

Proba Stats

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Chapitre 1

Probabilities : starter guide

1.1 Probabilities spaces and measures

1.1.1 Probability spaces

Définition 1

A **probability space** is a set of realisations denoted Ω , together with a probability measure on Ω . A **probability measure** on Ω is a function $\mathcal{P} : \mathcal{P}(\Omega) \rightarrow [0, 1]$ such that

1. $P(\emptyset) = 0, P(\Omega) = 1$.
2. If $A_i \in \mathcal{F}, i \in \mathbb{N}$ is a sequence of events with $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$P\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} P(A_i).$$

Remarque

- (1) The probability that something happens is 1 and that nothing happens is 0.
 - (2) The probability of events that cannot occur simultaneously is the sum of the probabilities of the events.
- From these properties, we can deduce the next theorems.

Théorème 1

Let P be a probability measure on some realisation set Ω . Then,

1. $P(\emptyset) = 0, P(\Omega) = 1$;
2. for any event $A, P(\Omega \setminus A) = 1 - P(A)$;
3. if two events A, B are such that $A \subset B, P(B) = P(A) + P(B \setminus A)$. In particular, $P(A) \leq P(B)$;
4. for two events A, B ,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B);$$

5. finite σ -additivity : if $n \geq 2$, and A_1, \dots, A_n are events such that $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i);$$

6. countable σ -additivity : if A_1, A_2, \dots are events such that $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i);$$

7. *finite σ -sub-additivity* : if $n \geq 2$, and A_1, \dots, A_n are events, then

$$P\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n P(A_i);$$

8. *countable σ -sub-additivity* : if A_1, A_2, \dots are events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i);$$

9. *monotone convergence, increasing sequences* : if A_1, A_2, \dots are events such that $A_i \subset A_{i+1}$ for all i 's then

$$\lim_{n \rightarrow \infty} P(A_n) = P\left(\bigcup_{n=1}^{\infty} A_n\right);$$

10. *monotone convergence, decreasing sequences* : if A_1, A_2, \dots are events such that $A_{i+1} \subset A_i$ for all i 's, then

$$\lim_{n \rightarrow \infty} P(A_n) = P\left(\bigcap_{n=1}^{\infty} A_n\right).$$

Remarque

It's not necessary to learn this list by heart

We will encounter two main types of probability spaces in these notes :

- **Discrete probability spaces** : in that case Ω is a finite or a countable set, and the set of events really is $\mathcal{F} = \mathcal{P}(\Omega)$.
- **Continuous probability spaces** : in that case, $\Omega = \mathbb{R}^d$ with $d \geq 1$ integer. We won't go into a formal definition of the *set of Borel sets*, and we will do as if we could take $\mathcal{F} = \mathcal{P}()$

1.1.2 Discrete probability measures

Définition 2

Let Ω be a finite countable set. A **probability mass function** on Ω is a function $p : \Omega \rightarrow [0, 1]$ such that

$$\sum_{\omega \in \Omega} p(\omega) = 1$$

The **probability measure** associated to a probability mass function p is the function $P_p : \mathcal{P}(\Omega) \rightarrow [0, 1]$ given by

$$P_p(A) = \sum_{\omega \in A} p(\omega).$$

1.1.3 Continuous probability measures

One cannot make sense of the probability that a drop of water falls at *precisely* one point x , but it is relatively easy to make sense of the probability that the drop falls *in a small disk* around x . This is the essence of the next definition.

Définition 3

Let $d \geq 1$. A **probability density function** on \mathbb{R}^d is a Riemann integrable function $f : \mathbb{R}^d \rightarrow [0, +\infty)$ such that

$$\int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{+\infty} dx_d f(x_1, \dots, x_d) = 1.$$

The **probability measure** associated to a probability density function f is the $[0, 1]$ -valued function P_f given by

$$P_f(A) = \int_{-\infty}^{+\infty} dx_1 \dots \int_{-\infty}^{+\infty} dx_d f(x_1, \dots, x_d) \mathbb{1}_A(x_1, \dots, x_d).$$

Remarque

This is the equivalent of **the second definition** but for a space where you can't delimit the element to sum : a continuous space.

