传递过程

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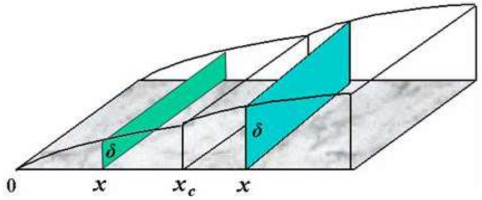
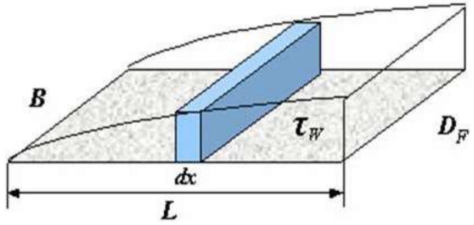
2022年秋季

2.3.2平板边界层阻力计算





临界雷诺数$Re_{xc}=5\times10^5$

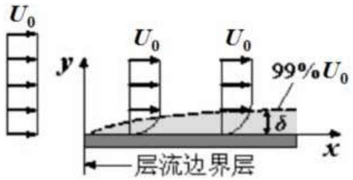
层流 $$\frac{\delta}{x}=\frac{4.96}{\sqrt{Re_x}}$$

$$D_F=\frac{1.328}{\sqrt{Re_L}}BL\frac{1}{2}\rho U_0^2$$

$$D_F=C_D A\frac{1}{2}\rho U_0^2 \qquad C_D=\frac{1.328}{\sqrt{Re_L}}$$

2.3.2.1普朗特边界层方程

层流边界层流动阻力规律



定常：$\dfrac{\partial u_x}{\partial t}=\dfrac{\partial u_y}{\partial t}=0$

$$
\text{二维流动：}\begin{cases} u_x\neq 0\\ u_y\neq 0\\ u_z=0\end{cases}
\begin{cases}
\dfrac{\partial u_x}{\partial x}\neq 0 & \dfrac{\partial u_y}{\partial x}\neq 0\\
\dfrac{\partial u_x}{\partial y}\neq 0 & \dfrac{\partial u_y}{\partial y}\neq 0\\
\dfrac{\partial u_x}{\partial z}=0 & \dfrac{\partial u_y}{\partial z}=0
\end{cases}
\begin{cases}
\dfrac{\partial^2 u_x}{\partial x^2}\neq 0 & \dfrac{\partial^2 u_x}{\partial x^2}\neq 0\\
\dfrac{\partial^2 u_x}{\partial y^2}\neq 0 & \dfrac{\partial^2 u_x}{\partial y^2}\neq 0\\
\dfrac{\partial^2 u_x}{\partial z^2}=0 & \dfrac{\partial^2 u_x}{\partial z^2}=0
\end{cases}
$$

忽略重力：$X=Y=0$　　无压差流动：$\dfrac{\partial p}{\partial x}=\dfrac{\partial p}{\partial y}=0$

问题探讨 边界层内流动是二维流动吗？

$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho X+\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)$$

奈维-斯托克斯方程简化可得：

$$u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}=\nu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}\right)$$

同理，y方向可得：

$$u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}=\nu\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}\right)$$

连续性方程：

$$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}=0$$

依据边界层流动特点，运用量级比较进一步简化方程。

边界层厚度薄， $\delta << x$ 。

量级表示： $\begin{cases} x \sim O(L) \\ y \sim O(\delta) \end{cases}$ $u_x \sim O(U_0)$

$$\frac{\partial u_x}{\partial x} \sim O\left(\frac{U_0}{L}\right)$$

由 $\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$ 可得： $\frac{\partial u_y}{\partial y} \sim O\left(\frac{U_0}{L}\right)$

$$u_y \sim O\left(\frac{\delta U_0}{L}\right)$$

量级表示

$$\begin{cases} u_x \sim O(U_0) \\ u_y \sim O\left(\dfrac{\delta U_0}{L}\right) \end{cases}$$

