Chap. 2 Basic Concepts of Prob. Theory

In this chapter we study:

* Set theory, used as a link betwee sample space and random experiment.

* Axioms of probability: these are properties that a probability law must satisfy.

* Counting methods (permutation and combinations)

* Conditional probability, ~~use~~ of partial information

* Statistical Independence

2.1 * Random experiments

It is an experiment that the outcome varies at random when the experiment is repeated under the same condition.

A random experiment is specified by stating an experimental procedure and a set of one or more measurements/observat

A random experiment may consist of the same procedure but differs in observation made

Ex. - Select a ball at random from an urn

- Tossing a coin

- measuring the time between two message arr

* outcome (or sample point) ξ

A result of random experiment that can not be decomposed into other results (represented by ξ).

* Sample space ; S

The set or collection of all possible outcomes of a specific experiment.

Discrete sample space S : outcome is discrete (countable) that is, its outcomes can be put into one-to-one correspondence with the positive integers.

Continuous sample space S : outcome is continuous (not countable)

* Events

- An event is a subset of ~~sample~~ space S that satisfies certain conditions.

- The event occurs iff the outcome of ~~[illegible]~~ experiment (ξ) is an element of the subset

- Certain events (S) consists of all outcomes, hence it always occurs.
- Impossible (Null) events ($\emptyset$) ; contains no outcomes, never occurs.
- Elementary events ; consists of a single ~~[illegible]~~ outcome from a

2-2

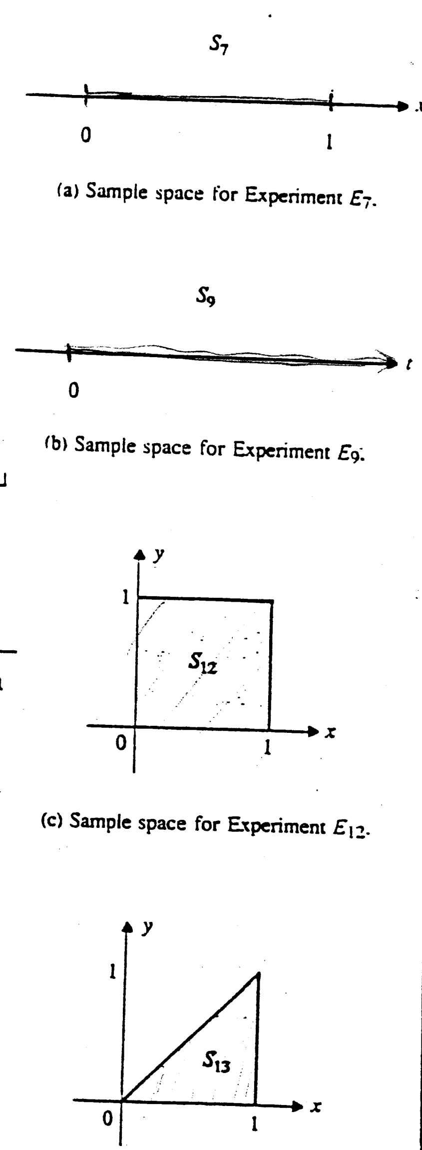
EXAMPLE 2.1

Experiment E_1: Select a ball from an urn containing balls numbered 1 to 50. Note the number of the ball.
Experiment E_2: Select a ball from an urn containing balls numbered 1 to 4. Suppose that balls 1 and 2 are black and that balls 3 and 4 are white. Note the number and color of the ball you select.
Experiment E_3: Toss a coin three times and note the sequence of heads and tails.
Experiment E_4: Toss a coin three times and note the number of heads.
Experiment E_5: Count the number of voice packets containing only silence produced from a group of N speakers in a 10-ms period.
Experiment E_6: A block of information is transmitted repeatedly over a noisy channel until an error-free block arrives at the receiver. Count the number of transmissions required.
Experiment E_7: Pick a number at random between zero and one.
Experiment E_8: Measure the time between two message arrivals at a message center.
Experiment E_9: Measure the lifetime of a given computer memory chip in a specified environment.
Experiment E_{10}: Determine the value of a voltage waveform at time t_1.
Experiment E_{11}: Determine the values of a voltage waveform at times t_1 and t_2.
Experiment E_{12}: Pick two numbers at random between zero and one.
Experiment E_{13}: Pick a number X at random between zero and one, then pick a number Y at random between zero and X.
Experiment E_{14}: A system component is installed at time $t = 0$. For $t \geq 0$ let $X(t) = 1$ as long as the component is functioning, and let $X(t) = 0$ after the component fails.

EXAMPLE 2.2

The sample spaces corresponding to the experiments in Example 2.1 are given below using set notation:

$S_1 = \{1, 2, \ldots, 50\}$
$S_2 = \{(1, b), (2, b), (3, w), (4, w)\}$
$S_3 = \{\text{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}\}$
$S_4 = \{0, 1, 2, 3\}$
$S_5 = \{0, 1, 2, \ldots, N\}$
$S_6 = \{1, 2, 3, \ldots\}$
$S_7 = \{x : 0 \leq x \leq 1\} = [0, 1]$ See Fig. 2.1(a).
$S_8 = \{t : t \geq 0\} = [0, \infty)$
$S_9 = \{t : t \geq 0\} = [0, \infty)$ See Fig. 2.1(b).
$S_{10} = \{v : -\infty < v < \infty\} = (-\infty, \infty)$
$S_{11} = \{(v_1, v_2) : -\infty < v_1 < \infty \text{ and } -\infty < v_2 < \infty\}$
$S_{12} = \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$ See Fig. 2.1(c).
$S_{13} = \{(x, y) : 0 \leq y \leq x \leq 1\}$ See Fig. 2.1(d).
S_{14} = set of functions $X(t)$ for which $X(t) = 1$ for $0 \leq t < t_0$ and $X(t) = 0$ for $t \geq t_0$, where $t_0 > 0$ is the time when the component fails.



(a) Sample space for Experiment E_7.
(b) Sample space for Experiment E_9.
(c) Sample space for Experiment E_{12}.
(d) Sample space for Experiment E_{13}.

FIGURE 2.1
Sample spaces for Experiments E_7, E_9, E_{12}, and E_{13}.

