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MEng Title

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Abstract

English

Afrikaans

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Nomenclature

Ocean Vessel Dynamics

$X,\ Y,\ Z$	Coordinates of force vector decomposed in the body-fixed frame(surge, sway and heave forces)
$K,\ M,\ N$	Coordinates of moment vector decomposed in the body-fixed frame(roll, pitch and yaw moment)
$u,\ v,\ w$	Coordinates of linear velocity vector decomposed in the body-fixed frame(surge, sway and heave velocities)
$p,\ q,\ r$	Coordinates of angular velocity vector decomposed in the body-fixed frame(roll, pitch and yaw angular velocities)
$x,\ y,\ z$	Coordinates of position vector decomposed in the body-fixed frame(surge, sway and heave positions)
$\phi,\ \theta,\ \psi$	Coordinates of Euler angle vector decomposed in the body-fixed frame(roll, pitch and yaw Euler angles)

Acronyms and abbreviations

SNAME	Society of Naval Architects and Engineers
CG	Center of gravity of the vessel

Chapter 1

Introduction

1.1. Background

1.2. Problem Statement

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Chapter 2

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Modeling of Ocean Vessels

This chapter models a standard ocean vessel in six degrees of freedom. It also introduces the definitions associated with movement in each direction of freedom. The chapter also take into account the forces and moments generated by hydrodynamics and restoration of an ocean vessel. The chapter continues to model the environmental disturbances experience by a semi-submerged ocean vessel. The environmental disturbances are wind, waves and ocean currents.

3.1. Standard Ocean Vessel Notation

An ocean vessels are modelled in six degrees of freedom, requiring six independent coordinates to determine its position and orientation. The first three coordinates corresponding to position (x, y, z) and their first time derivatives, translation motion along the x-, y-, and z-axes. The last three coordinates (ϕ, θ, ψ) and their first time derivatives describing orientation and rotational motion [1]. Figure 3.1 illustrates the motion variables of an ocean vessel with the six independent coordinates.

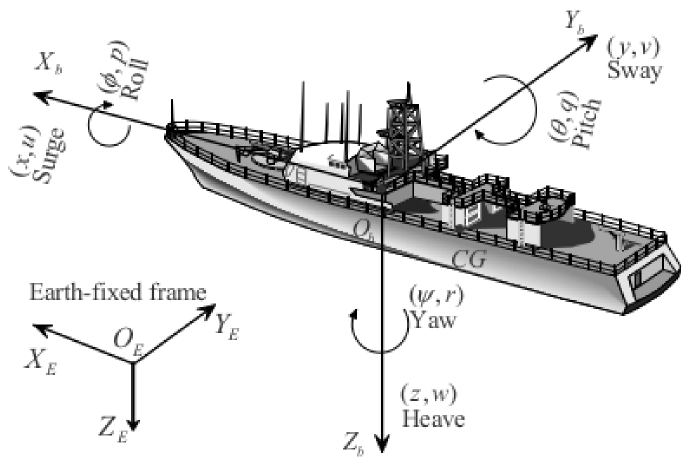

Figure 3.1: Motion variables for an ocean vessel

The SNAME(Society of Naval Architects and Marine Engineers) established the notation for the six different motion components as *surge*, *sway*, *heave*, *roll*, *pitch* and *yaw*. Table A.1 summarizes the SNAME notation for ocean vessels.

Two reference models are used to determine the equations of motion, namely the inertial to earth frame $O_eX_eY_eZ_e$ that may be displaced to overlap with the vessel's fixed coordinates in some initial condition and the body-fixed frame $O_bX_bY_bZ_b$, illustrated in Figure 3.1. The most common used position for the body-fixed frame results in symmetry about the $O_bX_bZ_b$-plane and approximate symmetry about the $O_bY_bZ_b$. The body axes coincides with the axes of inertia and are usually defines as follows: O_bX_b is the longitudinal axis, O_bY_b is the transverse axis and O_bZ_b is the normal axis. Below are the vectors used to describe the general motion of an ocean vessel:

$$\mathbf{n} = [\mathbf{n_1}\mathbf{n_2}]^T \tag{3.1}$$

$$\mathbf{v} = [\mathbf{v_1}\mathbf{v_2}]^T \tag{3.2}$$

$$\tau = [\tau_\mathbf{1}\tau_\mathbf{2}]^T \tag{3.3}$$

$$\mathbf{n_1} = [\,x \;\; y \;\; z\,]^T \tag{3.4}$$

$$\mathbf{n_2} = [\,\phi \;\; \theta \;\; \psi\,]^T \tag{3.5}$$

$$\mathbf{v_1} = [\,u \;\; v \;\; w\,]^T \tag{3.6}$$

$$\mathbf{v_2} = [\,p \;\; q \;\; r\,]^T \tag{3.7}$$

$$\tau_\mathbf{1} = [\,X \;\; Y \;\; Z\,]^T \tag{3.8}$$

$$\tau_\mathbf{2} = [\,K \;\; M \;\; N\,]^T \tag{3.9}$$

where $\mathbf{n}$ denotes the position and orientation vector with coordinates in the earth fixed frame, $\mathbf{v}$ denotes the linear and angular velocity vector with coordinates in the body-fixed frame and τ denotes the forces and moments acting on the vessel in the body-fixed frame. The vessel dynamics are divided into two parts known as *kinematics* and *kinetics*.

3.2. Kinematics

Kinematics looks at the motion of the vessel without directly considering the forces affecting the motion. The first time derivative of the position vectors $\mathbf{n_1}$ and $\mathbf{n_2}$ is related to the linear velocity vector $\mathbf{v_1}$ and $\mathbf{v_2}$ via the following transformations,

$$\dot{\mathbf{n}}_\mathbf{1} = \mathbf{J_1}(\mathbf{n_2})\mathbf{v_1} \tag{3.10}$$

$$\dot{\mathbf{n}}_\mathbf{2} = \mathbf{J_2}(\mathbf{n_2})\mathbf{v_2} \tag{3.11}$$

where $\mathbf{J_1}(\mathbf{n_2})$ and $\mathbf{J_2}(\mathbf{n_2})$ are transformation matrices, which is related through the functions of the Euler angles: roll(ϕ), pitch(θ) and yaw(ψ). The $\mathbf{J_1}$ transformation matrix is given by

$$\mathbf{J_1(n_2)} = \begin{bmatrix} \cos(\psi)\cos(\theta) & -\sin(\psi)\cos(\theta)+\sin(\phi)\sin(\theta)\cos(\psi) & \sin(\psi)\sin(\phi)+\sin(\theta)\cos(\psi)\cos(\phi) \\ \sin(\psi)\cos(\theta) & \cos(\psi)\cos(\phi)+\sin(\phi)\sin(\theta)\sin(\psi) & -\cos(\psi)\sin(\phi)+\sin(\theta)\sin(\psi)\cos(\phi) \\ -\sin(\theta) & \sin(\phi)\cos(\theta) & \cos(\phi)\cos(\theta) \end{bmatrix} \tag{3.12}$$

and the transformation matrix $\mathbf{J_2}$ is given by,

$$\mathbf{J_2(n_2)} = \begin{bmatrix} 1 & -\sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)/\cos(\theta) & \cos(\phi)/\cos(\theta) \end{bmatrix} \tag{3.13}$$