$$\begin{cases} \dfrac{\partial u_x}{\partial x} \sim O\left(\dfrac{U_0}{L}\right) \\ \dfrac{\partial u_x}{\partial y} \sim O\left(\dfrac{U_0}{\delta}\right) \\ \dfrac{\partial u_y}{\partial x} \sim O\left(\dfrac{\delta U_0}{L^2}\right) \\ \dfrac{\partial u_y}{\partial y} \sim O\left(\dfrac{U_0}{L}\right) \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 u_x}{\partial x^2} \sim O\left(\dfrac{U_0}{L^2}\right) \\ \dfrac{\partial^2 u_x}{\partial y^2} \sim O\left(\dfrac{U_0}{\delta^2}\right) \\ \dfrac{\partial^2 u_y}{\partial x^2} \sim O\left(\dfrac{\delta U_0}{L^3}\right) \\ \dfrac{\partial^2 u_y}{\partial y^2} \sim O\left(\dfrac{U_0}{\delta L}\right) \end{cases}$$

x方向 $$u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}=\nu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}\right)$$

量级表示 $$\frac{U_0^2}{L}\qquad\frac{U_0^2}{L}\qquad\frac{U_0}{L^2}<<\frac{U_0}{\delta^2}$$

根据边界层内惯性力与粘性力量级相当的特点，则有：

$$u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}=\nu\frac{\partial^2 u_x}{\partial y^2}$$

y方向 $$u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}=\nu\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}\right)$$

量级表示 $$\frac{\delta U_0^2}{L^2}\qquad\frac{\delta U_0^2}{L^2}\qquad\frac{\delta U_0}{L^3}<<\frac{U_0}{\delta L}$$

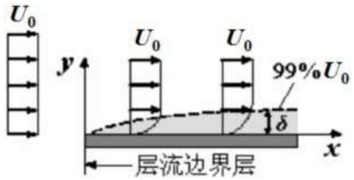
惯性力项　　粘性力项

x方向与y方向比较

$$\frac{\dfrac{U_0^2}{L}}{\dfrac{\delta U_0^2}{L^2}}=\frac{L}{\delta}>>1\qquad\frac{\dfrac{U_0}{\delta^2}}{\dfrac{U_0}{\delta L}}=\frac{L}{\delta}>>1$$

结论：忽略y方向的流动

量级比较简化可得
普朗特边界层方程：



$$\begin{cases} u_x \dfrac{\partial u_x}{\partial x} + u_y \dfrac{\partial u_x}{\partial y} = \nu \dfrac{\partial^2 u_x}{\partial y^2} \\ \dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y} = 0 \end{cases}$$

边界条件：$\begin{cases} y = 0, u_x = u_y = 0 \\ y \to \infty, u_x = U_0 \end{cases}$

引入流函数的目的是将速度变量 u_x, u_y 用一个变量 ψ 代替，从而使方程的求解得以简化。

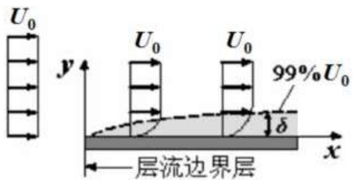
$$\begin{cases} u_x = \dfrac{\partial \psi}{\partial y} \\ u_y = -\dfrac{\partial \psi}{\partial x} \end{cases}$$

可得：$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}$$

边界条件：$\begin{cases} y = 0, \dfrac{\partial \psi}{\partial y} = \dfrac{\partial \psi}{\partial x} = 0 \\ y \to \infty, \dfrac{\partial \psi}{\partial y} = U_0 \end{cases}$

相似变换法求解

令速度分布为： $\dfrac{u_x}{U_0}=\varphi\left(\dfrac{y}{\delta}\right)$



根据边界层内惯性力与粘性力量级相当的特点，则有：

$$\frac{U_0^2}{L}\sim v\frac{U_0}{\delta^2}\Rightarrow\delta\sim\sqrt{\frac{vL}{U_0}}\Rightarrow\delta\sim\sqrt{\frac{vx}{U_0}}$$

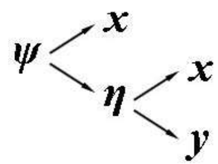
令 $\eta=\dfrac{y}{\delta}$ 则有：$\eta=y\sqrt{\dfrac{U_0}{vx}}$ $\dfrac{u_x}{U_0}=\varphi(\eta)=\varphi\left(y\sqrt{\dfrac{U_0}{vx}}\right)$

