纳维-斯托克斯方程

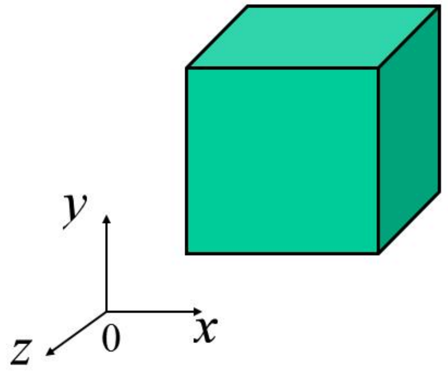
孙志仁

第四讲. 纳维-斯托克斯方程

1. 粘性流体运动方程
2. 纳维-斯托克斯方程
3. 平板库特流
4. 静止流体中的平板启动
5. 绕球爬流

1. 粘性流体运动方程

对非定常流体的动量守恒： $$\frac{\partial(m\vec{u})}{\partial t}=(w\vec{u})_1-(w\vec{u})_2+\Sigma\vec{F}$$



x，y，z 方向的分量守恒式:

$$\frac{\partial(m\vec{u}_x)}{\partial t}=(w\vec{u})_{1x}-(w\vec{u})_{2x}+\Sigma\vec{F}_x$$

$$\frac{\partial(m\vec{u}_y)}{\partial t}=(w\vec{u})_{1y}-(w\vec{u})_{2y}+\Sigma\vec{F}_y$$

$$\frac{\partial(m\vec{u}_z)}{\partial t}=(w\vec{u})_{1z}-(w\vec{u})_{2z}+\Sigma\vec{F}_z$$

选取流场中一微元体（直角坐标系）

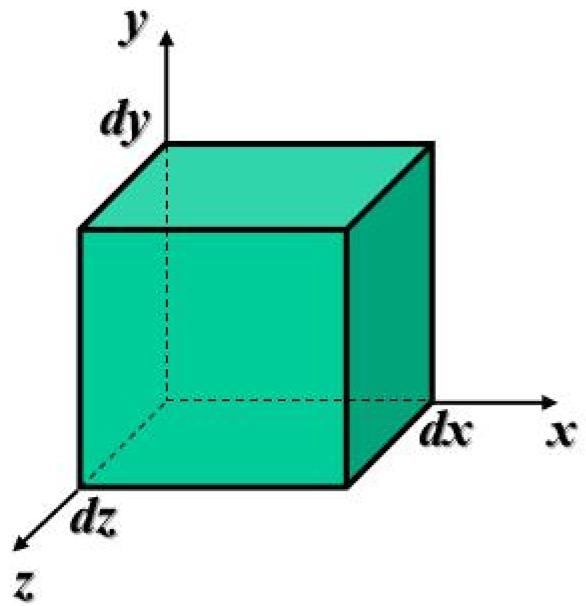
ΣF 包括压力 p 、剪切应力 τ 、体积力 X 、其它外力。

粘性流体运动方程

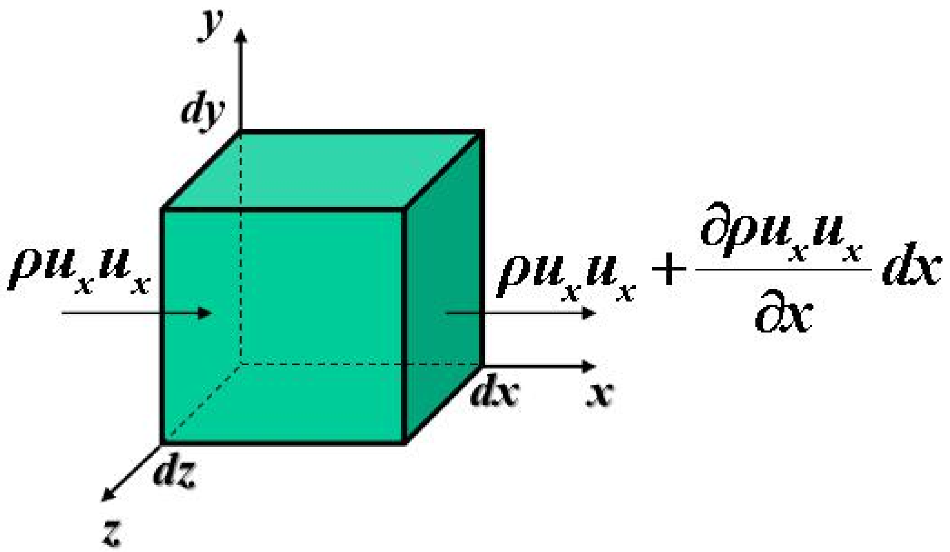
x 方向：
$$\frac{\partial\left(m\vec{u}_x\right)}{\partial t}=\left(w\vec{u}\right)_{1x}-\left(w\vec{u}\right)_{2x}+\Sigma\vec{F}_x$$

微元体内动量累积速率：

$$\frac{\partial\left(m\vec{u}_x\right)}{\partial t}=\frac{\partial\left(\rho u_x\right)}{\partial t}dxdydz$$



