Part I

Chap 5: Pairs of random Variables

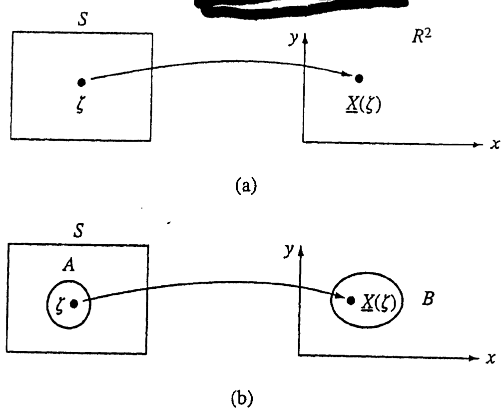


FIGURE 5.1
(a) A function assigns a pair of real numbers to each outcome in S. (b) Equivalent events for two random variables.

$\mathbf{X}(\zeta) = (X(\zeta), Y(\zeta))$ to each outcome ζ in S. Basically we are dealing with a vector function that maps S into R^2, the real plane, as shown in Fig. 5.1(a). We are ultimately interested in events involving the pair (X, Y).

Example 5.1

Let a random experiment consist of selecting a student's name from an urn. Let ζ denote the outcome of this experiment, and define the following two functions:

$$H(\zeta) = \text{height of student } \zeta \text{ in centimeters}$$

$$W(\zeta) = \text{weight of student } \zeta \text{ in kilograms}$$

$(H(\zeta), W(\zeta))$ assigns a pair of numbers to each ζ in S.

We are interested in events involving the pair (H, W). For example, the event $B = \{H \le 183, W \le 82\}$ represents students with height less that 183 cm (6 feet) and weight less than 82 kg (180 lb).

Example 5.3

Let the outcome ζ in a random experiment be the length of a randomly selected message. Suppose that messages are broken into packets of maximum length M bytes. Let Q be the number of full packets in a message and let R be the number of bytes left over. $(Q(\zeta), R(\zeta))$ assigns a pair of numbers to each ζ in S. Q takes on values in the range $0, 1, 2, \ldots$, and R takes on values in the range $0, 1, \ldots, M - 1$. An event of interest may be $B = \{R < M/2\}$, "the last packet is less than half full."

The events involving a pair of random variables (X, Y) are specified by conditions that we are interested in and can be represented by regions in the plane. Figure 5.2 shows three examples of events:

$$A = \{X + Y \leq 10\}$$
$$B = \{\min(X, Y) \leq 5\}$$
$$C = \{X^2 + Y^2 \leq 100\}.$$

Event A divides the plane into two regions according to a straight line. Note that the event in Example 5.2 is of this type. Event C identifies a disk centered at the origin and

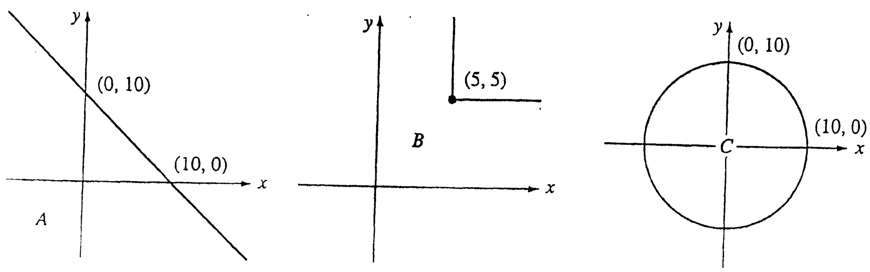


FIGURE 5.2
Examples of two-dimensional events.

it corresponds to the event in Example 5.4. Event B is found by noting that $\{\min(X, Y) \leq 5\} = \{X \leq 5\} \cup \{Y \leq 5\}$, that is, the minimum of X and Y is less than or equal to 5 if either X and/or Y is less than or equal to 5.

