普朗特边界层

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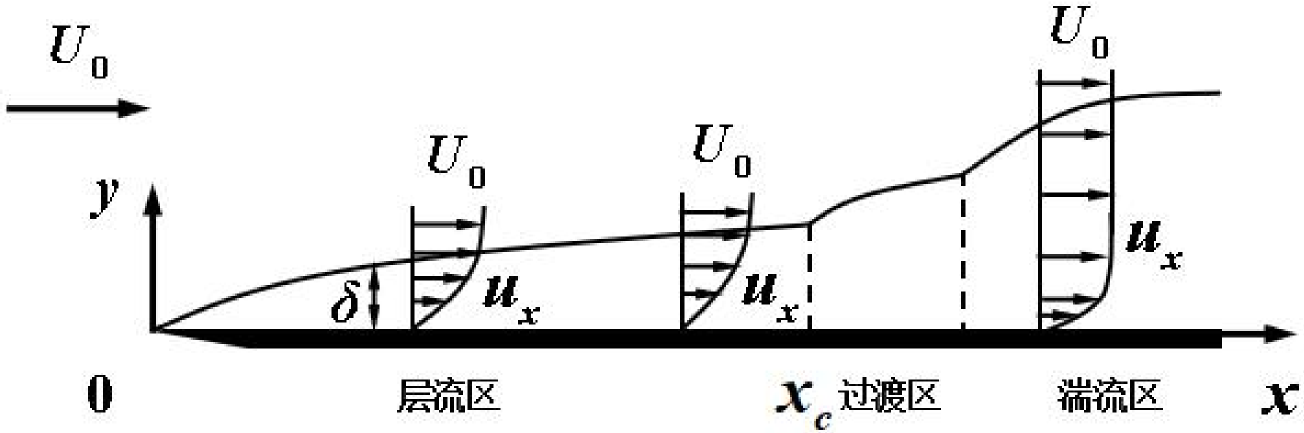
第六讲. 普朗特边界层

1. 边界层
2. 普朗特边界层方程
3. 边界层动量积分方程
4. 边界层计算

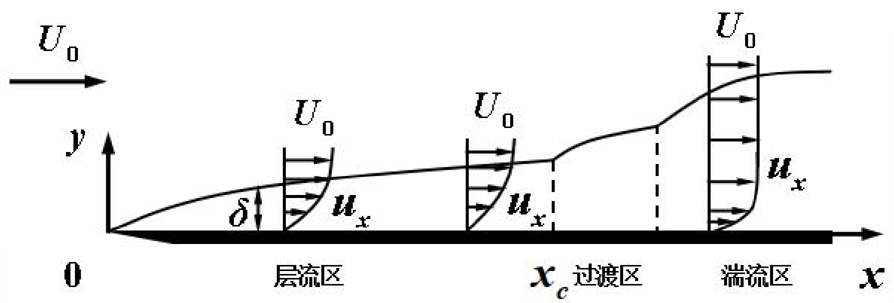
1. 边界层

1904年，普朗特提出“边界层”概念

高雷诺数下，流体区域以 $99\%U_0$ 作为边界，内层区为粘性流体 u_x，外层区为理想流体 U_0 。



边界层特点



①. 慢：边界层内 $u_x < U_0$，壁面 $u_x = 0$。

②. 薄：$\delta << x$。

③. 陡：$\dfrac{du_x}{dy}$ 很大。

④. 增：$x\uparrow$，$\delta\uparrow$。

⑤. 旋：微团有旋。

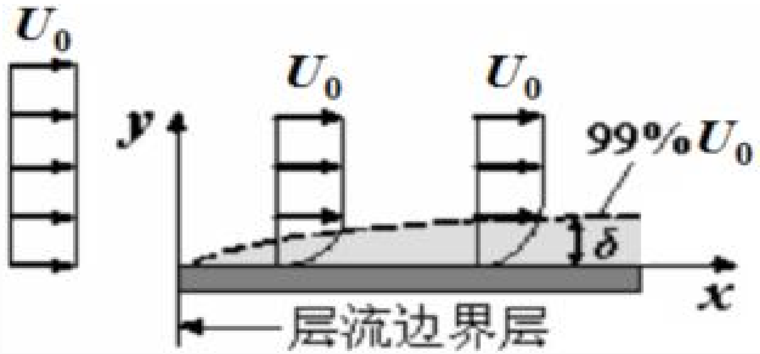
⑥. 惯、粘同量级：惯性力与粘性力在边界层内量级相当。

⑦. 截面等压力：无压差流动。

⑧. 流型会转变：$x > x_c$ 时，层流→湍流。

⑨. 逆压，失速会分离（绕曲面流动时的表现）。

2. 普朗特边界层方程



物理分析

问题探讨 边界层内流动是二维流动吗？

定常： $\dfrac{\partial u_x}{\partial t}=\dfrac{\partial u_y}{\partial t}=0$

二维流动：

$$\begin{cases} u_x \neq 0 \\ u_y \neq 0 \\ u_z = 0 \end{cases} \quad \begin{cases} \dfrac{\partial u_x}{\partial x}\neq 0 & \dfrac{\partial u_y}{\partial x}\neq 0 \\ \dfrac{\partial u_x}{\partial y}\neq 0 & \dfrac{\partial u_y}{\partial y}\neq 0 \\ \dfrac{\partial u_x}{\partial z}=0 & \dfrac{\partial u_y}{\partial z}=0 \end{cases} \quad \begin{cases} \dfrac{\partial^2 u_x}{\partial x^2}\neq 0 & \dfrac{\partial^2 u_x}{\partial x^2}\neq 0 \\ \dfrac{\partial^2 u_x}{\partial y^2}\neq 0 & \dfrac{\partial^2 u_x}{\partial y^2}\neq 0 \\ \dfrac{\partial^2 u_x}{\partial z^2}=0 & \dfrac{\partial^2 u_x}{\partial z^2}=0 \end{cases}$$