The **density function** is a function that shows where the variable like to be at most. The probability is the area under this curve.

$$\mathbb{1}_A = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

1.1.4 Genreal probability spaces and measures

Définition 4

Let Ω be a set. A **sigma-algebra** on Ω is a set $\mathcal{F} \subset \mathcal{P}(\Omega)$ which satisfies

1. \mathcal{F} contains the empty set ($\emptyset \in \mathcal{F}$).
2. \mathcal{F} is stable by taking the complement ($A \in \mathcal{F} \implies \Omega \setminus A \in \mathcal{F}$).
3. \mathcal{F} is stable by countable unions (if for all $i \in \mathbb{N}$, $A_i \in \mathcal{F}$, then $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$

Remarque

$\mathcal{P}(\Omega)$ is the set of all subset of Ω .

Now we can deduce the following properties.

Théorème 2

Let Ω be a set and \mathcal{F} a sigma-algebra on Ω . Then all of the following hold.

1. $\Omega \in \mathcal{F}$.
2. \mathcal{F} is stable by finite intersections : if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
3. If $A, B \in \mathcal{F}$, then $A \setminus B \in \mathcal{F}$.
4. \mathcal{F} stable by countable intersections : if $A_1, A_2, \dots \in \mathcal{F}$, then $\bigcap_{i \geq 1} A_i \in \mathcal{F}$.
5. \mathcal{F} is stable by increasing limits : if $A_i \in \mathcal{F}$, $i \geq 1$ is such that $A_i \subset A_{i+1}$, then, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.
6. \mathcal{F} is stable by decreasing limits : if $A_i \in \mathcal{F}$, $i \geq 1$ is such that $A_{i+1} \subset A_i$, then, $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

Définition 5

Let Ω be a set and \mathcal{F} a sigma-algebra on Ω . A **probability measure** on (Ω, \mathcal{F}) is a function $P : \mathcal{F} \rightarrow [0, 1]$ such that

1. $P(\Omega) = 1$.
2. If $A_i \in \mathcal{F}$, $i \in \mathbb{N}$ is a sequence of events with $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$P\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} P(A_i).$$

Remarque

The same as before but more precise.

Définition 6

A **probability space** is a triplet Ω, \mathcal{F}, P where Ω is a set (the set of realisations), \mathcal{F} is a sigma-algebra on Ω (the set of events), and P is a probability measure on (Ω, \mathcal{F}) .

Remarque

This is the most important definition of this “introduction”

1.1.5 Inclusion-Exclusion

It is a generalisation of the following fact that we encounter when counting objects : to count the number of objects with property A or property B, we can count the number of objects with property A add the number of objects with property B, and correct our over-counting by removing from this the number of objects with both property A and property B (which were counted twice).

Théorème 3

Let P be a probability measure on some realisation set Ω . Let $n \geq 1$ and A_1, \dots, A_n be events. Then,

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{k=1}^n (-1)^{k+1} \sum_{i_1 < \dots < i_k \leq n} P\left(\bigcap_{j=1}^k A_{i_j}\right).$$

Moreover, for $1 \leq l \leq \frac{n}{2}$ integer a ,

$$P\left(\bigcup_{i=1}^n A_i\right) = \begin{cases} \leq \sum_{k=1}^{2l-1} (-1)^{k+1} \sum_{1 \leq i_1 < \dots < i_k \leq n} P\left(\bigcap_{j=1}^k A_{i_j}\right) \\ \geq \sum_{k=1}^{2l} (-1)^{k+1} \sum_{1 \leq i_1 < \dots < i_k \leq n} P\left(\bigcap_{j=1}^k A_{i_j}\right) \end{cases}.$$

1.2 Random variables and expectation

1.2.1 Random variables

Random variables are therefore functions going from the set of realisations to the real numbers ; for example, if the “experiment ” is looking at all people born in 2000, one could make the measurement of the height of the first individual born that year.

Définition 7

A (real) **random variable** is a function from the realisation space Ω to \mathbb{R} . The probability that a random variable falls in a set A is

$$P(X \in A) := P(X^{-1}(A))$$

In words : it is the probability that the realisation of the experiment is such that the measurement X takes a value A .

