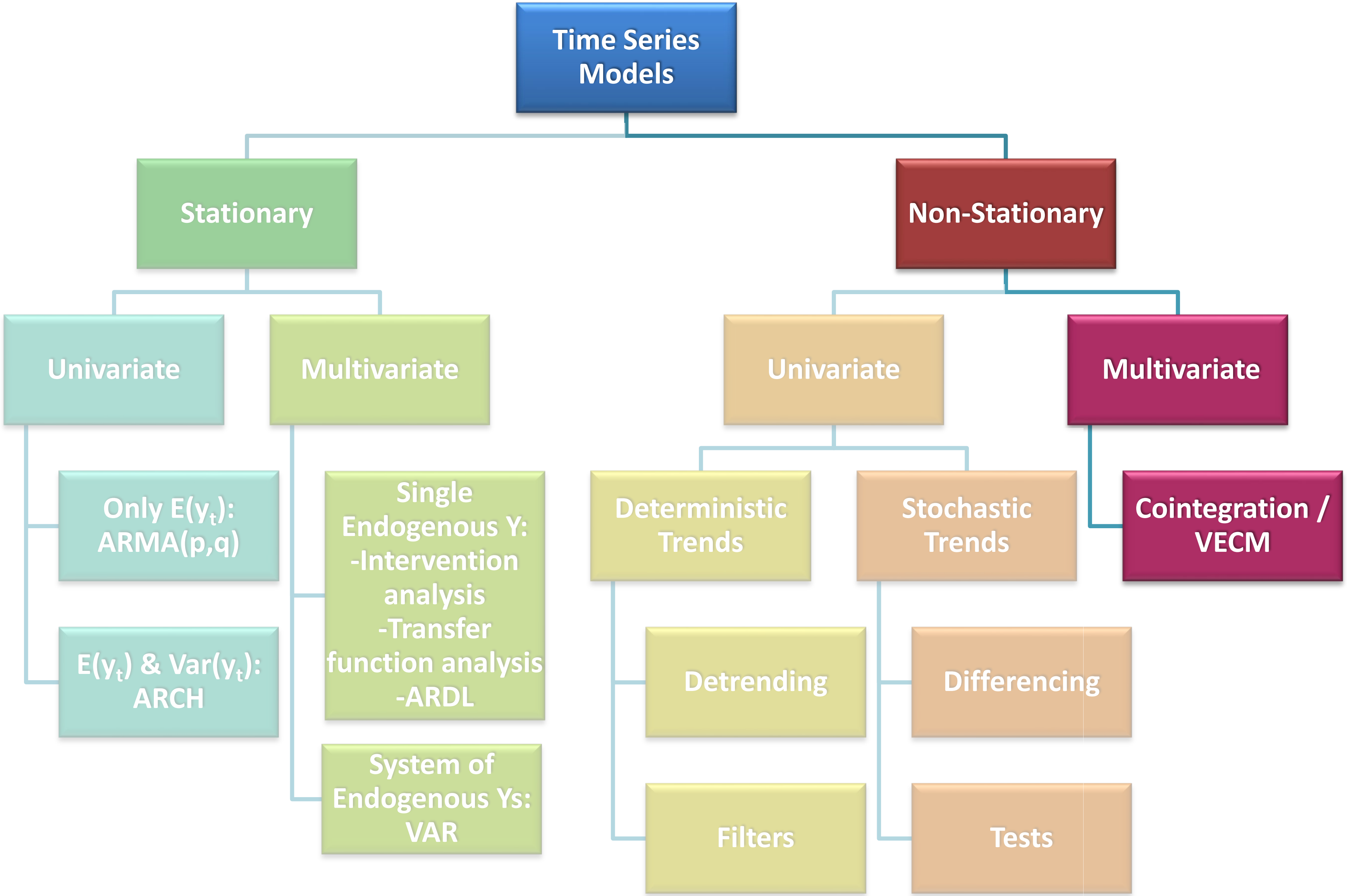
Advanced Econometrics: Time Series

TOPIC 5

Cointegration



Time Series Models
Stationary
Non-Stationary
Univariate
Multivariate
Univariate
Multivariate
Only $E(y_t)$: ARMA(p,q)
$E(y_t)$ & $Var(y_t)$: ARCH
Single Endogenous Y: -Intervention analysis -Transfer function analysis -ARDL
System of Endogenous Ys: VAR
Deterministic Trends
Stochastic Trends
Cointegration / VECM
Detrending
Filters
Differencing
Tests

Another resource:

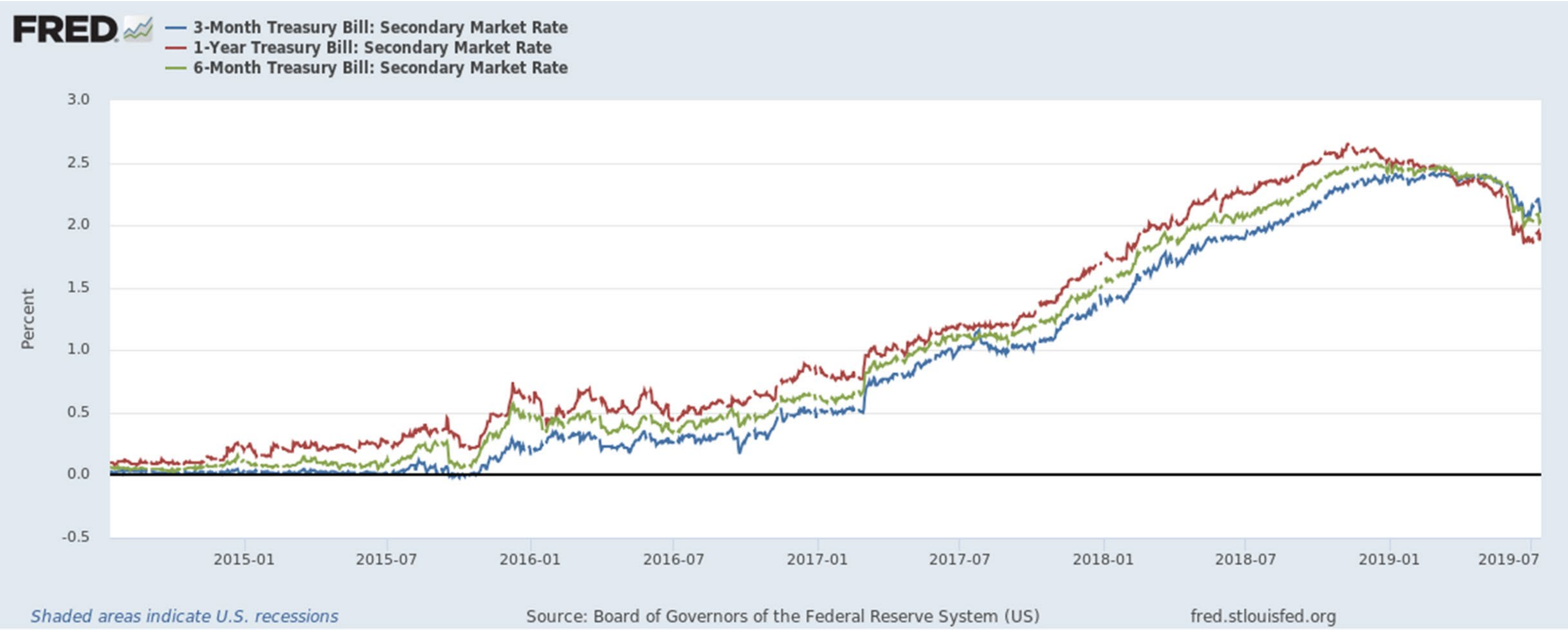
- I found another excellent set of lecture notes from Eric Zivot
  - He is co-author of one of the most used stationarity tests under an arbitrary structural break
  - https://faculty.washington.edu/ezivot/