E_1, E_2, E_3, E_4, E_5 have finite discrete sample space
E_6 has countably infinite discrete sample space
E_7 through E_{13} have continuous sample space (uncountably infinite)

Set operations

set operation can be used to combine events to obtain other events using multiple simple events.

(1) Union of two events A and B, denoted by $A \cup B$, is defined as the set of outcomes that are either in A or in B, or both.

For multiple events,

$$\bigcup_{k=1}^{n} A_k = A_1 \cup A_2 \cup A_3 \cdots \cup A_n$$

- The $A \cup B$ occurs if either A or B or both A and B occur.
- The event $\bigcup_{k=1}^{n} A_k$ occurs if one or more A_k occurs.

(2) Intersection of two events A, B, denoted by $A \cap B$, and is defined as the set of outcomes that are in both A and B.

- The event $A \cap B$ occurs when both A and B occurs.
- mutually exclusive: two events are said to be mutually exclusive if their intersection is null i.e. $A \cap B = \Phi$ (they can not occur simultaneously)

(3) Complement: The Complement of an event A, denoted by A^c is the set of all outcomes not in A.

(4) Subset — $A \subset B$, event B occurs whenever event A occurs.

(5) equality $A = B$: two events are equal if they contain the same outcomes

(6) Basic properties

Commutative: $A \cup B = B \cup A$, $A \cap B = B \cap A$

Associative: $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap$

Distributive: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(7) DeMorgan's rule $(A \cap B)^c = A^c \cup B^c$, $(A \cup B)^c = A^c \cap B^c$

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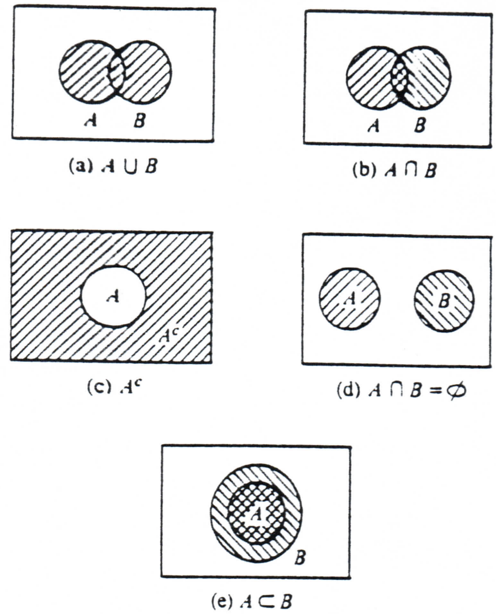
FIGURE 2.2
Set operations and set relations.



Distributive Properties:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \tag{2.3}$$

DeMorgan's Rules:

$$(A \cap B)^c = A^c \cup B^c \quad \text{and} \quad (A \cup B)^c = A^c \cap B^c. \tag{2.4}$$

2.7

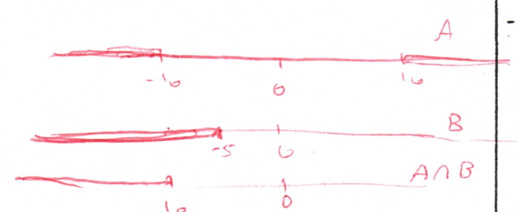
EXAMPLE 2.

For Experiment E_{10}, let the events A, B, and C be defined by

$A = \{v : |v| > 10\}$, "magnitude of v is greater than 10 volts,"
$B = \{v : v < -5\}$, "v is less than -5 volts," and
$C = \{v : v > 0\}$, "v is positive."

You should then verify that

$A \cup B = \{v : v < -5 \text{ or } v > 10\}$,
$A \cap B = \{v : v < -10\}$,
$C^c = \{v : v \le 0\}$,
$(A \cup B) \cap C = \{v : v > 10\}$,
$A \cap B \cap C = \varnothing$, and
$(A \cup B)^c = \{v : -5 \le v \le 10\}$.



2.2. The Axioms of Probability

probability : Numbers assigned to events that indicate the degree of likelihood (of occurrence) based on the probability law.

Let E be a Random Experiment
S be the Sample Space
A be the Event
then Probability of Event A : $P(A)$ that satisfy the following Axioms ;

Axiom 1 : $P(A) \geq 0$ (Non-Negativity)

2 : $P(S) = 1$ (Certain Event)

mutually exclusive events → 3 : If $A \cap B = \phi$, then
$$P(A \cup B) = P(A) + P(B)$$

4 : Generalization of Axiom 3,
If $A_i \cap A_j = \phi \ \forall \ i \neq j$ (Mutually Exclusive)
then $$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$$
$$\left(\text{OR} \quad P\left(\bigcup_{k=1}^{N} A_n\right) = \sum_{k=1}^{N} P(A_n) \right)$$

Corollary 1: $P(A^c) = 1 - P(A)$ note A and A^c are mutually exclusive, $A \cap A^c = \phi$, so using axiom 3 $\underbrace{P(A \cup A^c)}_{= P(S) = 1} = P(A) + P(A^c)$

Corollary 2 : $P(A) \leq 1$ → $P(A) = 1 - P(A^c) \leq 1$

$S \equiv$ certain event
$\phi \equiv$ impossible event

Corollary 3 : $P(\phi) = 0$, let $A = S$, $A^c = \phi$, from corollary 1, $P(\phi) = 1 - P(S) = 0$

Corollary 4 : If $A_1, A_2, \cdots, A_n$ are pairwise mutually exclusive, then
$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k) \quad \left(\text{General form of Axiom 4} \right)$$
for $n \geq 2$

Can be proved using mathematical induction
i.e. the result is true for $n=2$,
suppose it is true for some n
Then we need to prove it is true for $n+1$

express $A \cup B$ as union of 3 disjoint events $(A \cap B)$

$A \cup B = (A \cap B^c) \cup (B \cap A^c)$

$P(A \cup B) = P(A \cap B^c) + P(B \cap A^c) + P(A \cap B)$ *

But
$P(A) = P(A \cap B^c) + P(A \cap B)$
$P(B) = P(B \cap A^c) + P(A \cap B)$

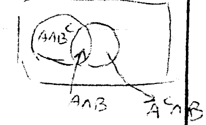
$P(A \cap B^c) = P(A) - P(A \cap B)$
$P(B \cap A^c) = P(B) - P(A \cap B)$

Plug in (1) you get

$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

Corollary 5 : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (For Not Necessarily Mutually Exclusive)