无压差流动： $\dfrac{\partial p}{\partial x}=\dfrac{\partial p}{\partial y}=0$

忽略重力： $X=Y=0$

$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho X+\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)$$

奈维-斯托克斯方程简化可得：

$$u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}=v\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}\right)$$

同理，y 方向可得：

$$u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}=v\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}\right)$$

连续性方程：

$$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}=0$$

量级比较

边界层厚度薄，$\delta << x$ 。

量级表示：$\begin{cases} x \sim O(L) \\ y \sim O(\delta) \end{cases}$

连续性方程：$\dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y} = 0$

$$u_x \sim O(U_0) \qquad \frac{\partial u_x}{\partial x} \sim O\left(\frac{U_0}{L}\right) \qquad \frac{\partial u_y}{\partial y} \sim O\left(\frac{U_0}{L}\right) \qquad u_y \sim O\left(\frac{\delta U_0}{L}\right)$$

量级表示

$$\begin{cases} u_x \sim O(U_0) \\ u_y \sim O\left(\dfrac{\delta U_0}{L}\right) \end{cases}$$

$$\begin{cases} \dfrac{\partial u_x}{\partial x} \sim O\left(\dfrac{U_0}{L}\right) \\ \dfrac{\partial u_x}{\partial y} \sim O\left(\dfrac{U_0}{\delta}\right) \\ \dfrac{\partial u_y}{\partial x} \sim O\left(\dfrac{\delta U_0}{L^2}\right) \\ \dfrac{\partial u_y}{\partial y} \sim O\left(\dfrac{U_0}{L}\right) \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 u_x}{\partial x^2} \sim O\left(\dfrac{U_0}{L^2}\right) \\ \dfrac{\partial^2 u_x}{\partial y^2} \sim O\left(\dfrac{U_0}{\delta^2}\right) \\ \dfrac{\partial^2 u_y}{\partial x^2} \sim O\left(\dfrac{\delta U_0}{L^3}\right) \\ \dfrac{\partial^2 u_y}{\partial y^2} \sim O\left(\dfrac{U_0}{\delta L}\right) \end{cases}$$

x 方向 $$u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}=\nu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}\right)$$

量级表示 $$\frac{U_0^2}{L}\qquad\frac{U_0^2}{L}\qquad\qquad\frac{U_0}{L^2}<<\frac{U_0}{\delta^2}$$

根据边界层内**惯性力与粘性力量级相当**，则有：

$$u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}=\nu\frac{\partial^2 u_x}{\partial y^2}$$

y 方向

$$u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}=\nu\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}\right)$$

量级表示

$$\frac{\delta U_0^2}{L^2}\qquad\frac{\delta U_0^2}{L^2}\qquad\frac{\delta U_0}{L^3}<<\frac{U_0}{\delta L}$$

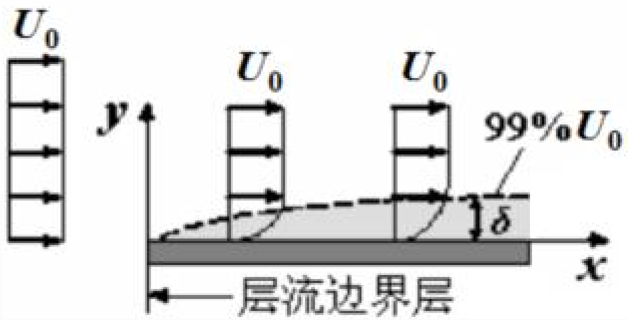
惯性力项　　粘性力项

x 方向与 y 方向比较

$$\frac{\frac{U_0^2}{L}}{\frac{\delta U_0^2}{L^2}}=\frac{L}{\delta}>>1\qquad\frac{\frac{U_0}{\delta^2}}{\frac{U_0}{\delta L}}=\frac{L}{\delta}>>1$$

结论：忽略 y 方向的流动

量级比较简化可得
普朗特边界层方程：



$$
\begin{cases}
u_x \dfrac{\partial u_x}{\partial x} + u_y \dfrac{\partial u_x}{\partial y} = \nu \dfrac{\partial^2 u_x}{\partial y^2} \\
\dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y} = 0
\end{cases}
$$

边界条件：
$$
\begin{cases}
y = 0, u_x = u_y = 0 \\
y \to \infty, u_x = U_0
\end{cases}
$$

引入流函数的目的是将速度变量 u_x, u_y 用一个变量 ψ 代替，从而使方程的求解得以简化。

$$\begin{cases} u_x = \dfrac{\partial \psi}{\partial y} \\ u_y = -\dfrac{\partial \psi}{\partial x} \end{cases}$$

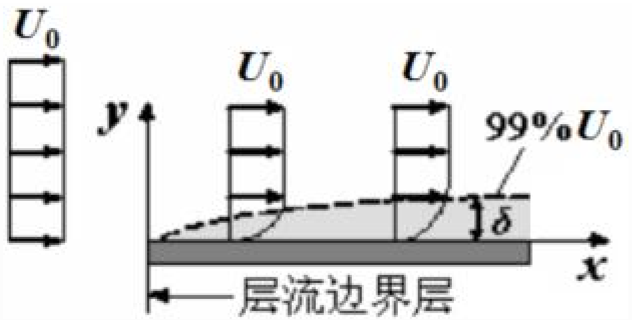
可得：

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = v\frac{\partial^3 \psi}{\partial y^3}$$

