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Theory and Methodology

Yacht velocity prediction using mathematical programming

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Abstract: A yacht velocity prediction program (VPP) is a computer model which can be used to determine the velocity of a given vessel in arbitrary sailing conditions. Such programs are becoming increasingly popular with yacht designers and those who are responsible for determining handicapping rules for yacht racing. We report here on the application of state-of-the-art mathematical programming software to a computer model of a yacht to give a velocity prediction program with which optimisation can be carried out. Such optimisation can be used to determine the best way of sailing a given design, and also to give indications of which trade-offs are important when one is designing a yacht to satisfy strict rule constraints.

Keywords: Sports; Nonlinear programming; Design

1. Introduction

The basic mechanics of sailing have been well known since the 1950's, and a thorough review may be found in the book by Marchaj (1979). A yacht velocity prediction program (VPP) is a computer program based on a model for the forces and moments acting on a sailing yacht, which computes a prediction of the speed of the yacht when it is sailed in a given fashion in a given set of wind conditions. Such programs evolved in the 1970's for the purposes of handicapping yachts, culminating in the International Measurement System which began with the 'H. Irving Pratt Project' (Kerwin, 1978) at MIT. A recent review of their development is given by Larsson (1990).

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Yacht designers were quick to see the advantages in being able to predict yacht performance at the design stage, so that VPP's of varying levels of sophistication are now widely used for design. They are also used by racing yachtsmen to provide guidance on optimum sailing speeds and angles for a completed boat.

The central principle underlying all velocity prediction programs is the following form of Newton's second law:
"For any body which is not accelerating the sum of forces in each coordinate direction, and the sum of moments around each coordinate axis must be zero."

This principle can be used to compute an estimate of the velocity of a sailing vessel in a given set of conditions providing that the forces and moments on the vessel can first be expressed as functions of its velocity. Once these functions are known (or approximated) then Newton's second law, as stated above, can be written down as six simultaneous equations in variables whose values determine the behaviour of the yacht.

If the values of all but six of the variables in these equations are specified then in principle the values of the remaining variables may be found, thereby determining the performance of the vessel at equilibrium. Thus in its most basic form a VPP can be viewed as a program which computes solutions to simultaneous nonlinear equations. In all previous VPP formulations only three equations have been used with the three variables of interest being the boat speed, its heel angle and its leeway angle (as defined below), but here the problem is formulated such that *any* of the numerous variables can be solved for. The method to be described also makes use of modern mathematical programming techniques, which allow a number of important improvements to the conventional VPP.

The factors that affect the motion of a sailing vessel in smooth water can be identified as:

(1) environmental factors (such as wind speed and direction);

(2) design factors (dimensions which are fixed for a completed yacht, such as waterline length);

(3) control variables (those which may be adjusted on a completed yacht, such as variables affecting sail trim).

The simplest models for velocity prediction fix all these variables at given values and solve for the remaining undetermined variables (principally the yacht speed). Since the principal aim of racing yacht designers is to maximise the performance of their design, values for the control variables in (3) should be chosen which maximise the velocity of the vessel (or velocity made good in a given direction). The optimal solution will indicate the best possible set of controls for a particular yacht in a particular situation. Although a number of attempts have been made to develop VPP's which perform this optimisation function by performing some form of unconstrained search (usually on factors which affect sail forces), these have generally been implemented by programmers with little knowledge of modern optimisation technology. We show below that recent advances in this field can be expected to produce VPP's which are more accurate, faster, and more flexible.

The optimisation approach is also extended to include the design variables as well as control variables in the optimisation. It is then possible, for example, for the solution process to choose values of design variables to maximise performance over a given range of weather conditions. Used in this fashion the VPP can be viewed as an automatic design tool.

Before proceeding to the more technical aspects of yacht velocity prediction, it is useful to review in layman's terms what control actions are available to the crew of a racing yacht to alter its speed. The helmsman of a yacht steers it with the *rudder*. He must keep the boat pointing in the direction of the immediate destination while the crew endeavour to maximise the driving forces on the yacht, and minimise the effects due to drag which slows the boat down. The direction that a vessel points is called its *heading*, and the helmsman must continually adjust the rudder angle to maintain the heading, since changes in forces on the boat tend to make it *yaw*, whereby the vessel experiences a change in heading due to rotation about an axis parallel to the mast. The task of the helmsman is further complicated by the fact that a yacht sailing in any direction except downwind slips sideways to some extent as it sails. It is said to have *leeway*, and the course that it follows is slightly different from its heading, by an angle called the leeway angle.

