Modeling and Nonlinear Heading Control for Sailing Yachts

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***Abstract*—This paper presents a study on the development and testing of a model-based heading controller for a sailing yacht. Using Fossen's compact notation for marine vehicles, we first describe a nonlinear 4-DOF dynamic model for a sailing yacht, including roll. Starting from this model, we then design a nonlinear heading controller using the integrator backstepping method, which asymptotically stabilizes the system to the heading/yaw dynamics. Additionally, we present a few simulation results to illustrate the behavior of our control designs.**

I. Introduction

Mankind has been sailing for thousands of years, and modern sailors benefit from a great deal of experience, in particular when it comes to the relatively complex task of simultaneously steering the rudder and trimming the sail to get the best performance for a specific course. This process can require great skills in the sense that it depends highly on the wind direction, but also the coupling between roll and yaw, the sail performance, etc.

Until now, there were only a few studies dedicated to autonomous control for such kinds of wind-propelled vehicles. In [1], a fuzzy logic-based control scheme for self-steering of a sailboat is presented so that the sailboat can automatically sail along the desired course and at the highest speed. A P/PI controller was employed in [2][3] in order to achieve tracking control. However, these works did not consider the rolling motion in their controller designs, while it actually plays an important role in the behaviour of yachts using the wind for propulsion. Therefore, as one of the contributions of the present paper, we propose to study the stabilization of the heading by considering a yacht model with 4 degrees of freedom (DOF), including roll motion.

In order to have an overall view on the yacht dynamics, we first present a maneuvering model of a generic keelboat equipped with a main sail and a rudder. Masuyama et. al have proposed a set of equations for the motion of sailing yachts in [4], which included hydrodynamic derivatives given by test results. This model was then tested by using parameters of an ancient Japanese sailing trader named "Naniwa-maru" [5][6], and the numerical simulation of a wearing maneuver were in agreement with measurements from actual sea trials. As an extension of Masuyama's mathematical model for a tacking maneuver of a sailing yacht, [7] reported a complete description of the forces and moments acting on the yacht considering the interactions in roll-sway-yaw. Based on the above investigations and following the way Fossen built dynamic models for marine vehicles[8][9], we derived a set of equations of motion for a class of sailing yachts using a vectorial state-space representation. As a preliminary assumption, the yachts we consider here can be seen as a composition of four parts, i.e. sail, rudder, keel, and hull, and the forces and moments acting on the yacht is the integration of the influence of each component taken separately. Additionally, every appendage is treated as a "thin foil" or a wing such that the basic aero-hydrodynamics can be used.

After this introduction, Section II will be dedicated to the complete derivation of the dynamic equations of motion for a sailing yacht. In this section, we also model the dynamic behaviour of the sail as it changes sides while tacking or wearing. In Section III, using the backstepping method, we introduce a model-based heading controller that exhibits an asymptotic stability property. Simulation results are briefly presented to illustrate the approach. Finally, a few concluding remarks end the paper.

II. Dynamic equations of motion

A list of the variables used in this paper is given in Table I.

A. A Vectorial representation in 4 DOF

Let the North-East-Down coordinate system be the inertial reference frame (n-frame) and the body-fixed frame (b-frame) $x_b y_b z_b$ is a rotating reference frame attached to the yacht(see Figure 1), with angular velocity $\boldsymbol{\omega} = [p, q, r]^T$ relative to the n-frame. The origin of the b-frame is assumed to coincide with the yacht's center of gravity (CG).

Assuming the yacht to be rigid and having 4 degrees of freedom, means we exclude both heaving and pitching motions, i.e. that the dynamics associated with the motions in heave and pitch are neglected and $w = q = 0$. The vector $\boldsymbol{\nu} = [u, v, p, r]^T$ denotes the generalized velocity vector decomposed in the b-frame and $\boldsymbol{\eta} = [x, y, \phi, \psi]^T$ is a vector describing respectively the position of the yacht in the n-frame and the roll and yaw angles.

Note that, in the following discussion, we consider the vehicle to evolve in calm waters, i.e. the motions due to waves and environmental disturbances are ignored.

