

# Proba Stats

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Lundi 15 septembre 2025

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# Chapitre 1

## Probabilities : starter guide

### 1.1 Probabilities spaces and measures

#### 1.1.1 Probability spaces

##### Définition 1

A **probability space** is a set of realisations denoted  $\Omega$ , together with a probability measure on  $\Omega$ . A **probability measure** on  $\Omega$  is a function  $P : \mathcal{P}(\Omega) \rightarrow [0, 1]$  such that

1.  $P(\emptyset) = 0, P(\Omega) = 1$ .
2. If  $A_i \in \mathcal{F}, i \in \mathbb{N}$  is a sequence of events with  $A_i \cap A_j = \emptyset$  for  $i \neq j$ , then

$$P\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} P(A_i).$$

*Remarque*

- (1) The probability that something happens is 1 and that nothing happens is 0.
  - (2) The probability of events that cannot occur simultaneously is the sum of the probabilities of the events.
- From these properties, we can deduce the next theorems.

##### Théorème 1

Let  $P$  be a probability measure on some realisation set  $\Omega$ . Then,

1.  $P(\emptyset) = 0, P(\Omega) = 1$ ;
2. for any event  $A, P(\Omega \setminus A) = 1 - P(A)$ ;
3. if two events  $A, B$  are such that  $A \subset B, P(B) = P(A) + P(B \setminus A)$ . In particular,  $P(A) \leq P(B)$ ;
4. for two events  $A, B$ ,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B);$$

5. finite  $\sigma$ -additivity : if  $n \geq 2$ , and  $A_1, \dots, A_n$  are events such that  $A_i \cap A_j = \emptyset$  for  $i \neq j$ , then

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i);$$

6. countable  $\sigma$ -additivity : if  $A_1, A_2, \dots$  are events such that  $A_i \cap A_j = \emptyset$  for  $i \neq j$ , then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i);$$

7. *finite  $\sigma$ -sub-additivity* : if  $n \geq 2$ , and  $A_1, \dots, A_n$  are events, then

$$P\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n P(A_i);$$

8. *countable  $\sigma$ -sub-additivity* : if  $A_1, A_2, \dots$  are events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i);$$

9. *monotone convergence, increasing sequences* : if  $A_1, A_2, \dots$  are events such that  $A_i \subset A_{i+1}$  for all  $i$ 's then

$$\lim_{n \rightarrow \infty} P(A_n) = P\left(\bigcup_{n=1}^{\infty} A_n\right);$$

10. *monotone convergence, decreasing sequences* : if  $A_1, A_2, \dots$  are events such that  $A_{i+1} \subset A_i$  for all  $i$ 's, then

$$\lim_{n \rightarrow \infty} P(A_n) = P\left(\bigcap_{n=1}^{\infty} A_n\right).$$

*Remarque*

It's not necessary to learn this list by heart

We will encounter two main types of probability spaces in these notes :

- **Discrete probability spaces** : in that case  $\Omega$  is a finite or a countable set, and the set of events really is  $\mathcal{F} = \mathcal{P}(\Omega)$ .
- **Continuous probability spaces** : in that case,  $\Omega = \mathbb{R}^d$  with  $d \geq 1$  integer. We won't go into a formal definition of the *set of Borel sets*, and we will do as if we could take  $\mathcal{F} = \mathcal{P}()$

### 1.1.2 Discrete probability measures

#### Définition 2

Let  $\Omega$  be a finite countable set. A **probability mass function** on  $\Omega$  is a function  $p : \Omega \rightarrow [0, 1]$  such that

$$\sum_{\omega \in \Omega} p(\omega) = 1$$

The **probability measure** associated to a probability mass function  $p$  is the function  $P_p : \mathcal{P}(\Omega) \rightarrow [0, 1]$  given by

$$P_p(A) = \sum_{\omega \in A} p(\omega).$$

### 1.1.3 Continuous probability measures

One cannot make sense of the probability that a drop of water falls at *precisely* one point  $x$ , but it is relatively easy to make sense of the probability that the drop falls *in a small disk* around  $x$ . This is the essence of the next definition.

#### Définition 3

Let  $d \geq 1$ . A **probability density function** on  $\mathbb{R}^d$  is a Riemann integrable function  $f : \mathbb{R}^d \rightarrow [0, +\infty)$  such that

$$\int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{+\infty} dx_d f(x_1, \dots, x_d) = 1.$$

The **probability measure** associated to a probability density function  $f$  is the  $[0, 1]$ -valued function  $P_f$  given by

$$P_f(A) = \int_{-\infty}^{+\infty} dx_1 \dots \int_{-\infty}^{+\infty} dx_d f(x_1, \dots, x_d) \mathbb{1}_A(x_1, \dots, x_d).$$

*Remarque*

This is the equivalent of **the second definition** but for a space where you can't delimit the element to sum : a continuous space.

The **density function** is a function that shows where the variable like to be at most. The probability is the area under this curve.

$$\mathbb{1}_A = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

### 1.1.4 Genreal probability spaces and measures

#### Définition 4

Let  $\Omega$  be a set. A **sigma-algebra** on  $\Omega$  is a set  $\mathcal{F} \subset \mathcal{P}(\Omega)$  which satisfies

1.  $\mathcal{F}$  contains the empty set ( $\emptyset \in \mathcal{F}$ ).
2.  $\mathcal{F}$  is stable by taking the complement ( $A \in \mathcal{F} \implies \Omega \setminus A \in \mathcal{F}$ ).
3.  $\mathcal{F}$  is stable by countable unions (if for all  $i \in \mathbb{N}$ ,  $A_i \in \mathcal{F}$ , then  $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$ ).

*Remarque*

$\mathcal{P}(\Omega)$  is the set of all subset of  $\Omega$ .

Now we can deduce the following properties.

#### Théorème 2

Let  $\Omega$  be a set and  $\mathcal{F}$  a sigma-algebra on  $\Omega$ . Then all of the following hold.

1.  $\Omega \in \mathcal{F}$ .
2.  $\mathcal{F}$  is stable by finite intersections : if  $A, B \in \mathcal{F}$ , then  $A \cap B \in \mathcal{F}$ .
3. If  $A, B \in \mathcal{F}$ , then  $A \setminus B \in \mathcal{F}$ .
4.  $\mathcal{F}$  stable by countable intersections : if  $A_1, A_2, \dots \in \mathcal{F}$ , then  $\bigcap_{i \geq 1} A_i \in \mathcal{F}$ .
5.  $\mathcal{F}$  is stable by increasing limits : if  $A_i \in \mathcal{F}$ ,  $i \geq 1$  is such that  $A_i \subset A_{i+1}$ , then,  $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ .
6.  $\mathcal{F}$  is stable by decreasing limits : if  $A_i \in \mathcal{F}$ ,  $i \geq 1$  is such that  $A_{i+1} \subset A_i$ , then,  $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$ .

#### Définition 5

Let  $\Omega$  be a set and  $\mathcal{F}$  a sigma-algebra on  $\Omega$ . A **probability measure** on  $(\Omega, \mathcal{F})$  is a function  $P : \mathcal{F} \rightarrow [0, 1]$  such that

1.  $P(\Omega) = 1$ .
2. If  $A_i \in \mathcal{F}$ ,  $i \in \mathbb{N}$  is a sequence of events with  $A_i \cap A_j = \emptyset$  for  $i \neq j$ , then

$$P\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} P(A_i).$$

*Remarque*

The same as before but more precise.

### Définition 6

A **probability space** is a triplet  $\Omega, \mathcal{F}, P$  where  $\Omega$  is a set (the set of realisations),  $\mathcal{F}$  is a sigma-algebra on  $\Omega$  (the set of events), and  $P$  is a probability measure on  $(\Omega, \mathcal{F})$ .

Remarque

This is the most important definition of this “introduction”

### 1.1.5 Inclusion-Exclusion

It is a generalisation of the following fact that we encounter when counting objects : to count the number of objects with property A or property B, we can count the number of objects with property A add the number of objects with property B, and correct our over-counting by removing from this the number of objects with both property A and property B (which were counted twice).

### Théorème 3

Let  $P$  be a probability measure on some realisation set  $\Omega$ . Let  $n \geq 1$  and  $A_1, \dots, A_n$  be events. Then,

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{k=1}^n (-1)^{k+1} \sum_{i_1 < \dots < i_k \leq n} P\left(\bigcap_{j=1}^k A_{i_j}\right).$$

Moreover, for  $1 \leq l \leq \frac{n}{2}$  integer  $a$ ,

$$P\left(\bigcup_{i=1}^n A_i\right) = \begin{cases} \leq \sum_{k=1}^{2l-1} (-1)^{k+1} \sum_{1 \leq i_1 < \dots < i_k \leq n} P\left(\bigcap_{j=1}^k A_{i_j}\right) \\ \geq \sum_{k=1}^{2l} (-1)^{k+1} \sum_{1 \leq i_1 < \dots < i_k \leq n} P\left(\bigcap_{j=1}^k A_{i_j}\right) \end{cases}.$$

## 1.2 Random variables and expectation

### 1.2.1 Random variables

Random variables are therefore functions going from the set of realisations to the real numbers ; for example, if the “experiment ” is looking at all people born in 2000, one could make the measurement of the height of the first individual born that year.

### Définition 7

A (real) **random variable** is a function from the realisation space  $\Omega$  to  $\mathbb{R}$ . The probability that a random variable falls in a set  $A$  is

$$P(X \in A) := P(X^{-1}(A))$$

In words : it is the probability that the realisation of the experiment is such that the measurement  $X$  takes a value  $A$ .

Example

We are throwing a dice :

- $\Omega = \{1, 2, 3, 4, 5, 6\}$
- $X(\omega) = 1$  if it's even,  $X(\omega) = 0$  if it's odd.
- So we are searching  $\omega$  in  $\Omega$  that gives  $X(\omega) = 1$  :

$$X^{-1}(A) = \{\omega \in \Omega : X(\omega) \in A\} = \{2, 4, 6\}$$

Then we apply the probability on these results

$$P(X \in A) = P(\{2, 4, 6\}) = \frac{3}{6} = 0.5$$

We will frequently use notations similar to the following :

$$\begin{aligned} P(X = x) &\equiv P(X \in \{x\}), \\ P(X \leq x) &\equiv P(X \in (-\infty, x]), \\ P(X > x) &\equiv P(X \in (x, +\infty)). \end{aligned}$$

### Définition 8

Let  $\Omega$  be a set of realisation, and let  $P$  be a probability measure on  $\Omega$ . Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable.  $X$  is **discrete** if there is  $\mathcal{D}_X \subset \Omega$  countable or finite such that  $P(X \in \mathcal{D}_X) = 1$ . The **law of  $X$**  is then the probability measure on  $\mathbb{R}$  given by

$$P_X(A) = \sum_{x \in A \cap \mathcal{D}_X} P(X = x)$$

In words, a **discrete random variable** is a variable that can take only finitely or countably many values with non-zero probability.

The second very important family of variables are **continuous random variables**.

### Définition 9

Let  $\Omega$  be a set of realisation, and let  $P$  be a probability measure on  $\Omega$ . Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable.  $X$  is **continuous random variable** if there is a density function  $f_X : \mathbb{R} \rightarrow [0, +\infty)$  such that

$$P(X \in A) = \int_{\mathbb{R}} \mathbb{1}_A f_X(x) dx.$$

The **law of  $X$**  is then the probability measure on  $\mathbb{R}$  given by  $P_{f_X}$ .

Remarque

Here we can't say : "the probability that  $X = 2$ ", because it will always be 0. Instead we use a density  $f_X(x)$  to compute on an interval. For example, if  $X$  is measuring the size of somebody,  $P(170 \leq X \leq 180)$  is compute with the density  $f_X$  by an integral, because the probability to have 170.000000cm is 0.

- Discrete : whe can say  $P(X = x)$ .
- Continue : values are infinite, we look at interval not precise points.

There is a similat notion of law for general random variables. Random variables allows us sometime to pass from some continuous probability to some discrete random variables.