The driving forces on a yacht can be altered by changing the *trim* of the sails, namely their size, shape and orientation to the wind. Throughout a race the sail trim is modified continuously by crew members who adjust ropes connected to the sails to maximise these driving forces. Changing the size of the sail carried can be effected by substituting smaller sails, or by an essentially equivalent process called *reefing*, which changes the size of the currently carried sails. On racing yachts there are also ropes which allow the crew to change the shape and orientation of the sails so as to alter the magnitude, direction, and point of action of the driving force.

Another important factor affecting the drive from the sails is the *heel angle*, namely the angle at which the boat is heeled over due to the wind. Since the amount of sail over which the wind flows is decreased as the yacht heels, such heeling decreases the driving force, and so every effort is made to reduce heel to the extent that crew members position themselves on the windward side of the yacht to produce a compensatory *righting moment*.

Since any control action by helmsman or crew has an effect not only on the driving force that one aims to improve, but also on other aspects of the yacht's behaviour (in particular the drag forces slowing the boat down) it is easy to see that finding the optimal settings of control variables is not straightforward. One of the purposes of a VPP is to shed some light on the relationships between these effects.

The paper is laid out according to the following plan. In the next section we give a brief description of the yacht model, identifying the force and moment balances on the vessel. In order to simplify the exposition, we choose not to discuss here details relating to the derivation of the functions involved. We make no claim to the originality of this model which is based principally on the work of Kerwin (1978). Any other model may be substituted without affecting the proposed optimisation procedure. In Section 3 the problem of maximising yacht performance is formulated as a mathematical programming problem. This is followed by a discussion of methods for determining the degree to which different factors in yacht design affect performance, leading naturally to the formulation of the optimal yacht design problem as a mathematical program.

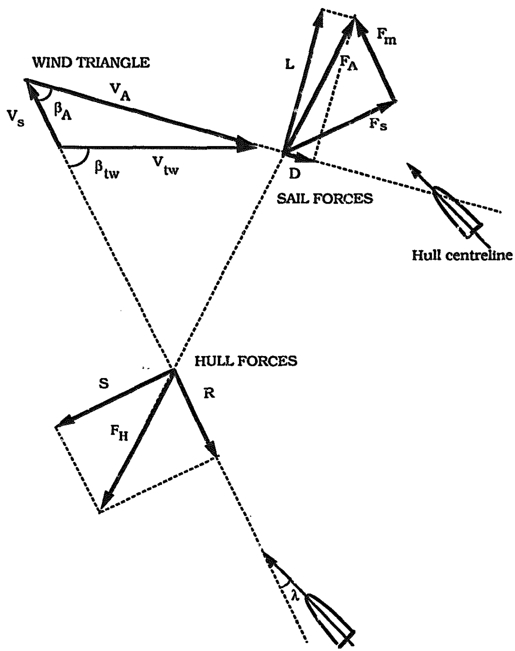

Figure 1. Force and velocity triangle

We conclude by presenting and discussing some of the results that have been produced by the VPP.

2. A model of a sailing vessel

2.1. Force and moment balances

In this section we outline the computer model of the yacht to which we shall apply the optimisation. The development of such a model requires careful judgement, since one must trade realism for computational tractability. While it is possible to identify the origin of all the forces and moments which must balance on a yacht in steady motion, in practice many of them have little influence on the speed of the yacht, and others are difficult to determine in terms of the performance factors mentioned above. A certain amount of approximation is therefore required in order to obtain a complete set of equations, but it is well known that well-founded VPP's give speed predictions which are accurate to a few tenths of a knot.

The model will consist of a number of equations which specify the forces and moments which must sum to zero. We start by assuming that two of these balances are always satisfied, which removes them from further consideration. The first is the vertical force balance which requires the yacht weight to equal the sum of buoyancy and vertical aerodynamic and hydrodynamic forces. While the motion of the yacht does indeed induce dynamic fluid forces, in practice these are totally dominated by the balance of buoyancy and weight. That is, the yacht motion causes a change in displacement (sinkage) which in practice is very small because of the rapid rate of change of buoyancy with displacement. It is therefore possible to assume a fixed displacement, and thereby eliminate further consideration of the vertical forces.

A similar assumption is made regarding the balance of pitching moments. The driving force from the sails tends to depress the bow and raise the stern, and this causes a fore/aft imbalance in buoyancy which resists this pitch. However again the restoring moment from buoyancy is such that the hull is very stiff in pitch, and it is sensible to assume that the mean hull pitch angle does not deviate from its value at rest. The effect of the unsteady pitching motion which is observed in rough water is clearly important, as rough water reduces yacht speed significantly – this may be accounted for later by adding an extra increment of drag to the still-water value.

