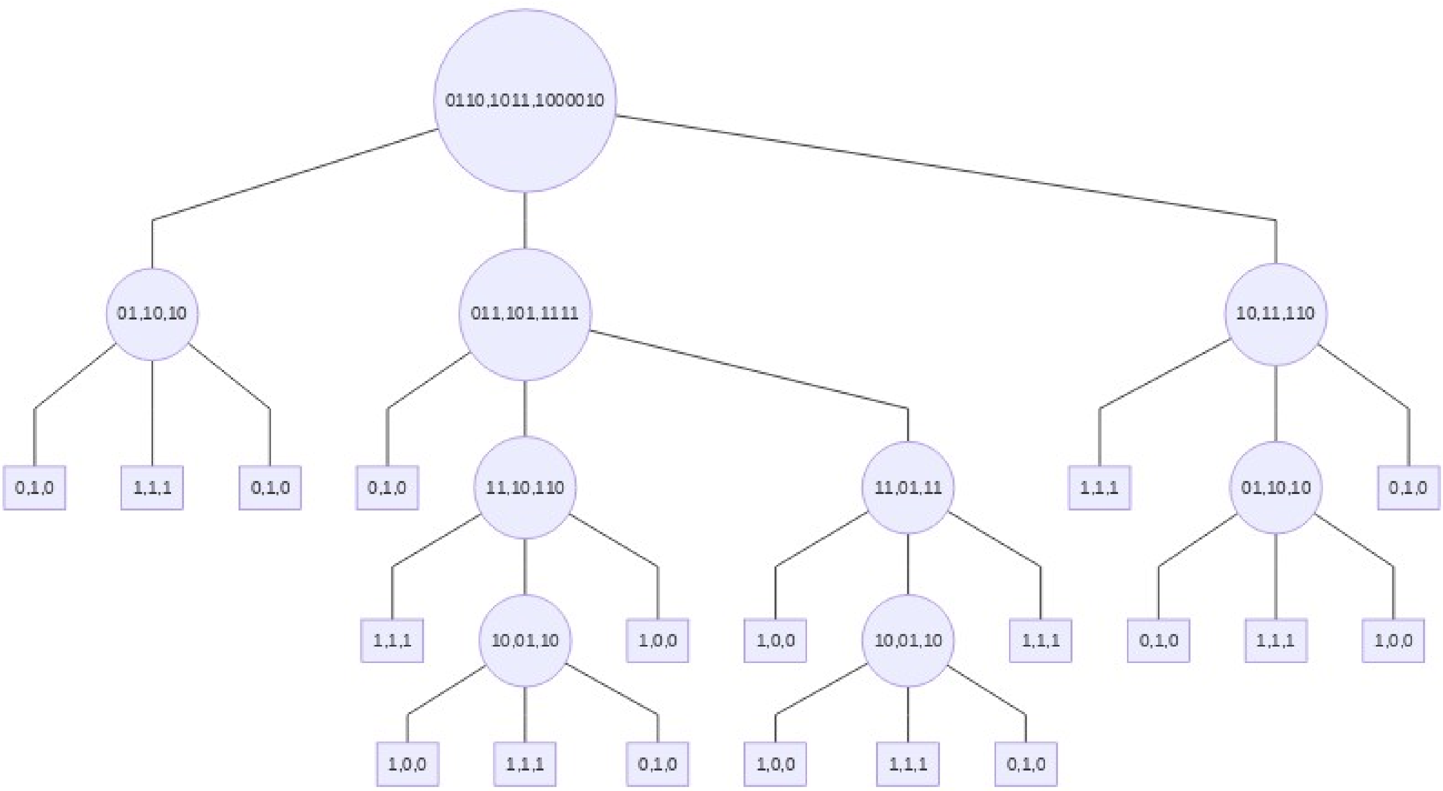
CSCI-GA 1170 Homework 2

Question 1



Question 2

(i)

