传递过程

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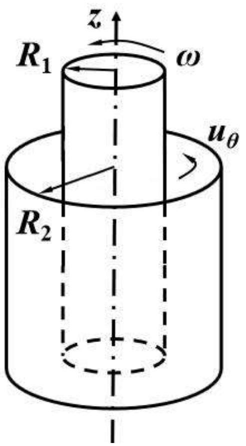
2.2.4旋转柱面间的流动—旋转粘度计原理

垂直的同心套筒环隙间充满了流体。内筒外径为R_1, 外筒内径为R_2, 当内筒以角速度 ω旋转时，环隙间的流体随之旋转。若圆筒足够长，端效应可以忽略。

定常：$\dfrac{\partial u_\theta}{\partial t}=0$

一维流动：$\begin{cases} u_r=0 \\ u_\theta \neq 0 \\ u_z=0 \end{cases}$ $\begin{cases} \dfrac{\partial u_\theta}{\partial r}\neq 0 \\ \dfrac{\partial u_\theta}{\partial \theta}=0 \\ \dfrac{\partial u_\theta}{\partial z}=0 \end{cases}$ $\begin{cases} \dfrac{\partial^2 u_\theta}{\partial \theta^2}=0 \\ \dfrac{\partial^2 u_\theta}{\partial z^2}=0 \end{cases}$

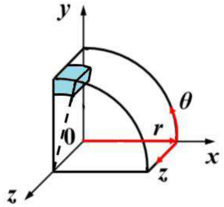


θ 方向无重力:$X_\theta=0$

θ方向无压差力：$\dfrac{\partial p}{\partial \theta}=0$

回顾：

柱坐标系—奈维-斯托克斯方程



r方向：$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)$

$$=-\frac{\partial p}{\partial r}+\rho X_r+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r u_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right]$$

θ方向：$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)$

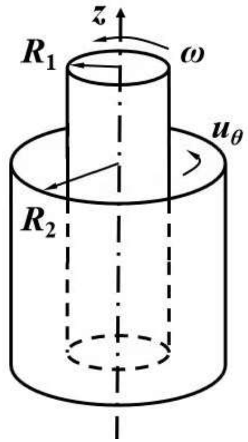
$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\rho X_\theta+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r u_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right]$$

z方向：$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)$

$$=-\frac{\partial p}{\partial z}+\rho X_z+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]$$

简化柱坐标系中的奈维-斯托克斯方程得：

$$\frac{d}{dr}\left(\frac{1}{r}\frac{dru_\theta}{dr}\right)=0$$



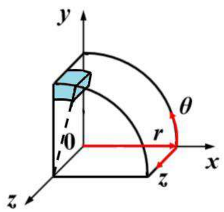
边界条件：$\begin{cases} r=R_1, u_\theta=\omega R_1 \\ r=R_2, u_\theta=0 \end{cases}$

解得速度分布：

$$u_\theta=\frac{\omega R_1^2}{r}\frac{R_2^2-r^2}{R_2^2-R_1^2}$$

回顾：

柱坐标系中剪切应力与形变的关系：



$$\tau_{r\theta}=\tau_{\theta r}=-\mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]\qquad \tau_{rr}=-2\mu\frac{\partial u_r}{\partial r}$$

$$\tau_{\theta z}=\tau_{z\theta}=-\mu\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right)\qquad \tau_{\theta\theta}=-2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)$$

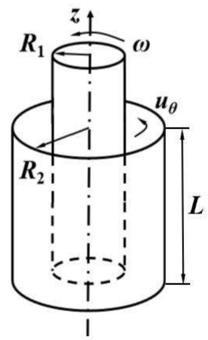
$$\tau_{zr}=\tau_{rz}=-\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)\qquad \tau_{zz}=-2\mu\frac{\partial u_z}{\partial z}$$

柱坐标系中剪切应力$\tau_{r\theta}$与形变的关系：

$$\tau_{r\theta}=-\mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]$$

因为$u_r=0$，所以：$\tau_{r\theta}=-\mu r\dfrac{\partial}{\partial r}\left(\dfrac{u_\theta}{r}\right)$

