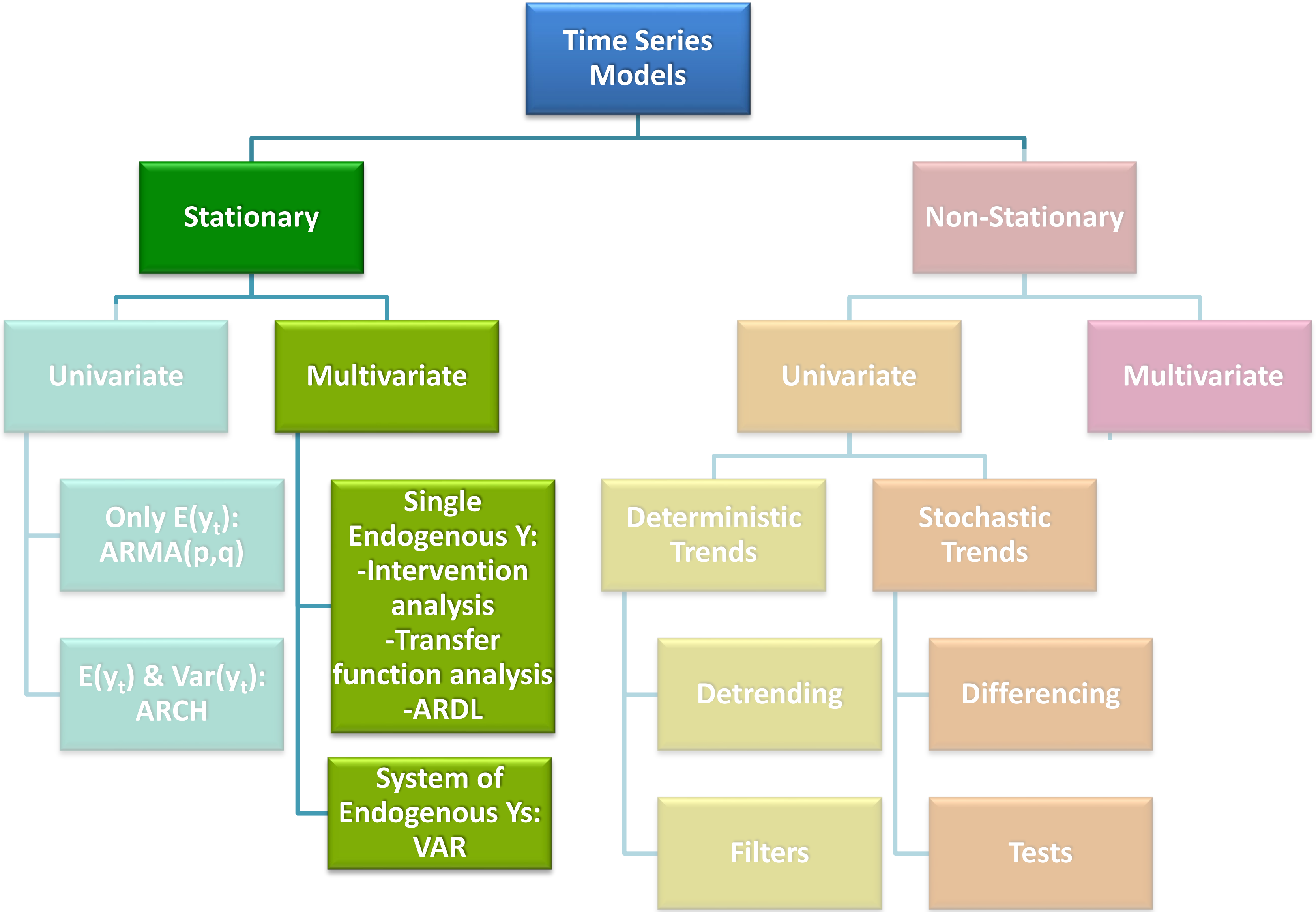
Econometrics 441 | 871

TOPIC 4:

Stationary Multivariate Models


Time Series Models
Stationary
Non-Stationary
Univariate
Multivariate
Univariate
Multivariate
Only $E(y_t)$: ARMA(p,q)
$E(y_t)$ & $Var(y_t)$: ARCH
Single Endogenous Y: -Intervention analysis -Transfer function analysis -ARDL
System of Endogenous Ys: VAR
Deterministic Trends
Stochastic Trends
Detrending
Filters
Differencing
Tests

Plan

- Introduction
- We will briefly study:
  - Intervention analysis
  - Transfer function Analysis
  - Autoregressive, distributed lag models (ADL,ARDL)
    - We will return to this approach later.
- Then in depth:
  - Vector Autoregression
    - Primitive vs Reduced form eq’ns: Identification
    - Stationarity
    - Analyzing the information in a VAR
    - Back to Identification – Various approaches
    - Estimation Methods
- Along the way we will do a joyful review of your love of linear algebra results

comment

- There is a lot of detail that will take time, effort and practice to internalize
- Focus on each of the aspects in the VAR lecture plan and get the essential point of each first
- Combined with applied tutorial, this will be sufficient for evaluation
- The rest of the detail is to make you aware of many aspects that will become interesting/important/sensible only when you need to apply the methods

Standard Macro Model

- Goal is to estimate standard macroeconomic models like the typical 3 equation New Keynesian model.

Phillips' Curve: $$\pi_t = \pi_{t-1} + \lambda y_t - \gamma r_t + \varepsilon_{\pi,t}$$

IS Curve: $$y_t = E_t\left[y_{t+1}\right] - \theta r_t + \varepsilon_{y,t}$$

Monetary Policy rule: $$r_t = \phi_\pi \pi_t + \phi_y y_t + \varepsilon_{r,t}$$

Problem:

- If we were to try to estimate these equations individually,
  - OLS will give inconsistent estimates
  - RHS variables *endogenous = correlated with error term*

Phillips' Curve: $\pi_t = \pi_{t-1} + \lambda y_t - \gamma r_t + \varepsilon_{\pi,t}$

IS Curve: $y_t = E_t\left[y_{t+1}\right] - \theta r_t + \varepsilon_{y,t}$

Monetary Policy rule: $r_t = \phi_\pi \pi_t + \phi_y y_t + \varepsilon_{r,t}$

Multivariate models

- I will present a sequence of models where the new variables we introduce range from fully exogenous to mutually endogenous
- Using the familiar OLS assumptions as an organizing tool
- We start with an endogenous variable of interest, which last week we modelled as a univariate process and add additional right hand side variables as explanatory features

OLS Assumptions

$$y = X\beta + \varepsilon$$

1. Linearity
2. No perfect multicollinearity
3. “Exogeneity of the X’s”
4. Zero mean, unsystematic errors
5. Normal errors

OLS Assumptions

1. Linearity
2. Full column rank X
3. “Exogeneity of the X’s”

- Different assumptions required for the β estimate to be
  - Unbiased, or
  - Consistent

Exogeneity: from strong to weak

- X is non-stochastic
- Mean independence
- “Same period” mean independence
- Predetermined variables on RHS

Intervention Analysis

- Formal test of the impact of a non-stochastic event or change on the mean of a time series:

$$y_t = a_0 + a_1 y_{t-1} + c_0 z_t + \varepsilon_t$$

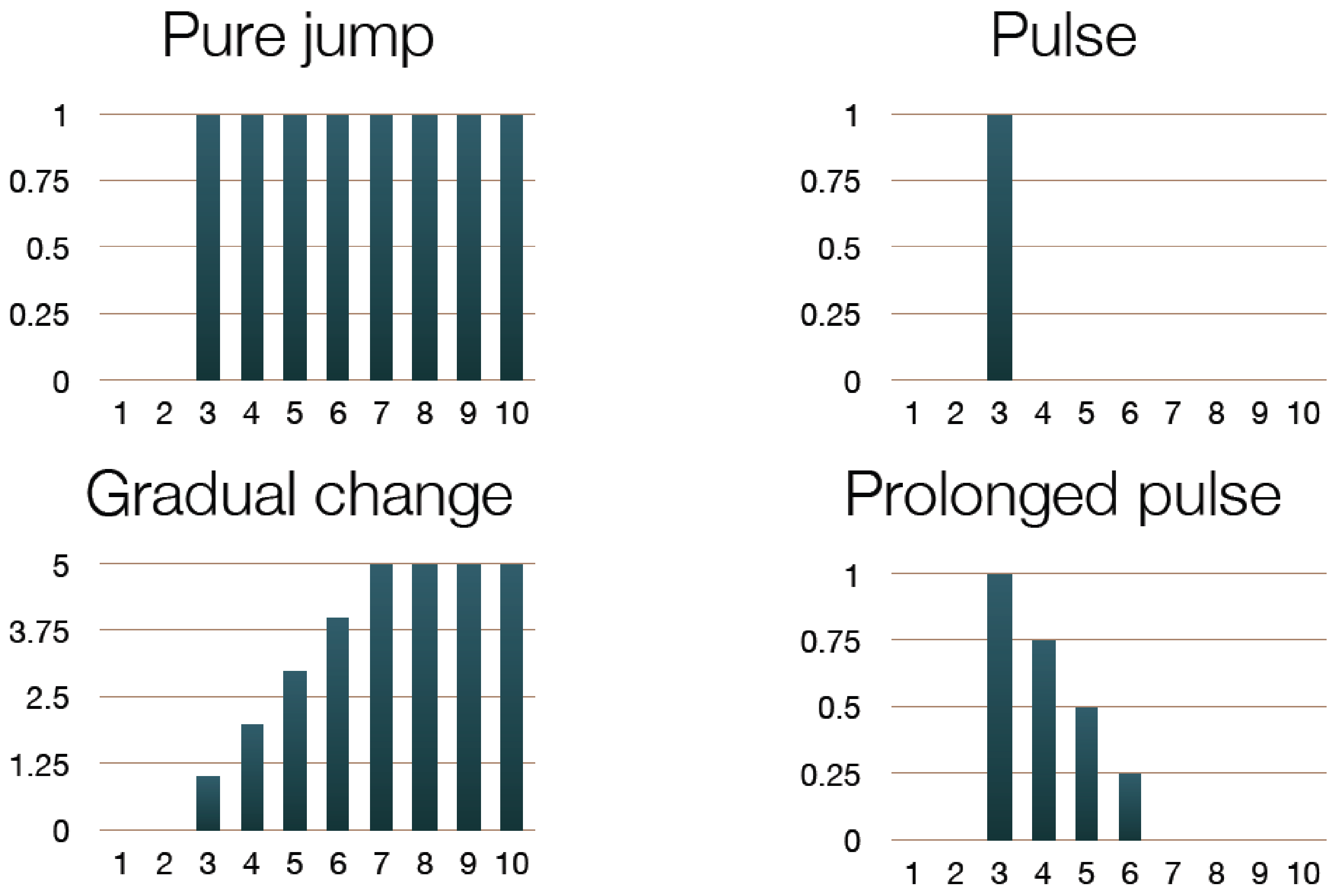
  - z_t represents an "impulse" that causes change
  - Modelled as a dummy variable – non stochastic

- Enders' example:
  - Impact of metal detectors on sky-jackings

- Impact, path and long run effects can be characterized
- One can also allow the impact to change the coefficients of the process
  - I.e. that induces a structural change

Some possible Impulse Patterns

Disappearance of MH350 on demand for Malaysian Airlines flights?



Problems with Intervention Analysis

Table 5.1 Metal Detectors and Skyjackings

	Pre-Intervention Mean	a_1	Impact Effect (c_0)	Long-Run Effect
Transnational $\{TS_t\}$	3.032	0.276	−1.29	−1.78
	(5.96)	(2.51)	(−2.21)	
US domestic $\{DS_t\}$	6.70		−5.62	−5.62
	(12.02)		(−8.73)	
Other skyjackings $\{OS_t\}$	6.80	0.237	−3.90	−5.11
	(7.93)	(2.14)	(−3.95)	

Notes:

[1] t-Statistics are in parentheses.

[2] The long-run effect is calculated as $c_0/(1 - a_1)$.

- Enders gives an excellent example that is detailed
  - What is required for the method to be reliable?
  - What are the limitations?
- Lucas-critique
  - Solution?
- Limited applications – requires change to be fully exogenous

Exogeneity: from strong to weak

- X is non-stochastic
- Mean independence

$$E(z_t \varepsilon_s) = 0 \; \forall \; t, s$$

- "Same period" mean independence
- Predetermined variables on RHS

Transfer function and ARDL models

- Impact of a stochastic, exogenous **process** on dependent variable
- I.e. the exogenous process is itself dynamic (i.e. itself an interesting ARMA(p,q) process
  - Not just an “impulse” or “step-change”
- And the variable of interest may itself have ARMA(p’,q’) properties
- Goal is to find a parsimonious model of the (ONE WAY) interaction

Note on the treatment in Enders

- New in this version
- Enders discusses this as a general ARDL but the application and derivations all clearly assume strictly exogenous additional variables
- How ARDL models are currently employed is not like this. Typically, mutually endogenous variables are modelled as a “single equation ARDL”, and endogeneity is dealt with in some other way (Instrumental Variables etc.)

Transfer function and ARDL models

- Impact of a stochastic, exogenous process on dependent variable
  - E.g. impact of **eruptions of Eyjafjallajökull** on airline revenue/profits, tourism, farming output in Iceland
  - Enders' example: impact of terrorism on tourism
- Impact may be dynamic, complicated
  - Patterns identified via **cross-correlogram**
- See the excellent example in Enders and the caveats

Exogeneity: from strong to weak

- X is non-stochastic
- Mean independence
- "Same period" mean independence

$$E(z_t \varepsilon_s) = 0 \;\forall\, s \geq t$$
$$E(z_t \varepsilon_s) \neq 0 \;\forall\, s < t$$

  – Predetermined variables on RHS

But

- Standard macro most interested in jointly endogenous processes:

Phillips' Curve: $\pi_t = \pi_{t-1} + \lambda y_t - \gamma r_t + \varepsilon_{\pi,t}$

IS Curve: $y_t = E_t\left[y_{t+1}\right] - \theta r_t + \varepsilon_{y,t}$

Monetary Policy rule: $r_t = \phi_\pi \pi_t + \phi_y y_t + \varepsilon_{r,t}$

Exogeneity: from strong to weak

- X is non-stochastic
- Mean independence
- "Same period" mean independence
- The most common situation in macroeconomics is a mutually endogenous set of variables
  - The errors of the process are pure *innovations* – future shocks are unpredictable based on current information:

$$E(z_t \varepsilon_{t+i}) = 0 \; \forall \; i > 0$$
$$E(z_t \varepsilon_{t-i}) \neq 0 \; \forall \; i \geq 0$$

History of Empirical Macro (pre 1980)

- Large structural economic models, loosely informed by theory
  - Estimate each equation individually
  - Aggregate results, forecast
- The process followed implied that the following is known with certainty:
  - The exact timing of relationships between variables (which variables affects which other variables in what sequence)
  - Lag structure of DGP

The contribution of Sims (1980)

