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End-Term Report

Option pricing models and their accuracy

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1 Introduction

The financial markets, characterized by their complexity and volatility, require robust mathematical models to price options and make informed investment decisions accurately. Among the plethora of models available, the Binomial Model and the Black-Scholes Model stand out for their widespread use and foundational contributions to financial theory.

This report compares the effectiveness of the Binomial and Black-Scholes models, illustrating their applications and performance through a detailed case study on the same assets.

1.1 Objectives

The primary objective of this study is to evaluate and compare the performance of the Binomial Model and the Black-Scholes Model in the context of option pricing. The key objectives include:

(a) Evaluating the efficacy of the Binomial and Black-Scholes Model in pricing European and American options.

(b) Conducting a comparative analysis using a case study on a specific asset to determine both models' accuracy and computational efficiency.

(c) Scrutinizing each model's relative strengths and limitations, considering both theoretical assumptions and practical application. This examination facilitates a practical comparison of their viability within real-world trading scenarios.

2 Options Pricing Models

2.1 The Binomial Options Pricing Model

2.1.1 Overview

The Binomial Model, introduced by Cox, Ross, and Rubinstein in 1979, is a discrete-time method for estimating the price of options. It constructs a binomial tree to represent possible future prices of the underlying asset. This model calculates the value of an option at time $t = 0$ and provides a payoff at a future date based on the value of non-dividend paying shares at that time.
The binomial option pricing model uses an iterative procedure, specifying nodes, or points in time, between the valuation date and the option's expiration date.

Assumptions:

- The key assumption is that there are only two possible outcomes for the stock price at each step: it can either move up or down.
- The model assumes no arbitrage opportunities exist in the market and that the stock price follows a random walk. The stock has a certain probability of moving up or down at each step.

By examining the binomial tree of values, a trader can determine when a decision on exercising the option may occur. If the option has a positive value, it might be exercised; if it has a value less than zero, it should be held longer.

2.1.2 How Does The Model Work?

Parameters:

- S_0: Price of the underlying asset at t_0
- X: Strike price of the option
- T: Time to maturity
- r: Risk-free interest rate
- σ: Volatility of the underlying asset
- N: Number of time steps in the binomial tree

Calculating the two possible outcomes:

- δt: Length of each time step, calculated as $\frac{T}{N}$
- u: Up factor, which is the factor by which the price increases in each step, calculated as $e^{\sigma\sqrt{\delta t}}$
- d: Down factor, which is the factor by which the price decreases in each step, calculated as $e^{-\sigma\sqrt{\delta t}}$
- p: Risk-neutral probability of an up move, calculated as $\frac{e^{r\delta t}-d}{u-d}$

Construct the Binomial Tree:

- **Stock Prices:** Build a binomial tree for stock prices from the initial price S_0 using the up and down factors. Each node in the tree represents a possible asset price at a specific time point.
- At node (i, j), the stock price is $S_0 \times u^j \times d^{i-j}$, where i is the step number and j is the number of up moves.

Calculate Option Payoff at Maturity:

- Calculate the option's payoff at each final node (N, j).
- For a call option: Payoff $= \max(S_N - X, 0)$
- For a put option: Payoff $= \max(X - S_N, 0)$

Backward Induction to Calculate Present Value:

- Starting from the final nodes, move backward through the tree to calculate the option value at each node.
- The option value at node (i, j) is the discounted expected value of the option values in the next step:
$$C_{i,j} = e^{-r\delta t} \left(p \times C_{i+1,j+1} + (1-p) \times C_{i+1,j}\right)$$
where $C_{i,j}$ is the option value at node (i, j).

Calculate the Option Value at the Root:

- Continue the backward induction until you reach the root of the tree $(0, 0)$, which gives the present value of the option.

