1. show that the function $f(z) = \begin{cases} \dfrac{x^3y(y-ix)}{x^6+y^2}, & z \neq 0 \\ 0 & \text{for } z=0 \end{cases}$

is not differentiable at origin though the cauchy-Rieman equations are satisfied at the origin.

Sol Given $f(z) = \begin{cases} \dfrac{x^3y(y-ix)}{x^6+y^2}, & z \neq 0 \\ 0, & z=0 \end{cases}$

Now $f'(0) = \lim\limits_{z \to 0} \dfrac{f(z)-f(0)}{z}$

$= \lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{x^3y^2 - ix^4y}{(x^6+y^2)(x+iy)}$

$= \lim\limits_{y \to 0} 0$

$= 0$

Along $y = mx^3$

$f'(0) = \lim\limits_{z \to 0} \dfrac{f(z)-f(0)}{z}$

$= \lim\limits_{x \to 0} \dfrac{x^3y^2 - ix^4y}{(x^6+y^2)(x+iy)}$

$= \lim\limits_{x \to 0} \dfrac{x^9m^2 - ix^7m}{x^7+y^2x+ix^6y+iy^3}$

$= \lim\limits_{x \to 0} \dfrac{x^9m^2 - imx^7}{x^7+m^2x^7+i(mx^9+m^3x^9)}$

$$= \lim_{x \to 0} \frac{x^7 [x^2 m^2 - im]}{x^7 [1 + m^2 + i(mx^2 + m^3 x^2)]}$$

$$f'(0) = \frac{-im}{1+m^2}$$

which depends on m. not a unique value hense $f'(0)$ is not differential

Here $u = \frac{x^3 y^2}{x^6 + y^2}$ $\quad v = \frac{x^4 y}{x^6 + y^2}$

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x}$$

$$= \lim_{x \to 0} 0$$

$$= 0$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y}$$

$$= \lim_{y \to 0} 0$$

$$= 0$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x}$$

$$= 0$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y}$$

$$= 0$$

since $u_x = v_y$ & $u_y = -v_x$, CR equations are satisfied.

Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at origin even though C.R equations are satisfied at the origin.

Given $f(z) = \sqrt{|xy|}$

Now $f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sqrt{xy}}{(x+iy)}$$

$$= 0$$

Along $y = mx$

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{x \to 0} \frac{\sqrt{x^2 m}}{(x + imx)}$$

$$= \lim_{x \to 0} \frac{\cancel{x}(\sqrt{m})}{\cancel{x}(1+im)}$$

$$f'(0) = \frac{\sqrt{m}}{1+im}$$

which depends on m, not a unique value

$f'(0)$ is not analytic.

Here $U = \sqrt{xy}$ $\quad V = 0$

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x \to 0} \frac{U(x,0) - U(0,0)}{x}$$

$$= 0$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y \to 0} \frac{u(0,y) - U(0,0)}{x}$$

$= 0$

$$\left(\frac{\partial v}{\partial x}\right)(0,0) = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x}$$

$$= 0$$

$$\left(\frac{\partial v}{\partial y}\right)(0,0) = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y}$$

$$= 0$$

since $u_x = v_y$ & $u_y = -v_x$ CR equations are satisfied.

3\. show that the function $f(z) = \begin{cases} \dfrac{x^2y^2 + ix^3y}{x^4+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

is not analytic at $z=0$ though the CR equations are satisfied at the origin

sol Given $f(z) = \begin{cases} \dfrac{x^2y^2 + ix^3y}{x^4+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2y^2 + ix^3y}{(x^4+y^2)(x+iy)}$$

$$= 0$$

Along $y = mx^2$

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{x \to 0} \frac{x^2y^2 + i x^3 y}{(x^4+y^2)(x+iy)}$$

$$= \lim_{x \to 0} \frac{x^6m^2 + imx^5}{x^5 + x^5m^2 + i(x^6m + m^3x^6)}$$

$$= \lim_{x \to 0} \frac{\cancel{x^5}[xm^2 + im]}{\cancel{x^5}[1+m^2 + i(xm + m^3x)]}$$

$$= \frac{im}{1+m^2}$$

which depends on m not a unique value hence $f'(0)$ is not analytic

Here $u = \frac{x^2y^2}{x^4+y^2}$ $\quad v = \frac{x^3y}{x^4+y^2}$

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x}$$

$$= \lim_{x \to 0} 0$$

$$= 0$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y}$$

$$= 0$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x \to 0} \frac{v(x,0) - u(0,0)}{x}$$

$$= 0$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y}$$

$$= 0$$

since $u_x = v_y$ & $u_y = -v_x$ CR equations are satisfies.

