Macroeconomics Assignment

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Abstract

Almost an abstract

I have used an R package from Mati (2019) and should give credit. This allowed me to code everything in R while still using Dynare.

1. Model Specification

Above could note be a heading on itself. Just a paragraph that it is a core rbc foundations and what that means.

Note that capital letters denote nominal amounts and lowercase denote real values.

Mainly adding capital to Sims (2024a), while also introducing adjustment costs from Sims (2024b) . Rotemberg prices was used while gleeming insights from European Central Bank (2022) .

1.1. Households

$$C_t \; = \; P_t \, c_t, \qquad I_t \; = \; P_t \, i_t \tag{1.1}$$

$$\max_{\{C_t, h_t, M_t, B_t, K_t, I_t\}} \mathbb{E}_0 \sum_{t=1}^{\infty} \beta^{t-1} \left[\frac{\left(\frac{C_t}{P_t} - \eta \, \frac{C_{t-1}}{P_{t-1}} \right)^{1-\theta}}{1-\theta} \; - \; \chi \, \frac{h_t^{1+\gamma}}{1+\gamma} \; + \; \psi \, \ln\!\left(\frac{M_t}{P_t} \right) \right] \tag{1.2}$$

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$$C_t \;+\; I_t \;+\; B_t \;+\; M_t \;\leq\; R^B_{t-1}\, B_{t-1} \;+\; M_{t-1} \;+\; W_t\, h_t \;+\; R^k_t\, K_{t-1} \;+\; \Pi_t \;-\; P_t\, \tau_t \tag{1.3}$$

$$K_t \;=\; (1-\delta)\, K_{t-1} \;+\; I_t \;-\; \frac{\phi}{2}\left(\frac{I_t}{K_{t-1}} - \delta\right)^2 K_{t-1} \tag{1.4}$$

From which we can derive the following equations, found in full form in Appendix 3.1.1

$$(c_t - \eta\, c_{t-1})^{-\theta} \;=\; \beta\, \mathbb{E}_t\!\Big[R^B_t\, \frac{P_t}{P_{t+1}}\, (c_{t+1} - \eta\, c_t)^{-\theta}\Big] \tag{1.5}$$

Equation 1.5 describes the household's optimal consumption decision over time. On the left-hand side, we have the marginal utility of consumption today, which takes into account *habit formation* — meaning that current satisfaction from consuming c_t is reduced if past consumption c_{t-1} was high. The right-hand side reflects the expected marginal benefit of postponing consumption to the next period: it combines the expected *real return* on bonds, $R^B_t \cdot \frac{P_t}{P_{t+1}}$, with the marginal utility of tomorrow's (habit-adjusted) consumption, $c_{t+1} - \eta\, c_t$.

Put simply, households balance the gain from consuming now against the expected value of saving and consuming later. Habit persistence, captured by η, introduces a kind of "inertia" in consumption preferences, while inflation, through the price ratio $\frac{P_t}{P_{t+1}}$, adjusts the real value of future returns.

$$\frac{W_t}{P_t} \;=\; \frac{\chi\, h_t^{\gamma}}{(c_t - \eta\, c_{t-1})^{-\theta} \;-\; \beta\, \eta\, \mathbb{E}_t\big[(c_{t+1} - \eta\, c_t)^{-\theta}\big]} \tag{1.6}$$

Equation 1.6 links the real wage $\frac{W_t}{P_t}$ to the household's labour supply decision. The right-hand side captures the trade-off between working more hours, h_t, and the (habit-adjusted) marginal utility of consumption. Stronger habit formation (η) lowers the perceived benefit of consumption, so households require a higher real wage to be willing to work the same amount — especially when the disutility of labour rises more steeply with hours (γ large).

$$M_t \;=\; \frac{\psi}{\beta\, \mathbb{E}_t\!\Big[(R^B_t - 1)\;\cdot\; \dfrac{(c_{t+1} - \eta\, c_t)^{-\theta} \;-\; \beta\, \eta\, \mathbb{E}_{t+1}\big[(c_{t+2} - \eta\, c_{t+1})^{-\theta}\big]}{P_{t+1}}\Big]} \tag{1.7}$$

Equation 1.7 characterises money demand as inversely related to the expected return on bonds - the higher the nominal rate $(R^B_t - 1)$, the greater the opportunity cost of holding money. The expression in the denominator reflects the *liquidity premium*, adjusted for how habits (η) and expected future consumption affect the marginal utility of spending. Inflation expectations, via P_{t+1}, also influence how attractive money is relative to interest-bearing assets.