Example

We are throwing a dice :

- $\Omega = \{1, 2, 3, 4, 5, 6\}$
- $X(\omega) = 1$ if it's even, $X(\omega) = 0$ if it's odd.
- So we are searching ω in Ω that gives $X(\omega) = 1$:

$$X^{-1}(A) = \{\omega \in \Omega : X(\omega) \in A\} = \{2, 4, 6\}$$

Then we apply the probability on these results

$$P(X \in A) = P(\{2, 4, 6\}) = \frac{3}{6} = 0.5$$

We will frequently use notations similar to the following :

$$\begin{aligned} P(X = x) &\equiv P(X \in \{x\}), \\ P(X \leq x) &\equiv P(X \in (-\infty, x]), \\ P(X > x) &\equiv P(X \in (x, +\infty)). \end{aligned}$$

Définition 8

Let Ω be a set of realisation, and let P be a probability measure on Ω . Let $X : \Omega \rightarrow \mathbb{R}$ be a random variable. X is **discrete** if there is $\mathcal{D}_X \subset \Omega$ countable or finite such that $P(X \in \mathcal{D}_X) = 1$. The **law of X** is then the probability measure on \mathbb{R} given by

$$P_X(A) = \sum_{x \in A \cap \mathcal{D}_X} P(X = x)$$

In words, a **discrete random variable** is a variable that can take only finitely or countably many values with non-zero probability.

The second very important family of variables are **continuous random variables**.

Définition 9

Let Ω be a set of realisation, and let P be a probability measure on Ω . Let $X : \Omega \rightarrow \mathbb{R}$ be a random variable. X is **continuous random variable** if there is a density function $f_X : \mathbb{R} \rightarrow [0, +\infty)$ such that

$$P(X \in A) = \int_{\mathbb{R}} \mathbb{1}_A f_X(x) dx.$$

The **law of X** is then the probability measure on \mathbb{R} given by P_{f_X} .

Remarque

Here we can't say : "the probability that $X = 2$ ", because it will always be 0. Instead we use a density $f_X(x)$ to compute on an interval. For example, if X is measuring the size of somebody, $P(170 \leq X \leq 180)$ is compute with the density f_X by an integral, because the probability to have 170.000000cm is 0.

- Discrete : whe can say $P(X = x)$.
- Continue : values are infinite, we look at interval not precise points.

There is a similat notion of law for general random variables. Random variables allows us sometime to pass from some continuous probability to some discrete random variables.

Définition 10

Let Ω be a set of realisation, and let P be a probability measure on Ω . Let $X : \Omega \rightarrow \mathbb{R}$. The **law of X** is the probability measure P_X on \mathbb{R} given by

$$P_X(A) = P(X \in A).$$

Définition 11

Let $X : \Omega \rightarrow \mathbb{R}$ be a random variable. The **cumulative distribution function of X** is defined by

$$F_X(t) = P(X \leq t).$$

Two random variables have the same law if and only if they have the same cumulative distribution functions. Note that if X is a continuous random variable with density f_X , one has that F_X is a primitive of f_X :

$$F'_X(t) = f_X(t) \quad \forall t \in \mathbb{R}.$$

Remarque

$$F_x : \mathbb{R} \rightarrow [0, 1]$$

Example 1 : If $P(X = 0) = P(X = 1) = \frac{1}{2}$

$$F_X(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2} & 0 \leq t < 1 \\ 1 & t \geq 1 \end{cases}$$

Théorème 4

Let $x \rightarrow \int_a^x f(t) dt$ is primitive of f

If X is a continuous random variable : there exist a density function

$$f_X : \mathbb{R} \rightarrow \mathbb{R}, P(X) = \int_{\mathbb{R}} \mathbb{1}_A(x) f_X(x) dx$$

So F_X is differentiable and

$$F'_X(t) = f_X(t) = \int_{-\infty}^t f_X(x) dx$$

Example

If X is a continuous random variable with density $f_X(x) = \mathbb{1}_{[0,4]}(x)$ then

$$P(X \leq t) = \int_{-\infty}^t f_X(x) dx = \frac{1}{4} \int_{-\infty}^t \mathbb{1}_{[0,4]}(x) dx = \begin{cases} 0 & \text{if } t < 0, \\ \frac{t}{4} & \text{if } t \in [0, 4], \\ 1 & \text{if } t > 4. \end{cases}$$

Remarque

to

Remarque importante

To resume an important point :

- for discrete random variables we use a **probability masse function**.
- for continue random variables we use a **density function**.

