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EXAMPLE 4.21

Show that the Gaussian pdf integrates to one. Consider the square of the integral of the pdf:

$$\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right]^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\int_{-\infty}^{\infty} e^{-y^2/2}\,dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dx\,dy.$$

Let $x = r\cos\theta$ and $y = r\sin\theta$ and carry out the change from Cartesian to polar coordinates, then we obtain:

$$\frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} e^{-r^2/2}\,r\,dr\,d\theta = \int_0^{\infty} re^{-r^2/2}\,dr$$

$$= \left[-e^{-r^2/2}\Big|_0^{\infty}\right]$$

$$= 1.$$

EXAMPLE 4.22

A communication system accepts a positive voltage V as input and outputs a voltage $Y = \alpha V + N$, where $\alpha = 10^{-2}$ and N is a Gaussian random variable with parameters $m = 0$ and $\sigma = 2$. Find the value of V that gives $P[Y < 0] = 10^{-6}$.

$N = N(0,2)$

The probability $P[Y < 0]$ is written in terms of N as follows:

$$P[Y < 0] = P[\alpha V + N < 0]$$

$F_N(-\alpha V) = \Phi\left(\frac{-\alpha V - 0}{2}\right)$

$$= P[N < -\alpha V] = \Phi\left(\frac{-\alpha V}{\sigma}\right) = Q\left(\frac{\alpha V}{\sigma}\right) = 10^{-6}.$$

$Q^{-1}(10^{-6}) = 4.753$

from table 3.4

From Table 3.4 we see that the argument of the Q-function should be $\alpha V/\sigma = 4.753$. Thus $V = (4.753)\sigma/\alpha = 950.6$.

EXAMPLE 4.23

Show that the pdf of a gamma random variable integrates to one.

The integral of the pdf is

$$\int_0^{\infty} f_X(x)\,dx = \int_0^{\infty} \frac{\lambda(\lambda x)^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)}\,dx$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty} x^{\alpha-1}e^{-\lambda x}\,dx.$$

Let $y = \lambda x$, then $dx = dy/\lambda$ and the integral becomes

$$\frac{\lambda^{\alpha}}{\Gamma(\alpha)\lambda^{\alpha}}\underbrace{\int_0^{\infty} y^{\alpha-1}e^{-y}\,dy}_{\Gamma(a)} = 1,$$

where we used the fact that the integral equals $\Gamma(\alpha)$.

EXAMPLE 4.26

Let the function $h(x) = (x)^+$ be defined as follows:

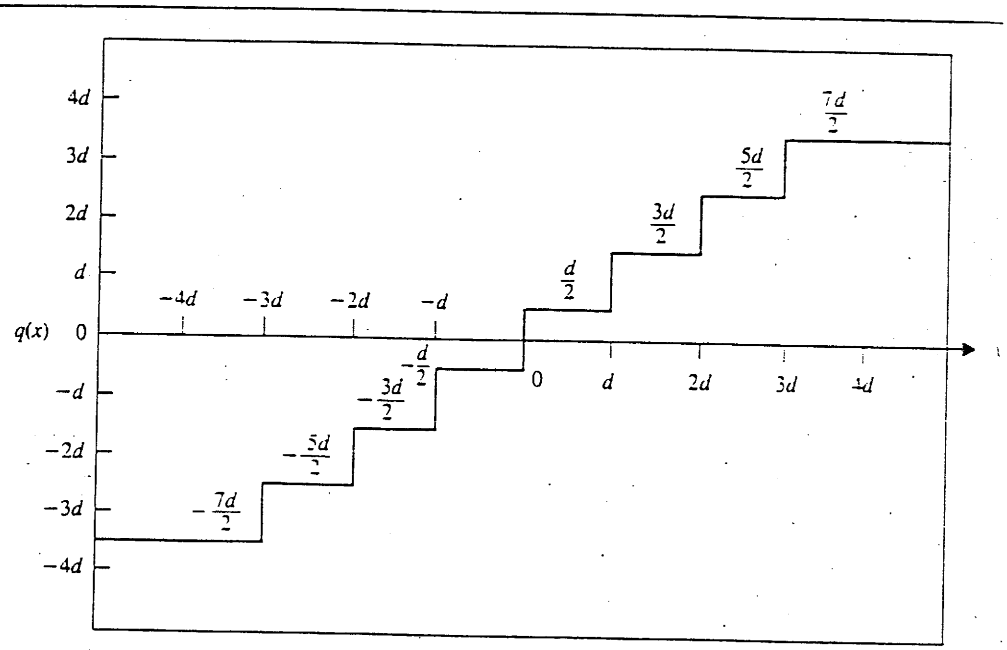
$$(x)^+ = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$$