$$q_t \equiv 1 - \phi\left(\frac{I_t}{K_{t-1}} - \delta\right)$$

$$\frac{\beta\,\mathbb{E}_t[\lambda_{t+1}\,R_t^B]}{q_t} = \beta\,\mathbb{E}_t\Big[\lambda_{t+1}\Big(R_{t+1}^k + \frac{1}{q_{t+1}}\Big(1 - \delta + \frac{\phi}{2}\big[(I_{t+1}/K_t)^2 - \delta^2\big]\Big)\Big)\Big] \tag{1.8}$$

Equation 1.8 defines Tobin's q_t - the value of an additional unit of capital - and describes how firms decide whether to invest. When the ratio of investment to existing capital $\frac{I_t}{K_{t-1}}$ exceeds depreciation δ, $q_t > 1$, signalling that it's profitable to expand the capital stock. The equation balances the bond-return-adjusted cost of investing (left-hand side) with the expected return on capital, including future capital gains and adjustment costs (right-hand side). The parameter ϕ governs these adjustment costs, meaning that rapid changes in investment are costly and create frictions.

1.2. Production

While the RBC model features a single representative firm under perfect competition, we now introduce two layers of firms to capture monopolistic competition and nominal rigidities:

1. **Final goods producer** (perfectly competitive aggregator)
2. **Intermediate goods producers** (monopolistically competitive with sticky prices)

1.2.1. Final Goods Producer

(Perfectly competitive, zero profits)
Role: Aggregates differentiated inputs into final output.

Key equations:

CES Production Function:

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}}\,dj\right)^{\frac{\epsilon}{\epsilon-1}}, \quad \epsilon > 1 \tag{1.9}$$

Demand for Good j:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t \tag{1.10}$$

Aggregate Price Index:

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}} \tag{1.11}$$

Equations 1.10 and 1.11 are derived from 1.9 in Appendix 3.2.1.

Economic Intuition:

- ϵ: Elasticity of substitution (lower ϵ implies more market power)
- Demand for j decreases with its relative price $\frac{P_t(j)}{P_t}$

1.2.2. Intermediate Goods Producers

(Monopolistic competitors, indexed by $j \in [0,1]$) Each firm j produces with Cobb-Douglas technology:

$$Y_t(j) = A_t K_t(j)^{\alpha} h_t(j)^{1-\alpha} \tag{1.12}$$

- Identical technology A_t (common TFP shock)
- Firms rent capital $K_t(j)$ and labor $h_t(j)$ from households
- Competitive factor markets: prices R_t^k, W_t

Minimizing costs yields nominal marginal cost (common to all firms - derived in Appendix 3.2.2):

$$\mathrm{MC}_t = \frac{1}{A_t} \left(\frac{R_t^k}{\alpha} \right)^{\alpha} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha} \tag{1.13}$$

Firms face Rotemberg (1982) price adjustment costs.

$$\mathrm{AdjCost}_t(j) = \frac{\psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t$$

The real profit function is:

$$\Pi_t(j) = \underbrace{\frac{P_t(j)}{P_t} Y_t(j)}_{\text{real revenue}} - \underbrace{\frac{MC_t}{P_t} \cdot Y_t(j)}_{\text{real cost}} - \underbrace{\frac{\psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t}_{\text{adjustment cost}} \tag{1.14}$$

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \Lambda_{t,t+s} \left[\frac{P_{t+s}(j)}{P_{t+s}} Y_{t+s}(j) - mc_{t+s} Y_{t+s}(j) - \frac{\psi}{2} \left(\frac{P_{t+s}(j)}{P_{t+s-1}(j)} - 1 \right)^2 Y_{t+s} \right]$$

where: - $\Lambda_{t,t+s} = \beta^s \frac{\lambda_{t+s}}{\lambda_t}$ = Stochastic discount factor (from households) - $mc_t = \frac{\mathrm{MC}_t}{P_t}$ = Real marginal cost (eq 3.12) - $Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t$ (demand curve from eq 1.10)