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TABLE I
DEFINITIONS OF VARIABLES

Notation	Description
A	Plan area of the foil
AR	Aspect ratio of the foil
$\boldsymbol{C}$, $\boldsymbol{C}_{RB}$, $\boldsymbol{C}_A$	System/rigid-body/added-mass Coriolis-centripetal matrix
C_D, C_L	Drag and lift coefficients
C_{Di}	Induced drag coefficient
CG	Vehicle's center of gravity
$\boldsymbol{D}$	Vector of damping
D, L	Drag and lift forces acting on the foils
F, M	External forces and moments acting on the vehicle in the b-frame
F_{rh}	Hull resistance
$\boldsymbol{g}$	Vector of restoring forces
$\boldsymbol{H}$	Total angular momentum of the body
$\boldsymbol{I}$	Principle moment of inertia in 4 degrees of freedom (DOF) in the b-frame
$\boldsymbol{J}$	Coordinate transformation matrix
$\boldsymbol{M}$, $\boldsymbol{M}_{RB}$, $\boldsymbol{M}_A$	System/rigid-body/added-mass inertia matrix
M_r	Static-righting moment
$M_{\phi d}$, $M_{\psi d}$	Heel/yaw-damping moment
m	Total mass of the yacht
p, q, r	Angular velocities in the b-frame
s	Span of the foil
sat	Saturation function
u, v, w	Linear velocities in the b-frame
$\boldsymbol{v}_{tw}^b$	Vector of true wind in the b-frame
v_a, α_a	Apparent velocity/angle in the b-frame
v_{au}, v_{av}	Apparent velocity decomposed in the longitudinal/lateral direction in the b-frame
v_{tw}, α_{tw}	True wind velocity/angle in the n-frame
$X/Y/K/N_{\dot{u}-\dot{r}}$	Added mass coefficients in the b-frame
x, y	Translation in the n-frame
x_m	x-coordinate of the mast in the b-frame
(x_s, y_s, z_s)	Center of effort (CoE) with the subscripts s, r, k, and h indicate respectively the sail, rudder, keel, and hull
x_{sm}	Distance between the mast and the sail's CoE
xyz	North-East-Down coordinate system (n-frame)
$x_b y_b z_b$	Body-fixed reference frame (b-frame)
α_e	Effective angle of attack in the b-frame
δ_r, δ_s	Rudder/sail angle in the b-frame
$\bar{\delta}_s$	Maximum sail angle
$\boldsymbol{\eta}$	Vector of position and orientation in the n-frame
$\boldsymbol{\nu}$	Velocity vector in the b-frame
ρ	Flow density
$\boldsymbol{\tau}$	Vector of control inputs
ϕ, ψ	Euler's angles in the n-frame
$\boldsymbol{\omega}$	Angular velocity vector in the b-frame

Applying Newton's second law and Euler's axioms, we derive the rigid-body dynamics expressed in the powerful compact vectorial setting introduced by Fossen [8][9]:

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}, \tag{1}$$

in which $\boldsymbol{\tau}_{RB}$ is the vector of external forces and moments acting on the yacht, $\boldsymbol{M}_{RB}$ and $\boldsymbol{C}_{RB}(\boldsymbol{\nu})$ are the rigid-body inertia matrix and Coriolis-centripetal matrix, which in the present paper are

$$\boldsymbol{M}_{RB} = \begin{bmatrix} m\mathbf{I}_{2\times 2} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \boldsymbol{I} \end{bmatrix}, \boldsymbol{I} = \begin{bmatrix} I_{xx} & -I_{xz} \\ -I_{xz} & I_{zz} \end{bmatrix}, \tag{2}$$

(where $\mathbf{I}_{2\times 2}$ is the identity matrix), and

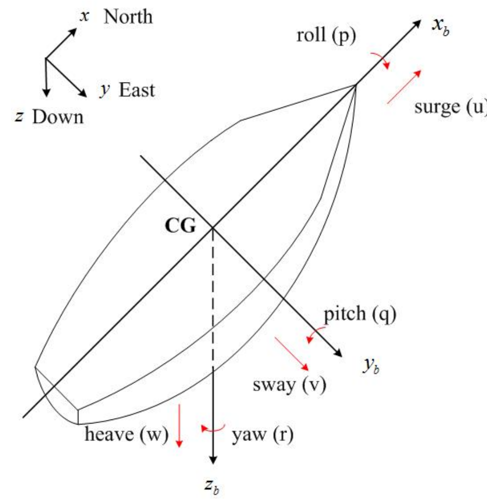

Fig. 1. Description of the coordinate systems.