The remaining forces are those which lie in the plane of the water, for which equilibrium requires the aerodynamic and hydrodynamic forces to be equal in magnitude and opposite in direction. This statement is expressed most effectively by the sketch in Figure 1, where the two kinds of force are shown separately for the same boat; the total aerodynamic force F_A is equal and opposite to the total hydrodynamic force F_H. The yacht is shown sailing at a speed V_S into a wind of *true* speed V_{tw} whose true direction is at an angle β_{tw} to the track of the yacht. In consequence, the yacht sees an *apparent* wind whose speed V_A and angle β_A are easily related to the true wind and yacht speed by the geometry of the velocity triangle shown. Observe also that the direction of the track of the yacht differs from its heading by what is known as the *leeway angle*, denoted λ. (Note that λ is shown exaggerated in Figure 1. It rarely exceeds 6 degrees.) Of course it is the apparent wind rather than the true one which must be used in estimating the aerodynamic force on the sails, and this is further complicated by the fact that the true wind varies with height. It is therefore convenient to use a reference wind at one particular height, which here is taken as that acting at the vertical centre of effort of the force on the sails (around 40% of the mast height).

2.2. Sail forces

By convention the aerodynamic force is divided into two components; the aerodynamic drag (D) which is directed along the reference apparent wind, and the aerodynamic lift (L) which is perpendicular to this. For a sail of area A, lift and drag force coefficients may be defined as follows, where each coefficient is known to depend only upon the sail shape (planform and trim), heel angle (ϕ), and angle of incidence:

$$L = qAC_\ell, \qquad D = qAC_d,$$

where each of C_ℓ and C_d are functions of β_A, ϕ, trim, and planform, and $q = \frac{1}{2}\rho V_A^2$ is the dynamic pressure of air of density ρ and velocity V_A. These statements do imbed some further assumptions, for which the reader is referred to any good aerodynamics text, but they are certainly accurate enough for their use here. Here the term *planform* refers to factors which determine the shape of the sail plan, and which are fixed by the yacht designer and sail maker (the number of sails, relative distance between them, aspect ratio and mainsail roach are examples). Sail *trim* refers to those factors which remain under the control of the crew such as reefing, sail twist and sail camber.

It remains to determine how the force coefficients depend upon the variables shown. The effect of heel angle can be neatly removed by using the following assumption which is suggested by classical aerodynamic theory: the sails generate a force which acts in a plane normal to the mast and which depends only on that component of apparent wind which lies in this plane. If we denote the velocity and direction of the *heeled* apparent wind by V_{aw} and β_{aw}, as distinct from V_A and β_A, and if V_{tw} and β_{tw} measure the true wind speed and direction at the centre of effort of the sail, then

$$V_{aw} = V_{aw}(V_{tw}, \beta_{tw}, \phi, \lambda), \tag{1a}$$

$$\beta_{aw} = \beta_{aw}(V_{tw}, \beta_{tw}, \phi, \lambda) \tag{1b}$$

where the actual relationships are easily found from trigonometry and the leeway angle λ enters because the apparent angle β_{aw} is now being defined relative to the centreline of the yacht (where previously β_A was relative to the track). With the above assumption regarding the forces, we now have lift and drag forces relative to the heeled apparent wind given by

$$L_h = qAC_\ell, \qquad D_h = qAC_d, \tag{2a}$$

$$C_\ell = C_\ell(\beta_{aw}, \text{trim, planform}), \tag{2b}$$

$$C_d = C_d(\beta_{aw}, \text{trim, planform}) \tag{2c}$$

where $q = \frac{1}{2}\rho V_{aw}^2$.

This has eliminated one of the variables affecting the force coefficients, but the dependence of these coefficients upon the other three variables remains to be found. This is undoubtedly the most difficult and least reliable part of all VPP's. Attempts have been made to recover the functional form of the coefficients using full-scale measurements and wind-tunnel modelling (Gerritsma, 1975; Marchaj, 1979) and modern aerodynamic theory (Milgram, 1968; Jackson, 1985) so that there is now some consensus on their proper form. The approach taken here is quite similar to that developed by Kerwin (1978) and Hazen (1980), and involves the use of a flattening factor $f \in [0, 1]$ which decreases the sail lift coefficient, and a reefing factor $r \in [0, 1]$ which gives the proportion of full sail carried. Both these factors alter the lift and drag and centre of effort of the sails in a manner suggested by classical aerodynamic theory. The drag of the hull and rig are added separately. Further details of the sail model are not really needed here, as the optimisation described below can cope with any model which has the general form of (2). (Complete details are given in Sullivan, 1989.)

