Sunday, 25 May 2025 22:20

1) Which of the following functions is/are continuous? **1 point**

- [ ] $\frac{1}{x-1}$
- [ ] $\frac{x^2-1}{x-1}$
- [ ] $sign(x-2)$
- [x] $sin(x)$

→ $\frac{1}{x-1} \neq$ continuous , $x=1$ = $1/0$ undefined

$\frac{x^2-1}{x-1} \neq$ continuous , $x=1$ $0/0$ undefined

$sign(x-2) \neq$ continuous , $x=2$ sing(0) not defined

$sin(x)$ = continous & well as differentiable

2) Regarding a d-dimensional vector x, which of the following four options is not equivalent to the rest three options? **1 point**

- ( ) $x^T x$
- ( ) $||x||^2$
- ( ) $\sum_{i=1}^{d} x_i^2$
- (✓) $x x^T$

→ Let take a vector $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$x^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

i) $x^T \cdot x \Rightarrow \underset{1\times3}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} \cdot \underset{3\times1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$

$Z = A_{m\times n} * B_{n\times m}$, Z order $= m \times m \Rightarrow 1\times1$

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Let's multiply:

$$[1 \ 2 \ 3] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow (1 \times 1) + (2 \times 2) + (3 \times 3) \Rightarrow [15] = Z_{1 \times 1}$

ii) $||x||^2 =$

$\Rightarrow \left(\sqrt{a^2 + b^2 + c^2}\right)^2$

$\Rightarrow \left(\sqrt{1^2 + 2^2 + 3^2}\right)^2 \Rightarrow \underline{\underline{15}}$ ($\therefore$ Ans)

iii) $\sum_{i=1}^{d} (x_i)^2$

$\Rightarrow (1)^2 + (2)^2 + (3)^2 = 15$ ($\therefore$ Ans)

iv) $x \cdot x^T =$

$$\underset{3 \times 1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \underset{1 \times 3}{[1 \ 2 \ 3]}$$

$x_{order} = 3 \times 3 = \begin{bmatrix} x_1x_1 & x_1x_2 & x_1x_3 \\ - \; - \; - & & - \; - \; - \\ - \; - \; - & & - \; - \; - \end{bmatrix} \neq 15$

So, xx^T is not equivalent to rest three

3) Consider the following function:

1 point

$$f(x) = \begin{cases} 3x+3, & \text{if } x \geq 3 \\ 2x+8, & \text{if } x < 3 \end{cases}$$

Which of the following is/are true?

- [ ] $f(x)$ is continuous at $x = 3$.
- [x] $f(x)$ is not continuous at $x = 3$.
- [ ] $f(x)$ is differentiable at $x = 3$.
- [x] $f(x)$ is not differentiable at $x = 3$.

→ Let's check continuity of $f(x)$

If derivative of $f(x)$ exists then it is continue

i) $3x+3$

$\frac{d}{dx} 3x+3 \Rightarrow \underline{\underline{3}}$

ii) $2x+8$

$\frac{d}{dx} 2x+8 \Rightarrow \underline{\underline{2}}$

Not equal so not continuous

If not continuous so not differentiable at 3

4) Approximate the value of $e^{0.011}$ by linearizing e^x around x=0.

1.011

1 point

$\rightarrow f(x) = f(x^*) + f'(x^*)(x - x^*)$

$x^* = 0$, because mentioned in question around $x = 0$

$f(x) = f(0) + f'(0)(x - 0)$

$= e^0 + e^0 (x - 0)$

$f(x) \Rightarrow (1 + x)$

$e^{0.011} \rightarrow$ 0.011 is close to 0

so $\Rightarrow (1 + 0.011) \Rightarrow 1.011$ ($\therefore$ Ans)

5) Approximate $\sqrt{3.9}$ by linearizing $\sqrt{x}$ around $x = 4$.

1.975

1 point

$\rightarrow f(x) = f(x^*) + f'(x^*)(x - x^*)$

Given around 4, so $x^* = 4$

$f(x) = f(4) + f'(4)(x - 4)$

$$= \sqrt{x^*} + \frac{d}{dx}\sqrt{x^*} \times (x - x^*)$$

$$\frac{d}{dx}\sqrt{x^*} = x^{*\,1/2} \Rightarrow \frac{1}{2} x^{*\,1/2 - 1}$$

$$\Rightarrow \frac{1}{2} x^{*\,-1/2} \Rightarrow \frac{1}{2\sqrt{x^*}}$$

$$\Rightarrow \sqrt{x^*} + \frac{1}{2\sqrt{x^*}} \cdot (x - x^*)$$

$$\Rightarrow 2 + \frac{1}{4}(x-4)$$

$$\Rightarrow 2 + \frac{x-4}{4} \Rightarrow \frac{8+x-4}{4} \Rightarrow \frac{x+4}{4}$$