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} 0 & -mr & 0 & 0 \\ mr & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

Due to the yacht's acceleration and the inertia of the surrounding flow, there are also forces and moments coming from the so-called added mass effect. Similarly to (2) and (3), define matrices $\boldsymbol{M}_A$ and $\boldsymbol{C}_A(\boldsymbol{\nu})$ [8][9] representing the added mass phenomenon:

$$\boldsymbol{M}_A = \begin{bmatrix} X_{\dot{u}} & X_{\dot{v}} & X_{\dot{p}} & X_{\dot{r}} \\ Y_{\dot{u}} & Y_{\dot{v}} & Y_{\dot{p}} & Y_{\dot{r}} \\ K_{\dot{u}} & K_{\dot{v}} & K_{\dot{p}} & K_{\dot{r}} \\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{p}} & N_{\dot{r}} \end{bmatrix}, \tag{4}$$

$$\boldsymbol{C}_A(\boldsymbol{\nu}) = \begin{bmatrix} \mathbf{0}_{2\times 2} & \boldsymbol{C}_{A12}(\boldsymbol{\nu}) \\ \boldsymbol{C}_{A21}(\boldsymbol{\nu}) & \boldsymbol{C}_{A22}(\boldsymbol{\nu}) \end{bmatrix}, \tag{5}$$

where, for example, the added mass coefficient $Y_{\dot{r}}$ represents the force in the y_b-axis due to the acceleration about the z_b-axis. The terms $\boldsymbol{C}_{A12}(\boldsymbol{\nu})$, $\boldsymbol{C}_{A21}(\boldsymbol{\nu})$, and $\boldsymbol{C}_{A22}(\boldsymbol{\nu})$ in (5) will be given later in this paper after simplifications of the added mass matrix (see eq. (30)).

Defining $\boldsymbol{M} = \boldsymbol{M}_{RB} + \boldsymbol{M}_A$, $\boldsymbol{C}(\boldsymbol{\nu}) = \boldsymbol{C}_{RB}(\boldsymbol{\nu}) + \boldsymbol{C}_A(\boldsymbol{\nu})$, and simultaneously considering the damping and restoring forces and moments influencing the motion of the yacht, we then have the following expression:

$$\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}(\boldsymbol{\nu},\boldsymbol{\eta}) + \boldsymbol{g}(\boldsymbol{\eta}) = \boldsymbol{\tau}, \tag{6}$$

where $\boldsymbol{D}(\boldsymbol{\nu},\boldsymbol{\eta})$ is the damping matrix, $\boldsymbol{g}(\boldsymbol{\eta})$ contains the restoring forces, while $\boldsymbol{\tau}$ is a vector related to the forces used to control the vehicle, i.e. the sail and the rudder. The vector $\boldsymbol{\eta}$ is then derived through a coordinate transformation [8][9], giving

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\boldsymbol{\eta})\boldsymbol{\nu}, \tag{7}$$

with $\boldsymbol{J}(\boldsymbol{\eta})$ defined as

$$\boldsymbol{J}(\boldsymbol{\eta}) = \begin{bmatrix} \boldsymbol{J}_1(\boldsymbol{\eta}) & \boldsymbol{0}_{2\times 2} \\ \boldsymbol{0}_{2\times 2} & \boldsymbol{J}_2(\boldsymbol{\eta}) \end{bmatrix}, \tag{8}$$

$$\boldsymbol{J}_1(\boldsymbol{\eta}) = \begin{bmatrix} \cos\psi & -\sin\psi\cos\phi \\ \sin\psi & \cos\psi\cos\phi \end{bmatrix}, \tag{9}$$

$$\boldsymbol{J}_2(\boldsymbol{\eta}) = \begin{bmatrix} 1 & 0 \\ 0 & \cos\phi \end{bmatrix}. \tag{10}$$

B. Forces and Moments

In this subsection, we now turn to the derivation of terms $\boldsymbol{g}(\boldsymbol{\eta})$, $\boldsymbol{D}(\boldsymbol{\nu},\boldsymbol{\eta})$ and $\boldsymbol{\tau}$ of representation (6).

In our case, $\boldsymbol{g}(\boldsymbol{\eta})$ comprises the static-righting moment $M_r(\phi)$, which represents the moment forcing the yacht upright at a given heel angle when it is stationary in the water. The heel-damping moment $M_{\phi d}(\dot{\phi})$ and the yaw-damping moment $M_{\psi d}(\dot{\psi})$, which prevent the yacht from oscillating endlessly when it is rotating in the roll and yaw motions, together with the forces from the keel and the hull account for the damping vector $\boldsymbol{D}$ in (6). Typically, terms $M_r(\phi)$, $M_{\phi d}(\dot{\phi})$ and $M_{\psi d}(\dot{\psi})$ can be obtained by fitting a quadratic in ϕ, $\dot{\phi}$, and $\dot{\psi}$, respectively (see for example [10]). As a consequence, we have

$$\boldsymbol{g}(\boldsymbol{\eta}) = \begin{bmatrix} 0 \\ 0 \\ M_r(\phi) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a\phi^2 + b\phi \\ 0 \end{bmatrix}, \tag{11}$$

with a and b the constant coefficients to be determined by the inclining test on the yacht. On the other hand, the quadratic damping is

$$\begin{aligned} \boldsymbol{D}_{heel}(\boldsymbol{\nu}) + \boldsymbol{D}_{yaw}(\boldsymbol{\nu},\boldsymbol{\eta}) &= \begin{bmatrix} 0 \\ 0 \\ M_{\phi d}(\dot{\phi}) \\ M_{\psi d}(\dot{\psi})\cos\phi \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ c\dot{\phi}|\dot{\phi}| \\ d\dot{\psi}|\dot{\psi}|\cos\phi \end{bmatrix}. \end{aligned} \tag{12}$$