To complete the discussion of the aerodynamic force it is helpful to calculate the components of the force which act in the direction of forward motion of the yacht and at right angles to this, as they are then in the same frame of reference as that used for the total hydrodynamic force on the hull. The drive force F_m and side force F_s can be expressed in terms of the forces above as follows:

$$F_m = (L_h \cos \beta_{aw} + D_h \sin \beta_{aw}) \cos \phi \sin \lambda + (L_h \sin \beta_{aw} - D_h \cos \beta_{aw}) \cos \lambda, \tag{3}$$

$$F_s = (L_h \cos \beta_{aw} + D_h \sin \beta_{aw}) \cos \phi \cos \lambda - (L_h \sin \beta_{aw} - D_h \cos \beta_{aw}) \sin \lambda. \tag{4}$$

Later we will need the *heel force* which is the sail force acting normal to the hull centreline (and normal to the mast by the assumption made above). The heel force is given by

$$F_h = L_h \cos \beta_{aw} + D_h \sin \beta_{aw}. \tag{5}$$

2.3. Hydrodynamic forces

The total hydrodynamic force on the yacht consists of a drag (or resistance, R) which opposes the forward motion of the vessel, and a lift (or sideforce, S) which opposes the sail side force, as shown in Figure 1. As the hull and keel are (usually) symmetric about the centreline the hull must proceed at a nonzero leeway angle in order to generate a sideforce. The resistance and sideforce forces can then be modelled in a fashion which is similar to their aerodynamic counterparts. The resistance is usually modelled by semi-theoretical formulae which are calibrated to fit data from towing hull models in a tank. In these formulae the yacht shape is described by a number of averaged dimensions and for ease of analysis the hull drag is split into three parts: upright resistance, heeled resistance, and induced resistance.

Upright resistance is the hull drag when proceeding upright and at zero leeway. It is usually further split into a frictional component due to the water viscosity, and a component called wave drag due to the propagation of energy away from the hull in the form of waves. The origin of this division and its justification, may be found in Newman (1977), for example. The theory of frictional resistance is relatively well understood, and one can derive the friction force,

$$F_f = \tfrac{1}{2}\rho_w C_f A_w V_s^2, \tag{6}$$

where A_w is the wetted surface area of the yacht, ρ_w is the density of water, and C_f is a resistance coefficient which depends on the yacht velocity V_s and its waterline length and the water viscosity.

In classical hydrodynamics the wave drag follows from

$$F_w = \tfrac{1}{2}\rho_w C_w A_w V_s^2, \tag{7}$$

where the wave drag coefficient C_w depends in turn only upon the hull shape and Froude number, Fr (a particular combination of boat speed and length). However the relationship between the wave drag coefficient and variables determining hull shape is not fully understood, so in practice the coefficient is often determined for particular shapes by model towing-tank tests. More general expressions between factors which describe hull shape have been derived by a number of researchers who fit curves to the tank data obtained for a range of hull shapes. Relevant examples may be found in van Oossanen (1981) and Gerritsma (1983) and but once again the details are not important provided that an expression is available for the wave drag in the above form.

Induced and heeled resistance are the extra drag supposedly arising from leeway and heel respectively. They are usually combined together into a single formula, $F_i(\lambda, \phi, V_s)$, the derivation of which is loosely based on classical aerodynamics. Once again different authors have arrived at different expressions, with the work of Letcher (1975) being probably the most thorough.

It remains to discuss the hydrodynamic side forces on the yacht hull. These will be made up of three parts arising from the hull canoe-body and from the lifting surfaces of the rudder and keel. Here we ignore the hull lift (as it is small at small leeway angles) and restrict attention to side forces produced by the keel and rudder only, where each of these surfaces is assumed to produce lift from the waterline to the depth that they draw. The side forces generated by these lifting surfaces will be perpendicular to their velocity through the water, and like the lift from the sails, may be expressed in terms of lift coefficients. Thus the total sideforce S on the yacht hull in the plane of the water will be given by

$$S = \tfrac{1}{2}\rho_w V_s^2 (C_r A_r + C_k A_k) \cos\phi. \tag{8}$$

Here A_r and A_k are the plan areas of rudder and keel respectively, and C_r and C_k their corresponding lift coefficients. These coefficients depend upon the shape of the rudder and keel and upon the angles of attack

$$\beta_{ar} = \tan^{-1}(\cos\phi \tan\lambda + \beta_r), \tag{9}$$

$$\beta_{ak} = \tan^{-1}(\cos\phi \tan\lambda). \tag{10}$$

Here as before λ is the leeway angle, and β_r is the angle the rudder makes with the centreline of the yacht. For small angles of attack the lift coefficients are known to depend linearly on the angles. Further details may be found in van Oossanen (1981), for example.

