, \tag{3.2}$$

and thus the corresponding real wage is

$$w^* = mc^*(1 - \alpha)Ak^\alpha. \tag{3.3}$$

Hours worked h^* then satisfy the modified labour-supply condition with habit formation:

$$(Ak^\alpha - \delta k)h^* - \bar{g} = \frac{1}{1 - \eta} \left(\frac{(1 - \beta\eta)w^*}{\chi} \right)^{1/\theta} (h^*)^{-\gamma/\theta} \tag{3.4}$$

Total capital, output, consumption, and investment follow immediately via $K^* = kh^*$, $Y^* = Ak^\alpha h^*$, $c^* = Y^* - \delta K^* - \bar{g}$, and $i^* = \delta K^*$,.

Finally, with inflation at its target $\pi^* = \pi_*$ and the mean monetary-policy shock zero, the Taylor rule collapses to

$$R^* = \pi_*/\beta \tag{3.5}$$

Here, the nominal interest rate just reflects the steady-state real rate. This shows that when prices are fully flexible and shocks are absent, monetary policy is neutral in the long run; it has no effect on the real allocation of resources. In the next Section I re-introduce price- and investment-adjustment costs and trace how an unexpected monetary shock $\varepsilon^R \neq 0$ drives dynamic deviations around these natural levels.

4. Monetary-Policy Transmission

Monetary policy transmits through the economy via intertemporal substitution and nominal rigidities. A monetary shock ε_t^R directly raises the nominal interest rate $\hat{R}_t$. Through the Fisher equation, this elevates the ex-ante real interest rate $\hat{r}_t^B$, as inflation expectations adjust sluggishly:

$$\hat{R}_t = \hat{r}_t^B + \mathbb{E}_t\hat{\pi}_{t+1} \tag{2.20}$$

Higher real rates incentivize saving over current consumption. The Euler condition captures this intertemporal substitution: $\hat{r}_t^B$ widens the gap between current and expected marginal utility of consumption:

$$\hat{r}_t^B = \hat{\lambda}_t - \mathbb{E}_t\hat{\lambda}_{t+1} \tag{2.10}$$

Reduced demand lowers output $\hat{Y}_t$, shown in the resource constraint:

$$\hat{Y}_t = \frac{\bar{C}}{\bar{Y}}\hat{c}_t + \frac{\bar{I}}{\bar{Y}}\hat{i}_t + \frac{\bar{G}}{\bar{Y}}\hat{g}_t \tag{2.18}$$

compressing marginal costs $\widehat{mc}_t$.

Nominal rigidities amplify and persist these effects. The New Keynesian Phillips Curve links marginal costs to inflation:

$$\hat{\pi}_t = \frac{\epsilon - 1}{\phi_p}\widehat{mc}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1} \tag{2.16}$$

Lower $\widehat{mc}_t$ reduces inflation, but expectations $\mathbb{E}_t\hat{\pi}_{t+1}$ anchor the response-greater price stickiness (higher ϕ_p) dampens and prolongs adjustment.

The initial monetary impulse originates from the Taylor rule:

$$\hat{R}_t = \rho_R\hat{R}_{t-1} + \frac{(1 - \rho_R)}{\bar{R}}\left(\kappa_\pi\hat{\pi}_t + \kappa_y\bar{Y}\hat{Y}_t\right) + \frac{\varepsilon_t^R}{\bar{R}} \tag{2.21}$$

where ρ_R governs interest-rate smoothing. Persistence arises from habits (η), investment frictions (ϕ_k), and nominal rigidities (ϕ_p).

5. Parameters and Results

The core parameterisation used in the model is summarised in Table 5.1, which compares calibrated values against established benchmarks from the United States (Sims (2024d)) and South Africa (Kemp & Hollander (2020)). Key divergences include South Africa's notably higher labour supply elasticity, with $\gamma = 5.0$ compared to 2.0 in the US, and a lower capital share, with α equal to 0.30 versus 0.33. These differences reflect deeper structural features of the South African economy. Additionally, the total factor productivity shock persistence is lower in South Africa, with ρ_a equal to compared to 0.98 in the US, implying faster mean reversion and, consequently, a need for more responsive monetary policy interventions.

Table 5.1: Comparison of Core DSGE Parameters: US vs South Africa

Parameter	Description	Sims (2024d) - US	Kemp & Hollander (2020) - SA
α	Capital's share in production	0.33	0.30
β	Discount factor	0.99	0.994
δ	Capital depreciation rate	0.02	0.015
θ	CRRA utility curvature	1.0 (log utility)	1.0 (assumed)
γ	Inverse Frisch elasticity of labour supply	2.0	5.0

Parameter	Description	Sims ([2024d](#)) - US	Kemp & Hollander ([2020](#)) - SA
ϵ	Elasticity of substitution across goods	≈ 6	≈ 6 (implied)
ρ_a	Persistence of TFP shocks	0.979	≈ 0.95

Turning to model performance, Table 5.2 highlights several significant discrepancies between the simulated moments and empirical targets. One key issue is the model's excessive smoothness. Output volatility is drastically understated, with a standard deviation of just 0.004 compared to Kemp and Hollander's empirical estimate of 1.875. Similarly, investment volatility is nearly flat (0.002) versus the observed 3.593. Nominal dynamics also appear anomalous. Inflation exhibits a weak and negative correlation with output, which contradicts the typical procyclical price behaviour observed. Furthermore, nominal interest rates display an implausible degree of procyclicality, which sharply deviates from Kemp's findings.

Table 5.2: Comparison of Simulated Moments: Model vs Empirical Estimates

		Standard Deviation			Autocorrelation			Correlation with Output		
Var	Description	SD (M)	SD (K)	SD (S)	AC (M)	AC (K)	AC (S)	CorY (M)	CorY (K)	CorY (S)
Y	Output	0.004	1.875	0.016	0.902	0.194	0.723	1.000	1.000	1.000
pi	Inflation	0.034	NA	NA	0.005	NA	NA	-0.110	NA	NA
R	Nominal interest rate	0.001	NA	NA	0.958	NA	NA	-0.939	NA	NA
rB	Real interest rate	0.000	NA	0.000	0.828	NA	0.714	-0.118	NA	0.972
c	Consumption	0.008	0.196	0.006	0.986	0.587	0.771	0.446	-0.127	0.955
i	Investment	0.002	3.593	0.053	0.996	0.713	0.715	0.109	-0.025	0.992
w	Real wage	0.147	1.802	0.010	0.979	0.186	0.739	0.076	0.128	0.992
h	Hours worked	0.073	1.613	0.007	0.979	0.312	0.714	0.074	0.882	0.984
K	Capital	0.002	NA	NA	1.000	NA	NA	0.068	NA	NA
g	Government spending	0.023	0.971	NA	0.900	0.523	NA	0.340	0.184	NA
tau	Taxes	0.023	NA	NA	0.900	NA	NA	0.000	NA	NA

Note:

M: Current Model, K: Kemp and Hollander (2020), S: Sims (2024)

