Machine Learning Assignment 2

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Problem 1

Let we have the following events :

E1 = Person is a terrorist

E2 = Person is not a terrorist

A = Person is identified as terrorist

P(E1) = $\frac{1}{100}$

P(E2) = $\frac{99}{100}$

P(A | E1) = 0.95 , P(A | E2) = 0.05

P(E1 | A) = $\frac{P(E1)P(A|E1)}{P(E1)P(A|E1)+P(E2)P(A|E2)} = \frac{0.01*0.95}{.01*0.95+.99*0.05} = \frac{19}{118}$

Problem 2

Let we have the following events :

E1 = HH (Both red balls are placed)

E2 = HT (First red and then white ball is placed)

E3 = TH (First white and then red ball is placed)

E4 = TT (Both white balls are placed)

A = All 3 red balls are drawn

P(E1) = P(E2) = P(E3) = P(E4) = $\frac{1}{4}$

P(A | E1) = 1 , P(A | E2) = P(A | E3) = $\frac{1}{8}$, P(A | E4) = 0

Using Baye's Theorem,

P(E1 | A) = $\frac{P(E1)P(A|E1)}{\sum_{n=1}^{4} P(E_i)P(A|E_i)} = \frac{\frac{1}{4}*1}{\frac{1}{4}*1+\frac{1}{4}*\frac{1}{8}+\frac{1}{4}*\frac{1}{8}+\frac{1}{4}*0} = \frac{4}{5}$

Problem 3

We stop the experiment when head shows up.
These are n bernoulli trials till first success occurs.
$p(success) = p(X = H) = \frac{1}{2} = p(say)$
$p(failure) = p(X = T) = \frac{1}{2} = 1 - p(say)$

For head(H), we have

Outcome	Head Count(H)	p(X=H)
H	1	1/2
TH	1	1/4
TTH	1	1/8
TTTH	1	1/16

$E[X=H] = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$
This is a G.P. with a = $\frac{1}{2}$ and r = $\frac{1}{2}$
$E[X=H] = \frac{a_1}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

For Tails(T) we have,

Outcome	Tail Count(H)	p(X=T)
H	0	1/2
TH	1	1/4
TTH	2	1/8
TTTH	3	1/16

$E[X=T] = 0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 2 \times \frac{1}{8} + 3 \times \frac{1}{16} + ... = \sum_{k=0}^{\infty} k(1-p)^k p = \frac{1-p}{p}$
$E[X=T] = \frac{1-p}{p} = \frac{1-\frac{1}{2}}{\frac{1}{2}} = 1$

Problem 4

The probability density function is given as :

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

The mean of this distribution is as

$\mu = E(X) = \int_a^b \frac{x}{b-a} dx = \frac{1}{2(b-a)} \quad x^2 \Big|_a^b = \frac{a+b}{2}$

The variance of this distribution is as
$\sigma^2 = E(X^2) - [E(X)]^2$

$E(X^2) = \int_a^b \frac{x^2}{b-a}dx = \frac{1}{3(b-a)} \quad x^3\Big|_a^b = \frac{a^2+b^2+ab}{3}$

$\sigma^2 = \frac{a^2+b^2+ab}{3} - (\frac{a+b}{2})^2 = \frac{a^2+b^2+ab}{3} - (\frac{a^2+b^2+2ab}{4}) = \frac{(b-a)^2}{12}$

Problem 5

Prove the following:

(1) $E[X] = E_Y[E_{X|Y}[X]]$

Proof :

$E[X] = \sum_x xp(x)$

$E_{X|Y}[X] = \sum_x xp(x|y) = \sum_x x\frac{p(x,y)}{p(y)}$

$E_Y[E_{X|Y}[X]] = \sum_x \sum_y p(y)x\frac{p(x,y)}{p(y)} = \sum_x \sum_y xp(x,y) = \sum_x x \sum_y p(x,y) = \sum_x xp(x)$

(2) $Var[X] = E_Y[Var_{X|Y}[X]] + Var_Y[E_{X|Y}[X]]$

$= E_Y[E_{X|Y}[X^2]] - E_Y[(E_{X|Y}[X])^2] + E_Y[(E_{X|Y}[X])^2] - (E_Y[E_{X|Y}[X]])^2$
$= E_Y[E_{X|Y}[X^2]] - (E_Y[E_{X|Y}[X]])^2$

Using results of part (1), we have

$= E[X^2] - (E[X])^2 = Var[X]$

Problem 6

Prove Weak Law Of Large Numbers

To Prove :

$$\lim_{N\to\infty} p(|\overline{X_N} - E[X]| > \epsilon) \to 0$$

where $\overline{X_N} = \frac{1}{N}\sum_{i=1}^{N} X_i$

We have a large population with mean $E[X]$ and a sample from this population with mean $\overline{X_N}$

As N $\to \infty$, $\overline{X_N} \to E[X]$

We know

$p(|X - E[X]| > \epsilon) \leq \frac{Var(X)}{\epsilon^2}$

Replacing a random variable X with $\overline{X_N}$ the relation still holds

$p(|\overline{X_N} - E[X]| > \epsilon) \leq \frac{Var(\overline{X_N})}{\epsilon^2}$

$$Var(\overline{X_N}) = \frac{\sum\limits_{1}^{N} Var(X_i)}{N^2} = \frac{N\sigma^2}{N^2} = \frac{\sigma^2}{N}$$

$$p(|\overline{X_N} - E[X]| > \epsilon) \leq \frac{\sigma^2}{N\epsilon^2}$$

As N $\rightarrow \infty$, $p(|\overline{X_N} - E[X]| > \epsilon) \rightarrow 0$