Because of the yacht's heeling, the yaw-damping moment $M_{\psi d}(\dot{\psi})$ has components about the y_b- and z_b-axes, hence the presence of term $\cos\phi$ in (12).

As mentioned in the introduction, each part of the yacht (sail, rudder, keel, and hull) is regarded as a 'thin foil' such that the basic aero-hydrodynamic theory can be applied. As a yacht is generally controlled by the motions of the sail and the rudder, we group in $\boldsymbol{\tau}$ in (6) the terms linked to the important variables that are the sail angle δ_s and the rudder angle δ_r.

From aero-hydrodynamics, the lift and drag forces acting on a foil are

$$L = \frac{1}{2}\,\rho\,A\,v_a^2\,C_L(\alpha_e) \tag{13}$$

$$D = \frac{1}{2}\,\rho\,A\,v_a^2\,C_D(\alpha_e), \tag{14}$$

where α_e is the effective angle of attack between the apparent incoming flow and the foil, C_L and C_D are the lift and drag coefficients as functions of α_e, both of which are obtained from appropriate measurements (eg. wind tunnel for an airfoil). Constant A represents the plan area of the foil.

The interaction between the sail and the wind generates the propulsive power of the yacht. As can be seen in Figure 2, the apparent wind $\boldsymbol{v}_{aw}$ as seen by the sail is the vector sum of the so-called true wind and the yacht velocity, i.e. $\boldsymbol{v}_{aw} = \boldsymbol{v}_{tw} - \boldsymbol{v}$, where the vectors $\boldsymbol{v}_{aw}$ and $\boldsymbol{v}$ are defined in the b-frame while $\boldsymbol{v}_{tw}$ is in the n-frame. In the present study, we assumed $\boldsymbol{v}_{tw}$ to be constant with, and without loss of generality, $\boldsymbol{v}_{tw} = [-v_{tw}, 0]^T$, i.e. the wind is coming from the north.

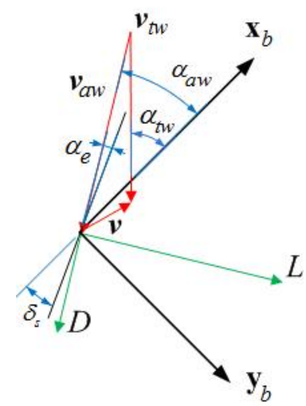

Fig. 2. Wind velocity triangle in 2-dimension.

However, in a 3-dimensional setting, the apparent velocity vector should also incorporate the effect induced by the rotation of the yacht:

$$\boldsymbol{v}_{aw} = \boldsymbol{v}_{tw} - \boldsymbol{v} - \boldsymbol{\omega} \times [x_s, y_s, z_s]^T, \tag{15}$$

(where, still without l.o.g., $\boldsymbol{v}_{tw} = [-v_{tw}, 0, 0]^T$). Thus, in the b-frame we have $\boldsymbol{v}_{tw}^b = \mathbf{R_2}\mathbf{R_1}\boldsymbol{v}_{tw}$ with $\mathbf{R_1}$ and $\mathbf{R_2}$:

$$\mathbf{R_1} = \begin{bmatrix} \cos(-\alpha_{tw}) & -\sin(-\alpha_{tw}) & 0 \\ \sin(-\alpha_{tw}) & \cos(-\alpha_{tw}) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{16}$$

$$\mathbf{R_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\phi) & -\sin(-\phi) \\ 0 & \sin(-\phi) & \cos(-\phi) \end{bmatrix}. \tag{17}$$