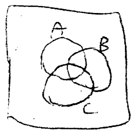
for three events



$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Corollary 6 : Generalization of Corollary 5

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{j=1}^{n} P(A_j) - \sum_{j<k} P(A_j \cap A_k) + \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n)]$$



From the Corollary 5 above,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

since $P(A \cap B) \geq 0$

$$P(A \cup B) \leq P(A) + P(B) : \text{Upper Bound}$$

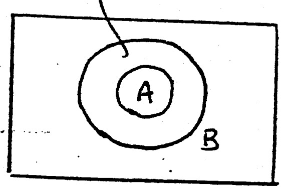
$P(A \cup B \cup C) = P(A \cup B) + P(C) - P[(A \cup B) \cap C)]$
$= P(A) + P(B) - P(A \cap B) + P(C) - [P((A \cap C) \cup (B \cap C))]$

$[P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)]$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

$A^c \cap B$

Corollary 7 : If $A \subset B$, then $P(A) \leq P(B)$



<proof> $P(B) = P(A) + P(A^c \cap B) \geq P(A)$

since $P(A^c \cap B) \geq 0$

Initial Probability Assignment

A. Discrete Sample Space Case : (Countable Sample Space S)

Prob. Law ~~for discrete~~ S ~~is giving the probs. of the elementary events~~ specifies the probability of an event in terms of the probabilities of the elementary events. let $S = \{a_1, a_2, \cdots, a_n\}$. if $B = \{a_1', a_2', \cdots a_m'\}$ is an event then, $P(B) = \sum_i P(a_i')$, for equally likely $P(a_i) = \frac{1}{n}$

B. Continuous Sample Space Case:

prob. Laws for Continuous S ~~is assigning~~ assigns numbers ~~to~~ to interval of the real line or rectangular region in the plane.

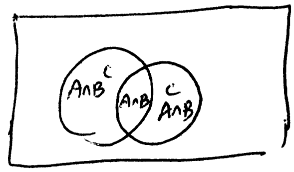
Here the outcome is not discrete. The outcome falls in an subinterval / region

e.g. $P([a, b]) = b - a$

Corollary 5

$$P(A \cup B) = P(A) + P(B) - P(A \cup B)$$

Proof:



$$A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)$$

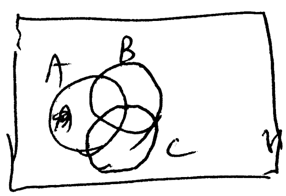
So $P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$

However $P(A) = P(A \cap B^c) + P(A \cap B)$

$P(B) = P(A^c \cap B) + P(A \cap B)$

Therefore $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Similary,



$$P(A \cup B \cup C) = P(A \cup B) + P(C) - P((A \cup B) \wedge C)$$

$$= P(A) + P(B) - P(A \cap B) + P(C) \cancel{- P(A \wedge C) \cup P(B \wedge C)}$$

$$- P[(A \wedge C) \cup (B \wedge C)]$$

$$- P(A \wedge C) - P(B \wedge C) + P(A \wedge B \wedge C)$$

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

$$\hookrightarrow \quad P(A \cap B) = P(A|B)\,P(B) \qquad (1)$$

Similarly $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

$$\hookrightarrow \quad P(A \cap B) = P(B|A)\,P(A) \qquad (2)$$

from 1 & 2 $\quad P(A|B)\,P(B) = P(B|A)\,P(A)$

Independent events

Two events A and B are independent iff.

$$P(A|B) = P(A) \quad \text{or} \quad P(B|A) = P(B)$$

Since $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = P(A)$

Therefor $\boxed{P(A \cap B) = P(A) \cdot P(B)}$

Three events are independent iff:

$$P(A \cap B) = P(A) \cdot P(B)$$
$$P(A \cap C) = P(A) \cdot P(C)$$
$$P(B \cap C) = P(B) \cdot P(C)$$
$$P((A \cap B) \cap C) = P(A) \cdot P(B) \cdot P(C) \longrightarrow$$

because $P(C|A \cap B) = P(C)$

$$= \frac{P(A \cap B \cap C)}{P(A \cap C)}$$

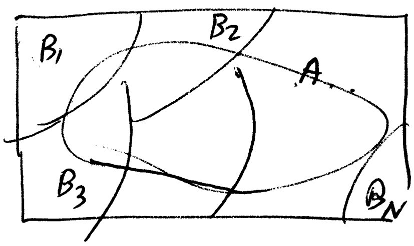
Total Probability Theorem

Let $B_1, B_2, \dots B_N$ be mutually exclusive events in S, i.e. $B_m \cap B_n = 0$, $m \neq n$ and $\bigcup_{n=1}^{N} B_n = S$, then any event A can be written,

$$A = A \cap S = A \cap (B_1 \cup B_2 \cup B_3 \cdots \cup B_N)$$
$$= (A \cap B_1) \cup (A \cap B_2) \cup \dots \cup (A \cap B_N)$$

$$\boxed{P(A) = \sum_{n=1}^{N} P(A \cap B_n = \sum_{n=1}^{N} P(A|B_n)\,P(B_n)}$$

Total prob. Thm.

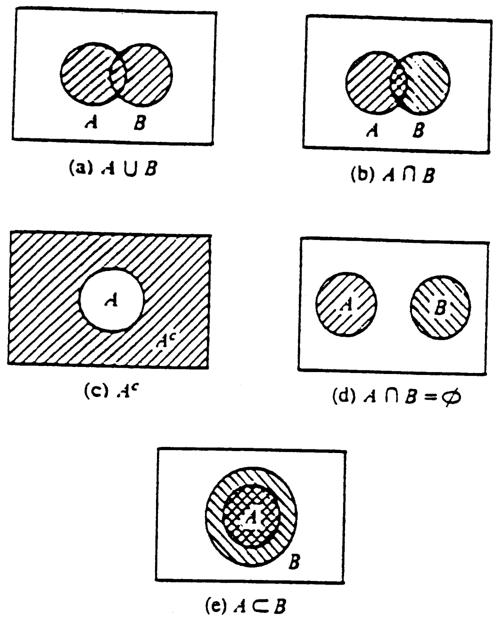


Bayes Rule

Let $B_1, B_2, \dots B_N$ be a partition of sample space S, suppose event A occurs, what is the prob. of event B_n?

$$P(B_n|A) = \frac{P(A \cap B_n)}{P(A)} = \frac{P(A|B_n)P(B_n)}{P(A)} = \frac{P(A|B_n)P(B_n)}{\sum_{n=1}^{N} P(A|B_n)P(B_n)}$$

FIGURE 2.2
Set operations and set relations.



