

$$\textcircled{4} \quad \langle cv_1, v_2 \rangle = c \langle v_1, v_2 \rangle$$

$$\textcircled{3} \quad \langle (y_1, y_2), (x_1, x_2) \rangle = c \left( \underset{\uparrow}{y_1} x_1 - \underset{\uparrow}{y_2} x_1 - \underset{\uparrow}{y_1} x_2 + 2 \underset{\uparrow}{y_2} x_2 \right)$$

$$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 - x_2 y_1 - x_1 y_2 + 2 x_2 y_2$$

$$(x_1, x_2), (x_1, x_2)$$

$$x_1^2 - x_2 x_1 - x_1 x_2 + 2 x_2^2$$

$$x_1^2 - 2 x_1 x_2 + 2 x_2^2$$

$$\underbrace{x_1^2 - 2 x_1 x_2 + x_2^2}_{(x_1 - x_2)^2} + x_2^2$$

$$\begin{matrix} (x_1 - x_2)^2 & x_2^2 \\ > 0 & > 0 \end{matrix}$$

When  $x_1 = x_2$   $x_2 = 0$   
 $x_1 = 0$

$$\langle v_1 + v_2, v_3 \rangle$$

$$\text{LHS} \quad \langle (x_1, x_2) + (y_1, y_2), (z_1, z_2) \rangle$$

RHS

$$\langle (x_1 + y_1, x_2 + y_2), (z_1, z_2) \rangle$$

$$(x_1 + y_1) z_1 - (x_2 + y_2) z_1 - (x_1 + y_1) z_2 + 2 (x_2 + y_2) z_2$$

$$\begin{matrix} x_1 z_1 & - & x_2 z_1 & - & x_1 z_2 & + & 2 x_2 z_2 & \rightarrow & \langle (x_1, x_2), (z_1, z_2) \rangle \\ + & + & + & + & & & & & + \\ y_1 z_1 & - & y_2 z_1 & - & y_1 z_2 & + & 2 y_2 z_2 & & \langle (y_1, y_2), (z_1, z_2) \rangle \end{matrix}$$

RHS

$$(0, 0) \rightarrow 0$$

$$V = M_{2 \times 2}(\mathbb{R})$$

$$\langle A, B \rangle = \text{Tr}(AB)$$

$$\langle B, A \rangle$$

$$AB \neq BA$$

$$V = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$a, b \in \mathbb{R}$$

$$\langle A, B \rangle = \text{Tr} \left( \begin{bmatrix} \phantom{a} \\ \phantom{b} \end{bmatrix} \begin{bmatrix} \phantom{a} & \phantom{b} \end{bmatrix} \right)$$

$$\langle A, A \rangle = \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}$$

$$\begin{bmatrix} a^2 & ab \\ ba & b^2 \end{bmatrix}$$

$$\langle A, A \rangle = a^2 + b^2 > 0$$

$$= 0 \text{ if } a=b=0$$

Outer product  $\rightarrow$  Matrix output

Inner product  $\rightarrow$  Scalar output

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 11$$

$$u^T v$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix}$$

$$\text{trace} = ac + bd$$

$$\begin{bmatrix} c \\ d \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}$$

$$ca + db$$

$$k \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right)$$

$$k \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

same

$$\left[ \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \right] \begin{bmatrix} p & q \end{bmatrix}$$

$$\begin{bmatrix} a+c \\ b+d \end{bmatrix} \begin{bmatrix} p & q \end{bmatrix}$$

$$ap + pc + bq + dq$$

LHS

$$\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} p & q \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \begin{bmatrix} p & q \end{bmatrix}$$

$$ap + bq + cp + dq$$

RHS

$$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_2 + x_2 y_1$$

$$+ \rightarrow x_1 x_2 + x_2 x_1 = 2x_1 x_2$$

$$\langle (1, 1), (1, 1) \rangle < 0 \quad \nearrow$$

$$(1, 0), (1, 0) = 0 \quad \text{but } (1, 0) \neq (0, 0)$$

$$(V, +, *, \langle \rangle)$$



Inner product space

$$\begin{pmatrix} 3 & 1 & \vdots & 4 \\ 0 & 0 & \vdots & 0 \end{pmatrix}$$



$$\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$3a + b = 4$$

$$0 = 0$$

$$\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\text{if } \begin{aligned} a &= t \\ b &= 4 - 3t \end{aligned}$$

$$(a, b) = (t, 4 - 3t)$$

$$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 + x_2 y_2$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$v \in \mathbb{R}^2$$
  

$$[v_1, v_2]$$

$$T_v: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$T_v = \langle u, v \rangle = \langle v, u \rangle$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

if rank = 1

What is nullity = 1

not 1-1

To be 1-1, nullity should be 0

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



0  $\mathbb{R}^2$  dim 1

Not onto.

$$T_v \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = T_v \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$



$$x_1 = y_1$$
  

$$\& x_2 = y_2$$

then 1-1

$$T_v = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

matrix multiplication

$$v \neq 0$$

Rank = 1

co-domain is  $\mathbb{R}$  dim = 1

Onto.