When $\theta = \pi/2$, the transformation matrix $\mathbf{J_2(n_2)}$ becomes singular, however this is unlikely to happen when practically testing an ocean vessel, because of the metacentric restoring forces. Combining Equation 3.12 and Equation 3.13 results in the kinematics of an ocean vessel.

$$\begin{bmatrix} \dot{\mathbf{n}}_\mathbf{1} \\ \dot{\mathbf{n}}_\mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{J_1(n_2)} & 0_{3\times3} \\ 0_{3\times3} & \mathbf{J_2(n_2)} \end{bmatrix} \begin{bmatrix} \mathbf{v_1} \\ \mathbf{v_2} \end{bmatrix} = \dot{\mathbf{n}} = \mathbf{J(n)v} \tag{3.14}$$

3.3. Kinetics

The Newton-Euler formulation [2] defines the balancing forces and moments for a rigid body with a mass of m as follows,

$$\mathbf{f}_{Ob} = \mathbf{m}[\dot{\mathbf{v}}^E_{Ob} + \dot{\mathbf{w}}^E_{Ob} \times \mathbf{r}_{Ob} + \mathbf{w}^E_{Ob} \times \mathbf{v}_O b + \mathbf{w}^E_{Ob} \times (\mathbf{w}^E_{Ob} \times \mathbf{r}_{Ob})] \tag{3.15}$$

$$\mathbf{m}_{Ob} = \mathbf{I}_o \mathbf{w}^E_{Ob} + \dot{\mathbf{w}}^E_{Ob} \times \mathbf{I}_o \mathbf{w}^E_{Ob} + m\mathbf{r}_{Ob} \times (\dot{\mathbf{v}}_O b + \mathbf{w}^E_{Ob} \times \mathbf{v}_{Ob}) \tag{3.16}$$

where $\mathbf{f}_{Ob}$ is the balancing forces,$\mathbf{m}_{Ob}$ the balancing moments and $\mathbf{I}_o$ is the inertia matrix about O_b defined as,

$$\mathbf{I_o} = \begin{bmatrix} I_x & -I_{xy} & -Ixz \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_Z \end{bmatrix} \tag{3.17}$$

I_x,I_y and I_z are the moments of inertia about the O_bX_b, O_bY_b and O_bZ_b axes. $I_{xy} = I_{yx}$, $I_{xz} = I_{zx}$ and $I_{yz} = I_{zy}$ are the products of inertia. These quantities are defined as

$$I_x = \int_V (y^2 + z^2)\rho_m dV \tag{3.18}$$

$$I_x y = \int_V xy\rho_m dV \tag{3.19}$$

$$I_y = \int_V (x^2 + z^2)\rho_m dV \tag{3.20}$$

$$I_x z = \int_V xz\rho_m dV \tag{3.21}$$

$$I_z = \int_V (z^2 + y^2)\rho_m dV \tag{3.22}$$

$$I_z y = \int_V zy\rho_m dV \tag{3.23}$$

where ρ_m are the mass density and V the volume of the rigid body. By substituting the definitions defined in Table A.2 into Equations 3.15 and 3.16, results in the equation below,

$$\mathbf{M_{RB}}\dot{\mathbf{v}} + \mathbf{C_{RB}(v)v} = \tau_{\mathbf{RB}} \tag{3.24}$$

where $\mathbf{v} = [u\ v\ w\ p\ q\ r]^T$ is the generalized velocity vector decomposed in the body-fixed frame and $\tau_{\mathbf{RB}} = [X\ Y\ Z\ K\ M\ N]^T$ is the generalized vector of external forces and moments. The rigid body system inertia matrix $\mathbf{M_{RB}}$ and the rigid body Coriolis and centripetal matrix $\mathbf{C_{RB}}$ is defined in Equation A.1 and A.2 The generalized external force and moment vector, $\tau_{\mathbf{RB}}$, is a sum of the hydrodynamic force and moment vector $\tau_{\mathbf{H}}$, external disturbance force and moment vector $\tau_{\mathbf{E}}$ and propulsion force and moment vector τ.

3.4. Hydrodynamic Forces and Moments

Hydrodynamic forces and moments can be defined as the forces and moments on a ocean body when the body is forced to oscillate with the wave excitation and no wave are incident on the body. As shown in [3], the hydrodynamic forces and moments acting on a rigid body can be assumed to be linearly superimposed. The forces and moments can be subdivided into three components,

1. Added mass due to the inertia of the surrounding fluid
2. Radiation-induced potential damping due to the energy carried away by the generated surface waves
3. Restoring forces due to Archimedian forces

The hydrodynamic forces and moments vector $\tau_{\mathbf{H}}$ is expressed in the equation below,

$$\tau_{\mathbf{H}} = -\mathbf{M_A}\dot{\mathbf{v}} - \mathbf{C_A}(\mathbf{v})\mathbf{v} - \mathbf{D}(\mathbf{v})\mathbf{v} - \mathbf{g}(\mathbf{n}) \tag{3.25}$$

where $\mathbf{M_A}$ is the added mass matrix, $\mathbf{C_A}(\mathbf{v})$ is the hydrodynamic Coriolis and centripetal matrix, $\mathbf{D}(\mathbf{v})$ is the damping matrix and $\mathbf{g}(\mathbf{n})$ is the position and orientation depending vector of restoring forces and moments. The added mass $\mathbf{M_A}$ is given below,

$$\mathbf{M_A} = \begin{bmatrix} X_{\dot{u}} & X_{\dot{v}} & X_{\dot{w}} & X_{\dot{p}} & X_{\dot{q}} & X_{\dot{r}} \\ Y_{\dot{u}} & Y_{\dot{v}} & Y_{\dot{w}} & Y_{\dot{p}} & Y_{\dot{q}} & Y_{\dot{r}} \\ Z_{\dot{u}} & Z_{\dot{v}} & Z_{\dot{w}} & Z_{\dot{p}} & Z_{\dot{q}} & Z_{\dot{r}} \\ K_{\dot{u}} & K_{\dot{v}} & K_{\dot{w}} & K_{\dot{p}} & K_{\dot{q}} & K_{\dot{r}} \\ M_{\dot{u}} & M_{\dot{v}} & M_{\dot{w}} & M_{\dot{p}} & M_{\dot{q}} & M_{\dot{r}} \\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{w}} & N_{\dot{p}} & N_{\dot{q}} & N_{\dot{r}} \end{bmatrix} \tag{3.26}$$

The hydrodynamic Coriolis and centripetal matrix is given below,

$$\mathbf{C_A}(\mathbf{v}) = \begin{bmatrix} 0 & 0 & 0 & 0 & -a_3 & a_2 \\ 0 & 0 & 0 & a_3 & 0 & -a_1 \\ 0 & 0 & 0 & -a_2 & a_1 & 0 \\ 0 & -a_3 & a_2 & 0 & -b_3 & b_2 \\ a_3 & 0 & -a_1 & b_3 & 0 & -b_1 \\ -a_2 & a_1 & 0 & -b_2 & b_1 & 0 \end{bmatrix} \tag{3.27}$$

where a_1, a_2, a_3, b_1, b_2 and b_3 are defined in Equations A.3, A.4, A.5, A.6, A.7 and A.8.