流函数 $\begin{cases} u_x=\dfrac{\partial\psi}{\partial y} \\ u_y=-\dfrac{\partial\psi}{\partial x}\end{cases}$

$$\psi=\int_0^y u_x dy=\int_0^y U_0\varphi\left(y\sqrt{\frac{U_0}{vx}}\right)dy=\sqrt{vxU_0}\int_0^\eta\varphi(\eta)d\eta$$

令 $\int_0^\eta\varphi(\eta)d\eta=f(\eta)$

则有 $\psi=\sqrt{vxU_0}\,f(\eta)$

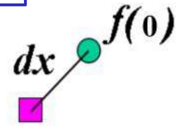


$$
\begin{cases}
\dfrac{\partial\psi}{\partial y}=\dfrac{\partial\psi}{\partial\eta}\dfrac{\partial\eta}{\partial y}=U_0 f'(\eta)\\
\dfrac{\partial\psi}{\partial x}=f(\eta)\dfrac{\partial\sqrt{\nu xU_0}}{\partial x}+\sqrt{\nu xU_0}\,f'(\eta)\dfrac{\partial\eta}{\partial x}=\dfrac{1}{2}\sqrt{\dfrac{\nu U_0}{x}}\left[f(\eta)-\eta f'(\eta)\right]\\
\dfrac{\partial^2\psi}{\partial x\partial y}=U_0\dfrac{\partial f'(\eta)}{\partial\eta}\dfrac{\partial\eta}{\partial x}=U_0 f''(\eta)\left(-\dfrac{1}{2}\dfrac{\eta}{x}\right)=-\dfrac{1}{2}\dfrac{U_0}{x}\eta f''(\eta)\\
\dfrac{\partial^2\psi}{\partial y^2}=U_0\dfrac{\partial f'(\eta)}{\partial\eta}\dfrac{\partial\eta}{\partial y}=U_0 f''(\eta)\sqrt{\dfrac{U_0}{\nu x}}=U_0\sqrt{\dfrac{U_0}{\nu x}}\,f''(\eta)\\
\dfrac{\partial^3\psi}{\partial y^3}=U_0\sqrt{\dfrac{U_0}{\nu x}}\dfrac{\partial f''(\eta)}{\partial\eta}\dfrac{\partial\eta}{\partial y}=U_0\sqrt{\dfrac{U_0}{\nu x}}\,f'''(\eta)\sqrt{\dfrac{U_0}{\nu x}}=\dfrac{U_0^2}{\nu x}f'''(\eta)
\end{cases}
$$

代入整理得： $2f'''(\eta)+f(\eta)f''(\eta)=0$

边界条件： $\begin{cases}\eta=0,\ f(\eta)=f'(\eta)=0\\ \eta\to\infty,\ f'(\eta)=1\end{cases}$

麦克劳林级数



$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots$$

布拉修斯求解： $2f'''(\eta)+f(\eta)f''(\eta)=0$

$$f(\eta)=f(0)+f'(0)\eta+\frac{f''(0)}{2!}\eta^2+\cdots+\frac{f^n(0)}{n!}\eta^n+\cdots$$

$$\frac{u_x}{U_0}=f'(\eta)=f'(0)+f''(0)\eta+\frac{f'''(0)}{2!}\eta^2+\cdots+\frac{f^n(0)}{(n-1)!}\eta^{n-1}+\cdots$$

在$\eta=0$附近展开，

解得速度分布： $\eta=0,\ f(\eta)=f'(\eta)=0$

$$\frac{u_x}{U_0}=f'(\eta)=f''(0)\eta-\frac{f''(0)^2}{2}\frac{\eta^4}{4!}+\frac{11f''(0)^3}{4}\frac{\eta^7}{7!}-\frac{375f''(0)^4}{8}\frac{\eta^{10}}{10!}+\cdots$$

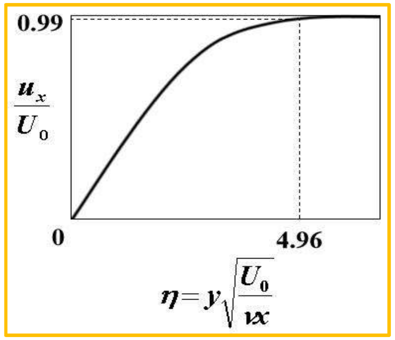
由 $\eta\to\infty, f'(\eta)=1$ 推得 $f''(0)=0.332$