In Karatsuba's algorithm, we have two numbers X, Y to multiply. If $|X| \neq |Y|$, we pad one of them so that $|X| = |Y| = n$. Next, we split X and get $|X_0| = \lceil n/2 \rceil$ and $|X_1| = \lfloor n/2 \rfloor$. For simplicity, let $|X_0| = |X_1| = \lceil n/2 \rceil$, and similar for Y_0 and Y_1. Then we compute $Z_0 = X_0 Y_0, Z_2 = X_1 Y_1$ and $Z_1 = (X_0 + X_1)(Y_0 + Y_1) - Z_0 - Z_2$, which includes three multiplications and we do recursively. Since $|X_0| = |X_1| = \lceil n/2 \rceil$, we have $|X_0 + X_1| = |n/2| + 1$. Thats where the constants 1, 2 and 3 come from in the recurrence. Besides, the addition of X_0 and X_1 costs $n/2 + O(1)$, and similar for adding Y_0 and Y_1. Since $|Z_0| = |Z_2| = n + O(1)$, subtracting Z_0 and Z_2 cost $2n + O(1)$. Finally, compute Z from Z_0, Z_1, Z_2 costs $2n$. So the total additional cost is $2(n/2 + O(1)) + (2n + O(1)) + 2n = 5n + O(1)$, which is the same as the recurrence.

(ii)

We use real induction to prove the upper bound.

[RB] Let $P(x)$ be the proposition

$$x \geq x_0 \Rightarrow T(x) \leq U(x)$$

Here we pick x_0 to be large enough such that $\frac{(x_0-3)^{\lg 3}}{x_0} \gg 10$.

Let $x_1 = 2x_0 - 3$. Invoke DIC to conclude that there exists a $C > 0$ such that

$$(\forall x)[x_0 \leq x < x_1 \Rightarrow T(x) \leq C]$$

Now, choose K such that $0 \leq K \leq \frac{(x-3)^{\lg 3} - C}{x}$. Since $\frac{(x_0-3)^{\lg 3}}{x_0} \gg 10$, thus we can pick $K = \frac{(x_0-3)^{\lg 3} - C}{x_0}$. So for all $x_0 \leq x < x_1$, we have $T(x) \leq C \leq (x_0 - 3)^{\lg 3} - Kx_0 \leq (x-3)^{\lg 3} - Kx$. Note that whether such K exists depends on the DIC.

For example, we can choose

$$T(x) = \begin{cases} 0, & \text{if } x < x_0 \\ 3T(\lceil x/2 \rceil + 1) + 5x & \text{o.w.} \end{cases}$$

So for all $x_0 \leq x < x_1 = 2x_0 - 3$, we have $\lceil x/2 \rceil + 1 \leq (x+3)/2 < (2x_0 - 3 + 3)/2 = x_0$, which means that $T(x) = 5x$. In this case, C can be $10x_0$. Since $\frac{(x_0-3)^{\lg 3}}{x_0} \gg 10$, K exists.

Hence, P(x) holds, this establishes the real basic step.

[RI]

Now, choose δ such that $\delta \leq \frac{x_1-3}{2}$. So for all $x \geq x_1$, we have $\frac{x+3}{2} = x - \frac{x-3}{2} \leq x - \frac{x_1-3}{2} = x - \delta$. Note that we have make $x_1 = 2x_0 - 3$. So for all $x \geq x_1$, $\frac{x+3}{2} \geq \frac{x_1+3}{2} = x_0$. Next, we must show that when $x \geq x_1, P_\delta^*(x)$ implies $P(x)$:

$$\begin{aligned}
T(x) &= 3T(\lceil x/2 \rceil + 1) + 5x \\
&\leq 3T(\frac{x+3}{2}) + 5x \\
&\leq 3(\frac{x-3}{2})^{\lg 3} - 3K\frac{x-3}{2} \\
&= 3(\frac{x-3}{2})^{\lg 3} - \frac{3}{2}Kx + \frac{9}{2}K + 5x \leq U(x) = (x-3)^{\lg 3} - Kx
\end{aligned}$$

To make the last inequality to be true, we must have $\frac{9}{2}K + 5x \leq \frac{1}{2}Kx$, which implies $K > 10$.

(iii)

There is "unrealistic" in the recurrence $T(n) = 3T(\lceil n/2 \rceil + 1) + 5n + O(1)$ except for the abstract $O(1)$ term. In the Karatsuba's algorithm, $|X_0| = \lceil n/2 \rceil$, and strictly speaking, $|X_1| = \lfloor n/2 \rfloor$. Besides, the "plus 1" only happens in the computation of Z_1. So a more "honest" recurrence should be $T(n) = T(\lceil n/2 \rceil + 1) + T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + 5n + O(1)$.