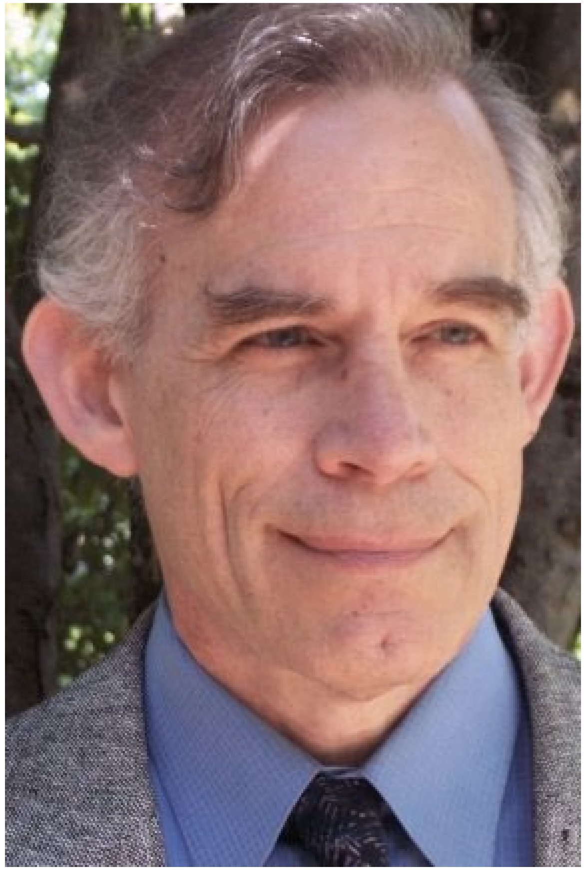
- Chris Sims publishes his seminal criticism of empirical macro models in 1980
- Key concern: "Incredible Restrictions"
  - Relevant variables
  - Precise timing of feedback

Brookings Quarterly Econometric Model of the United States, as reported by Suits and Sparks (p. 208, 1965):

$$C_{\mathrm{NF}} = \underset{(0.0165)}{0.0656} Y_{\mathrm{D}} - \underset{(2.49)}{10.93} (P_{\mathrm{CNF}}/P_{\mathrm{C}})_{t-1} + \underset{(0.0522)}{0.1889} (N + N_{\mathrm{ML}})_{t-1}$$

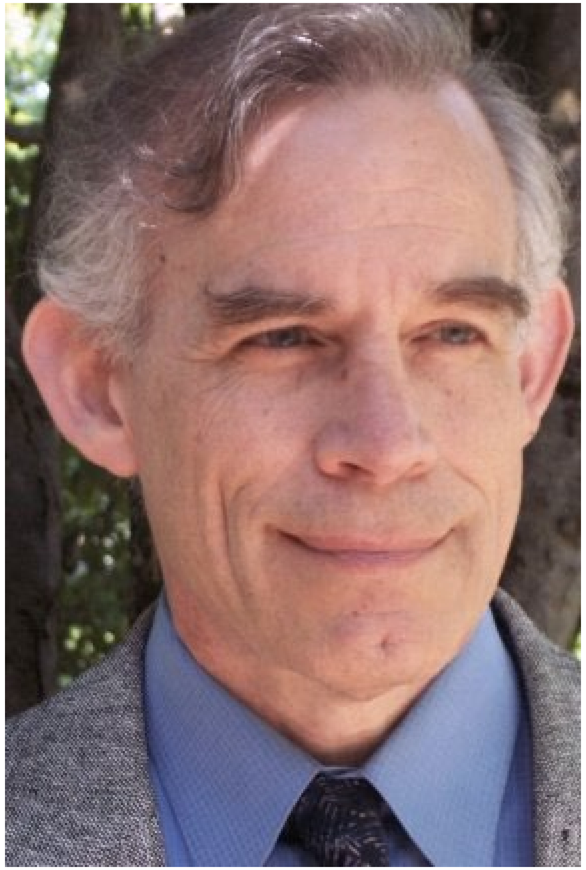
$$C_{\mathrm{NEF}} = 4.2712 + \underset{(0.0127)}{0.1691} Y_{\mathrm{D}} - \underset{(0.0213)}{0.0743} (ALQD_{\mathrm{HH}}/P_{\mathrm{C}})_{t-1}$$

where C_{NF} = personal consumption expenditures on food
Y_{D} = disposable personal income
P_{CNF} = implicit price deflator for personal consumption expenditures on food
P_{C} = implicit price deflator for personal consumption expenditures
N = civilian population
N_{ML} = military population including armed forces overseas
C_{NEF} = personal consumption expenditures for nondurables other than food
$ALQD_{\mathrm{HH}}$ = end-of-quarter stock of liquid assets held by households



Sims’ contribution

- Chris Sims’ publishes his seminal criticism of empirical macro models in 1980
- Key concern: “Incredible Restrictions”
  - Relevant variables
  - Precise timing of feedback
- Proposed:
  - Large estimated models with as few restrictions as possible
  - All variables endogenous, so simultaneously estimated system
- Nobel prize in 2011 for
  - "empirical research on cause and effect in the macro economy"



Plan

- Vector Autoregression
  - **Structural/Primitive vs Reduced form eq’ns**
  - The Identification Problem
  - Stationarity
  - Analyzing the information in a VAR
  - Back to Identification – Various approaches
  - Estimation Methods

VAR

- Structural form of the simplest VAR model:

$$
\begin{aligned}
y_t &= \gamma_{01} + b_{12} z_t + \gamma_{11} y_{t-1} + \gamma_{12} z_{t-1} + \varepsilon_{y,t} \\
z_t &= \gamma_{02} + b_{21} y_t + \gamma_{21} y_{t-1} + \gamma_{22} z_{t-1} + \varepsilon_{z,t}
\end{aligned}
$$

Where the **structural innovations** $\varepsilon_{y,t}$ and $\varepsilon_{z,t}$ are uncorrelated white noise processes

$$
E\begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{bmatrix} = E[\boldsymbol{\varepsilon}_t] = \mathbf{0}
$$

$$
E[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t'] = \begin{bmatrix} \sigma^2_{\varepsilon_y} & 0 \\ 0 & \sigma^2_{\varepsilon_z} \end{bmatrix}
$$

$$
E[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_s'] = \mathbf{0} \, \forall \, t \neq s
$$

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Where the **structural innovations** $\varepsilon_{y,t}$ and $\varepsilon_{z,t}$ are uncorrelated white noise processes

$$
E\begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{bmatrix} = E\left[\boldsymbol{\varepsilon}_t\right] = \mathbf{0}
$$

Importantly, this implies that they are unpredictable relative to the variables under analysis:

$$
E\left(\varepsilon_{i,t} | y_{t-1}\right) = 0 \text{ for } i = z, y \text{ and } E\left(\varepsilon_{i,t} | z_{t-1}\right) = 0 \text{ for } i = z, y
$$

VAR

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$$
\begin{aligned}
y_t &= \gamma_{01} + b_{12} z_t + \gamma_{11} y_{t-1} + \gamma_{12} z_{t-1} + \varepsilon_{y,t} \\
z_t &= \gamma_{02} + b_{21} y_t + \gamma_{21} y_{t-1} + \gamma_{22} z_{t-1} + \varepsilon_{z,t}
\end{aligned}
$$

- In matrix form:

$$
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \gamma_{01} \\ \gamma_{02} \end{bmatrix} + \begin{bmatrix} 0 & b_{12} \\ b_{21} & 0 \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{bmatrix}
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VAR

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$$\begin{bmatrix} 1 & -b_{12} \\ -b_{21} & 1 \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \gamma_{01} \\ \gamma_{02} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{bmatrix}$$

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Collect in matrix notation:

$$\underbrace{\begin{bmatrix} 1 & -b_{12} \\ -b_{21} & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} y_t \\ z_t \end{bmatrix}}_{\mathbf{x}_t} = \underbrace{\begin{bmatrix} \gamma_{01} \\ \gamma_{02} \end{bmatrix}}_{\Gamma_0} + \underbrace{\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}}_{\Gamma_1} \underbrace{\begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix}}_{\mathbf{x}_{t-1}} + \underbrace{\begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{bmatrix}}_{\boldsymbol{\varepsilon}_t}$$

$$B\mathbf{x}_t = \Gamma_0 + \Gamma_1 \mathbf{x}_{t-1} + \boldsymbol{\varepsilon}_t$$

VAR

- **Structural form** of the simplest model:

$$B\mathbf{x}_t = \Gamma_0 + \Gamma_1 \mathbf{x}_{t-1} + \boldsymbol{\varepsilon}_t$$

- The **Reduced form** of the model is:

$$\begin{aligned}\mathbf{x}_t &= B^{-1}\Gamma_0 + B^{-1}\Gamma_1 \mathbf{x}_{t-1} + B^{-1}\boldsymbol{\varepsilon}_t \\ &= A_0 + A_1 \mathbf{x}_{t-1} + \boldsymbol{e}_t\end{aligned}$$

VAR

- **Structural form** of the simplest model:

Structural errors

$$B\mathbf{x}_t = \Gamma_0 + \Gamma_1 \mathbf{x}_{t-1} + \boldsymbol{\varepsilon}_t$$

- The **Reduced form** of the model is:

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Reduced form errors

VAR

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- where

$$\begin{aligned} \boldsymbol{e}_t &= B^{-1}\boldsymbol{\varepsilon}_t = C^{(0)}\boldsymbol{\varepsilon}_t = \left[\begin{array}{cc} c_{11}^{(0)} & c_{12}^{(0)} \\ c_{21}^{(0)} & c_{22}^{(0)} \end{array}\right]\boldsymbol{\varepsilon}_t \\ &= \left[\begin{array}{c} c_{11}^{(0)}\varepsilon_{y,t} + c_{12}^{(0)}\varepsilon_{z,t} \\ c_{21}^{(0)}\varepsilon_{y,t} + c_{22}^{(0)}\varepsilon_{z,t} \end{array}\right] \end{aligned}$$

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- where

$$\begin{aligned}\boldsymbol{e}_t &= B^{-1}\boldsymbol{\varepsilon}_t = C^{(0)}\boldsymbol{\varepsilon}_t = \begin{bmatrix} c_{11}^{(0)} & c_{12}^{(0)} \\ c_{21}^{(0)} & c_{22}^{(0)} \end{bmatrix} \boldsymbol{\varepsilon}_t \\ &= \begin{bmatrix} c_{11}^{(0)}\varepsilon_{y,t} + c_{12}^{(0)}\varepsilon_{z,t} \\ c_{21}^{(0)}\varepsilon_{y,t} + c_{22}^{(0)}\varepsilon_{z,t} \end{bmatrix}\end{aligned}$$

Each reduced form error is a linear combination of **both** the structural errors

VAR

- **Structural form** of the simplest model:

$$B\mathbf{x}_t = \Gamma_0 + \Gamma_1 \mathbf{x}_{t-1} + \boldsymbol{\varepsilon}_t$$

- The **Reduced form** of the model is:

$$\begin{aligned} \mathbf{x}_t &= B^{-1}\Gamma_0 + B^{-1}\Gamma_1 \mathbf{x}_{t-1} + B^{-1}\boldsymbol{\varepsilon}_t \\ &= A_0 + A_1 \mathbf{x}_{t-1} + \boldsymbol{e}_t \end{aligned}$$

- Moreover:

$$E\left(\boldsymbol{e}_t \boldsymbol{e}_t'\right) = E\left(C^{(0)} \boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t' C^{(0)\prime}\right)$$

Plan

- Vector Autoregression
  - Structural vs Reduced form eq’ns
  - **The Identification Problem**
  - Stationarity
  - Analyzing the information in a VAR
  - Back to Identification – Various approaches
  - Estimation Methods

Identification Problem

- Identification
  - Definition in this literature: The recovery of Structural form from Reduced form
  - Let us count free parameters in each form
- Structural form

$$B\mathbf{x}_t = \Gamma_0 + \Gamma_1 \mathbf{x}_{t-1} + \boldsymbol{\varepsilon}_t$$

$$\begin{bmatrix} 1 & -b_{12} \\ -b_{21} & 1 \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \gamma_{01} \\ \gamma_{02} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{bmatrix} \quad E\left[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t'\right] = \begin{bmatrix} \sigma^2_{\varepsilon_y} & 0 \\ 0 & \sigma^2_{\varepsilon_z} \end{bmatrix}$$

- Reduced form

$$\begin{aligned} \mathbf{x}_t &= B^{-1}\Gamma_0 + B^{-1}\Gamma_1 \mathbf{x}_{t-1} + B^{-1}\boldsymbol{\varepsilon}_t \\ &= A_0 + A_1 \mathbf{x}_{t-1} + \boldsymbol{e}_t \end{aligned}$$

$$\begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} a_{01} \\ a_{02} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} e_{y,t} \\ e_{z,t} \end{bmatrix} \quad E\left(\boldsymbol{e}_t \boldsymbol{e}_t'\right) = \begin{bmatrix} \sigma^2_{e_1} & \sigma_{e_1 e_2} \\ \sigma_{e_1 e_2} & \sigma^2_{e_2} \end{bmatrix}$$

Identification

- Identification
  - Definition: Recovery of Structural form from Reduced form
  - A key issue in the literature is being able to *identify* exogenous shocks attributable to specific policies
  - Without this, an empirical model cannot be said to have any clear implication for policy analysis
- For an unrestricted VAR, the most general structural form is under-identified by the closest estimable reduced form
  - Identification requires a restriction on the **structural** parameters
  - Note that this (just) identifying restriction is **entirely untestable.** It is a restriction on the ***theory*** not the empirical specification
  - As such, many identification schemes have developed
  - We will consider the most basic one now, then more later

Identification Schemes (1)

Choleski decomposition (jargon borrowed from linear algebra)

- Sometimes called:
  - Recursive identification scheme
  - Triangular decomposition
- At the core, this scheme implies an assumption on timing of effects
  - i.e. an assumption on which variables *cannot* contemporaneously affect which other variables
  - A simple government expenditure example
    - Government observes a negative shock in GDP in one quarter, but can only react by spending more in the following quarter
    - But government expenditure is a component of GDP, so a shock to government expenditure is immediately part of GDP

Identification Schemes (1)

- Choleski decomposition
  - The unrestricted structural model was:

$$
\begin{aligned}
y_t &= \gamma_{01} + b_{12} z_t + \gamma_{11} y_{t-1} + \gamma_{12} z_{t-1} + \varepsilon_{y,t} \\
z_t &= \gamma_{02} + b_{21} y_t + \gamma_{21} y_{t-1} + \gamma_{22} z_{t-1} + \varepsilon_{z,t}
\end{aligned}
$$

  - If y is government expenditure and z is GDP, then the example could be imposed by setting $b_{12} = 0$

$$
\begin{aligned}
y_t &= \gamma_{01} + 0 \; z_t + \gamma_{11} y_{t-1} + \gamma_{12} z_{t-1} + \varepsilon_{y,t} \\
z_t &= \gamma_{02} + b_{21} y_t + \gamma_{21} y_{t-1} + \gamma_{22} z_{t-1} + \varepsilon_{z,t}
\end{aligned}
$$

Identification Schemes (1)

- Choleski decomposition
  - The unrestricted structural model was:

$$\begin{aligned} y_t &= \gamma_{01} + b_{12} z_t + \gamma_{11} y_{t-1} + \gamma_{12} z_{t-1} + \varepsilon_{y,t} \\ z_t &= \gamma_{02} + b_{21} y_t + \gamma_{21} y_{t-1} + \gamma_{22} z_{t-1} + \varepsilon_{z,t} \end{aligned}$$

  - If y is government expenditure and z is GDP, then the example could be imposed by setting $b_{12} = 0$

$$\begin{aligned} y_t &= \gamma_{01} + 0 \ z_t + \gamma_{11} y_{t-1} + \gamma_{12} z_{t-1} + \varepsilon_{y,t} \\ z_t &= \gamma_{02} + b_{21} y_t + \gamma_{21} y_{t-1} + \gamma_{22} z_{t-1} + \varepsilon_{z,t} \end{aligned}$$

$$\begin{bmatrix} 1 & 0 \\ -b_{21} & 1 \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \gamma_{01} \\ \gamma_{02} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -b_{21} & 1 \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \gamma_{01} \\ \gamma_{02} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -b_{21} & 1 \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \gamma_{01} \\ \gamma_{02} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{bmatrix}$$

$$\begin{aligned} B &= \begin{bmatrix} 1 & 0 \\ -b_{21} & 1 \end{bmatrix} \\ B^{-1} &= \begin{bmatrix} 1 & 0 \\ b_{21} & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \boldsymbol{e}_t &= B^{-1}\boldsymbol{\varepsilon}_t = \begin{bmatrix} 1 & 0 \\ b_{21} & 1 \end{bmatrix} \boldsymbol{\varepsilon}_t \\ &= \begin{bmatrix} \varepsilon_{y,t} \\ b_{21}\varepsilon_{y,t} + \varepsilon_{z,t} \end{bmatrix} \end{aligned}$$