### Définition 10

Let  $\Omega$  be a set of realisation, and let  $P$  be a probability measure on  $\Omega$ . Let  $X : \Omega \rightarrow \mathbb{R}$ . The **law of  $X$**  is the probability measure  $P_X$  on  $\mathbb{R}$  given by

$$P_X(A) = P(X \in A).$$

### Définition 11

Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable. The **cumulative distribution function of  $X$**  is defined by

$$F_X(t) = P(X \leq t).$$

Two random variables have the same law if and only if they have the same cumulative distribution functions. Note that if  $X$  is a continuous random variable with density  $f_X$ , one has that  $F_X$  is a primitive of  $f_X$  :

$$F'_X(t) = f_X(t) \quad \forall t \in \mathbb{R}.$$

*Remarque*

$$F_X : \mathbb{R} \rightarrow [0, 1]$$

Example 1 : If  $P(X = 0) = P(X = 1) = \frac{1}{2}$

$$F_X(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2} & 0 \leq t < 1 \\ 1 & t \geq 1 \end{cases}$$

### Théorème 4

Let  $x \rightarrow \int_a^x f(t) dt$  is primitive of  $f$

If  $X$  is a continuous random variable : there exist a density function

$$f_X : \mathbb{R} \rightarrow \mathbb{R}, \quad P(X \in A) = \int_A f_X(x) dx$$

So  $F_X$  is differentiable and

$$F'_X(t) = f_X(t) = \int_{-\infty}^t f_X(x) dx$$

*Example*

If  $X$  is a continuous random variable with density  $f_X(x) = \mathbb{1}_{[0,4]}(x)$  then

$$P(X \leq t) = \int_{-\infty}^t f_X(x) dx = \frac{1}{4} \int_{-\infty}^t \mathbb{1}_{[0,4]}(x) dx = \begin{cases} 0 & \text{if } t < 0, \\ \frac{t}{4} & \text{if } t \in [0, 4], \\ 1 & \text{if } t > 4. \end{cases}$$

*Remarque*

to

*Remarque  
importante*

To resume an important point :

- for discrete random variables we use a **probability masse function**.
- for continue random variables we use a **density function**.

## 1.2.2 Expectation

### Définition 12

**Discrete case :** If  $\Omega$  is a finite or countable set,  $p : \Omega \rightarrow [0, 1]$  is a probability mass function, and  $X : \Omega \rightarrow \mathbb{R}$  is a random variable such that

$$\sum_{\omega \in \Omega} |X(\omega)| p(\omega) < \infty,$$

the **expectation of X under p** is

$$E_p(X) := \sum_{\omega \in \Omega} X(\omega) p(\omega).$$

In this case, we say that X is  **$P_p$ -integrable**.

**Continuous case :** If  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is a probability density function, and  $X : \mathbb{R}^d \rightarrow \mathbb{R}$  is a random variable such that

$$\int_{\mathbb{R}^d} dx |X(x)| f(x) < \infty,$$

the **expectation of X under  $P_f$**  is

$$E_f(x) := \int_{\mathbb{R}^d} dx X(x) f(x).$$

In this case, we say that X is  **$P_f$ -integrable**.

**General case :** If  $\Omega$  is a set of realisation, and P is a probability measure on  $\Omega$ , we will denote  $E_p(X)$  the **expectation of X under P**.

*Remarque*

Note that the expectede value of a random variable depends only on its law

*Example*

Look at a fair 6-face dice roll :  $\Omega = \{F1, F2, \dots, F6\}$ ,  $p(\omega) = \frac{1}{6}$  for every  $\omega \in \Omega$ .

Take the random variable  $X(F1) = X(F3) = X(F5) = -1$ ,  $X(F2) = X(F4) = X(F6) = 2$ , then

$$\begin{aligned} E_p(X) &= p(F1) X(F1) + p(F2) X(F2) + p(F3) X(F3) + p(F4) X(F4) \\ &\quad + p(F5) X(F5) + p(F6) X(F6) = -\frac{1}{6} + \frac{1}{6} \cdot 2 - \frac{1}{6} + \frac{1}{6} \cdot 2 - \frac{1}{6} + \frac{1}{6} \cdot 2 = \frac{1}{2}. \end{aligned}$$

The properties of expectation are summarized in the next Theorem.

### Théorème 5

Let  $\Omega$  be a realisation set, and P a probabilisty measure on  $\Omega$ . Let  $X, Y : \Omega \rightarrow \mathbb{R}$  be two random variables. Then,

1. *linearity* : for any  $a, b \in \mathbb{R}$ ,  $E_P(aX + bY) = aE_P(X) + bE_P(Y)$  ;
2. *ordering* : if  $P(X \geq Y) = 1$ ,  $E_P(X) \geq E_P(Y)$ . In particular,
  - if  $P(x \geq 0) = 1$ , then  $E_P(X) \geq 0$  ;

- if  $P(a \leq X \leq b) = 1$ , then  $a \leq E_P(X) \leq b$ ;
- $|E_P(X)| \leq E_P(|X|)$ .

### 1.2.3 Transfer Theorem

The transfer theorem confirm us that the intuition that if we have a random variable that takes values  $x_1, x_2, x_3$ , with probabilities  $p_1, p_2, p_3$ , and we have a function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , then expectation of  $g(X)$  should be

$$E(g(X)) = p_1 g(x_1) + p_2 g(x_2) + p_3 g(x_3).$$

#### Théorème 6

Let  $X$  be a random variable. Then,

1. if  $X$  is a discrete random variable, for any  $g : \mathbb{R} \rightarrow \mathbb{R}$ ,  $g(X)$  is a discrete random variable, and

$$E(g(X)) = \sum_{x \in \text{Image}(X)} g(x) P(X = x)$$

as soon as the sum converges absolutely;

2. if  $X$  is a continuous random variable, for any  $g : \mathbb{R} \rightarrow \mathbb{R}$

$$E(g(X)) = \int_{-\infty}^{+\infty} f_X(x) g(x) dx$$

as soon as the integral converges absolutely.

### 1.2.4 Random vectors

#### Définition 13

Let  $\Omega$  be a realisation set, and  $P$  a probability measure on  $\Omega$ . Let  $d \geq 1$ . A **random vector of dimension  $d$**  is function  $X : \Omega \rightarrow \mathbb{R}^d$ . We will denote

$$X(\omega) = (X_1(\omega), \dots, X_d(\omega)).$$

The functions  $X_i : \Omega \rightarrow \mathbb{R}$  are random variables. They are called the **marginals** of  $X$ .

The **cumulative distribution function** (CDF) of a random vector  $X : \Omega \rightarrow \mathbb{R}^d$  is given by  $F_X : \mathbb{R}^d \rightarrow [0, 1]$ ,

$$F_X(t_1, \dots, t_d) = P(X_1 \leq t_1, \dots, X_d \leq t_d)$$

*Remarque*

Random vectors are just a list of random variables.

*Pas sur de l'utilité de celle la*

#### Définition 14

Let  $\Omega$  be a realisation set, and  $P$  a probability measure on  $\Omega$ . A **complex random variable** is a function  $X : \Omega \rightarrow \mathbb{C}$ . The real and imaginary parts of  $X$  are then random variables.

There is then also discrete and continuous random vectors

#### Définition 15

Let  $\Omega$  be a realisation set, and  $P$  a probability measure on  $\Omega$ . Let  $X : \Omega \rightarrow \mathbb{R}^d$  be a random vector. We say that

- $X$  is a **discrete random vector** if there is a finite of countable set  $\mathcal{D}_X \subset \mathbb{R}^d$  with

$$P(X \in \mathcal{D}_X) = 1;$$

- X is a **continuous random vector** if there is a density function  $f_X : \mathbb{R}^d \rightarrow [0, +\infty)$  such that

$$P(X \in A) = \int_{\mathbb{R}^d} \mathbb{1}_A(x) dx$$

*Example*

Consider  $(X, Y)$  a uniform random vector in the unit disc :

$$f_{(X,Y)}(x, y) = \frac{1}{\pi} \mathbb{1}_{[0,1]}(x^2 + y^2).$$

The first marginal, X, of this random vector is then a continuous random variable with density given by  $f_X(x) = 0$  for  $|x| > 1$ , and, for  $|x| \leq 1$ ,

$$f_X(x) = \int_{-\infty}^{\infty} f_{(X,Y)}(x, y) dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathbb{1}_{[0,1-x^2]}(y^2) dy = \frac{1}{\pi} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy = \frac{2}{\pi} \sqrt{1-x^2}.$$

**Hors sujet important**

Let a random vector have two variables, then his density function will be  $f_{XY}(x, y)$ . So the density of the random variable  $X_1$  will be  $f_X(x) = \int_{\mathbb{R}} f_{XY}(x, y) dy$  and the expected value of  $X_1$  will be  $E[X_1] = \int_{\mathbb{R}} x f_X(x) dx = \int_{\mathbb{R}} \int_{\mathbb{R}} x f_{XY}(x, y) dx dy$ .

### 1.2.5 Change of variable formula

Let  $d \geq 1, U \subset \mathbb{R}^d$  an open set, and  $\phi : U \rightarrow \mathbb{R}^d, \phi(y) = (\phi_1(y), \dots, \phi_d(y)), y = (y_1, \dots, y_d)$ . Then,

- we say that  $\phi$  is **continuously differential** on U if the partial derivative  $\frac{\delta \phi_i}{\delta y_j}$  exists and are continuous on U;
- we denote  $D_\phi(y)$  the **Jacobian matrix** of  $\phi$ ;
- we denot det the determinant.

#### Théorème 7

Let  $d \geq 1, U \subset \mathbb{R}^d$  an open set,  $V \subset \mathbb{R}^d$ , and  $\phi : U \rightarrow V$  a continuously differentiable bijection with  $\det D_\phi(y) \neq 0$  for all  $y \in U$ . Then, for any function  $f : V \rightarrow \mathbb{R}$ , we have :

$$\int_U dy_1 \dots dy_d f(\phi(y)) |\det D_\phi(y)| = \int_V dx_1 \dots dx_d f(x).$$

From this, we can deduce :

#### Théorème 8

Let  $\Omega$  be a realisation set,  $P$  be a probability measure on  $\Omega$ , and  $X : \Omega \rightarrow \mathbb{R}^d$  a random vector with a density  $f_X$ . Let  $U, V, \phi : U \rightarrow V$ , be as in **the theorem above**. Suppose that  $P(X \in U) = 1$ . We then have that  $Y = \phi \odot X : \Omega \rightarrow \mathbb{R}^d (Y = \phi(X))$  is a continuous random vector with density

$$f_Y(y) = f_X(\phi^{-1}(y)) |\det D_{\phi^{-1}}(y)| = \frac{1}{|\det D_\phi(\phi^{-1}(y))|} f_X(\phi^{-1}(y)).$$

In the case  $d = 1$ , this formula simplifies to

$$f_Y(y) = f_X(\phi^{-1}(y)) \left| (\phi^{-1})'(y) \right| = \frac{1}{|\phi'(\phi^{-1}(y))|} f_X(\phi^{-1}(y)).$$

*Example*

We take  $U$  a uniform random variable on  $[0, 1]$  :  $U$  is a continuous random variable with density  $f_U(x) = \mathbb{1}_{[0,1]}(x)$ . Then for  $a \in \mathbb{R}$  and  $r > 0$ , define  $X = a + rU$ . Using theorem 8 with  $\phi(x) = a + rx$ ,  $\phi^{-1}(x) = \frac{x-a}{r}$ , we get that  $X$  is a continuous random variable with density

$$f_X(x) = f_U(\phi^{-1}(x)) \frac{1}{\phi'(\phi^{-1}(x))} = \mathbb{1}_{[0,1]} \left( \frac{x-a}{r} \right) \frac{1}{r} = \frac{1}{r} \mathbb{1}_{[a, a+r]}(x).$$

So,  $X$  is a uniform random variable on  $[a, a+r]$ .

*Remarque*

With polar coordinates, we then have

$$\phi^{-1}(x, y) = \left( \text{atan2}(y, x), \sqrt{x^2 + y^2} \right)$$

where

$$\text{atan2}(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0, \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0, y \geq 0, \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0, y < 0, \\ \frac{\pi}{2} & \text{if } x = 0, y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0, y < 0, \\ \text{undefined} & \text{if } x = y = 0. \end{cases}$$

In this case

$$|\det D_\phi(\theta, r)| = r,$$

which leads to the formula

$$dxdy = rd\theta dr$$

## 1.2.6 Moments, and Moment Generating Function

We already saw the cumulative distribution function (CDF), there is an other useful object sometime : the **moment generating function**

### Définition 16

Let  $X$  be a random variable. Let  $p > 0$ . We say that  $X$  **admits a moment of order  $p$**  if

$$E(|X|^p) < \infty$$

When  $X$  admits a moment of order  $p$ , we define

- the  $p$ th moment of  $X$  :  $E(X^p)$ ;
- the  $p$ th absolute moment of  $X$  :  $E(|X|^p)$ .

### Définition 17

Let  $X$  be a random variable. we say that  $X$  **admits exponential moments of order  $\delta > 0$**  if

$$E(e^{\delta|X|}) < \infty$$

When  $X$  admits exponential moments, we define the **moment generating function of  $X$**  by

$$M_X(t) = E(e^{tx}), \quad t \in (-\delta, \delta).$$

### Théorème 9

Let  $X, Y$  be two random variables. Suppose that there is  $\delta > 0$  such that

$$E(e^{\delta|X|}) < \infty, \quad E(e^{\delta|Y|}) < \infty.$$

Then, we have the following properties.