4. find the analytic function $f(z)=u+iv$ if $u-v=e^x(\cos y-\sin y)$

Sol Let $f(z)=u+iv$ —① be the required analytic function.

Given $u-v=e^x[\cos y-\sin y]$

then $if(z)=iu-v$ —②

①+② =>

$(i+1)f(z)=(u-v)+i(u+v)$ —③

Let $F(z)=U+iV$

where $F(z)=(1+i)f(z)$

$U=u-v$

$V=u+v$

Now

$U_x=\frac{\partial u}{\partial x}=e^x[\cos y-\sin y]$

$U_y=\frac{\partial u}{\partial y}=e^x[-\sin y-\cos y]$

Now $F'(z)=U_x+iV_x$

$=U_x-iU_y$ [∵ CR equation]

$F'(z)=e^x[\cos y-\sin y]+ie^x[\sin y+\cos y]$ —④

By using Milne thomson method, we can express $F(z)$ in terms of z by putting $x=z$ and $y=0$

$$F'(z) = e^z + ie^z$$

$$= (1+i)e^z$$

$$\therefore F(z) = (1+i)e^z \quad \text{--- (5)}$$

W.K.T $\quad F(z) = (1+i)f(z)$

eq (5) becomes

$$(1+i)f(z) = (1+i)e^z$$

$$f(z) = e^z + c$$

5. Show that the function $U = 2\log(x^2+y^2)$ is harmonic and find its harmonic conjugate

sol Given that $U = 2\log(x^2+y^2)$

Now

$$\frac{\partial u}{\partial x} = 2\frac{1}{x^2+y^2}(2x)$$

$$= \frac{4x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = 4\left[\frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2}\right]$$

$$= 4\left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{4(y^2-x^2)}{(x^2+y^2)^2}$$

lly

$$\frac{\partial^2 u}{\partial y^2} = \frac{4(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{4}{(x^2+y^2)^2}\left[y^2-x^2+x^2-y^2\right]$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$\therefore$ u is harmonic

Let v be the conjugate harmonic of u

By C-R equations we have

$$\frac{\partial u}{\partial x}=\frac{-\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$$

Now $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$

$$v=\int\frac{-4y}{x^2+y^2}\,dx$$

$$=-4y\int\frac{1}{x^2+y^2}\,dx$$

$$=-4y\int\frac{1}{x^2+y^2}\,dx$$

$$=-4y\cdot\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)+c_1 \quad \left[\because \int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{y}\right)\right]$$

$$=-4\tan^{-1}\left(\frac{x}{y}\right)+c_1$$

$$=4\tan^{-1}\left(\frac{y}{x}\right)+c_1$$

$$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}$$

$$v=\int\frac{4x}{x^2+y^2}\,dy$$

$$= 4x \int \frac{1}{x^2+y^2} dy$$

$$= 4x \frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right) + c_2$$

$$= 4 \tan^{-1}\left(\frac{y}{x}\right) + c_2$$

$$\therefore v = 4 \tan^{-1}\left(\frac{y}{x}\right) + c.$$

6. Construct the Analytic function whose real part is $u(x,y) = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$.

Sol Given $u(x,y) = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$.