2.1.3 Advantages

- The multi-period view allows users to visualize changes in asset prices over time and evaluate the option based on decisions made at different points.
- The binomial model helps determine when exercising the option may be advisable and when it should be held longer.
- One major advantage of the binomial option pricing model is its mathematical simplicity, even though it can become complex in a multi-period model.
- The binomial model offers flexibility as users can alter inputs at each step to account for differences in the ability to exercise options with non-standard features.

2.1.4 Disadvantages

- Binomial models are complex to construct and, depending on the number of steps, can become unwieldy regarding spreadsheet size and the computing power needed.
- They require predicting future prices, which can be challenging and uncertain.

2.2 The Black-Scholes Options Pricing Model

2.2.1 Overview

In contrast to the Binomial Model, the Black-Scholes Model, developed by Fischer Black, Myron Scholes, and Robert Merton in 1973, provides a continuous-time framework for pricing European-style options. This model is renowned for its analytical elegance and computational efficiency, offering a closed-form solution under the assumption of constant volatility and a risk-free rate.

The model uses current stock prices, expected dividends, the option's strike price, expected interest rates, time to expiration, and expected volatility to calculate the theoretical value of an option contract. It assumes that stock prices follow a lognormal distribution with a random walk, exhibiting constant drift and volatility.

Assumptions:

- The underlying asset pays no dividends.
- Markets are random and not influenced by external factors.
- The risk-free interest rate and volatility are constant over the option's life.
- Asset returns are normally distributed.

2.2.2 How Does The Model Work?

The Black-Scholes Model provides a closed-form solution to calculate the theoretical value of option prices.

Formula:

$$C = S_0 N(d_1) - X e^{-rT} N(d_2)$$

where:

$$d_1 = \frac{ln(S_0/x) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Parameters:

- S_0: Current price of the asset
- X: Strike price
- T: Time to maturity
- r: Risk-free interest rate
- σ: Volatility of the asset
- $N(\cdot)$: Cumulative distribution function of the standard normal distribution

2.2.3 Advantages

- **Closed-form solution:** The Black-Scholes Model provides an explicit formula to calculate option prices, making it straightforward to use and implement.
- **Widely accepted:** It is a standard model in financial markets, streamlining the process of option pricing and trading.
- **Market risk optimization:** By providing theoretical values, the model helps compare market prices and identify mispriced options, aiding in risk management.
- **Portfolio optimization:** The model also helps understand the risks associated with different options, allowing for better portfolio management.

2.2.4 Disadvantages

- **Many assumptions:** The model relies on several assumptions, such as constant volatility and risk-free rate, which may not hold true in real markets.
- **European options only:** It is designed for European options, which can only be exercised at expiration, limiting its applicability to American options.
- **Implied volatility:** The model does not account for changes in implied volatility, which can affect option prices.
- **No dividends:** It assumes the underlying asset does not pay dividends, which is a limitation for stocks that do.

3 Case Study-Mid Term

In this case study, we will evaluate the call and put options of Apple Inc. (AAPL) using two different models: the Black-Scholes Model (BSM) and the Cox-Ross-Rubinstein (CRR) Binomial Model. We will compare the predicted option prices with the actual market prices to assess the accuracy of these models.

3.1 Data Collection

1. Selection of Ticker and Option Dates
    - Ticker: We selected Apple Inc. (AAPL) due to its high trading volume and availability of options data.
    - Option Expiry Dates: We selected options expiring on September 20, 2024 (calls) and December 18, 2026 (puts).

2. Data Sources
    - Yahoo Finance retrieves historical and current stock prices, options chain data, and treasury yield data.