Another notable issue is the labour-consumption mismatch. Although the model replicates the persistence of hours worked, its correlation with output is extremely weak (CorY: 0.074 versus the observed 0.882). This misalignment may stem from miscalibrated wage rigidities or an overly aggressive habit formation parameter η, both of which distort the transmission of shocks to household behaviour. These modelling anomalies also help explain the inverted impulse responses displayed in Figure 5.1.
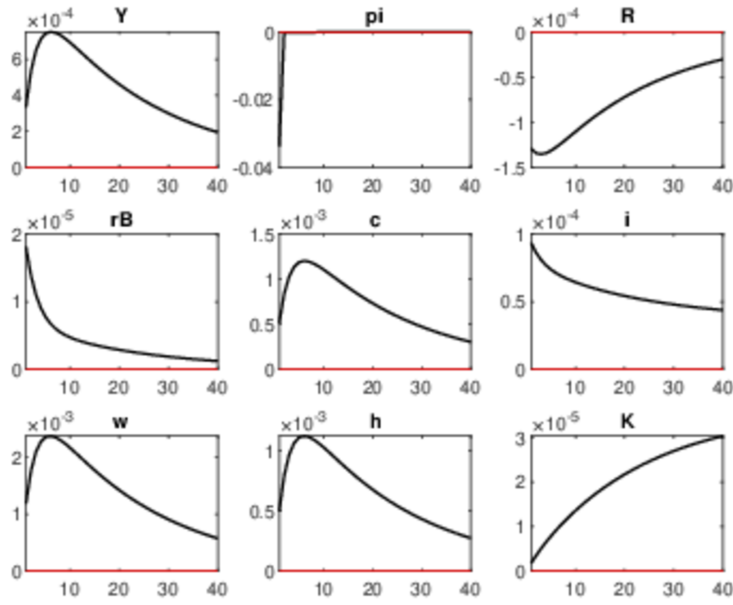
Figure 5.1: (mis-specified) impulse responses to a monetary policy shock in my New Keynesian model

One likely cause is the dominance of portfolio effects, which may arise from a large money-in-utility parameter ψ or overly restrictive capital adjustment costs, both of which can reverse standard monetary transmission mechanisms. Weak nominal rigidities, such as a near-zero ϕ_p, flatten the Phillips Curve, severing the typical link between inflation and marginal costs. Additionally, the high degree of Taylor rule inertia appears to delay the policy transmission process, reducing the immediate effectiveness of monetary interventions. These structural frictions, together with potential parameter miscalibration, may help explain why my impulse responses deviate so sharply from established benchmarks.
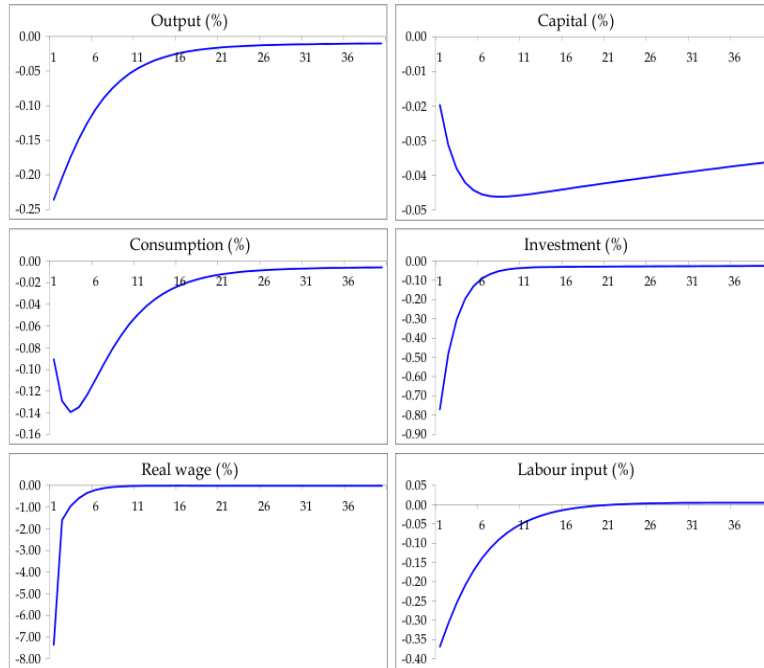
Figure 5.2: De Paoli, Scott & Weeken (2007)'s impulse responses in the sticky-price model following a monetary policy shock

I benchmark my model against De Paoli *et al.* (2007), which provides a well-structured New Keynesian framework with sticky prices, capital adjustment costs, and habit formation-features shared by my model. Their richer setup includes equities, multi-period bonds, and a second-order solution to capture risk premiums, making their results an ideal reference. In response to a standard monetary-policy tightening, their model delivers the textbook contraction: output, consumption, investment, real wages, hours, and inflation all fall before gradually recovering.

By contrast, my IRFs show the opposite: my "tightening" raises output, consumption, investment, hours, and capital, while inflation remains flat and the nominal rate falls. The only structural similarity lies in the hump-shaped capital adjustment-but with reversed sign. These inconsistencies suggest my model either misidentifies the shock or mis-specifies transmission mechanisms. While De Paoli's setup reinforces the contractionary logic through richer asset pricing, mine fails to deliver the intended dynamics - highlighting my poor MATLAB and Dynare skills.

6. Appendix

6.1. Households

Define Lagrangian

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=1}^{\infty} \beta^{t-1} \Bigg\{ \underbrace{\frac{(c_t - \eta\, c_{t-1})^{1-\theta}}{1-\theta}}_{\text{Consumption utility}} - \underbrace{\chi \frac{h_t^{1+\gamma}}{1+\gamma}}_{\text{Labour disutility}} + \underbrace{\psi \ln(m_t)}_{\text{Money utility}}$$

$$+ \underbrace{\lambda_t \Big[\frac{R_{t-1}^B b_{t-1}}{\pi_t} + \frac{m_{t-1}}{\pi_t} + w_t h_t + r_t^k K_{t-1} + \Pi_t^r - \tau_t - c_t - i_t - b_t - m_t \Big]}_{\text{Real flow constraint}} + \underbrace{\mu_t \Big[(1-\delta)\, K_{t-1} + i_t - \frac{\phi_k}{2} \left(\frac{i_t}{K_{t-1}} - \delta \right)^2 K_{t-1} - K_t}_{\text{Capital accumulation}}$$

6.1.1. First Order Conditions

FOC w.r.t. Consumption

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0$$

$$\left[(c_t - \eta c_{t-1})^{-\theta} - \lambda_t \right] - \beta\, \eta\, \mathbb{E}_t \left[(c_{t+1} - \eta c_t)^{-\theta} \right] = 0$$

Rearrange

$$\lambda_t = (c_t - \eta c_{t-1})^{-\theta} - \beta\, \eta\, \mathbb{E}_t \left[(c_{t+1} - \eta c_t)^{-\theta} \right]$$

$$\boxed{\lambda_t = (c_t - \eta c_{t-1})^{-\theta} - \beta\, \eta\, \mathbb{E}_t \left[(c_{t+1} - \eta c_t)^{-\theta} \right]} \tag{6.1}$$

FOC w.r.t. Labour

$$\frac{\partial \mathcal{L}}{\partial h_t} = 0$$

$$- \chi h_t^{\gamma} + \lambda_t w_t = 0$$

Rearrange

$$\lambda_t w_t = \chi h_t^{\gamma}$$

$$\boxed{\lambda_t w_t = \chi h_t^{\gamma}} \tag{6.2}$$

FOC w.r.t. Real Bonds:

$$\frac{\partial \mathcal{L}}{\partial b_t} = 0$$

$$- \beta^{t-1} \lambda_t + \beta^t \, \mathbb{E}_t \left[\lambda_{t+1} \cdot \frac{R_t^B}{\pi_{t+1}} \right] = 0$$

Divide by β^{t-1}

$$- \lambda_t + \beta \, \mathbb{E}_t \left[\lambda_{t+1} \frac{R_t^B}{\pi_{t+1}} \right] = 0$$

$$\boxed{\lambda_t = \beta \, \mathbb{E}_t \left[\lambda_{t+1} \frac{R_t^B}{\pi_{t+1}} \right]} \tag{6.3}$$

FOC w.r.t. Real Money Balances:

$$\frac{\partial \mathcal{L}}{\partial m_t} = 0$$

$$\beta^{t-1} \left[\frac{\psi}{m_t} - \lambda_t \right] + \beta^t \, \mathbb{E}_t \left[\lambda_{t+1} \cdot \frac{1}{\pi_{t+1}} \right] = 0$$

