

$$\textcircled{4} \quad \langle c v_1, v_2 \rangle = c \langle v_1, v_2 \rangle$$

$$\textcircled{3} \quad \langle (y_1, y_2)(x_1, x_2) \rangle = c \left( y_1 x_1 - y_2 x_1 - y_1 x_2 + \underset{\uparrow}{2 y_2 x_2} \right)$$

$$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 - x_2 y_1 - x_1 y_2 + 2 x_2 y_2$$

$$(x_1, x_2), (x_1, x_2)$$

$$x_1^2 - x_2 x_1 - x_1 x_2 + 2 x_2^2$$

$$x_1^2 - 2 x_1 x_2 + 2 x_2^2$$

$$\underbrace{x_1^2 - 2 x_1 x_2 + x_2^2}_{(x_1 - x_2)^2} + x_2^2$$

$$(x_1 - x_2)^2 + x_2^2 > 0$$

$$\langle v_1 + v_2, v_3 \rangle$$

$$\begin{aligned} & \left| \begin{array}{l} \text{LHS} \\ \langle ((x_1, x_2) + (y_1, y_2), (z_1, z_2)) \rangle \\ \langle (x_1 + y_1, x_2 + y_2), (z_1, z_2) \rangle \\ (x_1 + y_1) z_1 - (x_2 + y_2) z_1 - (x_1 + y_1) z_2 + 2(x_2 + y_2) z_2 \\ x_1 z_1 - x_2 z_1 - x_1 z_2 + 2 x_2 z_2 \rightarrow \langle (x_1, x_2), (z_1, z_2) \rangle \\ + + + + \\ y_1 z_1 - y_2 z_1 - y_1 z_2 + 2 y_2 z_2 \quad \langle (y_1, y_2), (z_1, z_2) \rangle \\ \text{RHS} \end{array} \right. \end{aligned}$$

when  $x_1 = x_2$        $x_2 = 0$        $(0, 0) \rightarrow 0$

$x_1 = 0$        $x_2 = 0$

$$V = M_{2 \times 2}(\mathbb{R}) \quad \langle A, B \rangle = \text{Tr}(AB) \quad AB \neq BA$$

$\uparrow$

$$\langle B, A \rangle$$

$\uparrow$

$$V = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$a, b \in \mathbb{R}$$

$$\langle A, B \rangle = \text{Tr}_{\text{eu}}([ ]^{\text{c } \text{ }})$$

$$\langle A, A \rangle = \begin{bmatrix} a \\ b \end{bmatrix} [a \ b]$$

$$\begin{pmatrix} a^2 & ab \\ ba & b^2 \end{pmatrix}$$

$$\langle A, A \rangle = a^2 + b^2 \geq 0$$

iff  $a=b=0$

Outer product  $\rightarrow$  Matrix output

Inner product  $\rightarrow$  Scalar output

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 11$$

$$U^T V$$

$$\begin{pmatrix} a \\ b \end{pmatrix} [c \ d]$$

$$\text{trace} = \cancel{ac} + \cancel{bd}$$

$$\cancel{k} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} [c \ d] \right)$$

$$\begin{pmatrix} c \\ d \end{pmatrix} [a \ b]$$

$$ca + db$$

$$\left[ \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}, \begin{pmatrix} p & q \end{pmatrix} \right]$$

$$\begin{pmatrix} ap+qc \\ bp+qd \end{pmatrix} [p \ q]$$

$$ap+qc + bp+qd$$

LHS

$$\begin{pmatrix} a \\ b \end{pmatrix} [p \ q] + \begin{pmatrix} c \\ d \end{pmatrix} [p \ q]$$

$$ap + bq + cp + dq$$

RHS

$$\langle (x_1 x_2), (y_1 y_2) \rangle = x_1 y_2 + x_2 y_1$$
$$(V, +, \cdot, \langle \cdot, \cdot \rangle)$$

$$+ \rightarrow x_1 x_2 + x_2 x_1 = 2 x_1 x_2$$

$$\langle (1, 1), (1, -1) \rangle < 0$$

$$(1, 0), (1, 0) = 0 \quad \text{but } (1, 0) \neq (0, 0)$$

Inner product space

$$\begin{pmatrix} 3 & 1 & : & 4 \\ 0 & 0 & : & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 3a + b &= 4 \\ 0 &= 0 \end{aligned}$$

if  $a = t$   
 $b = 4 - 3t$

$$(a, b) = (t, 4 - 3t)$$

$$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 + x_2 y_2$$

$\uparrow$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

if rank = 1

what is nullity = 1

To be 1-1, nullity should be 0

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$0 \quad \mathbb{R}^{\dim 1}$$