边界条件：

$$\begin{cases} y = 0, \dfrac{\partial \psi}{\partial y} = \dfrac{\partial \psi}{\partial x} = 0 \\ y \to \infty, \dfrac{\partial \psi}{\partial y} = U_0 \end{cases}$$

相似变换



令速度分布为：$$\frac{u_x}{U_0}=\varphi\left(\frac{y}{\delta}\right)$$

根据边界层内**惯性力与粘性力量级相当**，则有：

$$\frac{U_0^2}{L}\sim v\frac{U_0}{\delta^2}\Rightarrow\delta\sim\sqrt{\frac{vL}{U_0}}\Rightarrow\delta\sim\sqrt{\frac{vx}{U_0}}$$

令 $\eta=\dfrac{y}{\delta}$ 则有：$\eta=y\sqrt{\dfrac{U_0}{vx}}$ $\qquad \dfrac{u_x}{U_0}=\varphi(\eta)=\varphi\left(y\sqrt{\dfrac{U_0}{vx}}\right)$

流函数 $\begin{cases} u_x = \dfrac{\partial \psi}{\partial y} \\ u_y = -\dfrac{\partial \psi}{\partial x} \end{cases}$

$$\psi = \int_0^y u_x dy = \int_0^y U_0 \varphi\left(y\sqrt{\frac{U_0}{vx}} \right) dy = \sqrt{vxU_0} \int_0^\eta \varphi(\eta) d\eta$$

令 $\int_0^\eta \varphi(\eta) d\eta = f(\eta)$

则有 $\psi = \sqrt{vxU_0} f(\eta)$

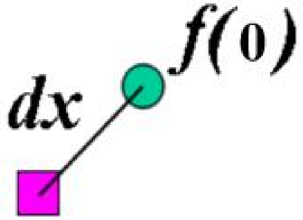
$$\begin{tikzcd} & x & \\ \psi \arrow[ur] \arrow[dr] & & x \\ & \eta \arrow[ur] \arrow[dr] & \\ & & y \end{tikzcd}$$

$$\begin{cases}\dfrac{\partial\psi}{\partial y}=\dfrac{\partial\psi}{\partial\eta}\dfrac{\partial\eta}{\partial y}=U_0f'(\eta)\\[2mm] \dfrac{\partial\psi}{\partial x}=f(\eta)\dfrac{\partial\sqrt{vxU_0}}{\partial x}+\sqrt{vxU_0}\,f'(\eta)\dfrac{\partial\eta}{\partial x}=\dfrac{1}{2}\sqrt{\dfrac{vU_0}{x}}\left[f(\eta)-\eta f'(\eta)\right]\\[2mm] \dfrac{\partial^2\psi}{\partial x\partial y}=U_0\dfrac{\partial f'(\eta)}{\partial\eta}\dfrac{\partial\eta}{\partial x}=U_0f''(\eta)\left(-\dfrac{1}{2}\dfrac{\eta}{x}\right)=-\dfrac{1}{2}\dfrac{U_0}{x}\eta f''(\eta)\\[2mm] \dfrac{\partial^2\psi}{\partial y^2}=U_0\dfrac{\partial f'(\eta)}{\partial\eta}\dfrac{\partial\eta}{\partial y}=U_0f''(\eta)\sqrt{\dfrac{U_0}{vx}}=U_0\sqrt{\dfrac{U_0}{vx}}f''(\eta)\\[2mm] \dfrac{\partial^3\psi}{\partial y^3}=U_0\sqrt{\dfrac{U_0}{vx}}\dfrac{\partial f''(\eta)}{\partial\eta}\dfrac{\partial\eta}{\partial y}=U_0\sqrt{\dfrac{U_0}{vx}}f'''(\eta)\sqrt{\dfrac{U_0}{vx}}=\dfrac{U_0^2}{vx}f'''(\eta)\end{cases}$$

代入整理得： $2f'''(\eta)+f(\eta)f''(\eta)=0$

边界条件： $\begin{cases}\eta=0,\ f(\eta)=f'(\eta)=0\\ \eta\to\infty,\ f'(\eta)=1\end{cases}$

麦克劳林级数



$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

布拉修斯求解： $$2f'''(\eta) + f(\eta)f''(\eta) = 0$$

$$f(\eta) = f(0) + f'(0)\eta + \frac{f''(0)}{2!}\eta^2 + \cdots + \frac{f^n(0)}{n!}\eta^n + \cdots$$

$$\frac{u_x}{U_0} = f'(\eta) = f'(0) + f''(0)\eta + \frac{f'''(0)}{2!}\eta^2 + \cdots + \frac{f^n(0)}{(n-1)!}\eta^{n-1} + \cdots$$

在 η=0 附近展开，解得速度分布： $\eta = 0$， $f(\eta) = f'(\eta) = 0$

$$\frac{u_x}{U_0}=f'(\eta)=f''(0)\eta-\frac{f''(0)^2}{2}\frac{\eta^4}{4!}+\frac{11f''(0)^3}{4}\frac{\eta^7}{7!}-\frac{375f''(0)^4}{8}\frac{\eta^{10}}{10!}+\cdots$$