3.2 Methodology

We used Python and various libraries for data acquisition and model implementation. We used the `yfinance` library to fetch historical stock prices and treasury yield data to obtain the data required for our analysis, which is as follows:

- We used the `yfinance` library to fetch the latest closing price of the stock using its ticker symbol
- The risk-free rate was obtained using the 10-year Treasury yield, fetched using its ticker symbol `T̂NX`
- The time to expiration was calculated by finding the difference between the current date and the option's expiration date
- We calculated the stock's annualized volatility based on its historical daily returns over the past year

3.2.1 Model Implementation

We implemented two models to predict option prices: the Black-Scholes model and the Cox-Ross-Rubinstein (CRR) Binomial Model

Black-Scholes Model

- The Black-Scholes model is a closed-form solution used to estimate the price of European call and put options
- We defined the `d1` and `d2` functions and used them to calculate the call and put option prices

Cox-Ross-Rubinstein (CRR) Binomial Model

- The CRR Binomial Model is a discrete-time model used to price options by building a binomial tree of possible stock prices
- We defined the binomial model functions for both call and put options

3.3 Results and Comparison

We compared the valuations from both models with the actual market prices of the options. Below are the key findings:
Call Options (Expiring on 2024-09-20):

- The mean absolute error (MAE) for Black-Scholes: 14.36
- The mean absolute error (MAE) for Binomial: 11.33

The Black-Scholes model predicted negative values for some put options, indicating potential inaccuracies due to the long time to maturity and high strike prices. The Binomial model showed more realistic valuations but still had significant deviations from the market prices.

3.4 Visualizing Results

The scatter plots and histograms below illustrate the differences between the actual market prices and the predicted prices for call options from both models.