The general hydrodynamic damping experienced by ocean vessels is the potential damping, skin friction, wave drift damping and damping due to vortex shedding. The hydrodynamic damping can be expressed in a general form as below,

$$\mathbf{D}(\mathbf{v}) = \mathbf{D} + \mathbf{D_n}(\mathbf{v}) \tag{3.28}$$

where the linear damping matrix $\mathbf{D}$ is given below,

$$\mathbf{D} = -\begin{bmatrix} X_u & X_v & X_w & X_p & X_q & X_r \\ Y_u & Y_v & Y_w & Y_p & Y_q & Y_r \\ Z_u & Z_v & Z_w & Z_p & Z_q & Z_r \\ K_u & K_v & K_w & K_p & K_q & K_r \\ M_u & M_v & M_w & M_p & M_q & M_r \\ N_u & N_v & N_w & N_p & N_q & N_r \end{bmatrix} \tag{3.29}$$

3.5. Restoring Forces and Moments

The restoring forces and moments are described by the symbol $\mathbf{g}(\mathbf{n})$. If ∇ is the volume of fluid displaced by the ocean vessel. The acceleration of gravity, g and the water density ρ. The submerged weight of the body and buoyancy forces are defined by

$$W = mg \tag{3.30}$$

$$B = \rho g \nabla \tag{3.31}$$

With the above definition for body and buoyancy forces, the restoring force and moment vector $\mathbf{g}(\mathbf{n})$ is due to gravity and buoyancy forces and is given by

$$\mathbf{g}(\mathbf{n}) = \begin{bmatrix} (W-B)sin(\theta) \\ -(W-B)cos(\theta)sin(\phi) \\ -(W-B)cos(\theta)cos(\phi) \\ -(y_gW - y_bB)cos(\theta)cos(\phi) + (z_gW - z_bB)cos(\theta)sin(\phi) \\ (z_gW - z_bB)sin(\theta) + (x_gW - x_bB)cos(\theta)cos(\phi) \\ -(x_gW - x_bB)cos(\theta)sin(\phi) - (y_gW - y_bB)sin(\theta) \end{bmatrix} \tag{3.32}$$

where (x_b, y_b, z_b) denote coordinates of the center of buoyancy(CB). The centre of buoyancy is the point at which the buoyancy forces acts on the body and is equivalent to the geometric center of the submerged portion of the hull [4]. This volume is often assumed since it is hard to define a mathematical formula for the ships geometry. Figure 3.2 shows the positions of the CG and CB. The CG and CB are approximated to be at a distance r form each other when all angles ϕ, θ, ψ are zero.

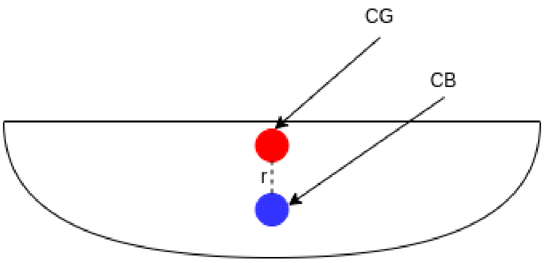

Figure 3.2: Centre of Gravity and Centre of Buoyancy

When an object is disturbed from a static position and returns to its original position is considered statically stable or in static equilibrium. Such ocean vessels are "self righting". If the vessel is unable to return to its equilibrium position and continues to turn over(capsize) it is considered unstable. Figure 3.3 illustrates the righting moment that will act on a stable vessel.

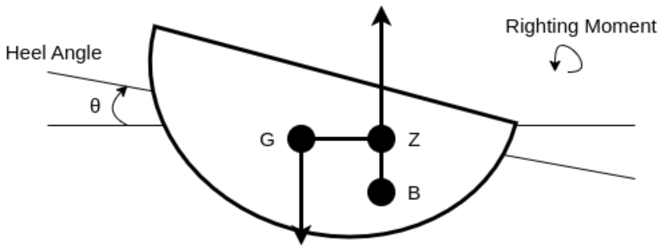

Figure 3.3: Stable vessel with righting moment

3.6. Environmental Disturbances

The forces and moments induced by the environmental disturbances is defined by the vector $\tau_{\mathbf{E}}$ and includes ocean currents, waves(wind generated) and wind.

$$\tau_{\mathbf{E}} = \tau_{\mathbf{E}}^{\mathbf{cu}} + \tau_{\mathbf{E}}^{\mathbf{wa}} + \tau_{\mathbf{E}}^{\mathbf{wi}} \tag{3.33}$$

where $\tau_{\mathbf{E}}^{\mathbf{cu}}$, $\tau_{\mathbf{E}}^{\mathbf{wa}}$ and $\tau_{\mathbf{E}}^{\mathbf{wi}}$ are vectors of forces and moments induced by ocean currents, waves and wind.

3.6.1. Current-induced Forces and Moments

The current induced forces and moments vector $\tau_{\mathbf{E}}^{\mathbf{cu}}$ is given by

$$\tau_{\mathbf{E}}^{\mathbf{cu}} = (\mathbf{M_{RB}} + \mathbf{M_A})\dot{\mathbf{v}}_{\mathbf{c}} + \mathbf{C}(\mathbf{v_r})\mathbf{v_r} - \mathbf{C}(\mathbf{v})\mathbf{v} + \mathbf{D}(\mathbf{v_r})\mathbf{v_r} - \mathbf{D}(\mathbf{v})\mathbf{v} \tag{3.34}$$

where $\mathbf{v_r} = \mathbf{v} - \mathbf{v_c}$ and $\mathbf{v_c} = [u_c, v_c, w_c, 0, 0, 0]^T$ is a vector irrotational body-fixed current velocities. Take the earth-fixed velocity vector denoted by $[u_c^E, v_c^E, w_c^E]^T$, then the body0fixed components $[u_c, v_c, w_c]^T$ can be calculated by

$$\begin{bmatrix} u_c \\ v_c \\ w_c \end{bmatrix} = \mathbf{J_1^T}(\mathbf{n_2}) \begin{bmatrix} u_C^E \\ v_c^E \\ w_C^E \end{bmatrix} \tag{3.35}$$

3.6.2. Wave-induced Forces and Moments

The vector $\tau_{\mathbf{E}}^{\mathbf{wa}}$ of the wave-induced forces and moments is given by

$$\tau_{\mathbf{E}}^{\mathbf{wa}} = \begin{bmatrix} \sum_{i=1}^{N} \rho g BLT cos(\beta) s_i(t) \\ \sum_{i=1}^{N} \rho g BLT sin(\beta) s_i(t) \\ 0 \\ 0 \\ 0 \\ \sum_{i=1}^{N} \frac{1}{24} \rho g BL(L^2 - B^2) sin(2\beta) s_i^2(t) \end{bmatrix} \tag{3.36}$$

where β is the vessel's heading(encounter) angle, illustrated in Figure , ρ is the water density, L is the length of the vessel, B is the breadth of the vessel and T is the draft of the vessel. Ignoring the higher-order terms of the wave amplitude, the wave slope $s_i(t)$ for the wave component i is defined by

$$s_i(t) = A_i \frac{2\pi}{\lambda_i} sin(\omega_{ei} t + \phi_i) \tag{3.37}$$

where A_i is the wave amplitude, λ_i is the wave length, ω_{ei} is the encounter frequency and ϕ_i is a random phase uniformly distributed and constant with time $[0\ 2\pi)$ corresponding to the wave component i.