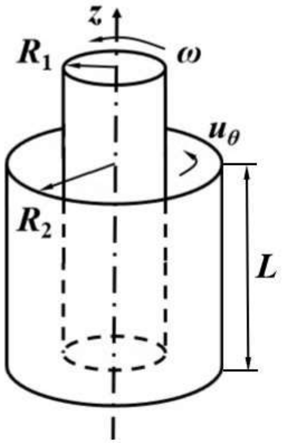
剪切应力分布：$\tau_{r\theta}=\dfrac{2\mu\omega R_1^2R_2^2}{R_2^2-R_1^2}\dfrac{1}{r^2}$



作用在内筒壁上的摩擦力F为：

$$F=\tau_{r\theta}\big|_{r=R_1}2\pi R_1L=\frac{4\pi\mu\omega R_1R_2^2L}{R_2^2-R_1^2}$$

作用在内外筒壁上的力矩相等，即：



$$M_{or}=FR_1=\frac{4\pi\mu\omega R_1^2R_2^2L}{R_2^2-R_1^2}$$

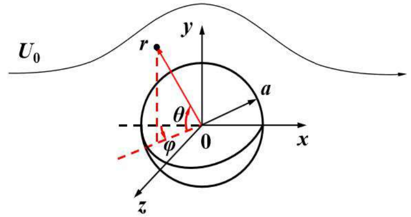
$$\mu=\frac{M_{or}\left(R_2^2-R_1^2\right)}{4\pi\omega R_1^2R_2^2L}$$

上式是旋转粘度计的测粘度的原理

2.2.5 低雷诺数下绕球爬流—斯托克斯阻力定律

爬流是$Re<1$的极慢运动 $Re=\dfrac{\text{惯性力}}{\text{粘性力}}<<1$

忽略惯性力，选用球坐标系



定常：$\dfrac{\partial(\ )}{\partial t}=0$

二维流动：$\begin{cases} u_r\neq 0 \\ u_\theta\neq 0 \\ u_\varphi=0 \end{cases}$ $\begin{cases} \dfrac{\partial(\ )}{\partial r}\neq 0 \\ \dfrac{\partial(\ )}{\partial \theta}\neq 0 \\ \dfrac{\partial(\ )}{\partial \varphi}=0 \end{cases}$

忽略重力： $X_r=X_\theta=X_\varphi=0$

简化球坐标系中的连续性方程和奈维-斯托克斯方程，可得：

连续性方程：

$$\frac{1}{r^2}\frac{\partial r^2 u_r}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial u_\theta \sin\theta}{\partial\theta}=0$$

奈维-斯托克斯方程 r 方向：

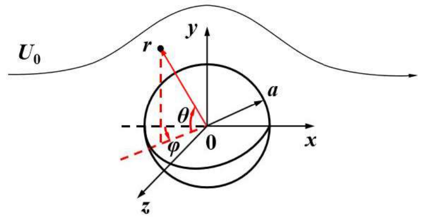
$$\frac{\partial p}{\partial r}=\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right)-\frac{2}{r^2}u_r-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2}{r^2}u_\theta\cot\theta\right]$$

奈维-斯托克斯方程 θ 方向：

$$\frac{1}{r}\frac{\partial p}{\partial\theta}=\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\theta}{\partial\theta}\right)+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r^2\sin^2\theta}\right]$$

3个方程，3个未知量 u_r, u_θ, p，可求出解析解

边界条件：

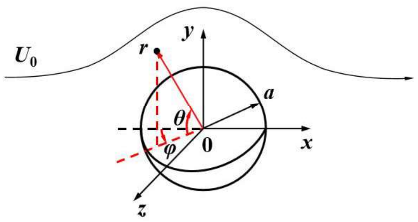


$$\begin{cases} r=a, u_r=0, u_\theta=0 \\ r\to\infty, u_r=U_0\cos\theta, u_\theta=-U_0\sin\theta, p=p_0 \end{cases}$$

方程组为线性偏微分方程组，可以用分离变量法解得速度分布和压力分布为：

$$\begin{cases} u_r=U_0\left[1-\frac{3}{2}\frac{a}{r}+\frac{1}{2}\left(\frac{a}{r}\right)^3\right]\cos\theta \\ u_\theta=-U_0\left[1-\frac{3}{4}\frac{a}{r}-\frac{1}{4}\left(\frac{a}{r}\right)^3\right]\sin\theta \\ p=p_0-\frac{3}{2}\frac{\mu U_0}{a}\left(\frac{a}{r}\right)^2\cos\theta \end{cases}$$