To determine the probability that the pair $\mathbf{X} = (X, Y)$ is in some region B in the plane, we proceed as in Chapter 3 to find the equivalent event for B in the underlying sample space S:

$$A = \mathbf{X}^{-1}(B) = \{\zeta: (X(\zeta), Y(\zeta)) \text{ in } B\}. \tag{5.1a}$$

The relationship between $A = \mathbf{X}^{-1}(B)$ and B is shown in Fig. 5.1(b). If A is in $\mathcal{F}$, then it has a probability assigned to it, and we obtain:

$$P[\mathbf{X} \text{ in } B] = P[A] = P[\{\zeta: (X(\zeta), Y(\zeta)) \text{ in } B\}]. \tag{5.1b}$$

The approach is identical to what we followed in the case of a single random variable. The only difference is that we are considering the *joint behavior of X and Y* that is induced by the underlying random experiment.

The joint probability mass function, joint cumulative distribution function, and joint probability density function provide approaches to specifying the probability law that governs the behavior of the pair (X, Y). Our general approach is as follows. We first focus on events that correspond to rectangles in the plane:

$$B = \{X \text{ in } A_1\} \cap \{Y \text{ in } A_2\} \tag{5.2}$$

where A_k is a one-dimensional event (i.e., subset of the real line). We say that these events are of **product form.** The event B occurs when both $\{X \text{ in } A_1\}$ and $\{Y \text{ in } A_2\}$ occur jointly. Figure 5.4 shows some two-dimensional product-form events. We use Eq. (5.1b) to find the probability of product-form events:

$$P[B] = P[\{X \text{ in } A_1\} \cap \{Y \text{ in } A_2\}] \triangleq P[X \text{ in } A_1, Y \text{ in } A_n]. \tag{5.3}$$

By defining A appropriately we then obtain the joint pmf, joint cdf, and joint pdf of (X, Y).

PAIRS OF DISCRETE RANDOM VARIABLES

Let the vector random variable $\mathbf{X} = (X, Y)$ assume values from some countable set $S_{X,Y} = \{(x_j, y_k), j = 1, 2, \ldots, k = 1, 2, \ldots\}$. The **joint probability mass function** of $\mathbf{X}$ specifies the probabilities of the event $\{X = x\} \cap \{Y = y\}$:

$$\begin{aligned} p_{X,Y}(x, y) &= P[\{X = x\} \cap \{Y = y\}] \\ &\triangleq P[X = x, Y = y] \qquad \text{for } (x, y) \in R^2. \end{aligned} \tag{5.4a}$$

The values of the pmf on the set $S_{X,Y}$ provide the essential information:

$$\begin{aligned} p_{X,Y}(x_j, y_k) &= P[\{X = x_j\} \cap \{Y = y_k\}] \\ &\triangleq P[X = x_j, Y = y_k] \quad (x_j, y_k) \in S_{X,Y}. \end{aligned} \tag{5.4b}$$

There are several ways of showing the pmf graphically: (1) For small sample spaces we can present the pmf in the form of a table as shown in Fig. 5.5(a). (2) We can present the pmf using arrows of height $p_{X,Y}(x_j, y_k)$ placed at the points $\{(x_j, y_k)\}$ in the plane, as shown in Fig. 5.5(b), but this can be difficult to draw. (3) We can place dots at the points $\{(x_j, y_k)\}$ and label these with the corresponding pmf value as shown in Fig. 5.5(c).

The probability of any event B is the sum of the pmf over the outcomes in B:

$$P[\mathbf{X} \text{ in } B] = \sum_{(x_j, y_k) \text{ in } B} \sum p_{X,Y}(x_j, y_k). \tag{5.5}$$

Frequently it is helpful to sketch the region that contains the points in B as shown, for example, in Fig. 5.6. When the event B is the entire sample space $S_{X,Y}$, we have:

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} p_{X,Y}(x_j, y_k) = 1. \tag{5.6}$$

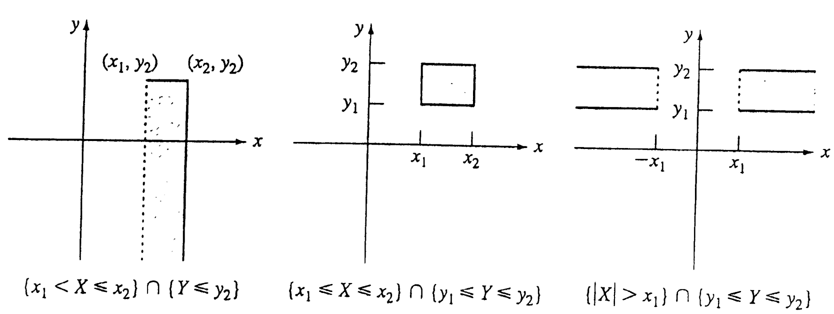


FIGURE 5.4
Some two-dimensional product-form events.