- $X$  admits moments of any integer order.
- $M_X$  is analytic in a neighbourhood of 0, and for any  $n \in \mathbb{N}$ ,
- $M_X, M_Y$  characterise  $X, Y$  :

$$M_X(t) = M_Y(t) \text{ for all } t \in (-\delta, \delta) \implies X = Y.$$

## 1.3 Conditional probability independence

### 1.3.1 Conditional probability

#### Définition 18

Let  $\Omega$  be a set of realisations, and  $P$  a probability measure on  $\Omega$ . Let  $A \subset \Omega$  be an event such that  $P(A) > 0$ . Define then the **probability measure  $P$  conditioned on  $A$** , denoted  $P(\cdot|A)$ , by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \forall \text{ event } B.$$

We can then define the **conditions expectation** of a random variable  $X : \Omega \rightarrow \mathbb{R}$  by

$$E_P(X|A) = E_{P_A}(X),$$

where  $P_A$  stands for  $P_A = P(\cdot|A)$ .

### 1.3.2 Independence

#### Définition 19

- Two events  $A, B$  are said to be **independent** if

$$P(A \cap B) = P(A)P(B).$$

- A family of events  $(A_i)_{i \in I}$  is said to be **two-by-two independent** if for any  $i \neq j$ ,  $A_i$  and  $A_j$  are independent.
- A family of events  $(A_i)_{i \in I}$  is said to be **an independent family** if for any  $J \subset I$  finite,

$$P\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} P(A_i).$$

#### Définition 20

- Two random variables  $X, Y$  are said to be **independent** if for any events  $A, B \subset \mathbb{R}$ ,

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B).$$

Equivalently,  $X, Y$  are independent if for any  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$E(f(X)g(Y)) = E(f(X))E(g(Y)).$$

- A family of random variables  $(X_i)_{i \in I}$  is said to be **two-by-two independent** if for any  $i, X_i$  and  $X_j$  are independent.
- A family of random variables  $(X_i)_{i \in I}$  is said to be **an independent family** if for any  $J \subset I$  finite, and any events  $A_i \subset \mathbb{R}, i \in J$ ,

$$P(\cap_{i \in J} \{X_i \in A_i\}) = \prod_{i \in J} P(X_i \in A_i).$$

Equivalently,  $(X_i)_{i \in I}$  is an independent family if for any  $J \subset I$  finite, and any functions  $f_i : \mathbb{R} \rightarrow \mathbb{R}, i \in J$ ,

$$E\left(\prod_{i \in J} f_i(X_i)\right) = \prod_{i \in J} E(f_i(X_i)).$$

The same definition holds with “random vectors” replacing “random variables”.

### Définition 21

A family  $(X_i)_{i \in I}$  of random variables is called an **independent identically distributed** family, abbreviated **i.i.d. family**, if the family  $(X_i)_{i \in I}$  is an independent family, and for any  $i, j \in I, X_i = X_j$ .

### Théorème 10

Let  $d, d' \geq 1$ . Let  $X : \Omega \rightarrow \mathbb{R}^d, Y : \Omega \rightarrow \mathbb{R}^{d'}$  be a random vector.

- If  $X, Y$  are **continuous random vector** :  $X$  and  $Y$  are independent if and only if the random vector  $(X, Y) : \Omega \rightarrow \mathbb{R}^{d+d'}$  has density

$$f_{(X,Y)}(x, y) = f_X(x) f_Y(y),$$

where  $f_X : \mathbb{R}^d \rightarrow \mathbb{R}$  is a density for  $X$ , and  $f_Y : \mathbb{R}^{d'} \rightarrow \mathbb{R}$  is a density for  $Y$ .

- If  $X, Y$  are **discrete random vectors** :  $X$  and  $Y$  are independent if and only if for any  $x \in \mathbb{R}^d, y \in \mathbb{R}^{d'}$ ,

$$P(X = x, Y = y) = P(X = x) P(Y = y).$$

- If  $X$  is **discrete** and  $Y$  is **continuous** :  $X$  and  $Y$  are independent if and only if for any  $x \in \mathbb{R}^d, A \subset \mathbb{R}^{d'}$ ,

$$P(X = x, Y \in A) = P(X = x) \int_A f_Y(y) dy$$

where  $f_Y : \mathbb{R}^{d'} \rightarrow \mathbb{R}$  is a density for  $Y$ .

### 1.3.3 Bayes law, formula of total probability

Bayes Law :

### Théorème 11

Let  $\Omega$  be a set of realisations, and let  $P$  be a probability measure on  $\Omega$ . Let  $A, B \subset \Omega$  be two events such that  $P(A), P(B) > 0$ . Then

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

### Théorème 12

Let  $\Omega$  be a set of realisations, and let  $P$  be a probability measure on  $\Omega$ . Let  $I$  be a finite or countable set. Let  $A_i, i \in I$  be a collection of events such that

- if  $i \neq j$ , then  $A_i \cap A_j = \emptyset$ ;
- $\cup_{i \in I} A_i = \Omega$ .

Suppose moreover that  $P(A_i) > 0$  for all  $i \in I$ . Then for any event  $B$ ,

$$P(B) = \sum_{i \in I} P(B \cap A_i) = \sum_{i \in I} P(B|A_i) P(A_i).$$

In the same fashion, for every random variable  $X$

$$E(X) = \sum_{i \in I} E(X|A_i) P(A_i).$$

Example

We throw some dices :

- $A_1$  : we throw an even dice
- $A_2$  : we throw an odd dice
- $B$  : the result is  $\leq 4$

Then

$$P(B) = P(B|A_1) P(A_1) + P(B|A_2) P(A_2).$$

### 1.3.4 Almost sure properties

#### Définition 22

An event  $A$  is said to occur **almost-surely** if

$$P(A) = 1.$$

## 1.4 Correlation

### 1.4.1 Variance, Covariance

Variance is a way to quantify “how far from its mean is typically my variable”. If every value than  $X$  can take is not far from the mean of every value of  $X$ , then the variance will be small.

#### Définition 23

Let  $\Omega$  be a realisation set and  $P$  a probability measure on  $\Omega$ . Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable. The **variance of  $X$**  is given by

$$\text{Var}_P(X) := E_P\left((X - E_P(X))^2\right).$$

Alternatively,  $\text{Var}_P(X) = E_P(X^2) - E_P(X)^2$ .

The inside of the expected value in the definition on  $\text{Var}(X - E_P(X))$  is called the standard deviation



### Définition 24

The **standard deviation** of a random variable  $X$ , often denoted as  $\sigma_X$ , is the square root of its variance :

$$\sigma_X = \sqrt{\text{Var}(X)}.$$

### Définition 25

Let  $\Omega$  be a realisation set and  $P$  a probability measure on  $\Omega$ . Let  $X, Y : \Omega \rightarrow \mathbb{R}$  be two random variables. The **covariance between  $X$  and  $Y$**  is given by

$$\text{Cov}_P(X, Y) := E_P(XY) - E_P(X)E_P(Y).$$

When  $\text{Cov}_P(X, Y) = 0$ , we say that  $X$  and  $Y$  are **uncorrelated**.

*Remarque*

The covariance between  $X$  and  $Y$  is a measure of how much “typical large values of  $X$ ” and “typical large values of  $Y$ ” are influencing each other. Two independent event have a covariance of 0 (the opposite isn’t true!). But there is one case where uncorrelated implies independent : it’s with Bernoulli random variables

### Théorème 13

Let  $x, Y$  be two random variables such that

$$P(X \in \{0, 1\}) \equiv P(Y \in \{0, 1\}) = 1.$$

Such variable are called **Bernoulli random variables**. Then,  $X$  and  $Y$  are independent if and only if  $\text{Cov}(X, Y) = 0$ .

### Théorème 14

Let  $X, Y, Y_1, Y_2$  be random variables and  $a, b \in \mathbb{R}$ . Then,

$$\text{Cov}(X, Y) = \text{Cov}(Y, X), \quad \text{Cov}(aX, bY) = ab\text{Cov}(x, Y),$$

$$\text{Cov}(X_1, Y_1 + Y_2) = \text{Cov}(X, Y_1) + \text{Cov}(X, Y_2).$$

In words : Cov is symmetric, and linear in each of its arguments.

## 1.4.2 Pearson correlation coefficient

In statistics, a coefficient obtained from the covariance and standard deviation is frequently used : the Pearson correlation coefficient.

### Définition 26

For two random variables  $X, Y$  define their **Pearson correlation coefficient** :

$$\rho_{X,Y} = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{\text{Cov}(X, Y)}{\sigma_Y \sigma_x} \in [-1, 1].$$

$|\rho_{X,Y}| = 1$  if and only if  $X$  and  $Y$  are related by an affine transformation (i.e : there are  $a, b \in \mathbb{R}$  such that  $Y = aX + b$ ).

*Remarque*

We normalize by the product of variance because “ the height of Bob influences the height of Alice” should not depend on unit we chose to measure height, but the covariance does, so it’s a way to correct this

## 1.5 Classical example of random variables

$\Omega$  will be an abstract space of realisation.

$P$  will be an abstract probability measure.

### 1.5.1 Discrete random variables

#### Constant random variable

$X : \Omega \rightarrow \mathbb{R}, \omega \rightarrow c$ . The law of  $X$  is a **Dirac measure**.

$$\delta_c(A) = \begin{cases} 1 & \text{if } c \in A \\ 0 & \text{else} \end{cases}$$

#### Bernoulli random variable

$X : \Omega \rightarrow \mathbb{R}$  is a random variables of Bernoulli of parameter  $p$  if

$$P(X = 1) = p = 1 - P(X = 0)$$

*Example*

$A \in \mathcal{P}(\Omega)$  event,

$$\mathbb{1}_A : \Omega \rightarrow \mathbb{R}, \mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{else} \end{cases}$$

is a random variable of Bernoulli parameter  $P(A)$ .

#### Binomiale random variable

A random variable  $X : \Omega \rightarrow \mathbb{R}$  is a binomial random variable with parameter  $n \in \mathbb{N}$ ,  $p \in [0, 1]$ , denoted  $X \sim \text{Bin}(n, p)$  if

$$P(X = k) = \mathbb{1}_{\{0, \dots, n\}}(k) \binom{n}{k} p^k (1-p)^{n-k}$$

In particular  $P(X \in \{0, \dots, n\}) = 1$ .

#### **Théorème 15**

Let  $n \in \mathbb{N}$ ,  $p \in [0, 1]$ . Let  $X_1, \dots, X_n$  be an independent family of Bernoulli random variables of parameter  $p$ . Define

$$Y = \sum_{k=1}^n X_k.$$

Then,  $Y \sim \text{Bin}(n, p)$ .

#### Geometric random variable

A random variable  $X : \Omega \rightarrow \mathbb{R}$  is a geometric random variable with parameter  $p \in [0, 1]$ , denoted  $X \sim \text{Geo}(p)$ , if

$$P(X = k) = \mathbb{1}_{k \in \mathbb{N}^*} (1-p)^{k-1} p.$$

#### **Théorème 16**

Let  $X_1, X_2, \dots$  be an i.i.d sequence of Bernoulli random variables with parameter  $p$ . Define

$$Y = 1 + \sum_{n \geq 1} \prod_{i=1}^n (1 - X_i),$$

the number of trials before getting a 1 in the sequence. Then,  $Y \sim \text{Geo}(p)$ .

### Théorème 17

Let  $X \sim \text{Geo}(p)$  be a geometric random variable. Then, for any  $n > k \in \mathbb{N}$ ,

$$P(X = n | X > k) = P(X = n - k).$$

In particular, under the law  $P(\cdot | X > k)$ ,  $X - k$  follows a geometric law of parameter  $p$ .

*Remarque*

We can see this “loss of memory” property as follows : a geometric random variable is the number of independent coin tosses needed to make a 1. If we pause after  $k$  tosses and that the first  $k$  coins all gave 0, the following coins being independent of the first  $k$ , we end up with simply a sequence of independent coins tosses, exactly as we started.