Plan

- Motivate the approach
- Define Cointegration
- Uniqueness and number of cointegrating relationships
- Work through properties of a cointegrated system
  - Common Trend Representation
  - Vector Error Correction Mechanism representation
- Estimation and Tests
- Weak exogeneity and single equation ARDL models

Stylized Facts about economic time series

6. Related economic series tend to co-move systematically



Multivariate Concern: Spurious Regression

$$y_{1t} = \beta_0 + \beta_1 y_{2t} + e_t$$

- Regressing two ***independent*** I(1) processes on each other gives misleading results
  - Easily find significant coefficients, high R^2
  - When the processes are pure random walks (no drift) the β_1 estimates can *still* be centred around the truth (i.e. the mode of the distribution is at zero, in my simulations)
  - The distribution is not standard (wider than a t-distribution), however, so the standard t tests have the wrong *size*
  - I.e. using standard t-tests, one will *under-reject* the null hypothesis of zero coefficient
  - Phillips (1986) shows that in the general case (allowing for drift and more than one I(1) regressor), the OLS t-stats diverge and the R-squared converges to 1

Monte Carlo Experiment: Spurious regression

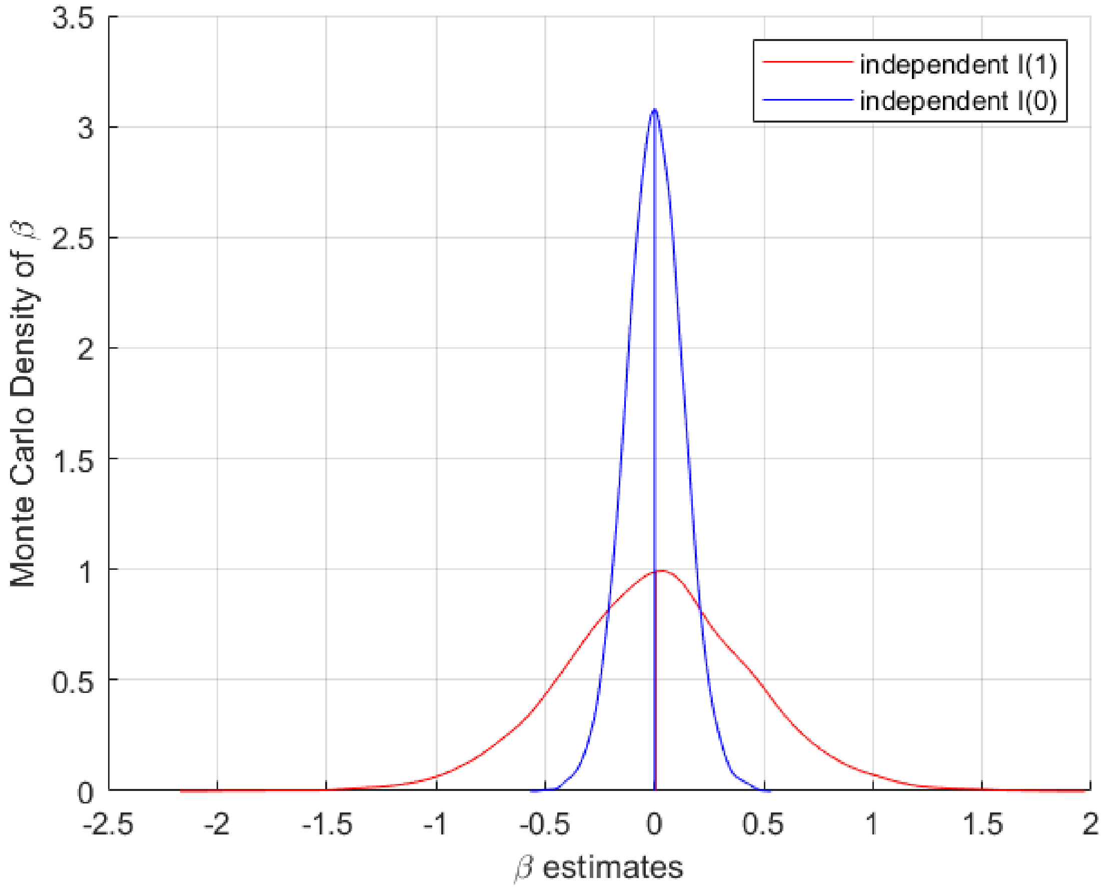
I generated 5000 independent samples of 100 periods:

- Two independent AR(1) processes with coefficient 0.5
- Two independent random walks with drift

Then performed regressions:

- the stationary variables on the stationary variables, plus constant and linear time trend
- the non-stationary variables on the non-stationary variables, plus constant and linear time trend

I plot the kernel density estimate of the distributions of each set of coefficients



Multivariate Tool: Cointegration

$$y_{1t} = \beta_0 + \beta_1 y_{2t} + e_t$$

- When two I(1) process are not independent, but ***cointegrated*** a regression of one on the other gives valid results
  - It is defined by a situation where there exists a linear combination of the I(1) variables that is stationary
  - Then a regression is valid, but *super-consistent* – to obtain the limiting distribution, one must scale with T rather than $\sqrt{T}$:
  - The limiting distribution is not standard

Monetary example

- Consider a central bank that follows a policy that implies a log money supply that is a random walk with drift

- How would aggregate money and prices behave if production was stationary and prices perfectly flexible?

Monetary example

- Consider a central bank that follows a policy that implies the log of the money supply that is a random walk with drift, i.e. m_t is an $I(1)$ process, or equivalently *integrated*

$$m_t = a_0 + m_{t-1} + \varepsilon_{M,t}$$

- How would aggregate money M_t and prices P_t behave if production Y_t was stationary, and prices perfectly flexible and the velocity of money is constant V?
  - If prices are perfectly flexible, it is optimal for firms to immediately internalize an increase in money supply by increasing their prices proportionally
  - In simple models we have the following prediction/equilibrium condition:

$$M_t V = P_t Y_t$$

Monetary example

- How would aggregate money M_t and prices P_t behave if production Y_t was stationary and prices perfectly flexible and the velocity of money is constant V?
  - If prices are perfectly flexible, it is optimal for firms to immediately internalize an increase in money supply by increasing their prices proportionally
  - In simple models we have the following prediction/equilibrium condition:

$$M_t V = P_t Y_t$$

  - In logarithms:

$$m_t + v = p_t + y_t$$
$$p_t = v + m_t - y_t$$

  - Thus, the log-price series “inherits” the unit root of the log money supply
  - Moreover, a linear combination of log money supply and log price level is stationary. They are called **co**integrated.

$$p_t - m_t = v - y_t$$

Plan

- Motivate the approach
- **Define Cointegration**
- Uniqueness and number of cointegrating relationships
- Work through properties of a cointegrated system
  - Common Trend Representation
  - Vector Error Correction Mechanism representation
- Estimation and Tests
- Weak exogeneity and single equation ARDL models

General Idea

- Many variables can be expected to have unit roots for economic reasons:
  - Money supply
  - Prices
  - Stock Prices
  - Nominal Exchange rates
- Economic forces induce equilibrium relationships between some of these variables
  - This implies that they can be cointegrated
- I.e. while the levels of the variables are non-stationary, some (linear) relationship among them will be stationary
  - E.g. while aggregate prices and nominal exchange rates inherit the monetary unit roots from the countries involved, the real exchange rate reflects real productivity changes which is mediated by forces of arbitrage

Cointegration Defined

- Simplest case:
  two processes y_{1t} and y_{2t} are defined to be linearly *cointegrated* if:
  - Each process has one unit root

$$y_{1t}, y_{2t} \sim I(1)$$

  - And there exists a linear combination (b_1, b_2) of them which is stationary:

$$b_1 y_{1t} + b_2 y_{2t} \sim I(0)$$

The typical notation for this is $CI(1,1)$

Cointegration Defined

Definition: the $I(p)$ process $\mathbf{y}_t$ is said to be *linearly cointegrated* of order q denoted as $\mathbf{y}_t \sim CI(p,q)$with $p,q \in \mathbb{Z}^{++}$, $p > q \geq 1$ if:
$\exists$ at least one non-zero vector $\mathbf{b} \in \mathbb{R}^n$, unique up to a scalar multiple, such that $\mathbf{b}\mathbf{y}_t$ is $I(p-q)$

What are we *not* looking at?