For example, let X be the number of active speakers in a group of N speakers, and let Y be the number of active speakers in excess of M, then $Y = (X - M)^+$. In another example, let X be a voltage input to a half-wave rectifier, then $Y = (X)^+$ is the output.

EXAMPLE 4.27

Let the function $q(x)$ be defined as shown in Fig. 4.8 where the set of points on the real line are mapped into the nearest representation point from the set $S_y = \{-3.5d, -2.5d, \ldots, -0.5d, 0.5d, \ldots, 2.5d, 3.5d\}$. Thus for example, all the points in the interval $(0, d)$ are mapped into the point $d/2$. The function $q(x)$ represents an eight-level uniform quantizer.

FIGURE 4.8
A uniform quantizer maps the input x into the closest point from the set $\{\pm d/2, \pm 3d/2, \pm 5d/2, \pm 7d/2\}$.



4.30

EXAMPLE

Let X be a sample voltage of a speech waveform, and suppose that X has a uniform distribution in the interval $[-4d, 4d]$. Let $Y = q(X)$, where the quantizer input-output characteristic is as shown in Fig. 4.10 Find the pmf for Y.

The event $\{Y = q\}$ for q in S_Y is equivalent to the event $\{X \text{ in } I_q\}$, where I_q is an interval of points mapped into the representation point q. The pmf of Y is therefore found by evaluating

$$P[Y = q] = \int_{I_q} f_X(t)\,dt = \int_{id}^{(i+1)d} \frac{1}{8d}\,dt = \frac{1}{8}$$

It is easy to see that the representation point has an interval of length d mapped into it. Thus the eight possible outputs are equiprobable, that is, $P[Y = q] = 1/8$ for q in S_Y.

A Linear Function Let the random variable Y be defined by

$$Y = aX + b,$$

where a is a nonzero constant. Suppose that X has cdf $F_X(x)$, then find $F_Y(y)$.

The event $\{Y \le y\}$ occurs when $A = \{aX + b \le y\}$ occurs. If $a > 0$, then $A = \{X \le (y - b)/a\}$ (see Fig. 3.16), and thus

$$F_Y(y) = P\left[X \le \frac{y-b}{a}\right] = F_X\left(\frac{y-b}{a}\right) \qquad a > 0.$$

On the other hand, if $a < 0$, then $A = \{X \ge (y - b)/a\}$, and

$$F_Y(y) = P\left[X \ge \frac{y-b}{a}\right] = 1 - F_X\left(\frac{y-b}{a}\right) \qquad a < 0.$$

We can obtain the pdf of Y by differentiating with respect to y. To do this we need to use the chain rule for derivatives:

$$\frac{dF}{dy} = \frac{dF}{du}\frac{du}{dy},$$

where u is the argument of F. In this case, $u = (y - b)/a$, and we then obtain

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \qquad a > 0$$

and

$$f_Y(y) = \frac{1}{-a} f_X\left(\frac{y-b}{a}\right) \qquad a < 0.$$

The above two results can be written compactly as

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right). \qquad (4.47)$$

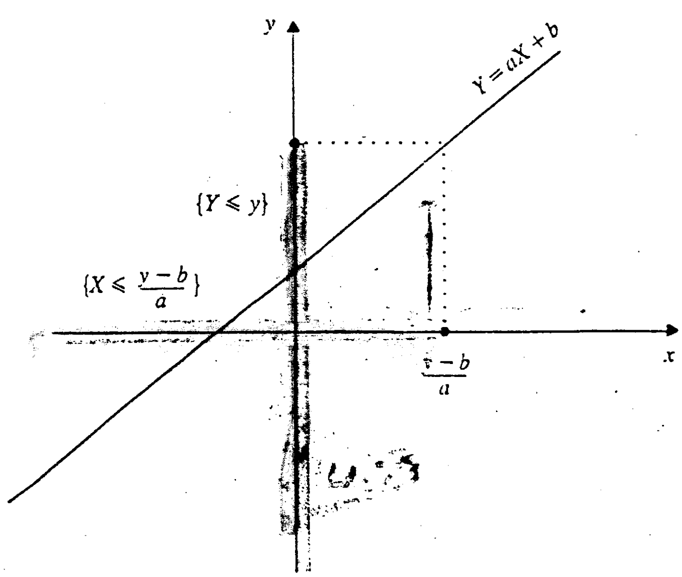


FIGURE 4.11 The equivalent event for $\{Y \le y\}$ is the event $\{X \le (y - b)/a\}$, if $a > 0$.

EXAMPLE 4.32
A Linear Function of a Gaussian Random Variable

Let X be a random variable with a Gaussian pdf with mean m and standard deviation σ:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-m)^2/2\sigma^2} \qquad -\infty < x < \infty. \tag{3.50}$$

Let $Y = aX + b$, then find the pdf of Y.
Substitution of Eq. (3.50) into Eq. (3.49) yields

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,|a\sigma|} e^{-(y-b-am)^2/2(a\sigma)^2}.$$

Note that Y also has a Gaussian distribution with mean $b + am$ and standard deviation $|a|\,\sigma$. Therefore *a linear function of a Gaussian random variable is also a Gaussian random variable.*

EXAMPLE 4.33

Let the random variable Y be defined by

$Y = X^2$,
Where X is a Continuous r.v. Find cdf, pdf of Y

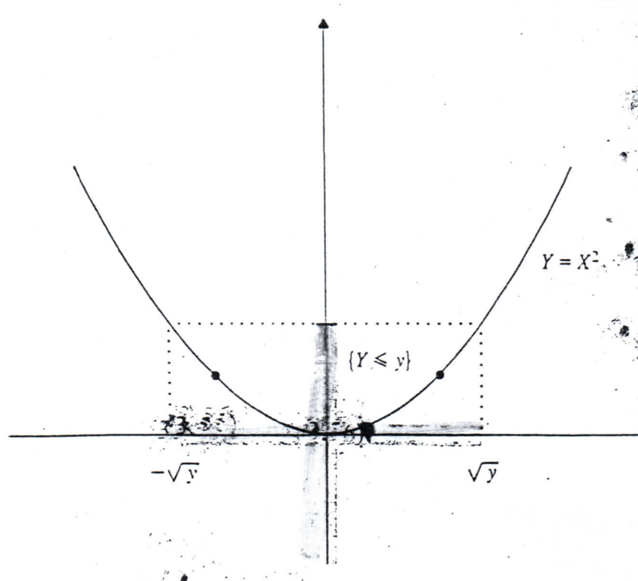
The event $\{Y \leq y\}$ occurs when $\{X^2 \leq y\}$ or equivalently when $\{-\sqrt{y} \leq X \leq \sqrt{y}\}$ for y nonnegative; see Fig. 3.17. The event is null when y is negative. Thus

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ F_X(\sqrt{y}) - F_X(-\sqrt{y}) & y > 0 \end{cases}$$

and differentiating with respect to y,

$$\begin{aligned} f_Y(y) &= \frac{f_X(\sqrt{y})}{2\sqrt{y}} - \frac{f_X(-\sqrt{y})}{-2\sqrt{y}} \qquad y > 0 \\ &= \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}. \end{aligned} \tag{3.51}$$

FIGURE 3.17
The equivalent event for $\{Y \leq y\}$ is the event $\{-\sqrt{y} \leq X \leq \sqrt{y}\}$, if $y \geq 0$.