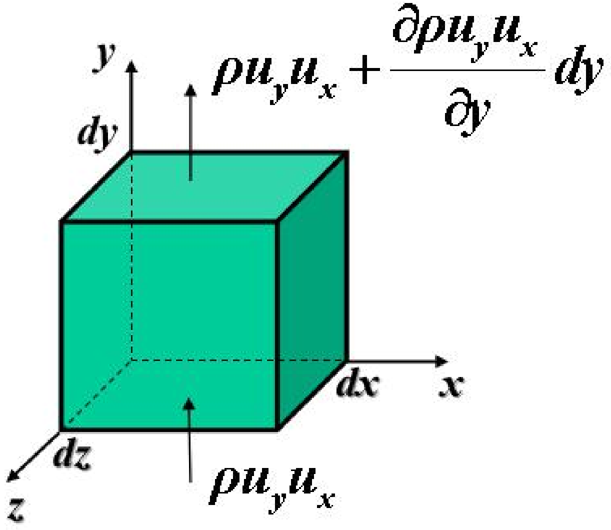
对流传递产生的动量净速率：



x 方向流动产生的 x 方向动量净速率：

$$\rho u_x u_x dydz - \left(\rho u_x u_x + \frac{\partial \rho u_x u_x}{\partial x} dx \right) dydz = -\frac{\partial \rho u_x u_x}{\partial x} dxdydz$$

y 方向流动产生的 x 方向动量净速率：



$$\rho u_y u_x dxdz - \left(\rho u_y u_x + \frac{\partial \rho u_y u_x}{\partial y} dy \right) dxdz = -\frac{\partial \rho u_y u_x}{\partial y} dxdydz$$

z 方向流动产生的 x 方向动量净速率：

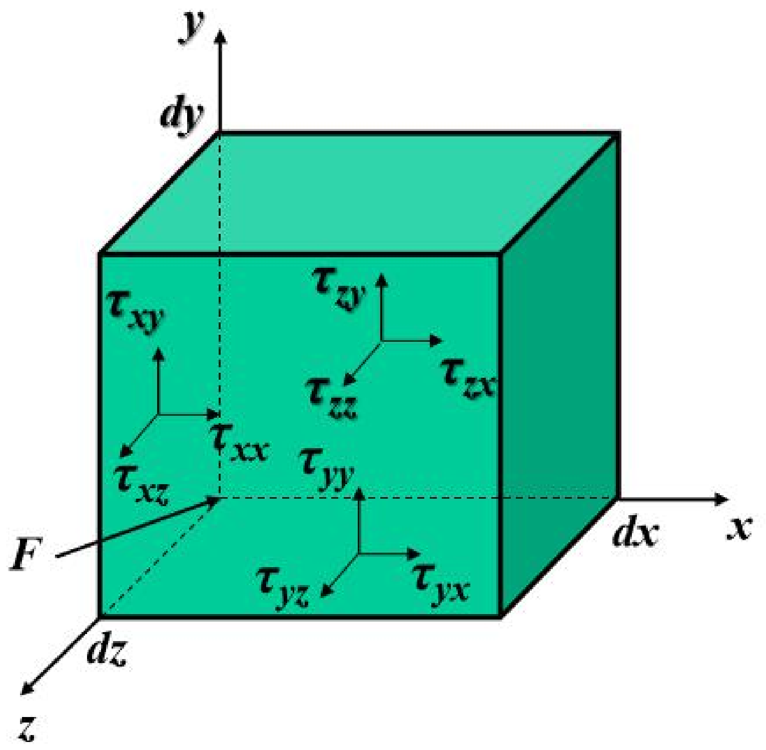
$$\rho u_z u_x dxdy - \left(\rho u_z u_x + \frac{\partial \rho u_z u_x}{\partial z} dz \right) dxdy = -\frac{\partial \rho u_z u_x}{\partial z} dxdydz$$

对流传递产生的 x 方向上动量净速率：

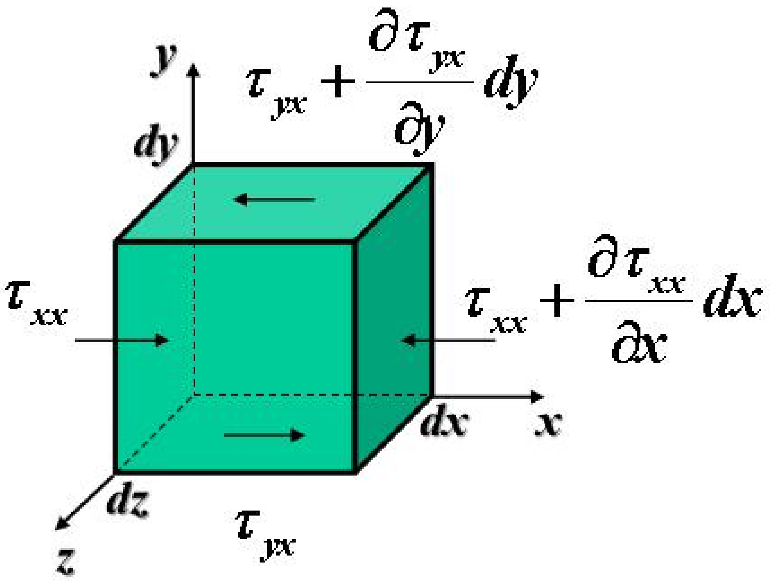
$$(w\vec{u})_{1x} - (w\vec{u})_{2x} = -\left(\frac{\partial \rho u_x u_x}{\partial x} + \frac{\partial \rho u_y u_x}{\partial y} + \frac{\partial \rho u_z u_x}{\partial z} \right) dxdydz$$