Optimal Price Setting

Each intermediate firm j maximizes discounted real profits, accounting for future adjustment costs. In symmetric equilibrium ($P_t(j) = P_t$, $\pi_t = P_t/P_{t-1}$), this yields the **Rotemberg Phillips Curve**:

$$(\pi_t - 1)\pi_t = \frac{\epsilon}{\psi} \left(mc_t - \frac{\epsilon - 1}{\epsilon} \right) + \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \frac{Y_{t+1}}{Y_t} (\pi_{t+1} - 1)\pi_{t+1} \right] \tag{1.15}$$

where $mc_t = \mathrm{MC}_t / P_t$ (real marginal cost) and λ_t is the household's marginal utility of consumption. This links inflation to:

1. Deviations of real marginal cost from its flexible-price level $\frac{\epsilon-1}{\epsilon}$
2. Expected future inflation (weighted by discounting and output growth)
3. Adjustment cost parameter ψ (higher ψ = stickier prices)

3.2.2

1.3. Government Sector

Fiscal rule: $T_t = \tau Y_t$ (lump-sum taxes)

Monetary authority ideas (not set in stone because unsure): - Taylor Rule: $R_t = \rho R_{t-1} + (1-\rho)[\phi_\pi \pi_t + \phi_y \hat{Y}_t] + \varepsilon_t^r$

-Money Growth Rule: $\ln \mu_t = \rho_\mu \ln \mu_{t-1} + \varepsilon_t^m$

1.4. Exogenous Processes

Unsure on the following before i look up if it at all breaks my model - TFP shock: $\ln A_t = (1 - \rho_A) \ln A_{ss} + \rho_A \ln A_{t-1} + \varepsilon_t^A$

- Monetary policy shocks ($\varepsilon_t^r, \varepsilon_t^m$

1.5. Equilibrium and Model Closure

- Output Gap: $\hat{Y}_t = Y_t - Y_t^n$ (natural rate output)
- Market clearing conditions
- Determinacy Requirements: Blanchard-Kahn conditions for policy rules

Market Clearing Aggregate production:

$$Y_t = A_t K_{t-1}^{\alpha} h_t^{1-\alpha} \tag{1.16}$$

Resource constraint (adjustment costs reduce output):

$$Y_t = C_t + I_t + \underbrace{\frac{\psi}{2}(\Pi_t - 1)^2 Y_t}_{\text{price adjustment cost}} \tag{1.17}$$

Factor markets clear:

$$\int_0^1 h_t(j)dj = h_t \tag{1.18}$$

$$\int_0^1 K_t(j)dj = K_{t-1} \tag{1.19}$$

2. Steady State

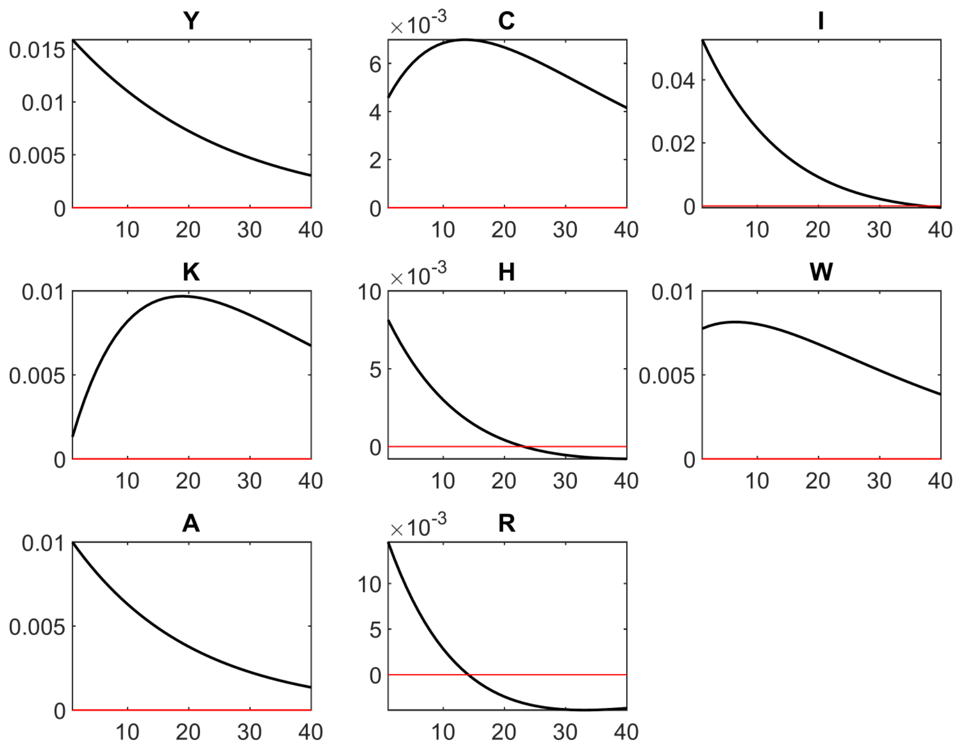

Figure 2.1: image

3. Appendix

3.1. Households

Define Lagrangian

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=1}^{\infty} \beta^{t-1} \Big\{ \underbrace{\frac{\left(\frac{C_t}{P_t} - \eta \frac{C_{t-1}}{P_{t-1}}\right)^{1-\theta}}{1-\theta}}_{\text{Consumption utility}} - \underbrace{\chi \frac{h_t^{1+\gamma}}{1+\gamma}}_{\text{Labor disutility}} + \underbrace{\psi \ln\left(\frac{M_t}{P_t}\right)}_{\text{Money utility}}$$

$$+ \underbrace{\lambda_t \left[R_{t-1}^B B_{t-1} + M_{t-1} + W_t h_t + R_t^k K_{t-1} + \Pi_t - P_t \tau_t - C_t - I_t - B_t - M_t\right]}_{\text{Nominal flow constraint}} + \underbrace{\mu_t \left[(1-\delta)K_{t-1} + I_t - \frac{\phi}{2}\left(\frac{I_t}{K_{t-1}}\right.\right.}_{\text{Capital accumulatio}}$$

3.1.1. First Order Conditions

FOC w.r.t. Consumption :

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0$$

$$\left[(c_t - \eta c_{t-1})^{-\theta}/P_t - \lambda_t\right] - \beta\, \mathbb{E}_t\left[\eta\,(c_{t+1} - \eta c_t)^{-\theta}/P_t\right] = 0$$

Combine terms over $1/P_t$

$$\frac{1}{P_t}\left[(c_t - \eta c_{t-1})^{-\theta} - \beta\eta\, \mathbb{E}_t[(c_{t+1} - \eta c_t)^{-\theta}]\right] - \lambda_t = 0$$

Multiply by P_t

$$(c_t - \eta c_{t-1})^{-\theta} - \beta\eta\, \mathbb{E}_t[(c_{t+1} - \eta c_t)^{-\theta}] - \lambda_t P_t = 0$$

$$\boxed{\lambda_t P_t = (c_t - \eta c_{t-1})^{-\theta} - \beta\eta\, \mathbb{E}_t\left[(c_{t+1} - \eta c_t)^{-\theta}\right]} \tag{3.1}$$

FOC w.r.t. Labour :

$$\frac{\partial \mathcal{L}}{\partial h_t} = 0$$

$$-\chi h_t^{\gamma} + \lambda_t W_t = 0$$

Rearrange

$$\lambda_t W_t = \chi h_t^{\gamma}$$

$$\boxed{\lambda_t W_t = \chi h_t^{\gamma}} \tag{3.2}$$

FOC w.r.t. Real Money Balances:

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial M_t} = 0 \\
&\quad \beta^{t-1}[\psi/M_t - \lambda_t] + \beta^t \mathbb{E}_t[\lambda_{t+1}] = 0 \\
&\text{Divide by } \beta^{t-1} \text{ and rearrange} \\
&\quad \psi/M_t - \lambda_t + \beta\, \mathbb{E}_t[\lambda_{t+1}] = 0
\end{aligned}$$

$$\boxed{\frac{\psi}{M_t} = \lambda_t - \beta\, \mathbb{E}_t[\lambda_{t+1}]} \tag{3.3}$$

FOC w.r.t. Bonds (3.4)):

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial B_t} = 0 \\
&\quad -\beta^{t-1}\lambda_t + \beta^t \mathbb{E}_t[\lambda_{t+1} R_t^B] = 0 \\
&\text{Divide by } \beta^{t-1} \text{ and simplify} \\
&\quad -\lambda_t + \beta\, \mathbb{E}_t[\lambda_{t+1} R_t^B] = 0
\end{aligned}$$

$$\boxed{\lambda_t = \beta\, \mathbb{E}_t[\lambda_{t+1} R_t^B]} \tag{3.4}$$

FOC w.r.t. Capital :

