传递过程

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不可压缩流体的奈维-斯托克斯方程　（ρ为常数）

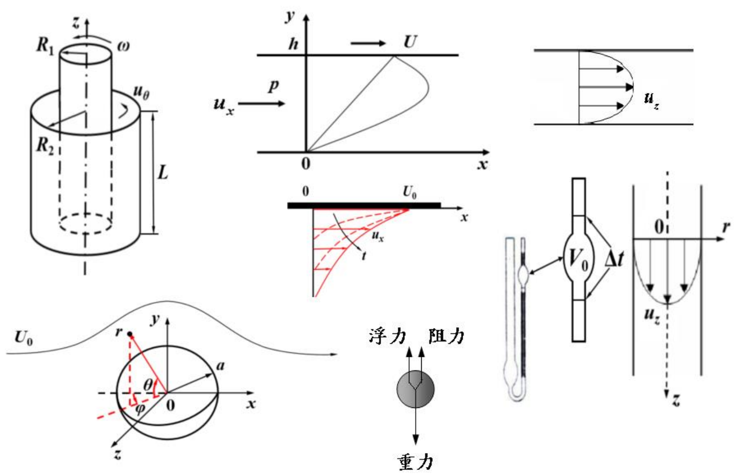
$$\rho\frac{Du_x}{Dt}=-\frac{\partial p}{\partial x}+\rho X+\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)$$

$$\rho\frac{Du_y}{Dt}=-\frac{\partial p}{\partial y}+\rho Y+\mu\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2}\right)$$

$$\rho\frac{Du_z}{Dt}=-\frac{\partial p}{\partial z}+\rho Z+\mu\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right)$$

惯性力　压力　体积力　粘性力

2.2奈维-斯托克斯方程的若干应用



2.2.1欧拉方程与伯努利方程

流体为理想流体，则 $\mu=0$，忽略奈维–斯托克斯方程中的粘性力项，可得：

$$\rho\frac{Du_x}{Dt}=-\frac{\partial p}{\partial x}+\rho X \qquad \rho\frac{Du_y}{Dt}=-\frac{\partial p}{\partial y}+\rho Y \qquad \rho\frac{Du_z}{Dt}=-\frac{\partial p}{\partial z}+\rho Z$$

欧拉方程

*x*方向：
$$-\frac{\partial p}{\partial x}+\rho X=\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right)$$

*y*方向：
$$-\frac{\partial p}{\partial y}+\rho Y=\rho\left(\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}\right)$$

*z*方向：
$$-\frac{\partial p}{\partial z}+\rho Z=\rho\left(\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right)$$

欧拉方程沿流线积分—伯努利方程

流体不可压缩 ρ 为常数，定常流动，X、Y、Z 为体积力

对欧拉方程分别乘以 dx、dy、dz, 得：

$$-\frac{\partial p}{\partial x}dx+\rho Xdx=\rho\left(u_x\frac{\partial u_x}{\partial x}dx+u_y\frac{\partial u_x}{\partial y}dx+u_z\frac{\partial u_x}{\partial z}dx\right)$$

$$-\frac{\partial p}{\partial y}dy+\rho Ydy=\rho\left(u_x\frac{\partial u_y}{\partial x}dy+u_y\frac{\partial u_y}{\partial y}dy+u_z\frac{\partial u_y}{\partial z}dy\right)$$

$$-\frac{\partial p}{\partial z}dz+\rho Zdz=\rho\left(u_x\frac{\partial u_z}{\partial x}dz+u_y\frac{\partial u_z}{\partial y}dz+u_z\frac{\partial u_z}{\partial z}dz\right)$$

引入流线方程：$\frac{dx}{u_x}=\frac{dy}{u_y}=\frac{dz}{u_z}$

$$u_ydx=u_xdy \qquad u_zdy=u_ydz \qquad u_zdx=u_xdz$$

可得：

$$-\frac{\partial p}{\partial x}dx+\rho Xdx=\rho\left(u_x\frac{\partial u_x}{\partial x}dx+u_x\frac{\partial u_x}{\partial y}dy+u_x\frac{\partial u_x}{\partial z}dz\right)$$

$$-\frac{\partial p}{\partial y}dy+\rho Ydy=\rho\left(u_y\frac{\partial u_y}{\partial x}dx+u_y\frac{\partial u_y}{\partial y}dy+u_y\frac{\partial u_y}{\partial z}dz\right)$$

$$-\frac{\partial p}{\partial z}dz+\rho Zdz=\rho\left(u_z\frac{\partial u_z}{\partial x}dx+u_z\frac{\partial u_z}{\partial y}dy+u_z\frac{\partial u_z}{\partial z}dz\right)$$