微元体上的作用力：$\Sigma \vec{F}_x = f(\tau) + f(p) + f(X)$

剪切应力 τ：



剪切应力 τ 产生的动量净速率：



x 面上 τ 产生的 x 方向动量净速率：

$$\tau_{xx}dydz-\left(\tau_{xx}+\frac{\partial\tau_{xx}}{\partial x}dx\right)dydz=-\frac{\partial\tau_{xx}}{\partial x}dxdydz$$

y 面上 τ 产生的 x 方向动量净速率：

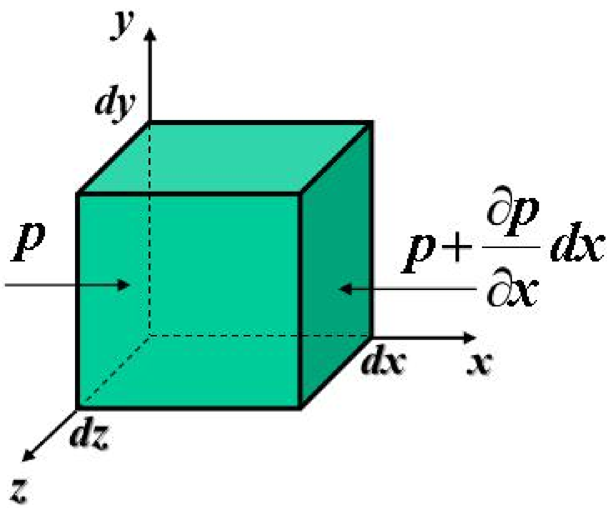
$$\tau_{yx}dxdz-\left(\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}dy\right)dxdz=-\frac{\partial\tau_{yx}}{\partial y}dxdydz$$

z 面上 τ 产生的 x 方向动量净速率：

$$\tau_{zx}dxdy-\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}dz\right)dxdy=-\frac{\partial\tau_{zx}}{\partial z}dxdydz$$

剪切应力 τ 产生的 x 方向上动量净速率：

$$f(\tau)=-\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)dxdydz$$



压力 p 产生的 x 方向上动量净速率：

$$f(p)=pdydz-\left(p+\frac{\partial p}{\partial x}dx\right)dydz=-\frac{\partial p}{\partial x}dxdydz$$

体积力 X 产生的 x 方向上动量净速率：

$$f(X)=\rho Xdxdydz$$

根据动量守恒原理

$$\frac{\partial(m\vec{u}_x)}{\partial t}=(w\vec{u})_{1x}-(w\vec{u})_{2x}+\Sigma\vec{F}_x$$

$$\frac{\partial(\rho u_x)}{\partial t}=-\left(\frac{\partial\rho u_x u_x}{\partial x}+\frac{\partial\rho u_y u_x}{\partial y}+\frac{\partial\rho u_z u_x}{\partial z}\right)-\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)-\frac{\partial p}{\partial x}+\rho X$$

引入连续性方程

$$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}=0$$

$$\frac{\partial\rho u_x}{\partial t}+u_x\frac{\partial\rho u_x}{\partial x}+u_y\frac{\partial\rho u_x}{\partial y}+u_z\frac{\partial\rho u_x}{\partial z}=-\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)-\frac{\partial p}{\partial x}+\rho X$$

ρ **为常数**

$$\rho\frac{Du_x}{Dt}=-\frac{\partial p}{\partial x}+\rho X-\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)$$

应力形式的运动方程

2. 纳维-斯托克斯方程

引入广义牛顿粘性定律

$$\tau_{xy} = \tau_{yx} = -\mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_x}{\partial y}\right) \qquad \tau_{xx} = -2\mu\frac{\partial u_x}{\partial x}$$

$$\tau_{yz} = \tau_{zy} = -\mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \qquad \tau_{yy} = -2\mu\frac{\partial u_y}{\partial y}$$

$$\tau_{zx} = \tau_{xz} = -\mu\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) \qquad \tau_{zz} = -2\mu\frac{\partial u_z}{\partial z}$$

纳维-斯托克斯方程

$$\rho \frac{Du_x}{Dt} = -\frac{\partial p}{\partial x} + \rho X + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right)$$

$$\rho \frac{Du_y}{Dt} = -\frac{\partial p}{\partial y} + \rho Y + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right)$$

$$\rho \frac{Du_z}{Dt} = -\frac{\partial p}{\partial z} + \rho Z + \mu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right)$$

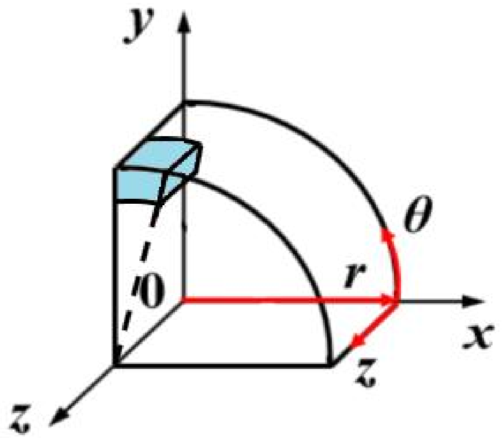
惯性力　压力项　重力　粘性力项

问题探讨

粘性力项是剪切应力吗?

参考资料　戴干策，陈敏恒.化工流体力学[M].北京:化学工业出版社，1988，868-872

1.柱坐标系中的纳维-斯托克斯方程



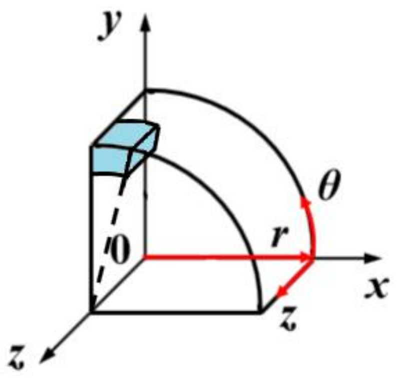
柱坐标系中剪切应力与形变的关系：

$$\tau_{r\theta}=\tau_{\theta r}=-\mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]\qquad \tau_{rr}=-2\mu\frac{\partial u_r}{\partial r}$$

$$\tau_{\theta z}=\tau_{z\theta}=-\mu\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right)\qquad \tau_{\theta\theta}=-2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)$$

$$\tau_{zr}=\tau_{rz}=-\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)\qquad \tau_{zz}=-2\mu\frac{\partial u_z}{\partial z}$$

柱坐标系—纳维-斯托克斯方程



***r* 方向：**

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)$$

$$=-\frac{\partial p}{\partial r}+\rho X_r+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r u_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right]$$