Question 3

Polynomial Sums

(i) $f(i) = i \log i$. Use the closure properties in the lecture notes. Since i is polynomial-type, so is $\log i$. Since the product of two polynomial-type functions is still polynomial-type, $i \log i$ is polynomial-type. Therefore, the summation $\sum_{i \geq 1}^{n} i \log i$ is polynomial-type.

(ii) From **(i)**, we know that the summation $\sum_{i \geq 1}^{n} \log i$ is polynomial-type.

(iii) Use the closure properties in the lecture notes. Raising a function of polynomial-type to any power $a \geq 0$ is still polynomial-type, so $i^a \, (a \geq 0)$ is polynomial-type. Therefore, the summation $\sum_{i \geq 1}^{n} i^a$ is polynomial-type.

Exponentially Increasing Sums

(i) We argue from first principles. Choose $C = b$ and $k = 1$, so we have $b^i \geq b^{i-1} \cdot b \text{ (ev.)}$. Therefore, the summation $\sum_{i \geq 1}^{n} b^i (b > 1)$ is exponentially increasing type.

(ii) We argue from first principles. Choose $C = 2$ and $k = 1$, we are going to show that $i^{-5} 2^{2^i} \geq (i-1)^{-5} 2^{2^{i-1}} \cdot 2 (\text{ev.})$. Let $g(i) = \frac{i^{-5} 2^{2^i}}{(i-1)^{-5} 2^{2^{i-1}} \cdot 2} = \frac{(i-1)^5 \cdot 2^{i-2}}{i^5}$. Clearly, $g(i) \to \infty$ as $i \to \infty$. Therefore, the summation $\sum_{i \geq 1}^{n} i^{-5} 2^{2^i}$ is exponentially increasing type.

(iii) We argue from first principles. Choose $C = 2$ and $k = 1$, so we have $i! \geq 2 \cdot (i-1)! \text{ (ev.)}$. Therefore, the summation is exponentially increasing type.

Exponentially Decreasing Sums

(i) We argue from first principles. Choose $C = b$ and $k = 1$, so we have $b^{-i} \leq b^{-i+1}/b \text{ (ev.)}$. Therefore, the summation is exponentially decreasing type.

(ii) We argue from first principles. Choose $C = 2$ and $k = 1$, we are going to show that $i^2 i^{-i} \leq (i-1)^2 (i-1)^{(-i+1)}/2 \text{ (ev.)}$. Let $g(i) = \frac{i^2 i^{-i}}{(i-1)^2 (i-1)^{(-i+1)}/2} \leq \frac{2i^2 \cdot i^{(i-1)}}{(i-1)^2 i^i} = \frac{2i}{(i-1)^2}$. Clearly, $g(i) \to 0$ as $i \to \infty$. Therefore, the summation is exponentially decreasing type.

(iii) Similar to **(ii)**. The summation $\sum_{i \geq 1}^{n} i^{-i}$ is exponentially decreasing type.

Question 4

(a) First, we use domain transformation. Let $x = \lg n$, then we have $t(x) = 4t(x-1) + \frac{4^x}{x^2}$. Next, by range transformation, let $s(x) = \frac{t(x)}{4^x}$, and we have $s(x) = s(x-1) + \frac{1}{x^2}$. By DIC, we assume that if $x \leq 0$, then $s(x) = 0$, so we have $s(x) = \sum_{i\geq 0}^{x} \frac{1}{i^2} = \Theta(1)$. Thus, $T(n) = t(x) = 4^x \cdot s(x) = \Theta(4^x) = \Theta(n^2)$

(b) Use the same techniques in part **(a)**, we have $s(x) = \Theta(x^{\frac{1}{2}})$, and $T(n) = \Theta(n^2\sqrt{\lg n})$.