7

Joint CDF of X and Y.

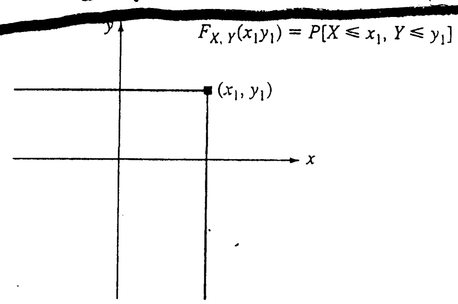


FIGURE 5.7
The joint cumulative distribution function is defined as the probability of the semi-infinite rectangle defined by the point (x_1, y_1).

A basic building block for events involving two-dimensional random variables is the semi-infinite rectangle defined by $\{(x, y): x \leq x_1 \text{ and } y \leq y_1\}$, as shown in Fig. 5.7. We also use the more compact notation $\{x \leq x_1, y \leq y_1\}$ to refer to this region. The **joint cumulative distribution function of X and Y** is defined as the probability of the event $\{X \leq x_1\} \cap \{Y \leq y_1\}$:

$$F_{X,Y}(x_1, y_1) = P[X \leq x_1, Y \leq y_1]. \tag{5.8}$$

In terms of relative frequency, $F_{X,Y}(x_1, y_1)$ represents the long-term proportion of time in which the outcome of the random experiment yields a point X that falls in the rectangular region shown in Fig. 5.7. In terms of probability "mass," $F_{X,Y}(x_1, y_1)$ represents the amount of mass contained in the rectangular region.

The joint cdf satisfies the following properties.

(i) The joint cdf is a nondecreasing function of x and y:

$$F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2) \qquad \text{if } x_1 \leq x_2 \text{ and } y_1 \leq y_2, \tag{5.9a}$$

(ii) $F_{X,Y}(x_1, -\infty) = 0, \qquad F_{X,Y}(-\infty, y_1) = 0, \qquad F_{X,Y}(\infty, \infty) = 1.$ (5.9b)

(iii) We obtain the **marginal cumulative distribution functions** by removing the constraint on one of the variables. The marginal cdf's are the probabilities of the regions shown in Fig. 5.8:

$$F_X(x_1) = F_{X,Y}(x_1, \infty) \quad \text{and} \quad F_Y(y_1) = F_{X,Y}(\infty, y_1). \tag{5.9c}$$

(iv) The joint cdf is continuous from the "north" and from the "east," that is,

$$\lim_{x \to a^+} F_{X,Y}(x, y) = F_{X,Y}(a, y) \quad \text{and} \quad \lim_{y \to b^+} F_{X,Y}(x, y) = F_{X,Y}(x, b). \tag{5.9d}$$

(v) The probability of the rectangle $\{x_1 < x \leq x_2, y_1 < y \leq y_2\}$ is given by:

$$P[x_1 < X \leq x_2, y_1 < Y \leq y_2] =$$
$$F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1). \tag{5.9e}$$

8

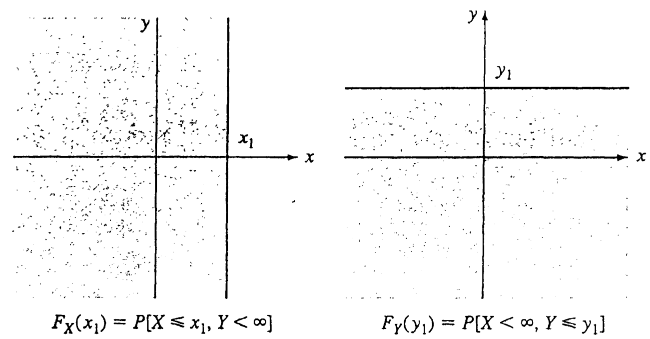


FIGURE 5.8
The marginal cdf's are the probabilities of these half-planes.

