Summary of introduction to information processing

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Table des matières

1 Linear algebra in Dirac notation

- A **Hilbert space** is a vector space on complex numbers with an inner product stucture.
- $\mathcal{H},\ dim\mathcal{H} = d,$ colon vectors : $\vec{\psi} = \begin{pmatrix} \psi_1 \\ \dots \\ \psi_d \end{pmatrix},\ \psi \in \mathbb{C},$ line vectors : $\vec{\psi}^{T,*} = (\psi_1^*, \psi_2^*, \dots, \psi_d^*),$

  scalar product : $\vec{\phi}^{T,*} \cdot \vec{\psi} = \sum_{i=1}^{d} \phi_i^* \psi_i$.

  (means conjugate)*

1.1 Dirac notation

Ket : $\vec{\psi} = |\psi\rangle$, Bra : $\vec{\psi}^{T,*} = \langle\psi|,\ \implies$ braket : $\vec{\phi}^{T,*}\vec{\psi} = \langle\phi|\psi\rangle$.

1.2 Propreties of Dirac

- **Linearity :** $(\alpha^* \langle\psi_1| + \beta^* \langle\psi_2|)\,|\phi\rangle = \alpha^* \langle\psi_1|\phi\rangle + \beta^* \langle\psi_2|\phi\rangle$.
- **Dirac conjugate :** $(|\phi\rangle)^{T,*} = \langle\phi|$.
- **Skew symmetry :** $\langle\phi|\psi\rangle^* = \langle\psi|\phi\rangle$.
- **Norm :** $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$ with properties :
  1. $\|\psi\| \geq 0, \|\psi\| = 0 \leftrightarrow |\psi\rangle = 0,$
  2. $\|\psi_1\| - \|\psi_2\| \leq ||\psi_1\rangle + |\psi_2\rangle| \leq \|\psi_1\| + \|\psi_2\|$,
  3. $|\langle\phi|\psi\rangle| \leq \|\phi\| \cdot \|\psi\|$.
- **Basis :**
  1. Orthonormal Basis : $\{|v_1\rangle, |v_2\rangle, \dots, |v_d\rangle\}$, then $\langle v_i|v_j\rangle = \delta_{ij} = \begin{cases} 1,\ i = j \\ 0,\ i \neq j \end{cases}$.
  2. Other basis : $\frac{|0\rangle+|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \equiv |+\rangle$, $\frac{|0\rangle-|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \equiv |-\rangle$.
- **Tensor product :**

  Linear : $(\alpha\,|0\rangle + \beta\,|1\rangle) \otimes (\gamma\,|0\rangle + \delta\,|1\rangle) = \alpha\gamma\,|0\rangle \otimes |0\rangle + \alpha\delta\,|0\rangle \otimes |1\rangle + \beta\gamma\,|1\rangle \otimes |0\rangle + \beta\delta\,|1\rangle \otimes |1\rangle$.

  Example

  Let $\mathcal{H} = \mathbb{C}^2,\ |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then

  $$|0\rangle \otimes |0\rangle = |00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ |0\rangle \otimes |1\rangle = |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

  $$|1\rangle \otimes |0\rangle = |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ |1\rangle \otimes |1\rangle = |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

- **Matrices :**
  - Dagger : $A^{T,*} = A^\dagger,\ (A\,|\psi\rangle)^{T,*} = (A\,|\psi\rangle)^\dagger = (|\psi\rangle)^\dagger A^\dagger = |\psi\rangle\, A^\dagger$.
  - Notation : Let $\{|v_1\rangle, \dots, |v_3\rangle\}$ an orthonormal basis and $A_{ij} = \langle v_i|\,A\,|v_j\rangle$ ((i,j)th element of the matrice), then $A = \sum_{i,j=1}^{d} A_{ij}\,|v_i\rangle\,\langle v_j|$.
  - Hermitian matrices : $A^\dagger = A$ (self-adjoint), i.e. $\begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix}$.
  - Unitary matrices : $U^\dagger U = UU^\dagger = \mathbb{1},\ U^\dagger = U^{-1}$, with properties :
    1. $\|U\,|\psi\rangle\| = \||\psi\rangle\|,$
    2. $\left(\langle\phi|\,U^\dagger\right)(U\,|\psi\rangle) = \langle\phi|\psi\rangle$.