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz$$

$\boxed{\text{全微分}}$

可得：

$$-\frac{\partial p}{\partial x}dx+\rho Xdx=\rho u_x du_x$$

$$-\frac{\partial p}{\partial y}dy+\rho Ydy=\rho u_y du_y$$

$$-\frac{\partial p}{\partial z}dz+\rho Zdz=\rho u_z du_z$$

三式相加，可得：

$$-dp+\rho\left(Xdx+Ydy+Zdz\right)=\rho\left(u_x du_x+u_y du_y+u_z du_z\right)$$

体积力只有重力，X=0、Y=0、Z=-g

$$-dp-\rho gdz=\rho\left(\frac{1}{2}du_x^2+\frac{1}{2}du_y^2+\frac{1}{2}du_z^2\right)$$

$$-dp-\rho gdz=\frac{1}{2}\rho d\left(u_x^2+u_y^2+u_z^2\right)$$

$$-dp-\rho gdz=\frac{1}{2}\rho du^2$$

$$\frac{1}{2}\rho du^2+dp+\rho gdz=0$$

积分得：

$$\boxed{\frac{1}{2}\rho u^2+p+\rho gz=C}$$

伯努利方程

同一根流线上C为常数
不同流线C可能不同

（同一根流线上）

欧拉方程无旋条件下积分—伯努利方程

$$-\frac{\partial p}{\partial x}+\rho X=\rho\left(\frac{\partial u_x}{\partial t}+\boxed{u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}}\right)$$

$$\boxed{u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}}=\boxed{\left(u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_y}{\partial x}+u_z\frac{\partial u_z}{\partial x}\right)}+u_y\boxed{\left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right)}+u_z\boxed{\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)}$$

$$\boxed{u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_y}{\partial x}+u_z\frac{\partial u_z}{\partial x}}=\frac{1}{2}\frac{\partial u^2}{\partial x}\qquad \omega_z=\frac{1}{2}\boxed{\left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right)}\qquad \omega_y=\frac{1}{2}\boxed{\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)}$$

$$-\frac{\partial p}{\partial x}+\rho X=\rho\left(\frac{\partial u_x}{\partial t}+\frac{1}{2}\frac{\partial u^2}{\partial x}-2u_y\omega_z+2u_z\omega_y\right)$$

流体微团作无旋运动 $\omega_x=\omega_y=\omega_z=0$　　定常 $\frac{\partial u_x}{\partial t}=0$

$$-\frac{\partial p}{\partial x}+\rho X=\frac{1}{2}\rho\frac{\partial u^2}{\partial x}$$

同理可得：

$$-\frac{\partial p}{\partial y}+\rho Y=\frac{1}{2}\rho\frac{\partial u^2}{\partial y}$$

$$-\frac{\partial p}{\partial z}+\rho Z=\frac{1}{2}\rho\frac{\partial u^2}{\partial z}$$

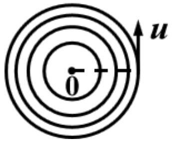
分别乘以dx、dy、dz, 三式相加，可得：

$$-dp+\rho\left(Xdx+Ydy+Zdz\right)=\frac{1}{2}\rho du^2$$

体积力只有重力，X=0、Y=0、Z=-g

$$-dp-\rho g=\frac{1}{2}\rho du^2$$

$$\frac{1}{2}\rho du^2+dp+\rho gdz=0$$



积分得：

$$\boxed{\frac{1}{2}\rho u^2+p+\rho gz=C}$$

伯努利方程

整个无旋运动流场中C为常数
即不同流线C相同

（不同流线之间）

伯努利方程

$$\boxed{\frac{1}{2}\rho u^2+p+\rho gz=C}$$

动能　静压能　位能 (单位体积)

一维流动　无换热　无外功　无支流　定常　不可压

三种能量之间可以相互转换，但总和不变。

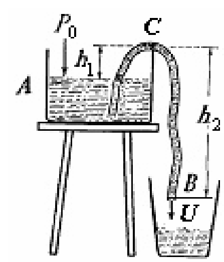
同一水平面上：$\frac{1}{2}\rho U^2+p=$常数

例2-1 虹吸

原理：伯努利方程　**控制面：*A—B***

$$\frac{1}{2}\rho U_1^2 + p_1 + \rho g z_1 = \frac{1}{2}\rho U_2^2 + p_2 + \rho g z_2$$

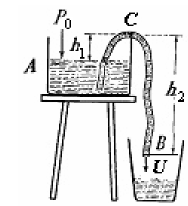
$$p_0 + \rho g\left(h_2 - h_1\right) = \frac{1}{2}\rho U^2 + p_0$$