Not onto.

$$\boxed{T_v \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = T_v \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}$$

{

$x_1 = y_1$   
 $x_2 = y_2$  then 1-1

$$v \in \mathbb{R}^2$$

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

$$T_v : \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$T_v = \langle u, v \rangle = \langle v, u \rangle$$

$$\uparrow$$

$$\begin{bmatrix} u & x_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$T_v = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

matrix multiplica-

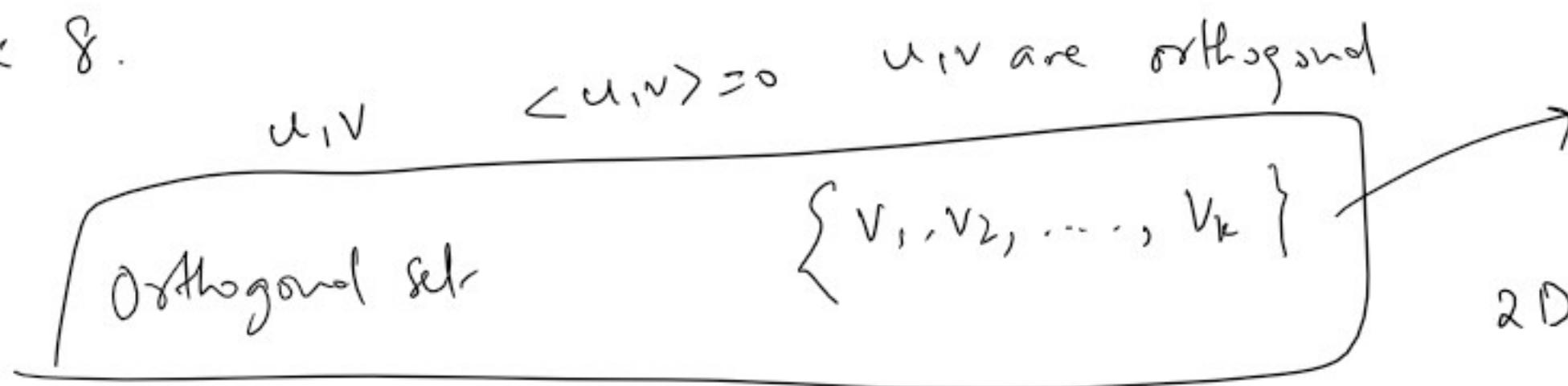
$$v \neq 0$$

$$\text{Rank} = 1$$

$\Leftrightarrow$  Co-domain  
in  $\mathbb{R}$   
 $\dim = 1$

onto.

Week 8.

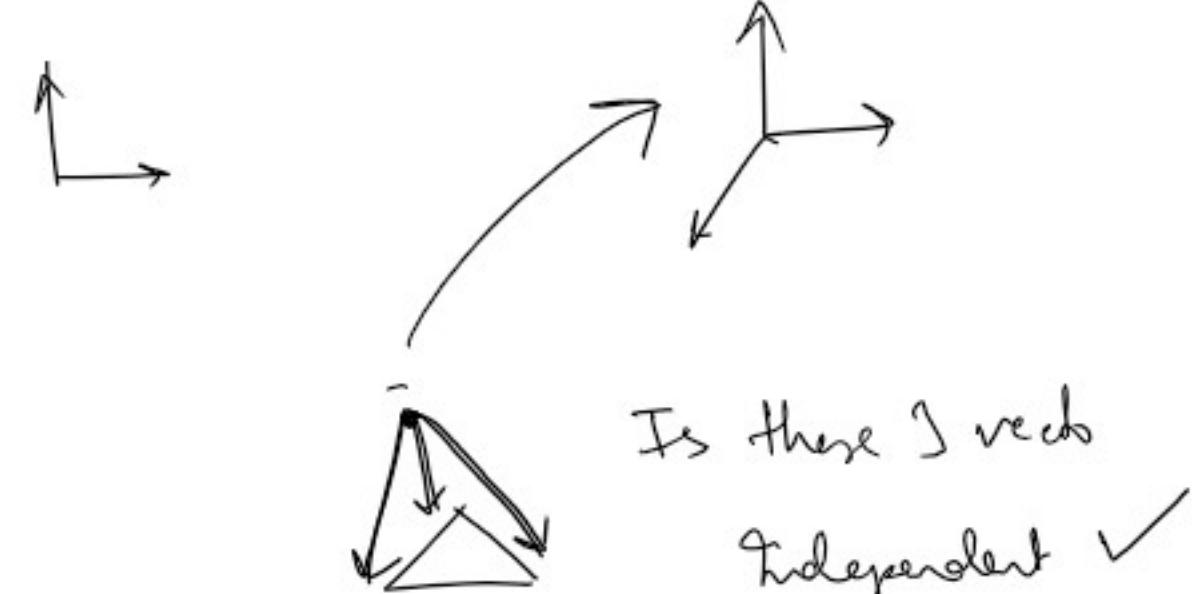


If orthogonal  
then they are independent.