Divide by β^{t-1}

$$\frac{\psi}{m_t} - \lambda_t + \beta \, \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\pi_{t+1}} \right] = 0$$

$$\boxed{\frac{\psi}{m_t} = \lambda_t - \beta \, \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\pi_{t+1}} \right]} \tag{6.4}$$

FOC w.r.t. Investment:

$$\frac{\partial \mathcal{L}}{\partial i_t} = 0$$

$$\beta^{t-1} \left[-\lambda_t + \mu_t \cdot \frac{\partial}{\partial i_t} \left(i_t - \frac{\phi_k}{2} \left(\frac{i_t}{K_{t-1}} - \delta \right)^2 K_{t-1} \right) \right] = 0$$

Compute derivative:

$$\frac{\partial}{\partial i_t} \left[i_t - \frac{\phi_k}{2} \left(\frac{i_t}{K_{t-1}} - \delta \right)^2 K_{t-1} \right] = 1 - \phi_k \left(\frac{i_t}{K_{t-1}} - \delta \right)$$

Substitute and simplify:

$$- \lambda_t + \mu_t \left(1 - \phi_k \left(\frac{i_t}{K_{t-1}} - \delta \right) \right) = 0$$

$$\boxed{\lambda_t = \mu_t \left(1 - \phi_k \left(\frac{i_t}{K_{t-1}} - \delta \right) \right)} \tag{6.5}$$

FOC w.r.t. Capital:

$$\frac{\partial \mathcal{L}}{\partial K_t} = 0$$

$$\beta^{t-1}(-\mu_t) + \beta^t \, \mathbb{E}_t \left[\lambda_{t+1} r_{t+1}^k + \mu_{t+1} \cdot \frac{\partial}{\partial K_t} \left((1-\delta)K_t + i_{t+1} - \frac{\phi_k}{2} \left(\frac{i_{t+1}}{K_t} - \delta \right)^2 K_t \right) \right] = 0$$

Compute derivative:

$$\frac{\partial}{\partial K_t} \left[(1-\delta)K_t - \frac{\phi_k}{2} \left(\frac{i_{t+1}}{K_t} - \delta \right)^2 K_t \right] = (1-\delta) - \frac{\phi_k}{2} \left(\delta^2 - \left(\frac{i_{t+1}}{K_t} \right)^2 \right)$$

Substitute and simplify:

$$-\mu_t + \beta \, \mathbb{E}_t \left[\lambda_{t+1} r_{t+1}^k + \mu_{t+1} \left((1-\delta) - \frac{\phi_k}{2} \left(\delta^2 - \left(\frac{i_{t+1}}{K_t} \right)^2 \right) \right) \right] = 0$$

$$\boxed{\mu_t = \beta \, \mathbb{E}_t \left[\lambda_{t+1} r_{t+1}^k + \mu_{t+1} \left((1-\delta) - \frac{\phi_k}{2} \left(\delta^2 - \left(\frac{i_{t+1}}{K_t} \right)^2 \right) \right) \right]} \tag{6.6}$$

6.2. Production

6.2.1. Final Good Producer

Derivation of Intermediate Goods Demand and Aggregate Price Index

Final goods producer's profit:

$$\Pi_t = P_t Y_t - \int_0^1 P_t(j) Y_t(j) dj$$

subject to $Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$

Substitute production function into profit:

$$\Pi_t = P_t \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 P_t(j) Y_t(j) dj$$

First-order condition for $Y_t(j)$:

$$\frac{\partial \Pi_t}{\partial Y_t(j)} = P_t \cdot \frac{\epsilon}{\epsilon-1} \left(\int_0^1 Y_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{1}{\epsilon-1}} \cdot \frac{\epsilon-1}{\epsilon} Y_t(j)^{-\frac{1}{\epsilon}} - P_t(j) = 0$$

$$\Rightarrow P_t \cdot Y_t^{\frac{1}{\epsilon}} Y_t(j)^{-\frac{1}{\epsilon}} = P_t(j)$$

Rearrange to obtain demand curve:

$$Y_t(j) = \left(\frac{P_t}{P_t(j)} \right)^{\epsilon} Y_t$$

Substitute demand into production function:

$$Y_t = \left(\int_0^1 \left[\left(\frac{P_t}{P_t(j)} \right)^{\epsilon} Y_t \right]^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

$$= Y_t \left(\int_0^1 \left(\frac{P_t}{P_t(j)} \right)^{\epsilon-1} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

Simplify to obtain price index:

$$1 = \left(\int_0^1 \left(\frac{P_t}{P_t(j)} \right)^{\epsilon - 1} dj \right)^{\frac{\epsilon}{\epsilon - 1}}$$

$$\Rightarrow P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon} dj$$

$$\Rightarrow P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}}$$

$$\boxed{\begin{array}{c} Y_t(j) = \left(\dfrac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t \\[2em] P_t = \left(\displaystyle\int_0^1 P_t(j)^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}} \end{array}}$$

(6.7)

6.2.2. Intermediate Goods Producers

Cost minimization for intermediate firm j :

$$\min_{K_t(j),h_t(j)} \left\{ r_t^k K_t(j) + w_t h_t(j) \right\}$$

subject to $Y_t(j) = A_t K_t(j)^\alpha h_t(j)^{1-\alpha}$

Lagrangian:

$$\mathcal{L} = r_t^k K_t(j) + w_t h_t(j) + \lambda_t \left[A_t K_t(j)^\alpha h_t(j)^{1-\alpha} - Y_t(j) \right]$$

First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial K_t(j)} = 0: \quad r_t^k = \lambda_t \alpha A_t K_t(j)^{\alpha-1} h_t(j)^{1-\alpha}$$

$$\frac{\partial \mathcal{L}}{\partial h_t(j)} = 0: \quad w_t = \lambda_t (1-\alpha) A_t K_t(j)^\alpha h_t(j)^{-\alpha}$$

Rearrange FOCs:

$$\lambda_t = \frac{r_t^k}{\alpha} \left(\frac{K_t(j)}{h_t(j)} \right)^{1-\alpha} \frac{1}{A_t}, \quad \lambda_t = \frac{w_t}{1-\alpha} \left(\frac{K_t(j)}{h_t(j)} \right)^\alpha \frac{1}{A_t}$$

Equate expressions:

$$\frac{r_t^k}{\alpha} \left(\frac{K_t(j)}{h_t(j)} \right)^{-\alpha} = \frac{w_t}{1-\alpha} \left(\frac{K_t(j)}{h_t(j)} \right)^{1-\alpha}$$

$$\Rightarrow \frac{K_t(j)}{h_t(j)} = \frac{\alpha}{1-\alpha} \frac{w_t}{r_t^k}$$

Substitute into capital FOC:

$$\lambda_t = \frac{r_t^k}{\alpha A_t} \left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t^k} \right)^{\alpha-1}$$

$$= \frac{1}{A_t} \left(\frac{r_t^k}{\alpha} \right)^\alpha \left(\frac{w_t}{1-\alpha} \right)^{1-\alpha}$$

$$\boxed{ mc_t = \frac{1}{A_t} \left(\frac{r_t^k}{\alpha} \right)^\alpha \left(\frac{w_t}{1-\alpha} \right)^{1-\alpha} } \tag{6.8}$$

Intermediate-goods producer's problem:

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\left[\left(\tfrac{P_{t+s}(j)}{P_{t+s}}\right)^{1-\epsilon} Y_{t+s} - mc_{t+s}\left(\tfrac{P_{t+s}(j)}{P_{t+s}}\right)^{-\epsilon} Y_{t+s} - \tfrac{\phi_p}{2}\left(\tfrac{P_{t+s}(j)}{P_{t+s-1}(j)} - 1\right)^2 Y_{t+s}\right]$$

subject to $\quad Y_t(j) = \left(\tfrac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t$