Week 8.

$u, v$   $\langle u, v \rangle = 0$   $u, v$  are orthogonal

Orthogonal set  $\{v_1, v_2, \dots, v_k\}$

If orthogonal then they are Independent.

2D



3D



$V = \text{Basis } (\beta_1, \beta_2, \beta_3)$

Orthogonal basis

Orthonormal set  $\{v_1, v_2, \dots, v_k\}$   
 $\|v_i\| = 1 \forall i$

Orthonormal basis  $v_1, v_2, \dots, v_k$   
 $\|v_i\| = 1 \forall i$   
 $\langle v_i, v_j \rangle = 0 \forall i \neq j$



Is these 3 vech Independent ✓

$V =$

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k$$

$\uparrow$   $\langle v, v_1 \rangle$   $\rightarrow$   $\langle v, v_k \rangle$

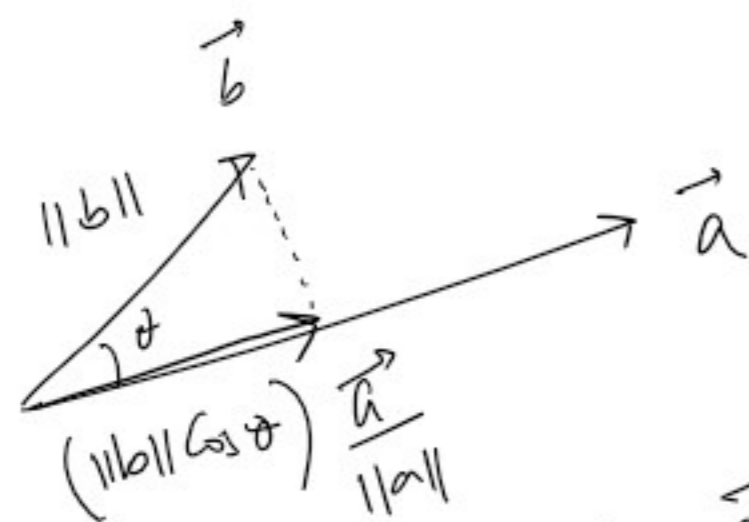
$$\frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle} = \langle v, v_i \rangle$$

$$\langle v, v_i \rangle = \langle \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k, v_i \rangle$$

$$\langle v, v_i \rangle = \alpha_1 \langle v_1, v_i \rangle + \alpha_2 \langle v_2, v_i \rangle + \dots + \alpha_k \langle v_k, v_i \rangle$$

$$\langle v, v_i \rangle = \alpha_i$$

# Projection



$$\cos \theta = \frac{\langle b, a \rangle}{\|b\| \|a\|}$$

$$\text{Shadow vector } ||b|| \cos \theta \frac{\vec{a}}{\|a\|} = ||b|| \frac{\langle b, a \rangle}{\|b\| \|a\|} \frac{\vec{a}}{\|a\|} = \frac{\langle b, a \rangle}{\langle a, a \rangle} \vec{a}$$