Distributive Properties:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
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DeMorgan's Rules:

$$(A \cap B)^c = A^c \cup B^c \quad \text{and} \quad (A \cup B)^c = A^c \cap B^c. \tag{2.4}$$

EXAMPLE 2.7

For Experiment E_{10}, let the events A, B, and C be defined by

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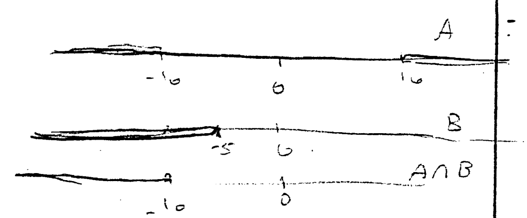
You should then verify that



$A \cup B = \{v : v < -5 \text{ or } v > 10\}$,
$A \cap B = \{v : v < -10\}$,
$C^c = \{v : v \leq 0\}$,
$(A \cup B) \cap C = \{v : v > 10\}$,
$A \cap B \cap C = \emptyset$, and
$(A \cup B)^c = \{v : -5 \leq v \leq 10\}$.

EXAMPLE [redacted]

An urn contains 10 identical balls numbered $0, 1, \ldots, 9$. A random experiment involves selecting a ball from the urn and noting the number of the ball. Find the probability of the following events:

A = "number of ball selected is odd,"

B = "number of ball selected is a multiple of 3,"

C = "number of ball selected is less than 5,"

and of $A \cup B$ and $A \cup B \cup C$.

The sample space is $S = \{0, 1, \ldots, 9\}$, so the sets of outcomes corresponding to the above events are

$$A = \{1, 3, 5, 7, 9\}, \qquad B = \{3, 6, 9\}, \qquad \text{and} \qquad C = \{0, 1, 2, 3, 4\}.$$

If we assume that the outcomes are equally likely, then

$$P[A] = P[\{1\}] + P[\{3\}] + P[\{5\}] + P[\{7\}] + P[\{9\}] = \frac{5}{10}.$$

$$P[B] = P[\{3\}] + P[\{6\}] + P[\{9\}] = \frac{3}{10}.$$

$$P[C] = P[\{0\}] + P[\{1\}] + P[\{2\}] + P[\{3\}] + P[\{4\}] = \frac{5}{10}.$$

From Corollary 5,

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] = \frac{5}{10} + \frac{3}{10} - \frac{2}{10} = \frac{6}{10},$$

where we have used the fact that $A \cap B = \{3, 9\}$, so $P[A \cap B] = 2/10$. From Corollary 6,

$$\begin{aligned} P[A \cup B \cup C] &= P[A] + P[B] + P[C] - P[A \cap B] \\ &\quad - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C] \\ &= \frac{5}{10} + \frac{3}{10} + \frac{5}{10} - \frac{2}{10} - \frac{2}{10} - \frac{1}{10} + \frac{1}{10} \\ &= \frac{9}{10}. \end{aligned}$$

You should verify the answers for $P[A \cup B]$ and $P[A \cup B \cup C]$ by enumerating the outcomes in the events.

EXAMPLE 2.10 Suppose that a coin is tossed three times. If we observe the sequence of heads and tails, then there are eight possible outcomes $S_3 = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$. If we assume that the outcomes of S_3 are equiprobable, then the probability of each of the eight elementary events is 1/8. This probability assignment implies that the probability of obtaining two heads in three tosses is, by Corollary 3,

$$P[\text{"2 heads in 3 tosses"}] = P[\{HHT, HTH, THH\}]$$

$$= P[\{HHT\}] + P[\{HTH\}] + P[\{THH\}] = \frac{3}{8}.$$

Now suppose that we toss a coin three times but we count the number of heads in three tosses instead of observing the sequence of heads and tails. The sample space is now $S_4 = \{0, 1, 2, 3\}$. If we assume the outcomes of S_4 to be equiprobable, then each of the elementary events of S_4 has probability 1/4. This second probability assignment predicts that the probability of obtaining two heads in three tosses is

$$P[\text{"2 heads in 3 tosses"}] = P[\{2\}] = \frac{1}{4}.$$

The first probability assignment implies that the probability of two heads in three tosses is 3/8, and the second probability assignment predicts that the probability is 1/4. Thus the two assignments are not consistent with each other. As far as the theory is concerned, either one of the assignments is acceptable. It is up to us to decide which assignment is more appropriate. Later in the chapter we will see that only the first assignment is consistent with the assumption that the coin is fair and that the tosses are "independent." This assignment correctly predicts the relative frequencies that would be observed in an actual coin tossing experiment.

EXAMPLE 2.11

A fair coin is tossed repeatedly until the first heads shows up; the outcome of the experiment is the number of tosses required until the first heads occurs. Find a probability law for this experiment.

It is conceivable that an arbitrarily large number of tosses will be required until heads occurs, so the sample space is $S = \{1, 2, 3, \ldots\}$. Suppose the experiment is repeated n times. Let N_j be the number of trials in which the jth toss results in the first heads. If n is very large, we expect N_1 to be approximately $n/2$ since the coin is fair. This implies that a second toss is necessarily about $n - N_1 \approx n/2$ times, and again we expect that about half of these—that is, $n/4$—will result in heads, and so on, as shown in Fig. 2.6. Thus for large n, the relative frequencies are

$$f_j \approx \frac{N_j}{n} = \left(\frac{1}{2}\right)^j \qquad j = 1, 2, \ldots .$$

We therefore conclude that a reasonable probability law for this experiment is

$$P[j \text{ tosses till first heads}] = \left(\frac{1}{2}\right)^j \qquad j = 1, 2, \ldots . \tag{2.13}$$

We can verify that these probabilities add up to one by using the geometric series with $\alpha = 1/2$:

$$\sum_{j=1}^{\infty} \alpha^j = \left.\frac{\alpha}{1 - \alpha}\right|_{\alpha=1/2} = 1.$$

EXAMPLE 2.13

Suppose that the lifetime of a computer memory chip is measured, and we find that "the proportion of chips whose lifetime exceeds t decreases exponentially at a rate α." Find an appropriate probability law.