The result in Example 4.33 suggests that if the equation $y_0 = g(x)$ has n solutions, $x_0, x_1, \ldots, x_n$, then $f_Y(y_0)$ will be equal to n terms of the type on the right-hand side of Eq. (3.51). We now show that this is generally true by using a method for directly obtaining the pdf of Y in terms of the pdf of X.

Consider a nonlinear function $Y = g(X)$ such as the one shown in Fig. 3.18. Consider the event $C_y = \{y < Y < y + dy\}$ and let B_y be its equivalent event. For y indicated in the figure, the equation $g(x) = y$ has three solutions x_1, x_2, and x_3, and the equivalent event B_y has a segment corresponding to each

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Consider a nonlinear function $Y = g(X)$ such as the one shown in Fig. 4.13. Consider the event $C_y = \{y < Y < y + dy\}$ and let B_y be its equivalent event. For y indicated in the figure, the equation $g(x) = y$ has three solutions x_1, x_2, and x_3, and the equivalent event B_y has a segment corresponding to each solution:

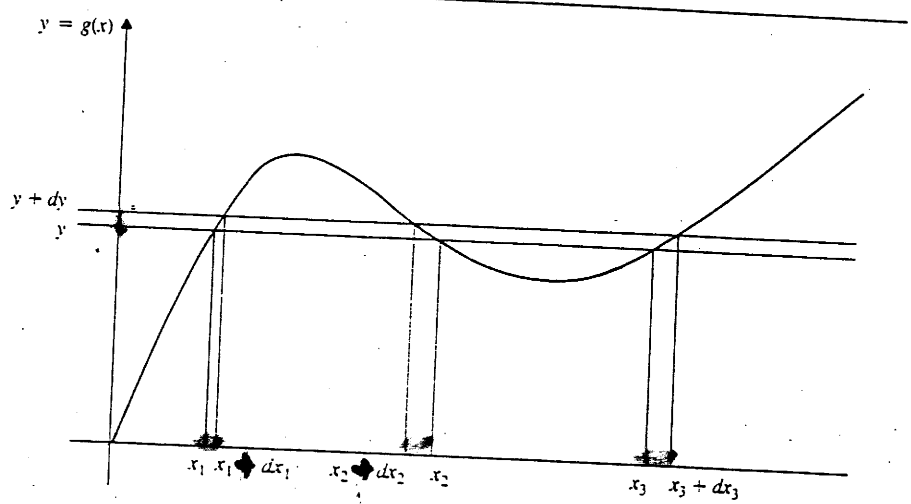

FIGURE 4.13 The equivalent event of $\{y < Y < y + dy\}$ is $\{x_1 < X < x_1 + dx_1\} \cup \{x_2 + dx_2 < X < x_2\} \cup \{x_3 < X < x_3 + dx_3\}$.

event $C_y = [y < Y < y + dy]$

The equivalent event B_y

$$B_y = \{x_1 < X < x_1 + dx_1\} \cup \{x_2 + dx_2 < X < x_2\} \cup \{x_3 < X < x_3 + dx_3\}.$$

The probability of the event C_y is approximately

$$P[C_y] = f_Y(y)\,|dy|, \tag{4.71}$$

where $|dy|$ is the length of the interval $y < Y \le y + dy$. Similarly, the probability of the event B_y is approximately

$$P[B_y] = f_X(x_1)\,|dx_1| + f_X(x_2)\,|dx_2| + f_X(x_3)\,|dx_3|. \tag{4.72}$$

Since C_y and B_y are equivalent events, their probabilities must be equal. By equating Eqs. (3.53) and (3.54) we obtain

$$f_Y(y) = \sum_k \frac{f_X(x)}{|dy/dx|}\bigg|_{x=x_k} \tag{4.73}$$

$$= \sum_k f_X(x)\left|\frac{dx}{dy}\right|\bigg|_{x=x_k} \tag{4.74}$$

It is clear that if the equation $g(x) = y$ has n solutions, the expression for the pdf of Y at that point is given by Eqs. 4.73 and (4.74) and contains n terms.