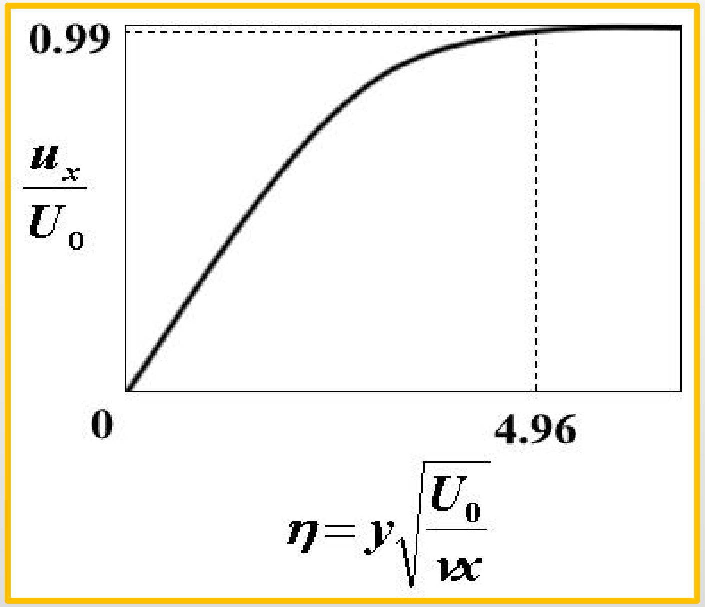
由 $\eta\to\infty,\quad f'(\eta)=1$ 推得 $f''(0)=0.332$

$$\frac{u_x}{U_0}=0.332\eta-2.2963\times10^{-3}\eta^4+1.9967\times10^{-5}\eta^7-1.5694\times10^{-7}\eta^{10}+\cdots$$

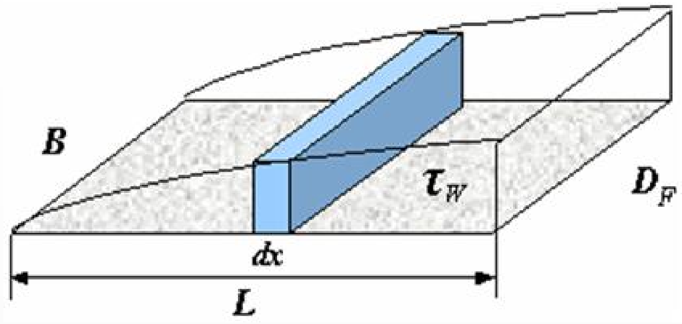
当 $\frac{u_x}{U_0}=0.99$ 时， $\eta=4.96$ 则有：

$$\eta=y\sqrt{\frac{U_0}{\nu x}}\Rightarrow y=\eta\sqrt{\frac{\nu x}{U_0}}\Rightarrow\delta=4.96\sqrt{\frac{\nu x}{U_0}}$$

$$\frac{\delta}{x}=\frac{4.96}{\sqrt{Re_x}}$$



壁面剪切应力



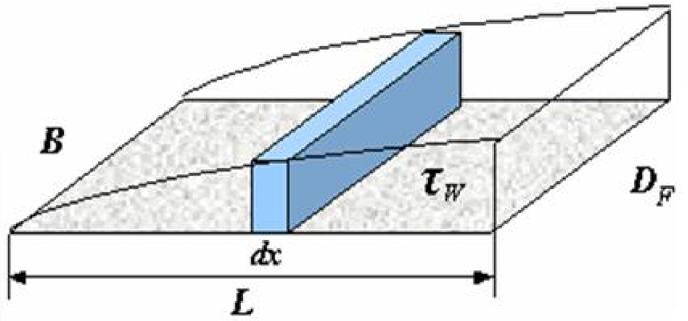
$$\tau_W = \mu \frac{\partial u_x}{\partial y}\bigg|_{y=0} = \mu U_0 \sqrt{\frac{U_0}{\nu x}} f''(0) = 0.332 \mu U_0 \sqrt{\frac{U_0}{\nu x}}$$

则壁面摩擦阻力 D_F：

$$D_F = \int_0^L \tau_W B dx = 0.332 \mu U_0 \sqrt{\frac{U_0}{\nu}} \int_0^L \frac{1}{\sqrt{x}} dx = 0.664 B U_0 \sqrt{\mu \rho L U_0}$$

$$D_F = \frac{1.328}{\sqrt{Re_L}} BL \frac{1}{2} \rho U_0^2$$

定义摩擦阻力系数 C_D：



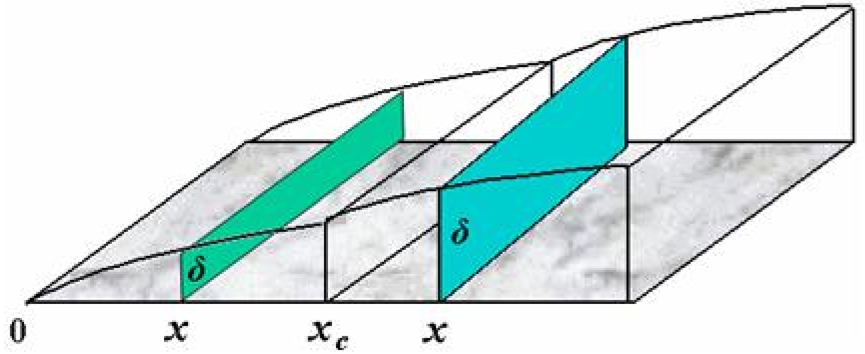
$$D_F = \frac{1.328}{\sqrt{Re_L}} BL \frac{1}{2}\rho U_0^2 = C_D A \frac{1}{2}\rho U_0^2$$