Now
$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y + \cos 2x)\cos 2x \cdot 2 - \sin 2x(-\sin 2x \cdot 2)}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cosh 2y \cos 2x + 2\cos^2 2x + 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cosh 2y \cos 2x + 2[\cos^2 2x + \sin^2 2x]}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cosh 2y \cos 2x + 2}{(\cosh 2y + \cos 2x)^2}$$

Let $f(z) = u + iv$ be the required function

then $f'(z)$ is

$$f'(z) = u_x + i v_x \quad - ①$$

By using C-R equation eq ① becomes

$$f'(z) = U_x - iU_y$$

Now

$$f'(z) = \frac{2\cos 2x\cosh 2y + 2}{(\cosh 2y + \cos 2x)^2} - i\ \frac{2\sin 2x\sinh 2y}{}$$

$$\frac{\partial u}{\partial y} = \frac{(\cos 2hy + \cos 2x)\,0 - \sin 2x(\sinh 2y \cdot 2)}{(\cosh 2y + \cos 2x)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2\sinh 2y\sin 2x}{(\cosh 2y + \cos 2x)^2}$$

Now

$$f'(z) = \frac{2\cos 2x\cosh 2y + 2}{(\cosh 2y + \cos 2x)^2} + i\ \frac{2\sinh 2y\sin 2x}{(\cosh 2y + \cos 2x)^2}$$

By using Milne thomson method we can express $f'(z)$ in terms of z. By putting $x = z$ and $y = 0$

$$\text{then } f'(z) = \frac{2\cos 2z + 2}{(1 + \cos 2z)^2}$$

$$= \frac{2(1 + \cancel{\cos 2z})}{(1 + \cos 2z)^{\cancel{2}}}$$

$$f'(z) = \frac{2}{1 + \cos 2z}$$

Now

$$f(z) = \int \frac{2}{1 + \cos 2z}\, dz.$$

$$f(z) = 2\int \frac{1}{2\cos^2 z}\, dz$$

$$= \frac{2}{2}\int \sec^2 z \, dz$$

$$\boxed{f(z) = \tan z + C}$$

7. Evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz$ where $C: |z-1| = 2$.

Sol Given $C: |z-1| = 2$

$z=1$ which lies inside the given curve.

$z=2$ which lies inside the given curve.

Now $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz$ can be written as

$$= \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\, dz - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\, dz$$

By using cauchy's integral formula we have

$$\int_C \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$$

$$\int \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\, dz - \int \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}$$

$$= 2\pi i f(2) - (2\pi i f(1))$$

$$= 2\pi i [\sin 4\pi + \cos 4\pi] - 2\pi i [\sin 4\pi + \cos \pi]$$

$= 2\pi i[0+1] - 2\pi i[0-1]$

$= 2\pi i + 2\pi i$

$= 4\pi i$

Evaluate $\oint_c \frac{e^{3iz}}{(z-\pi)^3} dz$ where c is $|z-\pi| = 3$

Given $c: |z-\pi| = 3$

$z = \pi$ which lies inside the given c

By using cauchy's integral formula we have

$$\int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$

Where $f(z) = e^{3iz}$

and $n = 2$

$$\therefore \int \frac{e^{3iz}}{(z-\pi)^3} dz = \frac{2\pi i}{2!} f''(\pi)$$

$f'(z) = e^{3iz} \times 3i$

$f''(z) = 3i \times e^{3iz} \times 3i$

$= -9e^{3iz}$

$$\int_c \frac{e^{3iz}}{(z-\pi)^3} dz = \frac{2\pi i}{2!} (-9e^{3i\pi})$$

$= -9\pi i[\cos 3\pi + i\sin 3\pi]$

$= 9\pi i$

9. Evaluate $\oint \frac{1}{z^2+2z+5} dz$ where C : is

i, $|z+1+i| = 2$ ii, $|z+i-1| = 2$

Sol Given

i, $|z+1+i| = 2$: C

$$z^2+2z+5 = (z+1)^2+4$$
$$= (z+1)^2+2^2$$
$$= (z+1+2i)(z+1-2i)$$
$$= [z-(-1-2i)][z-(-1+2i)]$$