We can see 3.9 is close to 4

$$\text{so} = \frac{3.9+4}{4} \Rightarrow \frac{7.9}{4} \Rightarrow 1.975$$

6) Which of the following pairs of vectors are perpendicular to each other? ***1 point***

- ☐ [2, 3, 5] and [-2, 3, -1]
- ☐ [1, 0, 1] and [0, 1, 1]
- ☐ [2, 3, 5] and [-2, 3, -5]
- ☑ [0, 1, 0] and [0, 0, 1]
- ☐ [2, -3, 5] and [-2, 3, -5]
- ☑ [1, 0, 0] and [0, 1, 0]

→ Two vectors are perpendicular when dot product

→ Two vectors are perpendicular when dot product of two vectors is zero

$$[a, b, c] \cdot [e, f, g] \Rightarrow [h]$$

$\quad 1\times3 \qquad\qquad 3\times1 \qquad\qquad 1\times1$

If $h=0$, then pair is perpendicular

$$[2,3,5] \cdot [-2,-3,-1] = -4-9-5 \Rightarrow -18 \neq 0$$

$$[1,0,1] \cdot [0,1,1] = 0+0+1 \Rightarrow 1 \neq 0$$

$$[2,3,5] \cdot [-2,3,-5] = -4+9-25 \Rightarrow -20 \neq 0$$

$$[0,1,0] \cdot [0,0,1] = 0+0+0 = 0 \checkmark$$

$$[2,-3,5] \cdot [-2,3,-5] \Rightarrow -4-9-25 \neq 0$$

$$[1,0,0] \cdot [0,1,0] = 0+0+0 = 0 \checkmark$$

7) What is the linear approximation of $f(x, y) = x^3 + y^3$ around (2, 2)? ***1 point***

- ◯ $4x + 4y - 8$
- ✓ $12x + 12y - 32$
- ◯ $12x + 4y - 8$
- ◯ $12x + 12y + 32$

→ Linear Approximation of multivariate

$$f(y_1, y_2) = f(v_1, v_2) + \nabla f(v)^T (y - v)$$

$$f(v_1, v_2) = x + y^3 = 8 + 8 \Rightarrow \underline{\underline{16}}$$

$$\nabla f(v)^T = \begin{bmatrix} \frac{df}{\delta x} \\ \frac{df}{dy} \end{bmatrix} \Rightarrow \frac{\delta f}{\delta x} x^3 + y^3 \Rightarrow 2x^2$$

$$\frac{\delta f}{\delta y} x^3 + y^3 \Rightarrow 2y^2$$

$$\nabla f(v) = \begin{bmatrix} 3x^2 \\ 3y^2 \end{bmatrix}^T \Rightarrow \begin{bmatrix} 3x^2 & 3y^2 \end{bmatrix}$$

$$(y - v) = \begin{bmatrix} x - 2 \\ y - 2 \end{bmatrix}$$

Linear approx $= f(v_1, v_2) + \nabla f(v)^T (y - v)$

$$\Rightarrow (x^3 + y^3) + \begin{bmatrix} 12 & 12 \end{bmatrix} \begin{bmatrix} x - 2 \\ y - 2 \end{bmatrix}$$

$$\Rightarrow 16 + (12(x-2) + 12(y-2))$$

$\Rightarrow 16 + 12x - 24 + 12y - 24$

$\Rightarrow 12x + 12y - 32$ (∴ Ans)

8) What is the gradient of $f(x, y) = x^3y^2$ at (1, 2)? *1 point*

- [x] [12, 4]
- [ ] [4, 12]
- [ ] [1, 4]
- [ ] [4, 1]

$\rightarrow \nabla f(v) = \begin{bmatrix} \frac{\delta f}{\delta x} \\ \frac{\delta f}{\delta y} \end{bmatrix} \Rightarrow \begin{bmatrix} 3x^2 y^2 \\ x^3 . 2y \end{bmatrix}$ around (1, 2)

$\Rightarrow \begin{bmatrix} 12 \\ 4 \end{bmatrix} \Rightarrow$ (∴ Ans)

9) The gradient of $f = x^3 + y^2 + z^3$ at $x = 0, y = 1$ and $z = 1$ is given by *1 point*

- [ ] [1, 2, 3]
- [ ] [-1, 2, 3]
- [ ] [0, 2, 3]
- [ ] [2, 0, 3]

$\rightarrow \nabla f = \begin{bmatrix} \frac{df}{dx} & \frac{\delta f}{\delta y} & \frac{\delta f}{\delta z} \end{bmatrix}$

$= \begin{bmatrix} 3x^2 & 2y & 3z^2 \end{bmatrix}$ (0, 1, 1)

$\Rightarrow \begin{bmatrix} 0 & 2 & 3 \end{bmatrix}$ (∴ Ans)

$$\Rightarrow \begin{bmatrix} 0 & 2 & 3 \end{bmatrix} \quad \therefore \text{Ans}$$

10) For two vectors **a** and **b**, which of the following is true as per Cauchy-Schwarz inequality? *1 point*