1.2.2 Expectation

Définition 12

Discrete case : If Ω is a finite or countable set, $p : \Omega \rightarrow [0, 1]$ is a probability mass function, and $X : \Omega \rightarrow \mathbb{R}$ is a random variable such that

$$\sum_{\omega \in \Omega} |X(\omega)| p(\omega) < \infty,$$

the **expectation of X under p** is

$$E_p(X) := \sum_{\omega \in \Omega} X(\omega) p(\omega).$$

In this case, we say that X is **P_p -integrable**.

Continuous case : If $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a probability density function, and $X : \mathbb{R}^d \rightarrow \mathbb{R}$ is a random variable such that

$$\int_{\mathbb{R}^d} dx |X(x)| f(x) < \infty,$$

the **expectation of X under P_f** is

$$E_f(x) := \int_{\mathbb{R}^d} dx X(x) f(x).$$

In this case, we say that X is **P_f -integrable**.

General case : If Ω is a set of realisation, and P is a probability measure on Ω , we will denote $E_P(X)$ the **expectation of X under P**.

Remarque

Note that the expected value of a random variable depends only on its law

Example

Look at a fair 6-face dice roll : $\Omega = \{F1, F2, \dots, F6\}$, $p(\omega) = \frac{1}{6}$ for every $\omega \in \Omega$.

Take the random variable $X(F1) = X(F3) = X(F5) = -1$, $X(F2) = X(F4) = X(F6) = 2$, then

$$\begin{aligned} E_p(X) &= p(F1)X(F1) + p(F2)X(F2) + p(F3)X(F3) + p(F4)X(F4) \\ &\quad + p(F5)X(F5) + p(F6)X(F6) = -\frac{1}{6} + \frac{1}{6} \cdot 2 - \frac{1}{6} + \frac{1}{6} \cdot 2 - \frac{1}{6} + \frac{1}{6} \cdot 2 = \frac{1}{2}. \end{aligned}$$

The properties of expectation are summarized in the next Theorem.

Théorème 5

Let Ω be a realisation set, and P a probability measure on Ω . Let $X, Y : \Omega \rightarrow \mathbb{R}$ be two random variables. Then,

1. *linearity* : for any $a, b \in \mathbb{R}$, $E_P(aX + bY) = aE_P(X) + bE_P(Y)$;
2. *ordering* : if $P(X \geq Y) = 1$, $E_P(X) \geq E_P(Y)$. In particular,
 - if $P(X \geq 0) = 1$, then $E_P(X) \geq 0$;
 - if $P(a \leq X \leq b) = 1$, then $a \leq E_P(X) \leq b$;
 - $|E_P(X)| \leq E_P(|X|)$.

1.2.3 Transfer Theorem

The transfer theorem confirm us that the intuition that if we have a random variable that takes values x_1, x_2, x_3 , with probabilities p_1, p_2, p_3 , and we have a function $g : \mathbb{R} \rightarrow \mathbb{R}$, then expectation of $g(X)$ should be

$$E(g(X)) = p_1 g(x_1) + p_2 g(x_2) + p_3 g(x_3).$$

Théorème 6

Let X be a random variable. Then,

1. if X is a discrete random variable, for any $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(X)$ is a discrete random variable, and

$$E(g(X)) = \sum_{x \in \text{Image}(X)} g(x) P(X = x)$$

as soon as the sum converges absolutely;

2. if X is a continuous random variable, for any $g : \mathbb{R} \rightarrow \mathbb{R}$

$$E(g(X)) = \int_{-\infty}^{+\infty} f_X(x) g(x) dx$$

as soon as the integral converges absolutely.