Property (i) follows by noting that the semi-infinite rectangle defined by (x_1, y_1) is contained in that defined by (x_2, y_2) and applying Corollary 7. Properties (ii) to (iv) are obtained by limiting arguments. For example, the sequence $\{x \leq x_1 \text{ and } y \leq -n\}$ is decreasing and approaches the empty set $\emptyset$, so

$$F_{X,Y}(x_1, -\infty) = \lim_{n\to\infty} F_{X,Y}(x_1, -n) = P[\emptyset] = 0.$$

For property (iii) we take the sequence $\{x \leq x_1 \text{ and } y \leq n\}$ which increases to $\{x \leq x_1\}$, so

$$\lim_{n\to\infty} F_{X,Y}(x_1, n) = P[X \leq x_1] = F_X(x_1).$$

For property (v) note in Fig. 5.9(a) that $B = \{x_1 < x \leq x_2, y \leq y_1\} = \{X \leq x_2, Y \leq y_1\} - \{X \leq x_1, Y \leq y_1\}$, so $P[B] = P[x_1 < X \leq x_2, Y \leq y_1] = F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_1)$. In Fig. 5.9(b), note that $F_{X,Y}(x_2, y_2) = P[A] + P[B] + F_{X,Y}(x_1, y_2)$. Property (v) follows by solving for $P[A]$ and substituting the expression for $P[B]$.

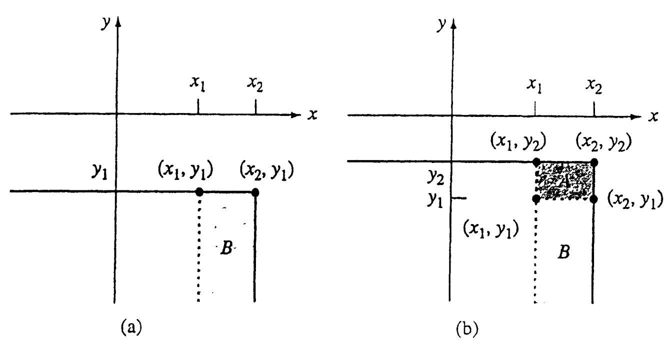


FIGURE 5.9
The joint cdf can be used to determine the probability of various events.

Page 2

Joint pdf of two jointly Continuous Random Va

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$$

properties:

1. $f_{XY}(x,y) \geq 0$ since $F_{XY}(x,y)$ is nondecreasing func.

2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1$

3. $F_{XY}(x,y) = P[X \leq x, Y \leq y]$

$$= \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(\xi_1, \xi_2)\, d\xi_1\, d\xi_2$$

4. $F_X(x) = \int_{-\infty}^{x} \underbrace{\int_{-\infty}^{\infty} f_{XY}(\xi_1,\xi_2)\, d\xi_2}_{f_X(\xi_1)}\, d\xi_1$

$F_Y(y) = \int_{-\infty}^{y} \underbrace{\int_{-\infty}^{\infty} f_{XY}(\xi_1,\xi_2)\, d\xi_1}_{f_{XY}(\xi_2)}\, d\xi_2$

Marginal CDF from Joint pdf

Marginal pdf can also be obtained from joint pdf

$$f_X(x) = \frac{d}{dx} \int_{-\infty}^{x} \left[\int_{-\infty}^{\infty} f_{XY}(x', y')\, dy' \right] dx'$$

$$= \int_{-\infty}^{\infty} f_{XY}(x, y')\, dy'$$

Similarly

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x', y)\, dx'$$

Example 5.12

The joint cdf for the vector of random variable $\mathbf{X} = (X, Y)$ is given by

$$F_{X,Y}(x, y) = \begin{cases} (1 - e^{-\alpha x})(1 - e^{-\beta y}) & x \geq 0, y \geq 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the marginal cdf's.

