Notes on Tropical Nets

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1 Definitions

Basic Setting
Inputs and Input Space $x \in \mathcal{X} = \mathbb{R}^{n_0}$
Network $\mathcal{N} : \mathcal{X} \mapsto \mathbb{R}$
Layer Indicies $l \in \{0\} \cup \{1, \ldots, d\} \cup \{d+1\}$ (0 for input; $d+1$ to output)
Hidden Layer Widths $n_1, \ldots, n_d$.
Total neurons $n = \sum_{l=1}^{d} n_l$
Weight matrices $W^l \in \mathbb{R}^{n_{l+1} \times n_l}$ for $l < d+1$.
Biases $b^l \in \mathbb{R}^{n_l}$ for $l > 0$.
Nonlinearity $R(x)_i = \max(0, x_i)$
Network definition:
$\mathcal{N}^1(x) = b^1 + W^0 x \qquad \mathcal{N}^{l+1}(x) = b^{l+1} + W^l R(\mathcal{N}^l(x)) \qquad \mathcal{N}(x) = \mathcal{N}^{d+1}(x)$

States and Linear Regions
Layer State Map $\sigma^l : \mathcal{X} \mapsto \{0,1\}^{n_l}$ where each coordinate, $\sigma^l(x)_i$, is an indicator for $\mathcal{N}^l(x)_i > 0$. ("Neuron State")
Network State Map $\sigma(x) = (\sigma^d(x), \ldots, \sigma^1(x))$
Network States $\Sigma = \{\sigma(x) : x \in \mathcal{X}\}$
As a shorthand, identify $\Sigma \subset \{0,1\}^n$ with flattened counterparts
For any $\mu = (\mu^d, \ldots, \mu^1) \in \{0,1\}^n$:
Identify each μ^l appearing in linear equations with diagonal matrix, $diag(\mu^l)$
    for example: $R(\mathcal{N}^l(x)) = \sigma^l(x)\mathcal{N}^l(x)$
Corresp. Linear Function: L_μ by replacing layer l with multilpication by μ^l.
Corresp. Region: $\mathcal{X}_\mu = \overline{\{x \in \mathcal{X} : \sigma(x) = \mu\}}$
    so that: $\mathcal{X}_\mu \subset \{x \in \mathcal{X} : \mathcal{N}(x) = L_\mu \left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)\}$ (Corresp. *Linear* Region)

Tropical Notation
For any vector b or matrix W,
    define $\quad b_\pm = R(\pm b) \qquad (W_\pm)_{i,j} = R(\pm W_{i,j})$
    such that $\quad b = b_+ - b_- \qquad W = W_+ - W_-$
    likewise define $|b| = b_+ + b_- \qquad |W| = W_+ + W_-$
Define recursively for $x \in \mathcal{X}$ $\mu, \tau \in \{0,1\}^n$:

Table 1: Example First Few Recursions (without biases)

layer l	$f^l_\mu(x)$	$g^l_\mu(x)$
1	W^0x	0
2	$W^1_+\mu^1W^0x$	$W^1_+\mu^1W^0x$
3	$W^2_+\mu^2W^1\mu^1W^0x + \lvert W^2\rvert W^1_-\mu^1W^0x$	$W^2_-\mu^2W^1\mu^1W^0x + \lvert W^2\rvert W^1_-\mu^1W^0x$

layer l	$p^l_\mu(x)$	$q^l_\mu(x)$
1	μ^1W^0x	W^0_-x
2	$\mu^2W^1\mu^1W^0$	$W^1_-\mu^1W^0x + \lvert W^1\rvert W^0_-x$
3	$\mu^3W^2\mu^2W^1\mu^1W^0x$	$W^2_-\mu^2W^1\mu^1W^0x + \lvert W^2\rvert(W^1_-\mu^1W^0x + \lvert W^1\rvert W^0_-x)$

Recursion 1

$f^1_\mu(x) = W^0x \qquad g^1_\mu(x) = 0$

$f^{l+1}_\mu(x) = b^{l+1} + W^l_+\mu^l(f^l_\mu(x) - g^l_\mu(x)) + |W^l|g^l_\mu(x)$

$g^{l+1}_\mu(x) = b^{l+1} + W^l_-\mu^l(f^l_\mu(x) - g^l_\mu(x)) + |W^l|g^l_\mu(x)$

$f_\mu(x) = f^{d+1}_\mu(x) \qquad g_\mu(x) = g^{d+1}_\mu(x)$

Recursion 2

$p^0_\mu(x) = x \qquad q^0_\mu(x) = 0$

$p^{l+1}_\mu(x) = \mu^{l+1}(b^{l+1} + W^{l+1}p^l_\mu(x))$

$q^{l+1}_\mu(x) = W^l_-p^l_\mu(x) + |W^l|q^l_\mu(x)$

Relation to Network

$\mathcal{N}^l(x) = f^l_{\sigma(x)}(x) - g^l_{\sigma(x)}(x)$ for $l = 1, \ldots, d+1$

$R(\mathcal{N}^l(x)) = p^l_{\sigma(x)}(x)$

Theorem 1. *Tropical Formulation*
For $l = 0, \ldots, d$

$$\mathcal{N}^{l+1}(x) = \left(\max_{\mu\in\{0,1\}^n} f^{l+1}_\mu(x)\right) - \left(\max_{\tau\in\{0,1\}^n} g^{l+1}_\tau(x)\right) \tag{1}$$

$$R(\mathcal{N}^l)(x) = \left(\max_{\mu\in\{0,1\}^n} p^l_\mu(x) + q^l_\mu(x)\right) - \left(\max_{\tau\in\{0,1\}^n} q^l_\tau(x)\right) \tag{2}$$

where each term is an affine function of x.