Rewriting (15) in component form in the $x_b y_b$-plane, we get

$$\begin{aligned} v_{awu} &= -v_{tw}\cos\alpha_{tw} - u + r y_s, \\ v_{awv} &= v_{tw}\sin\alpha_{tw}\cos\phi - v - r x_s + p z_s, \end{aligned} \tag{18}$$

where v_{awu} and v_{awv} are the apparent wind velocity along the longitudinal and lateral axes in the b-frame, and (x_s, y_s, z_s)

be the sail's center of effort, or CoE for short (i.e. the point at which the total moment produced from all forces over the foil can be represented by a single force producing the same moment). Similarly, the CoE of the rudder, keel, and hull will be (x_r, y_r, z_r), (x_k, y_k, z_k), and (x_h, y_h, z_h), respectively. Hence, we have $v_{aw} = \sqrt{v_{awu}^2 + v_{awv}^2}$, while the apparent wind angle in the b-frame is $\alpha_{aw} = \arctan 2(v_{awv}, -v_{awu})$ [1]. Then, from the definition of α_e, we have

$$\alpha_{es}(v_{tw}, \alpha_{tw}, \boldsymbol{\nu}, \boldsymbol{\eta}, \delta_s) = \alpha_{aw}(v_{tw}, \alpha_{tw}, \boldsymbol{\nu}, \boldsymbol{\eta}) - \delta_s, \quad (19)$$

which, in turn, gives the value of lift and drag coefficients $C_{Ls}(\alpha_{es})$ and $C_{Ds}(\alpha_{es})$. These can finally be used to get the lift and drag *forces* L_s and D_s, obtained from (13)-(14). Hereby, the forces and moments generated by the sail are expressed as

$$\boldsymbol{\tau}_s(v_{tw}, \alpha_{tw}, \boldsymbol{\nu}, \boldsymbol{\eta}, \delta_s) = \begin{bmatrix} F_{xs} \\ F_{ys} \\ M_{xs} \\ M_{zs} \end{bmatrix}$$

$$= \begin{bmatrix} L_s \sin\alpha_{aw} - D_s \cos\alpha_{aw} \\ L_s \cos\alpha_{aw} + D_s \sin\alpha_{aw} \\ -(L_s \cos\alpha_{aw} + D_s \sin\alpha_{aw}) z_s \\ -(L_s \sin\alpha_{aw} - D_s \cos\alpha_{aw}) x_{sm} \sin\delta_s + \\ (L_s \cos\alpha_{aw} + D_s \sin\alpha_{aw})(x_m - x_{sm}\cos\delta_s) \end{bmatrix}, \quad (20)$$

where x_{sm} is the distance between the mast and the sail's CoE, while x_m is the x-coordinate of the mast in the b-frame.

Similarly to (18), the apparent velocity of the water on the rudder is given by

$$\begin{aligned} v_{aru} &= -u + r y_r, \\ v_{arv} &= -v - r x_r + p z_r, \end{aligned} \quad (21)$$

and $\alpha_{er}(\boldsymbol{\nu}, \delta_r) = \alpha_{ar}(\boldsymbol{\nu}) - \delta_r$. As a noteworthy difference with sail expressions, the so-called induced drag effect is also included, so that we have $C_{Dr}(\alpha_{er}) = C_{Dr}(\alpha_{er}) + C_{Di}(\alpha_{er})$ [11] with

$$C_{Di}(\alpha_{er}) = \frac{C_{Lr}^2(\alpha_{er})}{\pi AR}, AR = \frac{s^2}{A}. \quad (22)$$

Equation (22) will also be used when calculating the drag from the keel. Accordingly, we have

$$\boldsymbol{\tau}_r(\boldsymbol{\nu}, \delta_r) = \begin{bmatrix} F_{xr} \\ F_{yr} \\ M_{xr} \\ M_{zr} \end{bmatrix} = \begin{bmatrix} L_r \sin\alpha_{ar} - D_r \cos\alpha_{ar} \\ L_r \cos\alpha_{ar} + D_r \sin\alpha_{ar} \\ -(L_r \cos\alpha_{ar} + D_r \sin\alpha_{ar}) z_r \\ (L_r \cos\alpha_{ar} + D_r \sin\alpha_{ar}) x_r \end{bmatrix}. \quad (23)$$

Consequently, combining (20) and (23), we can derive a complete expression for the input vector, i.e. $\boldsymbol{\tau}(v_{tw}, \alpha_{tw}, \boldsymbol{\nu}, \boldsymbol{\eta}, \delta_s, \delta_r) = \boldsymbol{\tau}_s(v_{tw}, \alpha_{tw}, \boldsymbol{\nu}, \boldsymbol{\eta}, \delta_s) + \boldsymbol{\tau}_r(\boldsymbol{\nu}, \delta_r)$.

[1] $arctan2(y, x)$ is a four-quadrant inverse tangent of the real parts of y and x, and $arctan2(y, x) \in [-\pi, \pi]$.