First-Order Condition w.r.t. $P_t(j)$:

$$\mathbb{E}_t\left[\frac{\partial \Pi_t(j)}{\partial P_t(j)} + \beta\,\Lambda_{t,t+1}\frac{\partial \Pi_{t+1}(j)}{\partial P_t(j)}\right] = 0$$

$$\frac{\partial \Pi_t}{\partial P_t(j)} = (1-\epsilon)\left(\tfrac{P_t(j)}{P_t}\right)^{-\epsilon}\tfrac{Y_t}{P_t} + \epsilon\, mc_t\left(\tfrac{P_t(j)}{P_t}\right)^{-\epsilon-1}\tfrac{Y_t}{P_t} - \phi_p\left(\tfrac{P_t(j)}{P_{t-1}(j)} - 1\right)\tfrac{Y_t}{P_{t-1}(j)}$$

$$\frac{\partial \Pi_{t+1}}{\partial P_t(j)} = \phi_p\left(\tfrac{P_{t+1}(j)}{P_t(j)} - 1\right)\tfrac{P_{t+1}(j)}{P_t(j)^2}\,Y_{t+1}$$

Impose symmetry: $P_t(j) = P_t,\ Y_t(j) = Y_t,\ \pi_t = \tfrac{P_t}{P_{t-1}}$.

$$(1-\epsilon) + \epsilon\, mc_t = \epsilon\left(mc_t - \tfrac{\epsilon-1}{\epsilon}\right), \quad \tfrac{P_t(j)}{P_{t-1}(j)} = \pi_t,\ \tfrac{P_{t+1}(j)}{P_t(j)} = \pi_{t+1}$$

$$0 = \epsilon\left(mc_t - \tfrac{\epsilon-1}{\epsilon}\right) - \phi_p\left(\pi_t - 1\right)\pi_t + \beta\,\mathbb{E}_t\left[\Lambda_{t,t+1}\,\phi_p\left(\pi_{t+1} - 1\right)\pi_{t+1}\,\tfrac{Y_{t+1}\,P_t}{Y_t\,P_{t+1}}\right]$$

Noting $\Lambda_{t,t+1} = \beta\,\tfrac{\lambda_{t+1}}{\lambda_t}$ and $P_{t+1}/P_t = \pi_{t+1}$, the bracket simplifies to $\beta\,\tfrac{\lambda_{t+1}}{\lambda_t}\,\tfrac{Y_{t+1}}{Y_t}$.

$$\boxed{0 = \epsilon\left(mc_t - \tfrac{\epsilon-1}{\epsilon}\right) - \phi_p\left(\pi_t - 1\right)\pi_t + \beta\,\mathbb{E}_t\left[\tfrac{\lambda_{t+1}}{\lambda_t}\,\phi_p\left(\pi_{t+1} - 1\right)\pi_{t+1}\,\tfrac{Y_{t+1}}{Y_t}\right]} \qquad (6.9)$$

6.3. Steady State

$$\max_{P_t(j)} (P_t(j)/P_t)^{1-\epsilon} Y_t \; - \; mc_t \, (P_t(j)/P_t)^{-\epsilon} Y_t$$

$$\frac{\partial}{\partial P_t(j)} : (1-\epsilon)(P_t(j)/P_t)^{-\epsilon}\frac{Y_t}{P_t} \; + \; \epsilon \, mc_t \, (P_t(j)/P_t)^{-\epsilon-1}\frac{Y_t}{P_t} = 0$$

$$\Rightarrow (1-\epsilon) + \epsilon \, mc_t \, (P_t(j)/P_t)^{-1} = 0$$

$$\Rightarrow P_t(j)/P_t = \frac{\epsilon}{\epsilon-1} \, mc_t$$

In equilibrium $P_t(j) = P_t$, so

$$1 = \frac{\epsilon}{\epsilon-1} \, mc_t$$

$$mc^* = \frac{\epsilon-1}{\epsilon} \tag{6.10}$$

Factor Prices and Technology Link

$$mc^* = \frac{\epsilon-1}{\epsilon}$$

$$r^{k,*} = \frac{1}{\beta} - (1-\delta)$$

$$mc^* = \frac{1}{A}\left(\frac{r^{k,*}}{\alpha}\right)^{\alpha}\left(\frac{w^*}{1-\alpha}\right)^{1-\alpha}$$

Substitute to get

$$\frac{\epsilon-1}{\epsilon} = \frac{1}{A}\left(\frac{r^{k,*}}{\alpha}\right)^{\alpha}\left(\frac{w^*}{1-\alpha}\right)^{1-\alpha}$$

Solve for Capital-labour Ratio

$$k = \left(\frac{mc^*\alpha A}{r^{k,*}}\right)^{\frac{1}{1-\alpha}}$$

Solve for Real Wage

$$w^* = mc^*(1-\alpha)Ak^{\alpha}$$

Labour Supply with Habit Formation

$$\lambda^* = [c^*(1-\eta)]^{-\theta}(1-\beta\eta)$$

$$[c^*(1-\eta)]^{-\theta}(1-\beta\eta)w^* = \chi(h^*)^\gamma$$

$$\Rightarrow (Ak^\alpha - \delta k)h^* - \bar{g} = \frac{1}{1-\eta}\left(\frac{(1-\beta\eta)w^*}{\chi}\right)^{1/\theta}(h^*)^{-\gamma/\theta}$$

Resource Constraints

$$K^* = kh^*$$

$$Y^* = Ak^\alpha h^*$$

$$c^* = Y^* - \delta K^* - \bar{g}$$

$$i^* = \delta K^*$$

Monetary Policy Steady State

$$R^* = \frac{\pi_*}{\beta}$$

$$\boxed{\begin{aligned}
k &= \left(\frac{mc^*\alpha A}{r^{k,*}}\right)^{1/(1-\alpha)}, \quad r^{k,*} = 1/\beta - (1-\delta), \\
w^* &= mc^*(1-\alpha)Ak^\alpha, \\
h^* : (Ak^\alpha - \delta k)&h^* - \bar{g} = \frac{1}{1-\eta}\left(\frac{(1-\beta\eta)w^*}{\chi}\right)^{1/\theta}(h^*)^{-\gamma/\theta}, \\
K^* &= kh^*, \quad Y^* = Ak^\alpha h^*, \\
c^* &= Y^* - \delta K^* - \bar{g}, \quad i^* = \delta K^*, \\
R^* &= \pi_*/\beta
\end{aligned}}$$

(6.11)

6.4. Log-Linearisation

Define real wealth: $\Omega_t \equiv r_{t-1}^B b_{t-1} + \dfrac{m_{t-1}}{\pi_t} + w_t h_t + r_t^k K_{t-1} + \Pi_t^r - \tau_t$

Take total differential around steady state

$$d(c_t) + d(i_t) + d(b_t) + d(m_t) = d\Omega_t$$

Expand differentials using product rules

$$dc_t + di_t + db_t + dm_t = b_{t-1}dr_{t-1}^B + r_{t-1}^B db_{t-1} + \frac{1}{\pi_t}dm_{t-1} + m_{t-1}d\left(\pi_t^{-1}\right)$$

$$+ h_t dw_t + w_t dh_t + K_{t-1}dr_t^k + r_t^k dK_{t-1} + d\Pi_t^r - d\tau_t$$

Apply $d(\pi_t^{-1}) = -\pi_t^{-2}d\pi_t$ and steady state $\bar{\pi} = 1$

$$dc_t + di_t + db_t + dm_t = \bar{b}dr_{t-1}^B + \bar{r}^B db_{t-1} + \bar{m}dm_{t-1} - \bar{m}d\pi_t$$

$$+ \bar{h}dw_t + \bar{w}dh_t + \bar{K}dr_t^k + \bar{r}^k dK_{t-1} + d\Pi_t^r - d\tau_t$$