- We are only considering *linear* cointegration. In principle a non-linear relationship between two $I(1)$ processes can be stationary. I have not seen this employed yet except in regime switching models.

- We will focus on $CI(1,1)$ processes. Enders gives some analysis and references of higher order applications in the literature.

Purchasing Price Parity

- The strongest form of the PPP hypothesis states that:

$$P_t = P_t^* E_t$$

  - The aggregate local price P_t should be equal to the foreign price level P_t^* once converted into local currency via the nominal exchange rate E_t (which is expressed in local currency units per foreign currency unit)
  - Equivalently, we can formulate it in terms of the real exchange rate Q_t:

$$Q_t = \frac{P_t}{P_t^* E_t} = 1$$

  - In natural logarithms:

$$q_t = p_t - p_t^* - e_t = 0$$

- This strongest version requires that the law of one price holds for *every* good
  - This makes sense for gold, but not for fresh hamburgers
  - Think in terms of arbitrage and transport times
  - Thus we expect the relationship in the data to be much weaker
- A weaker version of this prediction is that any deviation of the log real exchange from its long run mean should eventually fade out
  - I.e. that q_t is I(0) with $\mathrm{E}(q_t) = \mu$
  - If p_t, p_t^* and e_t are all I(1), PPP implies that they must be cointegrated

Purchasing Price Parity

- In matrix form:

$$[1 \quad -1 \quad -1]\begin{bmatrix} p_t \\ p_t^* \\ e_t \end{bmatrix} = q_t \sim I(0)$$

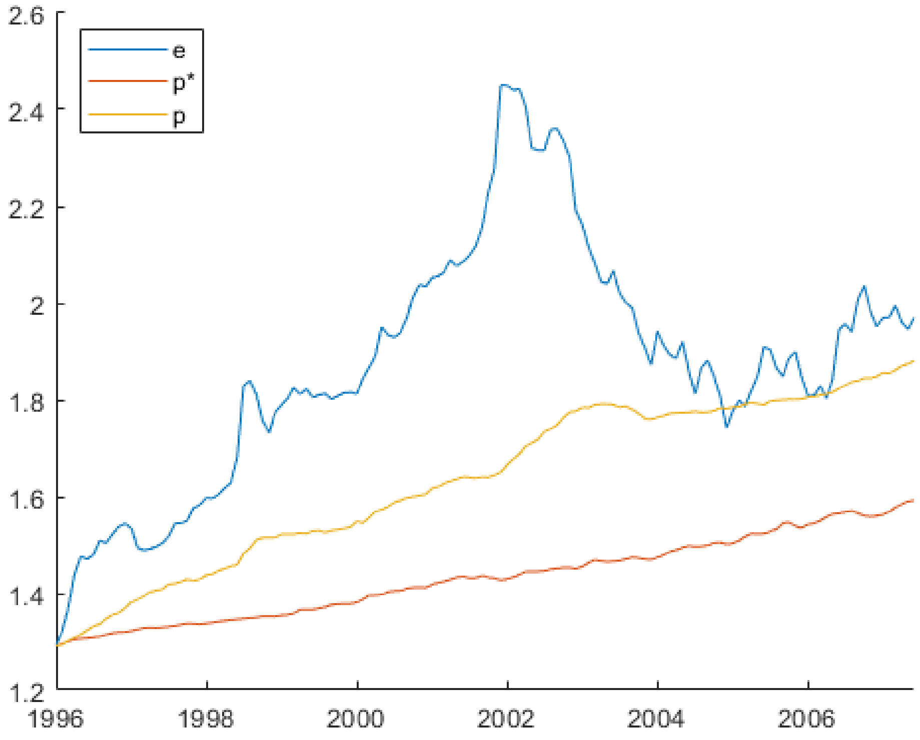
  - Where $[1 \quad -1 \quad -1]$ is the hypothesized cointegrating relationship that PPP predicts
- This is still a very strong restriction to expect to hold exactly:
  - Measurement error/issues
    - prices are arbitrary indices, the level of the exchange rates are measured directly. It is not obvious that the units are exactly conformable to expect unity coefficients.
  - Risk premia/Trade costs
    - Could drive a permanent wedge in this relationship (i.e. a constant in the cointegrating relationship)

- Thus, we can order our investigation as follows using a more general approach:

$$q_t = \beta_0 + \beta_1 p_t + \beta_2 p_t^* + \beta_3 e_t$$

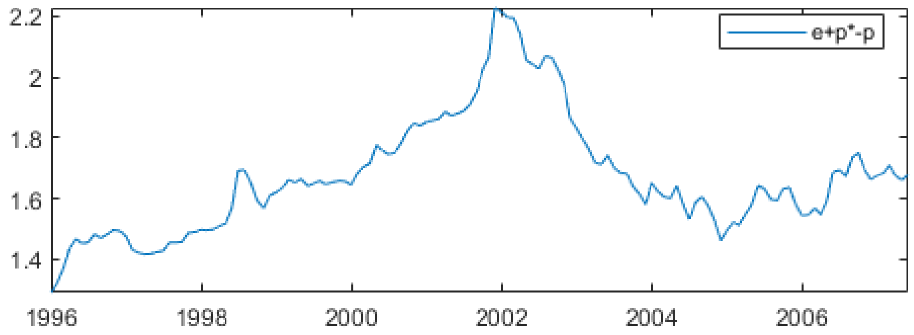
  - Does there exist a linear combination $\boldsymbol{\beta}$ of these variables that is stationary?
  - Are the signs of the relationship as predicted?
  - Is there a risk premium?
  - How close is the relationship to the PPP predicted one?

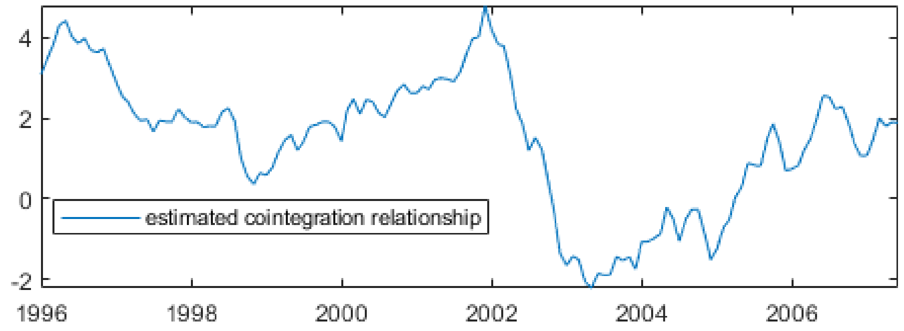
Data (in logarithms)



Results:

$$q_t = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} p_t \\ p_t^* \\ e_t \end{bmatrix}$$





$$q_t = 0.8 + \begin{bmatrix} 1 & -1.5 & -0.16 \end{bmatrix} \begin{bmatrix} p_t \\ p_t^* \\ e_t \end{bmatrix}$$

Uncovered Interest Parity

- In international finance, the Fischer model argues that the exchange rate adjusts to interest differentials across countries
  - If the foreign interest rate increases, the expectation is that the nominal exchange should depreciate:
$$E_t(e_{t+1}) - e_t = i_t^* - i_t$$
  - Where e_t is the log of the nominal exchange rate (in domestic currency units per foreign), i_t^* and i_t are the foreign and domestic interest rates respectively.
  - Exchange rates (and thus their expectations) are typically I(1) while economic theory strongly suggests that $s_t = E_t(e_{t+1}) - e_t$ should be stationary
  - Unfortunately, it is not possible to measure $E_t(e_{t+1})$ directly, with the best alternative being some forward rate (which is not easy to get historical series of)

- We can however do the following to test the theory:
  - If both i_t^* and i_t are I(1), UIP would imply that $i_t^* - i_t$ should be I(0), i.e. that the interest rates should be cointegrated
$$[1 \quad -1]\begin{bmatrix} i_t^* \\ i_t \end{bmatrix} = s_t \sim I(0)$$
  - Again, real world concerns may affect the predicted relationship, so we will start from the general option:
$$s_t = \beta_0 + \beta_1 i_t^* + \beta_2 i_t$$
  - Does there exist a cointegrating relationship? Are the signs as expected? How close to the strict version is it?