Similarly again, the forces and moments arising from the keel are

$$\boldsymbol{D}_k(\boldsymbol{\nu}) = \begin{bmatrix} F_{xk} \\ F_{yk} \\ M_{xk} \\ M_{zk} \end{bmatrix} = \begin{bmatrix} -L_k \sin\alpha_{ak} + D_k \cos\alpha_{ak} \\ -L_k \cos\alpha_{ak} - D_k \sin\alpha_{ak} \\ -(-L_k \cos\alpha_{ak} - D_k \sin\alpha_{ak}) z_k \\ (-L_k \cos\alpha_{ak} - D_k \sin\alpha_{ak}) x_k \end{bmatrix}, \quad (24)$$

with the apparent velocity

$$\begin{aligned} v_{aku} &= -u + r y_k, \\ v_{akv} &= -v - r x_k + p z_k, \end{aligned} \quad (25)$$

and $\alpha_{ek} = \alpha_{ak} = \arctan 2(v_{akv}, -v_{aku})$.

With regard to hull forces, the so-called Extended Keel Method is used, whereby the lift force generated by the hull is accounted for by extending the keel and rudder inside the yacht body to the waterline [12]. As a result, in the so-called hull resistance $F_{rh}(v_{ah})$, only the drag component is considered, and it is increasing with the apparent velocity of the flow on the hull in the vessel parallel coordinate system, i.e. the plane parallel to the water surface:

$$\begin{aligned} v_{ahx} &= -u + r y_h, \\ v_{ahy} &= (-v - r x_h + p z_h) \sec\phi, \end{aligned} \quad (26)$$

which gives $v_{ah} = \sqrt{v_{ahx}^2 + v_{ahy}^2}$ and $\alpha_{ah} = \arctan 2(v_{ahy}, -v_{ahx})$ leading to

$$\boldsymbol{D}_h(\boldsymbol{\nu}, \boldsymbol{\eta}) = \begin{bmatrix} F_{xh} \\ F_{yh} \\ M_{xh} \\ M_{zh} \end{bmatrix} = \begin{bmatrix} F_{rh}(v_{ah}) \cos\alpha_{ah} \\ -F_{rh}(v_{ah}) \sin\alpha_{ah} \cos\phi \\ -(-F_{rh}(v_{ah}) \sin\alpha_{ah} \cos\phi) z_h \\ (-F_{rh}(v_{ah}) \sin\alpha_{ah} \cos\phi) x_h \end{bmatrix}. \quad (27)$$

As a consequence, the damping vector is $\boldsymbol{D}(\boldsymbol{\nu}, \boldsymbol{\eta}) = \boldsymbol{D}_k(\boldsymbol{\nu}) + \boldsymbol{D}_h(\boldsymbol{\nu}, \boldsymbol{\eta}) + \boldsymbol{D}_{heel}(\boldsymbol{\nu}) + \boldsymbol{D}_{yaw}(\boldsymbol{\nu}, \boldsymbol{\eta})$.

C. Dynamic Behaviour of the Sail

In addition to the above few terms, another feature of our model is that it also includes the behavior of the sail as it changes sides while tacking or wearing. Indeed, most current models assume that the sail angle can be directly controlled, while in many real situations, the boom is relatively free to move on its own accord, in a tacking maneuver for example, mostly due to the sail luffing and changing sides.

Considering that the sail angle is constrained by a rope, denote by $\overline{\delta}_s$ the maximum sail angle can be obtained on the yacht given a particular rope length, i.e. $|\delta_s| \leq \overline{\delta}_s$. We assume that whenever the yacht tacks, the sail is loose and is aligned with the apparent wind until it catches the wind again, i.e. $\alpha_{es}(v_{tw}, \alpha_{tw}, \boldsymbol{\nu}, \boldsymbol{\eta}, \delta_s) = 0$ in (19). As a result, we propose a simple expression describing such dynamic behaviour:

$$\delta_s = sat_{-\overline{\delta}_s}^{\overline{\delta}_s}(\alpha_{aw}). \quad (28)$$

Note that online tuning of $\overline{\delta}_s$ can of course be considered, and is indeed what is usually done in practise.

D. Model Simplifications

In the added mass matrix given in (4), not all of the terms have evident impact on the forces and moments due to accelerations. In this case, it is possible to simplify (4) to a form which is more convenient computationally.