2 5 principles of quantum mechanics

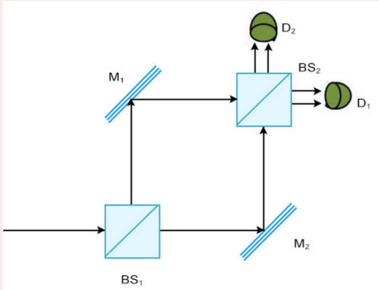
2.1 State of a system

The state of an **isolated** system is a vector $|\phi\rangle \in \mathcal{H}$ in Hilbert space $\mathcal{H}$ w.r.t the normalization condition $\langle\phi|\phi\rangle = 1$.

Remarque

$|\phi\rangle$ and $e^{i\lambda}|\phi\rangle, \lambda \in \mathbb{R}$ are physically equivalent.

Examples

1. Let $\mathcal{H} = \mathbb{C}^2 = \{\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \mid \alpha, \beta \in \mathbb{C}\}$ (qubit space), then $\mathbb{C} \implies |\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \ |\alpha|^2 + |\beta|^2 = 1 = \alpha\alpha^* + \beta\beta^*$,
2. Physical system : Mach-Zehnder interferometer : 

,
3. Let $\mathcal{H} = \mathbb{C}^d$, $\begin{pmatrix}\alpha_1 \\ \dots \\ \alpha_d\end{pmatrix} = |\psi\rangle = \sum_{i=1}^d \alpha_i |i\rangle, \ \sum_{i=1}^d (\alpha_i)^2 = 1$, (qudit space).

2.2 States evolve with time

As follows : $|\psi_t| = U_t |\psi_0\rangle$ where U_t is unitary matrix.

Example

We can desribe the transformation of the state of a particle by a prefecting reflecting mirror ($|H\rangle \to |V\rangle$ and $|V\rangle \to |H\rangle$), by

$$U = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}, \ |H\rangle = \begin{pmatrix}1 \\ 0\end{pmatrix}, \ |V\rangle = \begin{pmatrix}0 \\ 1\end{pmatrix},$$

$$\implies U(\alpha|H\rangle + \beta|V\rangle) = \beta|H\rangle + \alpha|V\rangle.$$

2.3 Observable quantities

Quantities that we measure : "observables" are given by Hermitian matrices of dimension $\mathcal{H} \cdot \mathcal{H}$

Example

Observable for 1 qubit $\left(\mathbb{C}^2\right)$:

$$A = a\mathbb{1} + bX + cY + dZ,\ A = a^{\dagger} \implies a,b,c,d \in \mathbb{R}$$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}.$$

2.4 Measurement give probability outcome

When observable A is measured, the outcome is a random eigenvalue of A, call it $\lambda_i, i = 1,\dots,d = dim\mathcal{H}$, the original state before measure $|\psi\rangle$ becomes after measurement an output state $|v_i\rangle$ with eigenvalues corresponding to λ_i. The probability distributioin associated to the output $\lambda_i, |v_i\rangle$ is $prob\,(i) = |\langle v_i|\psi\rangle|^2 \leftarrow$ **BORN RULE**.

- Results of the measurement are eigenvalues λ_i of observable A,
- Every eigenvalue is associated to an eigenvector $|v_i\rangle$,
- If the system is in the original state $|\psi\rangle$, the probability to obtain λ_i is

$$P\left(\lambda_i\right) = |\langle v_i|\psi\rangle|^2,$$

- After the measurement, the state collapse on the eigenvector : $|\psi\rangle \to |v_i\rangle$.

Théorème 1

Spectral Theorem :
Let A be an hermitian matrice $A = A^{\dagger}$ and let $A\,|v_i\rangle = \lambda_i\,|v_i\rangle\,, i = 1,\dots,d = dim\mathcal{H}$

- $\lambda_i \in \mathbb{R}$,
- $|v_1\rangle\,,\dots,|v_d\rangle$ *form an orthogonal basis,*

$$\implies A = \sum_{i=1}^{d} \lambda_i\,|v_i\rangle\,\langle v_i| = \begin{pmatrix} \lambda_1 & \dots & 0 \\ & \dots & \\ 0 & \dots & \lambda_d \end{pmatrix}.$$

- Lemma : $\sum_{i=1}^{d} |\langle v_i|\psi\rangle|^2 = 1$, because $\|\psi\| = 1$.
- Property :
  - $E\left(A\right) = \sum_{i=1}^{d} \lambda_i\,|\langle v_i|\psi\rangle|^2 = \langle\psi|\,A\,|\psi\rangle$
  - $\mathrm{Var}(A) = \langle\psi|\,A^2\,|\psi\rangle - \langle\psi|\,A\,|\psi\rangle^2$

2.5 Composite system and entanglement

The composite system of $\mathcal{H}_A$ and $\mathcal{H}_B$ is equal to $\mathcal{H}_{A\cup B} = \mathcal{H}_A \otimes \mathcal{H}_B$.