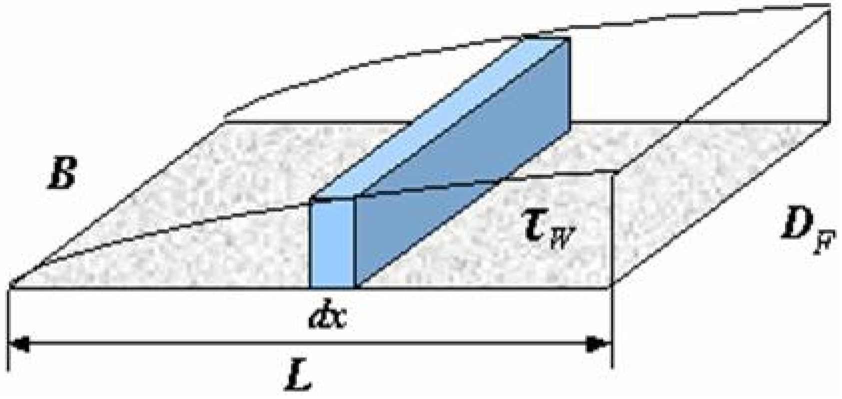
$$C_D = \frac{1.328}{\sqrt{Re_L}}$$

$$D_F = C_D A \frac{1}{2}\rho U_0^2$$

问题探讨 上述阻力公式只适用层流？

3. 边界层动量积分方程



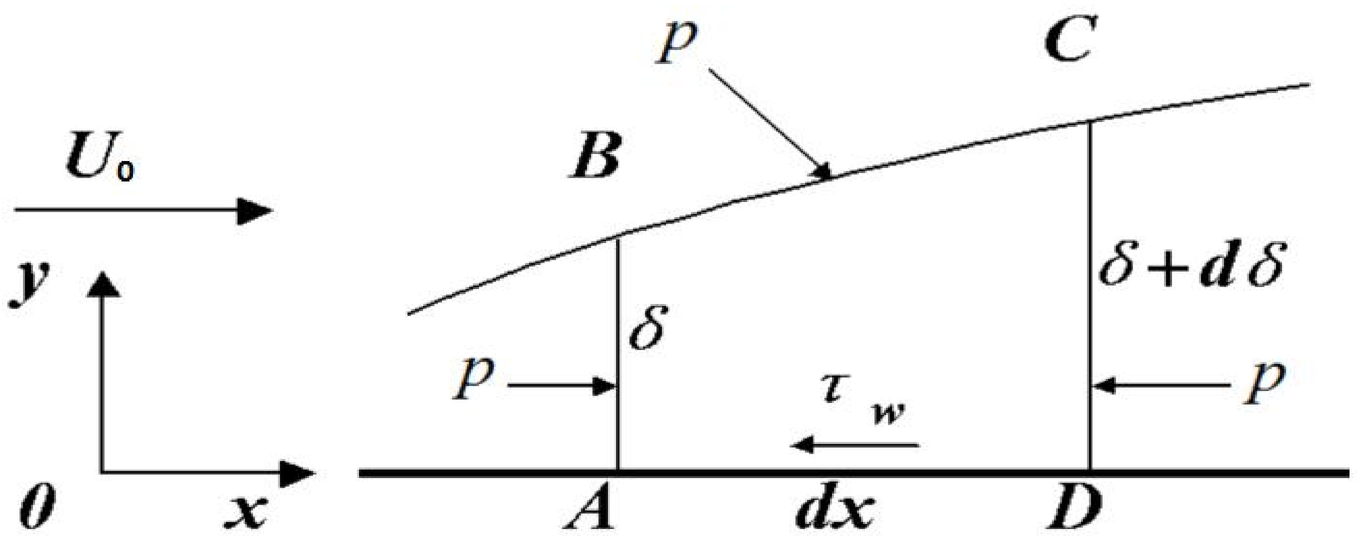


临界雷诺数 $Re_{xc}=5\times10^5$

层流 $$\frac{\delta}{x}=\frac{4.64}{\sqrt{Re_x}}$$ $$D_F=\frac{1.292}{\sqrt{Re_L}}\cdot BL\cdot\frac{1}{2}\rho U_0^2$$

湍流 $$\frac{\delta}{x}=\frac{0.376}{\sqrt[5]{Re_x}}$$ $$D_F=\frac{0.072}{\sqrt[5]{Re_L}}\cdot BL\cdot\frac{1}{2}\rho U_0^2$$

冯•卡门 边界层动量积分方程



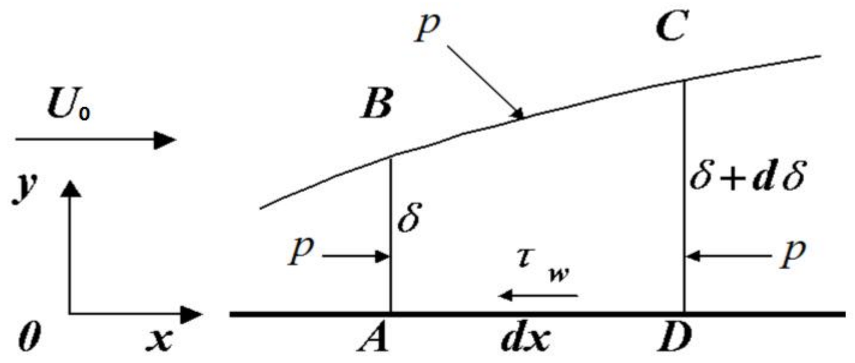
选取控制体 $ABCD$，单位宽度

流动为无压差流动 $\dfrac{\partial p}{\partial x} = 0$

动量守恒：

$$\frac{\partial \left(m\vec{u}_x\right)}{\partial t} = \left(w\vec{u}\right)_{1x} - \left(w\vec{u}\right)_{2x} + \Sigma \vec{F}_x$$