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With the identification assumption, we can now *identify* the structural error to y, and then compute the structural error to z

- How do we find b_{21}? It is not estimated

- It can be found from the estimated covariance matrix (of the reduced form), noting that the structural form has a diagonal covariance matrix
  - Details not crucial for us, automatically performed in estimation programs

Over identification

- Another assumption we can make is that *neither* variable contemporaneously affects the other
  - Then the reduced form is identical to the structural form
  - This is rarely a valid assumption, but is sometimes used
- This is an *over restricted* model.
  - We impose more restrictions than is required
  - We now impose a restriction on the *estimated* model as well as the theoretical model
  - This is a testable restriction as we *force* the covariance matrix to be diagonal, hence this must reduce the fit

$$\begin{bmatrix} 1 & 0 \\ -b_{21} & 1 \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \gamma_{01} \\ \gamma_{02} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{bmatrix}$$

$$\begin{aligned} B &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ B^{-1} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \boldsymbol{e}_t &= B^{-1}\boldsymbol{\varepsilon}_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{\varepsilon}_t \\ &= \begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{bmatrix} \end{aligned}$$

n Variable VAR(p)

- Primitive Form:

$$\underset{[n\times n]}{\mathbf{B}}\underset{[n\times 1]}{\mathbf{x}_t} = \underset{[n\times 1]}{\boldsymbol{\Gamma}_0} + \underset{[n\times n]}{\boldsymbol{\Gamma}_1}\underset{[n\times 1]}{\mathbf{x}_{t-1}} + \ldots + \boldsymbol{\Gamma}_p\mathbf{x}_{t-p} + \boldsymbol{\varepsilon}_t$$

$$E\left[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\right] = \underset{[n\times n]}{\boldsymbol{\Sigma}_\varepsilon}$$

- Reduced Form:

$$\underset{[n\times 1]}{\mathbf{x}_t} = \underset{[n\times 1]}{\mathbf{A}_0} + \underset{[n\times n]}{\mathbf{A}_1}\underset{[n\times 1]}{\mathbf{x}_{t-1}} + \ldots + \mathbf{A}_p\mathbf{x}_{t-p} + \boldsymbol{e}_t$$

$$E\left[\mathbf{e}_t\mathbf{e}_t'\right] = \underset{[n\times n]}{\boldsymbol{\Sigma}}$$

n Variable VAR(p)

- Primitive Form:

$n^2 - n$ unrestricted parameters

$$\underset{[n\times n]}{\mathbf{B}}\underset{[n\times 1]}{\mathbf{x}_t} = \underset{[n\times 1]}{\mathbf{\Gamma}_0} + \underset{[n\times n]}{\mathbf{\Gamma}_1}\underset{[n\times 1]}{\mathbf{x}_{t-1}} + \ldots + \mathbf{\Gamma}_p\mathbf{x}_{t-p} + \boldsymbol{\varepsilon}_t$$

$$E\left[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\right] = \underset{[n\times n]}{\mathbf{\Sigma}_\varepsilon}$$

n non-zero parameters

only the diagonal is non-zero

we *assume* that the structural errors are uncorrelated

i.e. "pure" innovations to only one variable

- Reduced Form:

$$\underset{[n\times 1]}{\mathbf{x}_t} = \underset{[n\times 1]}{\mathbf{A}_0} + \underset{[n\times n]}{\mathbf{A}_1}\underset{[n\times 1]}{\mathbf{x}_{t-1}} + \ldots + \mathbf{A}_p\mathbf{x}_{t-p} + \boldsymbol{e}_t$$

$$E\left[\mathbf{e}_t\mathbf{e}_t'\right] = \underset{[n\times n]}{\mathbf{\Sigma}}$$

$\frac{n(n+1)}{2}$ unique parameters

n Variable VAR(p)

- Primitive Form:

$n^2 - n$ unrestricted parameters

$$\underset{[n\times n]}{\mathbf{B}}\underset{[n\times 1]}{\mathbf{x}_t} = \underset{[n\times 1]}{\boldsymbol{\Gamma}_0} + \underset{[n\times n]}{\boldsymbol{\Gamma}_1}\underset{[n\times 1]}{\mathbf{x}_{t-1}} + \ldots + \boldsymbol{\Gamma}_p\mathbf{x}_{t-p} + \boldsymbol{\varepsilon}_t$$

$$E\left[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\right] = \underset{[n\times n]}{\boldsymbol{\Sigma}_\varepsilon} \longleftarrow$$

n non-zero parameters

only the diagonal is non-zero

we *assume* that the structural errors are uncorrelated

i.e. "pure" innovations to only one variable

- Reduced Form:

$$\underset{[n\times 1]}{\mathbf{x}_t} = \underset{[n\times 1]}{\mathbf{A}_0} + \underset{[n\times n]}{\mathbf{A}_1}\underset{[n\times 1]}{\mathbf{x}_{t-1}} + \ldots + \mathbf{A}_p\mathbf{x}_{t-p} + \mathbf{e}_t$$

$$E\left[\mathbf{e}_t\mathbf{e}_t'\right] = \underset{[n\times n]}{\boldsymbol{\Sigma}} \longleftarrow$$

$\frac{n(n+1)}{2}$ unique parameters

variance/covariance matrix is *symmetric*

$\frac{n^2-n}{2}$ restrictions necessary for a just-identified system

Plan

- Vector Autoregression
  - Structural vs Reduced form eq’ns
  - The Identification Problem
  - **Stationarity**
  - Analyzing the information in a VAR
  - Back to Identification – Various approaches
  - Estimation Methods

Stationarity of a VAR(1)

- We now extend our rule for stationarity of an AR(p) process to a VAR(p) process
- There are a myriad of ways of doing this, I will give a version that is not a proof but gives my favourite intuition. Enders does this another way, but it is equivalent.
- As in the univariate case, a process is stationary if the effect of shocks infinitely far in the past eventually fade out
- Again, we start with the simplest version, a 2 variable VAR(1)

$$\mathbf{x}_t = A_0 + A_1 \mathbf{x}_{t-1} + \boldsymbol{e}_t$$

Stationarity of a VAR(1)

- As with the univariate case, we can iterate backwards:

$$
\begin{aligned}
\mathbf{x}_t &= A_0 + A_1\mathbf{x}_{t-1} + \boldsymbol{e}_t \\
&= A_0 + A_1\left(A_0 + A_1\mathbf{x}_{t-2} + \boldsymbol{e}_{t-1}\right) + \boldsymbol{e}_t \\
&= A_0 + A_0A_1 + \boldsymbol{e}_t + A_1\boldsymbol{e}_{t-1} + A_1^2\left(A_0 + A_1\mathbf{x}_{t-3} + \boldsymbol{e}_{t-2}\right) \\
&= A_0 + A_0A_1 + A_0A_1^2 + \boldsymbol{e}_t + A_1\boldsymbol{e}_{t-1} + A_1^2\boldsymbol{e}_{t-2} + A_1^3\left(A_0 + A_1\mathbf{x}_{t-4} + \boldsymbol{e}_{t-3}\right) \\
&= A_0\sum_{i=0}^{\infty} A_1^i + \sum_{i=0}^{\infty} A_1^i\boldsymbol{e}_{t-i} + \lim_{i\to\infty} A_1^i\mathbf{x}_{t-i}
\end{aligned}
$$

- This may seem very intimidating, so let’s take a step back
  - A non-stationary/integrated/unit root process is characterized by infinite memory
    - The impact of shocks *never* fade out
  - A stationary process is one where the effect of shocks/initial conditions fade over time
  - So, what must happen in the above equation for the past to eventually become irrelevant?

Stationarity of a VAR(1)

- As with the univariate case, we can iterate backwards:

$$
\begin{aligned}
\mathbf{x}_t &= A_0 + A_1 \mathbf{x}_{t-1} + \boldsymbol{e}_t \\
&= A_0 + A_1 \left(A_0 + A_1 \mathbf{x}_{t-2} + \boldsymbol{e}_{t-1} \right) + \boldsymbol{e}_t \\
&= A_0 + A_0 A_1 + \boldsymbol{e}_t + A_1 \boldsymbol{e}_{t-1} + A_1^2 \left(A_0 + A_1 \mathbf{x}_{t-3} + \boldsymbol{e}_{t-2} \right) \\
&= A_0 + A_0 A_1 + A_0 A_1^2 + \boldsymbol{e}_t + A_1 \boldsymbol{e}_{t-1} + A_1^2 \boldsymbol{e}_{t-2} + A_1^3 \left(A_0 + A_1 \mathbf{x}_{t-4} + \boldsymbol{e}_{t-3} \right) \\
&= A_0 \sum_{i=0}^{\infty} A_1^i + \sum_{i=0}^{\infty} A_1^i \boldsymbol{e}_{t-i} + \lim_{i \to \infty} A_1^i \mathbf{x}_{t-i}
\end{aligned}
$$

- To evaluate these limits, we need the tools for integer powers of square matrices.
- For this, we detour into a review of eigenvalues
- What are eigenvalues? Intuitively? *evil smiling face*

Eigenvalues

- Let
  - A be a given $[n \times n]$ matrix
  - $\mathbf{w}$ be an arbitrary, non-zero $[n \times 1]$ vector, and
  - I be the $[n \times n]$ identity matrix
- Then λ is called an *eigenvalue* or a *characteristic root* of the matrix A if:

$$A\mathbf{w} = \lambda\mathbf{w}$$

$$(A - \lambda I)\,\mathbf{w} = 0$$

Eigenvalues

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- Then λ is called an *eigenvalue* or a *characteristic root* of the matrix A if:

$$A\mathbf{w} = \lambda\mathbf{w}$$

$$(A - \lambda I)\,\mathbf{w} = 0$$

- Since $\mathbf{w}$ is non-zero, this requires linear dependence in $(A - \lambda I)$, or equivalently, that the *determinant* of the matrix must be zero:

$$|A - \lambda I| = 0$$

Eigenvalues

- The eigenvalues of a matrix A are those values of λ that solve:

$$|A - \lambda I| = 0$$

- Consider the 2x2 case:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}$$

$$\left| \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \right| =$$

$$=$$

$$=$$

Eigenvalues

- The eigenvalues of a matrix A are those values of λ that solve:

$$|A - \lambda I| = 0$$

- Consider the 2x2 case:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}$$

$$\left| \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \right| = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$

$$=$$

$$=$$

Eigenvalues

- The eigenvalues of a matrix A are those values of λ that solve:

$$|A - \lambda I| = 0$$

- Consider the 2x2 case:

$$\begin{aligned}
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} &= \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \\
\left| \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \right| &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\
&= (\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} \\
&=
\end{aligned}$$

Eigenvalues

- The eigenvalues of a matrix A are those values of λ that solve:

$$|A - \lambda I| = 0$$

- Consider the 2x2 case:

$$\begin{aligned}
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} &= \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \\
\left| \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \right| &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\
&= (\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} \\
&= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}
\end{aligned}$$

Eigenvalues

- The eigenvalues of a matrix A are those values of λ that solve:

$$|A - \lambda I| = 0$$

- Thus:

$$\begin{aligned} |A - \lambda I| &= 0 \\ &\Updownarrow \\ \lambda^2 - (a_{11} + a_{22})\,\lambda + a_{11}a_{22} - a_{12}a_{21} &= 0 \end{aligned}$$

- The last expression is the *characteristic equation* of the matrix A.
- For a $[2 \times 2]$ matrix, this is a quadratic in the eigenvalues λ for which there are at most 2 distinct solutions, real, complex or zero
- For an $[n \times n]$ matrix, this will be an n^{th} order polynomial in the eigenvalues λ, with at most n distinct solutions.

Eigenvalues

- The eigenvalues of a matrix A are those values of λ that solve:

$$|A - \lambda I| = 0$$

- For an $[n \times n]$ matrix, this will be an n^{th} order polynomial in the eigenvalues λ, with at most n distinct solutions.
  - If there are n non-zero eigenvalues, A is invertible/has linearly independent rows/columns, and is of full rank: $\text{rank}(A) = n$
  - If there are only $q \in \{1, \ldots, n-1\}$ non-zero eigenvalues, A is non-invertible, has linearly dependent columns/rows and is rank deficient: $\text{rank}(A) = q$
  - If all eigenvalues are zero, the matrix A is the $[n \times n]$ zero matrix.
- We will only consider full rank matrices in this lecture.
  - Rank deficient matrices will pop up again in cointegration, so don't forget this.

Representation of powers of full rank square matrices

- If an $[n \times n]$ matrix A has n non-zero eigenvalues, there exists an invertible matrix T such that (the eigenvalue decomposition):

$$A = T \Lambda T^{-1}$$

- Where Λ is a matrix with the n non-zero eigenvalues on the diagonal:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Representation of powers of full rank square matrices

- If an $[n \times n]$ matrix A has n non-zero eigenvalues, there exists an invertible matrix T such that (the eigenvalue decomposition):

$$A = T\Lambda T^{-1}$$

- Moreover:

$$\begin{aligned} A^2 = AA &= T\Lambda T^{-1} T\Lambda T^{-1} \\ &= T\Lambda^2 T^{-1} \end{aligned}$$

- Where the standard result holds:

$$\Lambda^2 = \begin{bmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^2 \end{bmatrix}$$

Representation of powers of full rank square matrices

- If an $[n \times n]$ matrix A has n non-zero eigenvalues, there exists an invertible matrix T such that (the eigenvalue decomposition):

$$A = T\Lambda T^{-1}$$

- Moreover:

$$\begin{aligned} A^2 = AA &= T\Lambda T^{-1} T\Lambda T^{-1} \\ &= T\Lambda^2 T^{-1} \end{aligned}$$

- If an $[n \times n]$ matrix A has n non-zero eigenvalues, there exists an invertible matrix T such that (the eigenvalue decomposition):
- Moreover:

- Where the standard result holds:

- Where the standard result holds:

- Thus:

- Thus:

$$A^i = T\Lambda^i T^{-1}$$

Representation of powers of full rank square matrices

$$A^i = T\Lambda^i T^{-1}$$

- If an $[n \times n]$ matrix A has n non-zero eigenvalues, each less than one in absolute value:

$$\lim_{i\to\infty} A^i = 0$$

Eigenvalue intuition on data

- My favourite intuition on what eigenvalues mean in statistics
- A measure of the independent information in a data set
  - Consider an example of N observations:

$$x_{1i} \; i.i.d.$$
$$x_{2i} \; i.i.d.$$
$$x_{3i} = x_{1i} + x_{2i}$$
$$X = [\, x_1, x_2, x_3]$$

    - $X'X$ will have 2 non-zero eigenvalues and one zero, because there are only two columns of linearly independent information in X
  - Another example: if we have k variables in X (without perfect multicollinearity), and $X'X$ has one very large eigenvalue and the rest are much smaller, it means that the variables are all highly correlated, so that most of the variation in the data can be captured with one *latent* variable.
    This leads to Principal Component Analysis.