***θ* 方向：**

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)$$

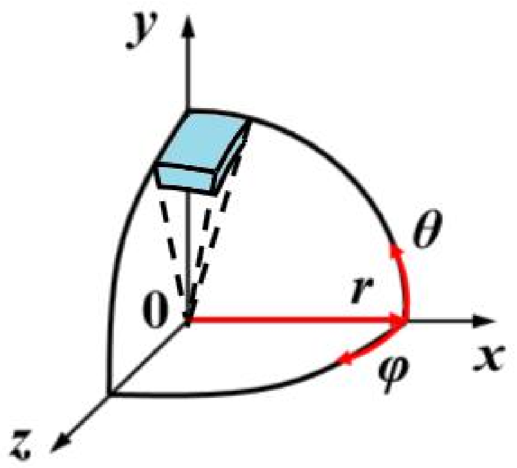
$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\rho X_\theta+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r u_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right]$$

***z* 方向：**

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)$$

$$=-\frac{\partial p}{\partial z}+\rho X_z+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]$$

2.球坐标系中的纳维-斯托克斯方程



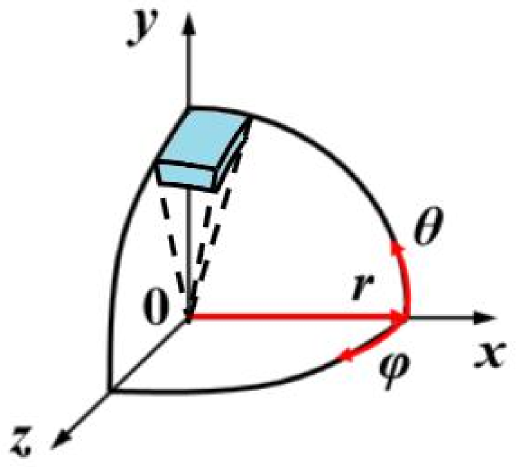
球坐标系中剪切应力与形变的关系：

$$\tau_{r\theta}=\tau_{\theta r}=-\mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] \qquad \tau_{rr}=-2\mu\frac{\partial u_r}{\partial r}$$

$$\tau_{\theta\varphi}=\tau_{\varphi\theta}=-\mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\varphi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi}\right] \qquad \tau_{\theta\theta}=-2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)$$

$$\tau_{\varphi r}=\tau_{r\varphi}=-\mu\left[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi}+r\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right)\right] \qquad \tau_{\varphi\varphi}=-2\mu\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi}+\frac{u_r}{r}+\frac{u_\theta\cot\theta}{r}\right)$$

球坐标系—纳维-斯托克斯方程



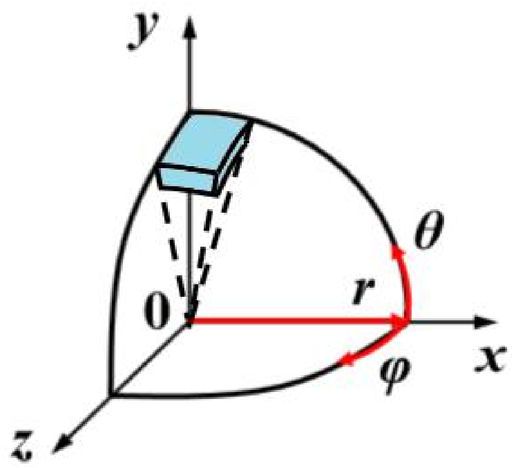
r 方向：

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+\frac{u_\varphi}{r\sin\theta}\frac{\partial u_r}{\partial \varphi}-\frac{u_\theta^2+u_\varphi^2}{r}\right)$$

$$=-\frac{\partial p}{\partial r}+\rho X_r+\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u_r}{\partial \theta}\right)\right.$$

$$\left.+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial \varphi^2}-\frac{2}{r^2}u_r-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}-\frac{2}{r^2}u_\theta\cot\theta-\frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial \varphi}\right]$$

θ 方向:

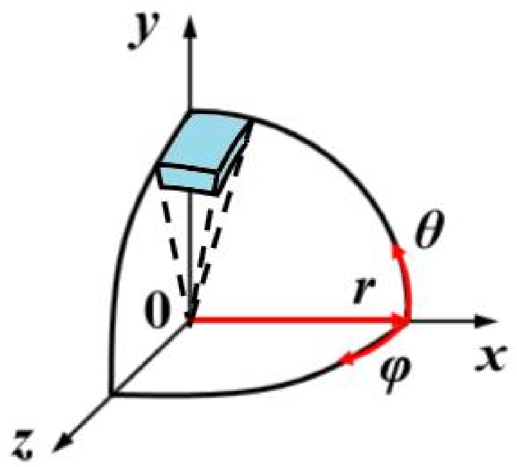


$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_\varphi}{r\sin\theta}\frac{\partial u_\theta}{\partial \varphi}+\frac{u_r u_\theta}{r}-\frac{u_\varphi^2\cot\theta}{r}\right)$$

$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\rho X_\theta+\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u_\theta}{\partial \theta}\right)\right.$$

$$\left.+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial \varphi^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial \varphi}\right]$$

φ 方向:

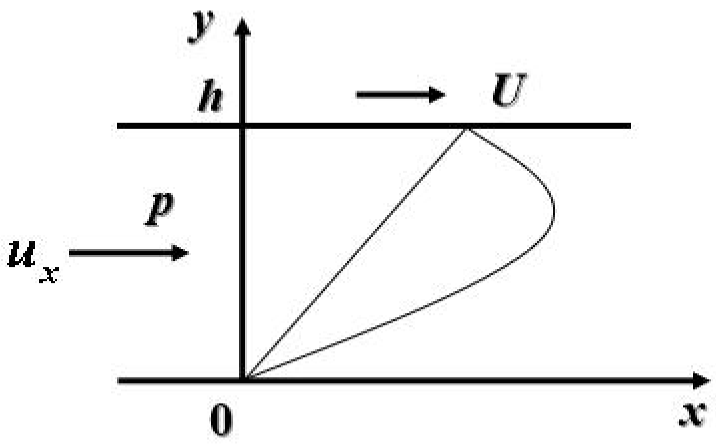


$$\rho\left(\frac{\partial u_\varphi}{\partial t}+u_r\frac{\partial u_\varphi}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\varphi}{\partial \theta}+\frac{u_\varphi}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi}+\frac{u_r u_\varphi}{r}+\frac{u_\theta u_\varphi\cot\theta}{r}\right)$$

$$=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \varphi}+\rho X_\varphi+\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\varphi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u_\varphi}{\partial \theta}\right)\right.$$

$$\left.+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\varphi}{\partial \varphi^2}-\frac{u_\varphi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial \varphi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial \varphi}\right]$$

3. 平板库特流



下板固定，上板以恒定速度 U 运动，板间流体在压差和上板拖动下作定常层流流动。

奈维-斯托克斯方程

$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho X+\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)$$

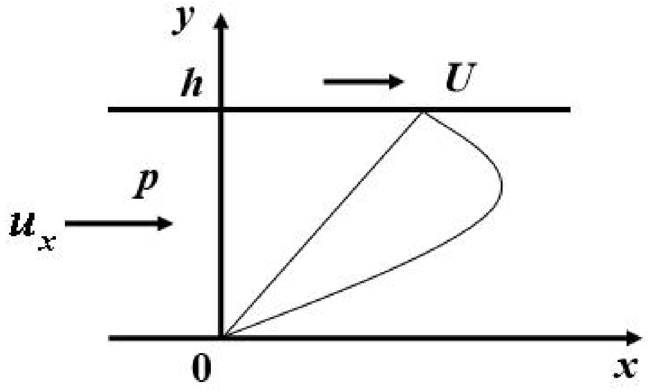
物理分析

定常： $\dfrac{\partial u_x}{\partial t} = 0$

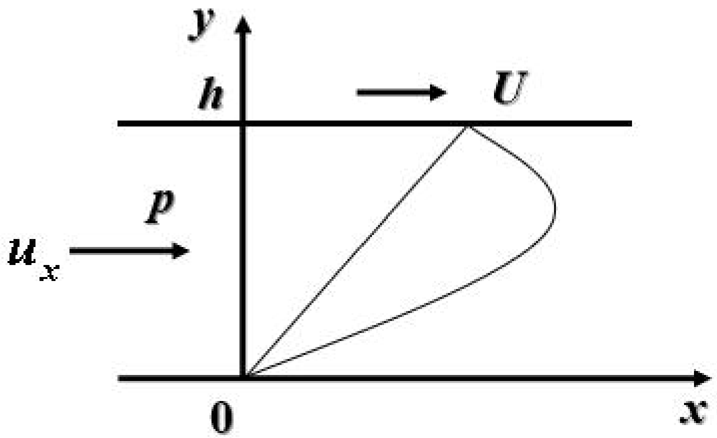
一维流动： $\begin{cases} u_x \neq 0 \\ u_y = 0 \\ u_z = 0 \end{cases}$ $\begin{cases} \dfrac{\partial u_x}{\partial x} = 0 \\ \dfrac{\partial u_x}{\partial y} \neq 0 \\ \dfrac{\partial u_x}{\partial z} = 0 \end{cases}$ $\begin{cases} \dfrac{\partial^2 u_x}{\partial x^2} = 0 \\ \dfrac{\partial^2 u_x}{\partial y^2} \neq 0 \\ \dfrac{\partial^2 u_x}{\partial z^2} = 0 \end{cases}$

x 方向无重力： $X = 0$

简化得： $\mu \dfrac{\partial^2 u_x}{\partial y^2} = \dfrac{\partial p}{\partial x}$



数学模型：$\mu \frac{d^2 u_x}{dy^2} = \frac{dp}{dx}$



边界条件：$\begin{cases} y = 0, u_x = 0 \\ y = h, u_x = U \end{cases}$

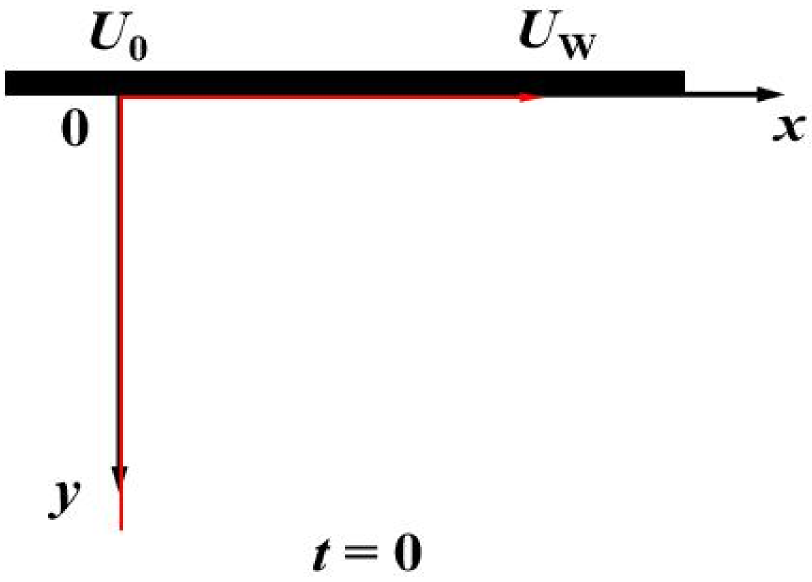
库特流速度分布：$u_x = \frac{1}{2\mu} \frac{dp}{dx} \left(y^2 - hy \right) + U \frac{y}{h}$