Convert to log-deviations $(\hat{x}_t \approx \dfrac{x_t - \bar{x}}{\bar{x}})$

$$\bar{C}\hat{c}_t + \bar{I}\hat{i}_t + \bar{B}\hat{b}_t + \bar{M}\hat{m}_t = \bar{r}^B \bar{B}\hat{r}_{t-1}^B + \bar{r}^B \bar{B}\hat{b}_{t-1} + \bar{M}\hat{m}_{t-1} - \bar{M}\hat{\pi}_t$$

$$+ \bar{w}\bar{H}\hat{w}_t + \bar{w}\bar{H}\hat{h}_t + \bar{r}^k \bar{K}\hat{r}_t^k + \bar{r}^k \bar{K}\hat{K}_{t-1} + \overline{\Pi}^r \widehat{\Pi}_t^r - \bar{T}\hat{\tau}_t$$

$$\boxed{\begin{aligned}
\bar{C}\hat{c}_t + \bar{I}\hat{i}_t &+ \bar{B}\hat{b}_t + \bar{M}\hat{m}_t = \\
&\bar{r}^B \bar{B}(\hat{r}_{t-1}^B + \hat{b}_{t-1}) + \bar{M}(\hat{m}_{t-1} - \hat{\pi}_t) \\
&+ \bar{w}\bar{H}(\hat{w}_t + \hat{h}_t) + \bar{r}^k \bar{K}(\hat{r}_t^k + \hat{K}_{t-1}) \\
&+ \overline{\Pi}^r \widehat{\Pi}_t^r - \bar{T}\hat{\tau}_t
\end{aligned}}$$

(6.12)

Linearisation of Capital Accumulation - Eq 1.3

Define investment ratio: $z_t \equiv i_t/K_{t-1}$

Rewrite equation: $K_t = (1-\delta)K_{t-1} + i_t - \dfrac{\phi_k}{2}(z_t - \delta)^2 K_{t-1}$

Steady state: $\bar{z} = \delta \implies \bar{K} = (1-\delta)\bar{K} + \bar{I}$

Take total differential:

$$dK_t = (1-\delta)dK_{t-1} + di_t - \phi_k(z_t - \delta)K_{t-1}dz_t - \dfrac{\phi_k}{2}(z_t - \delta)^2 dK_{t-1}$$

Evaluate at $\bar{z} = \delta$:

$$dK_t = (1-\delta)dK_{t-1} + di_t$$

Convert to log-deviations:

$$\bar{K}\hat{K}_t = (1-\delta)\bar{K}\hat{K}_{t-1} + \bar{I}\hat{i}_t$$

Divide by $\bar{K}$ and use $\delta \equiv \bar{I}/\bar{K}$:

$$\hat{K}_t = (1-\delta)\hat{K}_{t-1} + \delta\hat{i}_t$$

$$\boxed{\hat{K}_t = (1-\delta)\hat{K}_{t-1} + \delta\hat{i}_t} \tag{6.13}$$

Linearisation of Firm Profits - Eq 1.4

Define price adjustment cost: $\Phi_t \equiv \dfrac{\phi_p}{2}(\pi_t - 1)^2 Y_t$

Steady state: $\bar{\pi} = 1 \implies \bar{\Phi} = 0$

Take total differential:

$$d\Pi_t^r = dY_t - d(r_t^k K_{t-1}) - d(w_t h_t) - d\Phi_t$$

Expand factor payments:

$$= dY_t - (K_{t-1}dr_t^k + r_t^k dK_{t-1}) - (h_t dw_t + w_t dh_t) - d\Phi_t$$

Differentiate adjustment cost:

$$d\Phi_t = \phi_p(\pi_t - 1)d\pi_t Y_t + \dfrac{\phi_p}{2}(\pi_t - 1)^2 dY_t$$

Evaluate at $\bar{\pi} = 1$:

$$d\Phi_t = 0$$

Convert to log-deviations:

$$\overline{\Pi}^r \widehat{\Pi}_t^r = \bar{Y}\hat{Y}_t - \bar{r}^k\bar{K}\hat{r}_t^k - \bar{r}^k\bar{K}\hat{K}_{t-1} - \bar{w}\bar{H}\hat{w}_t - \bar{w}\bar{H}\hat{h}_t$$

$$\boxed{\widehat{\Pi}_t^r = \frac{\bar{Y}}{\overline{\Pi}^r}\hat{Y}_t - \frac{\bar{r}^k\bar{K}}{\overline{\Pi}^r}(\hat{r}_t^k + \hat{K}_{t-1}) - \frac{\bar{w}\bar{H}}{\overline{\Pi}^r}(\hat{w}_t + \hat{h}_t)}$$

(6.14)

Linearisation of FOC Consumption - Eq 1.5

Define habit-adjusted consumption:

$$s_t \equiv c_t - \eta c_{t-1}, \quad s_{t+1} \equiv c_{t+1} - \eta c_t$$

Original equation: $\lambda_t = s_t^{-\theta} - \beta\eta\mathbb{E}_t[s_{t+1}^{-\theta}]$

Steady state: $\bar{\lambda} = (1 - \beta\eta)\bar{s}^{-\theta}$

Take total differential:

$$d\lambda_t = -\theta s_t^{-\theta-1}ds_t + \beta\eta\theta\mathbb{E}_t[s_{t+1}^{-\theta-1}ds_{t+1}]$$

Expand differentials:

$$ds_t = dc_t - \eta dc_{t-1}, \quad ds_{t+1} = dc_{t+1} - \eta dc_t$$

Evaluate at steady state $(s_t = s_{t+1} = \bar{s})$:

$$d\lambda_t = -\theta\bar{s}^{-\theta-1}(dc_t - \eta dc_{t-1}) + \beta\eta\theta\bar{s}^{-\theta-1}\mathbb{E}_t[dc_{t+1} - \eta dc_t]$$

Convert to log-deviations:

$$\bar{\lambda}\hat{\lambda}_t = -\theta\bar{s}^{-\theta-1}\bar{C}(\hat{c}_t - \eta\hat{c}_{t-1}) + \beta\eta\theta\bar{s}^{-\theta-1}\bar{C}\mathbb{E}_t[\hat{c}_{t+1} - \eta\hat{c}_t]$$

Simplify using $\bar{s} = \bar{C}(1 - \eta)$ and $\bar{\lambda} = (1 - \beta\eta)\bar{s}^{-\theta}$:

$$\hat{\lambda}_t = -\frac{\theta}{(1-\eta)}\left[(\hat{c}_t - \eta\hat{c}_{t-1}) - \beta\eta\mathbb{E}_t(\hat{c}_{t+1} - \eta\hat{c}_t)\right]$$

$$\boxed{\hat{\lambda}_t = -\frac{\theta}{(1-\eta)}\left(\hat{c}_t - \eta\hat{c}_{t-1} - \beta\eta\mathbb{E}_t\left[\hat{c}_{t+1} - \eta\hat{c}_t\right]\right)}$$

(6.15)

Log-linearisation of FOC w.r.t. Labour (h_t) - Eq 1.6

Original equation: $\lambda_t w_t = \chi h_t^\gamma$

Take natural logarithm of both sides

$\ln(\lambda_t w_t) = \ln(\chi h_t^\gamma)$

Apply logarithmic identities

$\ln \lambda_t + \ln w_t = \ln \chi + \gamma \ln h_t$

Define log-deviations from steady state $(\hat{x}_t \equiv \ln x_t - \ln \bar{x})$

$(\ln \lambda_t - \ln \bar{\lambda}) + (\ln w_t - \ln \bar{w}) = \gamma(\ln h_t - \ln \bar{h})$

Recognize steady-state relationship: $\bar{\lambda}\bar{w} = \chi \bar{h}^\gamma$

$\Rightarrow \ln \bar{\lambda} + \ln \bar{w} = \ln \chi + \gamma \ln \bar{h}$

Subtract steady-state equation

$\hat{\lambda}_t + \hat{w}_t = \gamma \hat{h}_t$

$$\boxed{\hat{\lambda}_t + \hat{w}_t = \gamma \hat{h}_t} \tag{6.16}$$

Log-linearisation of FOC for Bonds (foc_B)