球表面压力分布：



$r = a \qquad p = p_0 - \frac{3}{2}\frac{\mu U_0}{a}\cos\theta$

剪切应力分布：

$$\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]$$

代入u_r, u_θ得

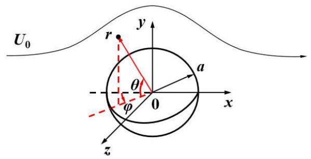
$$\tau_{r\theta} = -\frac{3}{2}\mu U_0 \frac{a^3}{r^4}\sin\theta$$

球表面剪切应力分布：

$r = a$

$$\tau_{r\theta} = -\frac{3}{2a}\mu U_0 \sin\theta$$

球表面总阻力：



$$D = \int_0^{2\pi} d\varphi \int_0^{\pi}\left(-p\cos\theta - \tau_{r\theta}\sin\theta\right)a^2\sin\theta d\theta$$

$$D = 2\pi\mu a U_0 + 4\pi\mu a U_0$$

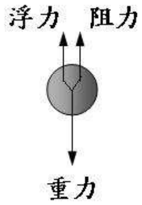
压差阻力　　摩擦阻力

$$D = 6\pi\mu a U_0$$

斯托克斯阻力定律

适用条件Re<1的爬流

例2-5 落球法测粘度



解：测定小球在静止流体中匀速下降速度u,根据力平衡有：

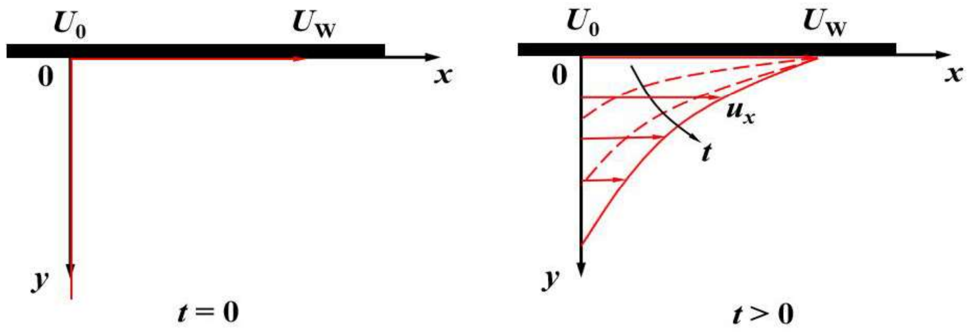
$$\frac{1}{6}\pi d^3\rho_s g=\frac{1}{6}\pi d^3\rho g+6\pi\mu\frac{d}{2}u$$

$$\mu=\frac{(\rho_s-\rho)gd^2}{18u}$$

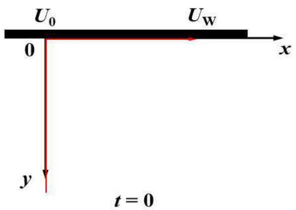
适用条件$Re<1$的爬流

2.2.6静止流体中的平板启动

静止的水面上有一块无限大平板，初始速度为U_0=0，突然以 U_W速度运动，并维持不变。平板下水中的速度分布u_x随时间也发生变化。

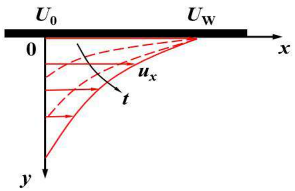


非定常：$\dfrac{\partial u_x}{\partial t}\neq 0$



$t=0$

一维流动：$\begin{cases} u_x \neq 0 \\ u_y = 0 \\ u_z = 0 \end{cases}$ $\begin{cases} \dfrac{\partial u_x}{\partial x} = 0 \\ \dfrac{\partial u_x}{\partial y} \neq 0 \\ \dfrac{\partial u_x}{\partial z} = 0 \end{cases}$ $\begin{cases} \dfrac{\partial^2 u_x}{\partial x^2} = 0 \\ \dfrac{\partial^2 u_x}{\partial y^2} \neq 0 \\ \dfrac{\partial^2 u_x}{\partial z^2} = 0 \end{cases}$



