I.6 Q1

$\lambda_1 = \cos\theta + i\sin\theta$

$\lambda_2 = \cos\theta - i\sin\theta$

$\lambda_1 + \lambda_2 = 2\cos\theta = Q_{11} + Q_{22}$

$\lambda_1\lambda_2 = \cos^2\theta + \sin^2\theta = \det(Q)$

$\bar{x}_1^T \cdot x_2 = \begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 0 \Rightarrow$ orthogonal

$$Q^{-1} = Q^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

It's eigenvalues are $\frac{1}{\lambda_1} = \cos\theta - i\sin\theta$ and $\frac{1}{\lambda_2} = \cos\theta + i\sin\theta$

I.6 Q2

$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} -\lambda & 2 \\ 1 & 1-\lambda \end{bmatrix}$

$\det(A - \lambda I) = -\lambda + \lambda^2 - 2 = (\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda_1 = -1 \quad \lambda_2 = 2$

$\lambda = -1$ has $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Eigenvector $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda = 2$ has $\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Eigenvector $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\text{trace}(A) = 1 = \lambda_1 + \lambda_2$

$$A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix} \qquad A^{-1} - \lambda I = \begin{bmatrix} -\frac{1}{2} - \lambda & 1 \\ \frac{1}{2} & -\lambda \end{bmatrix}$$

$$\det(A^{-1} - \lambda I) = \lambda^2 + \tfrac{1}{2}\lambda - \tfrac{1}{2} = \tfrac{1}{2}(2\lambda - 1)(\lambda + 1) = 0 \Rightarrow \lambda_1' = -1 \quad \lambda_2' = \tfrac{1}{2}$$

$\lambda = -1$ has $\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Eigenvector $\vec{V_1}' = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda = \frac{1}{2}$ has $\begin{bmatrix} -1 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Eigenvector $\vec{V_2}' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\text{trace}(A^{-1}) = -\tfrac{1}{2} = \lambda_1' + \lambda_2'$$

I.6 Q11

$$(A - \lambda I)^T = A^T - \lambda I$$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ have different eigenvectors

I.6 Q15

(a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix}$ $\det(A - \lambda I) = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$

$\Rightarrow \lambda_1 = 1 \quad \lambda_2 = 3$

$\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \vec{V_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \vec{V_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$B = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ $\lambda_1 = 0$ (rank 1) $\lambda_2 = 4$ (trace = 4)

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

I.6 Q16

$A = X\Lambda X^{-1}$

The eigenvalue matrix for $A+2I = \Lambda + 2I$

The eigenvector matrix doesn't change.

$$A + 2I = X(\Lambda + 2I)X^{-1} = X\Lambda X^{-1} + 2XX^{-1} = A + 2I$$

I.6 Q19

(a) True (no zero eigenvalues)

(b) False (repeated λ)

(c) False (repeated λ may have a full set of eigenvectors)