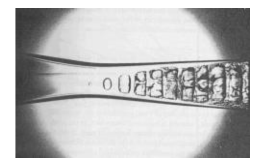
$$U = \sqrt{2g\Delta h}$$

$\Delta h \uparrow$　　$U \uparrow$

问题探讨



$$U = \sqrt{2g\Delta h}$$

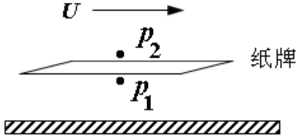


问题： $\Delta h \uparrow$，$U \uparrow$？

$$p_C = p_0 - \rho g h_2$$

20℃下饱和水蒸汽压强：2334Pa

例2-2 拍纸牌

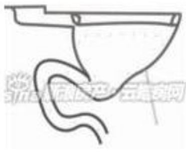


忽略位能变化，由伯努利方程得：$\frac{1}{2}\rho U_1^2 + p_1 = \frac{1}{2}\rho U_2^2 + p_2$

一拍，$U_2=U$，此时$U_1=0$，则：　　$p_1 - p_2 = \frac{1}{2}\rho U^2 > 0$

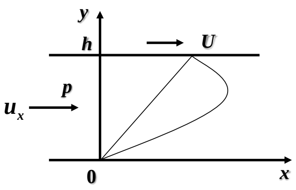
课后思考

1.你了解座便器的抽吸原理吗？
2.当列车进站时，为什么要站在安全线后？



2.2.2平板间流动—库特流

下板固定，上板以恒定速度U运动，板间流体在压差和上板拖动下作定常层流流动。



运用奈维–斯托克斯方程求解板间速度分布

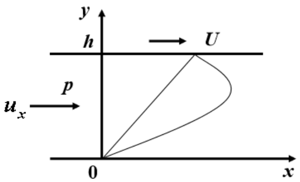
$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho X+\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)$$

定常：$\dfrac{\partial u_x}{\partial t}=0$

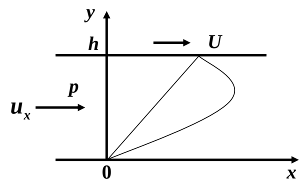
一维流动：$\begin{cases} u_x\neq 0 \\ u_y=0 \\ u_z=0 \end{cases}$ $\begin{cases} \dfrac{\partial u_x}{\partial x}=0 \\ \dfrac{\partial u_x}{\partial y}\neq 0 \\ \dfrac{\partial u_x}{\partial z}=0 \end{cases}$ $\begin{cases} \dfrac{\partial^2 u_x}{\partial x^2}=0 \\ \dfrac{\partial^2 u_x}{\partial y^2}\neq 0 \\ \dfrac{\partial^2 u_x}{\partial z^2}=0 \end{cases}$

x方向无重力：$X=0$

简化得：$\mu\dfrac{\partial^2 u_x}{\partial y^2}=\dfrac{\partial p}{\partial x}$



$$\mu \frac{d^2 u_x}{dy^2} = \frac{dp}{dx}$$



边界条件：$\begin{cases} y = 0, u_x = 0 \\ y = h, u_x = U \end{cases}$

库特流的速度分布：

$$u_x = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - hy\right) + U\frac{y}{h}$$

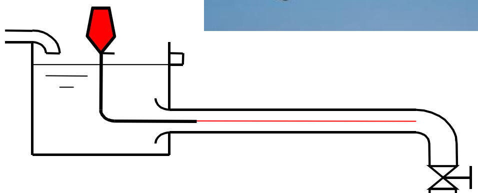
情形1:若上板也固定：$u_x = \dfrac{1}{2\mu}\dfrac{dp}{dx}\left(y^2 - hy\right)$ 抛物线分布

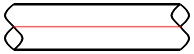
情形2:若无压差：$\dfrac{u_x}{U} = \dfrac{y}{h}$ 线性分布

2.2.3管内层流

2.2.3.1雷诺数Re

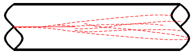
雷诺实验





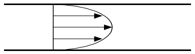
层流

分子传递 抛物面

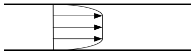


湍流

涡流传递 对数



$$\frac{U}{u_{max}} = \frac{1}{2}$$



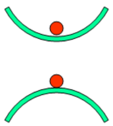
$$\frac{U}{u_{max}} \approx 0.8$$

流动状态的判别—雷诺数*Re*

判据：$$Re=\frac{\text{流体密度}\times\text{特征尺度}\times\text{特征速度}}{\text{流体粘度}}$$

对圆管流动：$$Re=\frac{\rho DU}{\mu}=\frac{DU}{\nu}$$

	$Re<2100$	层流	稳定
$2100<$	$Re<10000$	过渡流	介稳
	$Re>10000$	湍流	稳定



临界雷诺数Re_{xc}=2100

*Re*数的物理含义：$$Re=\frac{\text{惯性力}}{\text{粘性力}}$$

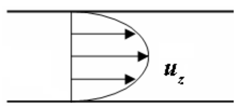
$$\text{惯性力}=Ma\propto L^3\rho\cdot\frac{U}{L/U}=L^2\rho U^2$$