### Poisson random variable

Let  $\lambda \geq 0$ . A random variable  $X : \Omega \rightarrow \mathbb{R}$  is a **Poisson random variable of parameter  $\lambda$** , denoted  $X \sim \text{Poi}(\lambda)$ , if

$$P(X = k) = \mathbb{1}_{k \in \mathbb{N}} e^{-\lambda} \frac{\lambda^k}{k!}.$$

### Théorème 18

Let  $X$  be a random variable. Then the two following points are equivalent :

- $X \sim \text{Poi}(\lambda)$  ;
- $P(X = 0) = e^{-\lambda}$  and for all  $k \in \mathbb{N}$ ,

$$\frac{P(X = k + 1)}{P(X = k)} = \frac{\lambda}{k + 1}.$$

### Uniform random variable (finite case)

Let  $J \subset \mathbb{R}$  be finite. A random variable  $X : \Omega \rightarrow \mathbb{R}$  is a **uniform random variable on  $J$**  denoted  $X \sim \text{Uni}(J)$  if

$$P(X = x) = \frac{1}{|J|} \forall x \in J.$$

In particular,  $P(X \in J) = 1$ . We will often look at  $J = \{0, 1, \dots, n\}$  or  $J = \{1, \dots, n\}$  for some  $n \geq 1$ .

### Théorème 19

Let  $J \subset \mathbb{R}$  be finite, and let  $X \sim \text{Uni}(J)$ . Let  $I \subset J$ . Then, for any  $A \subset I$ ,

$$P(X \in A | X \in I) = \frac{|A|}{|I|}.$$

## 1.5.2 Continuous random variables

### Uniform random variable on an interval

Let  $a < b \in \mathbb{R}$ . A random variable  $X : \Omega \rightarrow \mathbb{R}$  is a **uniform random variable on  $[a, b]$** , denoted  $X \sim \text{Uni}([a, b])$ , if it is a continuous random variable with probability density given by

$$f_X(x) = \frac{1}{b - a} \mathbb{1}_{[a, b]}(x).$$

### Théorème 20

Let  $a < b < c < d \in \mathbb{R}$ . Then, if  $X \in \text{Uni}([a, d])$ ,

$$P(t_1 \leq X \leq t_2 | b \leq X \leq c) = \frac{t_1 - t_2}{c - b}, \quad \forall b \leq t_1 \leq t_2 \leq c,$$

which is equivalent to say that under the conditioning  $\{X \in [b, c]\}$ ,  $X$  is a uniform random variable on  $[b, c]$ .

### Gaussian random variables

Let  $\mu \in \mathbb{R}, \sigma \geq 0$ . A random variable  $X : \Omega \rightarrow \mathbb{R}$  is a **Gaussian random variable with mean  $\mu$  and variance  $\mu^2$** , denoted  $X \sim \mathcal{N}(\mu, \sigma^2)$ , if it is a continuous random variable with density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

### Théorème 21

Let  $X, Y : \Omega \rightarrow \mathbb{R}$  be two independent Gaussian random variables. Suppose that  $x \sim \mathcal{N}(\mu_1, \sigma_1^2)$  and  $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$ . Then,

- the random variable  $\tilde{X} = (X - \mu_1) / \sigma_1$  is a centred and reduced Gaussian random variable :  $\tilde{X} \sim \mathcal{N}(0, 1)$  ;
- the random variable  $Z = X + Y$  is a Gaussian random variable with mean  $\mu_1 + \mu_2$  and variance  $\sigma_1^2 + \sigma_2^2$  :  $Z \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ .

### Exponential random variable

let  $\lambda > 0$ . A random variable  $X : \Omega \rightarrow \mathbb{R}$  is an **exponential random variable of parameter  $\lambda$** , denoted  $X \sim \text{Exp}(\lambda)$ , if  $X$  is a continuous random variable with density

$$f_X(x) = \mathbb{1}_{[0, \infty)}(x) \lambda e^{-\lambda x}$$

*Remarque*

The exponential random variable is the continuous version of the geometric random variable, it is therefore not a surprise that they share the “memory loss” property.

### Théorème 22

Let  $\lambda > 0$ , and  $X \sim \text{Exp}(\lambda)$ . Then for any  $0 < a < b$ ,

$$P(X \geq b | Y \geq a) = P(X \geq b - a).$$

In particular, under the conditioning  $\{X \geq \cdot\}$ , the variable  $X - a$  is an exponential random variable with parameter  $\lambda$ .

### Cauchy random variable

Let  $x_0 \in \mathbb{R}$  and  $\alpha > 0$ . A random variable  $X : \Omega \rightarrow \mathbb{R}$  is a **Cauchy random variable**, denoted  $X \sim \text{Cauchy}(x_0, \alpha)$ , if it is a continuous random variable with density

$$f_X(x) = \frac{\alpha}{\pi((x - x_0)^2 + \alpha^2)}.$$

## Summary of usual random variables

Variable	Expectation	Variance	Mom. Gen. Fct.
$\delta_c$	$c$	0	$e^{tc}$
Bern( $p$ )	$p$	$p(1-p)$	$1 + p(e^t - 1)$
Bin( $n, p$ )	$np$	$np(1-p)$	$(1 + p(e^t - 1))^n$
Geo( $p$ )	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Poi( $\lambda$ )	$\lambda$	$\lambda$	$\exp(\lambda(e^t - 1))$
Uni( $\{0, 1, \dots, n\}$ )	$\frac{n}{2}$	$\frac{n^2+1}{12}$	$\frac{e^{(n+1)t}-1}{(n+1)(e^t-1)}$
Uni( $\{1, \dots, n\}$ )	$\frac{n+1}{2}$	$\frac{n^2-1}{12}$	$\frac{e^{nt}-1}{n(1-e^{-t})}$
Uni( $[a, b]$ )	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{t(b-a)}$
$\mathcal{N}(\mu, \sigma^2)$	$\mu$	$\sigma^2$	$\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$
Exp( $\lambda$ )	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$
Cauchy( $x_0, \alpha$ )	N.D.	N.D.	N.D.

## 1.6 Probabilistic inequalities and applications

### 1.6.1 Markov's inequality

#### Théorème 23

Let  $X$  be a non-negative random variable ( $X : \Omega \rightarrow [0, +\infty)$ ). Then, for any  $a > 0$ ,

$$P(X \geq a) \leq \frac{E(X)}{a}.$$

### 1.6.2 First moment method

The first moment method is a simple observation : if we have a random variable  $X$  taking values in the non-negative integers,  $P(X \in \mathbb{N}) = 1$ , we can upper bound the probability that  $X$  is non-zero by using its mean :

$$P(X \neq 0) = P(X > 0) = E(\mathbb{1}_{X>0}X) \leq E(X).$$

*Remarque*

This means that if you have a small expectation it implies that you have a large probability to be 0.

Example

- We define  $M_n$  = the maximum length of a consecutive run of 1's in the  $n$  bits.
- To study  $M_n$ , we look at  $Y_k$  the number of runs of 1's of length  $k$ .
- We compute the expected value of  $Y_k$  :

$$E(Y_k) = (n - k + 1) \cdot 2^{-k}.$$

- Applying the first moment method

$$P(M_n \geq k) \leq E(Y_k) \leq n \cdot 2^{-k}$$

- if  $k > \log_2(n)$ , then  $n \cdot 2^{-k}$  becomes very small, so the probability of having such a long run of 1's is close to 0. In particular, we obtain that the longest run of 1's is at most of order  $\log_2(n)$

### 1.6.3 Chebychev's inequality

Chebychev's inequality can be seen as a re-phrasing of Markov's inequality.

#### Théorème 24

Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable. Let  $g : \mathbb{R} \rightarrow \mathbb{R}_+$  be an increasing function such that  $g(X)$  is a random variable. Then,

$$P(X \geq a) \leq \frac{E(g(X))}{g(a)}, \forall a \in \mathbb{R}.$$

The following cases are of particular interest.

- $p$ -th moment version : for all  $p \in (0, +\infty)$ ,

$$P(X \geq a) \leq \frac{E(|X|^p)}{a^p}, \forall a > 0.$$

- Exponential version : for all  $\delta \in \mathbb{R}_+$ ,

$$P(X \geq a) \leq e^{-\delta a} E(e^{\delta X}), \forall a > 0.$$

Remarque

Taking the function  $g(x) = x^2$  applied to the random variable  $|X - E(X)|$ , we obtain the useful particular case

$$P(|X - E(X)| \geq a) \leq \frac{\text{Var}(X)}{a^2}, \forall a > 0.$$

### 1.6.4 A weak Law of Large Numbers

This is an application of Chebychev's inequality. This law states that with high probability, the empirical average of independent identically distributed (i.i.d) random variables is close to its expectation.

#### Théorème 25

Let  $X_1, X_2, \dots$  be a sequence of **identically distributed** random variables. Suppose that

1. they admit a second moment :  $E(X_1^2) < \infty$ ,
2. they are **uncorrelated** :  $\text{Cov}(X_i, X_j) = 0$  if  $i \neq j$ .

Denote

$$E(X_1) = \mu, \text{Var}(X_1) = \sigma^2, \bar{S}_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

Then, for all  $\epsilon > 0$ ,

$$P(|\bar{S}_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2 n}.$$

In particular,  $\bar{S}_n$  converges in probability towards  $\mu$ .

### 1.6.5 Cauchy-Schwartz and Hölder's inequalities

#### **Théorème 26**

For any random variables  $X, Y$ ,

$$E(XY)^2 \leq E(X^2) E(Y^2).$$

A direct application is that Pearson correlation coefficient defined in 1.4.2

#### **Théorème 27**

Let  $X, Y$  be two random variables. Then,

$$|\text{Cov}(X, Y)| \leq \sqrt{\text{Var}(X)} \sqrt{\text{Var}(Y)}.$$

In particular,  $\rho_{XY} \in [-1, 1]$ .

The generalisation of this theorem is the Hölder's inequality.

#### **Théorème 28**

Let  $p, q \in (1, +\infty)$  be such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Then, for all random variables  $X, Y$ ,

$$E(|XY|) \leq (E(|X|^p))^{\frac{1}{p}} (E(|Y|^q))^{\frac{1}{q}}.$$

### 1.6.6 Second moment method

The second moment method is useful to show that a random variable is often positive. This is the complement of the first moment method. It relies on the following :

for  $X$  a  $\mathbb{N}$ -valued random variable,

$$P(X > 0) \geq \frac{E(X)^2}{E(X^2)}.$$

*Example*

Let define  $M_n$  = the length of the longest run of 1's,  $Y_k$  = number of consecutive blocks of length  $k$  consisting only of 1's

- $Y_k > 0 \leftrightarrow$  there exist a run of length  $\geq k$
- $P(M_n \geq k) = P(Y_k > 0)$

Step 1 : First Moment

We compute the expectation :

$$E(Y_k) = (n - k + 1) 2^{-k}.$$

This gives an upper bound, if this expectation is small, it's unlikely to have a run of 1's

Step 2 : Second Moment

To show that run actually exists, we use the second moment method :

$$Var(Y_k) = \sum_{i=1}^{n-k+1} \sum_{j=1}^{n-k+1} Cov(B_i, B_j),$$

Where  $B_i = \prod_{l=0}^{k-1} X_{i+l}$  is the indicator that the block starting at  $i$  is all 1's.

- If the block do not overlap ( $|i - j| \geq k$ ) they are independent  $\rightarrow$  covariance = 0.
- If they overlap ( $|i - j| < k$ ), the covariance is compute explicitly :  $Cov(B_i, B_j) = 2^{-j+i-k} - 2^{-2k}$ .

It gives :  $Var(Y_k) \leq 3(n - k + 1) 2^{-k}$

Step 3 : Apply the second moment formula

$$P(Y_k > 0) \geq \left(1 + \frac{Var(Y_k)}{(E[Y_k])^2}\right)^{-1} \geq \left(1 + \frac{3}{(n - k + 1) 2^{-k}}\right)^{-1}.$$

Step 4 : Asymptotic consequence

Choose  $k = (1 - \epsilon) \log_2 n$ . Then  $2^{-k} = n^{-(1-\epsilon)}$ , so :

$$(n - k + 1) 2^{-k} n^\epsilon$$

and thus

$$P(M_n \geq (1 - \epsilon) \log_2 n) = P(Y_k > 0) \geq \left(1 + \frac{6}{n^\epsilon}\right)^{-1} \lim_{n \rightarrow \infty} 1.$$

## 1.6.7 Jensen's inequality

### **Théorème 29**

Let  $I \subset \mathbb{R}$  be an interval. Let  $g : I \rightarrow \mathbb{R}$  be a convex functions. Then, for any random variable  $X$  taking values in  $I$  ( $P(X \in I) = 1$ ),

$$E(g(X)) \geq g(E(X)).$$

Moreover, for any  $h : I \rightarrow \mathbb{R}$  concave,

$$E(h(X)) \leq h(E(X)).$$



## Chapitre 2

# Convergence of random variables

### 2.1 What does converge mean ?

#### 2.1.1 Modes of converges

The weakest notion of convergence is the convergence in law. In words, convergence in law means that the statistical properties of the sequence approach the statistical properties of the limiting random variable as we go further and further in the sequence.