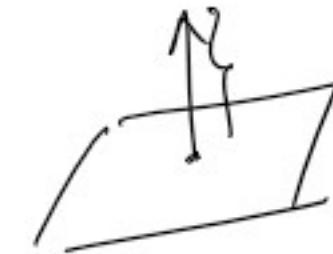
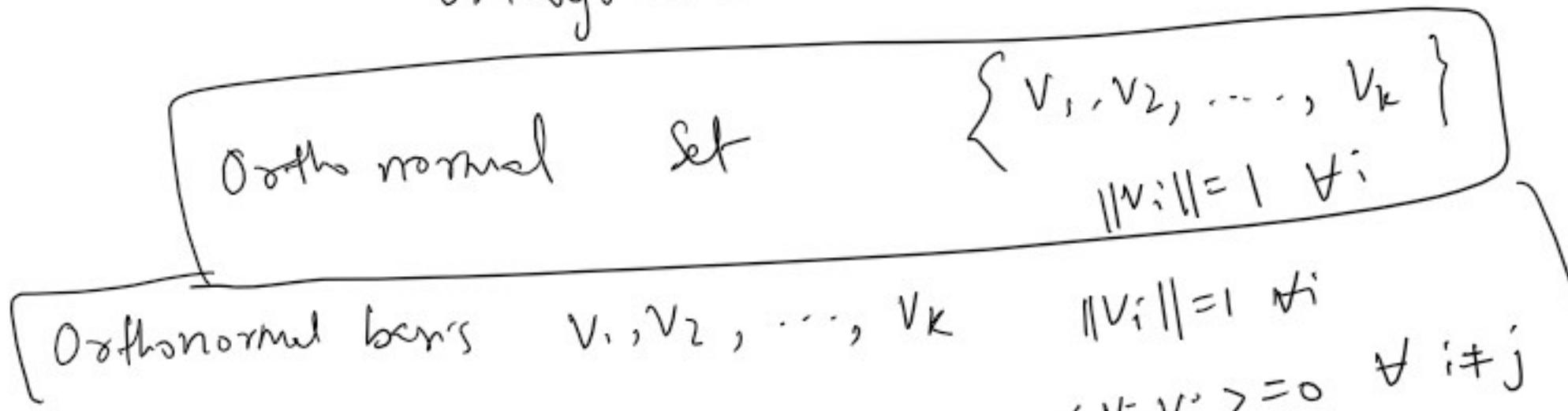
3D

$$V = \text{Span}(\beta_1, \beta_2, \beta_3)$$

orthogonal basis



Is these 3 vecs  
independent ✓



$V =$

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k$$



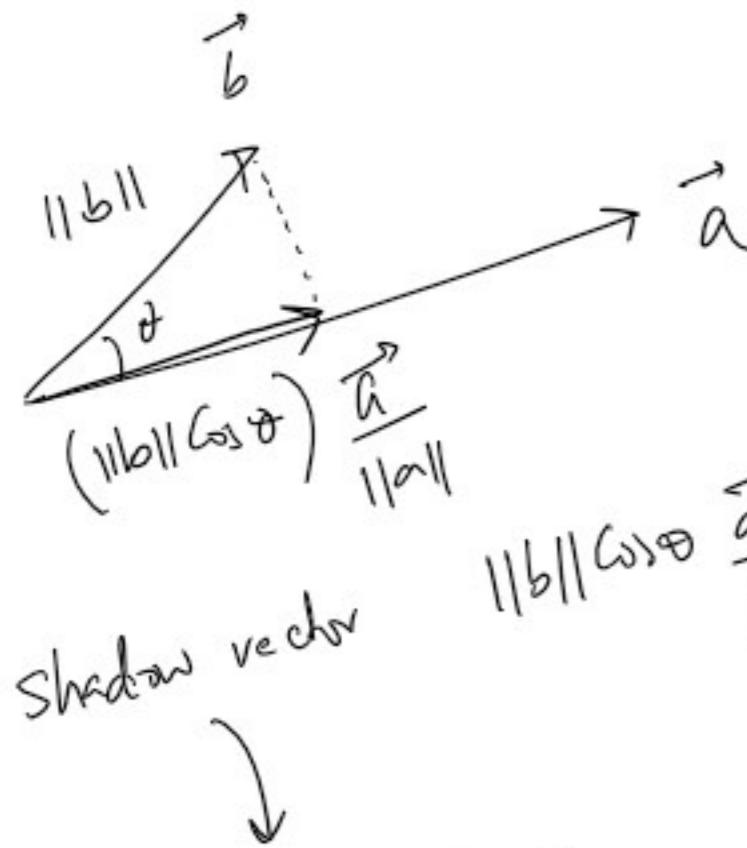
$$\frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle} = \frac{\langle v, v_i \rangle}{\alpha_i}$$