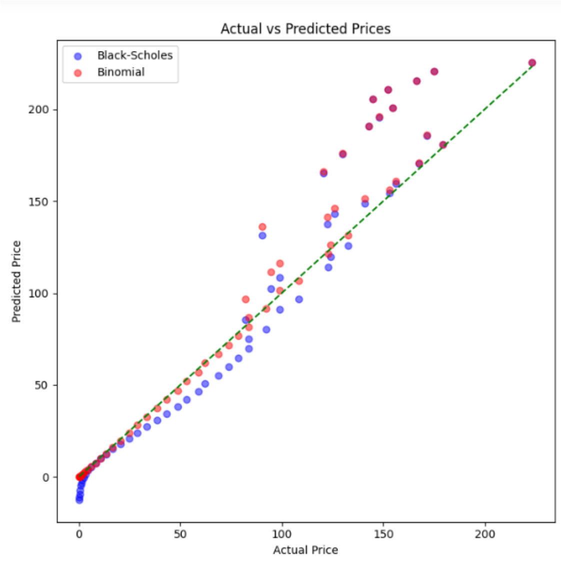

Figure 1: Actual vs. Predicted Prices

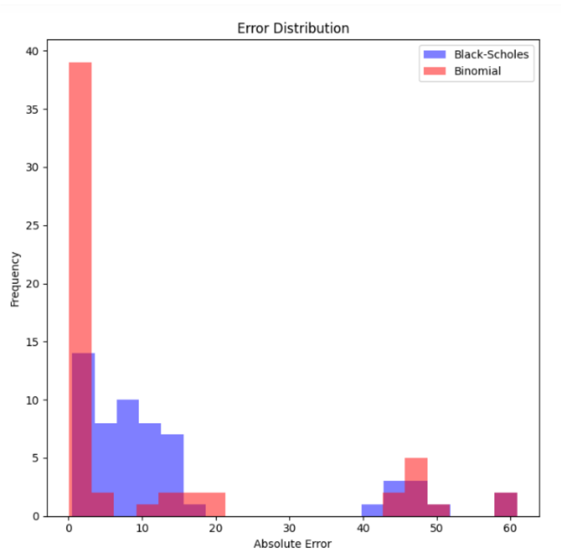

Figure 2: Error Distribution

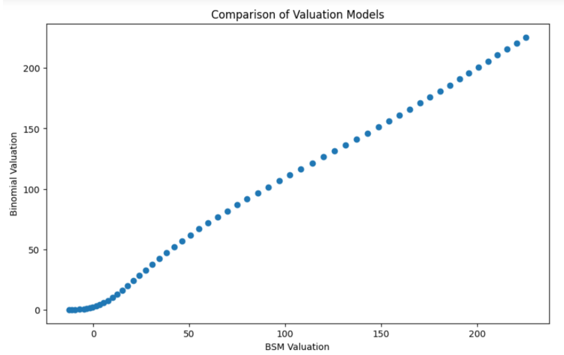

Figure 3: Comparison of Valuation Models

3.5 Discussion

In-The-Money Calls Options

rike	lastPrice	bid	ask	change	percentChange	impliedVolatility	inTheMoney	bsmValuation	binValuation
5.0	223.40	226.65	227.10	0.0	0.0	3.539064	True	226.859503	226.859503
10.0	175.16	180.50	184.15	0.0	0.0	0.000010	True	221.898999	221.898999
15.0	166.23	157.55	158.95	0.0	0.0	0.000010	True	216.938485	216.938495
20.0	152.13	144.00	146.25	0.0	0.0	0.000010	True	211.977824	211.977990
25.0	144.84	145.05	147.25	0.0	0.0	0.000010	True	207.016209	207.017486
30.0	154.79	160.70	164.35	0.0	0.0	0.000010	True	202.051103	202.056982
35.0	147.99	162.05	162.65	0.0	0.0	0.000010	True	197.077113	197.096478
40.0	142.89	157.15	157.70	0.0	0.0	0.000010	True	192.085444	192.135973
45.0	171.47	186.70	187.35	0.0	0.0	1.608400	True	187.064251	187.175469
50.0	179.19	181.75	182.40	0.0	0.0	1.543948	True	181.999767	182.214965

Figure 4: Call Options

The call options we examined were primarily in-the-money (ITM). For a growing company like Apple Inc., these ITM call options are expected to have high prices due to the positive outlook and anticipated returns. The Black-Scholes and Binomial models reflect this expectation, showing relatively high predicted prices for these options.

Out-of-the-Money Put Options

strike	lastPrice	bid	ask	change	percentChange	impliedVolatility	inTheMoney	bsmValuation	binValuation
50.0	0.18	0.15	0.27	0.00	0.000000	0.430181	False	0.000000e+00	0.000006
60.0	0.45	0.21	0.46	0.00	0.000000	0.409918	False	0.000000e+00	0.000107
70.0	0.45	0.36	0.58	0.00	0.000000	0.377936	False	0.000000e+00	0.000840
80.0	0.60	0.51	0.58	0.00	0.000000	0.338141	False	0.000000e+00	0.004474
85.0	0.67	0.60	0.87	0.00	0.000000	0.342292	False	2.557954e-13	0.009172
90.0	0.75	0.65	0.85	0.00	0.000000	0.322883	False	5.599077e-12	0.017428
95.0	0.80	0.67	1.00	0.00	0.000000	0.314948	False	9.191581e-11	0.030633
100.0	1.06	0.84	1.28	-0.06	-5.357148	0.312629	False	1.146304e-09	0.049902
105.0	1.27	1.03	1.49	0.00	0.000000	0.305427	False	1.122640e-08	0.083779
110.0	1.52	1.40	1.55	0.00	0.000000	0.291755	False	8.878604e-08	0.131019

Figure 5: Put Options

Conversely, the put options analyzed were mostly out-of-the-money (OTM). This results in significantly lower prices, as the market does not expect the stock price of Apple Inc. to decrease substantially in the near future. The Black-Scholes model, in particular, predicts very low prices for these OTM put options, sometimes even resulting in negative values, which indicates the model's limitation in handling certain market conditions. With its stepwise approach, the Binomial model provides a more realistic floor to these predictions but still reflects the low valuation in line with market expectations.

These observations underline the influence of market sentiment and the current stock trajectory on option pricing. The models' predictions align with the logical financial market behavior for a growth stock, where ITM calls are valued higher, and OTM puts hold little value.

3.6 Reasons for Inaccuracies and Variances

Model Assumptions
The Black-Scholes Model assumes constant volatility, interest rates, log-normally distributed returns, and continuous trading. Real market conditions often deviate from these assumptions. The Binomial Model provides more flexibility by allowing discrete time intervals and adjusting for early exercise in American options, leading to more accurate pricing in some cases.