##### Définition 27

Let  $X, X_1, X_2, \dots : \Omega \rightarrow \mathbb{R}$  be random variables. Say that  $(X_i)_{i \geq 1}$  **converges in law** towards  $X$ , denoted  $X_n \rightarrow X$ , if one of the two following equivalent conditions is fulfilled.

1.  $F_{X_n}(t) \rightarrow F_X(t)$  for all  $t \in \mathbb{R}$  such that  $F_X$  is continuous at  $t$ .
2.  $E_P(\phi(X_n)) \rightarrow E_P(\phi(X))$  for all  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  continuous and bounded.

*Remarque*

Note that in the particular case of  $X$  being a continuous random variable,  $F_X$  is continuous (primitive of the density function). Moreover, still for  $X$  continuous, as  $P(X = x) = 0$  for all  $x$ , we have that  $P(X \in (a, b)) = P(X \in [a, b])$  for all  $a < b \in \mathbb{R}$ . In particular, if  $X$  is a continuous random variable, convergence in law of  $X_n$  towards  $X$  is equivalent to : for any interval  $I \subset \mathbb{R}$ ,

$$P(X_n \in I) \xrightarrow{n \rightarrow \infty} P(X \in I).$$

##### Théorème 30

Let  $X, X_1, X_2, \dots : \Omega \rightarrow \mathbb{R}$  be random variables. Suppose that there is  $\delta > 0$  such that

$$\sup_{|t| < \delta} E(e^{tX}) < +\infty, \quad \sup_{|t| < \delta} E(e^{tX_i}) < +\infty \quad \forall i \geq 1,$$

$$E(e^{tX_n}) \xrightarrow{n \rightarrow \infty} E(e^{tX}) \quad \forall |t| < \delta.$$

Then,

$$X_n \xrightarrow{Law} X \text{ as } n \rightarrow \infty,$$

and for any  $p \geq 1$ ,

$$E(X_n^p) \xrightarrow{n \rightarrow \infty} E(X^p).$$

### Examples

1. Let  $X_n \sim \text{Bern}(p + n^{-1})$ , and  $X \sim \text{Bern}(p)$ . Then,  $X_n \xrightarrow{\text{Law}} X$ . Indeed, we have  $F_X(t) = (1 - p) \mathbb{1}_{t \geq 0} + p \mathbb{1}_{t \geq 1}$  is continuous everywhere except at 0 and 1.

For  $t \in \mathbb{R} \setminus \{0, 1\}$ ,

$$P(X_n \leq t) = (1 - p + n^{-1}) \mathbb{1}_{t \geq 0} + (p + n^{-1}) \mathbb{1}_{t \geq 1} \xrightarrow{n \rightarrow \infty} F_X(t).$$

2. Let  $X_n \sim \mathcal{N}(0, n^{-1})$ , and  $X \sim \delta_0$ . Then,  $X_n \xrightarrow{\text{Law}} X$ . Indeed, we have  $E(e^{tX}) = 1$ , and for any  $t \in \mathbb{R}$ ,

$$E(e^{tX_n}) = e^{t^2/(2n)} \xrightarrow{n \rightarrow \infty} 1.$$

So Theorem 30 implies  $X_n \xrightarrow{\text{Law}} X$ . This is a particular instance of : Gaussian random variables with 0 variance are Dirac random variables.

### Théorème 31

Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable with  $\text{Image}(X) \subset \mathbb{Z}$ . Let  $X_1, X_2, \dots : \Omega \rightarrow \mathbb{R}$  be random variables. Then, the two following conditions are equivalent.

1.  $X_n \xrightarrow{\text{Law}} X$ .
2. For every  $k \in \mathbb{Z}$ ,

$$\lim_{\epsilon \rightarrow 0^+} \lim_{n \rightarrow \infty} P(X_n \in (k - \epsilon, k + \epsilon)) = P(X = k).$$

Now, let's see the second mode of convergence, it is saying : if I allow a small probability of failure  $\epsilon$ , and a small error of approximation  $\epsilon'$ , I can find a rank in the sequence which approximate the limit object up to an error  $\epsilon'$  with probability of success at least  $1 - \epsilon$ .

### Définition 28

Let  $X, X_1, X_2, \dots : \Omega \rightarrow \mathbb{R}$  be random variables. Say that  $(X_n)_{n \geq 1}$  **converges in probability** towards  $X$ , denoted  $X_n \xrightarrow{\text{Proba}} X$ , if for any  $\epsilon > 0$ ,

$$\lim_{n \rightarrow \infty} P(|X - X_n| \geq \epsilon) = 0.$$

The third mode of convergence states that the sequence of random variables converge with probability one.

### Définition 29

Let  $X, X_1, X_2, \dots : \Omega \rightarrow \mathbb{R}$  be random variable. We say that the sequence  $(X_n)_{n \geq 1}$  **converges almost surely** towards  $X$ , denoted  $X_n \xrightarrow{\text{a.s.}} X$ , if

$$P\left(\lim_{n \rightarrow \infty} X_n = X\right) = 1.$$

### Théorème 32

$P(\lim_{n \rightarrow \infty} X_n = X) = 1$  is equivalent to

$$\forall \epsilon > 0, \lim_{n \rightarrow \infty} P(\sup_{m \geq n} |X_m - X| \geq \epsilon) = 0.$$

### Définition 30

Let  $p \in (0, +\infty)$ . Let  $X, X_1, X_2, \dots : \Omega \rightarrow \mathbb{R}$  be random variables admitting a moment of order  $p$ . We say that the sequence  $(X_n)_{n \geq 1}$  **converges in  $L^p$**  towards  $X$ , denoted  $X_n \xrightarrow{L^p} X$ , if

$$\lim_{n \rightarrow \infty} E(|X_n - X|^p) = 0.$$

And there is a  $p = \infty$  version of this definition :

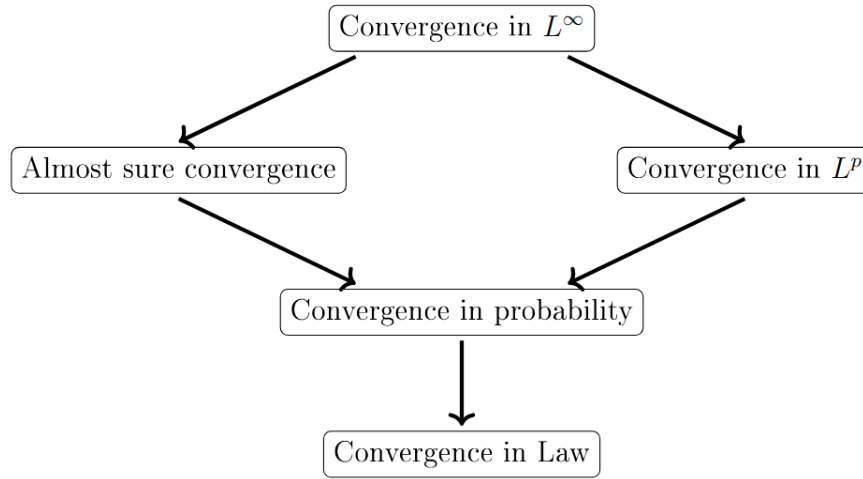
### Définition 31

Let  $X, X_1, X_2, \dots : \Omega \rightarrow \mathbb{R}$  be essentially bounded random variables. We say that  $(X_n)_{n \geq 1}$  **converges essentially uniformly** towards  $X$ , denoted  $X_n \xrightarrow{L^\infty} X$ , if

$$\lim_{n \rightarrow \infty} \|X - X_n\|_\infty = 0,$$

where  $\|Y\|_\infty = \inf\{M \geq 0 : P(|Y| \leq M) = 1\}$ .

## 2.1.2 Relations between convergence modes



## 2.2 Limit theorems

We will now study the two fundamental results on which most of statistics are based : the **law of large number** which formalizes the fact that the probability of an event represents its occurrence frequency ; and the **central limit theorem** which quantifies the typical deviations of frequency in a long sequence of experiments. Both will be about independent sequences of identically random variables : the setup will be as follows.

- Take a random variable  $X$ . For example, some measurement in a random experiment.
- Take a sequence  $X_1, X_2, \dots$  of random variables such that
  1.  $(X_i)_{i=1,2,\dots}$  is an independent family,
  2.  $X_i$  has the same law as  $X$  for every  $i \geq 1$ .

This will be the repeated measurement in sequence of repetitions of our random experiment.

- The goal : for  $n$  large, study
  1. whether  $\frac{1}{n} \sum_{i=1}^n X_i$  gets typically close or not to  $E(X)$  as  $n \rightarrow \infty$ ,
  2. quantify how far from  $E(X)$   $\frac{1}{n} \sum_{i=1}^n X_i$  typically is.

### 2.2.1 Weak Law of Large Numbers

#### Théorème 33

Let  $X, X_1, X_2, \dots$  be an independent family of identically distributed random variables. Then, if  $E(|X|) < \infty$ , for any  $\epsilon > 0$ ,

$$\lim_{n \rightarrow \infty} P \left( \left| \frac{1}{n} \sum_{i=1}^n X_i - E(X) \right| \geq \epsilon \right) = 0.$$

In other words,  $\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{\text{Proba}} E(X)$  as  $n \rightarrow \infty$ .

Another version of this theorem :

#### Théorème 34

Let  $X, X_1, X_2, \dots$  be an independent family of identically distributed random variables. Then, if  $E(X^2) < \infty$ , for any  $\epsilon > 0$ , and any  $n \geq 1$ ,

$$\lim_{n \rightarrow \infty} P \left( \left| \frac{1}{n} \sum_{i=1}^n X_i - E(X) \right| \geq \epsilon \right) \leq \frac{\text{Var}(X)}{\epsilon^2 n}.$$

### 2.2.2 Strong Law of Large numbers

#### Théorème 35

Let  $X, X_1, X_2, \dots$  be an independent family of identically distributed random variables. Then, if  $E(|X|) < \infty$ ,

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{\text{a.s.}} E(X)$$

as  $n \rightarrow \infty$ .

### 2.2.3 Central limit theorem

#### Théorème 36

Let  $X, X_1, X_2, \dots$  be an independent family of identically distributed random variables. Then, if  $E(X^2) < \infty$ ,

$$\frac{1}{\sqrt{\sigma_X^2 n}} \sum_{i=1}^n (X_i - \mu) \xrightarrow{\text{Law}} \mathcal{N}(0, 1)$$

as  $n \rightarrow \infty$ , where

$$\mu = E(X), \sigma_X^2 = \text{Var}(X).$$

Another version of this theorem :

#### Théorème 37

Let  $X, X_1, X_2, \dots$  be an independent family of identically distributed random variables. Then, if  $E(|X|^3) < \infty$ , for any  $n \geq 1$ ,

$$\sup_{t \in \mathbb{R}} \left| P \left( \frac{1}{\sqrt{\sigma_X^2 n}} \sum_{i=1}^n (X_i - \mu) \leq t \right) - P(Z \leq t) \right| \leq \frac{0.5 E(|X|^3)}{\sqrt{n}},$$

where  $Z \sim \mathcal{N}(0, 1)$ , and

$$\mu = E(X), \sigma_X^2 = \text{Var}(X).$$

Now, let's see a more restrictive version of CLT.

**Théorème 38**

*Let  $X, X_1, X_2, \dots$  be an independent family of identically distributed random variables. Then, if for some  $\delta > 0$ ,  $E(e^{\delta|X|}) < \infty$ ,*

$$\frac{1}{\sqrt{\sigma_X^2 n}} \sum_{i=1}^n (X_i - \mu) \xrightarrow{Law} \mathcal{N}(0, 1)$$

*as  $n \rightarrow \infty$ , where*

$$\mu = E(X), \sigma_X^2 = \text{Var}(X).$$

# Chapitre 3

## Statistics : starter guide

### 3.1 Basic objects

#### 3.1.1 Statistics

##### Définition 32

Let  $P$  be a probability measure on  $\mathbb{R}^d$ . Let  $n \geq 1$ . A **size  $n$  sample** of law  $P$  is an i.i.d. sequence  $X_1, \dots, X_n$  of random variables/vectors having law  $P$ . A **realisation** of a size  $n$  sample is a sequence of values of the random variables : that is a sequence  $(x_1, \dots, x_n) \in (\mathbb{R}^d)^n$ .

##### Définition 33

A **statistic** on a size  $n$  sample  $X_1, \dots, X_n$  with values in  $\mathbb{R}^d$  is a random variable  $Y : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$ .