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial K_t} = 0 \\
&\quad -\beta^{t-1}\mu_t + \beta^t \mathbb{E}_t\big[\lambda_{t+1} R_{t+1}^k + \mu_{t+1}(1 - \delta + \tfrac{\phi}{2}((I_{t+1}/K_t)^2 - \delta^2))\big] = 0 \\
&\text{Divide by } \beta^{t-1} \text{ and solve} \\
&\quad \mu_t = \beta\, \mathbb{E}_t\big[\lambda_{t+1} R_{t+1}^k + \mu_{t+1}(1 - \delta + \tfrac{\phi}{2}((I_{t+1}/K_t)^2 - \delta^2))\big]
\end{aligned}$$

$$\boxed{\mu_t = \beta\, \mathbb{E}_t\big[\lambda_{t+1} R_{t+1}^k + \mu_{t+1}(1 - \delta + \tfrac{\phi}{2}((I_{t+1}/K_t)^2 - \delta^2))\big]} \tag{3.5}$$

FOC w.r.t. Investment :

$$\frac{\partial \mathcal{L}}{\partial I_t} = 0$$

$$\beta^{t-1}\big[-\lambda_t + \mu_t(1 - \phi(\tfrac{I_t}{K_{t-1}} - \delta))\big] = 0$$

Divide by β^{t-1} and isolate

$$\lambda_t = \mu_t\big(1 - \phi(\tfrac{I_t}{K_{t-1}} - \delta)\big)$$

$$\boxed{\lambda_t = \mu_t\big(1 - \phi(\tfrac{I_t}{K_{t-1}} - \delta)\big)} \tag{3.6}$$

3.1.2. Household Final Equations

Consumption Euler Equation

Combines consumption–habit dynamics with bond returns (from (3.4) and (3.1)) :

Start with FOC for Bonds

$$\lambda_t = \beta\,\mathbb{E}_t[\lambda_{t+1} R_t^B] \quad \text{(Equation 3.4)}$$

Substitute λ_t and λ_{t+1} from FOC for Consumption

$$\lambda_t = \frac{(c_t - \eta\, c_{t-1})^{-\theta}}{P_t} \quad \text{(from Equation 3.1 rearranged)}$$

$$\lambda_{t+1} = \frac{(c_{t+1} - \eta\, c_t)^{-\theta}}{P_{t+1}} \quad \text{(time-shifted)}$$

Combine results

$$\frac{(c_t - \eta\, c_{t-1})^{-\theta}}{P_t} = \beta\,\mathbb{E}_t\left[R_t^B \cdot \frac{(c_{t+1} - \eta\, c_t)^{-\theta}}{P_{t+1}}\right]$$

Clear denominator

$$(c_t - \eta\, c_{t-1})^{-\theta} = \beta\,\mathbb{E}_t\left[R_t^B \cdot \frac{P_t}{P_{t+1}} \cdot (c_{t+1} - \eta\, c_t)^{-\theta}\right]$$

$$\boxed{(c_t - \eta\, c_{t-1})^{-\theta} \;=\; \beta\, \mathbb{E}_t\Big[R_t^B\, \frac{P_t}{P_{t+1}}\, (c_{t+1} - \eta\, c_t)^{-\theta}\Big]} \tag{3.7}$$

Labour Supply

Real wage equals the marginal rate of substitution between leisure and consumption (from (3.1) and (3.2)) :

Start with FOC for Hours Worked

$\lambda_t W_t = \chi\, h_t^{\gamma} \quad$ (Equation 3.2)

Solve for λ_t

$$\lambda_t = \frac{\chi\, h_t^{\gamma}}{W_t}$$

Equate to FOC of Consumption expression

$$\frac{\chi\, h_t^{\gamma}}{W_t} = \frac{(c_t - \eta\, c_{t-1})^{-\theta} - \beta\, \eta\, \mathbb{E}_t[(c_{t+1} - \eta\, c_t)^{-\theta}]}{P_t}$$

Solve for real wage (W_t/P_t)

$$\frac{W_t}{P_t} = \frac{\chi\, h_t^{\gamma}}{(c_t - \eta\, c_{t-1})^{-\theta} - \beta\, \eta\, \mathbb{E}_t[(c_{t+1} - \eta\, c_t)^{-\theta}]}$$

$$\boxed{\frac{W_t}{P_t} \;=\; \frac{\chi\, h_t^{\gamma}}{(c_t - \eta\, c_{t-1})^{-\theta} \;-\; \beta\, \eta\, \mathbb{E}_t\big[(c_{t+1} - \eta\, c_t)^{-\theta}\big]}} \tag{3.8}$$