Considering the yacht to be symmetrical about the $x_b z_b$-plane and $M_{Aij} = M_{Aji}$, gives [8][9]

$$\boldsymbol{M}_A = \begin{bmatrix} X_{\dot{u}} & 0 & 0 & 0 \\ 0 & Y_{\dot{v}} & Y_{\dot{p}} & Y_{\dot{r}} \\ 0 & Y_{\dot{p}} & K_{\dot{p}} & K_{\dot{r}} \\ 0 & Y_{\dot{r}} & K_{\dot{r}} & N_{\dot{r}} \end{bmatrix}. \tag{29}$$

Thus, the matrices in (5) are now

$$\begin{aligned} \boldsymbol{C}_{A12}(\boldsymbol{\nu}) &= \begin{bmatrix} 0 & -Y_{\dot{v}}v - Y_{\dot{p}}p - Y_{\dot{r}}r \\ 0 & X_{\dot{u}}u \end{bmatrix}, \\ \boldsymbol{C}_{A21}(\boldsymbol{\nu}) &= \begin{bmatrix} 0 & 0 \\ Y_{\dot{v}}v + Y_{\dot{p}}p + Y_{\dot{r}}r & -X_{\dot{u}}u \end{bmatrix}, \\ \boldsymbol{C}_{A22}(\boldsymbol{\nu}) &= \begin{bmatrix} \mathbf{0}_{2\times 2} \end{bmatrix}. \end{aligned} \tag{30}$$

For further simplification, we also considered that the smaller off-diagonal terms of (2) and (29) could be neglected for control design, while they can be kept to simulate the real plant dynamics (see also [13] for an interesting discussion on process plant and control plant models). As a result, we get $\boldsymbol{M}_{RB} = diag\{m, m, I_{xx}, I_{zz}\}$ and $\boldsymbol{M}_A = diag\{X_{\dot{u}}, Y_{\dot{v}}, K_{\dot{p}}, N_{\dot{r}}\}$, with $\boldsymbol{C}_A(\boldsymbol{\nu})$ being written as

$$\boldsymbol{C}_A(\boldsymbol{\nu}) = \begin{bmatrix} 0 & 0 & 0 & -Y_{\dot{v}}v \\ 0 & 0 & 0 & X_{\dot{u}}u \\ 0 & 0 & 0 & 0 \\ Y_{\dot{v}}v & -X_{\dot{u}}u & 0 & 0 \end{bmatrix}.$$

This simplification will be used throughout the next section.

III. Heading Control Using Backstepping

Our control objective is to find a feedback law that asymptotically stabilizes the heading ψ of the yacht around a chosen desired heading ψ_d. In this section, we investigate the heading/yaw sub-dynamics of system (6)-(7), and then design a heading controller using the backstepping method.

A. Stability Analysis and Controller Design

From (6)-(7), we extract the following heading/yaw sub-dynamics

$$\begin{aligned} \dot{\psi} &= r\cos\phi \\ \dot{r} &= f(\psi, r, \delta_r, u, v, p, \phi), \end{aligned} \tag{31}$$

where $[\psi, r]^T \in [-\pi, \pi] \times \mathbb{R}$ is the state and $\delta_r \in [-\pi/3, \pi/3]$ is the control input, while u, v, p, ϕ can be seen as disturbances to system (31). The function f is smooth. We then consider the stability of equilibrium points of the nonlinear dynamical system in (31), which is usually characterized in the sense of Lyapunov [14][15]. Additionally, we would like to prove that all solutions are asymptotically stable. To do so, we will make use of the integrator backstepping method [16][17].

Considering again system (31), perform now a change of coordinate and define the variable $\tilde{\psi} = \psi - \psi_d$ representing the error in heading, for which we obtain the new dynamics

$$\begin{aligned} \dot{\tilde{\psi}} &= r\cos\phi & (32) \\ \dot{r} &= f(\tilde{\psi}, r, \delta_r, u, v, p, \phi), & (33) \end{aligned}$$

where the equilibrium point is situated at the origin.

Using the backstepping technique, start with scalar system (32) with r viewed as the virtual input to stabilize the origin $\tilde{\psi} = 0$. Using the control Lyapunov function (CLF) $V_1 = \tilde{\psi}^2/2$, we obtain

$$\dot{V}_1 = \tilde{\psi}\dot{\tilde{\psi}} = \tilde{\psi} r\cos\phi, \tag{34}$$

in which the heeling angle is constrained within $\phi \in]-\pi/2, \pi/2[$. We then set $r = \alpha_1(\tilde{\psi}, \phi) = -\tilde{\psi}\sec\phi$ which gives $\dot{V}_1 = -\tilde{\psi}^2$, hence the origin of (32) is exponentially stable. Rewriting now (32) to make $\alpha_1(\tilde{\psi}, \phi)$ appear explicitly, we have

$$\dot{\tilde{\psi}} = \alpha_1(\tilde{\psi}, \phi) + z_2, z_2 = r\cos\phi - \alpha_1(\tilde{\psi}, \phi), \tag{35}$$

where z_2 is a new state variable to be stabilized.