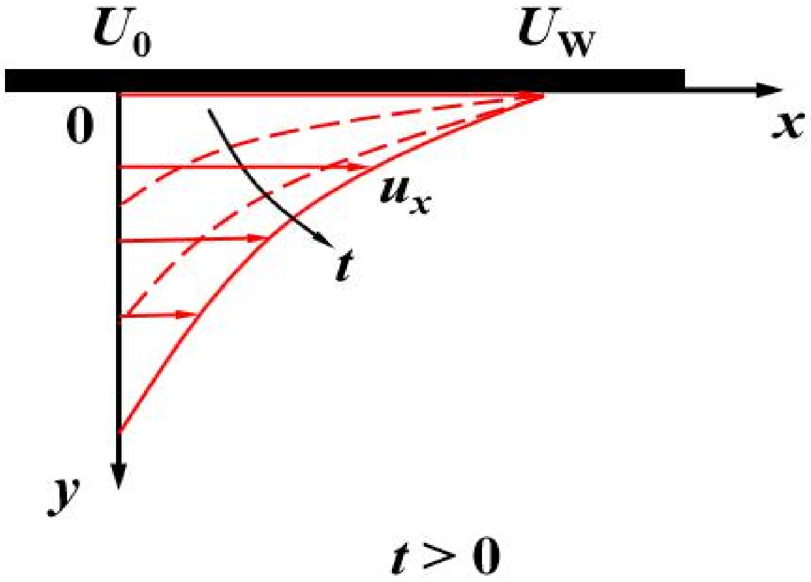
若上板也固定：$u_x = \frac{1}{2\mu} \frac{dp}{dx} \left(y^2 - hy \right)$ 抛物线分布

若无压差：$\frac{u_x}{U} = \frac{y}{h}$ 线性分布

4. 静止流体中的平板启动

静止的水面上有一块无限大平板，初始速度为 $U_0 = 0$，突然以 U_W 速度运动，并维持不变。平板下水中的速度分布 u_x 随时间也发生变化。



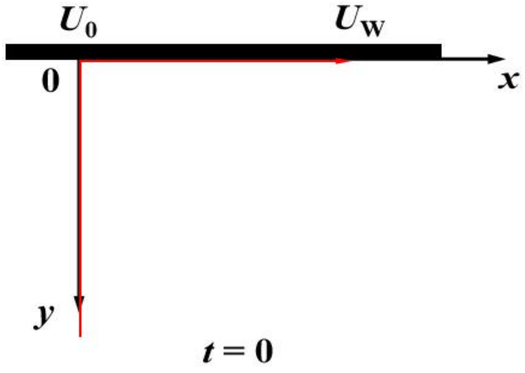


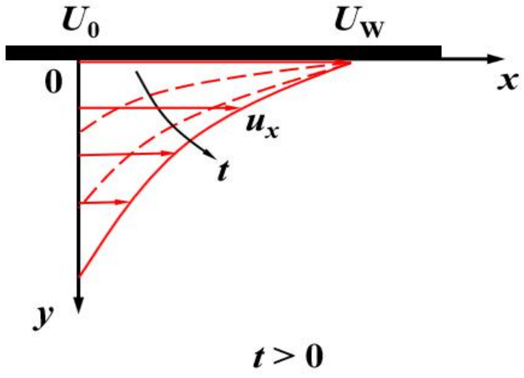
物理分析

非定常： $\frac{\partial u_x}{\partial t} \neq 0$

一维流动：

$$\begin{cases} u_x \neq 0 \\ u_y = 0 \\ u_z = 0 \end{cases} \quad \begin{cases} \frac{\partial u_x}{\partial x} = 0 \\ \frac{\partial u_x}{\partial y} \neq 0 \\ \frac{\partial u_x}{\partial z} = 0 \end{cases} \quad \begin{cases} \frac{\partial^2 u_x}{\partial x^2} = 0 \\ \frac{\partial^2 u_x}{\partial y^2} \neq 0 \\ \frac{\partial^2 u_x}{\partial z^2} = 0 \end{cases}$$





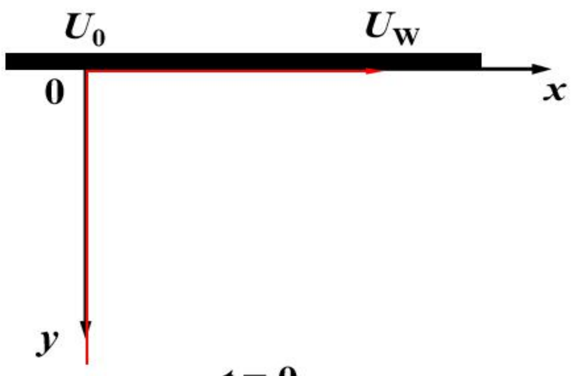
x 方向没有重力： $X = 0$

x 方向无压差力： $\frac{\partial p}{\partial x} = 0$

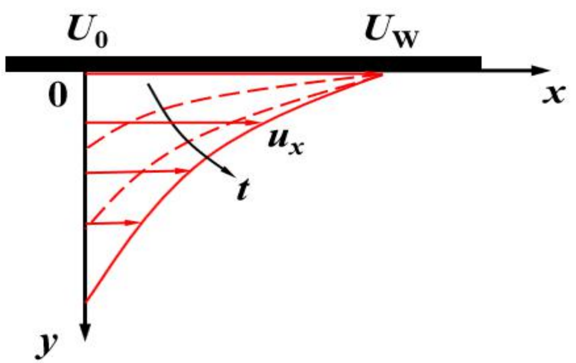
$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho X+\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)$$