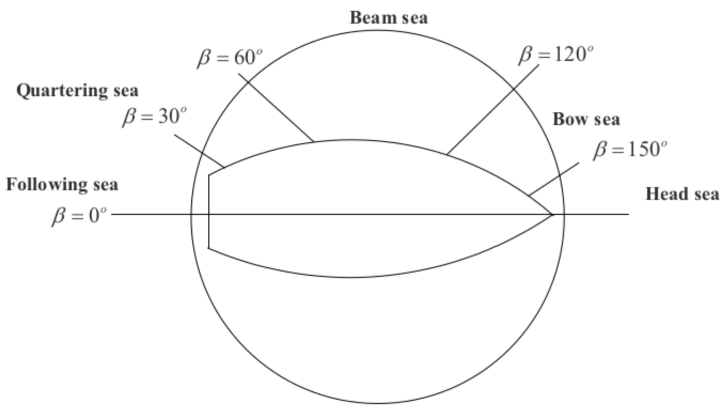

Figure 3.4: Ocean vessel's heading angle

3.6.3. Wind-induced Forces and Moments

When the ocean vessel is at rest the vector τ_E^{wi} of the wind induced forces and moments is given by

$$\tau_E^{\omega i} = \begin{bmatrix} C_X(\gamma_\omega) A_{F\omega} \\ C_Y(\gamma_\omega) A_{L\omega} \\ C_Z(\gamma_\omega) A_{F\omega} \\ C_K(\gamma_\omega) A_{L\omega} H_{L\omega} \\ C_M(\gamma_\omega) A_{F\omega} H_{F\omega} \\ C_N(\gamma_\omega) A_{L\omega} L_{oa} \end{bmatrix} \tag{3.38}$$

where V_ω is the wind speed, ρ_a is the air density, $A_{F\omega}$ is the frontal projected area, $A_{L\omega}$ is the lateral projected area, $H_{F=\omega}$ is the centroid of $A_{F\omega}$ above the water line, $H_{L\omega}$ is the centroid of $A_{l\omega}$ above the water line, L_{oa} is the over all length of the vessel, γ_ω is the angle of relative wind of the vessel bow, illustrated in Figure 3.5 and is given by

$$\gamma_\omega = \psi - \beta_\omega - \pi \tag{3.39}$$

where β_ω being the wind direction. All the wind coefficients(look-up tables) $C_X(\gamma_\omega)A_{F\omega}$, $C_Y(\gamma_\omega)A_{L\omega}$, $C_Z(\gamma_\omega)A_{F\omega}$, $C_K(\gamma_\omega)A_{L\omega}H_{L\omega}$, $C_M(\gamma_\omega)A_{F\omega}H_{F\omega}$ and $C_N(\gamma_\omega)A_{L\omega}L_{oa}$ are computed numerically or by experiments in a wind tunnel as shown in [5].

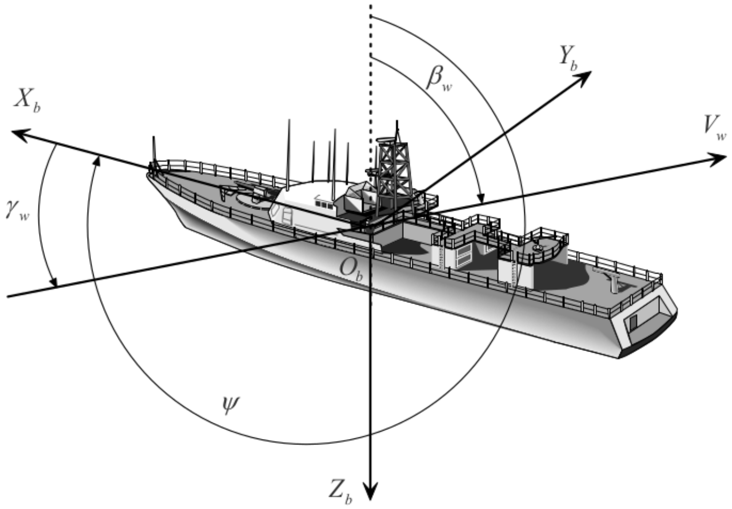

Figure 3.5: Wind angle on vessel

When the vessel is moving the vector $\tau_E^{\omega i}$ is given by

$$\tau_E^{\omega i} = \begin{bmatrix} C_X(\gamma_{r\omega})A_{F\omega} \\ C_Y(\gamma_{r\omega})A_{L\omega} \\ C_Z(\gamma_{r\omega})A_{F\omega} \\ C_K(\gamma_{r\omega})A_{L\omega}H_{L\omega} \\ C_M(\gamma_{r\omega})A_{F\omega}H_{F\omega} \\ C_N(\gamma_{r\omega})A_{L\omega}L_{oa} \end{bmatrix} \tag{3.40}$$

where

$$V_{r\omega} = \sqrt{u_{r\omega}^2 + v_{r\omega}^2} \tag{3.41}$$

$$\gamma_{r\omega} = -arctan2(v_{r\omega}, u_{r\omega}) \tag{3.42}$$

with

$$u_{r\omega} = u - V_\omega cos(\beta_\omega - \psi) \tag{3.43}$$

$$v_{r\omega} = v - V_\omega cos(\beta_\omega - \psi) \tag{3.44}$$

3.7. Simplifications of 6-DOF

3.7.1. Standard 3-DOF Horizontal Model

The horizontal motion of a surface ship in a horizontal plane is often described by the motion component in surge, sway and yaw. You choose $\mathbf{n} = [x,\ y,\ \psi]^T$ and $\mathbf{v} = [u,\ v,\ r]^T$. Figure 3.6 illustrates the motion variables in this case.