Thus, the z_2-dynamics is

$$\dot{z}_2 = f\cos\phi - pr\sin\phi - \dot{\alpha}_1(\tilde{\psi}, \phi). \tag{36}$$

Picking the CLF candidate for system (35)-(36) as $V_2 = \tilde{\psi}^2/2 + z_2^2/2$, we obtain

$$\dot{V}_2 = \tilde{\psi}(\alpha_1(\tilde{\psi}, \phi) + z_2) + z_2(f\cos\phi - pr\sin\phi - \dot{\alpha}_1(\tilde{\psi}, \phi). \tag{37}$$

Then, letting

$$\begin{aligned} f(\tilde{\psi}, r, \delta_r, u, v, p, \phi) &= -\frac{\tilde{\psi} + r + r\cos\phi - pr\sin\phi}{\cos\phi} \\ &\quad -\frac{\tilde{\psi}}{\cos^2\phi} - \frac{p\tilde{\psi}\sin\phi}{\cos^3\phi}, \end{aligned} \tag{38}$$

yields $\dot{V}_2 = -\tilde{\psi}^2\sec\phi - z_2^2$ such that the origin of (32)-(33) is asymptotically stable, which in turn implies asymptotic stability of original system (31) around any desired heading. The value of the control input, i.e. the rudder angle, is then obtained by solving equality (38), i.e. we get the control law $\delta_r(\tilde{\psi}, r, \delta_r, u, v, p, \phi)$.

B. Simulation Results

The result of the effect of control law $\delta_r(\tilde{\psi}, r, \delta_r, u, v, p, \phi)$ derived from (38) was simulated in Matlab Simulink for a 12-m class yacht, whose behavior is visualized in a 3D graphic environment (see Fig. 3), with parameters and lift and drag coefficients taken from a real yacht [18]. A standard Runge-Kutta method was employed to solve the equations of motion with a step size of 0.2 seconds. The wind velocity is chosen to be v_{tw} = 10m/s (i.e. approximately 20knots) and the maximum sail angle is set as $\overline{\delta}_s = \pi/9$. We started the simulation with initial values $[x, y, \phi, \psi, u, v, p, r]^T(0) = [0, 0, 0, \pi, 5, 0, 0, 0]^T$.

Fig. 3. Virtual representation of our yacht model.

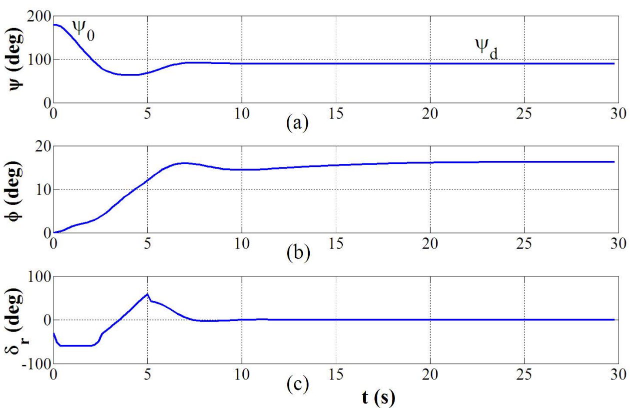

Fig. 4. Time evolutions of ψ, ϕ, and δ_r.

A few plots of this simulation are given in Fig. 4. As can be seen in Fig. 4(a), $\psi(t)$ eventually converges to the desired heading $\psi_d = 90\,\text{deg}$, thus showing the good behavior of the controller for large errors in initial heading. Fig. 4(b) shows that the final roll angle is not zero, as expected from the normal behavior of a yacht ($\phi(\infty) \approx 16.28\,\text{deg}$ in this simulation), while the evolution of rudder angle $\delta_r(t)$ can be seen in Fig. 4(c).

IV. Concluding remarks

In this paper, a 4 DOF mathematical model describing the dynamic motion of the sailing yacht was described using Fossen's vectorial representation, and the integrator backstepping approach is employed to achieve heading control for the sailing yacht. The states of the system are partially controlled using only the rudder as a control input and the asymptotic stabilization of the heading dynamics is realized. Our model is readily available for further testing and is applicable to other yachts by just replacing the parameters and data with that from wind tunnel and towing tank tests.

Stability issues for the entire dynamic model of the yacht and partial stability (stabilization) properties of the overall system are subjects of current research. Additionally, other directions of research include further developping our model in terms of the sail trimming and investigating more nonlinear feedback control laws for autonomous control of surface sailing vehicles.

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