Sunday, 29 June 2025 14:49

1) If P is a projection matrix, then the eigenvalue corresponding to every nonzero vector orthogonal to the column space of P is

- (selected) 0
- 1
- -1

→ Remember:

Projection of a vector which is already in column space (plane)

$Px = x$; fact

Imagine a plane, a vector x is already on plane, so projecting it again will end up at same place

Projection is more like projecting on subspace, what if we are already in subspace, even if we project we end up on place, on the plane up in place on the plane

that is $Px = x$

Now our question is telling eigenvalue of all orthogonal (perpendicular) to column space;

We know $Px = 0$ for orthogonality

we also know x is non-zero vector

so $Px = \lambda x$

If x is non zero then λ must be 0

so we get $Px = 0$

That means 0 is eigenvalue of vectors which are orthogonal to column space of P

2) Consider the following statements regarding a real symmetric matrix A **_1 point_**

The eigenvalues of A are always real.
The eigenvalues of A may be imaginary.
The eigenvectors corresponding to different eigenvalues of A are linearly independent.
The eigenvectors corresponding to different eigenvalues of A are not linearly independent.
A is orthogonally diagonalizable.
A is not diagonalizable.

Which of the above statements are true?

- ○ 2, 3 and 5
- ○ 1, 4 and 5
- ○ 2, 4 and 6
- ✓ 1, 3 and 5

→ Remember these are straight-forward facts: proof is really big watch lecture

3) The eigenvectors corresponding to distinct eigenvalues of a matrix **_1 point_**

- ✓ are linearly independent
- ☐ are linearly dependent
- ☐ have no relation

Fact.

4) The determinant of a 3×3 matrix having eigenvalues 1, -2 and 3 is **_1 point_**

- ○ 2
- ○ 0
- ○ 6
- ✓ -6
- ○ -2

→

Determinant = product of eigenvalues

$= \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

$= -6 \therefore$ Ans

5) The trace of a 2×2 matrix is -1 and its determinant is -6. Its eigenvalues will be **1 point**

- -1, 3
- 2, 3
- 2, -3
- -2, 3

→ Trace of (A) = sum of eigenvalues

Given $\det(A) = -6$

we know $\lambda_1 \cdot \lambda_2 = -6$

$\lambda_1 + \lambda_2 = -1$

check options:

i) $(-1, 3) \Rightarrow -1 \times 3 \neq -6$

ii) $(2, 3) \Rightarrow 2 \times 3 \neq -6$

iii) $(2, -3) \Rightarrow 2 \times -3 = -6$ ✓ Det (A)

$2 + (-3) = -1$ ✓ Trace (A) } (Ans)

iv) $(-2, 3) \Rightarrow -2 \times 3 = -6$ ✓ Det (A)

$-2 + 3 = 1$ ✗ Trace (A)

6) If the eigenvalues of a matrix are -1, 0 and 4 are

Find the Trace?

1 point

7) Find the Determinant?

1 point

→ eigenvalues = $\{-1, 0, 4\}$

we know

Trace (A) = Sum of eigenvalues

Trace = 3 (∴ Ans)

Det (A) = product of eigenvalues

Det(A) → 0 (∴ Ans)

8) The characteristic polynomial for the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ is

1 point

- $\lambda^2 - 4\lambda + 1$
- $\lambda^2 - 4\lambda$
- $\lambda^2 - 4\lambda - 2$
- $\lambda^2 + 4\lambda + 2$
- $\lambda^2 - 4\lambda + 2$

→ $A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

$\rightarrow A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

Remember this formula :

$$\det(A - \lambda I) = 0$$

$$\left| \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda) - 1 = 0$$

$$\Rightarrow 3 - \lambda - 3\lambda + \lambda^2 - 1 = 0$$

$\Rightarrow \lambda^2 - 4\lambda + 2 = 0$ ($\therefore$ Ans)

9) The eigenvalues of matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ are

1 point

- ✓ $2+\sqrt{3}, 2-\sqrt{3}$
- $\sqrt{3}, -\sqrt{3}$
- 0,1
- $\sqrt{5}, -\sqrt{5}$

$\rightarrow$ Formula $\det(A - \lambda I) = 0$

$$\left| \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{bmatrix} 1-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)(3-\lambda) - 2 = 0$$

$\lambda^2 - 4\lambda + 1 = 0$, solve this quad eq^n

$$\Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \frac{4 \pm \sqrt{12}}{2} \Rightarrow \frac{4 \pm \sqrt{2 \times 2 \times 3}}{2}$$

$$\Rightarrow \frac{4 \pm 2\sqrt{3}}{2} \Rightarrow \frac{\cancel{2}(2+\sqrt{3})}{\cancel{2}} \text{ or } \frac{\cancel{2}(2-\sqrt{5})}{\cancel{2}}$$

$\lambda_1 = 2+\sqrt{3}$, $\lambda_2 = 2-\sqrt{3}$ ($\therefore$ Ans)