$$\text{粘性力}=\tau A=A\mu\frac{dU}{dy}\propto L^2\mu\frac{U}{L}=L\mu U$$

$$\frac{\text{惯性力}}{\text{粘性力}}=\frac{L^2\rho U^2}{L\mu U}=\frac{L\rho U}{\mu}=Re$$

2.2.3.2 管内层流速度分布——抛物线分布

$$u_z = 2U\left(1 - \frac{r^2}{R^2}\right)$$



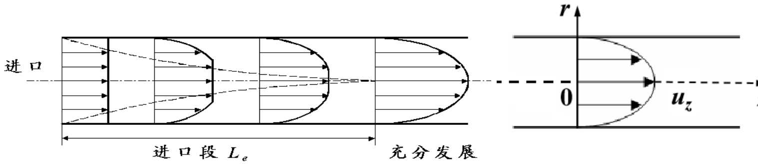
2.2.3.3 管道沿程阻力——压降

$$-\Delta p = \lambda \frac{L}{D}\frac{1}{2}\rho U^2$$

摩擦阻力系数 $\lambda = \dfrac{64}{Re}$

2.2.3.2 管内层流速度分布——抛物线分布

运用柱坐标系中的奈维–斯托克斯方程求解。



柱坐标系速度分布：

z 方向：

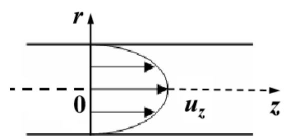
$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right)$$
$$= -\frac{\partial p}{\partial z} + \rho X_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right]$$

定常：$\dfrac{\partial u_z}{\partial t}=0$

一维流动：$\begin{cases} u_r=0 \\ u_\theta=0 \\ u_z\neq 0 \end{cases}$ $\begin{cases} \dfrac{\partial u_z}{\partial r}\neq 0 \\ \dfrac{\partial u_z}{\partial \theta}=0 \\ \dfrac{\partial u_z}{\partial z}=0 \end{cases}$ $\begin{cases} \dfrac{\partial^2 u_z}{\partial \theta^2}=0 \\ \dfrac{\partial^2 u_z}{\partial z^2}=0 \end{cases}$

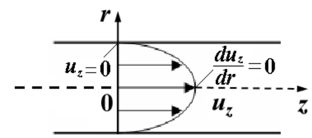
z方向无重力：$X_z=0$

简化得：$\mu\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u_z}{\partial r}\right)=\dfrac{\partial p}{\partial z}$



$$\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right)=\frac{dp}{dz}$$

边界条件：$\begin{cases} r=0,\dfrac{du_z}{dr}=0 \\ r=R,u_z=0 \end{cases}$



积分：$r\dfrac{du_z}{dr}=\dfrac{1}{2\mu}\dfrac{dp}{dz}r^2+C_1$

$\because r=0,\dfrac{du_z}{dr}=0$； $\therefore C_1=0$

再积分，代入边界条件得：$u_z=-\dfrac{1}{4\mu}\dfrac{dp}{dz}\left(R^2-r^2\right)$

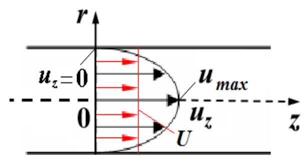
管内层流速度分布

速度分布：$$u_z = -\frac{1}{4\mu}\frac{dp}{dz}\left(R^2 - r^2\right)$$

流量：$$V = \int_A u_z dA = -\frac{\pi R^4}{8\mu}\frac{dp}{dz}$$ 哈根-泊谡叶方程

平均速度：$$U = \frac{V}{A} = -\frac{R^2}{8\mu}\frac{dp}{dz}$$

速度分布：$$u_z = 2U\left(1 - \frac{r^2}{R^2}\right)$$



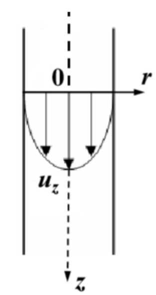
例2-3 毛细管粘度计


奥氏粘度计　乌氏粘度计

奥氏粘度计和乌氏粘度计测量粘度的原理？使用中应注意哪些问题？

定常：$\frac{\partial u_z}{\partial t} = 0$

一维流动：$\begin{cases} u_r = 0 \\ u_\theta = 0 \\ u_z \neq 0 \end{cases}$ $\begin{cases} \frac{\partial u_z}{\partial r} \neq 0 \\ \frac{\partial u_z}{\partial \theta} = 0 \\ \frac{\partial u_z}{\partial z} = 0 \end{cases}$ $\begin{cases} \frac{\partial^2 u_z}{\partial \theta^2} = 0 \\ \frac{\partial^2 u_z}{\partial z^2} = 0 \end{cases}$