The marginal cdf's are obtained by letting one of the variables approach infinity:

$$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x, y) = 1 - e^{-\alpha x} \quad x \geq 0$$

$$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x, y) = 1 - e^{-\beta y} \quad y \geq 0.$$

X and Y individually have exponential distributions with parameters α and β, respectively.

Example 5.13

Find the probability of the events $A = \{X \leq 1, Y \leq 1\}$, $B = \{X > x, Y > y\}$, where $x > 0$ and $y > 0$, and $D = \{1 < X \leq 2, 2 < Y \leq 5\}$ in Example 5.12.

The probability of A is given directly by the cdf:

$$P[A] = P[X \leq 1, Y \leq 1] = F_{X,Y}(1, 1) = (1 - e^{-\alpha})(1 - e^{-\beta}).$$

The probability of B requires more work. By DeMorgan's rule:

$$B^c = (\{X > x\} \cap \{Y > y\})^c = \{X \leq x\} \cup \{Y \leq y\}.$$

Corollary 5 in Section 2.2 gives the probability of the union of two events:

$$\begin{aligned} P[B^c] &= P[X \leq x] + P[Y \leq y] - P[X \leq x, Y \leq y] \\ &= (1 - e^{-\alpha x}) + (1 - e^{-\beta y}) - (1 - e^{-\alpha x})(1 - e^{-\beta y}) \\ &= 1 - e^{-\alpha x}e^{-\beta y}. \end{aligned}$$

Finally we obtain the probability of B:

$$P[B] = 1 - P[B^c] = e^{-\alpha x}e^{-\beta y}.$$

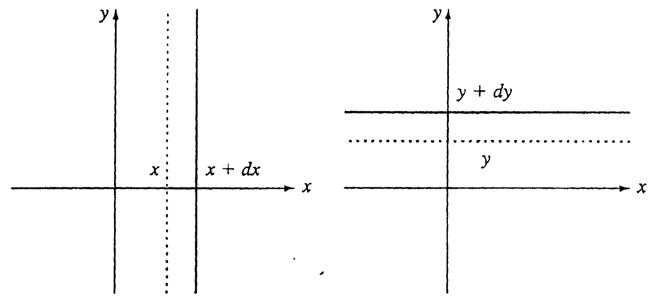
You should sketch the region B on the plane and identify the events involved in the calculation of the probability of B^c.

The probability of event D is found by applying property (vi) of the joint cdf:

$$\begin{aligned} &P[1 < X \leq 2, 2 < Y \leq 5] \\ &\quad = F_{X,Y}(2, 5) - F_{X,Y}(2, 2) - F_{X,Y}(1, 5) + F_{X,Y}(1, 2) \\ &\quad = (1 - e^{-2\alpha})(1 - e^{-5\beta}) - (1 - e^{-2\alpha})(1 - e^{-2\beta}) \\ &\qquad -(1 - e^{-\alpha})(1 - e^{-5\beta}) + (1 - e^{-\alpha})(1 - e^{-2\beta}). \end{aligned}$$

Part 2

15



$f_X(x)dx \cong P[x < X \leq x + dx, Y < \infty]$ $f_Y(y)dy \cong P[X < \infty, y < Y \leq y + dy]$

(a) (b)

FIGURE 5.14
Interpretation of marginal pdf's.

Example 5.15 Jointly Uniform Random Variables

A randomly selected point (X, Y) in the unit square has the uniform joint pdf given by

$$f_{X,Y}(x, y) = \begin{cases} 1 & 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

The scattergram in Fig. 5.3(a) corresponds to this pair of random variables. Find the joint cdf of X and Y.

The cdf is found by evaluating Eq. (5.13). You must be careful with the limits of the integral: The limits should define the region consisting of the intersection of the semi-infinite rectangle defined by (x, y) and the region where the pdf is nonzero. There are five cases in this problem, corresponding to the five regions shown in Fig. 5.15.