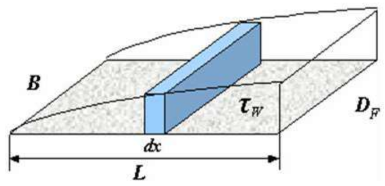
$$\frac{u_x}{U_0}=0.332\eta-2.2963\times10^{-3}\eta^4+1.9967\times10^{-5}\eta^7-1.5694\times10^{-7}\eta^{10}+\cdots$$

当 $\frac{u_x}{U_0}=0.99$ 时， $\eta=4.96$ 则有：

$$\eta=y\sqrt{\frac{U_0}{\nu x}}\Rightarrow y=\eta\sqrt{\frac{\nu x}{U_0}}\Rightarrow\delta=4.96\sqrt{\frac{\nu x}{U_0}}$$

$$\frac{\delta}{x}=\frac{4.96}{\sqrt{Re_x}}$$





壁面剪切应力

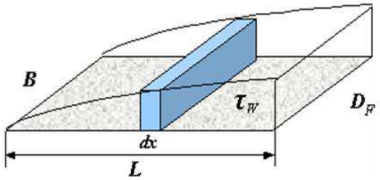
$$\tau_W=\mu\frac{\partial u_x}{\partial y}\bigg|_{y=0}=\mu U_0\sqrt{\frac{U_0}{\nu x}}f''(0)=0.332\mu U_0\sqrt{\frac{U_0}{\nu x}}$$

则壁面摩擦阻力D_F:

$$D_F=\int_0^L\tau_W B dx=0.332\mu U_0\sqrt{\frac{U_0}{\nu}}\int_0^L\frac{1}{\sqrt{x}}dx=0.664BU_0\sqrt{\mu\rho LU_0}$$

$$D_F=\frac{1.328}{\sqrt{Re_L}}BL\frac{1}{2}\rho U_0^2$$

定义摩擦阻力系数C_D:

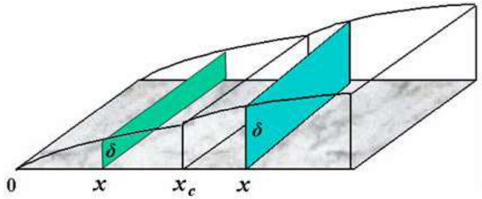


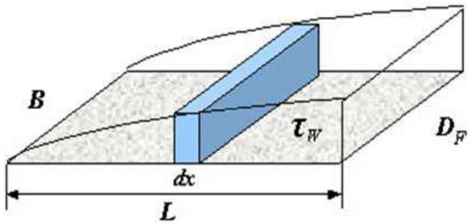
$$D_F = \frac{1.328}{\sqrt{Re_L}} BL \frac{1}{2} \rho U_0^2 = C_D A \frac{1}{2} \rho U_0^2$$

$$C_D = \frac{1.328}{\sqrt{Re_L}} \qquad D_F = C_D A \frac{1}{2} \rho U_0^2$$

问题探讨 上述阻力公式只适用层流？

2.3.2.2边界层动量积分方程



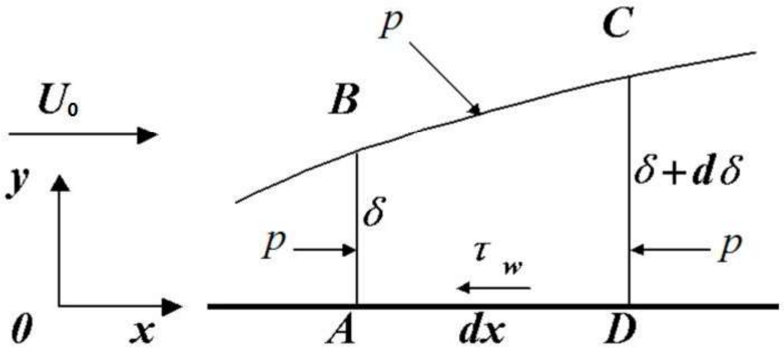


临界雷诺数Re_{xc}=5×10^5

层流 $\frac{\delta}{x} = \frac{4.64}{\sqrt{Re_x}}$ $D_F = \frac{1.292}{\sqrt{Re_L}} \cdot BL \cdot \frac{1}{2} \rho U_0^2$