Stationarity of a VAR(1)

- Returning to the backward iteration:

$$\begin{aligned}
\mathbf{x}_t &= A_0 + A_1\mathbf{x}_{t-1} + \boldsymbol{e}_t \\
&= A_0 + A_1\left(A_0 + A_1\mathbf{x}_{t-2} + \boldsymbol{e}_{t-1}\right) + \boldsymbol{e}_t \\
&= A_0 + A_0A_1 + \boldsymbol{e}_t + A_1\boldsymbol{e}_{t-1} + A_1^2\left(A_0 + A_1\mathbf{x}_{t-3} + \boldsymbol{e}_{t-2}\right) \\
&= A_0 + A_0A_1 + A_0A_1^2 + \boldsymbol{e}_t + A_1\boldsymbol{e}_{t-1} + A_1^2\boldsymbol{e}_{t-2} + A_1^3\left(A_0 + A_1\mathbf{x}_{t-4} + \boldsymbol{e}_{t-3}\right) \\
&= A_0\sum_{i=0}^{\infty} A_1^i + \sum_{i=0}^{\infty} A_1^i\boldsymbol{e}_{t-i} + \lim_{i\to\infty} A_1^i\mathbf{x}_{t-i}
\end{aligned}$$

- By the results above, if and only if the eigenvalues of matrix A_1 are less than one in absolute value:

$$\begin{aligned}
\mathbf{x}_t &= A_0 + A_1\mathbf{x}_{t-1} + \boldsymbol{e}_t \\
&= A_0\sum_{i=0}^{\infty} A_1^i + \sum_{i=0}^{\infty} A_1^i\boldsymbol{e}_{t-i} < \infty
\end{aligned}$$

Stationarity of a VAR(1)

- Using Lag operator:

$$\begin{aligned} \mathbf{x}_t &= A_0 + A_1\mathbf{x}_{t-1} + e_t \\ &= A_0 + A_1 L\mathbf{x}_t + e_t \\ (I - A_1 L)\,\mathbf{x}_t &= A_0 + e_t \end{aligned}$$

- The object $(I - A_1 L)$ is the *inverse characteristic matrix polynomial* of the VAR(1) process, and the process is stationary if the characteristic roots of this polynomial are all *larger* than 1 in absolute value
- Then the inverse $(I - A_1 L)^{-1}$ is well defined and implies that the VAR(1) has an Infinite order Vector Moving Average $VMA(\infty)$ representation

Extending to a VAR(p)

- Consider the generic VAR(p) process:

$$\mathbf{x}_t = A_0 + A_1\mathbf{x}_{t-1} + \cdots + A_p\mathbf{x}_{t-p} + e_t$$

- We can extend the rule we just derived by a simple recasting of the process into a more complicated VAR(1) process
- Define the following objects:

$$\underset{[np\times 1]}{\mathbf{X}_t} = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{x}_{t-1} \\ \vdots \\ \mathbf{x}_{t-p+1} \end{bmatrix}, \underset{[np\times 1]}{\mathbf{D}_0} = \begin{bmatrix} A_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \underset{[np\times np]}{\mathbf{D}_1} = \begin{bmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}, \underset{[np\times 1]}{\mathbf{U}_t} = \begin{bmatrix} e_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Extending to a VAR(p)

- Consider the generic VAR(p) process: $\mathbf{x}_t = A_0 + A_1\mathbf{x}_{t-1} + \cdots + A_p\mathbf{x}_{t-p} + \mathbf{e}_t$

- Define the following objects:

$$\underset{[np\times 1]}{\mathbf{X}_t} = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{x}_{t-1} \\ \vdots \\ \mathbf{x}_{t-p+1} \end{bmatrix}, \underset{[np\times 1]}{\mathbf{D}_0} = \begin{bmatrix} A_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \underset{[np\times np]}{\mathbf{D}_1} = \begin{bmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}, \underset{[np\times 1]}{\mathbf{U}_t} = \begin{bmatrix} \mathbf{e}_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- The VAR(p) is then equivalently cast into *companion form* as:

$$\mathbf{X}_t = \mathbf{D}_0 + \mathbf{D}_1\mathbf{X}_{t-1} + \mathbf{U}_t$$

- Thus the VAR(p) is stationary as long as the eigenvalues of matrix D_1 are less than one in absolute value.
- In sum, the rule stays the same as in the univariate case. There are characteristic polynomials whose roots determine stationarity

Plan

- Vector Autoregression
  - Structural vs Reduced form eq'ns
  - The Identification Problem
    - The closest estimable version of a fully general linear structural model is under-identified.
    - To identify structural innovations, the structural model must be restricted in some way. This is called an *identification strategy*.
    - The Choleski decomposition imposes a timing assumption by restricting contemporaneous effects
  - Stationarity
    - The rule for stationarity of a VAR extends directly from the univariate rule:
    - The eigenvalues of the inverse characteristic matrix equation must be outside the unit circle
    - There is a mathematically equivalent way that expresses stationarity in terms of eigenvalues that must be smaller than 1 in absolute value, and this is what most packages report
  - **Analyzing the information in a VAR**
  - Back to Identification – Various approaches
  - Estimation Methods

Summarizing the information in a VAR

- Impulse Response functions
- Variance Decompositions

Impulse Response Function

- Starting all processes at their mean,
- What is the predicted **time path** of each variable in response to

  an *impulse*

  in each of the innovations?

  (equivalently: a fundamental shock to only one variable)

Impulse Response Functions

- In words: how does the time path of an endogenous variable respond to an innovation in one of the variables
- Theoretical IRF is w.r.t an **Innovation**
  - Structural model shock
- Empirical IRF is w.r.t a
  - Residual (no restrictions imposed), or
  - Estimate of Innovation based on specific identification strategy.
    - Identification strategy may influence shape of IRF, hence the economic meaning of results

Impulse Response Functions - Univariate

- The definition of an IRF is clearest in the univariate case
- Consider the simplest case: a stationary AR(1) process:

$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$$

- Suppose the process starts at its expected value $y_0 = E(y_t) = \frac{a_0}{1-a_1} = \mu$, and there are no shocks (no impulse): i.e. $\varepsilon_t = 0 \ \forall \ t$

$$y_0 = \frac{a_0}{1 - a_1}$$

$$y_1 = a_0 + a_1 \left(\frac{a_0}{1 - a_1} \right)$$

$$y_1 = \frac{a_0 - a_0 a_1}{1 - a_1} + \frac{a_0 a_1}{1 - a_1}$$

$$y_1 = \frac{a_0}{1 - a_1}$$

Impulse Response Functions - Univariate

- Suppose the process starts at its expected value, $y_0 = E(y_t) = \frac{a_0}{1-a_1}$ and there is only a unit shock in period 1: $\varepsilon_1 = 1$, $\varepsilon_t = 0 \; \forall \; t > 1$

$$y_0 = \frac{a_0}{1 - a_1}$$

Impulse Response Functions - Univariate

- Suppose the process starts at its expected value, $y_0 = E(y_t) = \frac{a_0}{1-a_1}$ and there is only a unit shock in period 1: $\varepsilon_1 = 1, \varepsilon_t = 0 \ \forall \ t > 1$

$$y_0 = \frac{a_0}{1 - a_1}$$

$$y_1 = a_0 + a_1\left(\frac{a_0}{1 - a_1}\right) + \varepsilon_1 = \frac{a_0}{1 - a_1} + 1$$

Impulse Response Functions - Univariate

- Suppose the process starts at its expected value, $y_0 = E(y_t) = \frac{a_0}{1-a_1}$ and there is only a unit shock in period 1: $\varepsilon_1 = 1, \varepsilon_t = 0 \; \forall \; t > 1$

$$y_0 = \frac{a_0}{1 - a_1}$$

$$y_1 = a_0 + a_1 \left(\frac{a_0}{1 - a_1} \right) + \varepsilon_1 = \frac{a_0}{1 - a_1} + 1$$

$$y_2 = a_0 + a_1 \left(\frac{a_0}{1 - a_1} + 1 \right) + \varepsilon_2 = \frac{a_0}{1 - a_1} + a_1 + 0$$

Impulse Response Functions - Univariate

- Suppose the process starts at its expected value, $y_0 = E(y_t) = \frac{a_0}{1-a_1}$ and there is only a unit shock in period 1: $\varepsilon_1 = 1, \varepsilon_t = 0 \; \forall \; t > 1$

$$y_0 = \frac{a_0}{1 - a_1}$$

$$y_1 = a_0 + a_1\left(\frac{a_0}{1 - a_1}\right) + \varepsilon_1 = \frac{a_0}{1 - a_1} + 1$$

$$y_2 = a_0 + a_1\left(\frac{a_0}{1 - a_1} + 1\right) + \varepsilon_2 = \frac{a_0}{1 - a_1} + a_1 + 0$$

$$y_3 = a_0 + a_1\left(\frac{a_0}{1 - a_1} + a_1\right) + \varepsilon_3 = \frac{a_0}{1 - a_1} + a_1^2 + 0$$

Impulse Response Functions - Univariate

- Suppose the process starts at its expected value, $y_0 = E(y_t) = \frac{a_0}{1-a_1}$ and there is only a unit shock in period 1: $\varepsilon_1 = 1, \varepsilon_t = 0 \; \forall \; t > 1$

$$y_0 = \frac{a_0}{1 - a_1}$$

$$y_1 = a_0 + a_1\left(\frac{a_0}{1 - a_1}\right) + \varepsilon_1 = \frac{a_0}{1 - a_1} + 1$$

$$y_2 = a_0 + a_1\left(\frac{a_0}{1 - a_1} + 1\right) + \varepsilon_2 = \frac{a_0}{1 - a_1} + a_1 + 0$$

$$y_3 = a_0 + a_1\left(\frac{a_0}{1 - a_1} + a_1\right) + \varepsilon_3 = \frac{a_0}{1 - a_1} + a_1^2 + 0$$

$$y_4 = a_0 + a_1\left(\frac{a_0}{1 - a_1} + a_1^2\right) + \varepsilon_4 = \frac{a_0}{1 - a_1} + a_1^3 + 0$$

Impulse Response Functions - Univariate

- Suppose the process starts at its expected value, $y_0 = E(y_t) = \frac{a_0}{1-a_1}$ and there is only a unit shock in period 1: $\varepsilon_1 = 1, \varepsilon_t = 0 \; \forall \; t > 1$

$$
\begin{aligned}
&y_0 - E(y_t) = 0 \\
&y_1 - E(y_t) = 1 \\
&y_2 - E(y_t) = a_1 \\
&y_3 - E(y_t) = a_1^2 \\
&y_4 - E(y_t) = a_1^3 \\
&\ldots \\
&y_t - E(y_t) = a_1^{t-1}
\end{aligned}
$$

- This sequence is the Impulse Response Function for this process given a once-off pulse in the error sequence

Impulse Response Functions - Univariate

- This generalizes to any order stationary ARMA(p,q) process:

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + \cdots + a_p y_{t-p} + \varepsilon_t + b_1 \varepsilon_{t-1} + \cdots + b_q \varepsilon_{t-q}$$

- We can always represent a stationary ARMA(p,q) process as an MA(∞) process:

$$y_t = \mu + \sum_{i=0}^{\infty} c_i \varepsilon_{t-i}$$

  – With $c_0 = 1$

- Thus:

$$y_t - \mu = \varepsilon_t + c_1 \varepsilon_{t-1} + c_2 \varepsilon_{t-2} + \cdots$$

Impulse Response Functions - Univariate

- We can always represent a stationary ARMA(p,q) process as an MA(∞) process:

$$y_t = \mu + \sum_{i=0}^{\infty} c_i \varepsilon_{t-i}$$

- Thus:

$$y_t - \mu = \varepsilon_t + c_1 \varepsilon_{t-1} + c_2 \varepsilon_{t-2} + \cdots$$

- If $\{\varepsilon_t\}_{t=1}^{T} = [1,0,0,\ldots]$, then:

$$y_1 - \mu = \varepsilon_1 + c_1 \varepsilon_0 + c_2 \varepsilon_{-1} + \cdots$$
$$y_1 - \mu = 1 + c_1 0 + c_2 0 + \cdots = 1$$
$$y_2 - \mu = 0 + c_1 1 + c_2 0 + \cdots = c_1$$
$$y_3 - \mu = 0 + c_1 0 + c_2 1 + \cdots = c_2$$

- The IRF for this process is:

$$[1, c_1, c_2, c_3, \ldots]$$

- This extends directly to the multivariate case, except that if we have a jointly endogenous process with n shocks, there are n IRFs for each variable

IRF - derivation

- Rewrite reduced form with lag operators:

$$
\begin{aligned}
\mathbf{x}_t &= \mathbf{A}_0 + \mathbf{A}_1 \mathbf{x}_{t-1} + \ldots + \mathbf{A}_p \mathbf{x}_{t-p} + e_t \\
&= \mathbf{A}_0 + \mathbf{A}_1 L \mathbf{x}_t + \ldots + \mathbf{A}_p L^p \mathbf{x}_t + e_t \\
&= \mathbf{A}_0 + \mathbf{A}(L)\, \mathbf{x}_t + e_t \\
(\mathbf{I} - \mathbf{A}(L))\, \mathbf{x}_t &= \mathbf{A}_0 + e_t
\end{aligned}
$$

- $(\mathbf{I} - \mathbf{A}(L))$ is the inverse characteristic matrix polynomial of this process
- If the process is stationary, its inverse exists

IRF - derivation

- If the process is stationary:

$$\begin{aligned}(\mathbf{I}-\mathbf{A}(L))\,\mathbf{x}_t &= \mathbf{A}_0+e_t\\ \mathbf{x}_t &= (\mathbf{I}-\mathbf{A}(L))^{-1}(\mathbf{A}_0+e_t)\\ &= (\mathbf{I}-\mathbf{A}(L))^{-1}\mathbf{A}_0+(\mathbf{I}-\mathbf{A}(L))^{-1}e_t\\ &= (\mathbf{I}-\mathbf{A}(L))^{-1}\mathbf{A}_0+(\mathbf{I}-\mathbf{A}(L))^{-1}\mathbf{B}^{-1}\varepsilon_t\\ &= \boldsymbol{\mu}+\mathbf{C}(L)\,\varepsilon_t\end{aligned}$$

- Where:

$$\boldsymbol{\mu}=(\mathbf{I}-\mathbf{A}(1))^{-1}\mathbf{A}_0=E[\mathbf{x}_t]$$

IRF - derivation

- Unpacking the representation result:

$$\begin{aligned}
\mathbf{x}_t - \boldsymbol{\mu} &= \mathbf{C}\left(L\right)\varepsilon_t \\
&= \mathbf{C}^{(0)}\varepsilon_t + \mathbf{C}^{(1)}\varepsilon_{t-1} + \mathbf{C}^{(2)}\varepsilon_{t-2} + \ldots \\
&= \sum_{i=0}^{\infty} \mathbf{C}^{(i)}\varepsilon_{t-i} \\
&= \left(\sum_{i=0}^{\infty} \mathbf{C}^{(i)} L^i\right)\varepsilon_t
\end{aligned}$$

IRF - derivation

- $\mathbf{C}(L)$ is a convergent matrix valued polynomial in the lag operator
- What happens if we substitute L with 1?

$$\mathbf{C}(L) = \mathbf{C}^{(0)} + \mathbf{C}^{(1)}L + \mathbf{C}^{(2)}L^2 + \cdots$$
$$\mathbf{C}(1) = \mathbf{C}^{(0)} + \mathbf{C}^{(1)}1 + \mathbf{C}^{(2)}1^2 + \cdots$$

- Convergent means the [n x n] matrix:

$$\mathbf{C}(1) = \sum_{i=0}^{\infty} \mathbf{C}^{(i)}$$

has only finite valued entries

IRF - derivation

- Back to two variable case:

$$\mathbf{x}_t - \boldsymbol{\mu} = \begin{bmatrix} y_t - \mu_y \\ z_t - \mu_z \end{bmatrix} = \begin{bmatrix} c_{11}^{(0)} & c_{12}^{(0)} \\ c_{21}^{(0)} & c_{22}^{(0)} \end{bmatrix} \begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{bmatrix} + \begin{bmatrix} c_{11}^{(1)} & c_{12}^{(1)} \\ c_{21}^{(1)} & c_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \varepsilon_{y,t-1} \\ \varepsilon_{z,t-1} \end{bmatrix} + \ldots$$

