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Week - 3

1) The length of the vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is

1 point

- 2.342
- 2.308
- 2.440
- ✓ 2.449

→ Length of vector $= \sqrt{(a_1)^2 + (a_2)^2 + \dots + (a_n)^2}$

$= \sqrt{(1)^2 + (2)^2 + (-1)^2} \Rightarrow \sqrt{6}$

$= 2.449$ ($\therefore$ Ans)

Question - 2:

2) The inner product of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$ is

1 point

- 11
- 12
- 14
- ✓ 16

→ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} \Rightarrow a \cdot b = a_1 b_1 + a_2 b_2 + \dots a_n b_n$

$\Rightarrow (1 \times -1) + (2 \times 1) + (3 \times 5)$

$\Rightarrow -1 + 2 + 15 \Rightarrow 16$ ($\therefore$ Ans)

Question - 3.

3) The rank of a 4×3 matrix is 1, what is the dimension of its null space?

1 point

- 3
- 1
- ✓ 2
- 4

→ Rank - Nullity Theorem

$R(A) + N(A) = n$

Given $A = m \times n = 4 \times 3$

$n = 3$

Rank is given 1 ; so $n - \text{Rank}(A) = \text{Nullity}(A)$

$\Rightarrow 3 - 1 = 2$ (∴ Ans)

Question - 4.

4) Which of the following vector is orthogonal to $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$?

1 point

- [-1 1 -3]
- [1 2 1]
- [-1 1 -3]
- ✓ [-3 0 1]

→ Orthogonality test : $X^T Y = 0$;

Orthogonality Test : $X'Y = 0$;

Then x is orthogonal to Y.

Option 1: $[-1, 1, 3]$

$$[-1, 1, 3] \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \Rightarrow -1 - 1 + 9 \neq 0$$

So not orthogonal

Option 2: $[1, 2, 1]$

$$[1, 2, 1] \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \Rightarrow 1 - 2 + 3 \neq 0$$

so not orthogonal

Option 3: $[-1, 1, -3]$

$$[-1, 1, -3] \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \Rightarrow -1 - 1 - 9 \neq 0$$

So not orthogonal

Option 4: $[-3, 0, 1]$

$$[-3, 0, 1] \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \Rightarrow -3 + 0 + 3 = 0 \checkmark$$

Orthogonal

$\begin{bmatrix} 3 \end{bmatrix}$

Orthogonal

Question - 5.

5) The rank of the following matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ is

2 points

- [x] 1
- [ ] 2
- [ ] 3
- [ ] 4

→ Rank of Matrix ⇒ No. of independent columns

When a column is independent?

If the pivot point is non zero i.e. 1 after row reduction

What is pivot points?

Simply the diagonal of the matrix

Given = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$; Use Row Reduction Method

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\Big\downarrow R_3 \leftarrow R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As you can see diagonal points are zero except first diagonal point so Rank is 1 ($\therefore$ Ans)

Question - 6.

6) Which of the following would be the smallest subspace containing the first quadrant of the space $\mathbb{R}^2$? *1 point*

- The first quadrant
- The first and the third quadrant
- The first and second quadrant
- ✓ The whole space $\mathbb{R}^2$

→ A subspace must be closed under

→ A subspace must be closed under multiplication by all scalars including negative numbers.

If we multiply any vector from 1st Quadrant by any negative number, it sends us to third quadrant

Same under addition with negative vector we might end up in 2nd or 4th Quadrant

So basically no single quadrant can be subspace, that why whole $\mathbb{R}^2$ can only be a subspace.

Question - 7.

7) 5 peaches and 6 oranges cost 150 rupees. 10 peaches and 12 oranges cost 300 rupees. Form a matrix out of the given information and find its rank. **2 points**

- Rank = 2
- ✓ Rank = 1
- Rank = 0
- Rank = 4

→ Let peaches be x_1 & oranges be x_2

$5x_1 + 6x_2 = 150$ eqn ①

$10x_1 + 12x_2 = 300$

$$\left[\begin{array}{cc|c} 5 & 6 & 150 \\ 10 & 12 & 300 \end{array}\right]$$; Use Row reduction to find independent columns

Independent column = non zero diagonal points

$$\begin{bmatrix} 5 & 6 \\ 10 & 12 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 5 & 6 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1/5} \begin{bmatrix} \underline{1} & 6/5 \\ 0 & 0 \end{bmatrix}$$

Rank = 1 because only 1 non zero diagonal point

Question - 8.