湍流 $\frac{\delta}{x} = \frac{0.376}{\sqrt[5]{Re_x}}$ $D_F = \frac{0.072}{\sqrt[5]{Re_L}} \cdot BL \cdot \frac{1}{2} \rho U_0^2$

冯•卡门 边界层动量积分方程



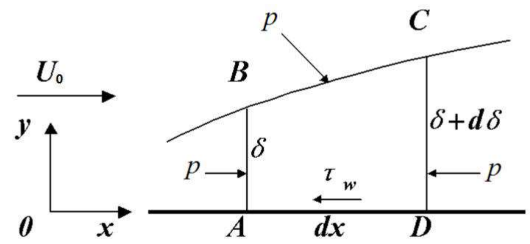
选取控制体*ABCD*，单位宽度

流动为无压差流动 $\frac{\partial p}{\partial x}=0$

动量守恒： $$\frac{\partial\left(m\bar{u}_x\right)}{\partial t}=\left(w\bar{u}\right)_{1x}-\left(w\bar{u}\right)_{2x}+\Sigma\vec{F}_x$$

对定常流体：

$$\Sigma\vec{F}_x=\left(w\bar{u}\right)_{2x}-\left(w\bar{u}\right)_{1x}$$



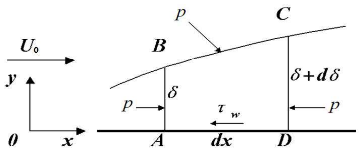
合力：

$$\Sigma\vec{F}_x=p\delta+pd\delta-p\left(\delta+d\delta\right)-\tau_w dx$$

$$\Sigma\vec{F}_x=-\tau_w dx$$

动量变化率：

进AB面	微元	AB面
质量流率	$\rho u_x dy$	$\int_0^{\delta}\rho u_x dy$
动量速率	$\rho u_x^2 dy$	$\int_0^{\delta}\rho u_x^2 dy$



出CD面：

质量流率 $$\int_0^\delta \rho u_x dy + \frac{\partial}{\partial x}\left(\int_0^\delta \rho u_x dy\right)dx$$

动量速率 $$\int_0^\delta \rho u_x^2 dy + \frac{\partial}{\partial x}\left(\int_0^\delta \rho u_x^2 dy\right)dx$$

进BC面： 质量流率 动量速率

$$\frac{\partial}{\partial x}\left(\int_0^\delta \rho u_x dy\right)dx \qquad U_0\frac{\partial}{\partial x}\left(\int_0^\delta \rho u_x dy\right)dx$$

动量变化率：

$$(w\vec{u})_{2x} - (w\vec{u})_{1x} = \frac{\partial}{\partial x}\left(\int_0^\delta \rho u_x^2 dy\right)dx - U_0\frac{\partial}{\partial x}\left(\int_0^\delta \rho u_x dy\right)dx$$

$$(w\vec{u})_{2x} - (w\vec{u})_{1x} = \rho dx\frac{\partial}{\partial x}\int_0^\delta (u_x - U_0)u_x dy$$

根据动量守恒：

$$\rho\frac{\partial}{\partial x}\int_0^\delta (U_0 - u_x)u_x dy = \tau_w$$

边界层动量积分方程

层流时，设速度分布：

$$\frac{u_x}{U}=a+b\left(\frac{y}{\delta}\right)+c\left(\frac{y}{\delta}\right)^2+d\left(\frac{y}{\delta}\right)^3$$

边界条件：
$$\begin{cases} y=0, u_x=0 & y=0, \dfrac{\partial^2 u_x}{\partial y^2}=0 \\ y=\delta, u_x=U_0 & y=\delta, \dfrac{\partial u_x}{\partial y}=0 \end{cases}$$

求得速度分布：$\dfrac{u_x}{U_0}=\dfrac{3}{2}\left(\dfrac{y}{\delta}\right)-\dfrac{1}{2}\left(\dfrac{y}{\delta}\right)^3$

$$\tau_w=\mu\frac{\partial u_x}{\partial y}\bigg|_{y=0}=\mu\frac{3}{2}\frac{U_0}{\delta}$$