Money Demand

Opportunity cost of holding money vs. bonds (from (3.3), (3.4) and (3.1)) :

Combine FOC for Money and Bonds

$$\frac{\psi}{M_t} = \lambda_t - \beta\,\mathbb{E}_t[\lambda_{t+1}] \quad \text{(Equation 3.3)}$$

$$\lambda_t = \beta\,\mathbb{E}_t[\lambda_{t+1} R_t^B] \quad \text{(Equation 3.4)}$$

Substitute λ_t into FOC of money

$$\frac{\psi}{M_t} = \beta\,\mathbb{E}_t[\lambda_{t+1} R_t^B] - \beta\,\mathbb{E}_t[\lambda_{t+1}]$$

$$\frac{\psi}{M_t} = \beta\,\mathbb{E}_t\left[\lambda_{t+1}(R_t^B - 1)\right]$$

Substitute λ_{t+1} from FOC of Consumption

$$\lambda_{t+1} = \frac{(c_{t+1} - \eta\,c_t)^{-\theta} - \beta\,\eta\,\mathbb{E}_{t+1}[(c_{t+2} - \eta\,c_{t+1})^{-\theta}]}{P_{t+1}}$$

Solve for M_t

$$M_t = \frac{\psi}{\beta\,\mathbb{E}_t\left[(R_t^B - 1)\cdot\dfrac{(c_{t+1} - \eta\,c_t)^{-\theta} - \beta\,\eta\,\mathbb{E}_{t+1}[(c_{t+2} - \eta\,c_{t+1})^{-\theta}]}{P_{t+1}}\right]}$$

$$\boxed{M_t \;=\; \frac{\psi}{\beta\,\mathbb{E}_t\Big[(R_t^B - 1)\;\cdot\;\dfrac{(c_{t+1} - \eta\,c_t)^{-\theta} \;-\; \beta\,\eta\,\mathbb{E}_{t+1}[(c_{t+2} - \eta\,c_{t+1})^{-\theta}]}{P_{t+1}}\Big]}} \tag{3.9}$$

Capital Euler Equation Defines Tobin's q and links required returns on capital to bond returns (from (3.6), (3.5) and (3.4)) :

Define Tobin's q from FOC for Investment

$\lambda_t = \mu_t q_t \quad \text{where} \quad q_t \equiv 1 - \phi\left(\frac{I_t}{K_{t-1}} - \delta\right)$

Rearrange FOC for Capital

$\mu_t = \beta\,\mathbb{E}_t\Big[\lambda_{t+1} R^k_{t+1} + \mu_{t+1}\Big(1 - \delta + \frac{\phi}{2}\left[(I_{t+1}/K_t)^2 - \delta^2\right]\Big)\Big]$

Substitute $\mu_t = \lambda_t / q_t$ and $\mu_{t+1} = \lambda_{t+1}/q_{t+1}$

$\dfrac{\lambda_t}{q_t} = \beta\,\mathbb{E}_t\left[\lambda_{t+1} R^k_{t+1} + \dfrac{\lambda_{t+1}}{q_{t+1}}\left(1 - \delta + \frac{\phi}{2}\left[(I_{t+1}/K_t)^2 - \delta^2\right]\right)\right]$

Factor λ_{t+1}

$\dfrac{\lambda_t}{q_t} = \beta\,\mathbb{E}_t\left[\lambda_{t+1}\left(R^k_{t+1} + \dfrac{1}{q_{t+1}}\left(1 - \delta + \frac{\phi}{2}\left[(I_{t+1}/K_t)^2 - \delta^2\right]\right)\right)\right]$

Substitute FOC for Bonds ($\lambda_t = \beta\,\mathbb{E}_t[\lambda_{t+1} R^B_t]$)