1.2.4 Random vectors

Définition 13

Let Ω be a realisation set, and P a probability measure on Ω . Let $d \geq 1$. A **random vector of dimension d** is function $X : \Omega \rightarrow \mathbb{R}^d$. We will denote

$$X(\omega) = (X_1(\omega), \dots, X_d(\omega)).$$

The functions $X_i : \Omega \rightarrow \mathbb{R}$ are random variables. They are called the **marginals** of X .

The **cumulative distribution function** (CDF) of a random vector $X : \Omega \rightarrow \mathbb{R}^d$ is given by $F_X : \mathbb{R}^d \rightarrow [0, 1]$,

$$F_X(t_1, \dots, t_d) = P(X_1 \leq t_1, \dots, X_d \leq t_d)$$

Remarque

Random vectors are just a list of random variables.

Pas sur de l'utilité de celle là

Définition 14

Let Ω be a realisation set, and P a probability measure on Ω . A **complex random variable** is a function $X : \Omega \rightarrow \mathbb{C}$. The real and imaginary parts of X are then random variables.

There is then also discrete and continuous random vectors

Définition 15

Let Ω be a realisation set, and P a probability measure on Ω . Let $X : \Omega \rightarrow \mathbb{R}^d$ be a random vector. We say that

- X is a **discrete random vector** if there is a finite or countable set $\mathcal{D}_X \subset \mathbb{R}^d$ with $P(X \in \mathcal{D}_X) = 1$;
- X is a **continuous random vector** if there is a density function $f_X : \mathbb{R}^d \rightarrow [0, +\infty)$ such

that

$$P(X \in A) = \int_{\mathbb{R}^d} \mathbb{1}_A(x) dx$$

Example

Consider (X, Y) a uniform random vector in the unit disc :

$$f_{(X,Y)}(x, y) = \frac{1}{\pi} \mathbb{1}_{[0,1]}(x^2 + y^2).$$

The first marginal, X , of this random vector is then a continuous random variable with density given by $f_X(x) = 0$ for $|x| > 1$, and, for $|x| \leq 1$,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{(X,Y)}(x, y) dy = \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \mathbb{1}_{[0,1-x^2]}(y^2) dy = \frac{1}{\pi} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy = \frac{2}{\pi} \sqrt{1-x^2}. \end{aligned}$$

Hors sujet important

Let a random vector have two variables, then his density function will be $f_{XY}(x, y)$. So the density of the random variable X_1 will be $f_X(x) = \int_{\mathbb{R}} f_{XY}(x, y) dy$ and the expected value of X_1 will be $E[X_1] = \int_{\mathbb{R}} x f_X(x) dx = \int_{\mathbb{R}} \int_{\mathbb{R}} x f_{XY}(x, y) dx dy$.

1.2.5 Change of variable formula

Let $d \geq 1, U \subset \mathbb{R}^d$ an open set, and $\phi : U \rightarrow \mathbb{R}^d, \phi(y) = (\phi_1(y), \dots, \phi_d(y)), y = (y_1, \dots, y_d)$. Then,

- we say that ϕ is **continuously differential** on U if the partial derivative $\frac{\partial \phi_i}{\partial y_j}$ exists and are continuous on U ;
- we denote $D_\phi(y)$ the **Jacobian matrix** of ϕ ;
- we denote \det the determinant.

Théorème 7

Let $d \geq 1, U \subset \mathbb{R}^d$ an open set, $V \subset \mathbb{R}^d$, and $\phi : U \rightarrow V$ a continuously differentiable bijection with $\det D_\phi(y) \neq 0$ for all $y \in U$. Then, for any function $f : V \rightarrow \mathbb{R}$, we have :

$$\int_U dy_1 \dots dy_d f(\phi(y)) |\det D_\phi(y)| = \int_V dx_1 \dots dx_d f(x).$$

From this, we can deduce :