Volatility Estimation
The volatility input is critical for both models. Historical volatility may not always predict future volatility well, leading to inaccuracies in model predictions.

Interest Rates
The risk-free rate was derived from the 10-year Treasury yield, which might not perfectly align with the specific time frames of the options analyzed.

4 Monte Carlo Simulations: An Overview

4.1 Introduction

Monte Carlo simulations are a computational technique used in finance to model the behavior of financial instruments, particularly in options pricing. By relying on repeated random sampling, these simulations estimate probabilities and predict potential outcomes, providing an alternative to the Black-Scholes-Merton formula. Despite their computational intensity, Monte Carlo simulations offer realistic depictions of scenarios such as stock prices, option prices, and risk probabilities.

4.2 Key Concepts

Monte Carlo simulations model the probability of different outcomes where random variables play a significant role, aiding in understanding the impact of risk and uncertainty. They are used in finance, investing, and various other fields to predict outcomes by averaging multiple values assigned to uncertain variables.

4.3 Historical Background

Named after the famous gambling destination, Monte Carlo simulations were developed by Stanislaw Ulam and refined with John Von Neumann during the Manhattan Project. The method highlights the role of randomness in modeling scenarios.

4.4 Applications in Finance

Monte Carlo simulations are widely used to assess financial risk, estimate project costs, and analyze investment outcomes. They are applied in various industries, including telecommunications, insurance, and financial planning, and are increasingly integrated with AI to enhance accuracy.

4.5 Monte Carlo Simulations in Options Pricing

Options Pricing: Options are derivatives that grant the holder the right to buy or sell an asset at a specified price before a certain date. The Monte Carlo method simulates potential future asset price paths and calculates the corresponding option payoffs. The average of these payoffs, discounted to present value, estimates the option price.

Steps in Monte Carlo Simulation:

- **Modeling Asset Price:** The asset price is modeled using Geometric Brownian Motion (GBM) represented by the stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

- **Simulating Price Paths:** The asset price is simulated over time using random values generated from a standard normal distribution.
- **Calculating Payoff:** For each simulated path, the option's payoff at expiration is calculated, e.g., for a European call option:

$$\text{Payoff} = \max(S_T - K, 0)$$

- **Discounting Payoffs:** Payoffs are discounted to present value using the risk-free rate.
- **Averaging Payoffs:** The option price is estimated by averaging the discounted payoffs across all simulated paths.

4.6 Advantages and Challenges

Monte Carlo simulations offer flexibility and accuracy in pricing complex options, such as exotic options. However, they can be computationally intensive, and their accuracy depends on the assumptions made about the asset's price dynamics.

4.7 Implementation in Python

Monte Carlo simulations can be efficiently implemented in Python using libraries like `numpy`. The Euler Discretization Scheme is applied to solve the stochastic differential equation, enabling fast and accurate simulation of price paths.

4.7.1 Example: Pricing a European Call Option

A European call option can be priced using the Black-Scholes-Merton SDE, where the stock price is modeled as $S(t)$ with the risk-free rate r, volatility σ, and Brownian motion $Z(t)$. Python's `numpy` library simulates the price paths and calculates the option price.

5 Backtesting on the Black-Scholes Model

5.1 Introduction

In this section, we performed backtesting on the Black-Scholes model using real market data from the Nifty50 index. This backtesting aimed to assess the accuracy of the Black-Scholes model in pricing both call and put options under real market conditions.

5.2 Data and Methodology

For this analysis, we have utilized historical options data from Nifty50, expiring on 22nd August 2024, including both call and put options. The Black-Scholes model was employed to calculate the theoretical prices of these options. The inputs for the model include the underlying asset's price, strike price, time to expiration, risk-free interest rate, and volatility.

All calculations were implemented in Python, leveraging libraries such as `numpy`, `scipy`, and `pandas` to compute the option prices and perform statistical analysis efficiently.