$t>0$

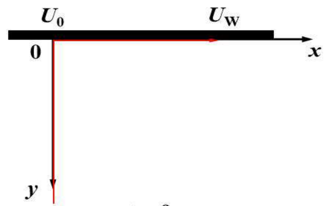
x方向没有重力：$X=0$

x方向无压差力：$\dfrac{\partial p}{\partial x}=0$

$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho X+\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)$$

简化奈维-斯托克斯方程，可得：

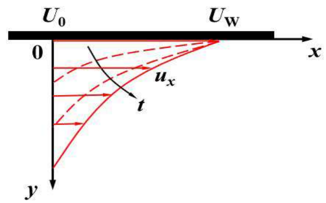
$$\frac{\partial u_x}{\partial t}=\nu\frac{\partial^2 u_x}{\partial y^2}$$



$t=0$

初始条件：$t=0, u_x=0$

边界条件：$t>0, \begin{cases} y=0, u_x = U_W \\ y\to\infty, u_x = U_0 = 0 \end{cases}$



$t>0$

方程为一维非定常偏微分方程 $\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial y^2}$

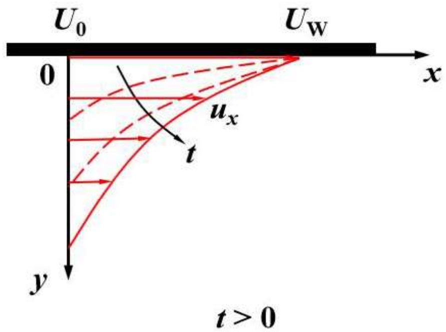
令 $\eta = \frac{y}{\sqrt{4\nu t}}$ 则 $\frac{\partial u_x}{\partial t} = \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial t} = -\frac{\eta}{2t}\frac{\partial u_x}{\partial \eta}$

$$\frac{\partial u_x}{\partial y} = \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{4\nu t}}\frac{\partial u_x}{\partial \eta}$$

$$\frac{\partial^2 u_x}{\partial y^2} = \frac{1}{\sqrt{4\nu t}}\frac{\partial \frac{\partial u_x}{\partial \eta}}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{4\nu t}\frac{\partial^2 u_x}{\partial \eta^2}$$

代入原方程可得： $\frac{d^2 u_x}{d\eta^2} + 2\eta\frac{du_x}{d\eta} = 0$

$$\frac{d^2 u_x}{d\eta^2} + 2\eta\frac{du_x}{d\eta} = 0$$



边界条件：$\begin{cases} \eta = 0, u_x = U_W \\ \eta \to \infty, u_x = U_0 = 0 \end{cases}$