#### 3.1.2 Estimators

##### Définition 34

We are interested in **parametric estimation**. We will denote  $\mathbb{P}_\theta$  the law that we are trying to approximate,  $\mathbb{P}_\theta$  is thus a probability measure on  $\mathbb{R}^d$ . We will suppose that it is entirely determined by a parameter  $\theta \in \Theta$ , where  $\Theta \subset \mathbb{R}^m$  for some  $m \geq 1$  is the **parameter space**.

##### Examples

We could for example look at

1. a squence of height measurements in the Swiss population,
2. a sequence of life duration for computers in a company ,
3. the number of blue cars exiting the highway at Ecublens during a day.

We can now chose an apprriate family of probability measures as candidates for approximating each of these situations.

1. One can use a Gaussian law  $\mathbb{P}_\theta = \mathcal{N}(\mu, \sigma^2)$ . The parameter defining the law is  $\theta = (\mu, \sigma)$ . It leads to the parameter space  $\Theta = \mathbb{R} \times [0, +\infty)$ .
2. One can use an exponential law  $\mathbb{P}_\theta = \text{Exp}(\lambda)$ . The parameter is then  $\theta = \lambda$ , and the parameter space is  $\Theta = (0, +\infty)$ .
3. One can use a Poisson law  $\mathbb{P}_\theta = \text{Poi}(\lambda)$ . The parameter is then  $\theta = \lambda$ , and the parameter space is  $\Theta = [0, +\infty)$ .

### Définition 35

Let  $X_1, \dots, X_n$  be a n-sample of law  $\mathbb{P}_\theta$ . For  $f : \Theta \rightarrow \mathbb{R}$ , an **estimator** of  $f(\theta)$  is a statistic  $\hat{f} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$ , which does not depend on the parameter  $\theta$ . In words, an estimator is a way to probe the parameter space without prior knowledge of the parameter value to approximate the parameter defining our probability law  $\mathbb{P}_\theta$ .

The first notion makes precise the idea that as we increase a sample size, we can find better and better approximations of our parameter.

### Définition 36

For  $n \geq 1$ , let  $X_1, \dots, X_n$  be a n-sample of law  $\mathbb{P}_\theta$ , and let  $\hat{f}_n$  be an estimator of  $f(\theta)$ . We say that the sequence of estimators  $(\hat{f}_n)_{n \geq 1}$  is **convergent** if  $\hat{f}_n$  converges in probability towards  $f(\theta)$  : for any  $\epsilon > 0$  and for any  $\theta \in \Theta$ ,

$$\lim_{n \rightarrow \infty} P \left( \left| \hat{f}_n(X_1, \dots, X_n) - f(\theta) \right| \geq \epsilon \right) = 0$$

The next notions are formalizations of the estimator gives the correct value on average.

### Définition 37

Let  $X_1, \dots, X_n$  be a n-sample of law  $\mathbb{P}_\theta$ . Let  $\hat{f}$  be an estimator of  $f(\theta)$ . The **bias of  $\hat{f}$**  is the difference between the expected value of  $\hat{f}$  and  $f(\theta)$  :

$$\text{Bias}_\theta(\hat{f}) = E \left( \hat{f}(X_1, \dots, X_n) \right) - f(\theta).$$

$\hat{f}$  is called **unbiased** if for any  $\theta \in \Theta$ ,  $\text{Bias}_\theta(\hat{f}) = 0$ . Otherwise, it is called **biased**.

### Définition 38

For  $n \geq 1$ , let  $X_1, \dots, X_n$  be a n-sample of law  $\mathbb{P}_\theta$ , and let  $\hat{f}_n$  be an estimator of  $f(\theta)$ . We say that the sequence of estimators  $(\hat{f}_n)_{n \geq 1}$  is **asymptotically unbiased** if for any  $\theta \in \Theta$ ,

$$\lim_{n \rightarrow \infty} \text{Bias}_\theta(\hat{f}_n) = 0.$$

### Définition 39

For  $n \geq 1$ , let  $X_1, \dots, X_n$  be a n-sample of law  $\mathbb{P}_\theta$ , and let  $\hat{f}_n$  be an estimator of  $f(\theta)$ . The **variance** of  $\hat{f}_n$ , denoted  $\text{Var}_\theta(\hat{f}_n)$ , is simply the variance of the random variable  $\hat{f}_n(X_1, \dots, X_n)$  :

$$\text{Var}_\theta(\hat{f}_n) = \text{Var} \left( \hat{f}_n(X_1, \dots, X_n) \right).$$

We can give a first example of criterion guaranteeing that a sequence of estimators converges.

### Théorème 39

For  $n \geq 1$ , let  $X_1, \dots, X_n$  be a n-sample of law  $\mathbb{P}_\theta$ , and let  $\hat{f}_n$  be an estimator of  $f(\theta)$ . Suppose that

1.  $\hat{f}_n$  is asymptotically unbiased,
2.  $\text{Var}(\hat{f}_n) \xrightarrow{n \rightarrow \infty} 0$ .

Then,  $\hat{f}_n$  is a convergent sequence of estimators.

### 3.1.3 Some examples of estimators

#### Empirical mean

The empirical mean is the estimator we already encountered several times. It is an estimator for the expected value and it is given by

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

It is so widely used that it got its own standard notation. It is direct to see that this estimator is unbiased :

$$E \left( \frac{1}{n} \sum_{i=1}^n X_i \right) = \frac{1}{n} \sum_{i=1}^n E(X_i) = E(X_1),$$

as the  $X_i$ 's are identically distributed. Moreover, the weak Law of Large Numbers implies that it is a convergent estimator as soon as  $E(X_1)$  is well defined.

#### Empirical median

A median for a random variable  $X$  is a number  $M$  such that

$$P(X) = P(X \geq M) = \frac{1}{2}.$$

The empirical median is obtained as follows.

1. Order the sample  $X_1, \dots, X_n$  in a non-decreasing fashion : let  $(\tilde{X}_1, \dots, \tilde{X}_n)$  be a permutation of  $X_1, \dots, X_n$  such that  $\tilde{X}_{i+1} \geq \tilde{X}_i$  for all  $i$ 's.
2. Define the empirical median of  $X_1, \dots, X_n$  to be

$$= \begin{cases} \tilde{X}_{\frac{n+1}{2}} & \text{if } n \text{ is odd,} \\ \frac{1}{2} \left( \tilde{X}_{\frac{n}{2}} + \tilde{X}_{1+\frac{n}{2}} \right) & \text{if } n \text{ is even.} \end{cases}$$

One can show that when the law  $\mathbb{P}_\theta$  is symmetric around its mean (if it is a continuous law with density  $f_\theta$ , this means  $f_\theta(\mu + x) = f_\theta(\mu - x)$  with  $\mu = \int_{-\infty}^{+\infty} x f_\theta(x) dx$ , the median is an unbiased estimator of the mean.

#### Empirical variance

We already have an estimator for the expectation :  $\bar{X}_n$ . A natural thing to do is to define the empirical variance as

$$\sigma_n^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2 = \frac{1}{n} \sum_{i=1}^n X_i^2 - \bar{X}_n^2 = \bar{X}_n^2 - \bar{X}_n^2.$$

I.e : take the difference between the empirical mean of  $X_1^2, \dots, X_n^2$  and the square of the empirical mean of  $X_1, \dots, X_n$ .

#### Empirical covariance

Consider an  $n$ -sample  $(X_1, Y_1), \dots, (X_n, Y_n)$  of random vectors of law  $\mathbb{P}_\theta$ . The goal is to estimate the covariance  $\text{Cov}(X, Y)$  where  $(X, Y) \sim \mathbb{P}_\theta$ . Define the estimator

$$\hat{\tau}_n((X_1, Y_1), \dots, (X_n, Y_n)) = \frac{1}{n} \sum_{i=1}^n X_i Y_i - \frac{1}{n} \left( \sum_{i=1}^n X_i \right) \frac{1}{n} \left( \sum_{i=1}^n Y_i \right) = \bar{X} \bar{Y}_n - \bar{X}_n \bar{Y}_n.$$



## 3.2 Constructing estimators

### 3.2.1 Moments method

Suppose one wants to estimate a parameter  $\theta$  from a sample  $X_1, \dots, X_n$  of law  $\mathbb{P}_\theta$ . The general idea is

1. find functions  $h, g$  such that we have the relation

$$\theta = h(E(g(X)))$$

with  $X \sim \mathbb{P}_\theta$ ;

2. use the empirical mean  $\frac{1}{n} \sum_{i=1}^n g(X_i)$  to estimate  $E(g(X))$ ;
3. use the estimator of  $\theta$  given by

$$\hat{\theta}_n = h\left(\frac{1}{n} \sum_{i=1}^n g(X_i)\right).$$

*Example*

We can look at the case of  $\mathbb{P}_\theta = \text{Geo}(p)$  (so that  $\theta = p, \Theta = [0, 1]$ ). We know that if  $X \sim \text{Geo}(p)$ ,

$$E(X) = \frac{1}{p}, \quad E(X^2) = \frac{2-p}{p^2}.$$

Each of these give rise to estimator via the moment method.

- First consider the first moment. We can use  $h(x) = x^{-1}$  and  $g(x) = x$ . This gives the estimator of  $p$

$$\hat{p}_n(X_1, \dots, X_n) = \frac{n}{\sum_{i=1}^n X_i}.$$

- Then consider the second moment. We can use  $h(x) = \frac{\sqrt{1+8x}-1}{2x}$  and  $g(x) = x^2$ .

This gives the estimator of  $p$

$$\hat{p}_n(X_1, \dots, X_n) = \frac{n}{2 \sum_{i=1}^n X_i^2} \left( \left( 1 + \frac{8}{n} \sum_{i=1}^n X_i^2 \right)^{\frac{1}{2}} - 1 \right).$$

### 3.2.2 Maximum likelihood estimator

This estimator is deeply linked to the Bayes formula : imagine that you have a realisation  $x_1, \dots, x_n$  of a sample  $X_1, \dots, X_n$  of law  $\mathbb{P}_\theta$ . The idea is to look for the value of  $\theta$  which maximises the probability to observe the realisation  $x_1, \dots, x_n$ .

#### Définition 40

Let  $n \geq 1$ , and a sample  $X_1, \dots, X_n$  of law  $\mathbb{P}_\theta$ . For a realisation  $x_1, \dots, x_n$  of  $X_1, \dots, X_n$ , the **likelihood** of  $x_1, \dots, x_n$  given  $\theta$ ,  $\mathcal{L}(x_1, \dots, x_n|\theta)$ , is defined by

- if  $\mathbb{P}_\theta$  is a discrete probability measure (i.e. : the  $X_i$ 's are discrete random variables),

$$\mathcal{L}(x_1, \dots, x_n|\theta) = P(X_1 = x_1, \dots, X_n = x_n) = \prod_{i=1}^n P(X_i = x_i) = \prod_{i=1}^n \mathbb{P}_\theta(\{x_i\})$$

as the  $X_i$ 's are independent and of law  $\mathbb{P}_\theta$ ;

- if  $\mathbb{P}_\theta$  is a continuous probability measure (i.e. : the  $X_i$ 's are continuous random variables),

$$\mathcal{L}(x_1, \dots, x_n|\theta) = \prod_{i=1}^n f_{X_i}(x_i) = \prod_{i=1}^n f_\theta(x_i)$$

where  $f_\theta$  is the density function associated to  $\mathbb{P}_\theta$ .

#### Définition 41

The **maximum likelihood estimator** of  $\theta$  is given by

$$MLE(X_1, \dots, X_n) = \operatorname{argmax}_\theta \mathcal{L}(X_1, \dots, X_n | \theta),$$

the point  $\theta$  at which  $\mathcal{L}(X_1, \dots, X_n | \theta)$  is maximized.

*Example*

Consider the case  $X_1, \dots, X_n$  is a n-sample of law  $\mathcal{N}(\mu, \sigma^2)$  (and take  $\theta = (\mu, \sigma^2)$ ). The likelihood function of a realisation  $x_1, \dots, x_n \in \mathbb{N}^*$  is given by

$$\mathcal{L}(x_1, \dots, x_n | \mu, \sigma^2) = \prod_{i=1}^n \frac{e^{-(x_i - \mu)^2 / 2\sigma^2}}{\sqrt{2\pi\sigma^2}}.$$

Maximizing  $\mathcal{L}$  is equivalent to minimizing  $-\ln(\mathcal{L})$ .

$$-\mathcal{L}(x_1, \dots, x_n | \mu, \sigma^2) = \frac{n}{2} \ln(2\pi\sigma^2) + \sum_{i=1}^n \frac{(x_i - \mu)^2}{2\sigma^2}.$$

For any fixed  $\sigma^2$ , the minima of this expression is reached at the empirical mean :  $\mu_* = \frac{1}{n} \sum_{i=1}^n x_i$  (which is found by finding the critical points as a function of  $\mu$ ). Then, the minima is reached at the empirical variance :  $\sigma^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \mu_*)^2$  (which is again found by finding the critical points as a function of  $\sigma^2$ ). The maximum likelihood estimator of  $(\mu, \sigma^2)$  is thus given by

$$MLE(X_1, \dots, X_n) = (\bar{X}_n, \hat{\sigma}_n^2).$$

### 3.3 Quantile tables

In the topic of the next section, we will often be confronted to problem of this form :

$$P(X \leq t) = 0.95,$$

To avoid doing computations every time, people have computed these values and made quantile tables.