8) Consider a set of 3 paired observations on $(x_i, b_i), i = 1,2,3$ as $((1,6),(-1,3),(3,15))$. For the closest line b to go through these points, which of the following is the least squares solution $(\hat{\theta})$? **2 points**

- (3,4)
- (4,3)
- (5,3)
- ✓ (3,5)

$\rightarrow (x_1, b_1) = (1,6)$

$(x_2, b_2) = (-1,3)$

$(x_3, b_3) = (3,15)$

We have to find a line which has least square solution.

$y = mx + c$, to find this line

Let's Assume each point has its own line

$b = \begin{bmatrix} 6 \\ 3 \\ 15 \end{bmatrix}$

$(x_1, b_1) \Rightarrow m + c = 6$

$(x_2, b_2) \Rightarrow -m + c = 3$

$(x_3, b_3) \Rightarrow 3m + c = 15$

$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 3 & 1 \end{bmatrix} = A$

$(x_3, b_3) \Rightarrow 3m + c = 15$ $\quad \begin{bmatrix} 3 & 1 \end{bmatrix}$

$3 \times 2 \quad m > n$

You can observe we have 3 equations and 2 unknowns

we formed matrix A with these eqn above ↑

if $m > n$ then no perfect solution exists

So we need to find a non perfect solution with minimum error

To find best fit line remember this formula $\quad A^T A \hat{x} = A^T b$

$$A^T = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 3 & 1 \end{bmatrix} \quad \hat{x} = \begin{bmatrix} m \\ c \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 11 & 3 \\ 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 3 \\ 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 48 \\ 24 \end{bmatrix}$$

$$A \quad x = b$$

Let convert this to agumented Matrix $[A|b]$

$$\left[\begin{array}{cc|c} 11 & 3 & 48 \\ 3 & 3 & 24 \end{array}\right] \longrightarrow \text{Do row reduction}$$

$$\Big\downarrow R_2/3$$

$$\left[\begin{array}{cc|c} 11 & 3 & 48 \\ 1 & 1 & 8 \end{array}\right] \xrightarrow{\text{Swap}} \left[\begin{array}{cc|c} 1 & 1 & 8 \\ 11 & 3 & 48 \end{array}\right]$$

$$\Big\downarrow R_2 \leftarrow R_2 - 11R_1$$

$$\left[\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & 1 & 5 \end{array}\right] \xleftarrow{R_2/-8} \left[\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & -8 & -40 \end{array}\right]$$

So we can say $m + c = 8$ & $c = 5$

$c = 5$, the $m = 8 - 5 \Rightarrow 3$

$m = 3$ & $c = 5$ $\Rightarrow$ $y = 3\hat{x} + 5$ → Best fit line

$(3, 5)$ ($\therefore$ Ans)

Question - 9

9) Which of the following represents the null space of the matrix $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix}$ ***3 points***

- ○ Span $\left\{ \begin{bmatrix} -9 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$
- ○ Span $\left\{ \begin{bmatrix} 9 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$
- ○ Span $\left\{ \begin{bmatrix} 9 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$
- ○ Span $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

→ What is null space of Matrix A ?

All those vectors x, when $Ax = 0$

Set of x vectors such that $Ax = 0$

$$\underset{3 \times 4}{\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix}} \underset{4 \times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let's do row reduction

$$\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix} \xrightarrow{R_1/2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix}$$

$$\downarrow R_2 - R_1$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & 3 & -1 \end{bmatrix} \xleftarrow{R_3 - R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -3 & 1 \\ 1 & 1 & 6 & 3 \end{bmatrix}$$

$$\downarrow R_3 + R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Identify pivot variables

x_1 & x_2

free variable x_3 & x_4

Express pivot variable in terms of free variable

$x_2 - 3x_3 + x_4 \Rightarrow x_2 = 3x_3 - x_4$

$x_1 + 2(3x_3 - x_4) + 3x_3 + 4x_4 = 0$

$$x_1 + 6x_3 - 2x_4 + 3x_3 + 4x_4 = 0$$

$$x_1 + 9x_3 + 2x_4 = 0$$

$$x_1 = -9x_3 - 2x_4$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 - 2x_4 \\ 3x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} -9x_3 - 2x_4 \\ 3x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} -9 \\ 3 \\ 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad (\therefore \text{Ans})$$

Question -10.