- The IRF is the impact on the time paths of the endogenous variables of a sequence of shocks:
  - Let the impulse be: $\varepsilon_{y,1} = 1$ and $\varepsilon_{z,1} = \varepsilon_{z,t} = \varepsilon_{y,t} = 0 \ \forall \ t > 1$
  - Both variables respond:

$$y_1 - \mu_y = \varepsilon_{y,1} + c_{11}^{(0)} \varepsilon_{y,0} + c_{11}^{(1)} \varepsilon_{y,-1} + \cdots$$
$$z_1 - \mu_z = \varepsilon_{y,1} + c_{12}^{(0)} \varepsilon_{y,0} + c_{12}^{(1)} \varepsilon_{y,-1} + \cdots$$

IRF - derivation

- Back to two variable case:

$$\mathbf{x}_t - \boldsymbol{\mu} = \begin{bmatrix} y_t - \mu_y \\ z_t - \mu_z \end{bmatrix} = \begin{bmatrix} c_{11}^{(0)} & c_{12}^{(0)} \\ c_{21}^{(0)} & c_{22}^{(0)} \end{bmatrix} \begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{bmatrix} + \begin{bmatrix} c_{11}^{(1)} & c_{12}^{(1)} \\ c_{21}^{(1)} & c_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \varepsilon_{y,t-1} \\ \varepsilon_{z,t-1} \end{bmatrix} + \ldots$$

- The IRF is the impact on the time paths of the endogenous variables of a sequence of shocks:
  - Let the impulse be: $\varepsilon_{y,1} = 1$ and$\varepsilon_{z,1} = \varepsilon_{z,t} = \varepsilon_{y,t} = 0 \;\forall\; t > 1$

$$\left\{\boldsymbol{\varepsilon}_{t+s}^{temporary}\right\}_{s=0}^{T} = \begin{bmatrix} 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \end{bmatrix}$$

$$E_t\left[\left\{\mathbf{x}_{t+s} - \boldsymbol{\mu}\right\}_{s=0}^{T} \middle| \left\{\boldsymbol{\varepsilon}_{t}^{temporary}\right\}\right] = \begin{bmatrix} c_{11}^{(0)} & c_{11}^{(1)} & c_{11}^{(2)} & \ldots \\ c_{21}^{(0)} & c_{21}^{(1)} & c_{21}^{(2)} & \ldots \end{bmatrix}$$

Two distinct processes:

- Process 1
  - No contemporaneous effect of x_{2t} on x_{1t}:

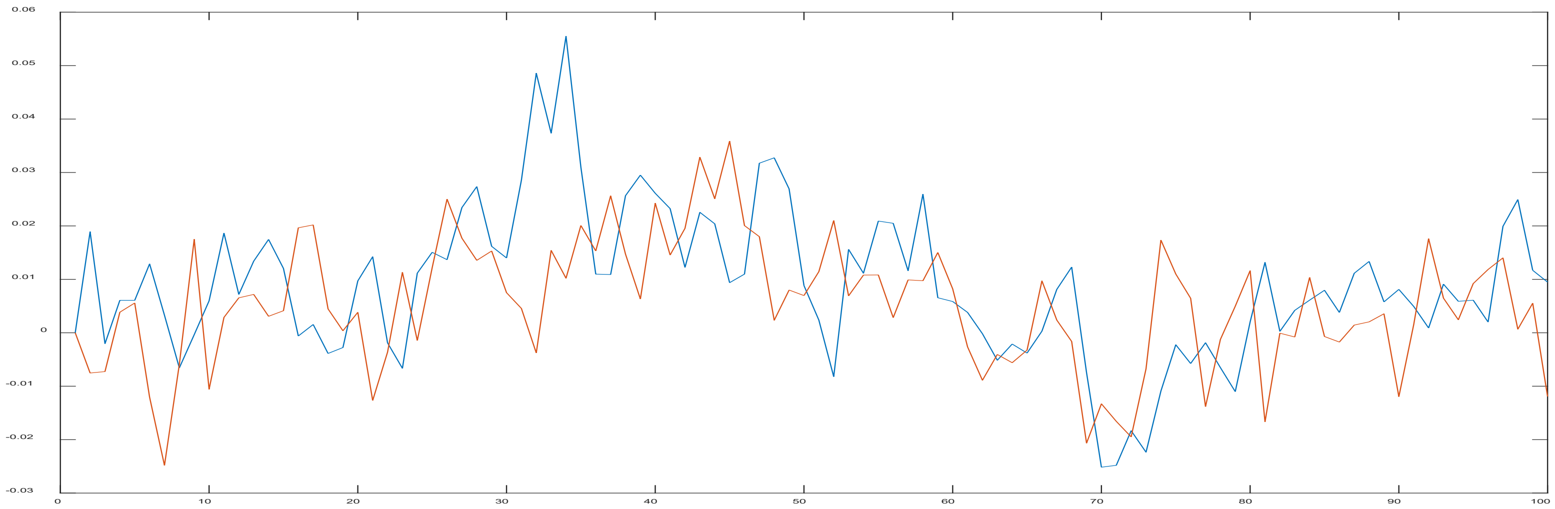
$$\begin{bmatrix} 1 & 0 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.2 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

- Process 2
  - Mutual contemporaneous effects:

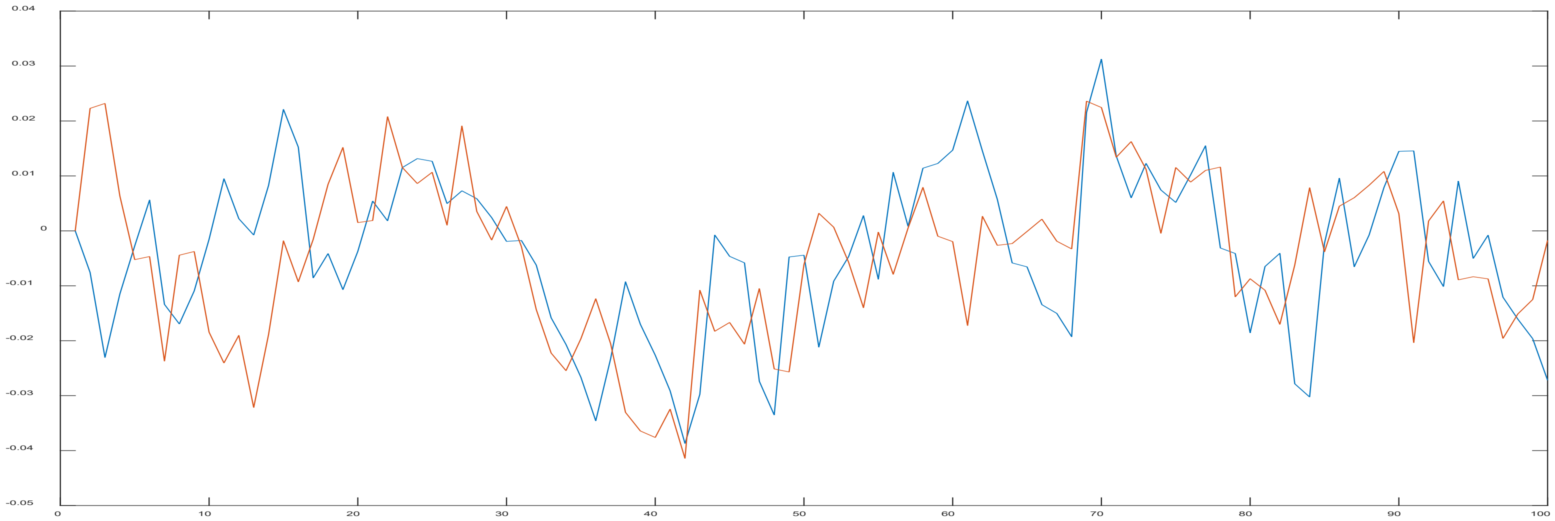
$$\begin{bmatrix} 1 & 0.1 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.2 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

Two distinct processes:

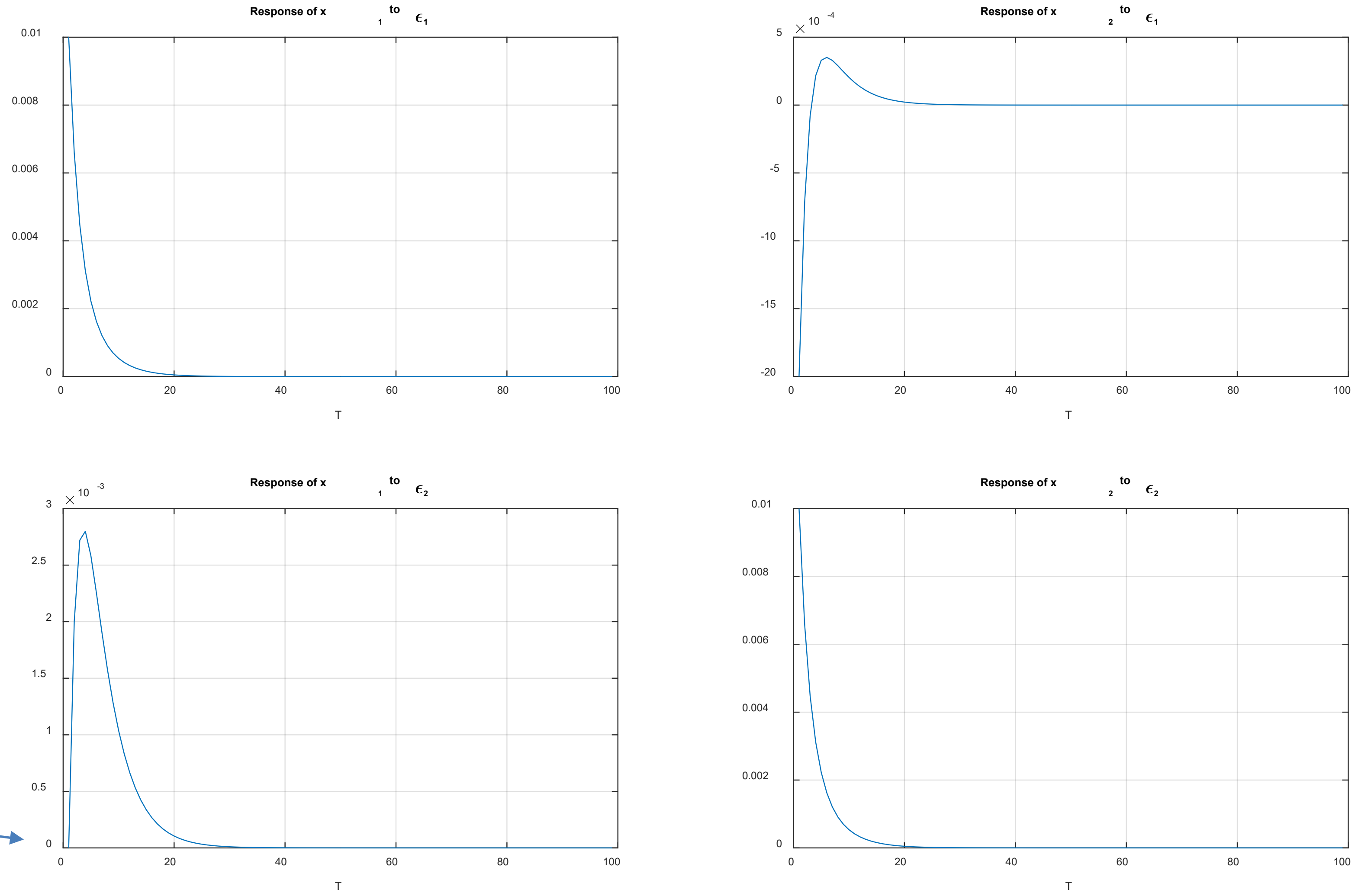
Process 1



Process 2

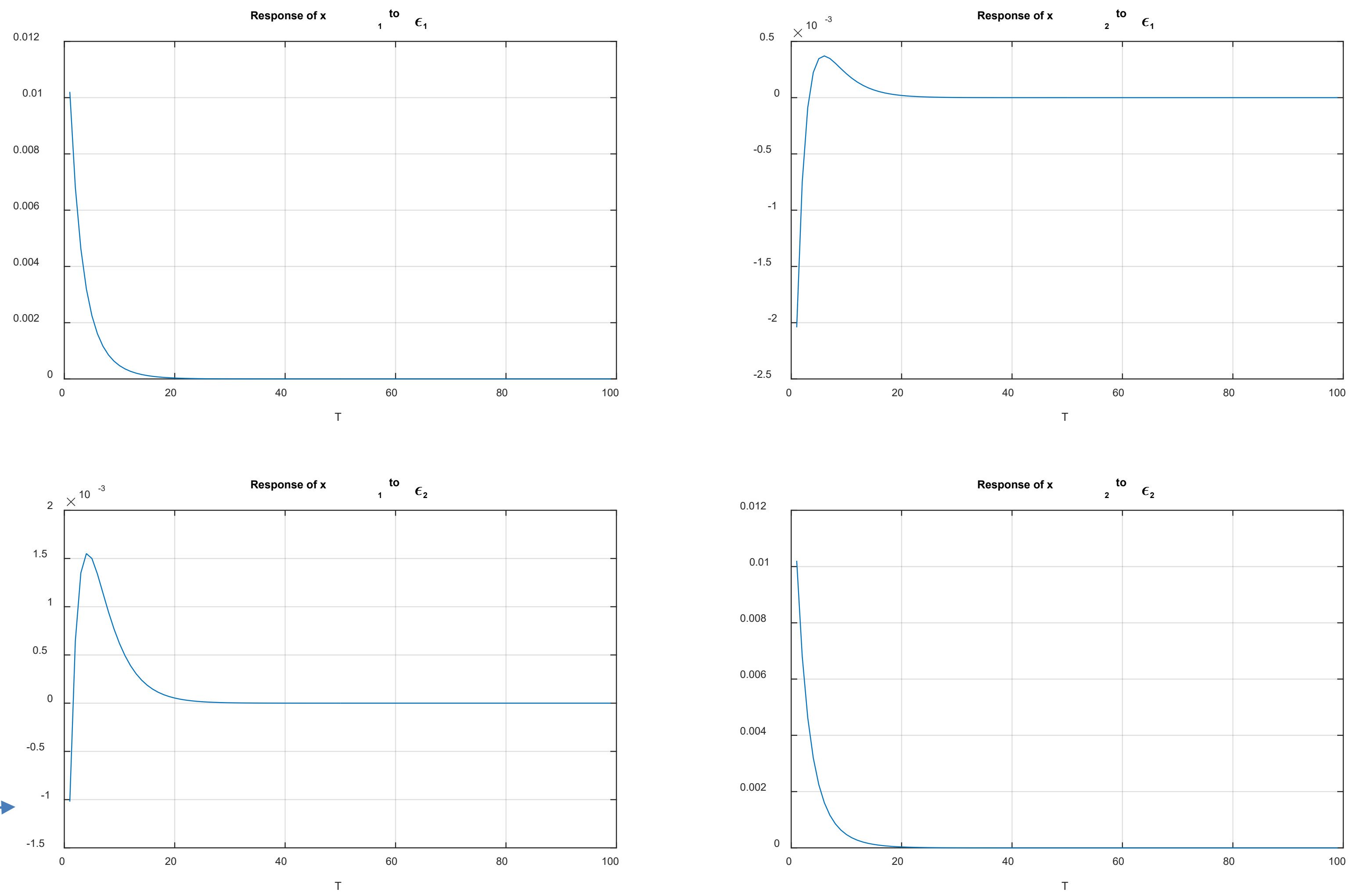


Theoretical IRF of Process 1: $\begin{bmatrix} 1 & 0 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.2 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$