(i) $a^Tb \leq ||a|| * ||b||$
(ii) $a^Tb \geq -||a|| * ||b||$
(iii) $a^Tb \geq ||a|| * ||b||$
(iv) $a^Tb \leq -||a|| * ||b||$

- ○ (i) only
- ○ (ii) only
- ○ (iii) only
- ○ (iv) only
- ✓ (i) and (ii)
- ○ (iii) and (iv)

→ Cauchy - Schwarz Inequality

→ $-||a|| * ||b|| \leq a^Tb \leq ||a|| * ||b||$

11) The directional derivative of $f(x, y, z) = x^3 + y^2 + z^3$ at (1, 1, 1) in the direction of unit vector along $v = [1, -2, 1]$ is (correct upto three decimal places)

0.8164

1 point

→ $D_u[f](v) = \nabla f(v)^T \cdot u$

$$\nabla f(v) = \begin{bmatrix} \frac{\delta f}{\delta x} & \frac{\delta f}{\delta y} & \frac{\delta f}{\delta z} \end{bmatrix}$$

$$= \begin{bmatrix} 3x^2 & 2y & 3z^2 \end{bmatrix}$$

given in Question

Unit vector $= \sqrt{x^2 + y^2 + z^2} = [1, -2, 1]$

$= \sqrt{(1)^2 + (-2)^2 + (1)^2}$

$$= \sqrt{(1)^2 + (-2)^2 + (1)^2}$$

$$\Rightarrow \sqrt{6}$$

Unit vector $\Rightarrow \dfrac{\text{each element}}{\text{magnitude}}$

$$= \left[\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right]$$

$\nabla f(v)$ at $(1,1,1) \Rightarrow [3x^2 \quad 2y \quad 3z^2] = [3 \quad 2 \quad 3]$

$$\Rightarrow [3 \quad 2 \quad 3] \left[\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right]$$

$$\Rightarrow \frac{3}{\sqrt{6}} - \frac{4}{\sqrt{6}} + \frac{3}{\sqrt{6}}$$

$$\Rightarrow \frac{6-4}{\sqrt{6}} \Rightarrow \frac{2}{\sqrt{6}} \Rightarrow 0.8164$$

14) Which of the following is the equation of the line passing through $[7, 8, 6]$ in the direction of vector $[1, 2, 3]$ **1 point**

- $[1, 2, 3] + \alpha[-6, -6, 3]$
- $[7, 8, 9] + \alpha[-6, -6, 3]$
- $[1, 2, 3] + \alpha[6, 6, 3]$
- $[7, 8, 6] + \alpha[6, 6, 3]$
- $[7, 8, 6] + \alpha[1, 2, 3]$
- $[1, 2, 3] + \alpha[7, 8, 6]$

→ line through $u \in \mathbb{R}^d$ along direction v

$x = u + \alpha v$

$= [7, 8, 6] + \alpha[1, 2, 3]$ (∴ Ans)

12) The direction of steepest ascent for the function $2x + y^3 + 4z$ at the point $(1, 0, 1)$ is **1 point**

- (selected) $\left[\frac{2}{\sqrt{20}}, \ 0 \ \frac{4}{\sqrt{20}},\right]$
- $\left[\frac{1}{\sqrt{29}}, \ 0 \ \frac{1}{\sqrt{29}},\right]$
- $\left[\frac{-2}{\sqrt{29}}, \ 0 \ \frac{4}{\sqrt{29}},\right]$
- $\left[\frac{2}{\sqrt{20}}, \ 0 \ \frac{-4}{\sqrt{20}},\right]$

→ We have to find the direction u

$\nabla f(v) = \begin{bmatrix} 2 & 3y^2 & 4 \end{bmatrix}$ at $(1, 0, 1)$

$= \begin{bmatrix} 2 & 0 & 4 \end{bmatrix}$

To find unit vector in gradient direction to get steepest ascent

$\|\nabla f\| \Rightarrow \sqrt{2^2+0^2+4^2} \Rightarrow \sqrt{20}$

To find unit vector divide each component by $\sqrt{20}$

$$\left[\frac{2}{\sqrt{20}}, 0, \frac{4}{\sqrt{20}}\right] \quad (\therefore \text{Ans})$$

13) The directional derivative of $f(x, y, z) = x + y + z$ at $(-1, 1, 0)$ in the direction of unit vector along [1, -1, 1] is (correct upto three decimal places)

0.577

1 point

$\rightarrow$ Direction derivative $= \nabla f(v)^T \cdot u$

$$\nabla f(v) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Unit vector $= \sqrt{1^2+(-1)^2+1^2} \Rightarrow \sqrt{3}$

unit vector $= \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\Rightarrow \frac{1}{\sqrt{3}} \Rightarrow 0.577 \quad (\therefore \text{Ans})$$

$\Rightarrow \frac{1}{\sqrt{3}} \Rightarrow 0.577$ (: Ans)