2.4. Heel and yaw moments

Finally, we address the moment balances that the yacht must satisfy. The necessary balance of heel moments about the centreline of the yacht is depicted in Figure 2. Here the sail and keel generate an overturning moment about a longitudinal axis on the waterline via the heel force derived above, and this is resisted by the hydrostatic righting moment on the hull and by a crew moment generated by the crew weight on the windward side of the centreline. Since expressions for the sail, keel, and rudder forces have all been found above only their lever arms are required to get their resulting moments about any fixed point in the hull. Again we follow Hazen (1980) and divide the sail heeling moment into reefable and unreefable components, M_1 and M_0. The height z_0 of the centre of effort of the unreefable heeling force is not reduced by changing r, whereas the height of the centre of effort of the reefable portion of the sail heeling force is reduced to rz_0. The components of the sail heeling moment M_s are then given by multilpying each heel force by the height of its centre of effort above the waterline, whence

$$M_s = M_0(V_{tw}, \beta_{tw}, \phi, \lambda, r) + M_1(V_{tw}, \beta_{tw}, \phi, \lambda, r). \tag{11}$$

Similarly the keel and rudder heeling moments can be calculated from the respective heeling forces acting at z_k and z_r, the vertical distances of the respective centres of effort of keel and rudder from the waterline, giving

$$M_2 = \tfrac{1}{2}\rho_w V_s^2 (C_r A_r z_r + C_k S_k z_k). \tag{12}$$

The sum of these heeling moments must be balanced by the righting moment which for a given boat is a function of heel angle, crew position, and yacht velocity. In the absence of crew, the righting moment at rest can be approximated by fitting a quadratic in ϕ to data from an inclining test on the hull. Alternatively, it can be found from hydrostatic calculations on the hull lines. The yacht motion tends to alter the righting moment slightly and a number of authors include a correction based on empirical tests to allow for this effect.

When the crew contributes to the righting moment, a variable y_c is included which ranges from 1 to -1 to give the crew position relative to the centreline of the yacht ($y_c = 0$ means all the crew are in the centre of the yacht). For small angles of heel ϕ, the contribution, M_c, of crew weight to the righting moment can be approximated by

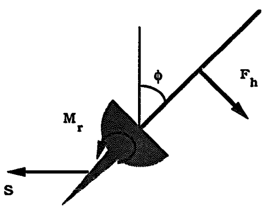

Figure 2. Heeling and righting moments

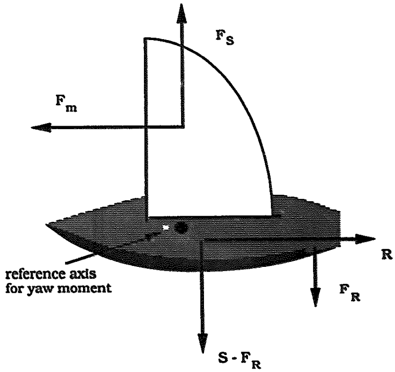

Figure 3. The yaw moment

$$M_c = y_c W_c Y_{max} \cos \phi, \tag{13}$$

where W_c is crew weight and Y_{max} is half the maximum yacht beam. Thus the total righting moment is $M_r(V_s, \phi, \lambda) + M_c$.

The remaining moment balance to consider is the yaw moment which causes rotation about an axis through the centre of the vessel and perpendicular to the surface of the sea. When the boat is sailing steadily this moment is zero. As shown by Figure 3 the yaw moment may be thought of as arising from the sail driving force, the side forces on different parts of the yacht, and the hull resistance, all of which have been calculated above. Although the lever arms for the forces contributing to the yaw moment will vary slightly as the velocity and heel of the vessel changes, we have assumed them to be fixed.