$$\langle v, v_i \rangle = (\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k, v_i)$$

$$\langle v, v_i \rangle = \alpha_1 \langle v_1, v_i \rangle + \alpha_2 \langle v_2, v_i \rangle + \dots + \alpha_k \langle v_k, v_i \rangle$$

$$\langle v, v_i \rangle = \alpha_i$$

Projection



$$\cos \theta = \frac{\langle b, a \rangle}{\|b\| \|a\|}$$

$$\text{shadow vector } \frac{\|b\| \cos \theta \frac{\vec{a}}{\|\vec{a}\|}}{\|\vec{a}\|} = \frac{\|b\| \frac{\langle b, a \rangle}{\|b\| \|\vec{a}\|} \frac{\vec{a}}{\|\vec{a}\|}}{\|\vec{a}\|} = \frac{\langle b, a \rangle \vec{a}}{\|\vec{a}\| \langle a, a \rangle}$$

$$P_{\text{proj}}_a(b) = \frac{\langle b, a \rangle \vec{a}}{\|\vec{a}\| \langle a, a \rangle}$$

$$P_w(P_w) = P_w$$

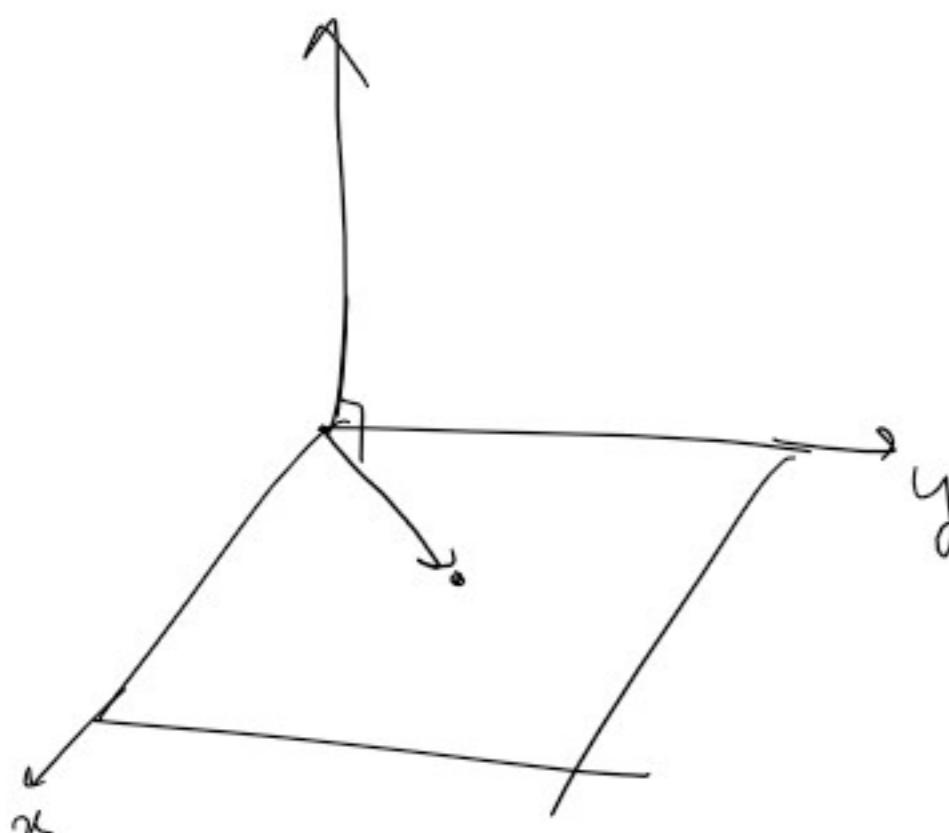
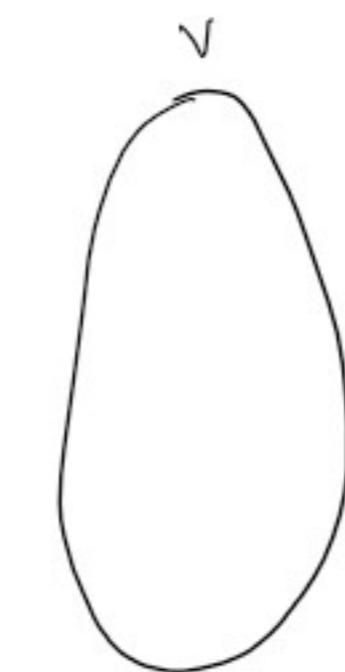
$$P_w^2 = P_w$$