Data Preprocessing The code begins with a function to clean a CSV file containing Nifty50 options data. The 'clean-csv-file' function removes leading and trailing commas from each file line. The 'process-nifty50-data' function reads the cleaned CSV file into a DataFrame, assigns appropriate column names for CALL and PUT options, and handles missing or invalid data entries.

Black-Scholes Model The 'black-Scholes' function implements the model for calculating the theoretical price of European call and put options. The model computes the option prices based on these inputs and the standard Black-Scholes formulas.

Backtesting The 'backtest-black-Scholes' function compares the theoretical prices generated by the Black-Scholes model with actual market prices from the dataset. It calculates the error between the models and observed market prices for both CALL and PUT options. This process involves:

- Parsing strike prices and implied volatilities from the dataset
- Calculating the Black-Scholes price for each option
- Computing the absolute error between the real price and the model price

The results are stored in a data frame for further analysis.

5.3 Error Metrics

To quantify the accuracy of the Black-Scholes model, the Mean Absolute Error (MAE) was calculated between the model's predicted prices and the actual market prices. The MAE is a widely used measure that indicates the average magnitude of errors in a set of predictions without considering their direction. It is given by:

$$\text{MAE} = \frac{1}{n} \sum_{i=1}^{n} |\text{Real Price}_i - \text{BS Price}_i|$$

Where n is the number of data points, Real Price_i is the actual market price of the option, and BS Price_i is the price predicted by the Black-Scholes model.

5.4 Visualization of Results

The backtesting results were visualized to compare the real market prices with the prices predicted by the Black-Scholes model. The plots depict the variation in pricing errors across different strike prices. Additionally, the graph was segmented to distinguish between In-The-Money (ITM), At-The-Money (ATM), and Out-of-The-Money (OTM) options for both calls and puts.

