Lecture 6: Singular Value Decomposition

Compare with $S = Q\Lambda Q^T$

Now $\underline{\underline{A}} = U\Sigma V^T$ — Singular vectors: U left, V right

$$\begin{bmatrix}\sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \\ & & & \sigma_r\end{bmatrix} \text{ singular values}$$

A^TA symmetric, semi-definite

$$[n\times m][m\times n] \quad n\times n$$

$$A^TA = V\Lambda V^T, \quad \underline{\underline{\Lambda}} \geq 0$$

V: orthogonal matrix

AA^T $m\times m$

$= U\Lambda U^T$

Looking For

$$\boxed{\begin{aligned} Av_1 &= \sigma u_1 \\ &\cdots \\ Av_r &= \sigma_r u_r \\ Av_{r+1} &= 0 \end{aligned}} \qquad u_1 = \frac{Av_1}{\sigma_1}, \quad u_r = \frac{Av_r}{\sigma_2}$$

$$A\begin{bmatrix} v_1 \cdots v_r \end{bmatrix} = \begin{bmatrix} u_1 \cdots u_r \end{bmatrix}\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_2\end{bmatrix}$$

$$AV = U\Sigma$$

$$\hookrightarrow A = U\Sigma V^T$$

$$A^TA = V\Sigma^T U^T U\Sigma V^T = V(\Sigma^T\Sigma)V^T$$

V – eigenvectors of A^TA

σ^2 – eigenvalues of A^TA

$$AA^T = U\Sigma V^T V\Sigma^T U^T = U(\Sigma\Sigma^T)U^T$$

U – eigenvectors of AA^T

$$u_1^T u_2 = 0$$

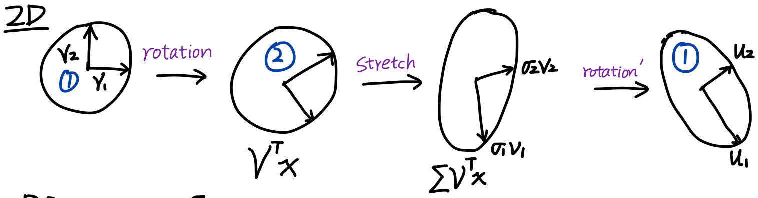
$$\left(\frac{Av_1}{\sigma_1}\right)^T\left(\frac{Av_2}{\sigma_2}\right) \overset{?}{=} 0$$

V's are orth evectors of A^TA

$$\frac{v_1^TA^TAv_2}{\sigma_1\sigma_2} = \frac{v_1^T\sigma_2^2 v_2}{\sigma_1\sigma_2} = \frac{\sigma_2}{\sigma_1}v_1^Tv = 0$$

$A = U\Sigma V^T x$ convention $\sigma_1 \geq \sigma_2 \geq \cdots > \sigma_r > 0$

2D



PD $Q\Lambda Q^T$

$A = U\Sigma V^T$

$$A = \begin{bmatrix} a & b \\ c & d\end{bmatrix} = \begin{bmatrix} \; 9 \; \end{bmatrix}$$

$$A = \underset{m\times r}{\begin{bmatrix} u_1 \cdots u_r \end{bmatrix}} \underset{r\times r}{\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{bmatrix}} \underset{r\times n}{\begin{bmatrix} v_1^T \\ \vdots \\ v_r^T\end{bmatrix}}$$

non-zeros

$$= \underset{m\times m}{\begin{bmatrix} u_1 \cdots u_m \end{bmatrix}} \underset{m\times n}{\begin{bmatrix}\sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0\end{bmatrix}} \underset{n\times n}{\begin{bmatrix} v_1^T \\ \vdots \\ v_n^T\end{bmatrix}}$$

Polar decomposition

$$A = U\Sigma V^T = SQ$$

$$\underbrace{(U\Sigma U^T)}_{S}\underbrace{(U\cdot V^T)}_{Q}$$

$u_1\sigma_1 v_1^T$