$$\underline{P_w^{100} = P_w}$$

Projection T is onto

$$\forall v \in W \quad P_w(v) = v$$

Projection on a Lin Transform



$W = xy$  plane,  $w^\perp = z$  axis

$$W^\perp = \{v \mid \text{such that } \langle v, w \rangle = 0 \quad \forall w \in W\}$$

Project the given vector to its each orthonormal basis vector and add them

$$P_{(1,2,3)} + P_{(1,2,3)}$$

$$\frac{\langle (1,2,3), (0,1,0) \rangle}{1} (0,1,0) + \frac{\langle (1,2,3), (1,0,0) \rangle}{2} (1,0,0)$$

$$2(0,1,0) + 1(1,0,0)$$

$$= (1,2,0)$$

$$T(x_1 + x_2) = T(x_1) + T(x_2)$$

$$[P]_W^V = P_{\text{proj}}_W(v) = \sum_{i=1}^n P_{\text{proj}}_{w_i}(v) = \sum_{i=1}^n \langle v, w_i \rangle w_i$$

$$P_{\text{proj}}_W(v_1 + v_2) = \sum_{i=1}^n \langle v_1 + v_2, w_i \rangle w_i$$

$$= \sum_{i=1}^n \langle v_1, w_i \rangle w_i + \sum_{i=1}^n \langle v_2, w_i \rangle w_i$$

$$P_{\text{proj}}_W(v_1 + v_2) = P_{\text{proj}}_W(v_1) + P_{\text{proj}}_W(v_2)$$

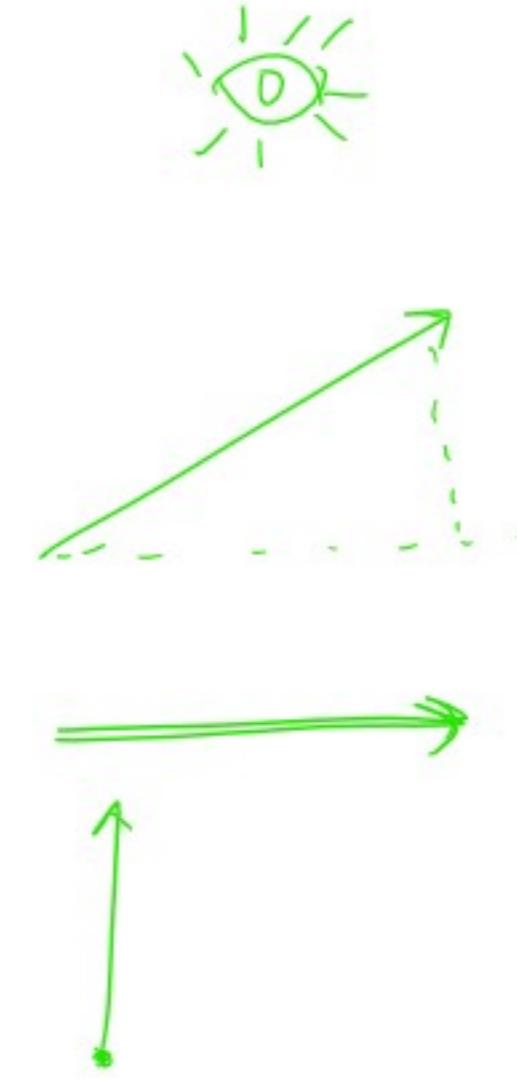
gram schmidt process.

①



Make them

② Rotation.



$$\|P_w(v)\| \leq \|v\|$$