Joint testing

- The beauty of the cointegration setting is that we can test *both* of these relationships simultaneously:

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} p_t \\ p_t^* \\ e_t \\ i_t^* \\ i_t \end{bmatrix} = \begin{bmatrix} q_t \\ s_t \end{bmatrix} \sim I(0)$$

- Again one can work from general to specific
  - Are there two cointegrating relationships?
  - Does the data accept the zero restrictions?
  - If so, are the signs of the unrestricted coefficients in the right direction?
  - How close are the restricted relationships to the predictions from the hypotheses?

Plan

- Motivate the approach
- Define Cointegration
- **Uniqueness and number of cointegrating relationships**
- Work through properties of a cointegrated system
  - Common Trend Representation
  - Vector Error Correction Mechanism representation
- Estimation and Tests
- Weak exogeneity and single equation ARDL models

General two variable case

Carefully consider the properties of the cointegrating vector

- Scaling
- Uniqueness

Let
$$\mathbf{y}_t = \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} \sim CI\left(1,1\right)$$

and let the cointegrating relationship be:
$$\mathbf{b} = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$$

Thus:
$$y_{1t} \sim I\left(1\right) \text{ and } y_{2t} \sim I\left(1\right), \text{ but } \mathbf{b}\mathbf{y}_t = b_1 y_{1t} + b_2 y_{2t} \sim I\left(0\right).$$

General two variable case

Define: $$u_t = b_1 y_{1t} + b_2 y_{2t}$$

By definition, u_t is stationary (note: NOT white noise)

- Note: we can allow for a constant in the cointegrating relationship, which will mean that the expected value of u_t could be a non-zero constant independent of time: $E(u_t) = \mu_u$

Thus:

$$E(u_t) = E(u_s) = \mu_u \quad \forall \quad t, s$$

$$E(u_t)^2 = E(u_s)^2 = \sigma_u^2 \quad \forall \quad t, s$$

$$E[u_t u_{t-s}] = \sigma_s \quad \forall \quad t, s$$

Now consider the properties of λu_t for any non-zero $|\lambda| < \infty$.

General two variable case

Define: $u_t = b_1 y_{1t} + b_2 y_{2t}$

We find that , λu_t is also stationary

$$E(\lambda u_t) = E(\lambda u_s) = \lambda \mu_u \quad \forall \quad t, s$$
$$E(\lambda u_t - \lambda \mu_u)^2 = E(\lambda u_s - \lambda \mu_u)^2 = \lambda^2 \sigma_u^2 \quad \forall \quad t, s$$
$$E[\lambda u_t \lambda u_{t-s}] = \lambda^2 \sigma_s \quad \forall \quad t, s$$

If $\mathbf{b}$ is a cointegrating relationship, so is $\lambda \mathbf{b}$

- Cointegrating relationships are unique only up to scale
- Thus, we will always normalize one of the coefficients to 1

Exercises (homework, self driven):

- Show that any linear combination of two stationary processes is also a stationary process
- Show/Argue that a non-stationary process cannot be made stationary by scaling with a non-zero constant

Cointegrating Relationships vs Common Trends

- We will show that
  - In a system $n > 1$ variables, there can be up to $(n-1)$ cointegrating relationships
    - If there are $r \in \{1,2,\ldots,n-1\}$ cointegrating relationships in a system
    - There are $n-r$ independent common stochastic trends in the system
  - if two processes are $CI(1,1)$, they share a common stochastic trend

Number of cointegrating relationships: 2 variables

Proof by contradiction:

Suppose $\mathbf{y}_t = \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} \sim CI(1,1)$ and there are two distinct cointegrating relationships: $\mathbf{b}_1 = \begin{bmatrix} 1 & \beta_1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 & \beta_2 \end{bmatrix}$.

Since they are distinct, we must have $\beta_1 \neq \beta_2$.

Since $\mathbf{b}_1$ and $\mathbf{b}_2$ are cointegrating relationships, both $\mathbf{b}_1\mathbf{y}_t \sim I(0)$ and $\mathbf{b}_2\mathbf{y}_t \sim I(0)$.

Recall that any linear combination of two stationary processes is also a stationary process.

Consider the following linear combination:

$$\left(\frac{\beta_2}{\beta_2 - \beta_1}\right)\mathbf{b}_1\mathbf{y}_t - \frac{\beta_1}{\beta_2}\left(\frac{\beta_2}{\beta_2 - \beta_1}\right)\mathbf{b}_2\mathbf{y}_t \quad \sim \quad I(0)$$

Number of cointegrating relationships: 2 variables

Proof by contradiction:

Suppose $\mathbf{y}_t = \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} \sim CI(1,1)$ and there are two distinct cointegrating relationships: $\mathbf{b}_1 = \begin{bmatrix} 1 & \beta_1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 & \beta_2 \end{bmatrix}$.

Since they are distinct, we must have $\beta_1 \neq \beta_2$.

Since $\mathbf{b}_1$ and $\mathbf{b}_2$ are cointegrating relationships, both $\mathbf{b}_1\mathbf{y}_t \sim I(0)$ and $\mathbf{b}_2\mathbf{y}_t \sim I(0)$.

Recall that any linear combination of two stationary processes is also a stationary process.

Consider the following linear combination:

$$\left(\frac{\beta_2}{\beta_2 - \beta_1}\right)\mathbf{b}_1\mathbf{y}_t - \frac{\beta_1}{\beta_2}\left(\frac{\beta_2}{\beta_2 - \beta_1}\right)\mathbf{b}_2\mathbf{y}_t = y_{1t} \sim I(0)$$

This contradicts the supposition that $y_{1t} \sim I(1)$. We conclude that there could not have been two distinct cointegrating relationships.

n-variable case

- We will show that there can be at most $n-1$ cointegrating relationships
- First: any linear combination of cointegrating relationships is also a cointegrating relationship
- Let the n variables $\mathbf{y}_t$ be individually $I(1)$, and let $\mathbf{b}_1$ and $\mathbf{b}_2$ be two distinct (i.e. linearly independent) cointegrating relationships. Then, for any $\alpha \in [0,1]$:

$$\alpha\mathbf{b}_1\mathbf{y}_t + (1-\alpha)\mathbf{b}_2\mathbf{y}_t \sim I(0)$$

$$[\alpha\mathbf{b}_1 + (1-\alpha)\mathbf{b}_2]\mathbf{y}_t \sim I(0)$$

- Thus $[\alpha\mathbf{b}_1 + (1-\alpha)\mathbf{b}_2]$ is also a cointegrating relationship

n-variable case

- Let the n variables $\mathbf{y}_t$ be individually I(1), and let $\mathbf{B}$ be an [n x n] matrix containing all the cointegrating relationships

whiteboard

- We are interested in the number of distinct (i.e. linearly independent) cointegrating relationships in $\mathbf{B}$
  - I.e. the number of independent rows of $\mathbf{B}$,
  - Or equivalently, the rank of $\mathbf{B}$
- Cases:
  - $rank(\mathbf{B}) = 0$
  - $rank(\mathbf{B}) = n$
  - $rank(\mathbf{B}) = r$ with $0 < r < n$

n-variable case

- Cases:
  - $rank(\mathbf{B}) = 0 \Rightarrow \mathbf{B} = \mathbf{0}$
    - no linear relationship is stationary
    - The only way to make $\mathbf{y}_t$ stationary by linear operation is to multiply by a matrix of zeros
    - $\mathbf{y}_t$ is not cointegrated

n-variable case

- Cases:
  - $rank(\mathbf{B}) = n$
    - $\mathbf{B}$ has n non-zero eigenvalues and $\mathbf{B}^{-1}$ exists,
    - If $\mathbf{B}\mathbf{y}_t$ is stationary, so is any linear combination, so $\mathbf{B}^{-1}\mathbf{B}\mathbf{y}_t$ is I(0)
    - So $\mathbf{I}\mathbf{y}_t$ is stationary, i.e. each component of $\mathbf{y}_t$ is stationary, which contradicts our starting point (cointegration requires that all elements of $\mathbf{y}_t$ be I(1))
    - Hence there cannot be n linearly independent cointegrating relationships
    - Intuition from Linear Algebra:

$\mathbf{B}$ forms a *basis* of $\mathbf{R}^n$ - any $x \in \mathbf{R}^n$ is a linear combination of rows of $\mathbf{B}$

n-variable case

- We conclude:
- If the n variables $\mathbf{y}_t$ are CI(1,1) and $\mathbf{B}$ is the [n x n] matrix containing all the cointegrating relationships:
  - $rank(\mathbf{B}) = r$ with $0 < r < n$
  - $\mathbf{B}$ has only r non-zero eigenvalues
  - There are r cointegrating relationships with $r \in \{1,2, \dots, n-1\}$

Plan

- Motivate the approach
- Define Cointegration
- Uniqueness and number of cointegrating relationships
- Work through properties of a cointegrated system
  - **Common Trend Representation**
  - Vector Error Correction Mechanism representation
- Estimation and Tests
- Weak exogeneity and single equation ARDL models

Common Trends Representation in 2 variables

- Let $\mathbf{y}_t = \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} \sim CI(1,1)$ with cointegrating relationship: $\mathbf{b} = \begin{bmatrix} 1 & \beta \end{bmatrix}$

- Since $y_{2t} \sim I(1)$ we know we can represent it as:

$$\begin{aligned} y_{2t} &= \mu_t + e_{2t} \\ \text{with} & \\ \mu_t &\sim I(1) \\ e_{2t} &\sim I(0) \text{ but not necessarily white noise} \end{aligned}$$

- By assumption $u_t = y_{1t} + \beta y_{2t}$ is stationary

- Now we substitute and rearrange:

Common Trends Representation in 2 variables

- Now we substitute and rearrange:

$$\begin{aligned} u_t &= y_{1t} + \beta y_{2t} \\ y_{1t} &= -\beta y_{2t} + u_t \\ &= -\beta\left(\mu_t + e_{2t}\right) + u_t \\ &= -\beta\mu_t - \beta e_{2t} + u_t \end{aligned}$$

- Thus two variables are cointegrated if (and only if) their stochastic trends are identical up to scaling
- Exercises:
  - Show the "only if" direction of the proof: if two variables share the same trend up to scaling, they are cointegrated
  - Show that if two I(1) processes have independent stochastic trends, they can be cointegrated with a third I(1) process under a specific linear relationship

Common Trends Representation

- In N variables, just the result:
  - If there are $0<r<n$ cointegrating relationship
  - There are n-r independent stochastic trends

Plan

- Motivate the approach
- Define Cointegration
- Uniqueness and number of cointegrating relationships
- Work through properties of a cointegrated system
  - Common Trend Representation
  - **Vector Error Correction Mechanism representation**
- Estimation and Tests
- Weak exogeneity and single equation ARDL models

The Vector Error Correction Mechanism (VECM) Representation

- Start from the n-variable case this time
- Consider a VAR(2)
  - Enders shows the same for a VAR(p)

The Vector Error Correction Mechanism (VECM) Representation

Let $\underset{[n\times 1]}{\mathbf{y}_t} \sim CI\,(1,1)$

Consider the standard reduced form VAR(2) (abstracting from constants):

$$\mathbf{y}_t = A_1\mathbf{y}_{t-1} + A_2\mathbf{y}_{t-2} + \boldsymbol{e}_t$$

Note:

- If the variables are cointegrated, the residuals from this regression must be stationary – the cointegrating relationship will be captured "somewhere" in the linear coefficients above
- Put differently: if the residuals are not stationary, there is no cointegrating relationship between all of the n variables
  - What about subsets?

The Vector Error Correction Mechanism (VECM) Representation

$$\mathbf{y}_t = A_1\mathbf{y}_{t-1} + A_2\mathbf{y}_{t-2} + \mathbf{e}_t$$

Add and subtract $A_2\mathbf{y}_{t-1}$ from the RHS:

$$\begin{aligned}
\mathbf{y}_t &= A_1\mathbf{y}_{t-1} + A_2\mathbf{y}_{t-2} - A_2\mathbf{y}_{t-1} + A_2\mathbf{y}_{t-1} + \mathbf{e}_t \\
&= (A_1 + A_2)\,\mathbf{y}_{t-1} - A_2\triangle\mathbf{y}_{t-1} + \mathbf{e}_t
\end{aligned}$$

Subtract $\mathbf{y}_{t-1}$ from both sides:

$$\begin{aligned}
\mathbf{y}_t - \mathbf{y}_{t-1} &= (A_1 + A_2)\,\mathbf{y}_{t-1} - \mathbf{y}_{t-1} - A_2\triangle\mathbf{y}_{t-1} + \mathbf{e}_t \\
\triangle\mathbf{y}_t &= -(I - A_1 - A_2)\,\mathbf{y}_{t-1} - A_2\triangle\mathbf{y}_{t-1} + \mathbf{e}_t \\
&= \Pi\mathbf{y}_{t-1} + C_1\triangle\mathbf{y}_{t-1} + \mathbf{e}_t
\end{aligned}$$

Vector Error Correction Mechanism (VECM) Representation

$$\Delta \mathbf{y}_t = \Pi \mathbf{y}_{t-1} + C_1 \Delta \mathbf{y}_{t-1} + \boldsymbol{e}_t$$

Here, Π contains the cointegrating relationships/vectors if they exist

Recall: $rank(\Pi) = r < n$

- We will use the estimated eigenvalues of Π to construct an empirical test of the rank of the matrix (i.e. the number of cointegrating relationships)
- Additionally, the rank condition implies the following decomposition exists

$$\underset{[n \times n]}{\Pi} = \underset{[n \times r]}{\boldsymbol{\alpha}} \; \underset{[r \times n]}{\boldsymbol{\beta}}$$

- Where $\boldsymbol{\beta}$ contains the cointegrating vectors in its r independent rows
- And $\beta \mathbf{y}_t = \underset{[r \times 1]}{\mathbf{u}_t}$ are the "equilibrium errors" or

  "deviations from the long run equilibrium" (what is "long run"?)

Vector Error Correction Mechanism (VECM) Representation

$$\Delta \mathbf{y}_t = \Pi \mathbf{y}_{t-1} + C_1 \Delta \mathbf{y}_{t-1} + \boldsymbol{e}_t$$

Additionally, this representation gives another result on appropriate time series modelling:

- *Unless,* both theoretically and empirically $rank(\Pi) = 0$, estimating a VAR in differences when the levels are integrated is a **misspecification**, and may yield inconsistent estimates.