There are three hydrodynamic contributions to the yaw moment. First, the total hull resistance

$$R = F_f + F_w + F_h$$

acts at an angle λ to the centreline of the boat, at a centre of resistance which moves across the vessel as it heels. If at rest z_y is the distance of this centre below the waterline, and x_y is the distance of this centre from the yaw axis, then the yaw moment produced by the hull resistance at an angle of heel of ϕ is

$$M_3 = R(z_y \cos\lambda \sin\phi - x_y \sin\lambda \cos\phi). \quad (14)$$

The keel and rudder side forces also contribute to the yaw moment acting at lever arms x_r and x_k respectively, giving an opposing moment of

$$M_4 = \tfrac{1}{2}\rho_w V_s^2 (C_r A_r x_r + C_k A_k x_k) \cos\phi \cos\lambda. \quad (15)$$

Finally, the contribution of sail forces to the yaw moment must be identified. If x_s is the yaw lever arms of the sails then the yaw moment M_5 from the sails comes from the side force $F_s \cos\lambda$ acting at x_s, and the motive force $F_m \cos\lambda$ acting at a lever arm of length $z_0 r \sin\phi$, whence

$$M_5 = (x_s F_s + z_0 r \sin\phi\, F_m) \cos\lambda. \quad (16)$$

We conclude this section by observing that since the leeway angle λ is usually small, many of the above equations may be simplified further by using linear approximations for $\cos\lambda$ and $\sin\lambda$.

3. The VPP

We are now in a position to summarise the force and moment balances described in the previous section, and incorporate them into an optimisation model. The forces and moments derived above are required to balance as follows:

$$R = F_f + F_w + F_i(V_s, \phi, \lambda) = F_m, \quad (17)$$

$$F_s = S, \quad (18)$$

$$M_s + M_2 = M_r(V_s, \phi, \lambda) + M_c, \quad (19)$$

$$M_3 + M_5 = M_4. \quad (20)$$

Here (17) states that the motive force from the sails equals the hull resistance, (18) states that the side forces from keel and rudder are equal to the sail side force, (19) states that the heeling moments equal the righting moment, and (20) states that the total yaw moments rounding the vessel into the wind balance the moments from keel and rudder turning it away from the wind.

The solution of these equations requires the provision of the additional equations outlined in Section 2, expressing each force and moment in terms of the primary variables of interest, namely the variables which describe the physical size and shape of the yacht, and its performance. These primary variables are of four kinds. First, there are *environmental* variables which describe the wind and sea conditions in which the yacht is sailing, of which the most important is the true wind speed. We shall denote the vector of environmental variables by $\boldsymbol{e}$.

Second there are variables which describe the size and shape of the components of the yacht. These are required in order to estimate the forces and moments when using the coefficients defined above. These may be thought of as *design* variables. Examples include the areas of the sails, keel and rudder, aspect ratios of the sails and keel, and length and beam of the hull. We shall denote the vector of design variables by $\boldsymbol{d}$.

Third, there are *control* variables whose values measure settings which may be altered on the vessel. In the model described above these are:
β_r = rudder angle,
y_c = crew position,
f = amount that sail is flattened,
r = amount the sail is reefed.
We shall denote this vector of four control variables by $\boldsymbol{c}$.

Finally there are the primitive variables with which we describe the actual *behaviour* of a given yacht in the given conditions. These variables were defined in Section 2:
λ = leeway angle,
β_{tw} = true wind angle,
ϕ = heel angle,
V_s = yacht speed.
We shall denote this vector of four behaviour variables by $\boldsymbol{b}$.

There is in fact another class of variables that it makes sense to define. These are the *auxiliary* variables in the model, defined in terms of the components of $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, and $\boldsymbol{e}$ by (1)–(16). We shall denote this vector of eighteen variables by $\boldsymbol{a}$. The components of $\boldsymbol{a}$ are

$$V_{aw}, \beta_{aw}, L_h, D_h, F_m, F_s,$$

$$F_h, F_f, F_w, S, \beta_{ak}, \beta_{ar},$$

$$M_s, M_c, M_2, M_3, M_4, M_5.$$

One can obtain a velocity prediction given values for the components of $\boldsymbol{e}$, $\boldsymbol{d}$ and $\boldsymbol{c}$ by computing values for the components of b and a which solve the 22 simultaneous nonlinear equations (1)–(20).We wish to take this a step further: our optimisation model computes those values for the components of c, b and a which satisfy (1)–(20) and optimise some performance objective.

In order to be able to do this one needs to specify a performance criterion, which may be different in different sailing situations. In particular, when sailing across the wind the usual objective is to maximise yacht velocity, V_s, but when sailing upwind the situation is slightly more complicated because one cannot sail directly into the wind. In this case the objective is to maximise $V_s \cos \beta_{tw}$, the component of velocity in the direction that the wind is coming from, which is known as *velocity made good* (V_{mg}). When racing to a destination point P one seeks to maximise V_{mg} at all points which lie inside the *laylines* of P for the yacht in question. These imaginary lines pass through P and make an angle of β_{tw} with the true wind, where β_{tw} takes the value which maximises V_{mg}. In what follows, we shall refer to the performance criterion as V, which can be interpreted as either V_{mg} or V_s depending on the context.