Let the sample space in this experiment be $S = (0, \infty)$. If we interpret the above finding as "the probability that a chip's lifetime exceeds t decreases exponentially at a rate α," we then obtain the following assignment of probabilities to events of the form (t, ∞):

$$P[(t, \infty)] = e^{-\alpha t} \qquad \text{for } t > 0, \tag{2.15}$$

where $\alpha > 0$. Note that the exponential is a number between 0 and 1 for $t > 0$, so Axiom I is satisfied. Axiom II is satisfied since

$$P[S] = P[(0, \infty)] = 1.$$

The probability that the lifetime is in the interval $(r, s]$ is found by noting in Fig. 2.7 that $(r, s] \cup (s, \infty) = (r, \infty)$, so by Axiom III,

$$P[(r, \infty)] = P[(r, s]] + P[(s, \infty)].$$

By rearranging the above equation we obtain

$$P[(r, s]] = P[(r, \infty)] - P[(s, \infty)] = e^{-\alpha r} - e^{-\alpha s}.$$

We thus obtain the probability of arbitrary intervals in S.

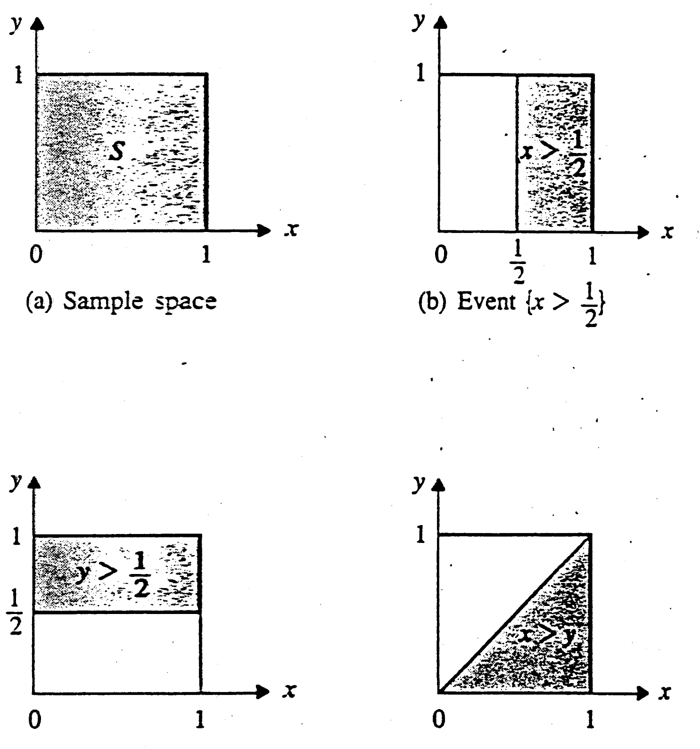
EXAMPLE 2.1[redacted]

Consider Experiment E_{12}, where we picked two numbers x and y at random between zero and one. The sample space is then the unit square shown in Fig. [redacted](a). If we suppose that all pairs of numbers in the unit square are equally likely to be selected, then it is reasonable to use a probability assignment in which the probability of any region R inside the unit square is equal to the area of R. Find the probability of the following events: $A = \{x > 0.5\}$, $B = \{y > 0.5\}$, and $C = \{x > y\}$.

Figures [redacted](b) through [redacted] show the regions corresponding to the events A, B, and C. Clearly each of these regions has area 1/2. Thus

$$P[A] = \frac{1}{2}, \qquad P[B] = \frac{1}{2}, \qquad P[C] = \frac{1}{2}.$$

GURE 2.[redacted]
wo-dimensional sample space
a three events.



(a) Sample space

(b) Event $\{x > \frac{1}{2}\}$

(c) Event $\{y > \frac{1}{2}\}$

(d) Event $\{x > y\}$

EXAMPLE 2.24

A ball is selected from an urn containing two black balls, numbered 1 and 2, and two white balls, numbered 3 and 4. The number and color of the ball is noted, so the sample space is $\{(1, b), (2, b), (3, w), (4, w)\}$. Assuming that the four outcomes are equally likely, find $P[A \mid B]$ and $P[A \mid C]$, where A, B, and C are the following events:

$A = \{(1, b), (2, b)\}$, "black ball selected,"

$B = \{(2, b), (4, w)\}$, "even-numbered ball selected," and

$C = \{(3, w), (4, w)\}$, "number of ball is greater than 2."

Since $P[A \cap B] = P[(2, b)]$ and $P[A \cap C] = P[\varnothing] = 0$, Eq. (2.21) gives

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} = \frac{.25}{.5} = .5 = P[A]$$

$$P[A \mid C] = \frac{P[A \cap C]}{P[C]} = \frac{0}{.5} = 0 \neq P[A].$$

In the first case, knowledge of B did not alter the probability of A. In the second case, knowledge of C implied that A had not occurred.

EXAMPLE 2.25

Conditional Probability

An urn contains two black balls and three white balls. Two balls are selected at random from the urn without replacement and the sequence of colors is noted. Find the probability that both balls are black.

We are looking for $P(B_1 \cap B_2)$

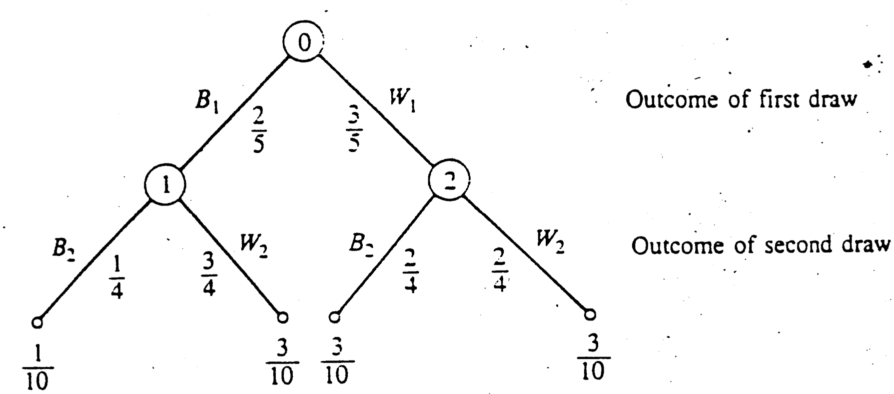
This experiment consists of a sequence of two subexperiments. We can imagine working our way down the tree shown in Fig. 2.10 from the topmost node to one of the bottom nodes. We reach node 1 in the tree if the outcome of the first draw is a black ball; then the next subexperiment consists of selecting a ball from an urn containing one black ball and three white balls. On the other hand, if the outcome of the first draw is white, then we reach node 2 in the tree and the second subexperiment consists of selecting a ball from an urn that contains two black balls and two white balls. Thus if we know which node is reached after the first draw, then we can state the probabilities of the outcome in the next subexperiment.