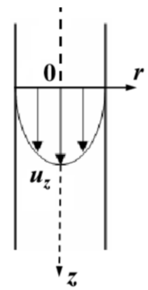


z方向有重力：$X_z = g$

z方向无压差力：$\frac{\partial p}{\partial z} = 0$

简化柱坐标系中的奈维−斯托克斯方程得：

$$\mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = -\rho g$$



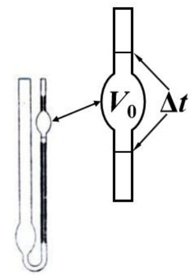
边界条件：$\begin{cases} r = 0, \dfrac{du_z}{dr} = 0 \\ r = R, u_z = 0 \end{cases}$

解得速度分布：$u_z = \dfrac{\rho g}{4\mu}\left(R^2 - r^2\right)$

$$\nu = \frac{\mu}{\rho} \qquad u_z = \frac{g}{4\nu}\left(R^2 - r^2\right)$$

流量：$V = \displaystyle\int_A u_z dA = \frac{\pi g R^4}{8\nu}$

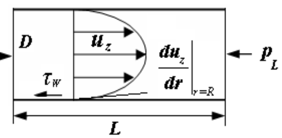
$$V_0 = V\Delta t = \frac{\pi g R^4 \Delta t}{8\nu}$$



标准样ν_1测定：流完V_0需要Δt_1时间
样品样ν_2测定：流完V_0需要Δt_2时间

$$\frac{\dfrac{\pi g R^4 \Delta t_1}{8\nu_1}}{\dfrac{\pi g R^4 \Delta t_2}{8\nu_2}} = 1 \qquad \nu_2 = \frac{\nu_1}{\Delta t_1}\Delta t_2$$

2.2.2.3 管道沿程阻力—压降



$$\mu \rightarrow u_z \rightarrow \left.\frac{du_z}{dr}\right|_{r=R} \rightarrow \tau_W \rightarrow -\Delta p$$

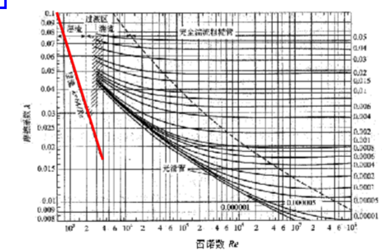
$$u_z = 2U\left(1-\frac{r^2}{R^2}\right) \rightarrow \left.\frac{du_z}{dr}\right|_{r=R} = -\frac{4U}{R} \rightarrow \tau_W = -\mu\left.\frac{du_z}{dr}\right|_{r=R} = \frac{4\mu U}{R}$$

$$(p_0 - p_L)\cdot\frac{1}{4}\pi D^2 = \tau_W \cdot \pi DL$$

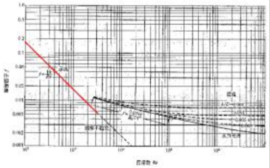
$$-\Delta p = p_0 - p_L = \frac{4L}{D}\tau_W = \frac{32\mu UL}{D^2} = \frac{64}{Re}\frac{L}{D}\frac{1}{2}\rho U^2$$

$$\boxed{-\Delta p = \lambda\frac{L}{D}\frac{1}{2}\rho U^2}$$

摩擦阻力系数 $\lambda = \frac{64}{Re}$



例2-4 范宁摩擦系数 f



圆管层流壁面切应力为：

$$\tau_W = \frac{4\mu U}{R}$$

$$\tau_W = \frac{4\mu U}{R} = \frac{16\mu}{\rho D U}\frac{1}{2}\rho U^2 = \frac{16}{Re}\frac{1}{2}\rho U^2$$

定义 $\boxed{\tau_W = f\frac{1}{2}\rho U^2}$

圆管层流时，范宁摩擦系数 $f = \frac{16}{Re}$

摩擦阻力系数 $\lambda = \frac{64}{Re} = 4\times\frac{16}{Re} = 4f$ 范宁摩擦系数