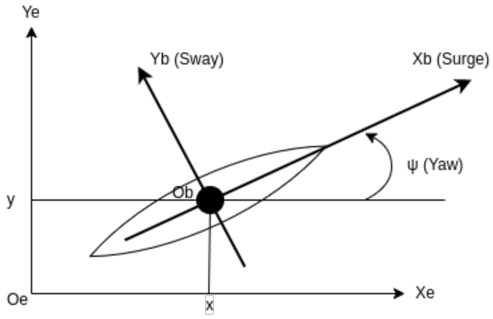

Figure 3.6: Standard 3-DOF Horizontal Model

Assumptions

1. The motion in roll, pitch and heave is ignored. This means that we ignore the dynamics associated with the motion in heave, roll and pitch, i.e., $z = 0$, $w = 0$, $\phi = 0$,$\theta = 0$ and $q = 0$.
2. The vessel has homogeneous mass distribution and xz-plane of symmetry so that
$$I_{xy} = I_{yz} = 0 \tag{3.45}$$
3. The center of gravity and center of buoyancy are located vertically on the z-axis

The vessel dynamics in a horizontal plane is simplified as follows:

$$\dot{\mathbf{n}} = \mathbf{J}(\mathbf{n})\mathbf{v} \tag{3.46}$$

$$\mathbf{M}\dot{\mathbf{v}} = -\mathbf{C}(\mathbf{v})\mathbf{v} - (\mathbf{D} + \mathbf{D_n}(\mathbf{v}))\mathbf{v} + +_E \tag{3.47}$$

where the matrices $\mathbf{J}(\mathbf{n})$, $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{D_n}(\mathbf{v})$ are given by

$$J(n) = \begin{bmatrix} cos(\psi) & -sin(\psi) & 0 \\ sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.48}$$

$$M = \begin{bmatrix} m - X_{\dot{u}} & 0 & 0 \\ 0 & m - Y_{\dot{v}} & mx_g - Y_{\dot{r}} \\ 0 & mx_g - Y_{\dot{r}} & I_z - N_{\dot{r}} \end{bmatrix} \tag{3.49}$$

$$C(v) = \begin{bmatrix} 0 & 0 & -m(x_g r + v) + Y_{\dot{v}v - Y_{\dot{r}}r} \\ 0 & 0 & mu - X_{\dot{u}u} \\ m(x_g r + v) - Y_{\dot{v}v - Y_{\dot{r}}r} & -mu + X_{\dot{u}u} & 0 \end{bmatrix} \tag{3.50}$$

$$D = \begin{bmatrix} X_u & 0 & 0 \\ 0 & Y_v & Y_r \\ 0 & N_v & N_r \end{bmatrix} \tag{3.51}$$

$$D_n(v) = \begin{bmatrix} X_{|u|u}|u| & 0 & 0 \\ 0 & Y_{|v|v}|v| + Y_{|r|v}|r| & Y_{|v|r}|v| \\ 0 & N_{|v|v}|v| + N_{|r|v}|r| & N_{|v|r}|v| + X_{|r|r}|r| \end{bmatrix} \tag{3.52}$$

The propulsion force and moment vector τ is given by

$$\tau = \begin{bmatrix} \tau_u \\ 0 \\ \tau_r \end{bmatrix} \tag{3.53}$$

The above propulsion force and moment vector τ implies that we are considering a surface vessel, which does not have an independent actuator in the sway. The environmental disturbance vector τ_E is given by

$$\tau = \begin{bmatrix} \tau_{uE} \\ \tau_{vE} \\ \tau_{rE} \end{bmatrix} \tag{3.54}$$

3.7.2. Simplified 3-DOF Horizontal Model

In some cases we can ignore the off-diagonal terms of the matrices $\mathbf{M}$ and $\mathbf{D}$, all elements of the nonlinear damping matrix $\mathbf{D_n}(\mathbf{v})$. These assumptions hold when the vessel has three planes of symmetry, for which the axes of the body-fixed reference frame are chosen to be parallel to the principle axis of the displaced fluid, which are equal to the principle axis of the vessel. Most ships have port/starboard symmetry and moreover, bottom/top symmetry is not required fore horizontal motion. Ship fore/aft nonsymmetry implies that the off-diagonal terms of the inertia and damping matrices are nonzero. However, these terms are small compared to the main diagonal terms. Furthermore, disturbances induced by the waves, wind and ocean currents are ignored. The dynamics of the vessel in the horizontal plane is simplified as follows:

$$\dot{\mathbf{n}} = \mathbf{J}(\mathbf{n})\mathbf{v} \tag{3.55}$$

$$\mathbf{M}\dot{\mathbf{v}} = -\mathbf{C}(\mathbf{v})\mathbf{v} - \mathbf{D}\mathbf{v} + \tau \tag{3.56}$$

where the matrices $\mathbf{J}(\mathbf{n})$, $\mathbf{M}$, $\mathbf{C}(\mathbf{v})$ and $\mathbf{D}$ are given by

$$J(n) = \begin{bmatrix} cos(\psi) & -sin(\psi) & 0 \\ sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.57}$$

$$M = \begin{bmatrix} m_{xy} & 0 & 0 \\ 0 & m_{xy} & 0 \\ 0 & 0 & m_{33} \end{bmatrix} \tag{3.58}$$

$$C(v) = \begin{bmatrix} 0 & 0 & -m_{xy}v \\ 0 & 0 & m_{xy}u \\ m_{xy}v & -m_{xy}u & 0 \end{bmatrix} \tag{3.59}$$

$$D = \begin{bmatrix} d_{xy} & 0 & 0 \\ 0 & d_{xy} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \tag{3.60}$$

with

$$m_{xy} = m - X_{\dot{u}} = m - Y_{\dot{v}} \tag{3.61}$$

$$m_{33} = I_z - N_{\dot{r}} \tag{3.62}$$

$$d_{xy} = -X_u = -Y_v \tag{3.63}$$

$$d_{33} = -N_r \tag{3.64}$$

3.8. Summary

The combined six degrees of freedom equations of motion is shown below:

$$\dot{n} = J(n)v \tag{3.65}$$

$$M\dot{v} = -C(v)v - D(v)v - g(n) + \tau + \tau_E \tag{3.66}$$

where

$$M = M_{RB} + M_A \tag{3.67}$$

$$C(v) = C_{RB}(v) + C_A(v) \tag{3.68}$$

$\mathbf{J}(\mathbf{n})$, equation 3.12 and 3.13, is the transformation matrix which translate $\mathbf{v_1}$ and $\mathbf{v_2}$ through the functions of the Euler angles to $\dot{\mathbf{n}}_\mathbf{1}$ and $\dot{\mathbf{n}}_\mathbf{2}$. $\mathbf{C}(\mathbf{v})$ is the linear combination of the rigid body Coriolis and centripetal matrix $\mathbf{C_{RB}}(\mathbf{v})$, equation A.2 and the hydrodynamic Coriolis and centripetal matrix $\mathbf{C_A}(\mathbf{v})$, equation 3.27. $\mathbf{D}(\mathbf{v})$, equation 3.29, is the hydrodynamic damping and $\mathbf{g}(\mathbf{n})$, equation 3.28, is the restoring forces and moments, equation 3.32. The propulsion forces and moments is modelled by τ and the environmental disturbances by τ_E.

3.9. Simulation Results

Chapter 4

Modeling of a Fixed-Wing Sail, Keel and Rudder

This chapter the modeling of a fixed-wing sail is considered. Traditional sails consist of a mainsail and a jib [6]. The sail considered in the chapter is a fixed-wing sail that is fully autonomous. The sail takes inspiration from a free rotating fixed wing sail [7] and fixed-wing sail [8]. The chapter models the forces experienced by adding a fixed-wing sail to the model described in Chapter 3. Also discussed in this chapter is the modelling of forces, moments and constraints caused by a rudder and keel.