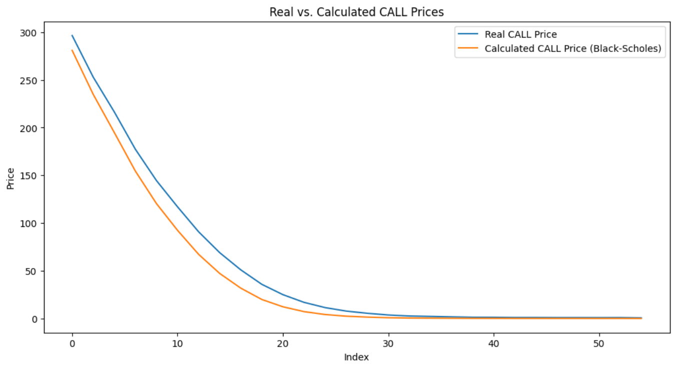

Figure 6: Real Prices vs Black-Scholes Prices for Nifty50 CALL Options

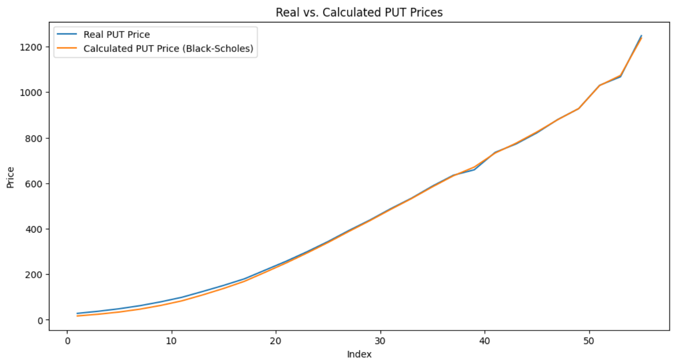

Figure 7: Real Prices vs Black-Scholes Prices for Nifty50 PUT Options

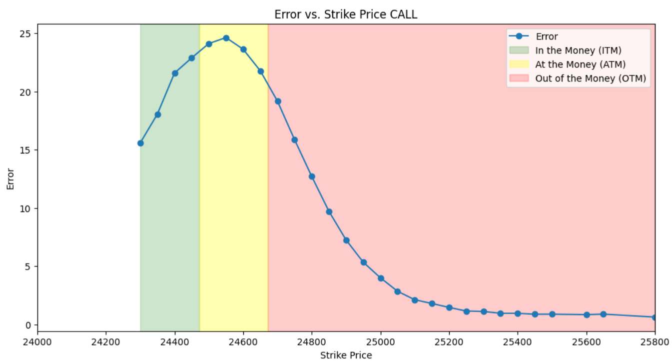

Figure 8: Error between Real Prices and Black-Scholes Prices vs. Strike Price for Nifty50 CALL Options

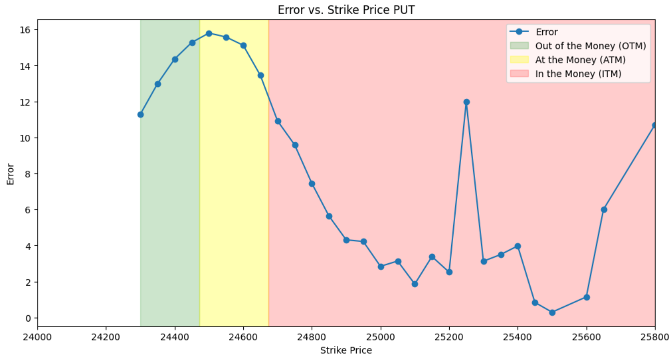

Figure 9: Error between Real Prices and Black-Scholes Prices vs. Strike Price for Nifty50 PUT Options

5.5 Results and Discussions

The Mean Absolute Error (MAE) for call options was 9.3844, while the MAE for put options was 7.5451. These values indicate the average deviation between the Black-Scholes model's theoretical and actual market prices.

For call options, the error was notably lower for out-of-the-money (OTM) options compared to at-the-money (ATM) options. This discrepancy could be due to the Black-Scholes model's assumptions, which include constant volatility and risk-free rates. In practice, these assumptions may not hold, particularly for ATM options where the impact of volatility changes is more pronounced.

Meanwhile, for put options, the pattern of errors was less consistent. Unlike the behavior observed for call options, the error did not exhibit a smooth trend with increasing strike prices. This suggests that the Black-Scholes model's assumptions might be less effective in capturing the pricing dynamics of put options across different strike prices.

Overall, the findings underscore the limitations of the Black-Scholes model, particularly in environments where volatility and interest rates are not constant. These factors contribute to the observed discrepancies in option pricing, highlighting the need for models that account for varying market conditions.

6 Conclusion

This report examined the performance of the Black-Scholes and the Binomial models for options pricing, including a case study comparing their effectiveness. The analysis also incorporated Monte Carlo simulations and backtesting to evaluate the robustness of the Black-Scholes model.

6.1 Comparison of Binomial and Black-Scholes Models

The Binomial model and the Black-Scholes model offer different approaches to options pricing. The Binomial model, which uses a discrete-time framework, provides flexibility by accommodating changing conditions and different underlying assumptions. In contrast, the Black-Scholes model relies on continuous-time assumptions, including constant volatility and interest rates.

6.2 Backtesting on the Black-Scholes Model

Backtesting involved comparing the theoretical prices generated by the Black-Scholes model with actual market prices for both calls and put options. The results revealed that the Black-Scholes model performed better for out-of-the-money call options but struggled with at-the-money options due to its constant volatility assumption.

6.3 Summary

Overall, the Binomial and Black-Scholes models have strengths and limitations. The Binomial model's flexibility and ability to handle varying conditions make it suitable for scenarios with significant changes in market variables. In contrast, the Black-Scholes model provides a closed-form solution that is efficient for European options but may not fully capture real-world complexities. The Monte Carlo simulations and backtesting highlighted the need for models accommodating dynamic market conditions and variable assumptions.

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