将 u_x，τ_w 代入动量积分方程求得：

$$\delta=\frac{4.64x}{\sqrt{Re_x}}+C$$

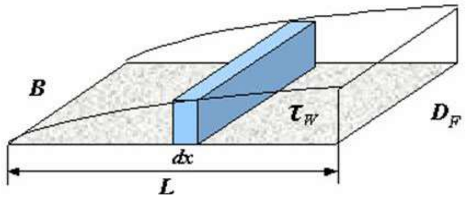
$\because x=0$， $\delta=0$； $\therefore C=0$ 。

$$\delta=\frac{4.64x}{\sqrt{Re_x}}$$

此式只适用 $x<x_c$

$$\tau_w=\mu\frac{3}{2}\frac{U_0}{\delta}=0.323\rho U^2 Re_x^{-\frac{1}{2}}$$

对$B\times L$壁面总阻力：



$$D_F=\int_0^L \tau_w dx\cdot B=0.646B\sqrt{\mu\rho L U_0^3}$$

阻力系数：$$C_D=\frac{\frac{D_F}{A}}{\frac{1}{2}\rho U_0^2}$$

层流时：$$C_D=\frac{1.292}{\sqrt{Re_L}}$$

湍流时，设速度分布：$$\frac{u_x}{U_0}=\left(\frac{y}{\delta}\right)^{\frac{1}{7}}$$

代入边界层动量积分方程 $$\rho\frac{\partial}{\partial x}\int_0^\delta (U_0-u_x)u_x dy=\tau_w$$

可得： $$\tau_w=\frac{7}{72}\rho U_0^2\frac{d\delta}{dx}$$

湍流的壁面剪切应力：

$$\tau_w=0.023\rho U_0^{\frac{7}{4}}\left(\frac{\nu}{\delta}\right)^{\frac{1}{4}}$$

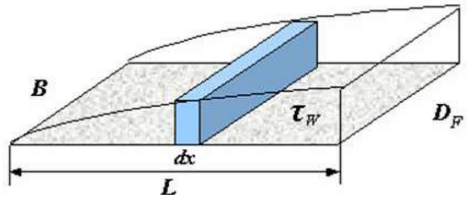
$$\frac{7}{72}\rho U_0^2\frac{d\delta}{dx}=0.023\rho U_0^{\frac{7}{4}}\left(\frac{\nu}{\delta}\right)^{\frac{1}{4}}$$

$$\int_0^{\delta}\delta^{\frac{1}{4}}d\delta=\int_0^{x}0.236\left(\frac{\nu}{U_0}\right)^{\frac{1}{4}}dx$$

注意 $x=0$， $\delta=0$ 。说明假定一开始就是湍流。

$$\frac{\delta}{x}=\frac{0.376}{\sqrt[5]{Re_x}}$$ 此式只适用$x>x_c$

对$B\times L$壁面总阻力：



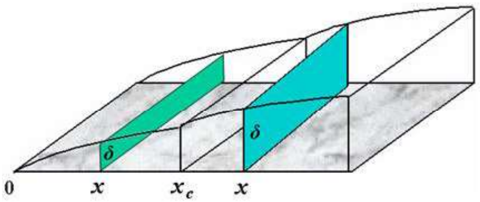
假定一开始就是湍流

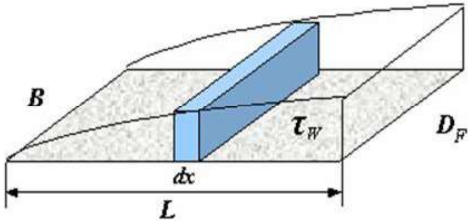
$$D_F=\int_0^L\tau_w dx\cdot B=\frac{0.073}{\sqrt[5]{Re_L}}BL\frac{1}{2}\rho U_0^2$$