Original equation: $1 = \beta \mathbb{E}_t \left[\dfrac{\lambda_{t+1}}{\lambda_t} r_t^B \right]$

Steady state: $1 = \beta \dfrac{\bar{\lambda}}{\bar{\lambda}} \bar{r}^B \implies \bar{r}^B = 1/\beta$

Take natural logarithm

$0 = \ln \beta + \ln \mathbb{E}_t \left[\dfrac{\lambda_{t+1}}{\lambda_t} r_t^B \right]$

Apply certainty equivalence $(\ln \mathbb{E}_t[X] \approx \mathbb{E}_t[\ln X])$

$0 = \ln \beta + \mathbb{E}_t \left[\ln \lambda_{t+1} - \ln \lambda_t + \ln r_t^B \right]$

Subtract steady state $(0 = \ln \beta + \ln \bar{\lambda} - \ln \bar{\lambda} + \ln \bar{r}^B)$

$0 = \mathbb{E}_t \left[(\ln \lambda_{t+1} - \ln \bar{\lambda}) - (\ln \lambda_t - \ln \bar{\lambda}) + (\ln r_t^B - \ln \bar{r}^B) \right]$

Convert to log-deviations

$0 = \mathbb{E}_t \left[\hat{\lambda}_{t+1} - \hat{\lambda}_t + \hat{r}_t^B \right]$

$$\boxed{\hat{r}_t^B = \hat{\lambda}_t - \mathbb{E}_t \hat{\lambda}_{t+1}} \tag{6.17}$$

Log-linearisation of FOC for Money (foc_M)

Original: $\lambda_t = \dfrac{\psi}{m_t} + \beta \mathbb{E}_t \left[\dfrac{\lambda_{t+1}}{\pi_{t+1}} \right]$

Steady state: $\bar{\lambda} = \dfrac{\psi}{\bar{m}} + \beta \dfrac{\bar{\lambda}}{\bar{\pi}}$ (with $\bar{\pi} = 1$)

Take total differential

$$d\lambda_t = -\dfrac{\psi}{m_t^2} dm_t + \beta \mathbb{E}_t \left[\dfrac{1}{\pi_{t+1}} d\lambda_{t+1} - \dfrac{\lambda_{t+1}}{\pi_{t+1}^2} d\pi_{t+1} \right]$$

Evaluate at steady state ($\bar{\pi} = 1$)

$$d\lambda_t = -\dfrac{\psi}{\bar{m}^2} dm_t + \beta \mathbb{E}_t \left[d\lambda_{t+1} - \bar{\lambda} d\pi_{t+1} \right]$$

Convert to log-deviations

$$\bar{\lambda} \hat{\lambda}_t = -\dfrac{\psi}{\bar{m}} \hat{m}_t + \beta \mathbb{E}_t \left[\bar{\lambda} \hat{\lambda}_{t+1} - \bar{\lambda} \hat{\pi}_{t+1} \right]$$

Divide by $\bar{\lambda}$ and define $\zeta \equiv \dfrac{\psi}{\bar{m}\bar{\lambda}}$

$$\hat{\lambda}_t = -\zeta \hat{m}_t + \beta \mathbb{E}_t \left[\hat{\lambda}_{t+1} - \hat{\pi}_{t+1} \right]$$

From steady state: $\zeta = 1 - \beta$

$$\boxed{\hat{\lambda}_t = -(1-\beta)\hat{m}_t + \beta \mathbb{E}_t \left[\hat{\lambda}_{t+1} - \hat{\pi}_{t+1} \right]} \tag{6.18}$$

Log-linearisation of FOC for Investment (foc_I)

$$\text{Original: } \mu_t = \lambda_t \left(1 - \phi_k \left(\frac{i_t}{K_{t-1}} - \delta \right) \right)^{-1}$$

$$\text{Steady state: } \bar{\mu} = \bar{\lambda} \quad (\text{since } i_t/K_{t-1} = \delta)$$

$$\text{Rewrite as: } \frac{\mu_t}{\lambda_t} = (1 - \phi_k z_t)^{-1} \quad \left(z_t \equiv \frac{i_t}{K_{t-1}} - \delta \right)$$

Take natural logarithm

$$\ln \mu_t - \ln \lambda_t = -\ln(1 - \phi_k z_t)$$

Linearize RHS (first-order Taylor):

$$-\ln(1 - \phi_k z_t) \approx \phi_k z_t \quad (\text{since } \bar{z} = 0)$$

Take total differential

$$\frac{d\mu_t}{\mu_t} - \frac{d\lambda_t}{\lambda_t} = \phi_k dz_t$$

Convert to log-deviations

$$\hat{\mu}_t - \hat{\lambda}_t = \phi_k d\left(\frac{i_t}{K_{t-1}} \right)$$

Expand differential:

$$d\left(\frac{i_t}{K_{t-1}} \right) = \frac{1}{\bar{K}} di_t - \frac{\bar{i}}{\bar{K}^2} dK_{t-1} = \delta \left(\frac{di_t}{\bar{i}} - \frac{dK_{t-1}}{\bar{K}} \right)$$

$$= \delta(\hat{i}_t - \hat{K}_{t-1})$$

$$\boxed{\hat{\mu}_t = \hat{\lambda}_t + \phi_k \delta \left(\hat{i}_t - \hat{K}_{t-1} \right)} \tag{6.19}$$

Log-linearisation of FOC for Capital (foc_K)

Original: $1 = \beta \mathbb{E}_t \left[\dfrac{\lambda_{t+1}}{\mu_t} r_{t+1}^k + \dfrac{\mu_{t+1}}{\mu_t} \Gamma_{t+1} \right]$

where $\Gamma_{t+1} \equiv (1 - \delta) - \dfrac{\phi_k}{2} \left(\delta^2 - \left(\dfrac{i_{t+1}}{K_t} \right)^2 \right)$

Steady state: $\dfrac{i_{t+1}}{K_t} = \delta \implies \bar{\Gamma} = 1 - \delta$

Steady state equation:

$$1 = \beta \left[\dfrac{\bar{\lambda}}{\bar{\mu}} \bar{r}^k + \dfrac{\bar{\mu}}{\bar{\mu}} (1 - \delta) \right] = \beta \left[\bar{r}^k + 1 - \delta \right] \quad \text{(since } \bar{\mu} = \bar{\lambda})$$

Take total differential

$$0 = \beta \mathbb{E}_t \left[d \left(\dfrac{\lambda_{t+1} r_{t+1}^k}{\mu_t} \right) + d \left(\dfrac{\mu_{t+1} \Gamma_{t+1}}{\mu_t} \right) \right]$$

First term: $d \left(\dfrac{\lambda_{t+1} r_{t+1}^k}{\mu_t} \right) = \dfrac{1}{\mu_t} d(\lambda_{t+1} r_{t+1}^k) - \dfrac{\lambda_{t+1} r_{t+1}^k}{\mu_t^2} d\mu_t$

Second term: $d \left(\dfrac{\mu_{t+1} \Gamma_{t+1}}{\mu_t} \right) = \dfrac{1}{\mu_t} d(\mu_{t+1} \Gamma_{t+1}) - \dfrac{\mu_{t+1} \Gamma_{t+1}}{\mu_t^2} d\mu_t$

Evaluate at steady state ($\bar{\Gamma} = 1 - \delta$)

$$0 = \beta \mathbb{E}_t \left[\dfrac{1}{\bar{\mu}} (\bar{r}^k d\lambda_{t+1} + \bar{\lambda} dr_{t+1}^k) - \dfrac{\bar{\lambda} \bar{r}^k}{\bar{\mu}^2} d\mu_t \right.$$

$$\left. + \dfrac{1}{\bar{\mu}} ((1 - \delta) d\mu_{t+1} + \bar{\mu} d\Gamma_{t+1}) - \dfrac{\bar{\mu}(1 - \delta)}{\bar{\mu}^2} d\mu_t \right]$$