对定常流体： $\Sigma \vec{F}_x = (w\vec{u})_{2x} - (w\vec{u})_{1x}$

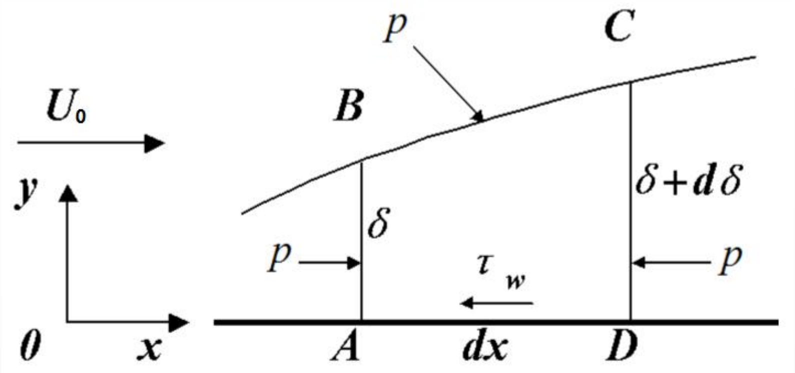


合力： $\Sigma \vec{F}_x = p\delta + pd\delta - p(\delta + d\delta) - \tau_w dx$

$$\Sigma \vec{F}_x = -\tau_w dx$$

动量变化率：	进 AB 面	微元	AB 面
	质量流率	$\rho u_x dy$	$\int_0^\delta \rho u_x dy$
	动量速率	$\rho u_x^2 dy$	$\int_0^\delta \rho u_x^2 dy$

出 CD 面:



质量流率 $$\int_0^\delta \rho u_x dy + \frac{\partial}{\partial x}\left(\int_0^\delta \rho u_x dy\right)dx$$

动量速率 $$\int_0^\delta \rho u_x^2 dy + \frac{\partial}{\partial x}\left(\int_0^\delta \rho u_x^2 dy\right)dx$$

进 BC 面:

质量流率 $$\frac{\partial}{\partial x}\left(\int_0^\delta \rho u_x dy\right)dx$$

动量速率 $$U_0\frac{\partial}{\partial x}\left(\int_0^\delta \rho u_x dy\right)dx$$

动量变化率：

$$(w\vec{u})_{2x}-(w\vec{u})_{1x}=\frac{\partial}{\partial x}\left(\int_0^{\delta}\rho u_x^2dy\right)dx-U_0\frac{\partial}{\partial x}\left(\int_0^{\delta}\rho u_xdy\right)dx$$

$$(w\vec{u})_{2x}-(w\vec{u})_{1x}=\rho dx\frac{\partial}{\partial x}\int_0^{\delta}(u_x-U_0)u_xdy$$

根据动量守恒：

$$\rho\frac{\partial}{\partial x}\int_0^{\delta}(U_0-u_x)u_xdy=\tau_w$$

边界层动量积分方程

层流边界层，设速度分布：

$$\frac{u_x}{U_0}=a+b\left(\frac{y}{\delta}\right)+c\left(\frac{y}{\delta}\right)^2+d\left(\frac{y}{\delta}\right)^3$$

边界条件：

$$\begin{cases} y=0,\ \ u_x=0; & y=0,\ \ \dfrac{\partial^2 u_x}{\partial y^2}=0 \\ y=\delta,\ \ u_x=U_0; & y=\delta,\ \ \dfrac{\partial u_x}{\partial y}=0 \end{cases}$$

速度分布：

$$\frac{u_x}{U_0}=\frac{3}{2}\left(\frac{y}{\delta}\right)-\frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

壁面切应力：

$$\tau_w=\mu\frac{\partial u_x}{\partial y}\bigg|_{y=0}=\mu\frac{3}{2}\frac{U_0}{\delta}$$

将 u_x，τ_w 代入动量积分方程求得：

$$\delta = \frac{4.64x}{\sqrt{Re_x}} + C$$

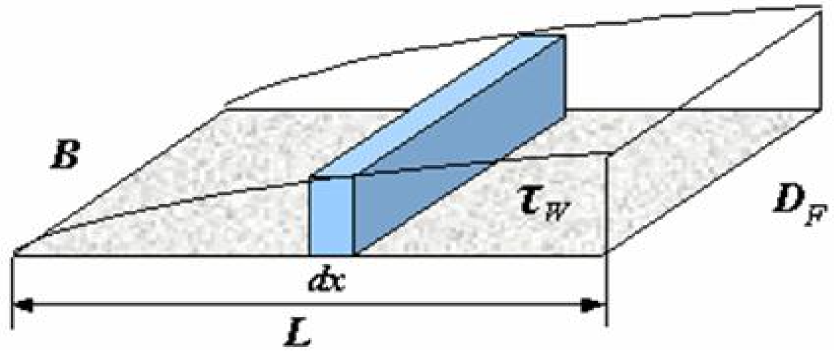
$\because x = 0$，$\delta = 0$；$\therefore C = 0$。

$$\delta = \frac{4.64x}{\sqrt{Re_x}}$$

此式只适用 $x < x_c$

壁面切应力:

$$\tau_w = \mu \frac{3}{2}\frac{U_0}{\delta} = 0.323\rho U^2 Re_x^{-\frac{1}{2}}$$



层流

对 $B\times L$ 壁面总阻力:

$$D_F = \int_0^L \tau_w dx \cdot B = 0.646B\sqrt{\mu\rho L U_0^3}$$

阻力系数:

$$C_D = \frac{\frac{D_F}{A}}{\frac{1}{2}\rho U_0^2}$$

层流:

$$C_D = \frac{1.292}{\sqrt{Re_L}}$$

湍流边界层，设速度分布：

$$\frac{u_x}{U_0}=\left(\frac{y}{\delta}\right)^{\frac{1}{7}}$$

代入**边界层动量积分方程**

$$\rho\frac{\partial}{\partial x}\int_0^{\delta}(U_0-u_x)u_x dy=\tau_w$$

湍流的壁面剪切应力：

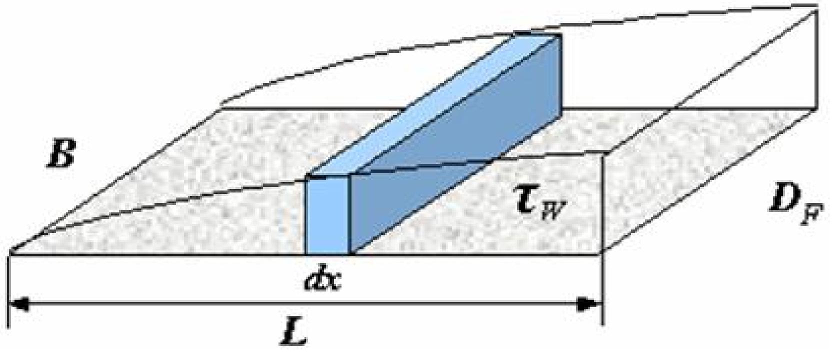
得： $$\tau_w=\frac{7}{72}\rho U_0^2\frac{d\delta}{dx}$$

$$\tau_w=0.023\rho U_0^{\frac{7}{4}}\left(\frac{\nu}{\delta}\right)^{\frac{1}{4}}$$

问题探讨 上述湍流 τ_W 公式如何求得？

$$\frac{7}{72}\rho U_0^2\frac{d\delta}{dx}=0.023\rho U_0^{\frac{7}{4}}\left(\frac{v}{\delta}\right)^{\frac{1}{4}}$$

$$\int_0^{\delta}\delta^{\frac{1}{4}}d\delta=\int_0^{x}0.236\left(\frac{v}{U_0}\right)^{\frac{1}{4}}dx$$



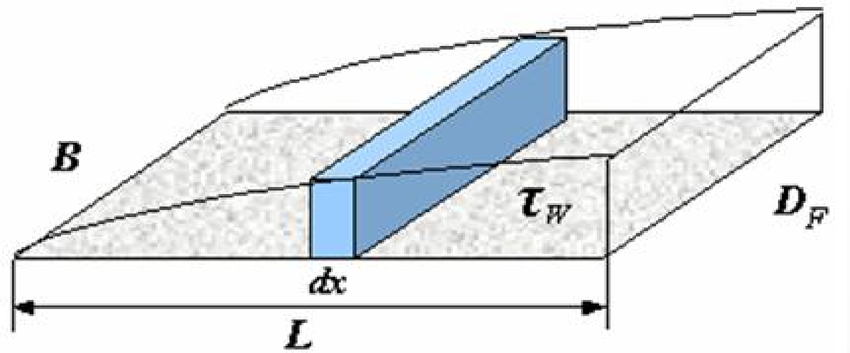
湍流

注意 $x=0$， $\delta=0$ 。说明假定一开始就是湍流。

$$\frac{\delta}{x}=\frac{0.376}{\sqrt[5]{Re_x}}$$

此式只适用 $x>x_c$

对 $B\times L$ 壁面总阻力：



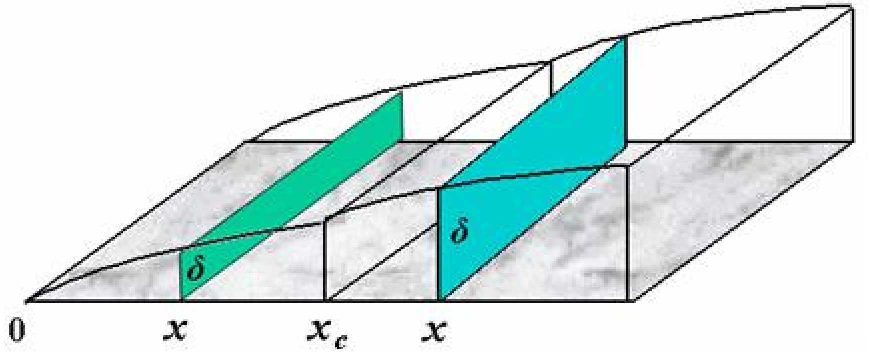
假定一开始就是湍流

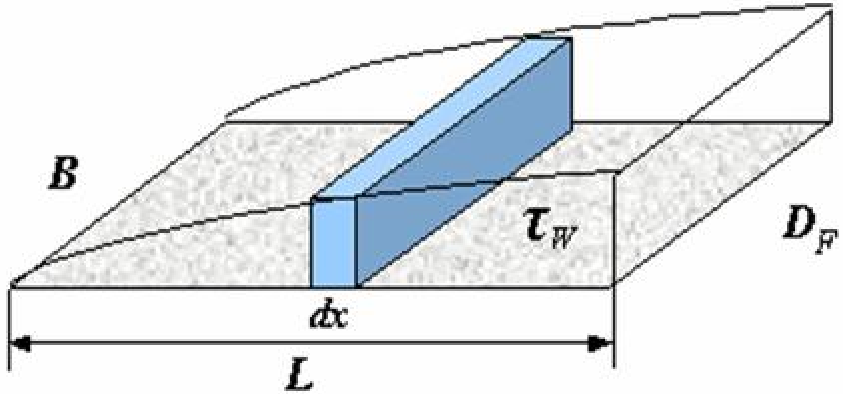
$$D_F = \int_0^L \tau_w dx \cdot B = \frac{0.073}{\sqrt[5]{Re_L}} BL \frac{1}{2}\rho U_0^2$$