设：$\frac{du_x}{d\eta} = p$

$$\frac{dp}{d\eta} + 2\eta p = 0$$

积分： $p = C_1 e^{-\eta^2}$ $\frac{du_x}{d\eta} = C_1 e^{-\eta^2}$

再积分：$\int_{U_W}^{u_x} du_x = C_1 \int_0^{\eta} e^{-\eta^2} d\eta$

$$u_x - U_W = C_1\int_0^{\eta} e^{-\eta^2} d\eta$$

边界条件：$\eta \to \infty, u_x = U_0 = 0$

$$C_1 = \frac{U_0 - U_W}{\int_0^{\infty} e^{-\eta^2} d\eta}$$

其中：$\int_0^{\infty} e^{-\eta^2} d\eta = \frac{\sqrt{\pi}}{2}$

解方程得速度分布：

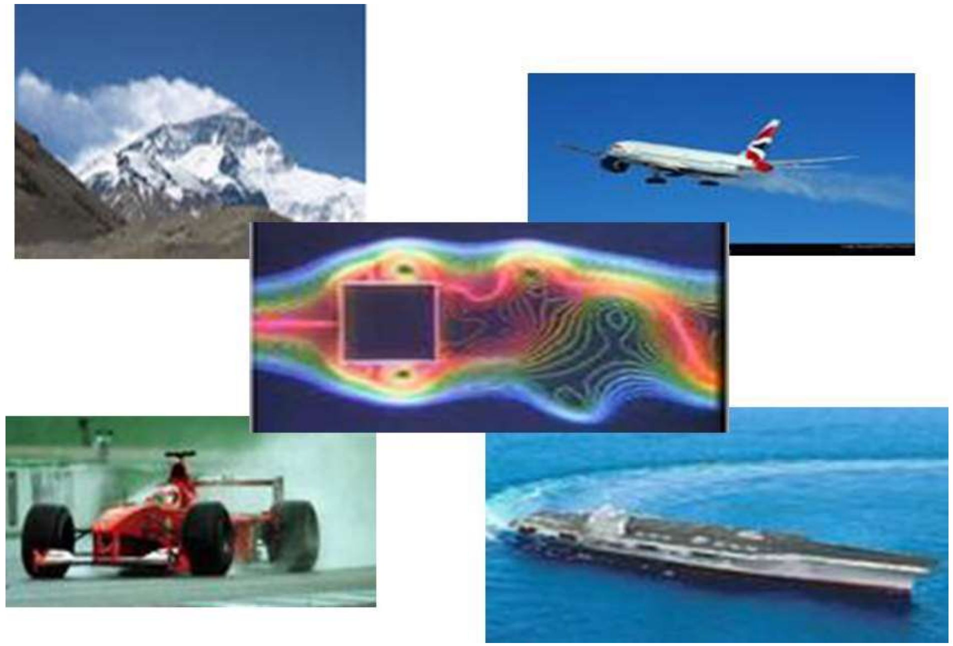
$$\frac{u_x - U_W}{U_0 - U_W} = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2} d\eta$$

高斯误差函数

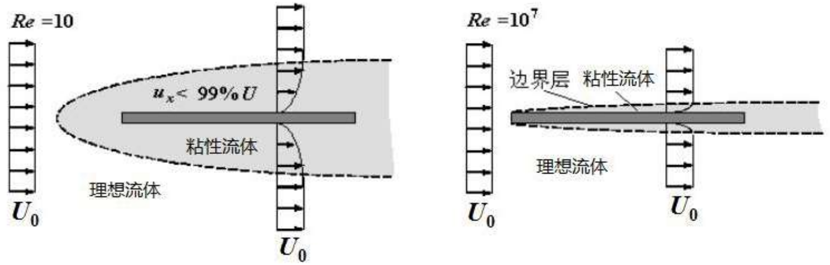
$$\frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2} d\eta = erf\left(\eta\right) = \frac{u_x - U_W}{U_0 - U_W}$$

其中：$\eta = \frac{y}{\sqrt{4\nu t}}$

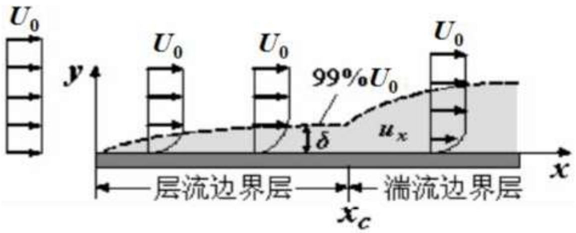
2.3边界层理论



2.3.1 概念和特点



1904年，普朗特提出“边界层”概念

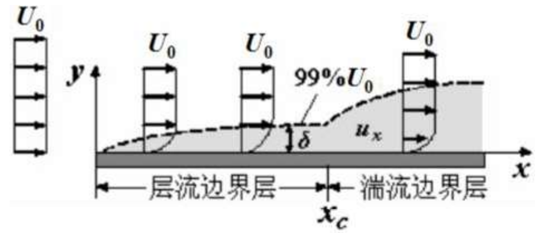


流体区域以99%U作为边界，内层区为粘性流体u_x，外层区为理想流体U_0。

特点

①. 慢：边界层内 $u_x < U_0$，壁面 $u_x = 0$。

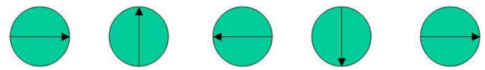
②. 薄： $\delta << x$ 。



③. 陡： $\dfrac{du_x}{dy}$ 很大。

④. 增：$x\uparrow$， $\delta\uparrow$。

⑤. 旋：微团有旋。



⑥. 惯、粘同量级：惯性力与粘性力在边界层内量级相当。

⑦. 截面等压力：无压差流动。

⑧. 流型会转变：$x > x_c$时，层流→湍流。

⑨. 逆压，失速会分离（绕曲面流动时的表现）。