MLF Term 2 2024
GA : 6

Q1. The function $f(x,y) = x^2+y^2$

(A) has no stationary point.

(B) has a stationary point at $(0,0)$.

(C) has a stationary point at $(1,1)$

(D) has a stationary point at $(-1,-1)$

Sol: Find the gradient of $f(x,y)$ and set to zero to get the stationary point.

$$\nabla f(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix} = 0 \Rightarrow x=0,\ y=0$$

$\therefore$ $(0,0)$ is a stationary point.

Q2. If $A = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, then $A+B$ is a positive definite matrix.

(A) Yes, it is true.

(B) No, it is not true.

Sol: $$A+B = \begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix} = P \text{ (let)}$$

The principal submatrices & their determinants.

$P_1 = [5]$ $\quad$ $\det(P_1) = 5 > 0$

$$P_2 = \begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix} = P, \quad \det(P_2) = 20-9 = 11 > 0$$

$\because$ All principal submatrices are positive, the matrix $P = A+B$ is positive definite.

Alt:

Compute the eigenvalues of P:

$$\det \begin{bmatrix} 5-\lambda & 3 \\ 3 & 4-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (5-\lambda)(4-\lambda) - 9 = 0 \Rightarrow \lambda^2 - 9\lambda + 20 - 9 = 0$$

$$\Rightarrow \lambda^2 - 9\lambda + 11 = 0$$

$$\lambda = \frac{9 \pm \sqrt{81-44}}{2}$$

$$= \frac{9 \pm \sqrt{37}}{2}$$

$$\approx 7.5,\ 1.5$$

$\because$ both the eigenvalues are positive,

$\therefore$ $A+B$ is positive definite.

Q3. The matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ is

(A) positive definite ✓
(B) positive semi-definite
(C) negative definite
(D) negative semi-definite

Sol: The sub matrices and their determinants

$A_1 = [2]$, $\det(A_1) = 2 > 0$

$A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $\det(A_2) = 4 - 1 = 3 > 0$

$A_3 = A$, $\det(A) = 2(4-1) + 1(-2+1) + 1(1-2)$
$= 6 - 1 - 1 = 4 > 0$

$\because$ all determinants are positive,
$\therefore$ A is positive definite.

Q4) The function $f(x,y) = 2x^2 + 2xy + 2y^2 - 6x$ has a stationary point at

(A) $(2,1)$ (B) $(1,2)$
(C) $(-1,2)$ (D) $(2,-1)$ ✓

Sol: $\nabla f(x,y) = \begin{bmatrix} 4x + 2y - 6 \\ 2x + 4y \end{bmatrix} = 0$

$\Rightarrow \left.\begin{array}{l} 4x + 2y - 6 = 0 \\ 2x + 4y = 0 \end{array}\right\}$

$2x + y = 3$
$\underline{\underset{-}{2x} \underset{-}{+4y} = \underset{-}{0}}$
$-3y = 3$
$\Rightarrow y = -1$

$\therefore$ $2x = 3 - y = 3 - (-1)$
$\Rightarrow 2x = 4 \Rightarrow x = 2$

$\therefore$ $(2,-1)$ is a stationary point.

Q5) The correct representation of $x^2 + y^2 - z^2 - xy + yz + zx$ in the matrix form is

(A) ✓ $[x \ y \ z] \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(B) $[x \ y \ z] \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(C) $[x \ y \ z] \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(D) $[x \ y \ z] \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Sol: The diagonal elements must contain the coefficients of x^2, y^2, z^2 in that order. Hence, the matrix should be of the form

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$$

This is satisfied for only the matrix in option (A). Just to make sure, let's expand

$$[x \quad y \quad z] \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x \quad y \quad z] \begin{bmatrix} x-y \\ y \\ x+y-z \end{bmatrix}$$

$$= x(x-y) + y^2 + z(x+y-z)$$

$$= x^2 - xy + y^2 + zx + yz - z^2$$

$$= x^2 + y^2 - z^2 - xy + yz + xz$$

which matches the given quadratic form.

Q 6) Given $f(x,y) = 3x^2 + 4xy + 2y^2$, the point $(0,0)$ is a

(A) maxima

✓ (B) minima

(C) Saddle point

(D) None of these.

Sol:

$$f(x,y) = [x \quad y] \underbrace{\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

Sub matrices & their determinants

$A_1 = [3]$, $\det(A_1) = 3 > 0$

$A_2 = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$, $\det(A_2) = 2 > 0$

$\therefore$ A is positive definite.

$\therefore$ $f(x,y)$ is a strictly convex function.

Also, $(0,0)$ is a stationary point, as

$$\nabla f(0,0) = \begin{bmatrix} 6x+4y \\ 4x+4y \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ $(0,0)$ is a minima.

Q7) Which of the following statements are true about the matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$?