Vector Error Correction Mechanism (VECM) Representation

$$
\begin{aligned}
\triangle \mathbf{y}_t &= \Pi \mathbf{y}_{t-1} + C_1 \triangle \mathbf{y}_{t-1} + \boldsymbol{e}_t \\
&= \boldsymbol{\alpha}\left(\beta \mathbf{y}_{t-1}\right) + C_1 \triangle \mathbf{y}_{t-1} + \boldsymbol{e}_t \\
&= \boldsymbol{\alpha}\left(\mathbf{u}_{t-1}\right) + C_1 \triangle \mathbf{y}_{t-1} + \boldsymbol{e}_t
\end{aligned}
$$

The matrix $\boldsymbol{\alpha}$ contains the *loading factors* on the previous long run error

- E.g. the entry α_{ij} measures the size of the response of Δy_{it} to the deviation from long run equilibrium relationship j in period $t-1$
- Hence: how much of the change in y_{it} is a *correction* of the j^{th} long run error from the previous period
- For the process to be cointegrated, one or more of the variables *must* adjust to deviation from any long run equilibrium, otherwise the process will drift arbitrarily far from the proposed long run relationship

Vector Error Correction Mechanism (VECM) Representation

$$\begin{bmatrix} \triangle y_{1t} \\ \triangle y_{2t} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} (y_{1t-1} + \beta y_{2t-1}) + \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix}$$
$$= \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} u_{t-1} + \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix}$$

- In the two variable system, or when there is only one cointegrating relationship, the $\boldsymbol{\alpha}$ vector is called the *speed of adjustment* coefficients
  - Some authors always call them that, but it doesn't quite work as cleanly in higher order systems.

- Some texts also argue that one must consider the values of these coefficients to ensure stability:
  - Suppose $\beta < 0$, then $u_{t-1} > 0$ implies that either y_{1t-1} was too large, or y_{2t-1} was too small, so either y_{1t} must decrease, i.e. $\alpha_1 < 0$ or y_{2t} must increase, i.e. $\alpha_2 > 0$ or both, otherwise the errors are not corrected
  - In practice, if there is strong empirical cointegration, this holds naturally – since the system *tries* to find stationary errors by construction
  - If the errors from the VECM are stationary, the cointegration relationships are necessarily stable

Vector Error Correction Mechanism (VECM) Representation

$$\Delta \mathbf{y}_t = \Pi \mathbf{y}_{t-1} + C_1 \Delta \mathbf{y}_{t-1} + \boldsymbol{e}_t$$

Note that, like in the two variable case, the decomposition is not unique:

$$\underset{[n \times n]}{\Pi} = \underset{[n \times r]}{\boldsymbol{\alpha}} \quad \underset{[r \times n]}{\boldsymbol{\beta}}$$

- Consider any $[r \times r]$ matrix H with $rank(H) = r$ then

$$\Pi = \alpha\beta = (\alpha H)(H^{-1}\beta) = \alpha^* \beta^*$$

- β^* is still a valid set of cointegrating relationships, since any linear combination of stationary processes is also a stationary process

- Thus, we usually impose a **normalization** on β

Vector Error Correction Mechanism (VECM) Representation

Typical normalization of β:

- Consider an arbitrary set of cointegrating relationships in a three variable CI(1,1) system:

$$\beta = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \end{bmatrix}$$

- If the first [2x2] entries $H = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}$ are full rank, then

$$\beta^* = H^{-1}\beta = \begin{bmatrix} 1 & 0 & \beta_{13}^* \\ 0 & 1 & \beta_{23}^* \end{bmatrix}$$

- Is also a valid set of cointegrating vectors,

- Note that this is a **normalization**, ***not a restriction*** – the first true restriction we impose on the cointegrating space is an *additional* one over and above this normalization

- Since the order of the variables are arbitrary at this point, this normalization can make it difficult to a link the estimated cointegrating relationships to theoretical economic equilibria of interest if there is more than one cointegrating relationship.

Plan

- Motivate the approach
- Define Cointegration
- Uniqueness and number of cointegrating relationships
- Work through properties of a cointegrated system
  - Common Trend Representation
  - Vector Error Correction Mechanism representation
- **Estimation and Tests**
- Weak exogeneity and single equation ARDL models

The Engle and Granger (1987) approach

- The first practical approach to estimating a cointegration relationship
  - Eventually got them a Nobel, although it probably should have been shared with David Hendry…
- We will just briefly consider this approach as the Johansen (1988,1992) method is a more general, simultaneous ML approach that is currently the most used
- The key problems with the EG method:
  - One has to pick the “LHS” variable explicitly, and the results may depend on this choice
  - It is a two step process – any error in the first stage carries over and influences the second step

The Engle and Granger (1987) approach

1. Pre-test variables for order of integration.
   - If all I(1), proceed. (Enders also gives clear coverage of multi-cointegration)
2. Estimate the long run relationship
$$y_{1t} = \beta_0 + \beta_1 y_{2t} + e_t$$
   - Remember if there is cointegration, OLS estimates are *super consistent,* they converge faster than I(0) regressions.
   - Test whether $\hat{e}_t$ series is I(0), if yes – conclude cointegration
     - Note, this is *not* just a standard ADF test, as the estimation above attempts to minimize squared residuals – thus biased towards stationarity
     - Engle and Granger constructed new tabulations for this test
3. Estimate the error correction model
$$\Delta \mathbf{y}_t = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 \hat{e}_{t-1} + A(L)\Delta \mathbf{y}_{t-1} + \boldsymbol{\varepsilon}_t$$
   - This is just a VAR in first differences with an additional term: the lagged deviation from the long run relationship estimated in step 2
4. Assess Model Adequacy
5. Test economic hypotheses

The Johansen (1988, 1992) method

- Johansen developed a maximum likelihood based test of cointegration in a VAR setting
  - Since it is maximum likelihood based, it avoids the small sample issues by construction
  - BUT, maximum likelihood is sensitive to the assumption of the correct density function
  - Probably still subject to small sample issues

$$\triangle \mathbf{y}_t = \Pi \mathbf{y}_{t-1} + C_1 \triangle \mathbf{y}_{t-1} + \boldsymbol{e}_t$$

- The core of the method is a test of the number of cointegrating relationships, which is equivalent to the number of non-zero eigenvalues of the matrix Π
  - In an empirical estimation (with noise), Π will be *algebraically* of full rank – a square matrix with randomly drawn real entries is almost always invertible.
  - Johansen derived tests for the number of *statistically significant* eigenvalues of the Π matrix
  - These boil down to a multivariate version of the Dickey-Fuller test
  - Again – a non-standard distribution tabulated via Monte Carlo methods

The Johansen (1988, 1992) method

$$\Delta \mathbf{y}_t = \Phi \mathbf{D}_t + \mathbf{\Pi} \mathbf{y}_{t-1} + \mathbf{C}(L)\Delta \mathbf{y}_{t-1} + e_t$$

The steps:

0. Choose the specification of the deterministic parts: Here the $\mathbf{\Phi D}_t = \mu_t = \mu_0 + \mu_1 t$ term contains all deterministic parts, i.e. constants and trends. As in the ADF test, the critical values of the tests depend on these terms.

   – In the cointegration setting there are 5 cases:

The Johansen (1988, 1992) method

No constant: $\mu_t = 0$:
$\mathbf{y}_t$ is I(1) without drift, $\beta\mathbf{y}_t$ is mean zero (no LR constant)

$$\Delta\mathbf{y}_t = \boldsymbol{\alpha\beta}\mathbf{y}_t + C(L)\Delta\mathbf{y}_{t-1} + \boldsymbol{e}_t$$

Restricted constant: $\mu_t = \mu_0 = \boldsymbol{\alpha\rho}_0$
$\mathbf{y}_t$ is I(1) without drift, $\beta\mathbf{y}_t$ has non-zero mean

$$\Delta\mathbf{y}_t = \boldsymbol{\alpha}(\boldsymbol{\beta}\mathbf{y}_t + \boldsymbol{\rho}_0) + C(L)\Delta\mathbf{y}_{t-1} + \boldsymbol{e}_t$$

Unrestricted constant: $\mu_t = \mu_0$
$\mathbf{y}_t$ is I(1) with drift vector μ_0, $\beta\mathbf{y}_t$ may have non-zero mean

$$\Delta\mathbf{y}_t = \mu_0 + \boldsymbol{\alpha\beta}\mathbf{y}_t + C(L)\Delta\mathbf{y}_{t-1} + \boldsymbol{e}_t$$

Restricted trend: $\mu_t = \mu_0 + \boldsymbol{\alpha\rho}_1 t$
$\mathbf{y}_t$ is I(1) with drift vector μ_0, $\beta\mathbf{y}_t$ has linear trend