Let B_1 and B_2 be the events that the outcome is a black ball in the first and second draw, respectively. From Eq. (2.2[illegible]b) we have

$$P[B_1 \cap B_2] = P[B_2 \mid B_1]P[B_1]. = 1/4 \times \frac{2}{5} = \frac{1}{10}$$

In terms of the tree diagram in Fig. 2.12, $P[B_1]$ is the probability of reaching node 1 and $P[B_2 \mid B_1]$ is the probability of reaching the leftmost bottom node from node 1. Now $P[B_1] = 2/5$ since the first draw is from an urn containing two black balls and three white balls; $P[B_2 \mid B_1] = 1/4$ since, given B_1, the second draw is from an urn containing one black ball and three white balls.

FIGURE 2.10
The paths from the top node to a bottom node correspond to the possible outcomes in the drawing of two balls from an urn without replacement. The probability of a path is the product of the probabilities in the associated transitions.



EXAMPLE 2.3[illegible] 3.10

Quality Control

Cond. prob. / Bayes' rule

Consider the memory chips discussed in Example 2.2[illegible]. Recall that a fraction p of the chips are bad and tend to fail much more quickly than good chips. Suppose that in order to "weed out" the bad chips, every chip is tested for t seconds prior to leaving the factory. The chips that fail are discarded and the remaining chips are sent out to customers. Find the value of t for which 99% of the chips sent out to customers are good.

There are two kinds of chips that can function after t seconds good & Bad

Let C be the event "chip still functioning after t seconds," and let G be the event "chip is good," and B the event "chip is bad." The problem requires that we find the value of t for which

$$P[G \mid C] = .99.$$

We find $P[G \mid C]$ by applying Bayes' rule:

$$\begin{aligned} P[G \mid C] &= \frac{P[C \mid G]P[G]}{P[C \mid G]P[G] + P[C \mid B]P[B]} \\ &= \frac{(1-p)e^{-\alpha t}}{(1-p)e^{-\alpha t} + pe^{-\alpha 1000t}} \\ &= \frac{1}{1 + \dfrac{pe^{-\alpha 1000t}}{(1-p)e^{-\alpha t}}} = .99. \end{aligned}$$

The above equation can then be solved for t:

$$t = \frac{1}{999\alpha} \ln\left(\frac{99p}{1-p}\right).$$

For example, if $1/\alpha = 20{,}000$ hours and $p = .10$, then $t = 48$ hours.

EXAMPLE 2.31

(Independent events)

(discrete case)

A ball is selected from an urn containing two black balls, numbered 1 and 2, and two white balls, numbered 3 and 4. Let the events A, B, and C be defined as follows:

$A = \{(1, b), (2, b)\}$, "black ball selected";

$B = \{(2, b), (4, w)\}$, "even-numbered ball selected"; and

$C = \{(3, w), (4, w)\}$, "number of ball is greater than 2."

Are events A and B independent? Are events A and C independent?

First, consider events A and B. The probabilities required by Eq. (2.31) are

$$P[A] = P[B] = \frac{1}{2},$$

and

$$P[A \cap B] = P[\{(2, b)\}] = \frac{1}{4}.$$

Thus

$$P[A \cap B] = \frac{1}{4} = P[A]P[B],$$

and the events A and B are independent. Equation (2.32b) gives more insight into the meaning of independence:

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} = \frac{P[\{(2, b)\}]}{P[\{(2, b), (4, w)\}]} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$P[A] = \frac{P[A]}{P[S]} = \frac{P[\{(1, b), (2, b)\}]}{P[\{(1, b), (2, b), (3, w), (4, w)\}]} = \frac{1/2}{1}.$$

These two equations imply that $P[A] = P[A \mid B]$ because *the proportion of outcomes in S that lead to the occurrence of A is equal to the proportion of outcomes in B that lead to A*. Thus knowledge of the occurrence of B does not alter the probability of the occurrence of A.

Events A and C are not independent since $P[A \cap C] = P[\varnothing] = 0$ so $P[A \mid C] = 0 \neq P[A] = .5$.

In fact, A and C are mutually exclusive since $A \cap C = \varnothing$, so the occurrence of C implies that A has definitely not occurred.

EXAMPLE 2.32

Independent events (continuous case)

Two numbers x and y are selected at random between zero and one. Let the events A, B, and C be defined as follows:

$$A = \{x > 0.5\}, \qquad B = \{y > 0.5\}, \qquad \text{and } C = \{x > y\}.$$

Are the events A and B independent? Are A and C independent?

Figure 2.13 shows the regions of the unit square that correspond to the above events. Using Eq. (2.32a), we have

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} = \frac{1/4}{1/2} = \frac{1}{2} = P[A],$$

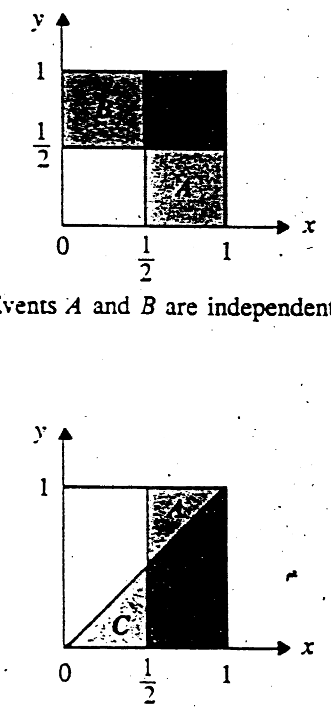
so events A and B are independent. Again we have that the "proportion" of outcomes in S leading to A is equal to the "proportion" in B that lead to A.