湍流边界层阻力系数：

$$C_D = \frac{0.073}{\sqrt[5]{Re_L}}$$

4. 边界层计算





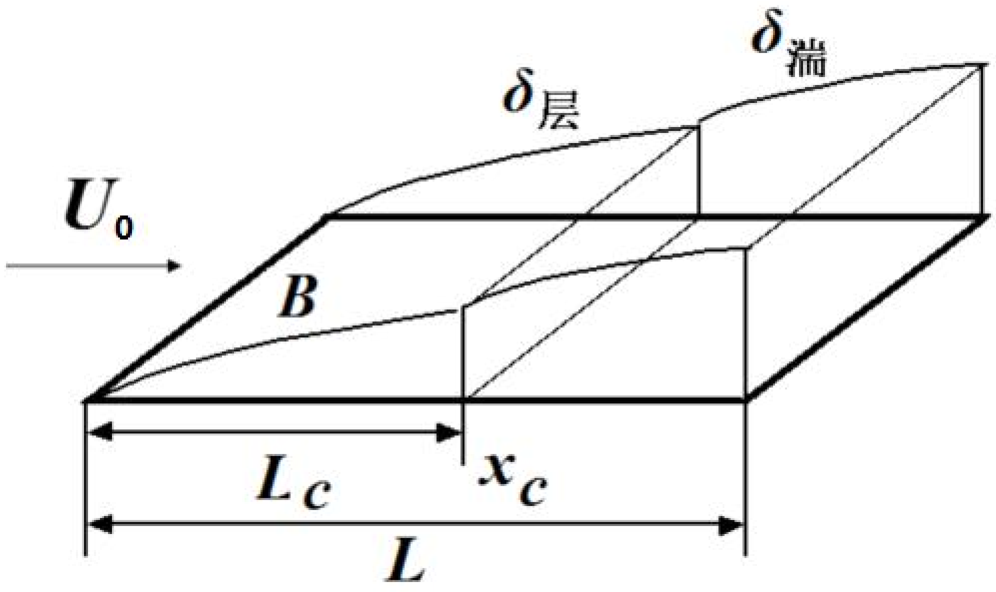
临界雷诺数 Re_{xc}=5×10^5

层流 $$\frac{\delta}{x}=\frac{4.64}{\sqrt{Re_x}}$$ $$D_F=\frac{1.292}{\sqrt{Re_L}}\cdot BL\cdot\frac{1}{2}\rho U_0^2$$

湍流 $$\frac{\delta}{x}=\frac{0.376}{\sqrt[5]{Re_x}}$$ $$D_F=\frac{0.072}{\sqrt[5]{Re_L}}\cdot BL\cdot\frac{1}{2}\rho U_0^2$$

当边界层达到湍流时

当 $L > x_c$ 时，求 D_F



层流段 $L_c = x_c$：$D_{F层，Lc}$

全部以湍流计算：$D_{F湍，L}$

误算作湍流的 $D_{F湍，Lc}$

$$\therefore D_F = D_{F层，Lc} + D_{F湍，L} - D_{F湍，Lc}$$

平板阻力计算

已知：流体 v=10^{-6}m^2/s，U_0=2.4m/s， Re_{xc}=5×10^5，ρ=1000 kg/m^3。

求：①. x_c ②. $\delta_{x=3\ \mathrm{m}}$ ③. $D_{F,\ L=2\ \mathrm{m},\ B=1\mathrm{m}}$

解：①. $$Re_{xc}=\frac{x_c U_0}{v} \qquad x_c=\frac{v\,Re_{xc}}{U_0}=0.208m$$

②. x=3m>x_c 湍流 $$\delta=\frac{0.376x}{\sqrt[5]{Re_x}}=0.048m$$

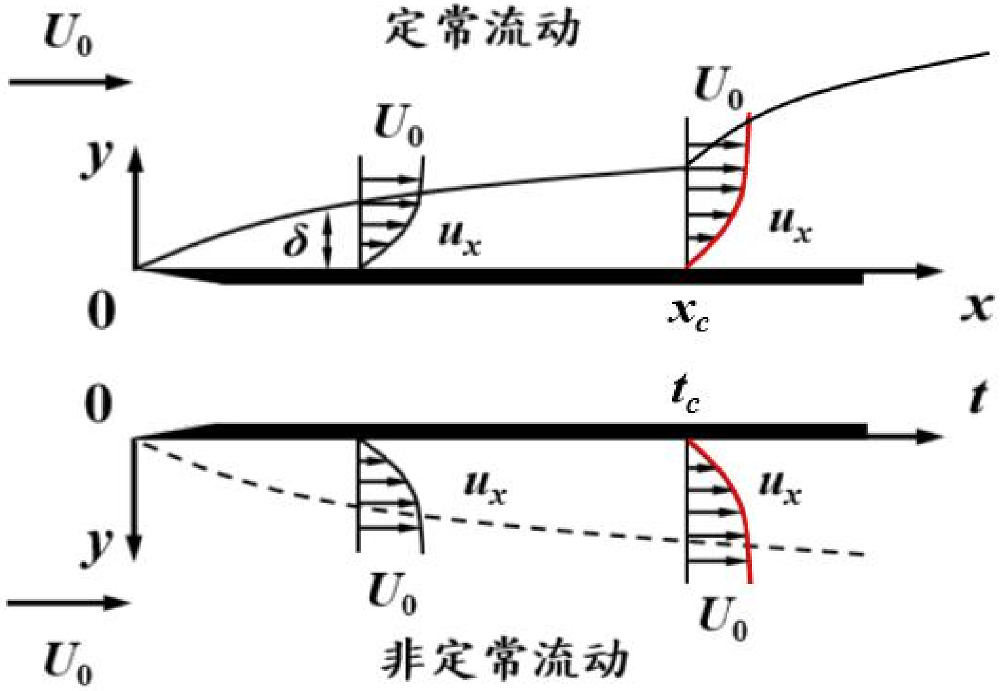
③. L=2m > L_c 湍流 $$\therefore D_F=D_{F层,\ Lc}+D_{F湍,\ L}-D_{F湍,\ Lc}=17.08\mathrm{N}$$

课后思考

1.还记得静止流体中的平板启动吗?

平板边界层流动

静止流体中的平板启动



临界距离　空间场

临界时间　时间场