#### Définition 42

Let  $X$  be a random variable,  $q > 0$  be an integer. For integer  $k \geq 1, t \in \mathbb{R}$  is a **kth q-quantile** of  $X$  if

$$P(X < t) \leq \frac{k}{q} \text{ AND } P(X \geq t) \leq \frac{k}{q}.$$

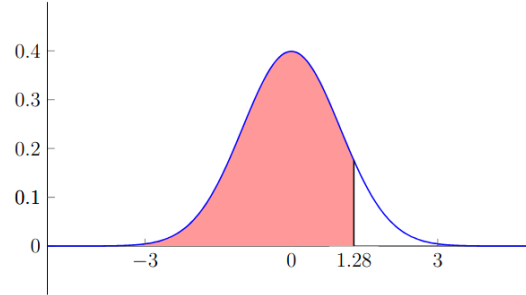
Of  $X$  is a continuous random variable with strictly positive density, there is only one kth q-quantile of  $X$ . Sometimes, one speaks about v-quantiles with  $v \in (0, 1)$ . In this case, a kth v-quantile of  $X$  is a number  $t \in \mathbb{R}$  such that

$$P(X < t) \leq kv \text{ AND } P(X \leq t) \geq kv.$$

Example

The 10-quantiles of the  $\mathcal{N}(0,1)$  distribution are given in the next table. See the graph after for an illustration of the concept. In the table,  $t_k$  is the number such that  $P(X \leq t) = \frac{k}{10}$  where  $X \sim \mathcal{N}(0,1)$ .

$k$	1	2	3	4	5	6	7	8	9	10
$t_k$	-1.282	-0.841	-0.524	-0.253	0	0.253	0.524	0.842	1.282	$+\infty$



### 3.4 Confidence intervals

Confidence intervals will allow to state things such as “given the measurements  $X_1, \dots, X_n$ , the parameter  $\theta$  belongs to the interval  $I(X_1, \dots, X_n)$  with probability  $p(X_1, \dots, X_n)$ ”, where  $I, p$ , are to be specified. One is typically interested in finding intervals with very short length (precise estimation) with  $p$  close to 1 (small probability of error).

#### Définition 43

Let  $X_1, \dots, X_n$  be a sample of law  $\mathbb{P}_\theta$ . Let  $\alpha \in (0, 1)$ . A random interval  $I = I(X_1, \dots, X_n)$  depending on  $\theta$  is called a **level  $1 - \alpha$  confidence interval for  $f(\theta)$**  if for every  $\theta \in \Theta$ ,

$$P(f(\theta) \in I(X_1, \dots, X_n)) = 1 - \alpha.$$

$1 - \alpha$  is called the **confidence level** of the estimation.

#### Définition 44

A confidence interval  $I = I(X_1, \dots, X_n)$  is an **excess confidence interval for  $f(\theta)$  at level  $1 - \alpha$**  if

$$P(f(\theta) \in I(X_1, \dots, X_n)) \geq 1 - \alpha.$$

### 3.5 Hypothese testing

#### 3.5.1 General principle

Let's start with an example.

Example

Let

- $p_1$  be the probability that a pipe from company 1 breaks,
- $p_2$  be the probability that a pipe from company 2 breaks.

The parameter is  $\theta = (p_1, p_2)$  with parameter space  $\Theta = [0, 1]^2$ . Our goal is not to estimate  $\theta$  precisely but to determine which region of the parameter space it belongs to.

**Hypotheses :**

- Null hypothesis  $H_0 : \theta \in \Theta_0 = \{(p_1, p_2) : p_1 > p_2\}$ ,  
(company 1 produces less safe pipes).
- Alternative hypothesis  $H_1 : \theta \in \Theta = \{(p_1, p_2) : p_1 \leq p_2\}$ .  
(company 1 is at least as safe as company 2).

Based on the data, we decide whether to reject  $H_0$ . Two types of error may occur :

- **Type I error** : Rejecting  $H_0$  when it is true  $\rightarrow$  serious consequence.
- **Type II error** : not rejecting  $H_0$  when it is false  $\rightarrow$  minor consequence.

**General framework :** we are trying to decide whether a parameter  $\theta$  belongs to a region  $\Theta_0 \subset \Theta$  or not, based on a sample  $X_1, \dots, X_n$  of law  $\mathbb{P}_\theta$ .

**Définition 45**

The hypotheses " $\theta \in \Theta_0$ ", usually denotes  $H_0$ , is called the **null hypotheses**. Its complement, the hypotheses " $\theta \in \Theta \setminus \Theta_0$ ", usually denoted  $H_1$ , is called the **alternative hypotheses**.

**Définition 46**

A **rejection region D** is an event for the random variables  $X_1, \dots, X_n$ . I.e : if the  $X_i$  take values in  $\mathbb{R}^d$ ,  $D \subset (\mathbb{R}^d)^n$ . In practice, one usually takes

$$D = \{(x_1, \dots, x_n) : T(x_1, \dots, x_n) \in [a, b]\}$$

for some statistic  $T$  and real numbers  $a \leq b$ .

**Définition 47**

Given  $D$  a rejection region, and  $H_0, H_1$  two hypotheses that are tested one against the other, a **test procedure** corresponds to

1. reject  $H_0$  if  $(X_1, \dots, X_n) \in D$ ;
2. do not reject  $H_0$  if  $(X_1, \dots, X_n) \notin D$ .

Failure of prediction are divided into two classes :

- **Type-I error** : we reject  $H_0$  whereas it was correct.
- **Type-II error** : we do not reject  $H_0$  whereas it was false.

**Définition 48**

Let  $\alpha \in [0, 1]$ . We say that the test procedure has a **risk level  $\alpha$** , or a **confidence level  $1 - \alpha$**  if

$$\sup_{\theta \in \Theta_0} P((X_1, \dots, X_n)) = \alpha.$$

**Définition 49**

The **power** of a test is given by

$$\inf_{\theta \in \Theta_1} P((X_1, \dots, X_n) \in D) = 1 - \beta.$$

In particular,

$$\beta = \sup_{\theta \in \Theta_1} P((X_1, \dots, X_n) \notin D).$$

In words : the risk level  $\alpha$  is the “worst case scenario” of the probability to fall in the rejection region whilst having a value of the parameter satisfying  $H_0$  (type-I error), and the  $\beta$  in the power of a test is the “worst case scenario” of the probability to fall outside of the rejection region whilst having a value of the parameter not satisfying  $H_0$  (type-II error).

**3.5.2 Chi-square distribution****Définition 50**

Let  $k \in \mathbb{N}^*$  be a positive integer. A random variable  $X$  follows the  $\mathcal{X}^2$  **distribution with k degrees of freedom**, denoted  $X \sim \mathcal{X}_k^2$ , if it is a continuous random variable with density given by

$$f_X(x) = \mathbb{1}_{[0, +\infty)}(x) \frac{x^{\frac{k}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right)}.$$

Recall the Gamma function  $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ . For  $n \in \mathbb{N}^*$ , we have  $\Gamma(n) = (n-1)!$

This law have links with the Gaussian random variables,

- if  $k \geq 1$  is an integer, and  $X_1, \dots, X_k$  are independent  $\mathcal{N}(0, 1)$  random variables, then

$$\sum_{i=1}^k X_i^2 \sim \mathcal{X}_k^2.$$

- if  $k \geq 1$  is an integer,  $\mu \in \mathbb{R}, \sigma^2 > 0$ , and  $X_1, \dots, X_k$  are independent  $\mathcal{N}(\mu, \sigma^2)$  random variables, then the following rescaling of the empirical variance follows a  $\mathcal{X}_{k-1}^2$  law :

$$\frac{1}{\sigma^2} \sum_{i=1}^k \left( X_i - \frac{1}{k} \sum_{i=1}^k X_i \right)^2 \sim \mathcal{X}_{k-1}^2.$$

**Définition 51**

Let  $k \geq 2$  be a positive integer. Let  $n \geq 1$ . Let  $p_1, \dots, p_k \in [0, 1]$  be such that  $\sum_{i=1}^k p_i = 1$ . A random vector  $X = (X_1, \dots, X_k)$  follows the **multinomial distribution with parameters**  $(k; n; p_1, \dots, p_k)$  if it is a discrete random vector with mass function given by

$$P(X = (x_1, \dots, x_k)) = \begin{cases} \frac{n!}{x_1! \dots x_k!} p_1^{x_1} \dots p_k^{x_k} & \text{if } x_1, \dots, x_k \in \mathbb{N}, \sum_{i=1}^k x_i = n, \\ 0 & \text{else.} \end{cases}$$

When  $k = 2$ , its reduces to the binomial distribution.

The convergence result we will use is the following.

**Théorème 40**

Let  $k \geq 2$ ,  $p_1, \dots, p_k \in (0, 1)$  such that  $p_1 + \dots + p_k = 1$ . For  $n \geq 1$ , let  $(N_{n,1}, \dots, N_{n,k})$  be a multinomial random vector with parameters  $(n, k; p_1, \dots, p_k)$ . Then the random variable

$$\sum_{i=1}^k \frac{(N_{n,i} - np_i)^2}{p_i}$$

converges in law to a  $\chi^2$  with  $k - 1$  degrees of freedom :

$$\sum_{i=1}^k \frac{(N_{n,i} - np_i)^2}{np_i} \xrightarrow{Law} \chi_{k-1}^2, \text{ as } n \rightarrow \infty.$$

This is used as follows. Let  $X_1, \dots, X_n : \Omega \rightarrow \mathbb{R}$  be some independent sequence of i.i.d. Let  $A_1, \dots, A_k \subset \mathbb{R}$  be a partition of  $\mathbb{R}$  :

$$A_i \cap A_j = \emptyset \text{ if } i \neq j, \cup_{i=1}^k A_i = \mathbb{R}.$$

Then, the random vector

$$\left( \sum_{i=1}^n \mathbb{1}_{A_1}(X_i), \sum_{i=1}^n \mathbb{1}_{A_2}(X_i), \dots, \sum_{i=1}^n \mathbb{1}_{A_k}(X_i) \right),$$

which simply counts the number of measurements which fell into each class, follows a multinomial distribution with parameters  $(n, k; P(X_1), \dots, P(X_1 \in A_k))$ .

### 3.5.3 $\chi^2$ tests

#### Adequacy tests

Adequacy tests are a way to check whether the observed realisation of the sample was obtained from a given law or not. Consider an experiment with a certain realisation space  $\Omega$ . Suppose that you can partition  $\Omega$  into disjoint classes  $\Omega_1, \dots, \Omega_k$ . If we let  $\mathbb{P}$  be the law of the sample (which we do not know), we then have the probability of falling in each of the classes is a number  $p_i = \mathbb{P}(\Omega_i) \in [0, 1]$ , and that  $\sum_{i=1}^k p_i = 1$ . Observing a realisation of a n-sample  $X_1, \dots, X_n$ , we can associate the numbers of times we observed a realisation in the class  $\Omega_i$  :

$$N_{n,i}(x_1, \dots, x_n) = \sum_{j=1}^n \mathbb{1}_{\Omega_i}(x_j).$$

If we assume that the law of the sample is  $\mathbb{P}$ , we have that the vector  $(N_{n,1}, \dots, N_{n,k})$  follows a **multinomial distribution** of parameters  $(n, k; p_1, \dots, p_k)$  : recall that it is a discrete random vector with, for  $m_1, \dots, m_k \in \mathbb{N}$ ,

$$P((N_{n,1}, \dots, N_{n,k}) = (m_1, \dots, m_k)) = \mathbb{1}_{m_1 + \dots + m_k = n} \frac{n!}{m_1! \dots m_k!} p_1^{m_1} \dots p_k^{m_k}.$$

We can now describe the idea of the test. Our goal is to test whether  $\mathbb{P}$ , the law of the sample, is a certain law  $\mathbb{Q}$  or not. We thus introduce

$$q_i = \mathbb{Q}(\Omega_i), \quad i = 1, \dots, k.$$

We then have that if  $\mathbb{P} = \mathbb{Q}$ ,  $p_i = q_i$  for every  $i = 1, \dots, k$  ... but if  $p_i \neq q_i$  for some  $i$ , then we know for sure that  $\mathbb{P} \neq \mathbb{Q}$ ! We will therefore test the null hypotheses  $H_0$  : “ $p_i = q_i$  for  $i = 1, \dots, k$ ” against the alternative hypotheses  $H_1$  “there is  $i \in \{1, \dots, k\}$  such that  $p_i \neq q_i$ ”.