简化纳维-斯托克斯方程：

$$\frac{\partial u_x}{\partial t}=\nu\frac{\partial^2 u_x}{\partial y^2}$$



初始条件： $t=0, u_x=0$



边界条件： $t>0,\begin{cases} y=0, u_x=U_W \\ y\to\infty, u_x=U_0=0 \end{cases}$

数学模型： $$\frac{\partial u_x}{\partial t} = v\frac{\partial^2 u_x}{\partial y^2}$$ **一维非定常偏微分方程**

令 $\eta = \dfrac{y}{\sqrt{4vt}}$ **则** $$\frac{\partial u_x}{\partial t} = \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial t} = -\frac{\eta}{2t}\frac{\partial u_x}{\partial \eta}$$

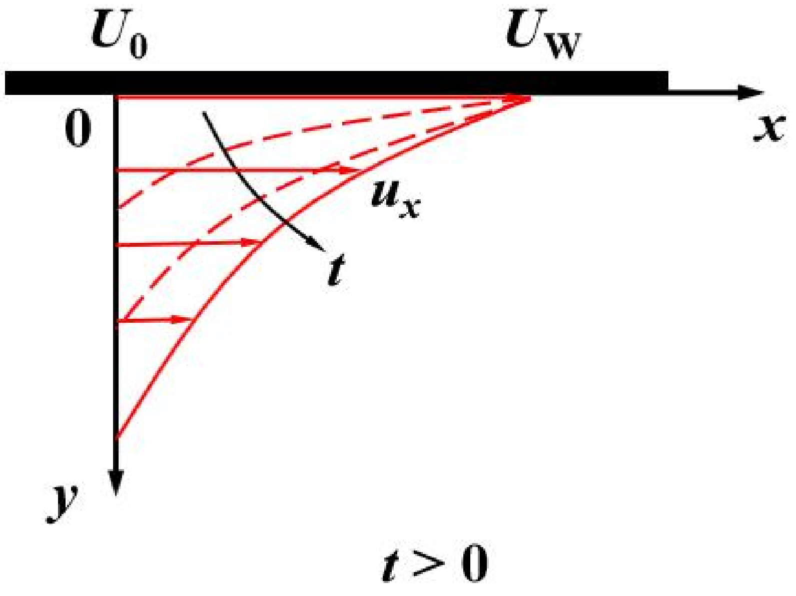
相似变换

$$\frac{\partial u_x}{\partial y} = \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{4vt}}\frac{\partial u_x}{\partial \eta}$$

$$\frac{\partial^2 u_x}{\partial y^2} = \frac{1}{\sqrt{4vt}}\frac{\partial \dfrac{\partial u_x}{\partial \eta}}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{4vt}\frac{\partial^2 u_x}{\partial \eta^2}$$

代入原方程可得： $$\frac{d^2 u_x}{d\eta^2} + 2\eta\frac{du_x}{d\eta} = 0$$

$$\frac{d^2u_x}{d\eta^2}+2\eta\frac{du_x}{d\eta}=0$$



边界条件：
$$\begin{cases}\eta=0,u_x=U_W\\ \eta\to\infty,u_x=U_0=0\end{cases}$$

设： $\frac{du_x}{d\eta}=p$

$$\frac{dp}{d\eta}+2\eta p=0$$

积分： $p=C_1e^{-\eta^2}$ $\frac{du_x}{d\eta}=C_1e^{-\eta^2}$

再积分： $\int_{U_W}^{u_x}du_x=C_1\int_0^{\eta}e^{-\eta^2}d\eta$

$$u_x - U_W = C_1 \int_0^{\eta} e^{-\eta^2} d\eta$$

边界条件： $\eta \to \infty, u_x = U_0 = 0$

$$C_1 = \frac{U_0 - U_W}{\int_0^{\infty} e^{-\eta^2} d\eta}$$

其中： $\int_0^{\infty} e^{-\eta^2} d\eta = \frac{\sqrt{\pi}}{2}$

速度分布：

$$\frac{u_x - U_W}{U_0 - U_W} = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta$$

高斯误差函数

$$\frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta = erf(\eta) = \frac{u_x - U_W}{U_0 - U_W}$$

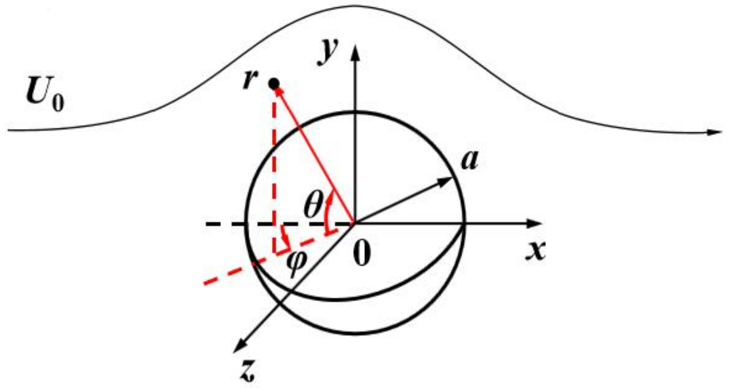
其中： $\eta = \frac{y}{\sqrt{4\nu t}}$

问题探讨 **平板下流体运动规律都符合该速度分布吗？**

5. 绕球爬流

爬流 $Re<1$ 极慢运动 $Re=\frac{\text{惯性力}}{\text{粘性力}}<<1$ 忽略惯性力，选用球坐标系

物理分析 **量级比较**



定常： $\frac{\partial(\ )}{\partial t}=0$

二维流动： $\begin{cases} u_r \neq 0 \\ u_\theta \neq 0 \\ u_\varphi = 0 \end{cases}$ $\begin{cases} \frac{\partial(\ )}{\partial r} \neq 0 \\ \frac{\partial(\ )}{\partial \theta} \neq 0 \\ \frac{\partial(\ )}{\partial \varphi} = 0 \end{cases}$