Note: no immediate response of x_{1t} to ε_{2t}

Theoretical IRF of Process 2: $\begin{bmatrix} 1 & 0.1 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.2 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$



Note: immediate response of x_{1t} to ε_{2t}

Two versions of impulse responses

- Since we are considering stationary VARs, all processes must eventually go back to their respective expected values
  - This means that IRFs from a once off shock must return to zero
  - This provides a visual "test" – if the impulse does not return to zero fast enough, suspect non-stationarity
- We might also be interested in the cumulative effect of a shock
  - Then we can construct the **cumulative IRF**
  - Equivalent to an impulse response to a permanent step change
  - If the IRF is $[1, c_1, c_2, c_3, \ldots]$,
    the cumulative IRF is $[1, 1 + c_1, 1 + c_1 + c_2, 1 + c_1 + c_2 + c_3, \ldots]$

Summarizing the information in a VAR

- Impulse Response functions
- **Variance Decompositions**
  - A more complete name is *Forecast Error Variance Decomposition*
  - Unlike the impulse response function, the variance decomposition is not interesting in the univariate case
  - In the multivariate case this is interesting:
    - Given that a set of variables are jointly endogenous, the uncertainty in forecasts in an individual variable is due to *all* the shocks in the system of equations
    - The variance decomposition determines what proportion of the uncertainty is due to each of the shocks for each forecast horizon

Variance Decomposition

- Consider the process in period $t + h$ in VMA form:

$$\mathbf{x}_{t+h} = \boldsymbol{\mu} + \sum_{i=0}^{\infty} \mathbf{C}^{(i)} \boldsymbol{\varepsilon}_{t+h-i}$$

- The Expected value conditional on period t information is:

$$E_t(\mathbf{x}_{t+h}) = \boldsymbol{\mu} + \sum_{i=h}^{\infty} \mathbf{C}^{(i)} \boldsymbol{\varepsilon}_{t+h-i}$$

- Note that the only difference is in the indices of the summation
  - The expected value of future shocks is zero
  - If these equations are not obvious to you, set h=1 and write out the summation

Variance Decomposition

- This defines the n-period ahead **Forecast Error**:

$$\mathbf{x}_{t+h} - E_t(\mathbf{x}_{t+h}) = \sum_{i=0}^{h-1} \mathbf{C}^{(i)} \boldsymbol{\varepsilon}_{t+h-i}$$

- Extracting the row that corresponds to y:

$$y_{t+h} - E_t(y_{t+h}) = c_{11}^{(0)} \varepsilon_{y,t+h} + c_{11}^{(1)} \varepsilon_{y,t+h-1} + \cdots + c_{11}^{(h-1)} \varepsilon_{y,t+1} + c_{12}^{(0)} \varepsilon_{z,t+h} + c_{12}^{(1)} \varepsilon_{z,t+h-1} + \cdots + c_{12}^{(h-1)} \varepsilon_{z,t+1}$$

- Note: uncertainty grows as we go further into the future

Variance Decomposition

- The **variance decomposition** is the proportions of this uncertainty that is due to each structural shock
- The h-period ahead total forecast error variance of the y process is:

$$E_t[y_{t+h} - E_t(y_{t+h})]^2 = \sigma_y^2\left[\left(c_{11}^{(0)}\right)^2 + \left(c_{11}^{(1)}\right)^2 + \cdots + \left(c_{11}^{(h-1)}\right)^2\right] +$$
$$\sigma_z^2\left[\left(c_{12}^{(0)}\right)^2 + \left(c_{12}^{(1)}\right)^2 + \cdots + \left(c_{12}^{(h-1)}\right)^2\right]$$

Variance Decomposition

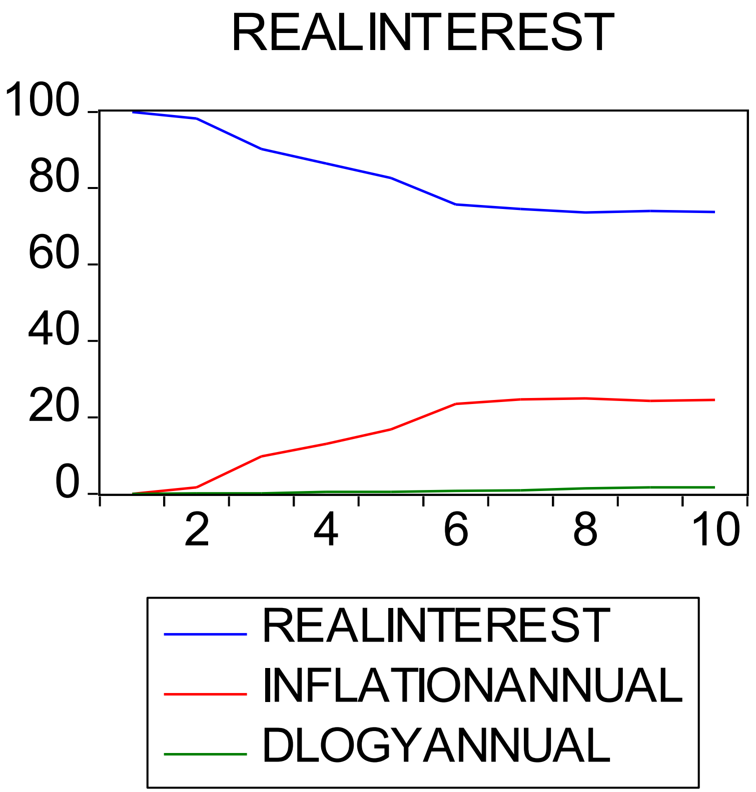
- Thus we can decompose the total forecast error variance into two parts:
  - Due to the structural shock to y:

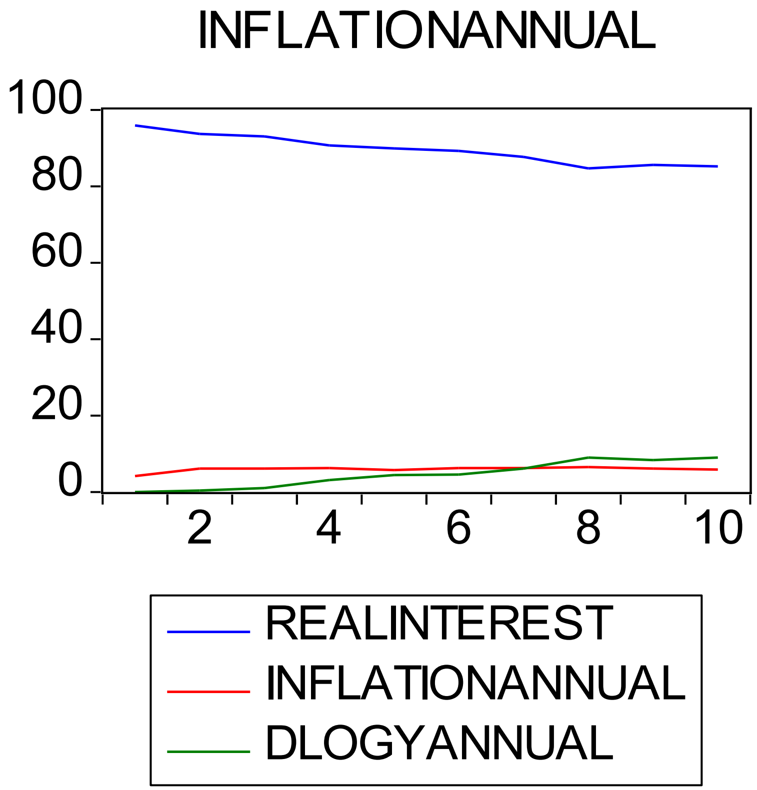
$$\frac{\sigma_y^2\left[\left(c_{11}^{(0)}\right)^2+\left(c_{11}^{(1)}\right)^2+\cdots+\left(c_{11}^{(h-1)}\right)^2\right]}{E_t[y_{t+h}-E(y_{t+h})]^2}$$

  - Due to the structural shock to z:

$$\frac{\sigma_z^2\left[\left(c_{12}^{(0)}\right)^2+\left(c_{12}^{(1)}\right)^2+\cdots+\left(c_{12}^{(h-1)}\right)^2\right]}{E_t[y_{t+h}-E(y_{t+h})]^2}$$

Example:


REALINTEREST
100
80
60
40
20
0
2
4
6
8
10
REALINTEREST
INFLATIONANNUAL
DLOGYANNUAL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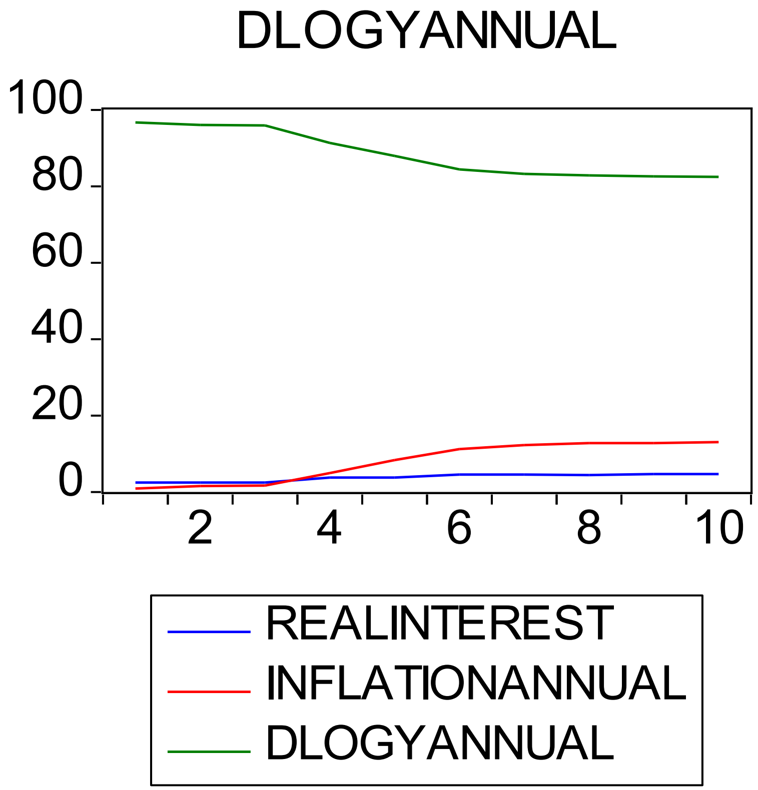
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INFLATIONANNUAL
DLOGYANNUAL

Plan

- Vector Autoregression
  - Structural vs Reduced form eq'ns
  - The Identification Problem
  - Stationarity
  - **Analyzing the information in a VAR**
  - Estimation Methods
  - Model Adequacy
  - Back to Identification – Various approaches

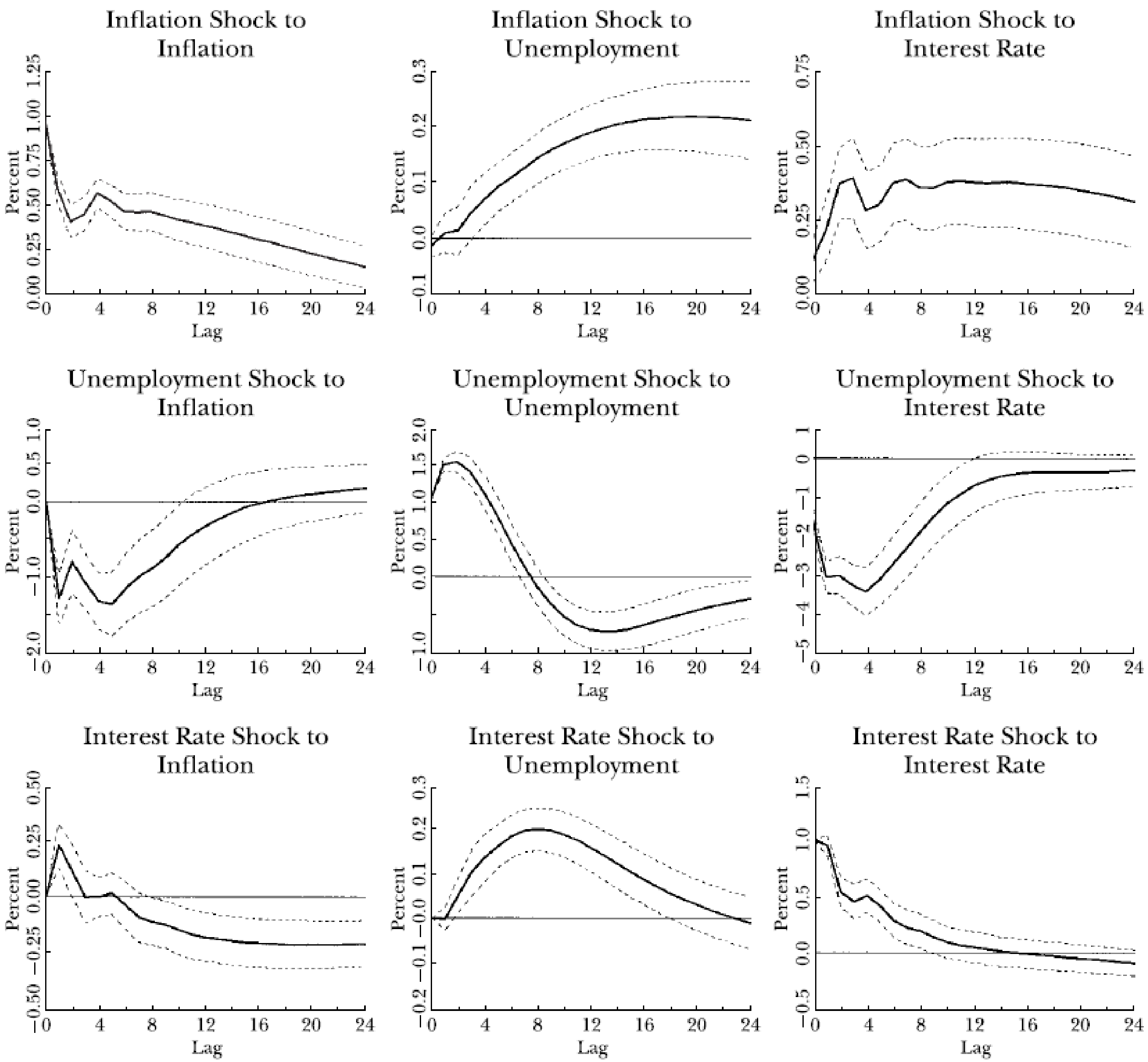
Summary so far

- A VAR is used to analyse the dynamic relationships between a set of jointly determined variables
- Monetary policy example:
  - how well does interest rate changes do in controlling inflation?
    - For this we need to model the relationship between prices, output/unemployment and interest rates
      - Production over potential/optimal (or unemployment below “natural rate”) tends to drive prices up
      - Higher interest rates tend to reduce demand and hence mitigate price pressures
- In the tutorial, we will attempt to replicate some results in the the following paper:

  Stock, J.H. and Watson, M.W., 2001. Vector autoregressions. *Journal of Economic perspectives*, *15*(4), pp.101-115.

