Lect 3 Orthonormal Columns in Q

Give $Q'Q=I$

Orthonormal columns $\quad Q$

$$Q^TQ=I \qquad = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$$

$$\begin{bmatrix} - & q_1^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

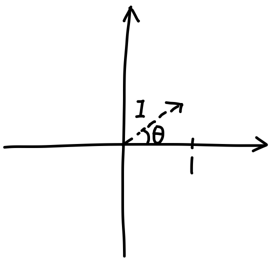
$QQ^T=I$?

Yes, if $Q=$ [square] $\quad Q$ is "orthogonal matrix"

$Q^T=Q^{-1}$

Square

eg.1 $\quad Q=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$



Any $x \quad \|Qx\|^2=\|x\|^2$

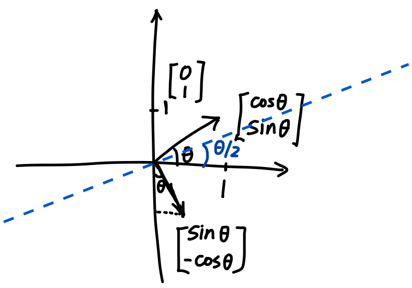
$$(Qx)^T(Qx)=x^Tx$$

$$x^T\underbrace{Q^TQ}_{I}x=x^TIx$$

eg.2

$$Q=\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

reflection matrix



Householder reflections

Start with $u^Tu=1$ (u is an unit vector)

$H=I-2uu^T$ (Obviously symmetric $Q^T=Q$)

Check $H^TH=I$

$$I-4uu^T+4u\underbrace{u^Tu}_{1}u^T=I$$

Hadamard

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} \qquad = H_4$$

$$H_8 = \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix}$$

H_{12} ?? Yes

Always possible if $N/4$ is a whole number

$$W_4=\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}$$

Haar wavelets

$$W_8=\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\frac{1}{\sqrt{8}} \qquad \frac{1}{\sqrt{4}} \qquad \frac{1}{\sqrt{2}}$$

Eigenvectors of $S^T=S$

are $Q^TQ=I$ are orthogonal

Eigenvectors of $\underline{\underline{Q}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ Discrete Fourier Transform

(permutation matrix)

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

$$\begin{cases} Q^TQ=I \\ Qx=\lambda x \\ Qy=\mu y \\ \bar{x}^T\cdot y=0 \quad \text{orthogonality} \end{cases}$$