10) If the eigenvalues of a matrix A are 0 -1 and 5, then the eigenvalues of A^3 are **2 points**

- [ ] 0, -1 and 5
- [x] 0, -1 and 125
- [ ] 0, 1 and -125
- [ ] 0, 1 and -5

$\rightarrow$

$A^k u_0 = \Lambda^k S c$

In short remember

If power of A increases by k,
the power of eigenvalue also increases by k

$A^3 \Rightarrow \lambda^3 \Rightarrow \{0, -1, 125\}$ ($\therefore$ Ans)

11) The 110th term of Fibonacci sequence is approximately given by **1 point**

- ✓ $\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{110}$
- $\frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{110}$
- $\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{-110}$
- $\frac{-1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{110}$

→ Watch lecture of this proof

$A^k u_0 = \Lambda^{100} S c$

12) Let A be an $n \times n$ matrix. Which of the following statements is/are false? **1 point**

- ✓ If A has r non-zero eigenvalues, then rank of A is at least r.
- If one of the eigenvalues of A are zero, then $|A| \neq 0$.
- ✓ If x is an eigenvector of A, then so are all the multiples of x.
- ✓ If 0 is an eigenvalue of A, then A cannot be inverted.

Facts.

13) The eigenvalues of the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ are **2 points**

- ✓ 0
- 2
- 3
- ✓ 5

→ $\det(A - \lambda I) = 0$

$\Rightarrow \left| \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(4-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 4 - 4 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda = 0$$

$$\Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \frac{5 \pm \sqrt{25 - 0}}{2} \Rightarrow \frac{5+5}{2} \Rightarrow \frac{5-5}{2}$$

$$\lambda_1 = 5 \quad , \quad \lambda_2 = 0 \quad (\therefore \text{Ans})$$

14) For the matrix given in the previous question, which of the following vectors is/are its eigenvector(s)? **2 points**

- ✔ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
- ✔ $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
- ☐ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- ☐ $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

→ We previously found $\lambda_1 = 5$, $\lambda_2 = 0$

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$$\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$$ let's put $\lambda_1 = 5$

$$\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -4x_1 + 2x_2 = 0$

$\Rightarrow \quad x_2 = \dfrac{4x_1}{2} \Rightarrow x_2 \Rightarrow 2x_1$

$\Rightarrow \quad \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Substitute any real number in $\begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix}$

You get vector - eigen

You get vector - eigen

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is first vector.

Now substitute $\lambda_2 = 0$ in

$$\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0$$

$$x_1 = -2x_2 \Rightarrow \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix}$$

$= \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ second eigenvector

$\Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ ($\therefore$ Ans)

15) Suppose that A, P are 3 × 3 matrices, and P is an invertible matrix. **3 points**

If $P^{-1}AP = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 3 & 8 \\ 0 & 0 & 4 \end{bmatrix}$, then the eigenvalues of the matrix A^2 are

- [x] 1
- [ ] 12
- [x] 9
- [x] 16

→ Recall the property

$AS = S\Lambda$ so $S^{-1}AS = \Lambda$

watch gilbert Strang lecture 22 for this

$P^{-1}AP = \Lambda \Rightarrow \lambda$ matrix

For upper triangular matrix eigenvalues are diagonal elements

so λ's $= \{-1, 3, 4\}$

we already know from previous problems $A^k \Rightarrow$ implies λ^k

problems ... implies ...

so eigenvalues of $A^2 = \lambda^2$

$= \{1, 9, 16\}$ (∴ Ans)

16) The best second degree polynomial that fits the data set **2 points**

x	y
0	0
1.3	1.5
4	1.2

is

- $1.35x^2 + 0.3x$
- $1.25x^2 + 0.45x$
- ✓ $-0.316x^2 + 1.56x$
- $-0.25x^2 + 0.5$

→ Option are already given instead of finding polynomial test the option:

i) $1.35x^2 + 0.3x = y$ put $(1.3, 1.5)$

$1.35 \times (1.3)^2 + 0.3 \times 1.3 \stackrel{?}{=} 1.5$

$1.35 \times 1.69 + 0.39 \stackrel{?}{=} 1.5$

$2.2815 + 0.39 \neq 1.5$

ii) $1.25x^2 + 0.45x = y$ put $(1.3, 1.5)$

$1.25 \times 1.69 + 0.45 \times 1.3 \overset{?}{=} 1.5$

$2.1125 + 0.585 \neq 1.5$

iii) $-0.316x^2 + 1.56x = y$ put $(1.3, 1.5)$

$-0.316 \times 1.69 + 1.56 \times 1.3 \overset{?}{=} 1.5$

$-0.53404 + 2.028 = 1.49 \approx 1.5$ ✓

Now put $(4, 1.2)$

$-0.316x^2 + 1.56x = y$

$-0.316 \times 16 + 1.56 \times 4 \overset{?}{=} 1.2$

$-5.056 + 6.24 = 1.18 \approx 1.2$ ✓ ($\therefore$ Ans)

Option 3 is correct