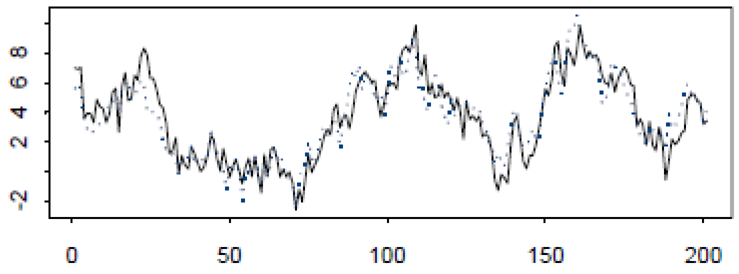
$$\Delta\mathbf{y}_t = \mu_0 + \boldsymbol{\alpha}(\boldsymbol{\beta}\mathbf{y}_t + \boldsymbol{\rho}_1 t) + A(L)\Delta\mathbf{y}_{t-1} + \boldsymbol{e}_t$$

Unrestricted trend: $\mu_t = \mu_0 + \mu_1 t$
$\mathbf{y}_t$ is I(1) with linear trend and drift vector μ_0 - quadratic trend in levels,
$\beta\mathbf{y}_t$ has linear trend

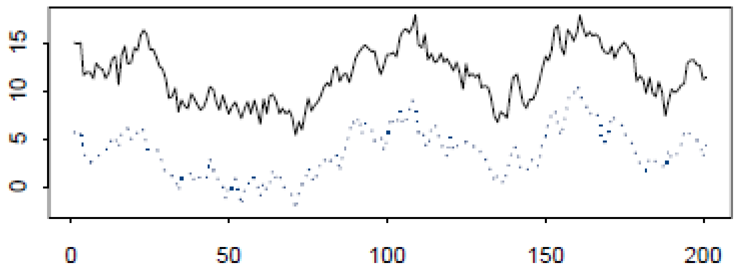
$$\Delta\mathbf{y}_t = \mu_0 + \mu_1 t + \boldsymbol{\alpha\beta}\mathbf{y}_t + A(L)\Delta\mathbf{y}_{t-1} + \boldsymbol{e}_t$$

The Johansen (1988, 1992) method

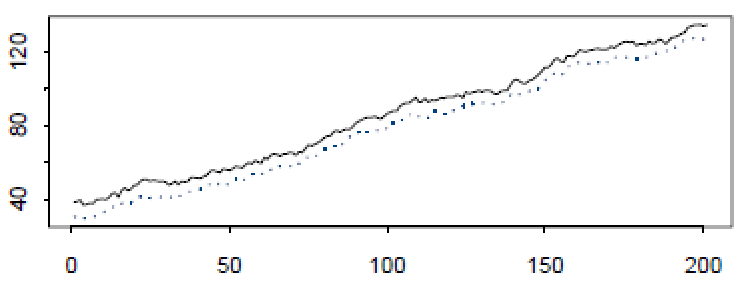
Case 1: No constant



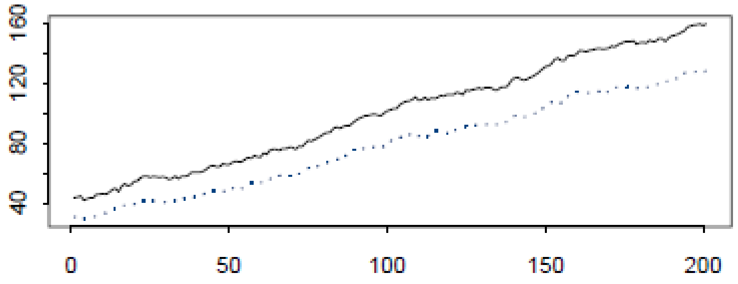
Case 2: Restricted constant



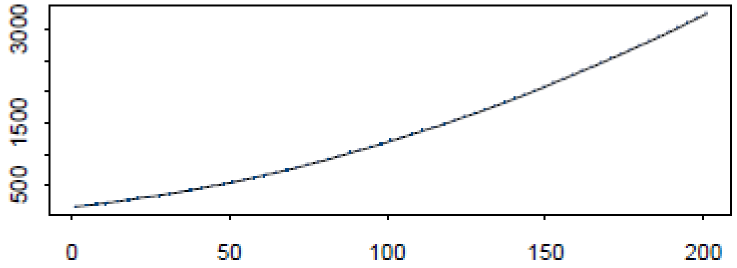
Case 3: Unrestricted constant



Case 4: Restricted trend



Case 5: Unrestricted trend

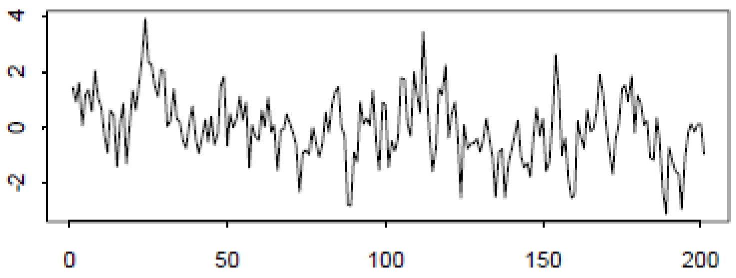


Simulated $\mathbf{y}_t$

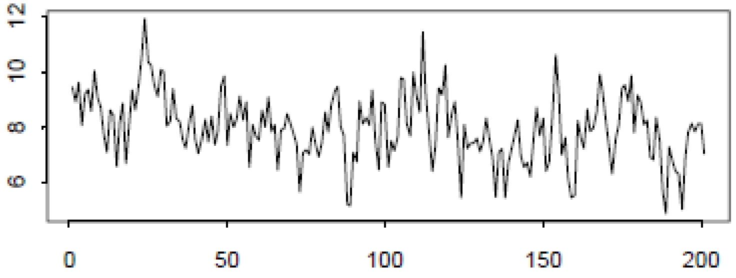
Source: https://faculty.washington.edu/ezivot/econ584/notes/cointegration.pdf

The Johansen (1988, 1992) method

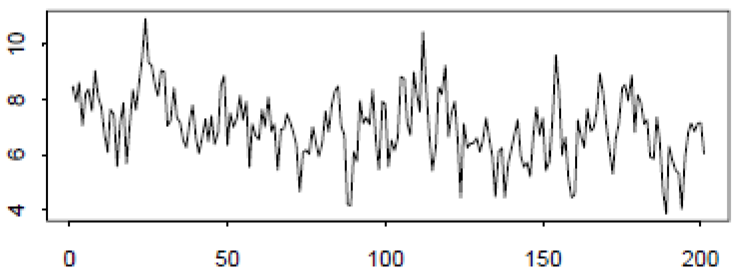
Case 1: No constant



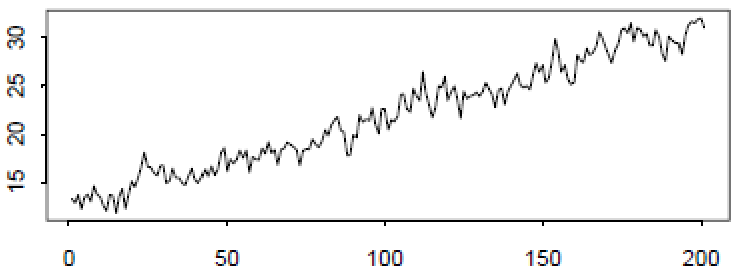
Case 2: Restricted constant



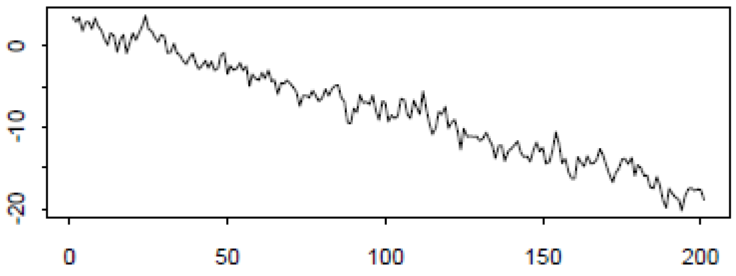
Case 3: Unrestricted constant



Case 4: Restricted trend



Case 5: Unrestricted trend



Simulated $\boldsymbol{\beta y}_t$

Source: https://faculty.washington.edu/ezivot/econ584/notes/cointegration.pdf

The Johansen (1988, 1992) method

The steps:

1. Pre-test the variables to conclude that they are (or may be) I(1)
   - Note that there are also tests for individual unit roots *within* the system that have different properties

The Johansen (1988, 1992) method

The steps:

2. Estimate an unrestricted VAR in levels and check the adequacy of specification of the model

$$\mathbf{y}_t = \Phi \mathbf{D}_t + \mathbf{A}(L)\mathbf{y}_{t-1} + e_t$$

- If there is cointegration, some linear combination(s) of the endogenous variables are stationary, and this will be captured in the coefficient matrices
- A critical point is to test that errors are **white noise**
  - The Johansen Method is a ML method, i.e. limiting distributions are derived assuming **normal** errors
  - Robust to some deviations: some non-normality, heteroscedasticity
  - **Requires:** i.i.d. errors with finite variance
  - **Hence the following is unacceptable:** autocorrelated residuals, time-varying parameters, structural breaks

The Johansen (1988, 1992) method

The steps:

3. Estimate the VECM form and Determine Cointegration Rank (number of cointegrating relationships)
   Perform Likelihood based tests on the eigenvalues of the Π matrix

$$\triangle \mathbf{y}_t = \Phi \mathbf{D}_t + \mathbf{\Pi} \mathbf{y}_{t-1} + \mathbf{C}(L) \triangle \mathbf{y}_{t-1} + e_t$$

- Johansen shows how a consistent estimate of Π can be recovered from the covariance matrices of two regressions: the level of Y on its lagged differences, and the difference of Y on its lagged differences. Thus, you should use packages you trust to make sure they are constructed correctly.
- Find the estimated eigenvalues and order them from large to small:

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n$$

- There are two tests:
  - The trace test
  - The maximum eigenvalue test
- Johansen shows that the following iterative scheme is consistent:
  Start from the hypothesis that there are zero significant eigenvalues, if rejected, there must be at least one. Then test for 1 against the alternative of more, and so on.

The Johansen (1988, 1992) method

The steps:

3.

- The trace test

$$H_0(r) : r = r_0 \text{ vs. } H_1(r_0) : r > r_0$$

$$LR_{trace}(r_0) = -T \sum_{i=r_0+1}^{n} \ln(1 - \hat{\lambda}_i)$$

  Intuition: if there are r_0 non-zero eigenvalues, the sum of the rest should be small

- The maximum eigenvalue test

$$H_0(r_0) : r = r_0 \text{ vs. } H_1(r_0) : r_0 = r_0 + 1$$

$$LR_{\max}(r_0) = -T \ln(1 - \hat{\lambda}_{r_0+1})$$

  Intuition: if there are r_0 non-zero eigenvalues, the eigenvalue $r_0 + 1$ should be small

The Johansen (1988, 1992) method

The steps:

4. Impose the number of cointegrating relationships r and perform a reduced rank regression of:

$$\triangle \mathbf{y}_t \quad = \quad \Phi \mathbf{D}_t + \mathbf{\Pi} \mathbf{y}_{t-1} + \mathbf{C}(L)\triangle \mathbf{y}_{t-1} + e_t$$

   – Again, Johansen shows how to obtain consistent ML estimates.
   – Note that *now* we are imposing restrictions on the Π matrix, so the estimates will differ from the unrestricted VAR/VECM we started with
      - Again evaluate the model adequacy

The Johansen (1988, 1992) method

The steps:

5. Normalize the cointegrating relationships, evaluate/interpret the economic content/forecasts of the model. Test further hypothesized restrictions
   - As in previous models, restrictions are tested analogously to an F test:
     - Estimate the restricted model, compute the reduction in fit. If the restriction is valid, the reduction in fit should be small
     - Again – the critical values are found by simulation
   - A key question is usually whether a variable is weakly exogenous (in terms of the loading factor)
   - Impulse Response Functions have been derived
     - But now there are two types of shock: permanent and temporary

Practical Implementation: An unpleasant reality

- The power of the cointegration paradigm is unquestionable to me
- The basics are widely available
- The ability to test very general restrictions, however, is complicated and the lack of a formal implementation across coding paradigms is deeply frustrating and disappointing
  - In 2004, Boswijk and Doornik published a unifying framework for testing generic hypotheses in a cointegration framework
  - To date (to my knowledge) this functionality is available in only three forms:
    - Eviews (a very expensive program, kept up as a corporate venture serving the professional sector) – fully generic implementation allow both equality and inequality restrictions
    - Boswijk and Doornik provide implementations in PCGive (a proprietary matrix language developed at Oxford) – from the paper, presumably only equality constraints.
    - Johansen and Juselius in a program called Microfit that I know nothing about, and an earlier implementation than anyone before (they developed the technology).

Plan

- Motivate the approach
- Define Cointegration
- Uniqueness and number of cointegrating relationships
- Work through properties of a cointegrated system
  - Common Trend Representation
  - Vector Error Correction Mechanism representation
- Estimation and Tests
- **Weak exogeneity and single equation ARDL models**

Weak Exogeneity and ARDL models

$$\begin{bmatrix} \triangle y_{1t} \\ \triangle y_{2t} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} u_{t-1} + \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

- If $\alpha_2 = 0$ all the "adjusting to the long run error" occurs through the dynamics of y_{1t}
- If, in addition, there are no contemporaneous effects from y_{1t} to y_{2t}, i.e. when $f_{21} = 0$, we call y_{2t} **weakly exogenous**
- Note that the first condition is testable, whereas the second is not

- When one is confident that *all* variables, other than one of interest, are weakly exogenous, one can validly estimate the first equation as a cointegrated ARDL model (i.e. a single equation model)
  - Then one validly can test for cointegration in a single equation, even with a mixture of I(0) and I(1) variables via the Pesaran, Shin and Smith (2001) **bounds test**
  - Be careful – many applications just do it without considering the necessary conditions for the method to yield consistent results.

Weak Exogeneity and ARDL models

$$\begin{bmatrix} \triangle y_{1t} \\ \triangle y_{2t} \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \end{bmatrix} u_{t-1} + \begin{bmatrix} f_{11} & f_{12} \\ 0 & f_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

Or:

$$\begin{aligned} \triangle y_{1t} &= a_1 \left(y_{1t-1} + \beta y_{2t-1} \right) + \left[f_{11} \varepsilon_{1t} + f_{12} \varepsilon_{2t} \right] \\ \triangle y_{2t} &= f_{22} \varepsilon_{2t} \end{aligned}$$

Then noting: $\varepsilon_{2t} = \frac{\triangle y_{2t}}{f_{22}}$ and defining $c = \frac{f_{12}}{f_{22}}$ we obtain:

$$\triangle y_{1t} = a_1 \left(y_{1t-1} + \beta y_{2t-1} \right) + c \triangle y_{2t} + f_{11} \varepsilon_{1t}$$

Which can be generalized to:

$$\triangle y_{1t} = \beta_1 y_{1t-1} + \beta_2 y_{2t-1} + C_1 (L) \triangle y_{1t-1} + C_2 (L) \triangle y_{2t} + e_{1t}$$

- Again, for this to be a valid approach y_{2t} **must** be weakly exogenous: $\mathrm{E}(y_{2t} e_{1t}) = 0$, which requires both $\alpha_2 = 0$ and $f_{21} = 0$

Summary

- Motivate the approach
- Define Cointegration
- Uniqueness and number of cointegrating relationships
- Work through properties of a cointegrated system
  - Common Trend Representation
  - Vector Error Correction Mechanism representation
- Estimation and Tests
- Weak exogeneity and single equation ARDL models