Using Eq. (2.32b), we have

$$P[A \mid C] = \frac{P[A \cap C]}{P[C]} = \frac{3/8}{1/2} = \frac{3}{4} \neq \frac{1}{2} = P[A],$$

so events A and C are not independent. Indeed from Fig. 2.15(b) we can see that knowledge of the fact that x is greater than y increases the probability that x is greater than 0.5.

FIGURE 2.13
Examples of independent and nonindependent events.



(a) Events A and B are independent.

(b) Events A and C are not independent.

$$P(A \cap C) = \tfrac{1}{2}\left(1+\tfrac{1}{2}\right)\tfrac{1}{2} = 3/8$$

two events are independent if $P(A \cap B) = P(A)P(B)$

3 events A, B, C are independent if $P(A \cap B \cap C) = P(A)P(B)P(C)$

EXAMPLE 2.35

Consider the experiment discussed in Example 2.32 where two numbers are selected at random from the unit interval. Let the events B, D, and F be defined as follows:

$$B = \left\{y > \frac{1}{2}\right\}, \qquad D = \left\{x < \frac{1}{2}\right\}$$

$$F = \left\{x < \frac{1}{2} \text{ and } y < \frac{1}{2}\right\} \cup \left\{x > \frac{1}{2} \text{ and } y > \frac{1}{2}\right\}.$$

The three events are shown in Fig. 2.14. It can be easily verified that any pair of these events is independent:

$$P[B \cap D] = \frac{1}{4} = P[B]P[D],$$

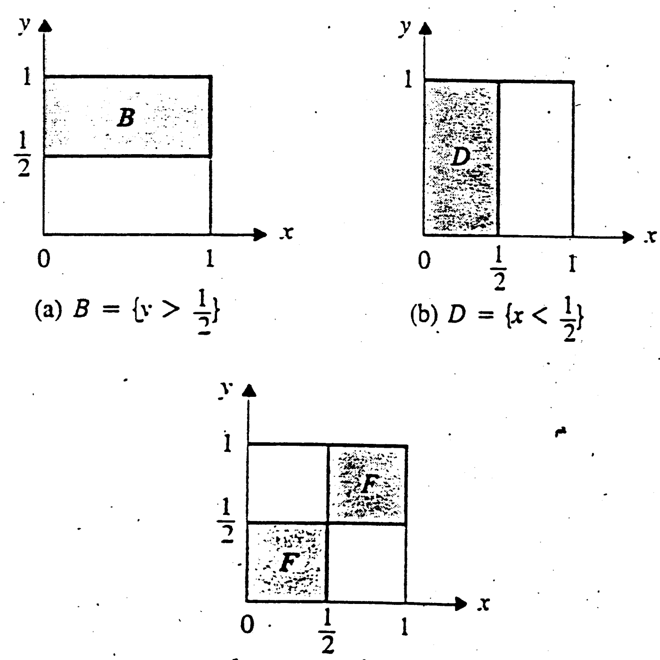
$$P[B \cap F] = \frac{1}{4} = P[B]P[F], \text{ and}$$

$$P[D \cap F] = \frac{1}{4} = P[D]P[F].$$

However, the three events are not independent, since $B \cap D \cap F = \varnothing$, so

$$P[B \cap D \cap F] = P[\varnothing] = 0 \neq P[B]P[D]P[F] = \frac{1}{8}.$$

FIGURE 2.14
Events B, D, and F are pairwise independent, but the triplet B, D, F are not independent events.



(a) $B = \{y > \frac{1}{2}\}$ (b) $D = \{x < \frac{1}{2}\}$

(c) $F = \{x < \frac{1}{2} \text{ and } y < \frac{1}{2}\} \{x > \frac{1}{2} \text{ and } y > \frac{1}{2}\}$

in dependant event

EXAMPLE 2.3

Suppose a fair coin is tossed three times and we observe the resulting sequence of heads and tails. Find the probability of the elementary events.

The sample space of this experiment is $S = \{\text{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}\}$. The assumption that the coin is fair means that the outcomes of a single toss are equiprobable, that is, $P[\text{H}] = P[\text{T}] = 1/2$. If we assume that the outcomes of the coin tosses are independent, then

$$P[\{\text{HHH}\}] = P[\{\text{H}\}]P[\{\text{H}\}]P[\{\text{H}\}] = \frac{1}{8},$$

$$P[\{\text{HHT}\}] = P[\{\text{H}\}]P[\{\text{H}\}]P[\{\text{T}\}] = \frac{1}{8},$$

$$P[\{\text{HTH}\}] = P[\{\text{H}\}]P[\{\text{T}\}]P[\{\text{H}\}] = \frac{1}{8},$$

$$P[\{\text{THH}\}] = P[\{\text{T}\}]P[\{\text{H}\}]P[\{\text{H}\}] = \frac{1}{8},$$

$$P[\{\text{TTH}\}] = P[\{\text{T}\}]P[\{\text{T}\}]P[\{\text{H}\}] = \frac{1}{8},$$

$$P[\{\text{THT}\}] = P[\{\text{T}\}]P[\{\text{H}\}]P[\{\text{T}\}] = \frac{1}{8},$$

$$P[\{\text{HTT}\}] = P[\{\text{H}\}]P[\{\text{T}\}]P[\{\text{T}\}] = \frac{1}{8}, \text{ and}$$

$$P[\{\text{TTT}\}] = P[\{\text{T}\}]P[\{\text{T}\}]P[\{\text{T}\}] = \frac{1}{8}.$$

2.35

EXAMPLE 2.35
System Reliability

<independent events>

A system consists of a controller and three peripheral units. The system is said to be "up" if the controller and at least two of the peripherals are functioning. Find the probability that the system is up, assuming that all components fail independently.

Define the following events: A is "controller is functioning" and B_i is "peripheral i is functioning" where $i = 1, 2, 3$. The event F, "two or more peripheral units are functioning," occurs if all three units are functioning or if exactly two units are functioning. Thus

$$F = (B_1 \cap B_2 \cap B_3^c) \cup (B_1 \cap B_2^c \cap B_3) \cup (B_1^c \cap B_2 \cap B_3) \cup (B_1 \cap B_2 \cap B_3).$$

Note that the events in the above union are mutually exclusive. Thus

$$\begin{aligned} P[F] &= P[B_1]P[B_2]P[B_3^c] + P[B_1]P[B_2^c]P[B_3] \\ &\quad + P[B_1^c]P[B_2]P[B_3] + P[B_1]P[B_2]P[B_3] \\ &= 3(1-a)^2a + (1-a)^3, \end{aligned}$$

where we have assumed that each peripheral fails with probability a, so that $P[B_i] = 1 - a$ and $P[B_i^c] = a$.