Simplify using $\bar{\mu} = \bar{\lambda}$:

$$0 = \beta \mathbb{E}_t \left[\dfrac{\bar{r}^k}{\bar{\lambda}} d\lambda_{t+1} + dr_{t+1}^k - \dfrac{\bar{r}^k}{\bar{\lambda}} d\mu_t + \dfrac{1 - \delta}{\bar{\lambda}} d\mu_{t+1} + d\Gamma_{t+1} - \dfrac{1 - \delta}{\bar{\lambda}} d\mu_t \right]$$

Differentiate Γ_{t+1} :

$$d\Gamma_{t+1} = \phi_k \dfrac{i_{t+1}}{K_t} d \left(\dfrac{i_{t+1}}{K_t} \right) = \phi_k \delta \cdot d \left(\dfrac{i_{t+1}}{K_t} \right)$$

$$d \left(\dfrac{i_{t+1}}{K_t} \right) = \dfrac{1}{\bar{K}} di_{t+1} - \dfrac{\bar{i}}{\bar{K}^2} dK_t = \delta(\hat{i}_{t+1} - \hat{K}_t)$$

Convert to log-deviations:

$$0 = \beta \mathbb{E}_t \left[\bar{r}^k \hat{\lambda}_{t+1} + \bar{r}^k \hat{r}_{t+1}^k - \bar{r}^k \hat{\mu}_t + (1 - \delta) \hat{\mu}_{t+1} - (1 - \delta) \hat{\mu}_t + \phi_k \delta^2 (\hat{i}_{t+1} - \hat{K}_t) \right]$$

Divide by β and use steady-state relation $1 = \beta(\bar{r}^k + 1 - \delta)$

$$\boxed{ \mathbb{E}_t \left[\bar{r}^k (\hat{\lambda}_{t+1} + \hat{r}_{t+1}^k) + (1 - \delta) \hat{\mu}_{t+1} + \phi_k \delta^2 (\hat{i}_{t+1} - \hat{K}_t) \right] = (\bar{r}^k + 1 - \delta) \hat{\mu}_t }$$ (6.20)

Log-linearisation of Aggregate Production Function

Original: $Y_t = A_t K_{t-1}^{\alpha} h_t^{1-\alpha}$

Take natural logarithm

$\ln Y_t = \ln A_t + \alpha \ln K_{t-1} + (1-\alpha) \ln h_t$

Define log-deviations $\hat{x}_t \equiv \ln x_t - \ln \bar{x}$

$(\ln Y_t - \ln \bar{Y}) = (\ln A_t - \ln \bar{A}) + \alpha(\ln K_{t-1} - \ln \bar{K}) + (1-\alpha)(\ln h_t - \ln \bar{h})$

Recognize steady state: $\bar{Y} = \bar{A}\bar{K}^{\alpha}\bar{h}^{1-\alpha}$

Substitute deviation variables

$$\hat{Y}_t = \hat{A}_t + \alpha \hat{K}_{t-1} + (1-\alpha)\hat{h}_t \qquad (6.21)$$

Log-linearisation of Marginal Cost

Original: $mc_t = \dfrac{1}{A_t} \left(\dfrac{r_t^k}{\alpha} \right)^{\alpha} \left(\dfrac{w_t}{1-\alpha} \right)^{1-\alpha}$

Take natural logarithm

$\ln mc_t = -\ln A_t + \alpha \ln r_t^k - \alpha \ln \alpha + (1-\alpha) \ln w_t - (1-\alpha) \ln(1-\alpha)$

Define log-deviations

$(\ln mc_t - \ln \overline{mc}) = -(\ln A_t - \ln \bar{A}) + \alpha(\ln r_t^k - \ln \bar{r}^k) + (1-\alpha)(\ln w_t - \ln \bar{w})$

Recognize steady state: $\overline{mc} = \dfrac{1}{\bar{A}} \left(\dfrac{\bar{r}^k}{\alpha} \right)^{\alpha} \left(\dfrac{\bar{w}}{1-\alpha} \right)^{1-\alpha}$

Substitute deviation variables

$$\widehat{mc}_t = -\hat{A}_t + \alpha \hat{r}_t^k + (1-\alpha)\hat{w}_t \qquad (6.22)$$

Log-linearisation of NK Phillips Curve

Original: $(\pi_t - 1)\pi_t = \dfrac{\epsilon}{\phi_p}\left(mc_t - \dfrac{\epsilon - 1}{\epsilon}\right) + \beta\mathbb{E}_t\left[\dfrac{\lambda_{t+1}}{\lambda_t}\dfrac{Y_{t+1}}{Y_t}(\pi_{t+1} - 1)\pi_{t+1}\right]$

Define auxiliary variable: $\Theta_t \equiv (\pi_t - 1)\pi_t$

Steady state: $\bar{\pi} = 1 \implies \bar{\Theta} = 0, \quad \overline{mc} = \dfrac{\epsilon - 1}{\epsilon}$

First-order Taylor expansion around steady state:

LHS: $\Theta_t \approx 0 + \left.\dfrac{\partial\Theta_t}{\partial\pi_t}\right|_{\bar{\pi}}(\pi_t - 1)$

$\quad \dfrac{\partial\Theta_t}{\partial\pi_t} = 2\pi_t - 1 \implies \left.\dfrac{\partial\Theta_t}{\partial\pi_t}\right|_{\bar{\pi}=1} = 1$

$\quad \Theta_t \approx \hat{\pi}_t$

First term RHS: $\dfrac{\epsilon}{\phi_p}\left(mc_t - \dfrac{\epsilon - 1}{\epsilon}\right) \approx \dfrac{\epsilon}{\phi_p}\overline{mc}\widehat{mc}_t = \dfrac{\epsilon}{\phi_p}\cdot\dfrac{\epsilon - 1}{\epsilon}\widehat{mc}_t = \dfrac{\epsilon - 1}{\phi_p}\widehat{mc}_t$

Second term RHS: $\beta\mathbb{E}_t\left[\dfrac{\lambda_{t+1}}{\lambda_t}\dfrac{Y_{t+1}}{Y_t}\Theta_{t+1}\right]$

$\quad$ Steady state value: $\beta\mathbb{E}_t\left[1\cdot 1\cdot 0\right] = 0$

$\quad$ Differential: $\beta\mathbb{E}_t\left[d\left(\dfrac{\lambda_{t+1}}{\lambda_t}\right)\cdot 1\cdot 0 + 1\cdot d\left(\dfrac{Y_{t+1}}{Y_t}\right)\cdot 0 + 1\cdot 1\cdot d\Theta_{t+1}\right]$

$\quad = \beta\mathbb{E}_t\left[d\Theta_{t+1}\right] \approx \beta\mathbb{E}_t\hat{\pi}_{t+1}$

Combine results

$$\boxed{\hat{\pi}_t = \dfrac{\epsilon - 1}{\phi_p}\widehat{mc}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1}} \tag{6.23}$$

Log-linearisation of Government Budget Constraint

Assume equality in steady state: $\bar{g} + \bar{r}^B \bar{d} = \bar{d} + \bar{\tau} + \dfrac{\bar{m} - \bar{m}}{\bar{\pi}}$

Take total differential:

$$dg_t + d(r_{t-1}^B d_{t-1}) = dd_t + d\tau_t + d\left(\frac{m_{t-1} - m_t}{\pi_t}\right)$$

Expand compound terms:

$$dg_t + d_{t-1}dr_{t-1}^B + r_{t-1}^B dd_{t-1} = dd_t + d\tau_t + \frac{1}{\pi_t}d(m_{t-1} - m_t) + (m_{t-1} - m_t)d\left(\frac{1}{\pi_t}\right)$$