4.1. Rudder Forces and Moments

The forces and moments experienced by a sailboat due to the rudder are defined in [9]. The equations formulated for the forces and moments are illustrated below

$$X_{rud} = C_{X\delta_R} sin(\alpha_R) sin(\delta_R) \times \frac{1}{2}\rho_w v_B^2 L_{WL} D_K \tag{4.1}$$

$$Y_{rud} = C_{Y\delta_R} sin(\alpha_R) cos(\delta_R) cos(\phi) \times \frac{1}{2}\rho_w v_B^2 L_{WL} D_K \tag{4.2}$$

$$K_{rud} = C_{K\delta_R} sin(\alpha_R) sin(\delta_R) \times \frac{1}{2}\rho_w v_B^2 L_{WL} D_K \tag{4.3}$$

$$N_{rud} = C_{N\delta_R} sin(\alpha_R) cos(\delta_R) cos(\phi) \times \frac{1}{2}\rho_w v_B^2 L_{WL} D_K \tag{4.4}$$

where $C_{X\delta_R}$, $C_{Y\delta_R}$, $C_{K\delta_R}$, $C_{N\delta_R}$ are non-dimensional coefficients, V_B is the boat velocity, ρ_ω is the water density, L_{WL} is the length on design waterline, D_K is the design draft length, δ_R is the physical rudder angle and α_R is the effective angle of attack on the rudder as defined below

$$\alpha_R = \delta_R - \epsilon_y \gamma - tan^{-1}\left(\frac{x_R R}{U}\right) \tag{4.5}$$

$$\epsilon = \frac{d\epsilon}{d\gamma} \times \gamma = \epsilon_\gamma \gamma \tag{4.6}$$

where γ is the leeway angle the sailboat is sailing and ϵ is the angle of inflow from the downwash generated by the keel and x_R is the longitudinal distance of the quarter-chord point of the rudder to the CG of the boat.

The angles α_R, δ_R, γ and ϵ are illustrated in Figure 4.1

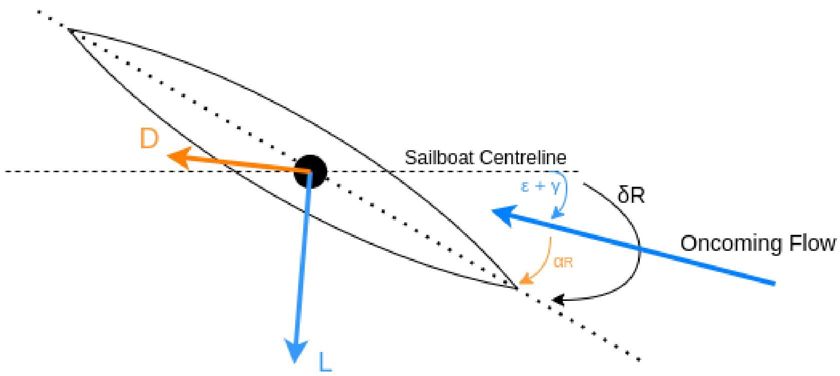

Figure 4.1: Definition of Rudder Angles

4.2. Sail Forces and Moments

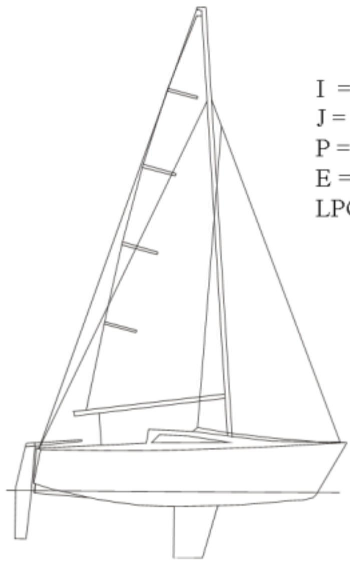

Figure 4.2: Example of Sailboat

The sail forces and moments are expressed in their horizontal components. When the sail is in the upright position the aerodynamic coefficients $X^{'}_{s0}$ and $Y^{'}_{s0}$ are expressed using lift coefficient $L^{'}_{s0}$ and drag coefficient $D^{'}_{s0}$ as shown in [10].

$$X^{'}_{s0} = L^{'}_{s0} sin(\gamma_A) - D^{'}_{s0} cos(\gamma_A) \tag{4.7}$$

$$Y^{'}_{s0} = -L^{'}_{s0} cos(\gamma_A) - D^{'}_{s0} sin(\gamma_A) \tag{4.8}$$

where the subscript of 0 means value at the upright condition.

In the heeled condition, when the vessel is rotated due to the sail lift and drag force, the effect of the heel on the aerodynamic forces is produced by the reduction of both the apparent wind angle and apparent wind speed. The apparent wind angle in the heeled condition $\gamma_{A\phi}$ is expressed as follows using the apparent wind angle γ_A and apparent wind speed U_A:

$$\gamma_{A\phi} = tan^{-1}\left(\frac{U_A sin(\gamma_A)cos(\phi)}{U_A cos(\gamma_A)}\right) = tan^{-1}(tan(\gamma_A)cos(\phi)) \tag{4.9}$$

The apparent wind speed in the heeled condition $U_{A\phi}$ is also expressed as:

$$U_{A\phi} = \sqrt{((U_A cos(\gamma_A))^2 + (U_A sin(\gamma_A)cos(\phi))^2} = U_A\sqrt{1-(sin(\gamma_A)sin(\phi))^2} \tag{4.10}$$

For the close-hauled condition, tacking behaviour of sailboat due to not being able to sail directly into the wind, the sail may not stall due to the small attack angle. Therefore, the lift force will decrease proportionally to the reduction of both the apparent wind angle and the dynamic pressure of flow(square of the apparent wind speed). Hence the decreasing ratio of lift force by the heel angle ϕ can be described as:

$$\left(\frac{\gamma_{A\phi}}{\gamma_A}\right)\left(\frac{U_{A\phi}}{U_A}\right) = \left(\frac{tan^{-1}(tan(\gamma_A)cos(\phi}{\gamma_A}\right)\left(1-(sin(\gamma_A)sin(\phi))^2\right) \tag{4.11}$$

The vector of lift force inclines with heel angle and rotates in the normal plane to the apparent wind axis. Since the angle between the apparent wind axis and the boat center line(heeling axis) is γ_A, the rotating angle of the lift force vector ϕ' in the normal plane to the apparent wind axis is given by:

$$\phi' = sin^{-1}(cos(\gamma_A)sin(\phi)) \tag{4.12}$$

Therefore, the decreasing ratio of horizontal component of the lift force is expressed as:

$$\left(\frac{\gamma_{A\phi}}{\gamma_A}\right)\left(\frac{U_{A\phi}}{U_A}\right)cos(\phi') = \left(\frac{tan^{-1}(tan(\gamma_A)cos(\phi}{\gamma_A}\right)\left(1-(sin(\gamma_A)sin(\phi))^2\right)cos(sin^{-1}(cos(\gamma_A)sin(\phi))) \tag{4.13}$$