(A) A is positive definite
(B) A is positive semi-definite
(C) A is neither positive definite nor positive semi-definite.
(D) can not be determined

Sol: $A = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$

The submatrices & their determinants

$A_1 = [4]$, $\det(A_1) = 4 > 0$

$A_2 = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$, $\det(A_2) = 8 > 0$

$\therefore$ A is positive definite.

Q8) The matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is positive definite.

(A) True (B) False.

Sol: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$A_1 = [1]$, $\det(A_1) = 1 > 0$

$A_2 = A$, $\det(A_2) = -3 < 0$

$\therefore$ It is not +ve definite.

Q9) Which of the following statements are true about the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$?

(A) A is positive definite
(B) A is positive semi-definite
(C) A is neither positive definite nor positive semi-definite.
(D) can not be determined

Sol: Being a diagonal matrix, the diagonal elements are the eigenvalues are 3, 5, 7 all of which are > 0. As such the matrix A is a positive definite.

Q 10) The non-zero singular values of the matrix $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ are

A) $3+\sqrt{3}$, $3-\sqrt{3}$

(B) $\sqrt{3+\sqrt{3}}$, $\sqrt{3-\sqrt{3}}$

(C) $2+\sqrt{2}$, $2-\sqrt{2}$

(D) $\sqrt{2+\sqrt{2}}$, $\sqrt{2-\sqrt{2}}$

Sol: $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

A^TA is 4×4 matrix & AA^T is 3×3.
Since the non-zero eigenvalues of A^TA & AA^T are same, it is easier to work with AA^T

$$AA^T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

The characteristic eqn
$\det(AA^T - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{vmatrix} = 0$$

Expanding about the first row:

$$(3-\lambda)(1-\lambda)(2-\lambda) - 1(2-\lambda) + 2(0 - 2(1-\lambda)) = 0$$
$$\Rightarrow (3-\lambda)(\lambda^2 - 3\lambda + 2) - 2 + \lambda - 4 + 4\lambda = 0$$
$$\Rightarrow 3\lambda^2 - 9\lambda + 6 - \lambda^3 + 3\lambda^2 - 2\lambda - 6 + 5\lambda = 0$$
$$\Rightarrow -\lambda^3 + 6\lambda^2 - 6\lambda = 0$$
$$\Rightarrow -\lambda(\lambda^2 - 6\lambda + 6) = 0$$
$$\lambda = 0, \quad \lambda^2 - 6\lambda + 6 = 0 \Rightarrow \lambda = \frac{6 \pm \sqrt{36-24}}{2}$$
$$\Rightarrow \lambda = \frac{6 \pm \sqrt{12}}{2} = 3 \pm \sqrt{3}$$

$\therefore$ The non zero eigenvalues of A^TA are $3+\sqrt{3}$ & $3-\sqrt{3}$

$\therefore$ The non-zero singular values of A are $\sqrt{3+\sqrt{3}}$, $\sqrt{3-\sqrt{3}}$

Q 11) The SVD of the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ is

A) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}$

B) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

C) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

D) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

Sol: Notice that we can rule out options A), B) and D), solely on the fact that the columns of the right matrix are not of unit norm and hence the answer is option C.

But for the sake of completion, we derive the SVD of the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

Consider

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

& $$AA^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\because$ the non-zero eigenvalues of $A^T A$ are same as that of AA^T, it is easier to work with the second matrix. Further, since AA^T is already diagonal, its eigenvalues are 2 and 2. As such, the non-zero eigenvalues of $A^T A$ are also 2 and 2, giving us the singular values as $\sqrt{2}, \sqrt{2}$.

So the structure of Σ is known

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}$$

We first find V.

The eigenvector associated with the eigenvalue 2 :

$$\begin{bmatrix} 1-2 & 0 & 1 & 0 \\ 0 & 1-2 & 0 & 1 \\ 1 & 0 & 1-2 & 0 \\ 0 & 1 & 0 & 1-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} -x + z &= 0 \\ -y + t &= 0 \\ x - z &= 0 \\ y - t &= 0 \end{aligned}\right\} \Rightarrow \quad z = x, \quad y = t$$

So the eigenvector is of the form $\begin{bmatrix} x \\ y \\ x \\ y \end{bmatrix}$.

Two orthogonal eigenvectors for the eigenvalue 2, can therefore taken as

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Normalizing we get $v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

For eigenvalue 0, we get

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 0$$

giving $x + z = 0, \quad y + t = 0$

$\therefore z = -x, \quad t = -y$

$\therefore$ The eigenvectors are of the form

$$\begin{bmatrix} x \\ y \\ -x \\ -y \end{bmatrix}.$$

So we can choose two eigenvectors that are orthogonal as

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \;\&\; \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

Normalize to get $V_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $V_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$.

So that $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

To find u_1, use the fact that

$$A v_1 = \sigma_1 u_1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \sqrt{2}\, u_1$$

$$\Rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \sqrt{2}\, u_1 \Rightarrow \quad u_1 = \frac{1}{2} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\therefore$ we can take $u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

As a result

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad A = U \Sigma V^T$$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

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