The yacht performance optimisation problem can therefore be posed as follows:

YPP: Choose values for the components of c, a and b, so as to maximise V subject to the constraints (1)–(20).

Given that solutions to YPP are easily computable, one can in principle determine the optimal sailing performance of a given yacht over a range of wind conditions. It is possible then to investigate the effect on performance of altering some of the design parameters of the yacht. A naive approach is to change these parameters one at a time in an interactive fashion, and recompute the optimal solution. Alternatively, one can allow the design variables d to vary between simple bounds in YPP, and compute the values of the Lagrange multipliers of these bound constraints at optimality. These give a measure of the effect of each design parameter on yacht performance in the assumed wind condition.

Of course one would like a yacht to perform well over a range of weather conditions and racing circumstances, and it is the problem of choosing design variables d to make this happen which is of ultimate concern. The specification of a performance criterion in this case is a less straightforward problem, since it is often the case that design features which give a yacht good performance in light winds are detrimental to its performance in heavy winds. The same could be said for performance when sailing downwind as compared with sailing towards the wind. Furthermore the performance criteria will change with the type of yacht race (short course racing versus ocean racing), the race venue, and the time of year that the race is scheduled. For the purposes of this paper we restrict attention to short course racing and seek to maximise a weighted sum of expected velocity and expected velocity made good, where the weights are determined by the course to be sailed, and the expectations are taken over some discrete probability distribution function for wind speeds at the race venue. This amounts to saying that the environmental variables e may take only a finite number of values, e^i, $i = 1, 2, \dots, p$, each with a known probability π^i.

The yacht design problem can therefore be posed as follows:

DP: Choose values for the components of d and for each wind condition i, $i = 1, \dots, p$, values a^i, b^i, and c^i for the components of a, b and c so as to maximise

$$\sum_{i=1}^{p} \pi^i u^i V_s^i + \sum_{i=1}^{p} \pi^i w^i V_{mg},$$

subject to the $22p$ constraints $(1)^i$–$(20)^i$, $i = 1, 2, \dots, p$.

Here the weight w^i measures the amount of sailing into the wind required in wind condition i and u^i measures the amount of remaining sailing required in wind condition i. The constraints $(1)^i$–$(20)^i$ are (1)–(20) with variables a^i, b^i, c^i and e^i substituted componentwise for a, b, c and e. This model admits the inclusion of additional constraints on the components of d, such as those which would arise from a rating rule.

4. Results

In this section we give some preliminary results from applying a nonlinear programming code NPSOL Version 4.0 to the models YPP and DP of the previous section. NPSOL 4.0 is a FORTRAN implementation of a sequential quadratic programming (SQP) algorithm developed by members of SOL at Stanford University (see Gill, Murray, Saunders and Wright, 1986). These algorithms solve nonlinear optimisation problems by solving a sequence of quadratic programming problems, each of which gives a search direction for the original problem. SQP algorithms are most suitable for relatively small scale, dense problems, of which YPP is an example. The details of SQP do not concern us here: the reader is referred to Gill, Murray, and Wright (1981).

A set of 30 test problems was run for YPP on a VaxStation 2000 with randomly chosen starting points. The solutions took between 1 and 3 seconds of CPU time to compute, and the code failed on five problems due to a poor choice of starting point. In all five cases the correct solution was found after changing the starting point. (Since, for YPP, a good choice of starting point is easy to determine intuitively, choosing a poor starting solution is unlikely to occur in practice.)

As a test of the YPP model a comparison was made between the actual and predicted performances of KZ5, a 12 metre yacht designed in New Zealand prior to the 1987 America's Cup. The ratio of predicted to observed velocity made good over a range of wind speeds is plotted in Figure 4.

A similar test was carried out using DP. Here the model was applied to KZ7, the 12 metre yacht which was New Zealand's entrant in the 1987 America's Cup. For simplicity the yacht design variables were limited to length, beam, draught, mast height and boom length. The values of these design variables influence the parameters of the model according to relationships which can be determined empirically. The rating rule which constrains a boat to be in the 12 metre class can be expressed by the following inequality constraint

$$\frac{1}{2.37}\Big(L_2 + 1.5Gc_f - 0.18 + \tfrac{1}{3}Gc_a - \tfrac{2}{3}h_a + 4B_p + 2L_p + 2Gs_m - 2Gc_m - F + \sqrt{S}\Big) + 3T_p \leq 12.$$

The details of the terms in this inequality do not concern us here. However, contrary to what is usually reported in the popular press, each of these terms does not represent a simple measurement of the vessel, but is a parameter which has been derived from measurements and adjusted by penalties if the measurements do not fall within certain bounds. Thus the constraint above really represents a set of inequality constraints on the design variables. Since the best tradeoffs with respect to these constraints are not immediately obvious, it seems that a design approach based on mathematical programming should yield promising results.