湍流时阻力系数：

$$C_D=\frac{0.073}{\sqrt[5]{Re_L}}$$

平板边界层阻力计算公式：





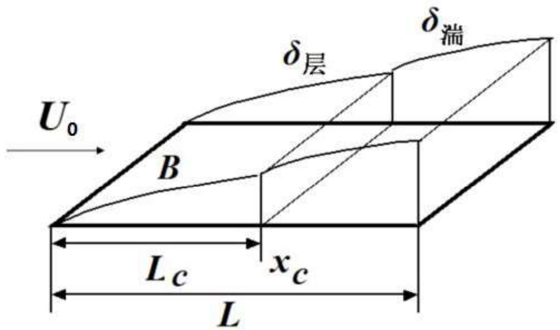
临界雷诺数Re_{xc}=5×10^5

层流 $$\delta = \frac{4.64x}{\sqrt{Re_x}} \qquad D_F = \frac{1.292}{\sqrt{Re_L}} \cdot BL \cdot \frac{1}{2}\rho U_0^2$$

湍流 $$\frac{\delta}{x} = \frac{0.376}{\sqrt[5]{Re_x}} \qquad D_F = \frac{0.073}{\sqrt[5]{Re_L}} \cdot BL \cdot \frac{1}{2}\rho U_0^2$$

当边界层达到湍流时

当$L > x_c$时，求D_F



层流段 L_c=x_c：$D_{F层，Lc}$

全部以湍流计算：$D_{F湍，L}$

误算作湍流的 $D_{F湍，Lc}$

$$\therefore D_F = D_{F层，Lc} + D_{F湍，L} - D_{F湍，Lc}$$

例2-6 平板阻力

已知：流体υ=10^{-6}m^2/s，U_0=2.4m/s， Re_{xc}=5×10^5,

ρ=1000 kg/m^3。

求：①. x_c ②. $\delta_{x=3\text{ m}}$ ③. $D_{F,\ L=2\text{ m},B=1\text{m}}$

解：①. $Re_{xc}=\dfrac{x_c U_0}{v}$ $x_c=\dfrac{v\,Re_{xc}}{U_0}=0.208m$

②. x=3m>x_c 湍流 $\delta=\dfrac{0.376x}{\sqrt[5]{Re_x}}=0.048m$

③. L=2m > L_c湍流

$$L_c=x_c$$

$$D_{F层,\ L_c}=\frac{1.292}{\sqrt{Re_{xc}}}\times BL_c\times\frac{1}{2}\rho U_0^2=1.095N$$

$$Re_x=Re_{xc}\times\frac{2}{0.208}=4807692$$

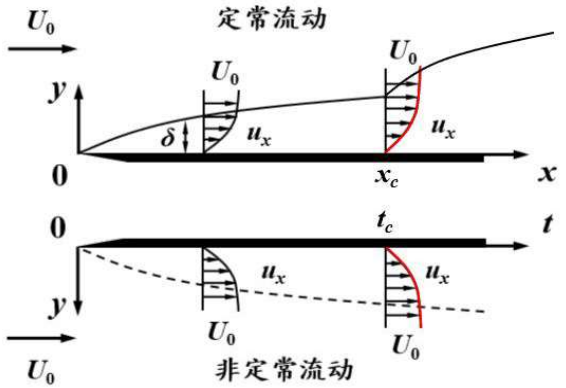
$$D_{F湍,\ L}=\frac{0.073}{\sqrt[5]{Re_x}}\times BL\times\frac{1}{2}\rho U_0^2=19.380N$$

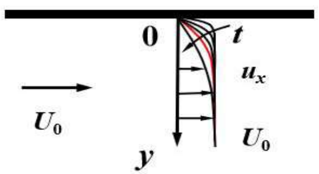
$$D_{F湍,\ L_C}=\frac{0.073}{\sqrt[5]{Re_{xc}}}\times BL_c\times\frac{1}{2}\rho U_0^2=3.169N$$

$$D_F=D_{F层,\ L_c}+D_{F湍,\ L}-D_{F湍,\ L_c}=17.31\text{N}$$

课后思考？

1.还记得静止流体中的平板启动吗？其中：问题探讨
平板下流体运动规律都符合该速度分布吗？





例2-6中 $U_0 = 2.4m/s$

$$x_c = 0.208m$$

$$\delta = \frac{4.96x_c}{\sqrt{5\times10^5}} = 0.00146m$$

临界时间：平板启动后，y=0.00146m处的速度达到u_x=99%U_0=2.376m/s时经历的时间。

由平板启动速度公式得：$erf(\eta) = \frac{u_x}{U_0} = 0.99$

查附录五得：$\eta = \frac{y}{\sqrt{4\nu t}} = 1.8$

$t_c = 0.164s$