Now $\oint \frac{1}{z^2+2z+5} dz = \int \frac{1}{(z+1+2i)(z+1-2i)} dz$

Here $z=-1-2i$ which lies ~~in~~ inside the C

$z=-1+2i$ which lies outside the C

$$\therefore \int \frac{\frac{1}{z+1-2i}}{z+1+2i} dz$$

Here $f(z) = \frac{1}{z+1-2i}$

By using cauchy's integral formula we have

$$\int \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\int \frac{\frac{1}{z+1-2i}}{z+1+2i} = 2\pi i f(-1-2i)$$

$$= 2\pi i \times \frac{1}{-4i} = \frac{-\pi}{2}$$

ii, Given $C: |z+i-1| = 2$

$\cancel{z^2+2z+5 = C}$

Now $\int \frac{1}{z^2+2z+5} = \frac{1}{[z-(-1-2i)][z-(-1+2i)]}$

there $z=-1-2i$ which lies outside the C

$z=-1+2i$ which lies inside the C

$\therefore \int \frac{\frac{1}{z+1+2i}}{z+1-2i} dz$

$f(z) = \frac{1}{z+1+2i}$

By using cauchy's integral formula we have

$$\int \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\int \frac{\frac{1}{z+1+2i}}{z+1-2i} dz = 2\pi i \times f(-1+2i)$$

$$= 2\pi i \times \frac{1}{4i}$$

$$= \frac{\pi}{2}$$

10. Evaluate $\int_C \frac{\log z}{(z-1)^3} dz$ where c is $|z-1| = \frac{1}{2}$

Sol Given $c: |z-1| = \frac{1}{2}$

$z=1$ which lies inside the c

By cauchy's integral formula we have

$$\int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$

where $f(z) = \log z$

$n=2$

$f'(z) = \frac{1}{z}$

$f''(z) = -\frac{1}{z^2}$

$$\therefore \int \frac{\log z}{(z-1)^3} dz = \frac{2\pi i}{2!} f''(1)$$

$$= \pi i(-1)$$

$$= -\pi i$$

11. Evaluate $\int_{1-i}^{2+i} (2x+iy+1) dz$ along the straight line joining $(1,-i)$ and $(2,i)$.

Sol Given $(1,-i)$ and $(2,i)$

Equation of straight line AB is

$y - y_1 = m(x - x_1)$

w.k.T $dz = dx + i dy$.

$y + 1 = \frac{1+1}{2-1}(x-1)$

$y+1 = 2(x-1)$

$y = 2x-2-1$

$y = 2x-3$

$dy = 2dx$

$$\int_{AB}(2x-iy+1)dz = \int_1^2 (2x+i(2x-3)+1)(dx+i2dx)$$

$$= \int_1^2 (2x+2ix \mp 3i+1)(1+i2)dx$$

$$= (1+2i)\int_1^2 (2x+2ix-3i+1)\,dx$$

$$= (1+2i)\left[\frac{2}{2}(x^2)_1^2 + \frac{2i}{2}(x^2)_1^2 - 3i(x)_1^2 + (x)_1^2\right]$$

$$= (1+2i)[3+\cancel{3i}-\cancel{3i}+1]$$

$$= (1+2i)[4\cancel{-2i}]$$

$$= 4+8i$$

12. Evaluate $\oint_0^{1+i}(x^2-iy)\,dz$ along the path $y=x^2$

sol Along $y=x^2$

then $dy = 2x\,dx$

NOW $dz = dx+idy$

$= dx + i2x\,dx$

$dz = (1+2ix)dx$

$$\int_0^{1+i} (x^2 - iy)\,dz = \int_0^1 (x^2 - ix^2)(1+2ix)\,dx$$

$$= \int_0^1 x^2 + 2x^3 + i(2x^3 - x^2)\,dx$$

$$= \frac{1}{3}(x^3)_0^1 + \frac{2}{4}(x^4)_0^1 + \frac{2i}{4}(x^4)_0^1 - \frac{i}{3}(x^3)_0^1$$

$$= \frac{1}{3} + \frac{1}{2} + \frac{i}{2} - \frac{i}{3}$$

$$= \frac{5}{6} + \frac{i}{6}$$

$$= \frac{5+i}{6}$$