1. If $x < 0$ or $y < 0$, the pdf is zero and Eq. (5.14) implies

$$F_{X,Y}(x, y) = 0.$$

2. If (x, y) is inside the unit interval,

$$F_{X,Y}(x, y) = \int_0^x \int_0^y 1\, dx'\, dy' = xy.$$

3. If $0 \leq x \leq 1$ and $y > 1$,

$$F_{X,Y}(x, y) = \int_0^x \int_0^1 1\, dx'\, dy' = x.$$

4. Similarly, if $x > 1$ and $0 \leq y \leq 1$,

$$F_{X,Y}(x, y) = y.$$

'16

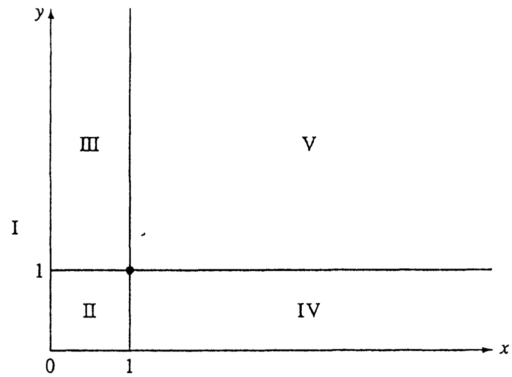


FIGURE 5.15
Regions that need to be considered separately in computing cdf in Example 5.15.

5. Finally, if $x > 1$ and $y > 1$,

$$F_{X,Y}(x, y) = \int_0^1 \int_0^1 1\, dx'\, dy' = 1.$$

We see that this is the joint cdf of Example 5.11.

Example 5.16

Find the normalization constant c and the marginal pdf's for the following joint pdf:

$$f_{X,Y}(x, y) = \begin{cases} ce^{-x}e^{-y} & 0 \le y \le x < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

The pdf is nonzero in the shaded region shown in Fig. 5.16(a). The constant c is found from the normalization condition specified by Eq. (5.12):

$$1 = \int_0^\infty \int_0^x ce^{-x}e^{-y}\, dy\, dx = \int_0^\infty ce^{-x}(1 - e^{-x})\, dx = \frac{c}{2}.$$

Therefore $c = 2$. The marginal pdf's are found by evaluating Eqs. (5.17a) and (5.17b):

$$f_X(x) = \int_0^\infty f_{X,Y}(x, y)\, dy = \int_0^x 2e^{-x}e^{-y}\, dy = 2e^{-x}(1 - e^{-x}) \qquad 0 \le x < \infty$$

and

$$f_Y(y) = \int_0^\infty f_{X,Y}(x, y)\, dx = \int_y^\infty 2e^{-x}e^{-y}\, dx = 2e^{-2y} \qquad 0 \le y < \infty.$$

You should fill in the steps in the evaluation of the integrals as well as verify that the marginal pdf's integrate to 1.

17

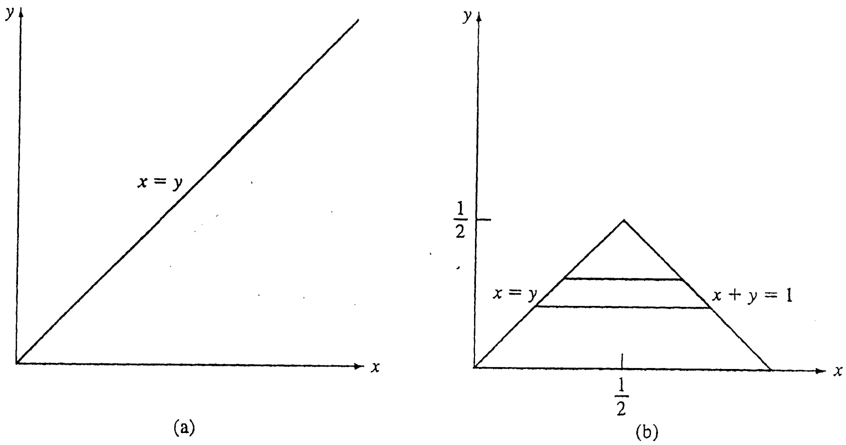


FIGURE 5.16
The random variables X and Y in Examples 5.16 and 5.17 have a pdf that is nonzero only in the shaded region shown in part (a).