$\dfrac{\beta\,\mathbb{E}_t[\lambda_{t+1} R^B_t]}{q_t} = \beta\,\mathbb{E}_t\left[\lambda_{t+1}\left(R^k_{t+1} + \dfrac{1}{q_{t+1}}\Gamma_{t+1}\right)\right]$

where $\Gamma_{t+1} \equiv 1 - \delta + \frac{\phi}{2}\left[(I_{t+1}/K_t)^2 - \delta^2\right]$

$$\boxed{\begin{aligned} q_t &\equiv 1 - \phi\left(\tfrac{I_t}{K_{t-1}} - \delta\right) \\[2ex] \frac{\beta\,\mathbb{E}_t[\lambda_{t+1}\,R^B_t]}{q_t} &= \beta\,\mathbb{E}_t\Big[\lambda_{t+1}\Big(R^k_{t+1} + \frac{1}{q_{t+1}}\Big(1 - \delta + \tfrac{\phi}{2}\left[(I_{t+1}/K_t)^2 - \delta^2\right]\Big)\Big)\Big] \end{aligned}} \tag{3.10}$$

3.2. Production

3.2.1. Final Good Producer

Derivation of Intermediate Goods Demand and Aggregate Price Index

Final goods producer's profit:

$$\Pi_t = P_t Y_t - \int_0^1 P_t(j) Y_t(j) dj$$

subject to $Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$

Substitute production function into profit:

$$\Pi_t = P_t \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 P_t(j) Y_t(j) dj$$

First-order condition for $Y_t(j)$:

$$\frac{\partial \Pi_t}{\partial Y_t(j)} = P_t \cdot \frac{\epsilon}{\epsilon - 1} \left(\int_0^1 Y_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{1}{\epsilon-1}} \cdot \frac{\epsilon - 1}{\epsilon} Y_t(j)^{-\frac{1}{\epsilon}} - P_t(j) = 0$$

$$\Rightarrow P_t \cdot Y_t^{\frac{1}{\epsilon}} Y_t(j)^{-\frac{1}{\epsilon}} = P_t(j)$$

Rearrange to obtain demand curve:

$$Y_t(j) = \left(\frac{P_t}{P_t(j)} \right)^{\epsilon} Y_t$$

Substitute demand into production function:

$$Y_t = \left(\int_0^1 \left[\left(\frac{P_t}{P_t(j)} \right)^{\epsilon} Y_t \right]^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

$$= Y_t \left(\int_0^1 \left(\frac{P_t}{P_t(j)} \right)^{\epsilon-1} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

Simplify to obtain price index:

$$1 = \left(\int_0^1 \left(\frac{P_t}{P_t(j)} \right)^{\epsilon - 1} dj \right)^{\frac{\epsilon}{\epsilon - 1}}$$

$$\Rightarrow P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon} dj$$

$$\Rightarrow P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}}$$

$$\boxed{\begin{gathered} Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t \\ \\ P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}} \end{gathered}} \tag{3.11}$$

3.2.2. Intermediate Goods Producers

Cost minimization for intermediate firm j :

$$\min_{K_t(j),h_t(j)} \left\{R_t^k K_t(j) + W_t h_t(j)\right\}$$

subject to $Y_t(j) = A_t K_t(j)^{\alpha} h_t(j)^{1-\alpha}$

Lagrangian:

$$\mathcal{L} = R_t^k K_t(j) + W_t h_t(j) + \lambda_t \left[A_t K_t(j)^{\alpha} h_t(j)^{1-\alpha} - Y_t(j)\right]$$

First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial K_t(j)} = 0 : \quad R_t^k = \lambda_t \alpha A_t K_t(j)^{\alpha-1} h_t(j)^{1-\alpha}$$
$$\frac{\partial \mathcal{L}}{\partial h_t(j)} = 0 : \quad W_t = \lambda_t (1-\alpha) A_t K_t(j)^{\alpha} h_t(j)^{-\alpha}$$

Rearrange FOCs:

$$\lambda_t = \frac{R_t^k}{\alpha}\left(\frac{K_t(j)}{h_t(j)}\right)^{1-\alpha}\frac{1}{A_t}, \quad \lambda_t = \frac{W_t}{1-\alpha}\left(\frac{K_t(j)}{h_t(j)}\right)^{\alpha}\frac{1}{A_t}$$

Equate expressions:

$$\frac{R_t^k}{\alpha}\left(\frac{K_t(j)}{h_t(j)}\right)^{-\alpha} = \frac{W_t}{1-\alpha}\left(\frac{K_t(j)}{h_t(j)}\right)^{1-\alpha}$$
$$\Rightarrow \frac{K_t(j)}{h_t(j)} = \frac{\alpha}{1-\alpha}\frac{W_t}{R_t^k}$$