Figure 1

Impulse Responses in the Inflation-Unemployment-Interest Rate Recursive VAR



Summary so far

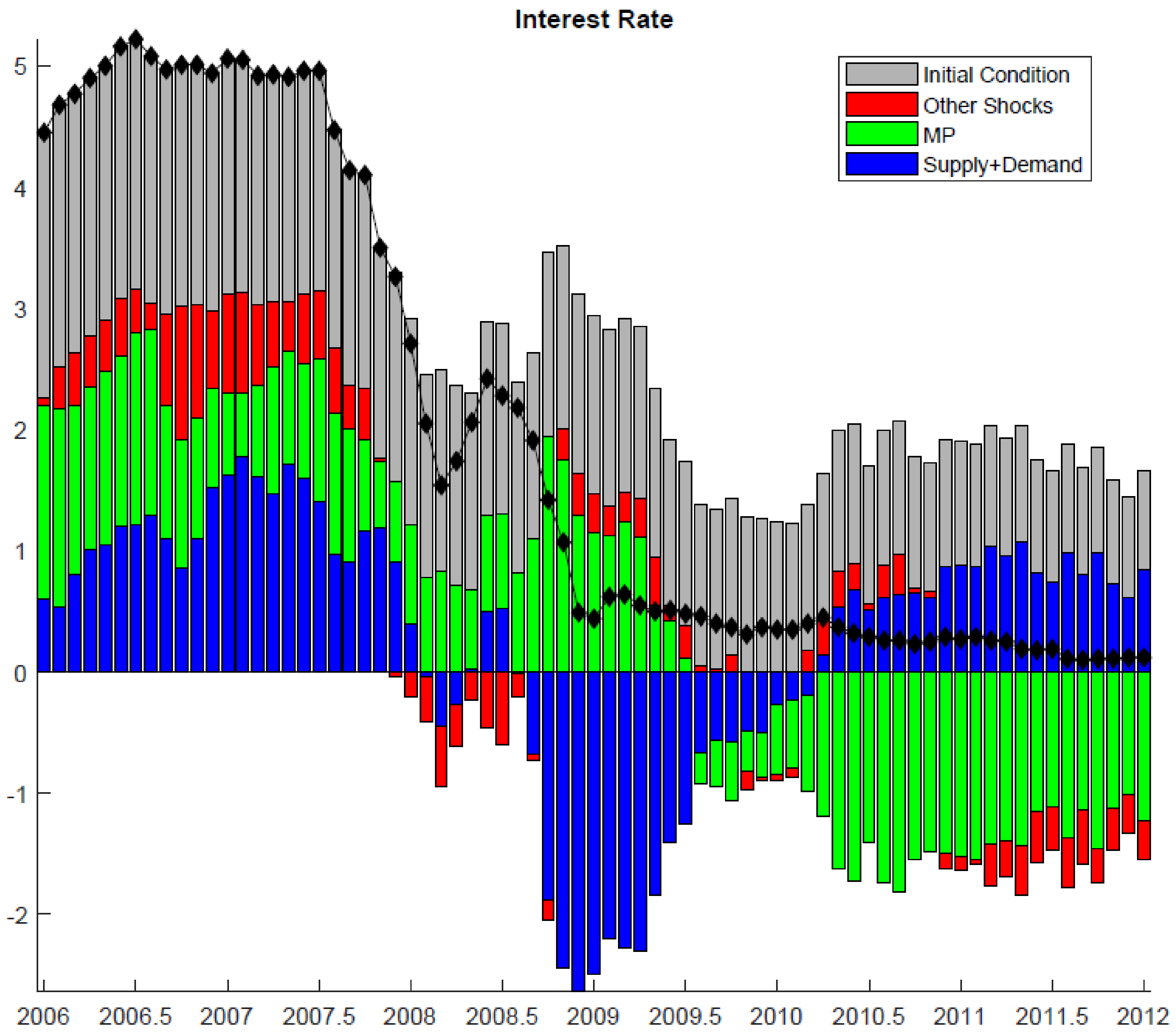
- A VAR is used to analyse the dynamic relationships between a set of jointly determined variables
  - We would like to find out what the impact a “pure” shock is
  - I.e. distinguish monetary policy shocks from supply and demand shocks
- If we allow contemporaneous effects, the most general model we can estimate is a restriction of the most general theoretical model
  - To recover the pure structural shocks, we must impose **identification assumptions** on the structural model
  - There are various different approaches, each designed for a specific purpose
- We analyse/summarize the information in an estimated VAR using:
  - **Impulse response functions:** time paths of deviations from expected value for each variable of interest after a shock to one or more of the variables in the system
  - **Forecast error variance decomposition**: describes what proportion of the forecast uncertainty is due to each of the shocks for each forecast horizon
  - **Historical decompositions**: how much of the observed variation in a variable can be attributedto each of the estimated structural shocks over time

Summary so far

- For the results to be reliable, the estimated VAR model must be a valid representation of the underlying Data Generating Process
- If the process is postulated to be stationary, the estimated VAR must satisfy the stationarity conditions
  - The eigenvalues/characteristic roots of the VAR in companion form must be smaller than 1 in absolute value
  - EQUIVALENTLY: the roots of the *inverse* characteristic equation of the VAR must be larger than 1 in absolute value
- Additionally, the VAR must be congruent with the data generating process
  - This means that all residuals must be white noise
- As we must estimate many parameters, the VAR should also be as parsimonious (small) as possible so that the coefficient estimates are as precise as possible

Analysing the information in a VAR

- Impulse response functions:
  – the time path of an endogenous variable (i.t.o. deviation from expected value) in response to an innovation/shock
- Variance decompositions:
  – How much of the forecast error variance of each variable is due to each innovation/shock at different horizons
- Historical decompositions:
  – how much of the observed variation in a variable can be attributed, in each period, to each of the estimated structural shocks



Interest Rate
Initial Condition
Other Shocks
MP
Supply+Demand
5
4
3
2
1
0
-1
-2
2006
2006.5
2007
2007.5
2008
2008.5
2009
2009.5
2010
2010.5
2011
2011.5
2012

Historical Decompositions

- Goal: split the observed variation in a variable into components due to the different innovations in the system
- Start with the reduced form of the VAR(1) and recall the VMA representation:

$$
\begin{aligned}
\mathbf{x}_t &= A_0 + A_1 \mathbf{x}_{t-1} + \boldsymbol{e}_t \\
\mathbf{x}_t &= A_0 + A_1 L \mathbf{x}_t + \boldsymbol{e}_t \\
(I - A_1 L)\, \mathbf{x}_t &= A_0 + \boldsymbol{e}_t \\
\mathbf{x}_t &= (I - A_1 L)^{-1} A_0 + (I - A_1 L)^{-1} \boldsymbol{e}_t \\
&= (I - A_1 L)^{-1} A_0 + (I - A_1 L)^{-1} B^{-1} \boldsymbol{\varepsilon}_t \\
&= \boldsymbol{\mu} + C(L)\, \boldsymbol{\varepsilon}_t
\end{aligned}
$$

- Or in summation form:

$$
\begin{aligned}
\mathbf{x}_t - \boldsymbol{\mu} &= C(L)\,\boldsymbol{\varepsilon}_t \\
&= C^{(0)}\boldsymbol{\varepsilon}_t + C^{(1)}\boldsymbol{\varepsilon}_{t-1} + C^{(2)}\boldsymbol{\varepsilon}_{t-2} + \ldots \\
&= \begin{bmatrix} c_{11}^{(0)} & c_{12}^{(0)} \\ c_{21}^{(0)} & c_{22}^{(0)} \end{bmatrix} \begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{bmatrix} + \begin{bmatrix} c_{11}^{(1)} & c_{12}^{(1)} \\ c_{21}^{(1)} & c_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \varepsilon_{y,t-1} \\ \varepsilon_{z,t-1} \end{bmatrix} + \ldots \\
&= \sum_{i=0}^{\infty} C^{(i)}\boldsymbol{\varepsilon}_{t-i}
\end{aligned}
$$

- Now consider the last value of our sample:

$$\mathbf{x}_T - \boldsymbol{\mu} = \underbrace{\sum_{i=0}^{T-2} C^{(i)} \boldsymbol{\varepsilon}_{T-i}}_{\text{In observed sample}} + \underbrace{\sum_{i=T-1}^{\infty} C^{(i)} \boldsymbol{\varepsilon}_{T-i}}_{\text{Before the observed sample – no information}}$$

In observed sample
- We can get estimates of the structural innovations (given an identification scheme)

Before the observed sample – no information

Equivalent to initial conditions

Or: $\mathbf{x}_1 - \mu$

- Now consider the last value of our sample:

$$\mathbf{x}_T - \boldsymbol{\mu} \quad = \quad \sum_{i=0}^{T-2} C^{(i)} \boldsymbol{\varepsilon}_{T-i} + \sum_{i=T-1}^{\infty} C^{(i)} \boldsymbol{\varepsilon}_{T-i}$$

$$\mathbf{x}_T - \mathbf{x}_1 \quad = \quad \sum_{i=0}^{T-2} C^{(i)} \boldsymbol{\varepsilon}_{T-i}$$

- I.e. we can express the deviation of the variables from their initial conditions as a weighted sum of (estimated) innovations

Empirical Historical Decomposition

- We can now consider any variable at any date in our system:

$$y_2 - y_1 = \widehat{c}_{11}^{(0)}\widehat{\varepsilon}_{y,2} + \widehat{c}_{12}^{(0)}\widehat{\varepsilon}_{z,2}$$

$$y_3 - y_1 = \widehat{c}_{11}^{(0)}\widehat{\varepsilon}_{y,3} + \widehat{c}_{12}^{(0)}\widehat{\varepsilon}_{z,3} + \widehat{c}_{11}^{(1)}\widehat{\varepsilon}_{y,2} + \widehat{c}_{12}^{(1)}\widehat{\varepsilon}_{z,2}$$

$$y_4 - y_1 = \widehat{c}_{11}^{(0)}\widehat{\varepsilon}_{y,4} + \widehat{c}_{12}^{(0)}\widehat{\varepsilon}_{z,4} + \widehat{c}_{11}^{(1)}\widehat{\varepsilon}_{y,3} + \widehat{c}_{12}^{(1)}\widehat{\varepsilon}_{z,3} + \widehat{c}_{11}^{(2)}\widehat{\varepsilon}_{y,2} + \widehat{c}_{12}^{(2)}\widehat{\varepsilon}_{z,2}$$

- Collecting terms according to the different structural shocks:

$$y_4 - y_1 = \underbrace{\left[\widehat{c}_{11}^{(0)}\widehat{\varepsilon}_{y,4} + \widehat{c}_{11}^{(1)}\widehat{\varepsilon}_{y,3} + \widehat{c}_{11}^{(2)}\widehat{\varepsilon}_{y,2}\right]}_{\text{Cumulative impact of y shock on y variable by period 4}} + \underbrace{\left[\widehat{c}_{12}^{(0)}\widehat{\varepsilon}_{z,4} + \widehat{c}_{12}^{(1)}\widehat{\varepsilon}_{z,3} + \widehat{c}_{12}^{(2)}\widehat{\varepsilon}_{z,2}\right]}_{\text{Cumulative impact of z shock on y variable by period 4}}$$

Plan

- Vector Autoregression
  - Structural vs Reduced form eq’ns
  - The Identification Problem
  - Stationarity
  - Analyzing the information in a VAR
  - **Estimation Methods**
  - Evaluation of Fit
  - Back to Identification – Various approaches

Estimating a VAR

- A VAR is readily estimable via OLS – linear in parameters
- Consider the generic VAR(p) process:

$$\mathbf{x}_t = A_1\mathbf{x}_{t-1} + \cdots + A_p\mathbf{x}_{t-p} + \boldsymbol{e}_t$$

- Suppose we have T+p observations on each of the n variables
- Define:

$$\underset{[np\times 1]}{\mathbf{X}_t} = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{x}_{t-1} \\ \vdots \\ \mathbf{x}_{t-p+1} \end{bmatrix}$$

$$\underset{[n\times T]}{Y} = \begin{bmatrix} \mathbf{x}_1, & \ldots, & \mathbf{x}_T \end{bmatrix}$$

$$\underset{[np\times T]}{Z} = \begin{bmatrix} \mathbf{X}_0, & \ldots, & \mathbf{X}_{T-1} \end{bmatrix}$$

$$\underset{[n\times np]}{A} = \begin{bmatrix} A_1, & \ldots, & A_p \end{bmatrix}$$

$$\underset{[n\times T]}{U} = \begin{bmatrix} \boldsymbol{e}_1, & \ldots, & \boldsymbol{e}_T \end{bmatrix}$$

Estimating a VAR

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$$\underset{[np\times 1]}{\mathbf{X}_t} = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{x}_{t-1} \\ \vdots \\ \mathbf{x}_{t-p+1} \end{bmatrix}$$

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$$\underset{[n\times np]}{A} = \begin{bmatrix} A_1, & \dots, & A_p \end{bmatrix}$$

$$\underset{[n\times T]}{U} = \begin{bmatrix} e_1, & \dots, & e_T \end{bmatrix}$$

$$Y = AZ + U$$

$$\widehat{A} = YZ'\left(ZZ'\right)^{-1}$$

Estimating a VAR

- A VAR is linear in parameters and variables, so can easily be estimated by OLS
  - Although many implementations use Maximum Likelihood
- Hence the estimator has the standard properties under suitable conditions:
  - Consistent (why not unbiased?)
  - Asymptotically normal

Plan

- Vector Autoregression
  - Structural vs Reduced form eq'ns
  - The Identification Problem
  - Stationarity
  - Analyzing the information in a VAR
  - Estimation Methods
  - **Evaluation of Fit**
  - Back to Identification – Various approaches

Evaluating Fit

- As in the univariate case, an adequate model must have white noise residuals
  - Check autocorrelation functions of residuals
  - Tests for serial correlation in residuals

Which Variables belong in a VAR?

- Up to now we've assumed there is some given simultaneous system we need to uncover.
  - Theory would suggest the variables

- How do we know which variables belong in a VAR?

- Standard tool:
  - Granger Causality
  - Block Exogeneity test

Which Variables belong in a VAR?

- Economic Theory is usually where we start
  - But macro models can get very large
  - Imagine simultaneously testing Purchasing Price Parity, Uncovered Interest Parity and the Term Structure hypotheses
  - This leads to the curse of dimensionality:
  - Suppose we have 5 variables and 4 lags
    - 100 slope coefficients:
    - 5 constants
    - 15 variance/covariance terms

$$\underset{[n \times np]}{A} = \left[\ A_1, \quad \ldots, \quad A_p\ \right]$$

  - Curse of Dimensionality: require large data sets for any kind of accurate estimate
  - Typically: use a limited number of variables and lags to answer a narrow question

Which Variables belong in a VAR?

- Other approaches to deal with dimensionality problem
  - Bayesian VARs (to some extent)
  - Factor Augmented VARs

- FAVARs augment a VAR with Factors extracted from a large dataset
  - Factors can be estimated as the first principle components of a large data set, after which the VAR is estimated
  - Or jointly estimated with VAR coefficients using Bayesian methods
  - Bernanke, Boivin and Eliasz (2004) augment a basic monetary VAR with the factors extracted from 120 disaggregated variables

Which Variables belong in a VAR?

- Given a set of variables that represent some economic process of interest, we are also interested if the variables *empirically* belong in the model of the DGP

- Standard tool:
  - Granger Causality
  - Block Exogeneity test

Granger Causality Test

- This is a simple F test:

$$\mathbf{x}_t = A_0 + A(L)\,\mathbf{x}_{t-1} + \boldsymbol{e}_t$$

$$\begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} a_{01} \\ a_{02} \end{bmatrix} + \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \ldots + \begin{bmatrix} a_{11}^{(p)} & a_{12}^{(p)} \\ a_{21}^{(p)} & a_{22}^{(p)} \end{bmatrix} \begin{bmatrix} y_{t-p} \\ z_{t-p} \end{bmatrix} + \begin{bmatrix} e_{y,t} \\ e_{z,t} \end{bmatrix}$$

The null-hypothesis that z_t does not Granger-cause y_t is:

$$H_0:\ a_{12}^{(1)} = a_{12}^{(2)} = \ldots = a_{12}^{(p)} = 0$$

Correct lag length of a VAR?

- Even given that we know the variables to include, what lag length should we use?
  - More = better in sample fit, imprecise coefficients
  - Less = worse in sample fit, more precise coefficients, more precise forecasts.