Example 5.17

Find $P[X + Y \leq 1]$ in Example 5.16.

Figure 5.16(b) shows the intersection of the event $\{X + Y \leq 1\}$ and the region where the pdf is nonzero. We obtain the probability of the event by "adding" (actually integrating) infinitesimal rectangles of width dy as indicated in the figure:

$$P[X + Y \leq 1] = \int_0^{.5} \int_y^{1-y} 2e^{-x}e^{-y}\,dx\,dy = \int_0^{.5} 2e^{-y}[e^{-y} - e^{-(1-y)}]\,dy$$
$$= 1 - 2e^{-1}.$$

Example 5.18 Jointly Gaussian Random Variables

The joint pdf of X and Y, shown in Fig. 5.17, is

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{1 - \rho^2}} e^{-(x^2 - 2\rho xy + y^2)/2(1-\rho^2)} \qquad -\infty < x, y < \infty. \tag{5.18}$$

We say that X and Y are jointly Gaussian.[1] Find the marginal pdf's.

The marginal pdf of X is found by integrating $f_{X,Y}(x, y)$ over y:

$$f_X(x) = \frac{e^{-x^2/2(1-\rho^2)}}{2\pi\sqrt{1 - \rho^2}} \int_{-\infty}^{\infty} e^{-(y^2 - 2\rho xy)/2(1-\rho^2)}\,dy.$$

[1]This is an important special case of jointly Gaussian random variables. The general case is discussed in Section 5.9.

found from

):

∞

ie marginal

20

$$= P[Q = q, R = r] \qquad \text{for all } q = 0, 1, \ldots \quad r = 0, \ldots, M - 1.$$

Therefore Q and R are independent.

In general, it can be shown that the random variables *X and Y are independent if and only if their joint cdf is equal to the product of its marginal cdf's*:

$$F_{X,Y}(x, y) = F_X(x)F_Y(y) \qquad \text{for all } x \text{ and } y. \tag{5.22}$$

Similarly, if X and Y are jointly continuous, then *X and Y are independent if and only if their joint pdf is equal to the product of the marginal pdf's*:

$$f_{X,Y}(x, y) = f_X(x)f_Y(y) \qquad \text{for all } x \text{ and } y. \tag{5.23}$$

Equation (5.23) is obtained from Eq. (5.22) by differentiation. Conversely, Eq. (5.22) is obtained from Eq. (5.23) by integration.

Example 5.21

Are the random variables X and Y in Example 5.16 independent?

Note that $f_X(x)$ and $f_Y(y)$ are nonzero for all $x > 0$ and all $y > 0$. Hence $f_X(x)f_Y(y)$ is nonzero in the entire positive quadrant. However $f_{X,Y}(x, y)$ is nonzero only in the region $y < x$ inside the positive quadrant. Hence Eq. (5.23) does not hold for all x, y and the random variables are not independent. You should note that in this example the joint pdf appears to factor, but nevertheless it is not the product of the marginal pdf's.

Example 5.22

Are the random variables X and Y in Example 5.18 independent? The product of the marginal pdf's of X and Y in Example 5.18 is

$$f_X(x)f_Y(y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2} \qquad -\infty < x, y < \infty.$$

By comparing to Eq. (5.18) we see that the product of the marginals is equal to the joint pdf if and only if $\rho = 0$. Therefore the jointly Gaussian random variables X and Y are independent if and only if $\rho = 0$. We see in a later section that ρ is the *correlation coefficient* between X and Y.

Example 5.23

Are the random variables X and Y independent in Example 5.12? If we multiply the marginal cdf's found in Example 5.12 we find

$$F_X(x)F_Y(y) = (1 - e^{-\alpha x})(1 - e^{-\beta y}) = F_{X,Y}(x, y) \qquad \text{for all } x \text{ and } y.$$

Therefore Eq. (5.22) is satisfied so X and Y are independent.

If X and Y are independent random variables, then the random variables defined by any pair of functions $g(X)$ and $h(Y)$ are also independent. To show this, consider the