Substitute into capital FOC:

$$\lambda_t = \frac{R_t^k}{\alpha A_t}\left(\frac{\alpha}{1-\alpha}\frac{W_t}{R_t^k}\right)^{\alpha-1}$$
$$= \frac{1}{A_t}\left(\frac{R_t^k}{\alpha}\right)^{\alpha}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}$$

$$\boxed{MC_t = \frac{1}{A_t}\left(\frac{R_t^k}{\alpha}\right)^{\alpha}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}} \tag{3.12}$$

Intermediate-goods producer's problem:

$$\max_{P_t(j)} \; \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\Big[\big(\tfrac{P_{t+s}(j)}{P_{t+s}}\big)^{1-\epsilon} Y_{t+s} - mc_{t+s}\,\big(\tfrac{P_{t+s}(j)}{P_{t+s}}\big)^{-\epsilon} Y_{t+s} - \tfrac{\psi}{2}\big(\tfrac{P_{t+s}(j)}{P_{t+s-1}(j)} - 1\big)^2 Y_{t+s}\Big]$$

subject to $\quad Y_t(j) = \big(\frac{P_t(j)}{P_t}\big)^{-\epsilon} Y_t$

First-Order Condition w.r.t. $P_t(j)$:

$$\mathbb{E}_t\Big[\frac{\partial \Pi_t(j)}{\partial P_t(j)} + \beta\,\Lambda_{t,t+1}\frac{\partial \Pi_{t+1}(j)}{\partial P_t(j)}\Big] = 0$$

$$\frac{\partial \Pi_t}{\partial P_t(j)} = (1-\epsilon)\big(\tfrac{P_t(j)}{P_t}\big)^{-\epsilon}\tfrac{Y_t}{P_t} + \epsilon\, mc_t\big(\tfrac{P_t(j)}{P_t}\big)^{-\epsilon-1}\tfrac{Y_t}{P_t} - \psi\big(\tfrac{P_t(j)}{P_{t-1}(j)} - 1\big)\tfrac{Y_t}{P_{t-1}(j)}$$

$$\frac{\partial \Pi_{t+1}}{\partial P_t(j)} = \psi\big(\tfrac{P_{t+1}(j)}{P_t(j)} - 1\big)\,\tfrac{P_{t+1}(j)}{P_t(j)^2}\,Y_{t+1}$$

Impose symmetry: $P_t(j) = P_t,\; Y_t(j) = Y_t,\; \pi_t = \frac{P_t}{P_{t-1}}$.

$$(1-\epsilon) + \epsilon\, mc_t = \epsilon\big(mc_t - \tfrac{\epsilon-1}{\epsilon}\big), \qquad \tfrac{P_t(j)}{P_{t-1}(j)} = \pi_t,\; \tfrac{P_{t+1}(j)}{P_t(j)} = \pi_{t+1}$$

$$0 = \epsilon\big(mc_t - \tfrac{\epsilon-1}{\epsilon}\big) - \psi\,(\pi_t - 1)\,\pi_t + \beta\,\mathbb{E}_t\Big[\Lambda_{t,t+1}\,\psi\,(\pi_{t+1}-1)\,\pi_{t+1}\,\tfrac{Y_{t+1}P_t}{Y_tP_{t+1}}\Big]$$

Noting $\Lambda_{t,t+1} = \beta\,\frac{\lambda_{t+1}}{\lambda_t}$ and $P_{t+1}/P_t = \pi_{t+1}$, the bracket simplifies to $\beta\,\frac{\lambda_{t+1}}{\lambda_t}\,\frac{Y_{t+1}}{Y_t}$.

$$\boxed{0 = \epsilon\Big(mc_t - \tfrac{\epsilon-1}{\epsilon}\Big) - \psi\,(\pi_t - 1)\,\pi_t + \beta\,\mathbb{E}_t\Big[\tfrac{\lambda_{t+1}}{\lambda_t}\,\psi\,(\pi_{t+1}-1)\,\pi_{t+1}\,\tfrac{Y_{t+1}}{Y_t}\Big]} \tag{3.13}$$

4. Old Stuff

Components breakdown:

Expenditures:

Consumption: $P_t c_t$

Investment: $P_t i_t$

Bonds: B_t

Money holdings: M_t

Income sources:

Bond returns: $(1 + i_{t-1})B_{t-1}$

Money carryover: M_{t-1}

Labor income: $W_t h_t$

Capital returns: $R_t^k K_{t-1}$ (Key addition missing in Sims)

Firm profits: Π_t

Net transfers: $-P_t \tau_t$

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