The environmental variables in DP were chosen to represent 2 different points of sail and 11 wind conditions. Cases 1 – 11 are sailing upwind in wind speeds ranging from about 3m/s to 13m/s and cases 12 – 22 are sailing downwind over the same wind speed range. The values of u^i, w^i, $i = 1,\dots,11$, were chosen to be 1. DP was run twice, each time with a different choice of discrete probability distribution for weather conditions given by π^i, $i = 1,\dots,11$. The first run used the probability distribution of wind speeds between 1pm and 5pm in Freemantle forecast by the New Zealand syndicate in the 1987 America's Cup races; the second run was carried out in order to investigate the effect of stronger than expected winds. Here the wind speeds are uniformly greater by 2m/s than those used in the first run.

The results of the two runs are shown in Figure 5 which shows the percentage change in objective function with respect to that given by the model YPP using the actual design parameters of KZ7. With the forecast wind speed probabilities (run #1) the optimal design decreased the waterline length of KZ7 and increased its sail

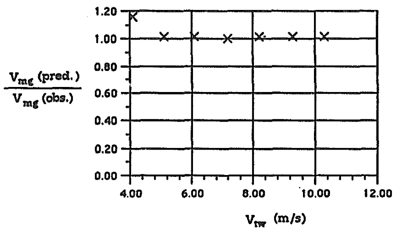

Figure 4. Observed and predicted performance of KZ5

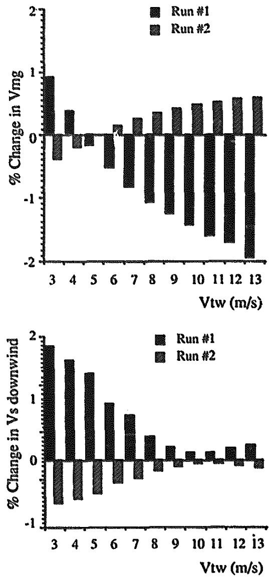

Figure 5. Changes in performance of KZ7 after design optimisation

area. However for the second probability distribution which assumed higher wind speeds, a longer waterline length and smaller sail area is optimal.

As one may observe from Figure 5, a shorter length gives improved downwind performance but results in a loss of upwind performance at higher wind speeds. A longer boat has a better performance at higher wind speeds especially upwind but does not perform so well downwind.

5. Conclusions

In the preceding sections we have outlined an optimisation model to predict yacht performance. This model has a number of weaknesses, arising principally because many of the physical phenomena which give rise to the forces on a yacht are not very well understood.

Two areas where this is evident are the hull-drag modelling and the sail modelling. One must still resort to tank tests to deduce a relationship between hull shape and residuary drag in still water. The formulae generated by this process are at best empirical in nature, and because the expense of tank testing precludes large numbers of observations, one finds that different researchers postulate quite different formulae. Furthermore any theoretical relationship deduced from tank test is only valid in smooth water; for a real yacht the effects of rough water must be taken into account. With regard to the sails, the relationship between lift and drag coefficients as functions of wind velocity and the orientation of the sails to the wind is not well understood and needs further investigation.

It is hoped that further research will give better quantitative models of the wind and water forces on a yacht. Even if large computation times mean that such models are too complicated as they stand to be used effectively in a VPP program, they will allow the construction of less accurate versions which are fast to compute and still give reasonable approximations to reality. Even with the current approximations made in our model we have shown that it is possible to produce results which give a reasonably good agreement with observed data.

It is clear from the ease with which convergence is obtained for the sequential quadratic programming routine that the use of state-of-the-art numerical optimisation software has considerable advantages over the ad hoc methods adopted by researchers in this field in the past. Of course this software can never entirely take over the role of the yacht designer by automatically generating a single optimal design; indeed in its current form the VPP program must be supplied with data from a base hull shape which must come from the designer's board. Nevertheless, the accurate, fast VPPs which modern optimisation techniques make possible, and the ability to quickly and exhaustively explore design tradeoffs, will give those designers who adopt this technology an advantage that they should be keen to exploit.

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