Expanding the above mention equation in a power series and assuming that γ_A is small, results in

$$\left(\frac{\gamma_{A\phi}}{\gamma_A}\right)\left(\frac{U_{A\phi}}{U_A}\right)cos(\phi') \approx \left(cos^2(\phi)+\frac{1}{2}sin^2(\phi)\right)cos(\phi) = \frac{1}{2}\left(cos(\phi)+cos^3(\phi)\right) \tag{4.14}$$

Equation 4.14 can be further expanded in terms of ϕ and results in

$$\left(\frac{\gamma_{A\phi}}{\gamma_A}\right)\left(\frac{U_{A\phi}}{U_A}\right)cos(\phi') \approx 1-\phi^2 \tag{4.15}$$

Equation 4.15 is incidentally equal to the first two terms of the power series for the $cos^2(\phi)$ function. Hence the curve of the $cos^2(\phi)$ was compared with the calculated results in Equation 4.13 for three γ_A cases. The calculated results show agreement with the curve of $cos^2(\phi)$ in spite of the large γ_A. Therefore, we adopted the formula of $cos^2(\phi)$ to express the decreasing ratio of the horizontal component of the lift force in place of Equation 4.13.

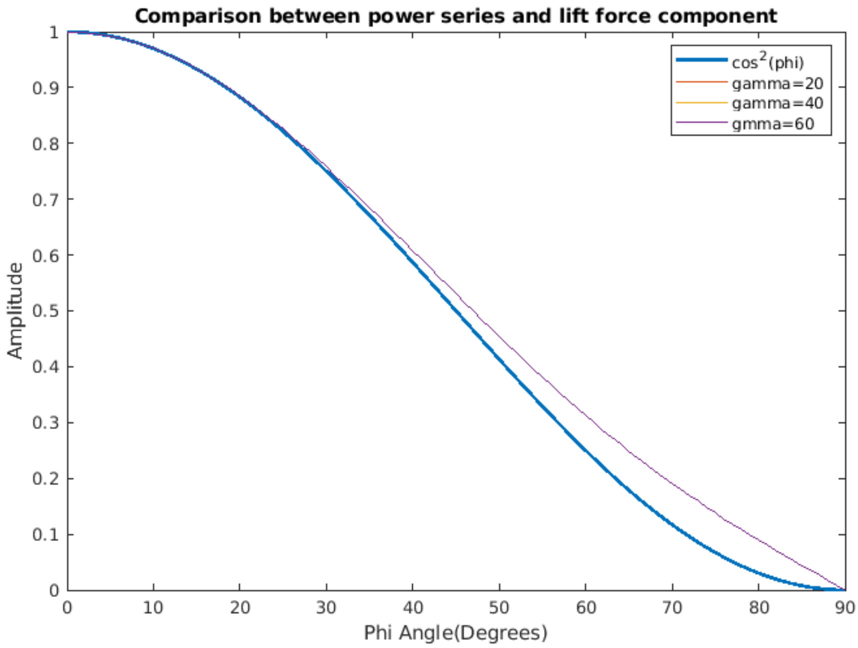

Figure 4.3: Lift force component represented as a power curve

Finally, when the lift coefficient represents the variation of the lift force including the contribution of dynamic pressure of apparent wind speed, the horizontal component of lift coefficient in the heeled condition $L_s^{'}$ is described as:

$$L_S^{'} = L_{s0}^{'}cos^2(\phi) \tag{4.16}$$

The main part of the drag is caused by the induced drag, which is in proportion to the square of the lift force. The reduction of lift force expressed by Equation 4.11 is also approximated by $cos(\phi)$. The vector of the drag force is in line with the apparent wind axis and does not incline by the heel angle. Therefore the horizontal component of the drag coefficient $D_s^{'}$ is described as:

$$D_s^{'} = D_{s0}^{'}cos^2(\phi) \tag{4.17}$$

From these results, the aerodynamic coefficients in the horizontal components $X_s^{'}$ and $Y_s^{'}$ are then expressed as follows using the coefficients at the upright condition $L_{s0}^{'}$ and $D_{s0}^{'}$:

$$X_s^{'} = L_S^{'}sin(\gamma_A) - D_S^{'}cos(\gamma_A) = L_{s0}^{'}cos^2(\phi)sin(\gamma_A) - D_{s0}^{'}cos^2(\phi)cos(\gamma_A) = X_{s0}^{'}cos^2(\phi) \tag{4.18}$$

$$Y_s^{'} = L_S^{'}cos(\gamma_A) - D_S^{'}sin(\gamma_A) = L_{s0}^{'}cos^2(\phi)cos(\gamma_A) - D_{s0}^{'}cos^s in(\gamma_A) = Y_{s0}^{'}cos^2(\phi) \tag{4.19}$$

The moment K_s is generated mainly by the Y_s force, however it is also affected by the component normal to the mast, hence

$$K_s^{'} = -Y_s^{'}\left(\frac{z_{GCE}^G}{\sqrt{S_A}}\right)/cos(\phi) \tag{4.20}$$

where z_{GCE}^G is the z-coordinate of the geometric center of effort of the sail from the CG of the boat and negative upwards.

The moment N_s is also generated mainly by the Y_s force, however, it is well known that the N_s is also affected by the heel angle ϕ due to the application point of the thrust force X_s moving outboard to lee side. Therefore $N_s^{'}$ can be written, including the effect of $X_{s0}^{'}$ as

$$N_s^{'} = \left(Y_{s0}^{'} \frac{x_{GCE}^G}{\sqrt{S_A}} + X_{s0}^{'} \frac{z_{GCE}^G}{\sqrt{S_A}} sin(\phi) \right) cos^2(\phi) \tag{4.21}$$

where x_{GCE}^G is x-coordinate of the geometric center of effort of the sail from CG of the boat.

4.3. Fixed-wing Sail

A simplified model, all the acting forces are assumed to be acting in the horizontal plane, are shown in Figure 4.4. The aerodynamic force has both a side force component F_{ax} and a forward force component F_{ay}. These forces need to be balanced by the corresponding hydrodynamic side force F_{Hx} and the resistance F_{Hy}. In order to generate the hydrodynamic side force, the hull needs to have a velocity relative to the water and an angle of attack, the leeway angle λ relative to the water flow field. The rudder of the boat is used to generate side force with the aim of controlling the longitudinal position of the sailboat.

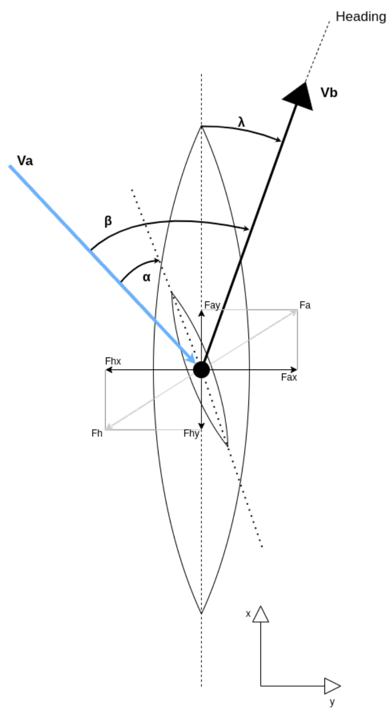

Figure 4.4: Top view of sailboat

Wind is both described in the n-frame and in the b-frame corresponding to true wind(tw) and apparent wind(aw) respectively. The definition of true wind and apparent wind is illustrated in Figure 4.5.