Théorème 8

Let Ω be a realisation set, P be a probability measure on Ω , and $X : \Omega \rightarrow \mathbb{R}^d$ a random vector with a density f_X . Let $U, V, \phi : U \rightarrow V$, be as in **the theorem above**. Suppose that $P(X \in U) = 1$. We then have that $Y = \phi \odot X : \Omega \rightarrow \mathbb{R}^d (Y = \phi(X))$ is a continuous random vector with density

$$f_Y(y) = f_X(\phi^{-1}(y)) |\det D_{\phi^{-1}}(y)| = \frac{1}{|\det D_\phi(\phi^{-1}(y))|} f_X(\phi^{-1}(y)).$$

In the case $d = 1$, this formula simplifies to

$$f_Y(y) = f_X(\phi^{-1}(y)) |(\phi^{-1})'(y)| = \frac{1}{|\phi'(\phi^{-1}(y))|} f_X(\phi^{-1}(y)).$$

Example

We take U a uniform random variable on $[0, 1]$: U is a continuous random variable with density $f_U(x) = \mathbb{1}_{[0,1]}(x)$. Then for $a \in \mathbb{R}$ and $r > 0$, define $X = a + rU$. Using theorem 8 with $\phi(x) = a + rx$, $\phi^{-1}(x) = \frac{x-a}{r}$, we get that X is a continuous random variable with density

$$f_X(x) = f_U(\phi^{-1}(x)) \frac{1}{\phi'(\phi^{-1}(x))} = \mathbb{1}_{[0,1]} \left(\frac{x-a}{r} \right) \frac{1}{r} = \frac{1}{r} \mathbb{1}_{[a, a+r]}(x).$$

So, X is a uniform random variable on $[a, a+r]$.

Remarque

With polar coordinates, we then have

$$\phi^{-1}(x, y) = \left(\text{atan2}(y, x), \sqrt{x^2 + y^2} \right)$$

where

$$\text{atan2}(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0, \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0, y \geq 0, \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0, y < 0, \\ \frac{\pi}{2} & \text{if } x = 0, y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0, y < 0, \\ \text{undefined} & \text{if } x = y = 0. \end{cases}$$

In this case

$$|\det D_\phi(\theta, r)| = r,$$

which leads to the formula

$$dxdy = rd\theta dr$$

1.2.6 Moments, and Moment Generating Function

We already saw the cumulative distribution function (CDF), there is an other useful object some-time : the **moment generating function**

Définition 16

Let X be a random variable. Let $p > 0$. We say that X **admits a moment of order p** if

$$E(|X|^p) < \infty$$

When X admits a moment of order p , we define

- the *p th moment* of X : $E(X^p)$;
- the *p th absolute moment* of X : $E(|X|^p)$.

Définition 17

Let X be a random variable. we say that X **admits exponential moments of order $\delta > 0$** if

$$E\left(e^{\delta|X|}\right) < \infty$$

When X admits exponential moments, we define the **moment generating function of X** by

$$M_X(t) = E\left(e^{tx}\right), \quad t \in (-\delta, \delta).$$

Théorème 9

Let X, Y be two random variables. Suppose that there is $\delta > 0$ such that

$$E\left(e^{\delta|X|}\right) < \infty, \quad E\left(e^{\delta|Y|}\right) < \infty.$$

Then, we have the following properties.

- X admits moments of any integer order.
- M_X is analytic in a neighbourhood of 0, and for any $n \in \mathbb{N}$,
- M_X, M_Y characterise X, Y :

$$M_X(t) = M_Y(t) \text{ for all } t \in (-\delta, \delta) \implies X = Y.$$

1.3 Conditional probability independence

1.3.1 Conditional probability

Définition 18

Let Ω be a set of realisations, and P a probability measure on Ω . Let $A \subset \Omega$ be an event such that $P(A) > 0$. Define then the **probability measure P conditioned on A** , denoted $P(\cdot|A)$, by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \forall \text{ event } B.$$

We can then define the **conditions expectation** of a random variable $X : \Omega \rightarrow \mathbb{R}$ by

$$E_P(X|A) = E_{P_A}(X),$$

where P_A stands for $P_A = P(\cdot|A)$.