$$Proj_a(b) = \frac{\langle b, a \rangle}{\langle a, a \rangle} \vec{a}$$

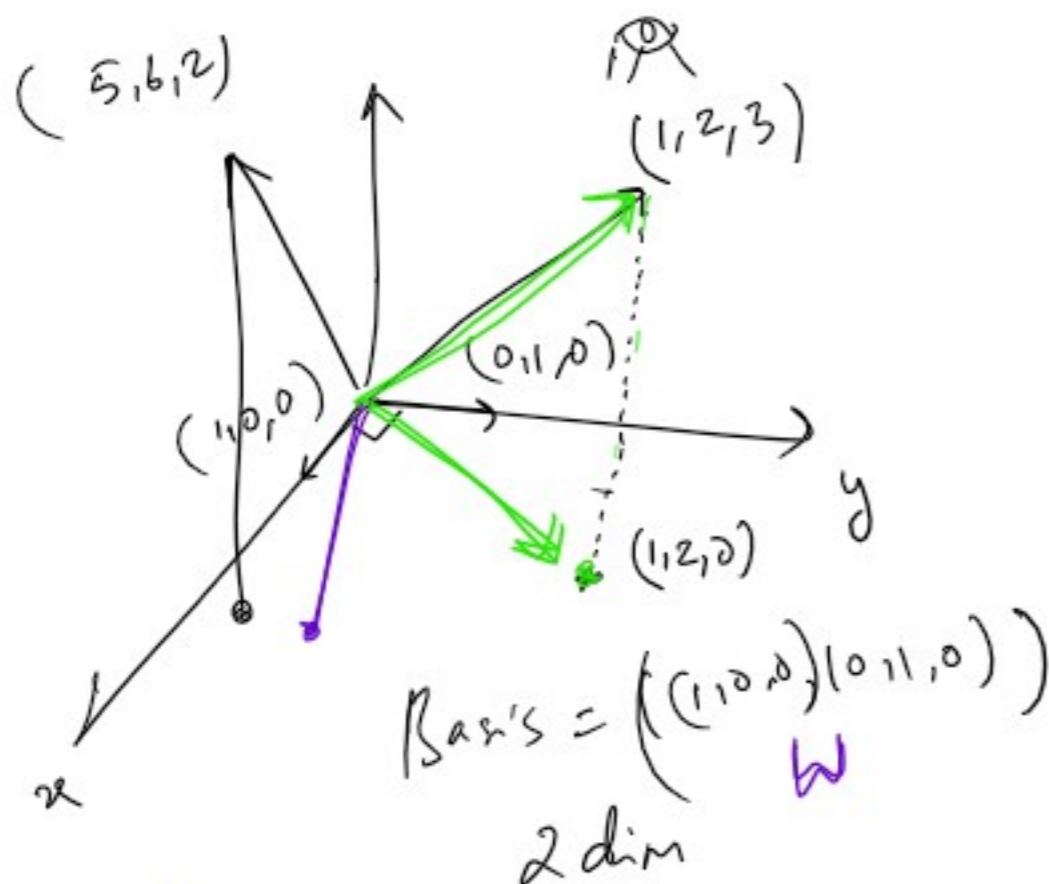
$$P_W(P_W) = P_W$$

$$P_W^2 = P_W$$

$$P_W^{100} = P_W$$

Projection T is idempotent

$$\forall v \in W \quad P_W(v) = v$$



Project the given vector to its each orthonormal basis vector and add them

$$P_{(0,1,0)}(1,2,3) + P_{(1,0,0)}(1,2,3)$$

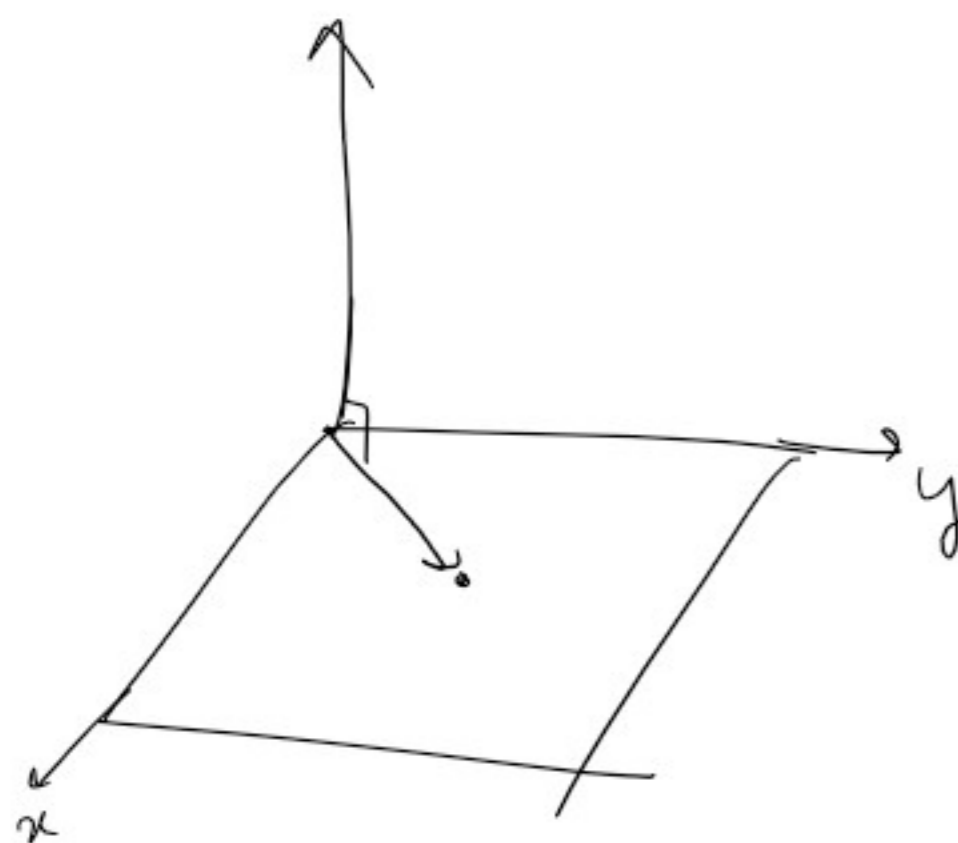
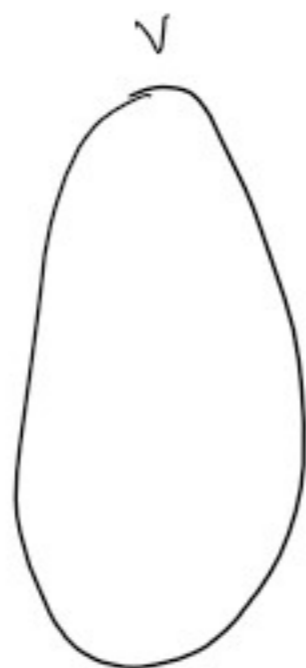
$$\frac{\langle (1,2,3), (0,1,0) \rangle}{1} (0,1,0) + \frac{\langle (1,2,3), (1,0,0) \rangle}{1} (1,0,0)$$

$$2(0,1,0) + 1(1,0,0)$$

$$= (1, 2, 0)$$

$$T(x_1 + x_2) = T(x_1) + T(x_2)$$

Projection as a Lin Transform