Question 5

$E_1 = (2^n)^{\frac{1}{\lg n}} = 2^{n/\lg n}, E_2 = (2^n)^{\lg \frac{1}{n}} = n^{-n}, E_3 = (\lg n)^{2^{1/n}} \to \lg n (n \to \infty), E_4 = (\lg n)^{1/2^n} \to 1(n \to \infty), E_5 = (1/n)^{2^{\lg n}} = n^{-n}, E_6 = (1/n)^{\lg(2^n)} = n^{-n}$

Therefore, the conclusion is $E_1 \succeq E_3 \succeq E_4 \succeq E_2 = E_5 = E_6$.

Question 6

(d) $<$ (e) $<$ (a) $<$ (c) $<$ (b)

Question 7

(i)

We define

$$F(x) = \begin{cases} 0 & \text{if } x < x_0 \\ d(x) + G(x, F(b_1(x)), F(b_2(x))) & \text{o.w.} \end{cases}$$

Let $P(x) : (\forall x)[x \geq x_0 \to F(x) \text{ is well-defined}]$. We use real induction to prove $P(x)$.

[RB] Pick $x_1 = x_0 + \gamma$. Then for all $x_0 \leq x < x_1$, we have $b_1(x) < x - \gamma < x_1 - \gamma = x_0$ and similarily, $b_2(x) < x_0$. So $F(x) = d(x) + G(x, 0, 0)$, which is well-defined.

[RI] Let $\delta = \gamma$. Then for all $x > x_1$, we have $b_1(x) < x - \gamma = x - \delta$ and similar for $b_2(x)$. According to our induction hypothesis, in the definition of $F(x) = d(x) + G(x, F(b_1(x)), F(b_2(x)))$, $F(b_1(x))$ and $F(b_2(x))$ are both well-defined. So $F(x)$ is well-defined. **QED.**

(ii)

We pick

$$F(x) = \begin{cases} 0 & \text{if } x < x_0 \\ d(x) + G(x, F(b_1(x)), F(b_2(x))) & \text{o.w.} \end{cases}$$

and show that $F(x)$ is eventually increasing.

[RB] Pick $x_1 = x_0 + \gamma$. Then for all $x_0 \leq x < x_1$, we have $b_1(x) < x - \gamma < x_1 - \gamma = x_0$ and similarily, $b_2(x) < x_0$. So $F(x) = d(x) + G(x, 0, 0)$. Given $d(x)$ is non-decreasing and $G(x, x_1, x_2)$ is increasing in x, we conclude that $F(x)$ is increasing. This establishes the real basic step.

[RB] Let $\delta = \gamma$. Then for all $x \geq x_1$, we have $b_1(x) < x - \delta$ and same for $b_2(x)$. Therefore,

$$
\begin{aligned}
F(n) &= d(n) + G(n, F(b_1(n)), F(b_2(n))) \\
&\le d(n) + G(n, F(b_1(n')), F(b_2(n'))) \\
&< d(n') + G(n', F(b_1(n')), F(b_2(n'))) \\
&= F(n')
\end{aligned}
$$

This proves the real induction step. **QED.**

(iii)

First, we use real induction to prove $T = O(T')$. i.e. $(\exists K > 0)(\exists x_0)(\forall x)[x \ge x_0 \to T(x) \le KT'(x)]$.

[RB] We pick $x_0 \ge n_0$ and $x_1 > x_0$. We assume that for all $x_0 \le x < x_1$, we have $T(x) \le a$ and $T'(x) \ge b$. We pick $K = \frac{a}{b}$, so we have $T(x) \le a = Kb \le KT'(x)$. This establishes the real basic step.

[RI] Let $\delta = \gamma$. Then for all $x \ge x_1$, we have $b_1(x) < x - \delta$ and same for $b_2(x)$. Therefore,

$$
\begin{aligned}
T(x) &= d(x) + G(x, T(b_1(x)), T(b_2(x))) \\
&\le d(x) + G(x, KT'(b_1(x)), KT'(b_2(x))) \\
&\le d(x) + K \cdot G(x, T'(b_1(x)), T'(b_2(x))) \\
&\le KT'(x) = K \cdot d(x) + K \cdot G(x, T'(b_1(x)), T'(b_2(x)))
\end{aligned}
$$

where the last inequality is guaranteed provided $K \ge 1$. This proves the real induction step. **QED.**