- Lag exclusion tests
  - F tests of the exclusion of “Furthest lag” from **all** equations

- Lag Length Criteria
  - Compare alternatives with measures that penalizing larger models
    - AIC, SBC/BIC, Hannan-Quin, Maximized Likelihood,
  - Or based on forecast performance

- Excluding arbitrary lags from **individual** equations
  - Only if VERY good economic reasons, say if all subsamples have an insignificant coefficient
  - I haven’t encountered such “good reasons” yet

Plan

- Vector Autoregression
  - Structural vs Reduced form eq'ns
  - The Identification Problem
  - Stationarity
  - Analyzing the information in a VAR
  - Estimation Methods
  - Evaluation of Fit
  - **Back to Identification – Various approaches**

Identification

- We need at least $\frac{n^2-n}{2}$ restrictions to identify the structural innovations/IRFs in an n-variable VAR from the reduced form
- If we impose exactly $\frac{n^2-n}{2}$ restrictions, the system is called *just-identified.*
  - The estimated reduced form is *not* affected by this. Only the mapping between the structural and reduced form.
- If we impose more than $\frac{n^2-n}{2}$ restrictions, the system is called *over-identified.*
  - The estimated reduced form *is* affected by this. As we reduce the number of coefficients, the fit will be worse
  - We can evaluate whether the restriction is statistically supported by checking whether the fit falls statistically significantly. (F test)

Structural VAR

- As soon as we use economic theory to restrict parameters in a VAR, it becomes a “structural VAR”

- Economic theory might suggest *more* than the necessary identification restrictions
  - Additional restrictions will worsen the in-sample fit
  - The degree to which this happens can be used as a test of the additional restrictions
  - These are commonly called *over-identifying* restrictions

Choleski Decomposition

- In the unrestricted primitive form,

$$Bx_t = \Gamma_0 + \Gamma_1 x_{t-1} + \cdots + +\Gamma_p x_{t-p} + \varepsilon_t$$

- The Choleski decomposition implies choosing a temporal ordering:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -b_{21} & 1 & 0 & \cdots & 0 \\ -b_{31} & -b_{32} & 1 & \vdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & 0 \\ -b_{n1} & -b_{n2} & \cdots & -b_{n,n-1} & 1 \end{bmatrix}$$

Variance Restrictions

- We have shown:

$$\begin{aligned} \boldsymbol{e}_t &= C^{(0)}\boldsymbol{\varepsilon}_t \\ &= \begin{bmatrix} c_{11}^{(0)} & c_{12}^{(0)} \\ c_{21}^{(0)} & c_{22}^{(0)} \end{bmatrix} \begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{bmatrix} \end{aligned}$$

- From this we can obtain three (non-linear) equations in 4 unknowns

Variance Restrictions

- From this we can obtain three (non-linear) equations in 4 unknowns

$$
\begin{aligned}
E\left(\boldsymbol{e}_t \boldsymbol{e}_t'\right) &= E\left(C^{(0)} \boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t' C^{(0)\prime}\right) \\
\begin{bmatrix} \sigma_{e_1}^2 & \sigma_{e_1 e_2} \\ \sigma_{e_1 e_2} & \sigma_{e_2}^2 \end{bmatrix} &= C^{(0)} I C^{(0)\prime} \\
&= C^{(0)} C^{(0)\prime}
\end{aligned}
$$

In this identification scheme, we can know nothing about the primitive variances, so we can just set them to 1

Variance Restrictions

- From this we can obtain three (non-linear) equations in 4 unknowns

$$
\begin{aligned}
\sigma_{e_1}^2 &= \left(c_{11}^{(0)}\right)^2 + \left(c_{12}^{(0)}\right)^2 \\
\sigma_{e_2}^2 &= \left(c_{21}^{(0)}\right)^2 + \left(c_{22}^{(0)}\right)^2 \\
\sigma_{e_1 e_2} &= c_{11}^{(0)} c_{21}^{(0)} + c_{12}^{(0)} c_{22}^{(0)}
\end{aligned}
$$

Variance Restrictions

- From this we can obtain three (non-linear) equations in 4 unknowns

- An additional restriction will allow us (potentially) to recover the C(0) coefficients and hence the structural shocks

- Rarely used in isolation in practice = no strong ex ante reason to restrict variances.

Other Identification Approaches:

- Blanchard-Quah decomposition
- Pesaran and Shin’s Generalized IRF
- Uhlig’s Sign Restrictions on IRF

- First an understanding check: What will the impact of different choices for a *just-identified* system be on forecasting?

Blanchard – Quah Identification Strategy

- Extends the Nelson and Plosser (1982) exercise that split the variation in GDP in permanent and temporary components
  - Showed that most of the variation can be explained by permanent shocks
  - Beginning of the Real Business Cycle literature
  - No unique way of doing this
- Blanchard and Quah (1989) use variance relations combined with a long run impact restrictions to identify a permanent and a temporary innovation

Blanchard – Quah Identification Strategy

- Blanchard and Quah (1989) long run impact restrictions:
- Recall that a stationary VAR is equivalent to a convergent series of coefficients in the VMA representation:

$$\mathbf{C}(1) = \sum_{i=0}^{\infty} \mathbf{C}^{(i)}$$

- The elements of this matrix may still be positive (or negative) so that a sequence of innovations may have a permanent cumulative effect
- The BQ approach involves setting one or more of the entries in $\mathrm{C}(1)$ to **zero**
  - Which implies the assumption that of these shocks have no permanent effect on the relevant variables

Blanchard – Quah Identification Strategy

- Blanchard and Quah (1989) setup:
- Endogenous Variables:

Log real GDP - y_t - unit root, non stationary

Growth in real GDP - Δy_t stationary

Unemployment - u_t stationary

- Structural Innovations:

Demand shock - $\varepsilon_{d,t}$ - temporary effects on both variables

Supply shock - $\varepsilon_{s,t}$ - permanent effect on y_t, temporary effect on u_t

Blanchard – Quah Identification Strategy

- Blanchard and Quah (1989) setup:
- VAR in VMA form:

$$\mathbf{x}_t - \mu = \begin{bmatrix} \Delta y_t - \mu_{\Delta y} \\ u_t - \mu_u \end{bmatrix} = \begin{bmatrix} c_{\Delta y,d}^{(0)} & c_{\Delta y,s}^{(0)} \\ c_{u,d}^{(0)} & c_{u,s}^{(0)} \end{bmatrix} \begin{bmatrix} \varepsilon_{d,t} \\ \varepsilon_{s,t} \end{bmatrix} + \begin{bmatrix} c_{\Delta y,d}^{(1)} & c_{\Delta y,s}^{(1)} \\ c_{u,d}^{(1)} & c_{u,s}^{(1)} \end{bmatrix} \begin{bmatrix} \varepsilon_{d,t-1} \\ \varepsilon_{s,t-1} \end{bmatrix} + \ldots$$

- Demand shock has no long run impact for a given sequence of shocks on y if:

$$\sum_{k=0}^{\infty} c_{\Delta y,d}^{(k)} \varepsilon_{d,t-k} = 0$$

- For all possible sequences of shocks:

$$\sum_{k=0}^{\infty} c_{\Delta y,d}^{(k)} = 0$$

Pesaran and Shin’s generalized IRF

Pesaran and Shin's generalized IRF

- Recall our original primitive model:

$$\underset{[n\times n]}{\mathbf{B}}\underset{[n\times 1]}{\mathbf{x}_t} = \underset{[n\times 1]}{\mathbf{\Gamma}_0} + \underset{[n\times n]}{\mathbf{\Gamma}_1}\underset{[n\times 1]}{\mathbf{x}_{t-1}} + \ldots + \mathbf{\Gamma}_p\mathbf{x}_{t-p} + \varepsilon_t \qquad E\left[\varepsilon_t\varepsilon_t'\right] = \underset{[n\times n]}{\mathbf{\Sigma}_\varepsilon}$$

  - Where:
    - B is unrestricted (non-diagonal, non-symmetric)
    - Σ_ε is diagonal (pure uncorrelated innovations)

- This is not directly estimable, and lead to the reduced form:

$$\underset{[n\times 1]}{\mathbf{x}_t} = \underset{[n\times 1]}{\mathbf{A}_0} + \underset{[n\times n]}{\mathbf{A}_1}\underset{[n\times 1]}{\mathbf{x}_{t-1}} + \ldots + \mathbf{A}_p\mathbf{x}_{t-p} + \boldsymbol{e}_t \qquad E\left[\mathbf{e}_t\mathbf{e}_t'\right] = \underset{[n\times n]}{\mathbf{\Sigma}}$$

- Where Σ is non-diagonal but symmetric, and "identification" was about recovering the B matrix

Pesaran and Shin's generalized IRF

- Pesaran and Shin start with the primitive model:

$$\mathbf{x}_t = \Gamma_0 + \Gamma_1 \mathbf{x}_{t-1} + \ldots + \Gamma_p \mathbf{x}_{t-p} + \varepsilon_t$$

$$E\left[\varepsilon_t \varepsilon_t'\right] = \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} & \cdots & \sigma_{2n} \\ \sigma_{12} & \sigma_{23} & \sigma_{33} & \cdots & \sigma_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \sigma_{3n} & \cdots & \sigma_{nn} \end{bmatrix}$$

$$E\left[\varepsilon_t \varepsilon_{t+s}'\right] = 0 \forall s \neq 0$$

  - Implying:
    - B is restricted – assumed to be identity matrix
    - Σ_ε is positive definite, symmetric
      - Fundamental errors may be correlated

Pesaran and Shin’s generalized IRF

- This means that there can be no “pure impulse” to one innovation. Since they are correlated, a shock to one is a shock to all

Pesaran and Shin’s generalized IRF

- Evaluation:
  – Using Pesaran and Shin’s definition allows mutual “contemporaneous” effects in a VAR
  – However, these effects are all via the estimated Var-Covar of the residuals
  – Thus necessarily symmetric
  – Cannot identify fundamental contemporaneous feedback that might be asymmetric
  – This is identification of a sort, but different from the program we began with.

Uhlig’s Sign Restricted IRF

- Motivation:
  - Sims’ unrestricted VAR approach lead to empirical “puzzles”
    - Price Puzzle: prices tend to rise after contractionary monetary policy
    - This is a “puzzle” because it does not fit theory
  - even with Cholesky ordering, some “implicit” theorizing is always done
  - Residuals from reduced form VAR estimate are always a linear combination of structural innovations

- Do the theorizing explicitly to impose “what we really think we know”
  - Force (some) IRFs to go in the “right” direction
  - Test the model by leaving IRFs of unknown response (theoretically) unrestricted.

Uhlig’s Sign Restricted IRF

- Operationalized Method:
  - Brute force simulation
  - Draw (random) impulse vectors for restricted responses, impose sign restrictions
  - Fit unrestricted impulse response functions
  - Repeat 10000 times
  - Plot distribution of impulse response functions
- Details beyond this course
  - Uhlig shows that the asymptotics are reliable

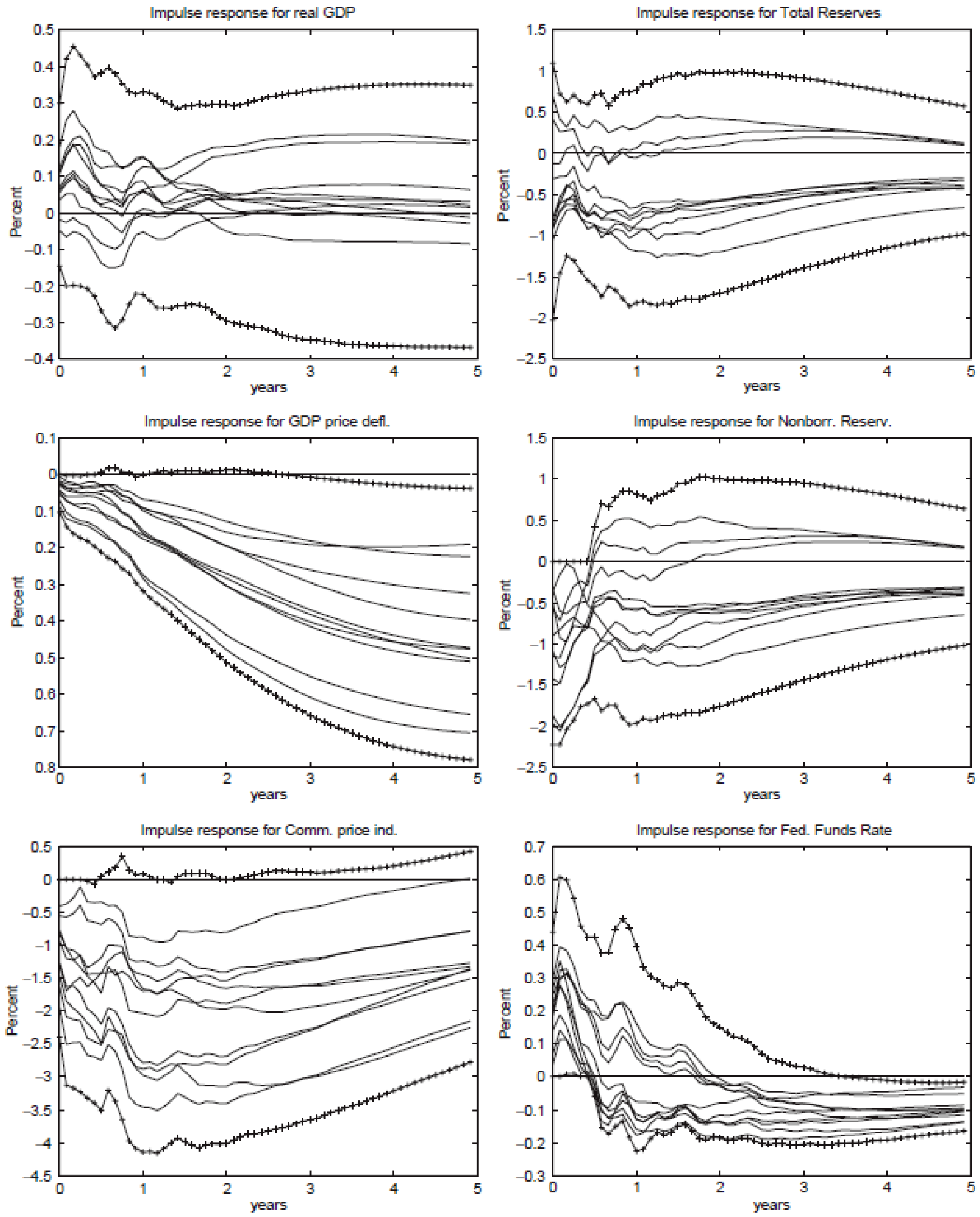

Fig. 2. This figure shows the possible range of impulse response functions when imposing the sign restrictions for $K = 5$ at the OLSE point estimate for the VAR.

Uhlig’s Sign Restricted IRF

- Concerns:
  - One has to choose “how long” an IRF cannot violate the sign imposed
  - Recent studies (Wolf 2016) shows that, theoretically, it may still be impossible to know that what was identified was a monetary policy shock
  - Could be a linear combination of other shocks

Estimating a VAR and SVAR

1. Test for Stationarity
2. Estimate Unrestricted VAR
    1. Test for Block Exogeneity/Granger Causality

3. and investigate lag length
4. Estimate SVAR
    1. Test for congruency, parsimony
    2. Test economic hypotheses

Conclusion: what do we do?

- My opinion:
  - Best identification strategy depends on the economic story you want to tell
  - There is no short-cut, no all-convincing strategy
  - Use the one that fits the question you are trying to pose to the data
  - Do LOTS of robustness checks
    - Be honest about the outcomes
    - Argue for your preferred interpretation but acknowledge alternatives