$W = xy \text{ plane}, W^\perp = z \text{ axis}$

$$W^\perp = \left\{ v \mid \text{such that } \langle v, w \rangle = 0 \quad \forall w \in W \right\}$$

$$[P_W]^v = Proj_W(v) = \sum_{i=1}^n Proj_{w_i}(v) = \sum_{i=1}^n \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i$$

$$Proj_W(v_1 + v_2) = \sum_{i=1}^n \langle v_1 + v_2, w_i \rangle w_i = \sum_{i=1}^n \langle v_1, w_i \rangle w_i + \sum_{i=1}^n \langle v_2, w_i \rangle w_i$$

$$Proj_W(v_1 + v_2) = Proj_W(v_1) + Proj_W(v_2)$$

①



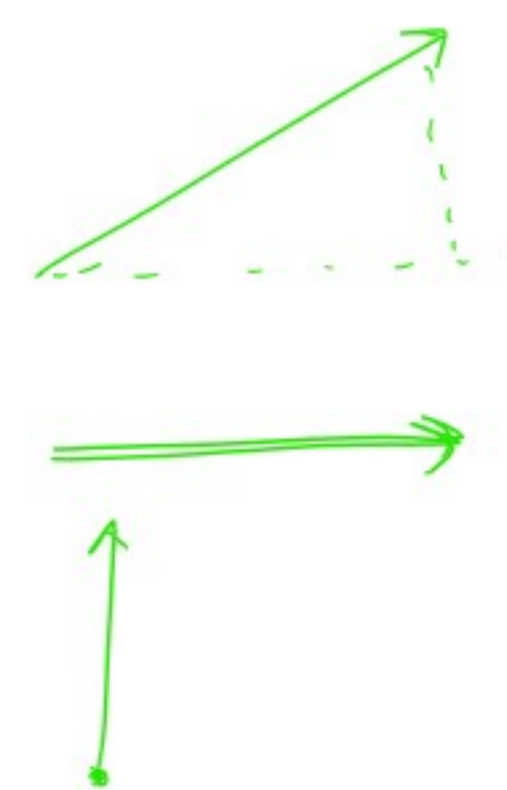
Make them



gram schmidt process.

②

Rotation.



$$\|P_W(v)\| \leq \|v\|$$

$$P_W(v) = \sum_{i=1}^n P_{w_i}(v)$$