Under  $H_0$ , the  $q_i$ 's should be well approximated by the empirical frequencies :  $q_i \approx N_{n,i}/n$ . We just transformed a non-parametric question “is  $\mathbb{P}$  equal to  $\mathbb{Q}$ ?” into a parametric one about the parameters of a multinomial law!

We now need to construct our rejection region. Introduce the statistic

$$Z_n(X_1, \dots, X_n) = \sum_{i=1}^k \frac{(N_{n,i}(X_1, \dots, X_n) - nq_i)^2}{nq_i} = n \sum_{i=1}^k \frac{\left( \frac{N_{n,i}(X_1, \dots, X_n)}{n} - q_i \right)^2}{q_i}.$$

It is some measure of the difference between the observed frequencies  $\frac{N_{n,i}(X_1, \dots, X_n)}{n}$  and the ones we should see under the null hypotheses. The reason to choose this particular statistic is the lemma that, in our setup, says that, under the null hypotheses  $H_0$ ,

$$Z_n(X_1, \dots, X_n) \xrightarrow{Law} \chi_{k-1}^2.$$

From this Lemma, we have on the one hand under  $H_0$ ,  $Z_n$  follows asymptotically a  $\chi^2$  law with  $k - 1$  degrees of freedom. On the other hand, under  $H_1$ , there is  $i_* \in \{1, \dots, k\}$  such that

$$\left( \frac{N_{n,i_*}}{n} - q_{i_*} \right)^2 \xrightarrow{\text{a.s.}} (p_{i_*} - q_{i_*})^2 > 0,$$

as  $n \rightarrow \infty$ , which implies that  $Z_n \xrightarrow{\text{a.s.}} +\infty$ . If we want a risk level  $\alpha$  for our test, we can thus take a rejection region of the form  $D = \{Z_n > C\}$  with  $C$  such that, with  $Z \sim \chi_{k-1}^2$ ,

$$P(Z > C) = \alpha.$$

This gives a region with risk level  $\alpha$  asymptotically as, under  $H_0$ ,

$$P(Z_n > C) \xrightarrow{n \rightarrow \infty} P(Z > C).$$

### Independence test

We will only consider an example of this test and not describe the general theory. The goal of the test is to test the hypotheses “are property A and property B independent?”. Here are the measures of hairs and eyes colour in a group of people.

eyes \ hairs	Blond	Brown	Black	Ginger	Total	Freq.
Blue	25	9	3	7	44	44/124
Grey	13	17	10	7	47	47/124
Brown	7	13	8	5	33	33/124
Total	45	39	21	19	124	
Freq	45/124	39/124	21/124	19/124		

We then want to test whether the colour of the eyes is independent of the colour of the hairs. Let the null hypotheses be “the eyes and hairs colours are independent” against the alternative hypotheses.

Denote the eyes colours by

$$1 \equiv \text{blue}, 2 \equiv \text{grey}, 3 \equiv \text{brown}.$$

and the hair colours by

$$1 \equiv \text{blond}, 2 \equiv \text{brown}, 3 \equiv \text{black}, 4 \equiv \text{ginger}.$$

Denote then  $p_{ij}$  the probability that an individual has eyes colour  $i$  and hair colour  $j$  and set

$$p_{i*} = \sum_{k=1}^4 p_{ik}, p_{*j} = \sum_{i=1}^3 p_{ij},$$

the probability that an individual has eye colour  $i$ , hair colour  $j$ . Under the null hypotheses, we have that hair and eyes colours are independent, which translates into

$$p_{ij} = p_{i*}p_{*j}, i = 1, 2, 3, j = 1, 2, 3, 4.$$

We therefore have that the null hypotheses can be re-phrased “ $p_{ij} = p_{i*}p_{*j}$  for every  $i \in \{1, 2, 3\}$ , and every  $j \in \{1, 2, 3, 4\}$ ”, which turns out to be a parametric hypotheses.

Under the null hypotheses, we have that the number of individual with eye colour  $i$  and hair colour  $j$  in a sample of  $n$  people is on average

$$np_{ij} = np_{i*}p_{*j}.$$

Also, we know from the LLN that  $p_{i*}$  is well approximated by the empirical frequency. We can then test this hypotheses as before by introducing the statistic

$$Z_n = \sum_{i=1}^3 \sum_{j=1}^4 \frac{\left( N_{ij} - n \frac{N_{i*}N_{*j}}{n^2} \right)^2}{n \frac{N_{i*}N_{*j}}{n^2}} = n \sum_{i=1}^3 \sum_{j=1}^4 \frac{\left( \frac{N_{ij}}{n} - \frac{N_{i*}N_{*j}}{n^2} \right)^2}{\frac{N_{i*}N_{*j}}{n^2}}$$

Where  $N_{ij}$  counts the number of individual eye colour  $i$  and hair colour  $j$ ,  $N_{i*}$  counts the number of individual with eye colour  $i$ , and  $N_{*j}$  with hair colour  $j$ . Under the null hypotheses,  $Z_n$  converges to a  $\chi_6^2$  law.

Using our data in  $Z_{124}$  with a rejection region at risk 5%, we obtain that we should reject the null hypotheses (so no independence).

### 3.6 t-test

#### Définition 52

Let  $v \in (0, +\infty)$  be a positive real. A random variable  $X$  follows the **Student's t-distribution with parameter  $\nu$** , denoted  $X \sim \text{Student}_t(\nu)$ , if it is a continuous random variable with density given by

$$f_X(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}.$$

Often, one takes  $\nu \in \mathbb{N}^*$ . In this case, we call  $\text{Student}_t(\nu)$  the **Student's t-distribution with  $\nu$  degrees of freedom**.

#### Théorème 41

Let  $Z, V$  be two independent random variables with

$$Z \sim \mathcal{N}(0, 1), \quad V \sim \mathcal{X}_k^2, \quad k \in \{2, 3, 4, \dots\}.$$

Then, the random variable  $Z\sqrt{\frac{k}{V}}$  follows a Student's t-distribution with parameter  $k$  :

$$Z\sqrt{\frac{k}{V}} \sim \text{Student}_t(k).$$

#### One-sample t-test

It is used to test the null Hypotheses that the mean of a sample is given by a fixed value.

**Setup :** we have a sample  $X_1, \dots, X_n$  of a law  $\mathbb{P}$ . We want to test the Hypotheses  $H_0: E(X_1) = \mu_0$  for a fixed  $\mu_0 \in \mathbb{R}$ .

**Assumptions :** to be an exact test, we assume that there are  $\mu \in \mathbb{R}$ ,  $\sigma^2 > 0$  such that  $\mathbb{P} = \mathcal{N}(\mu, \sigma^2)$ . One can make approximate versions of this in applications.

**Running the test :** we want to test the null Hypotheses " $E(X_1) = \mu_0$ " using the test statistic

$$\frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sqrt{\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2}}.$$

Under the null Hypotheses, our assumptions and (3.2) imply that  $\frac{\sqrt{n}}{\sigma^2}(\bar{X}_n - \mu_0) \sim \mathcal{N}(0, 1)$ , and thus that

$$T = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sqrt{\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2}} \sim \text{Student}_t(n-1).$$

The rejection region is then taken to be of the form  $\{|T| \geq C\}$  with  $C > 0$  depending on the wanted confidence level.

#### Two-samples t-test

It is used to check whether two independent samples have the same mean or not.

**Setup :** we have two samples of the same size  $X_1, \dots, X_n$  of law  $\mathbb{P}$ ,  $Y_1, \dots, Y_n$  of a law  $\mathbb{Q}$ . We want to test the Hypotheses  $H_0: E(X_1) = E(Y_1)$ .

**Assumptions :** to be an exact test, we assume that  $\mathbb{P} = \mathcal{N}(\mu_X, \sigma^2)$ , and that  $\mathbb{Q} = \mathcal{N}(\mu_Y, \sigma^2)$  for some  $\mu_X, \mu_Y \in \mathbb{R}$ ,  $\sigma^2 > 0$ . Note that we in particular assume that the variance is the same in both sample ( $\sigma^2$ ).

**Running the test :** we will use the test statistic

$$T_n(X_1, \dots, X_n, Y_1, \dots, Y_n) = \frac{\sqrt{n}(\bar{X}_n - \bar{Y}_n)}{\sqrt{s_X^2 + s_Y^2}}$$

where

$$s_X^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2, \quad s_Y^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y}_n)^2,$$



are the un-biased versions of the empirical variances. Under our assumptions, if  $H_0$  holds (i.e : if  $\mu_X = \mu_Y$ ),

$$\frac{\sqrt{n}}{\sigma^2} (\bar{X}_n - \bar{Y}_n) \sim \mathcal{N}(0, 1), \quad \frac{n-1}{\sigma^2} (s_X^2 + s_Y^2) \sim \mathcal{X}_{2n-2}^2,$$

and  $T_n \sim \text{Student}_t(2n-2)$ . On the other hand, if  $H_0$  does not hold,  $\bar{X}_n - \bar{Y}_n \xrightarrow{\text{a.s.}} \delta \neq 0$  as  $n \rightarrow \infty$ , which implies that  $T_n$  takes extremely large values for large  $n$ . We thus take a rejection region of the form  $\{|T_n| \geq C\}$  with  $C$  to be chosen depending on the wanted confidence level.

### 3.6.1 F-tests

#### Fisher distribution

F-tests are designed to compare variances.

##### Définition 53

Let  $k_1, k_2 > 0$  be two positive integers. A random variable  $X$  follows the **Fisher's law with degrees of freedom**  $(k_1, k_2)$ , denoted  $X \sim F(k_1, k_2)$ , if it is a continuous random variable with density given by

$$f_X(x) = \mathbb{1}_{[0, +\infty)}(x) \frac{1}{xB(k_1/2, k_2/2)} \left( \frac{k_1 x}{k_1 x + k_2} \right)^{k_1/2} \left( 1 - \frac{k_1 x}{k_1 x + k_2} \right)^{k_2/2}.$$

If  $X \sim \mathcal{X}_{k_1}^2$  and  $Y \sim \mathcal{X}_{k_2}^2$  are independent, then,

$$\frac{X/k_1}{Y/k_2} \sim F(k_1, k_2).$$

## 3.7 Comparing estimators

### 3.7.1 Mean square error

##### Définition 54

Let  $n \geq 1, X_1, \dots, X_n$  be an  $n$ -sample of law  $\mathbb{P}_\theta$ , and let  $\hat{f}$  be an estimator of  $f(\theta)$ . The **mean square error** of  $\hat{f}$  is

$$MSE_\theta(\hat{f}) = E \left( \left( \hat{f}(X_1, \dots, X_n) - f(\theta) \right)^2 \right).$$

*Remarque*

$$MSE_\theta(\hat{f}) = Var_\theta(\hat{f}) + \left( Bias_\theta(\hat{f}) \right)^2.$$

Minimizing the mean square error is yet another way to construct an estimator.

##### Définition 55

Let  $n \geq 1, X_1, \dots, X_n$  be an  $n$ -sample of law  $\mathbb{P}_\theta$ , and  $f(\theta)$  be a quantity to estimate. The **minimal mean square error estimator** of  $f(\theta)$  is

$$MMSE_{f(\theta)}(X_1, \dots, X_n) = \operatorname{argmin}_{\hat{f}} E \left( \left( \hat{f}(X_1, \dots, X_n) - f(\theta) \right)^2 \right).$$

where the minimum is over all estimators  $\hat{f}(X_1, \dots, X_n)$  of  $f(\theta)$ . The existence of this estimator is a non-trivial fact of probability theory. It is in general not so easy to compute.

### 3.7.2 Asymptotic normality

#### Définition 56

For  $n \geq 1$ , let  $X_1, \dots, X_n$  be an n-sample of law  $\mathbb{P}_\theta$ , and let  $\hat{f}_n$  be an estimator of  $f(\theta)$ . We say that  $\hat{f}_n$  is an **asymptotically normal** sequence of estimators if

1.  $\hat{f}_n$  is convergent ;
2. there is  $C \geq 0$  such that  $\sqrt{n} \left( \hat{f}_n - f(\theta) \right) \xrightarrow{\text{Law}} \mathcal{N}(0, C)$  as  $n \rightarrow \infty$ .