Apply derivative rules:

$$d(1/\pi_t) = -\pi_t^{-2}d\pi_t$$

Evaluate at steady state $(\bar{\pi} = 1, \bar{m}_{t-1} = \bar{m}_t = \bar{M})$:

$$dg_t + \bar{d}dr_{t-1}^B + \bar{r}^B dd_{t-1} = dd_t + d\tau_t + \frac{1}{1}(dm_{t-1} - dm_t) + (0) \cdot (-\pi_t^{-2}d\pi_t)$$

Convert to log-deviations $(\hat{x}_t \equiv \dfrac{x_t - \bar{x}}{\bar{x}})$:

$$\bar{G}\hat{g}_t + \bar{d} \cdot \bar{r}^B \hat{r}_{t-1}^B + \bar{r}^B \bar{d}\hat{d}_{t-1} = \bar{D}\hat{d}_t + \bar{T}\hat{\tau}_t + \bar{M}\hat{m}_{t-1} - \bar{M}\hat{m}_t$$

Log-linearisation of Resource Constraint

Original: $Y_t = c_t + i_t + g_t + \dfrac{\phi_k}{2}\left(\dfrac{i_t}{K_{t-1}} - \delta\right)^2 K_{t-1} + \dfrac{\phi_p}{2}(\pi_t - 1)^2 Y_t$

Steady state: $\dfrac{\bar{i}}{\bar{K}} = \delta, \bar{\pi} = 1 \implies$ adjustment costs $= 0$

Take total differential:

$$dY_t = dc_t + di_t + dg_t + d\left[\frac{\phi_k}{2}\left(\frac{i_t}{K_{t-1}} - \delta\right)^2 K_{t-1}\right] + d\left[\frac{\phi_p}{2}(\pi_t - 1)^2 Y_t\right]$$

Evaluate at steady state:

(Adjustment cost terms vanish)

$$dY_t = dc_t + di_t + dg_t$$

Convert to log-deviations:

$$\bar{Y}\hat{Y}_t = \bar{C}\hat{c}_t + \bar{I}\hat{i}_t + \bar{G}\hat{g}_t$$

$$\boxed{\hat{Y}_t = \frac{\bar{C}}{\bar{Y}}\hat{c}_t + \frac{\bar{I}}{\bar{Y}}\hat{i}_t + \frac{\bar{G}}{\bar{Y}}\hat{g}_t} \tag{6.24}$$

Log-linearisation of Bond Market Clearing

$$\text{Original: } b_t = d_t$$

Take total differential:

$$db_t = dd_t$$

Convert to log-deviations:

$$\bar{B}\hat{b}_t = \bar{D}\hat{d}_t$$

Since $\bar{B} = \bar{D}$ in steady state:

$$\hat{b}_t = \hat{d}_t$$

$$\boxed{\hat{b}_t = \hat{d}_t} \tag{6.25}$$

Log-linearisation of Fisher Equation

$$\text{Original: } R_t = r_t^B \cdot \mathbb{E}_t[\pi_{t+1}]$$

Take natural logarithm:

$$\ln R_t = \ln r_t^B + \ln \mathbb{E}_t[\pi_{t+1}]$$

Apply certainty equivalence: $\ln \mathbb{E}_t[\pi_{t+1}] \approx \mathbb{E}_t[\ln \pi_{t+1}]$

$$\ln R_t = \ln r_t^B + \mathbb{E}_t[\ln \pi_{t+1}]$$

Subtract steady state $(\ln \bar{R} = \ln \bar{r}^B + \ln 1)$:

$$(\ln R_t - \ln \bar{R}) = (\ln r_t^B - \ln \bar{r}^B) + \mathbb{E}_t[(\ln \pi_{t+1} - \ln 1)]$$

Recognize log-deviations:

$$\boxed{\hat{R}_t = \hat{r}_t^B + \mathbb{E}_t\hat{\pi}_{t+1}} \tag{6.26}$$

Log-linearisation of Taylor Rule

Define output gap: $\hat{Y}_t^g \equiv \dfrac{Y_t - Y_*}{Y_*} \approx \hat{Y}_t$ (since $Y_* = \bar{Y}$ in steady state)

Steady state: $\bar{R} = R_*, \quad \bar{\pi} = \pi_*$

Express in deviations:

$$R_t - \bar{R} = \rho_R(R_{t-1} - \bar{R}) + (1 - \rho_R)\left[\kappa_\pi(\pi_t - \bar{\pi}) + \kappa_y \hat{Y}_t\right] + \varepsilon_t^R$$

Convert to log-deviations $(\hat{R}_t \equiv \ln R_t - \ln \bar{R} \approx \dfrac{R_t - \bar{R}}{\bar{R}})$:

$$\bar{R}\hat{R}_t = \rho_R \bar{R}\hat{R}_{t-1} + (1 - \rho_R)\left[\kappa_\pi \bar{\pi}\hat{\pi}_t + \kappa_y \bar{Y}\hat{Y}_t\right] + \varepsilon_t^R$$

Divide by $\bar{R}$ and use $\bar{\pi} = 1$:

$$\boxed{\hat{R}_t = \rho_R \hat{R}_{t-1} + \frac{(1 - \rho_R)}{\bar{R}}\left(\kappa_\pi \hat{\pi}_t + \kappa_y \bar{Y}\hat{Y}_t\right) + \frac{\varepsilon_t^R}{\bar{R}}} \tag{6.27}$$

Log-linearisation of Tax Rule

Define deviation: $\hat{\tau}_t \equiv \dfrac{\tau_t - \bar{\tau}}{\bar{T}}$

Original: $\tau_t - \bar{\tau} = \rho_\tau(\tau_{t-1} - \bar{\tau}) + \varepsilon_t^\tau$

Divide by steady-state taxes $\bar{T} = \bar{\tau}$:

$$\frac{\tau_t - \bar{\tau}}{\bar{T}} = \rho_\tau \frac{\tau_{t-1} - \bar{\tau}}{\bar{T}} + \frac{\varepsilon_t^\tau}{\bar{T}}$$

$$\boxed{\hat{\tau}_t = \rho_\tau \hat{\tau}_{t-1} + \varepsilon_t^\tau} \text{ where } \varepsilon_t^\tau \text{ is redefined as } \frac{\text{original shock}}{\bar{T}} \tag{6.28}$$

Log-linearisation of Government Spending Process

Original: $g_t = (1 - \rho_g)\bar{g} + \rho_g g_{t-1} + \varepsilon_t^g$

Define log-deviation: $\hat{g}_t \equiv \dfrac{g_t - \bar{g}}{\bar{G}}$

Subtract steady state:

$$g_t - \bar{g} = \rho_g(g_{t-1} - \bar{g}) + \varepsilon_t^g$$

Divide by $\bar{G}$ (where $\bar{G} = \bar{g}$) :

$$\frac{g_t - \bar{g}}{\bar{G}} = \rho_g \frac{g_{t-1} - \bar{g}}{\bar{G}} + \frac{\varepsilon_t^g}{\bar{G}}$$

$$\boxed{\hat{g}_t = \rho_g \hat{g}_{t-1} + \varepsilon_t^g} \tag{6.29}$$

Log-linearisation of Technology Process

$$\text{Original: } \ln A_t = \rho_a \ln A_{t-1} + \varepsilon_t^a$$

$$\text{Define log-deviation: } \hat{A}_t \equiv \ln A_t - \ln \bar{A}$$

$$\text{Steady state: } \ln \bar{A} = \rho_a \ln \bar{A} \implies \ln \bar{A}(1 - \rho_a) = 0$$

Subtract steady state:

$$(\ln A_t - \ln \bar{A}) = \rho_a(\ln A_{t-1} - \ln \bar{A}) + \varepsilon_t^a$$

$$\boxed{\hat{A}_t = \rho_a \hat{A}_{t-1} + \varepsilon_t^a} \tag{6.30}$$

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