1.3.2 Independence

Définition 19

- Two events A, B are said to be **independent** if

$$P(A \cap B) = P(A)P(B).$$

- A family of events $(A_i)_{i \in I}$ is said to be **two-by-two independent** if for any $i \neq j$, A_i and A_j are independent.
- A family of events $(A_i)_{i \in I}$ is said to be **an independent family** if for any $J \subset I$ finite,

$$P\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} P(A_i).$$

Définition 20

- Two random variables X, Y are said to be **independent** if for any events $A, B \subset \mathbb{R}$,

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B).$$

Equivalently, X, Y are independent if for any $f, g : \mathbb{R} \rightarrow \mathbb{R}$,

$$E(f(X)g(Y)) = E(f(X))E(g(Y)).$$

- A family of random variables $(X_i)_{i \in I}$ is said to be **two-by-two independent** if for any i, X_i and X_j are independent.

- A family of random variables $(X_i)_{i \in I}$ is said to be **an independent family** if for any $J \subset I$ finite, and any events $A_i \subset \mathbb{R}, i \in J$,

$$P(\cap_{i \in J} \{X_i \in A_i\}) = \prod_{i \in J} P(X_i \in A_i).$$

Equivalently, $(X_i)_{i \in I}$ is an independent family if for any $J \subset I$ finite, and any functions $f_i : \mathbb{R} \rightarrow \mathbb{R}, i \in J$,

$$E\left(\prod_{i \in J} f_i(X_i)\right) = \prod_{i \in J} E(f_i(X_i)).$$

The same definition holds with “random vectors” replacing “random variables”.

Définition 21

A family $(X_i)_{i \in I}$ of random variables is called an **independent identically distributed** family, abbreviated **i.i.d. family**, if the family $(X_i)_{i \in I}$ is an independent family, and for any $i, j \in I, X_i = X_j$.

Théorème 10

Let $d, d' \geq 1$. Let $X : \Omega \rightarrow \mathbb{R}^d, Y : \Omega \rightarrow \mathbb{R}^{d'}$ be a random vector.

- If X, Y are **continuous random vector** : X and Y are independent if and only if the random vector $(X, Y) : \Omega \rightarrow \mathbb{R}^{d+d'}$ has density

$$f_{(X,Y)}(x, y) = f_X(x) f_Y(y),$$

where $f_X : \mathbb{R}^d \rightarrow \mathbb{R}$ is a density for X , and $f_Y : \mathbb{R}^{d'} \rightarrow \mathbb{R}$ is a density for Y .

- If X, Y are **discrete random vectors** : X and Y are independent if and only if for any $x \in \mathbb{R}^d, y \in \mathbb{R}^{d'}$,

$$P(X = x, Y = y) = P(X = x) P(Y = y).$$

- If X is **discrete** and Y is **continuous** : X and Y are independent if and only if for any $x \in \mathbb{R}^d, A \subset \mathbb{R}^{d'}$,

$$P(X = x, Y \in A) = P(X = x) \int_A f_Y(y) dy$$

where $f_Y : \mathbb{R}^{d'} \rightarrow \mathbb{R}$ is a density for Y .

1.3.3 Bayes law, formula of total probability

Bayes Law :

Théorème 11

Let Ω be a set of realisations, and let P be a probability measure on Ω . Let $A, B \subset \Omega$ be two events such that $P(A), P(B) > 0$. Then

$$P(A|B) = \frac{P(B|A) P(A)}{P(B)}.$$

Théorème 12

Let Ω be a set of realisations, and let P be a probability measure on Ω . Let I be a finite or countable set. Let $A_i, i \in I$ be a collection of events such that

- if $i \neq j$, then $A_i \cap A_j = \emptyset$;
- $\cup_{i \in I} A_i = \Omega$.

Suppose moreover that $P(A_i) > 0$ for all $i \in I$. Then for any event B ,

$$P(B) = \sum_{i \in I} P(B \cap A_i) = \sum_{i \in I} P(B|A_i) P(A_i).$$

In the same fashion, for every random variable X

$$E(X) = \sum_{i \in I} E(X|A_i) P(A_i).$$

Example

We throw some dices :

- A_1 : we throw an even dice
- A_2 : we throw an odd dice
- B : the result is ≤ 4

Then

$$P(B) = P(B|A_1) P(A_1) + P(B|A_2) P(A_2).$$