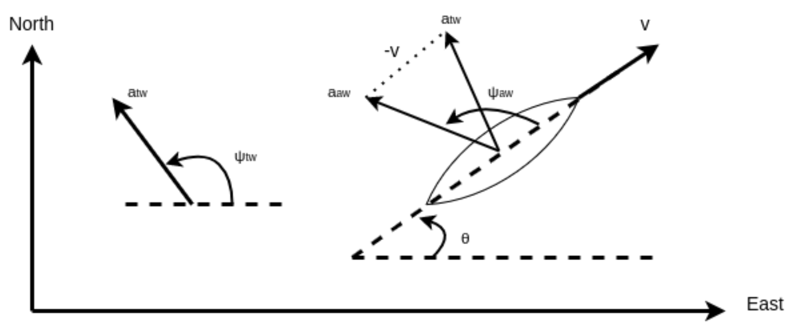

Figure 4.5: Definition of true and apparent wind

The apparent wind in Cartesian relative to the direction of the boat, i.e., the first coordinate corresponding to the heading of the boat can be calculated from true wind by

$$\mathbf{W}_{c,aw} = \begin{bmatrix} a_{tw}cos(\psi_{tw} - \theta) - v \\ a_{tw}sin(psi_{tw} - \theta) \end{bmatrix} \tag{4.22}$$

The corresponding polar coordinates are thus

$$\mathbf{W}_{p,aw} = \begin{bmatrix} a_{aw} \\ \psi_{aw} \end{bmatrix} = \begin{bmatrix} |\mathbf{W_{c,aw}}| \\ atan2(\mathbf{W}_{c,aw}) \end{bmatrix} \tag{4.23}$$

The angle of attack, α_s, on the sail is determined by the direction of the apparent wind and the angle of the sail, $\alpha_s = \psi_{aw} - \lambda$. The force on the sail is this given by,

$$F_s = -La_{aw}sin(\lambda - \psi_{aw}) \tag{4.24}$$

where L is the lift force and D is the drag force and are defined by:

$$L = \frac{1}{2}\rho A v_a^2 C_L \tag{4.25}$$

$$D = \frac{1}{2}\rho A v_a^2 C_D \tag{4.26}$$

where ρ is the air density, A is the plane area of the foil, v_a is the apparent wind velocity defined in equation, C_L and C_D are the lift and drag coefficients respectively. These coefficients depends on the winds angle of attack, α and the foils shape. In [11] it is shown that the coefficients can be approximated as,

$$C_L = k_1 sin(2\alpha) \tag{4.27}$$

$$C_L = k_1(1 - cos(2\alpha)) \tag{4.28}$$

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Appendix A

Additional Modelling Information

A.1. Notation and Vector Definitions

Table A.1: SNAME Notation for ocean vessels

Degree of freedom		Force and moment	Linear and angular velocity	Position and Euler angles
1	Surge	X	u	x
2	Sway	Y	v	y
3	Heave	Z	w	z
4	Roll	K	p	ϕ
5	Pitch	M	q	θ
6	Yaw	N	r	ψ

Table A.2: Rigid body motion vectors

Vector	Components	Definition
$\mathbf{f}_{Ob}$	$[X\ Y\ Z]^T$	force decomposed in the body-fixed frame
$\mathbf{m}_{Ob}$	$[K\ M\ N]^T$	moment decomposed in the body-fixed frame
$\mathbf{v}_{Ob}$	$[u\ v\ w]^T$	linear velocity decomposed in the body-fixed frame
$\mathbf{w}_{Ob}^E$	$[p\ q\ r]^T$	angular velocity of the body-fixed relative to the earth-fixed frame
$\mathbf{r}_{Ob}$	$[x_g\ y_g\ z_g]^T$	vector from O_b to CG decomposed in the body-fixed frame

A.2. Modeling Equations

$$\mathbf{M_{RB}} = \begin{bmatrix} m & 0 & 0 & mz_g & mz_g & -my_g \\ 0 & m & 0 & 0 & 0 & mx_g \\ 0 & 0 & m & -mx_g & -mx_g & 0 \\ 0 & -mz_g & -my_g & I_x & -Ixy & -I_xz \\ mz_g & 0 & -mx_g & -I_{xy} & I_y & -I_yz \\ -my_g & mx_g & 0 & -I_{zx} & -I_{zy} & I_z \end{bmatrix} \tag{A.1}$$

$$\mathbf{C_{RB}}(\mathbf{v}) = \begin{bmatrix} 0 & 0 & 0 & m(y_gq + z_gr) & -m(x_gq - w) & -m(x_gr + v) \\ 0 & 0 & 0 & -m(y_gp + w) & m(z_gr + x_gp) & -m(y_gr - u) \\ 0 & 0 & 0 & -m(z_gp - v) & -m(z_gq + u) & m(x_gp + y_gq) \\ -m(y_gq + z_gr) & m(y_gp + w) & m(y_gp - v) & 0 & -I_{yz}q - I_{xz}q + I_zr & I_{yz}r + I_{xy}p - I_yq \\ m(x_gp - w) & -m(z_gr - x_gp) & m(z_gq + u) & I_{yz}q + I_{xz}p - I_zr & 0 & -I_{xz}r - I_{xy}q + I_Xp \\ m(x_gr + v) & m(y_gr - u) & -m(x_gp + y_gq) & -I_{yz}r - I_{xy}p + I_yq & I_{xz}r + I_{xy}q - I_xp & 0 \end{bmatrix} \tag{A.2}$$

$$a_1 = X_{\dot{u}}u + X_{\dot{v}}v + X_{\dot{w}}w + X_{\dot{p}}p + X_{\dot{q}}q + X_{\dot{r}}r \tag{A.3}$$

$$a_2 = Y_{\dot{u}}u + Y_{\dot{v}}v + Y_{\dot{w}}w + Y_{\dot{p}}p + Y_{\dot{q}}q + Y_{\dot{r}}r \tag{A.4}$$

$$a_3 = Z_{\dot{u}}u + Z_{\dot{v}}v + Z_{\dot{w}}w + Z_{\dot{p}}p + Z_{\dot{q}}q + Z_{\dot{r}}r \tag{A.5}$$

$$b_1 = K_{\dot{u}}u + K_{\dot{v}}v + K_{\dot{w}}w + K_{\dot{p}}p + K_{\dot{q}}q + K_{\dot{r}}r \tag{A.6}$$

$$b_2 = M_{\dot{u}}u + M_{\dot{v}}v + M_{\dot{w}}w + M_{\dot{p}}p + M_{\dot{q}}q + M_{\dot{r}}r \tag{A.7}$$

$$b_3 = N_{\dot{u}}u + N_{\dot{v}}v + N_{\dot{w}}w + N_{\dot{p}}p + N_{\dot{q}}q + N_{\dot{r}}r \tag{A.8}$$