Remarque $\quad dim\left(\mathcal{H}_A \otimes \mathcal{H}_B\right) = \left(dim\mathcal{H}_A\right)\left(dim\mathcal{H}_B\right)$.

Example

$$\mathcal{H}_A = \mathbb{C}^2,\ \mathcal{H}_B = \mathbb{C}^2,\ \mathbb{C}^2 \otimes \mathbb{C}^2 = \begin{cases} |00\rangle = |0\rangle \otimes |0\rangle \\ |01\rangle = |0\rangle \otimes |1\rangle \\ |10\rangle = |1\rangle \otimes |0\rangle \\ |11\rangle = |1\rangle \otimes |1\rangle \end{cases}$$

Product states : $|\psi\rangle = |\phi_A\rangle \otimes |\chi_B\rangle$, $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$, $\phi \in \mathcal{H}_A$, $\chi \in \mathcal{H}_B$.

Entangled states : $\nexists$ a factorisation of the state.

Example

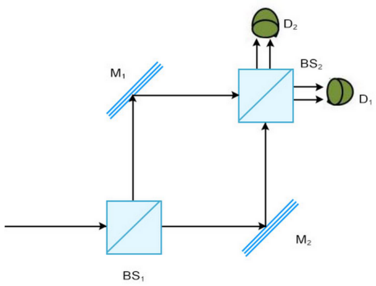
$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \neq (\alpha |0\rangle + \beta |1\rangle) \otimes (\gamma |0\rangle + \delta |1\rangle).$$

Aparté

Bloch sphere of a qubit state vector :
$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$.

3 Applications of principles

3.1 Mach-Zehnder interferometer



- **Space :** $\mathcal{H} = \mathbb{C}^2 = \{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \alpha\beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1\}$.
- **Basis :** $\begin{pmatrix}1\\0\end{pmatrix} = |H\rangle$, $\begin{pmatrix}0\\1\end{pmatrix} = |V\rangle$, $\alpha |H\rangle + \beta |V\rangle = |\psi\rangle$.
- **Beam splitter :** $U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ (Hadamard matrix).
- **State after the beam splitter :** $U|H\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$.

Remarque

It's a very srange state that is at the same time horizontal and vertical. If it wasn't a qubit, it would either be completely reflected or it would go through.

- **Perfect mirror :** $R = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $RR^\dagger = R^\dagger R = 1$ (**important condition !**) , $R^\dagger = R$ (coincidence).
- **State after the perfect mirror :** $RU|H\rangle = \frac{1}{\sqrt{2}}(R|H\rangle + R|V\rangle) = \frac{1}{\sqrt{2}}(|V\rangle + |H\rangle)$.
- **State after the second beam splitter :** $URU|H\rangle = \frac{1}{\sqrt{2}}(U|V\rangle + U|H\rangle)$.
- **State before the detector :** $\psi_{\text{before detector}} = 2 \cdot \frac{1}{2}|H\rangle = |H\rangle$.
- **Measurement :** model with orthogonal basis of $\mathcal{H} = \mathbb{C}^2 = \{|H\rangle \text{ and } |V\rangle\}$:
  At the end of the interferometer, we measure state $|H\rangle$ or state $|V\rangle$,
  if we measure $|H\rangle \implies$ clic in $D_1 \implies$ register $+1$,
  if we measure $|V\rangle \implies$ clic in $D_1 \implies$ register -1.
  By the Born rule :

$$prob\,(+1) = |\langle H|\psi_{\text{before detector}}\rangle|^2 = 1$$
$$prob\,(-1) = |\langle V|\psi_{\text{before detector}}\rangle|^2 = 0$$

If we follow the 4^{th} *principle*, then we know that $|H\rangle$ and $|V\rangle$ are the eigenvectors of our observable and $(+1)$ and (-1) are the eigenvalues of our observable.

By spectral theorem, we define our observable : $(+1)\,|H\rangle\,\langle H| + (-1)\,|V\rangle\,\langle V| = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} = Z$.

Expectated value of (Z) : $\langle \psi_{\text{before detector}}|\, Z\,|\psi_{\text{before detector}}\rangle = \langle H|\,\{(+1)\,|H\rangle\,\langle H| + (-1)\,|V\rangle\,\langle V|\}\,|H\rangle = (+1)\,\langle H|H\rangle\,\langle H|H\rangle + (-1)\,\langle H|V\rangle\,\langle H|V\rangle = (+1)$.
Var(Z) : $\langle \psi|\, Z^2\,|\psi\rangle - \langle \psi|\, Z\,|\psi\rangle^2 = 0$.