10) Which of the two vectors are orthogonal to each other? **1 point**

- [1 2 3],[-1 -2 3]
- [1 2 1],[0 -1 2]
- [1 2 5],[1 2 3]
- [1 2 3],[2 4 6]
- [1 2 1],[-1 0 -1]

→ Two vector are orthogonal if $X^T Y = 0$

$$[1\ 2\ 1] \cdot \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \Rightarrow -2 + 2 = 0$$

Second option is correct

Question - 11

11) Find projection of [5,-4,1] along [3,-2,4] **1 point**

- $\begin{bmatrix} \frac{27}{29} & \frac{-18}{29} & \frac{36}{29} \end{bmatrix}$
- $\begin{bmatrix} \frac{27}{29} & \frac{18}{29} & \frac{36}{29} \end{bmatrix}$
- ✓ $\begin{bmatrix} \frac{81}{29} & \frac{-54}{29} & \frac{108}{29} \end{bmatrix}$
- $\begin{bmatrix} \frac{81}{29} & \frac{54}{29} & \frac{108}{29} \end{bmatrix}$

→ $$\text{proj}_u v = \left(\frac{v \cdot u}{u \cdot u} \right) \cdot u$$

$$u = [3, -2, 4] , \quad v = [5, -4, 1]$$

$$= \frac{[5, -4, 1] \cdot [3, -2, 4]}{[3, -2, 4] \cdot [3, -2, 4]} \cdot u$$

$$\Rightarrow \quad \frac{15+8+4}{9+4+16} \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

$$\Rightarrow \quad \frac{27}{29} \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

$$\left[\frac{81}{29}, -\frac{54}{29}, \frac{108}{29} \right] \quad (\therefore \text{Ans})$$

Question-12.

12) **2 points**

The projection matrix for the matrix $v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ is

- $\frac{1}{14}\begin{bmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 9 \end{bmatrix}$
- $\frac{1}{14}\begin{bmatrix} 4 & 3 & 6 \\ 2 & 2 & 3 \\ 9 & 3 & 9 \end{bmatrix}$
- $\frac{1}{14}\begin{bmatrix} 4 & -2 & 6 \\ 3 & 1 & 3 \\ 6 & 6 & 9 \end{bmatrix}$
- $\frac{1}{14}\begin{bmatrix} 2 & 2 & -6 \\ 2 & -1 & 5 \\ 5 & 7 & 9 \end{bmatrix}$

$\rightarrow$ Projection matrix $= \dfrac{aa^T}{a^Ta}$

$$a = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad \Rightarrow \quad \frac{\underset{3\times 1}{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} 2 & 1 & 3 \end{bmatrix}}}{\begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}}$$

$$\Rightarrow \frac{\begin{bmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 9 \end{bmatrix}}{14} \Rightarrow \frac{1}{14}\begin{bmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 9 \end{bmatrix} \quad (\therefore \text{Ans})$$

Question - 13.

13) Find projection of [2,-4,4] along [2,-2,1] **1 point**

- $\begin{bmatrix} \frac{-32}{9} & \frac{-32}{9} & \frac{16}{9} \end{bmatrix}$
- $\begin{bmatrix} \frac{32}{9} & \frac{-32}{9} & \frac{16}{9} \end{bmatrix}$
- $\begin{bmatrix} \frac{32}{9} & \frac{16}{9} & \frac{-32}{9} \end{bmatrix}$
- $\begin{bmatrix} \frac{32}{9} & \frac{-16}{9} & \frac{32}{9} \end{bmatrix}$

$\rightarrow$ $\text{Projection}_u v = \left(\frac{v \cdot u}{u \cdot u}\right) \cdot u$

$$= \frac{[2,-4,4]\,[2,-2,1]}{[2,-2,1]\,[2,-2,1]} \cdot u$$

$$= \frac{4+8+4}{4+4+1} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$= \frac{16}{9}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$\Rightarrow \left[\frac{32}{9}, \frac{-32}{9}, 16/9 \right]$ ($\because$ Ans)