忽略重力： $X_r = X_\theta = X_\varphi = 0$

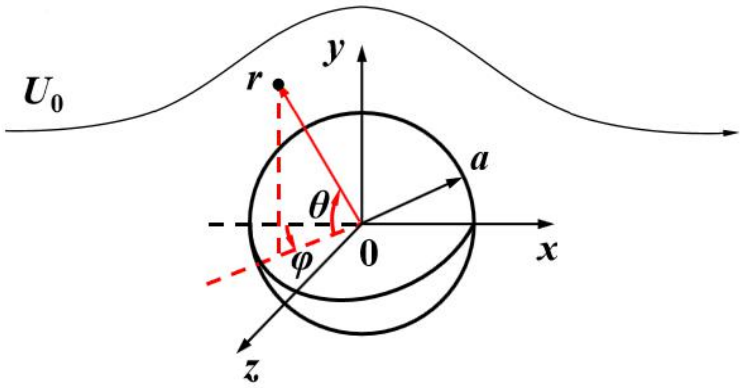
简化球坐标系中的连续性方程和纳维-斯托克斯方程：

$$\frac{1}{r^2}\frac{\partial r^2 u_r}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial u_\theta \sin\theta}{\partial\theta}=0$$

$$\frac{\partial p}{\partial r}=\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right)-\frac{2}{r^2}u_r-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2}{r^2}u_\theta\cot\theta\right]$$

$$\frac{1}{r}\frac{\partial p}{\partial\theta}=\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\theta}{\partial\theta}\right)+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r^2\sin^2\theta}\right]$$

边界条件:

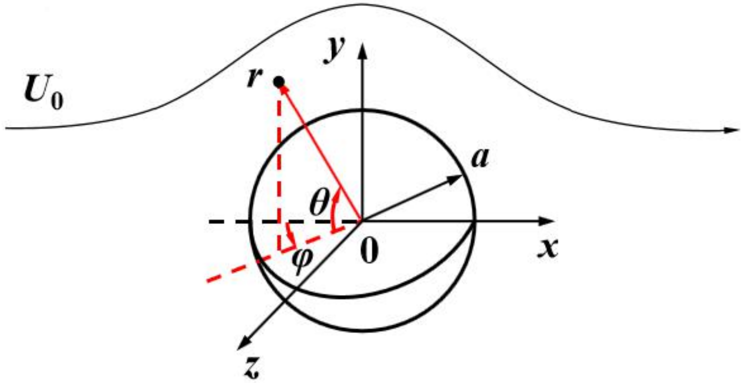


$$\begin{cases} r = a, u_r = 0, u_\theta = 0 \\ r \to \infty, u_r = U_0 \cos\theta, u_r = -U_0 \sin\theta, p = p_0 \end{cases}$$

方程组为线性偏微分方程组，可以用分离变量法解得速度分布和压力分布为:

$$\begin{cases} u_r = U_0\left[1 - \frac{3}{2}\frac{a}{r} + \frac{1}{2}\left(\frac{a}{r}\right)^3\right]\cos\theta \\ u_\theta = -U_0\left[1 - \frac{3}{4}\frac{a}{r} - \frac{1}{4}\left(\frac{a}{r}\right)^3\right]\sin\theta \\ p = p_0 - \frac{3}{2}\frac{\mu U_0}{a}\left(\frac{a}{r}\right)^2 \cos\theta \end{cases}$$

球表面压力分布：



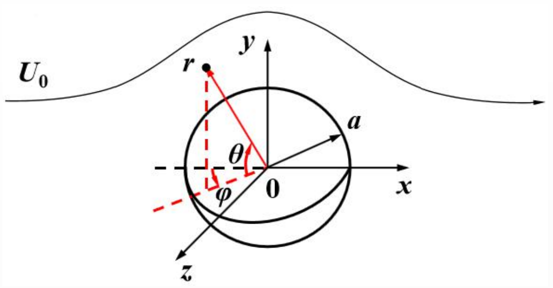
$$r = a \qquad p = p_0 - \frac{3}{2}\frac{\mu U_0}{a}\cos\theta$$

剪切应力分布： $$\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]$$

代入 u_r, u_θ 得： $$\tau_{r\theta} = -\frac{3}{2}\mu U_0\frac{a^3}{r^4}\sin\theta$$

球表面剪切应力分布： $$\tau_{r\theta} = -\frac{3}{2a}\mu U_0\sin\theta$$

球表面总阻力：



$$D=\int_0^{2\pi}d\varphi\int_0^{\pi}\left(-p\cos\theta-\tau_{r\theta}\sin\theta\right)a^2\sin\theta d\theta$$

$$D=2\pi\mu aU_0+4\pi\mu aU_0$$

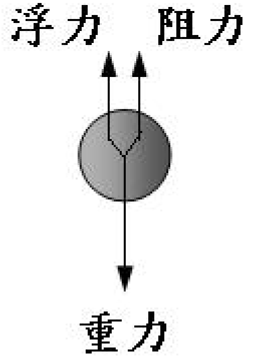
压差阻力　　摩擦阻力

$$D=6\pi\mu aU_0$$

斯托克斯阻力定律　　　适用条件 $Re<1$ 的爬流

落球法测粘度

测定小球在静止流体中匀速下降速度 u，根据力平衡有：



$$\frac{1}{6}\pi d^{3}\rho_{s}g=\frac{1}{6}\pi d^{3}\rho g+6\pi\mu\frac{d}{2}u$$

$$\mu=\frac{\left(\rho_{s}-\rho\right)gd^{2}}{18u}$$

适用条件 $Re<1$ 的爬流

课后思考

1.已知绕流阻力定义式：

$$D = C_D A \frac{1}{2} \rho U_0^2$$

A 迎流投影面

试推导绕球爬流阻力系数公式：

$$C_D = \frac{24}{Re}$$