The event "system is up" is then $A \cap F$. If we assume that the controller fails with probability p, then

$$\begin{aligned} P[\text{"system up"}] &= P[A \cap F] = P[A]P[F] \\ &= (1-p)P[F] \\ &= (1-p)\{3(1-a)^2a + (1-a)^3\}. \end{aligned}$$

Let $a = 10\%$, then all three peripherals are functioning $(1-a)^3 = 72.9\%$ of the time and two are functioning and one "down" $3(1-a)^2a = 24.3\%$ of the time. Thus two or more peripherals are functioning 97.2% of the time. Suppose that the controller is not very reliable, say $p = 20\%$, then the system is up only 77.8% of the time, mostly because of controller failures.

Suppose a second identical controller with $p = 20\%$ is added to the system, and that the system is "up" if at least one of the controllers is functioning and if two or more of the peripherals are functioning. In Problem 69, you are asked to show that at least one of the controllers is functioning 96% of the time, and that the system is up 93.3% of the time. This is an increase of 16% over the system with a single controller.

EXAMPLE 2.37

Suppose that a coin is tossed three times. If we assume that the *tosses are independent* and the probability of heads is p, then the probability for the sequences of heads and tails is

Binomial distribution deals with the no. of successes in n trials.

$$P[\{HHH\}] = P[\{H\}]P[\{H\}]P[\{H\}] = p^3,$$
$$P[\{HHT\}] = P[\{H\}]P[\{H\}]P[\{T\}] = p^2(1-p),$$
$$P[\{HTH\}] = P[\{H\}]P[\{T\}]P[\{H\}] = p^2(1-p),$$
$$P[\{THH\}] = P[\{T\}]P[\{H\}]P[\{H\}] = p^2(1-p),$$
$$P[\{TTH\}] = P[\{T\}]P[\{T\}]P[\{H\}] = p(1-p)^2,$$
$$P[\{THT\}] = P[\{T\}]P[\{H\}]P[\{T\}] = p(1-p)^2,$$
$$P[\{HTT\}] = P[\{H\}]P[\{T\}]P[\{T\}] = p(1-p)^2, \text{ and}$$
$$P[\{TTT\}] = P[\{T\}]P[\{T\}]P[\{T\}] = (1-p)^3;$$

where we used the fact that the tosses are independent. Let k be the number of heads in three trials, then

General case $p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

$$P[k=0] = P[\{TTT\}] = (1-p)^3,$$
$$P[k=1] = P[\{TTH, THT, HTT\}] = 3p(1-p)^2,$$
$$P[k=2] = P[\{HHT, HTH, THH\}] = 3p^2(1-p), \text{ and}$$
$$P[k=3] = P[\{HHH\}] = p^3.$$

EXAMPLE 2.38

Verify that Eq. (2.35) gives the probabilities found in Example 2.37.

In Example 2.37, let "toss results in heads" correspond to a "success," then

$$p_3(0) = \frac{3!}{0!\,3!}p^0(1-p)^3 = (1-p)^3, \qquad \longrightarrow p(0) = \binom{3}{0} p^0 (1-p)^3$$

$$p_3(1) = \frac{3!}{1!\,2!}p^1(1-p)^2 = 3p(1-p)^2, \qquad \longrightarrow p(1) = \binom{3}{1} p^1 (1-p)^2$$

$$p_3(2) = \frac{3!}{2!\,1!}p^2(1-p)^1 = 3p^2(1-p), \text{ and} \qquad \longrightarrow p(2) = \binom{3}{2} p^2 (1-p)^1$$

$$p_3(3) = \frac{3!}{0!\,3!}p^3(1-p)^0 = p^3, \qquad \longrightarrow p(3) = \binom{3}{3} p^3 (1-p)^0$$

which are in agreement with our previous results.

Counting

Permutations

A permutation is an arrangement of a set of objects in a given order

e.g. There are 6 arrangement letters A, B, C

A B C
A C B
B A C
B C A
C A B
C B A

each of these arrangement is called a permutation of the letters A, B, C.

* e.g. Find the number of permutations of the four letters W, X, Y, Z

Solution

4	3	2	1

$4 \cdot 3 \cdot 2 \cdot 1 = 24$
$= 4!$

* e.g. How many permutations can be found from the letters in word MONDAY using all 6 letters?

answer $6!$

How many permutations can be formed from the letters in word Monday using only 3 letters?

$6 \cdot 5 \cdot 4 = 120$

Permutations taken r at a time.

n	$n-1$	$n-2$	$n-3$	$\cdots$	$n-(r-1)$

$${}_nP_r = \underbrace{n(n-1)(n-2)\cdots\cdots(n-(r-1))}_{r \text{ factors}} = \frac{n!}{(n-r)!}$$

note ${}_nP_n = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$

Permutations with repetitions

The number of permutations of n objects of which n_1 are alike, n_2 are alike, … n_r are alike is

$$\frac{n!}{n_1!\, n_2! \cdots n_r!}$$

e.g. DADDY

~~So think of distinguishable~~

So there are $\frac{5!}{3!}$ different 5 letter words that can be formed from the ~~letter~~ word DADDY

note all permutations, $D_1 D_2 D_3 AY$ $D_2 D_1 D_3 AY$, $D_1 D_3 D_2$, $D_2 D_3 D_1 AY$, $D_3 D_2 D_1 AY$ produce the same word when subscripts are

Combinations

The number of Combinations of n objects taken r at a time is the no. of subsets, each of size r, which can be formed from the n objects. This is denoted by ${}_nC_r$.

-binations of four letters taken 3 letters at a time

Combinations	Permutations
abc	abc, acb, bac, bca, cab, cba
abd	
acd	
bcd	

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!\ r!} = \binom{n}{r} $$

e.g. Find the no. of ways of selecting two applicants out of 5.

$$ \binom{5}{2} = \frac{5!}{2!\,3!} = 10 $$