第九章

多元函数微分法及其应用

习题 9－1　多元函数的基本概念

1. 判定下列平面点集中哪些是开集、闭集、区域、有界集、无界集？并分别指出它们的聚点所成的点集（称为导集）和边界.

(1) $\{(x,y) \mid x \neq 0, y \neq 0\}$；　(2) $\{(x,y) \mid 1 < x^2 + y^2 \leqslant 4\}$；

(3) $\{(x,y) \mid y > x^2\}$；

(4) $\{(x,y) \mid x^2 + (y-1)^2 \geqslant 1\} \cap \{(x,y) \mid x^2 + (y-2)^2 \leqslant 4\}$.

解　(1) 集合是开集，无界集；导集为 $\mathbf{R}^2$，边界为 $\{(x,y) \mid x = 0 \text{ 或 } y = 0\}$.

(2) 集合既非开集，又非闭集，是有界集；导集为 $\{(x,y) \mid 1 \leqslant x^2 + y^2 \leqslant 4\}$，边界为 $\{(x,y) \mid x^2 + y^2 = 1\} \cup \{(x,y) \mid x^2 + y^2 = 4\}$.

(3) 集合是开集，区域，无界集；导集为 $\{(x,y) \mid y \geqslant x^2\}$，边界为 $\{(x,y) \mid y = x^2\}$.

(4) 集合是闭集，有界集；导集为集合本身，边界为 $\{(x,y) \mid x^2 + (y-1)^2 = 1\} \cup \{(x,y) \mid x^2 + (y-2)^2 = 4\}$.

2. 已知函数 $f(x,y) = x^2 + y^2 - xy\tan\dfrac{x}{y}$，试求 $f(tx,ty)$.

解
$$f(tx,ty) = (tx)^2 + (ty)^2 - (tx)(ty)\tan\frac{tx}{ty}$$
$$= t^2\left(x^2 + y^2 - xy\tan\frac{x}{y}\right)$$
$$= t^2 f(x,y).$$

3. 试证函数 $F(x,y) = \ln x \cdot \ln y$ 满足关系式
$$F(xy,uv) = F(x,u) + F(x,v) + F(y,u) + F(y,v).$$

证
$$F(xy,uv) = \ln(xy) \cdot \ln(uv) = (\ln x + \ln y)(\ln u + \ln v)$$
$$= \ln x \cdot \ln u + \ln x \cdot \ln v + \ln y \cdot \ln u + \ln y \cdot \ln v$$
$$= F(x,u) + F(x,v) + F(y,u) + F(y,v).$$

4. 已知函数 $f(u,v,w) = u^w + w^{u+v}$，试求 $f(x+y, x-y, xy)$.

解　$f(x+y,x-y,xy) = (x+y)^{xy} + (xy)^{(x+y)+(x-y)} = (x+y)^{xy} + (xy)^{2x}$.

5. 求下列各函数的定义域：

(1) $z = \ln(y^2 - 2x + 1)$；　(2) $z = \dfrac{1}{\sqrt{x+y}} + \dfrac{1}{\sqrt{x-y}}$；

$$\frac{\partial s}{\partial v}=\frac{\frac{\partial}{\partial v}(u^2+v^2)\cdot uv-(u^2+v^2)\cdot\frac{\partial}{\partial v}(uv)}{(uv)^2}$$

$$=\frac{2uv^2-(u^2+v^2)u}{u^2v^2}$$

$$=\frac{1}{u}-\frac{u}{v^2}.$$

(3) $\frac{\partial z}{\partial x}=\frac{1}{2}\cdot\frac{1}{\sqrt{\ln(xy)}}\cdot\frac{1}{xy}\cdot y=\frac{1}{2x\sqrt{\ln(xy)}}$,

$\frac{\partial z}{\partial y}=\frac{1}{2}\cdot\frac{1}{\sqrt{\ln(xy)}}\cdot\frac{1}{xy}\cdot x=\frac{1}{2y\sqrt{\ln(xy)}}$.

(4) $\frac{\partial z}{\partial x}=y\cos(xy)+2\cos(xy)\cdot[-\sin(xy)]\cdot y$

$=y[\cos(xy)-\sin(2xy)]$,

$\frac{\partial z}{\partial y}=x\cos(xy)+2\cos(xy)\cdot[-\sin(xy)]\cdot x$

$=x[\cos(xy)-\sin(2xy)]$.

(5) $\frac{\partial z}{\partial x}=\cot\frac{x}{y}\cdot\sec^2\frac{x}{y}\cdot\frac{1}{y}=\frac{2}{y}\csc\frac{2x}{y}$,

$\frac{\partial z}{\partial y}=\cot\frac{x}{y}\cdot\sec^2\frac{x}{y}\cdot\left(-\frac{x}{y^2}\right)=-\frac{2x}{y^2}\csc\frac{2x}{y}$.

(6) $\frac{\partial z}{\partial x}=y^2(1+xy)^{y-1}$,

$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}[e^{y\ln(1+xy)}]=(1+xy)^y\left[\ln(1+xy)+\frac{xy}{1+xy}\right]$.

(7) $\frac{\partial u}{\partial x}=\frac{y}{z}x^{\frac{y}{z}-1}$, $\frac{\partial u}{\partial y}=\frac{1}{z}x^{\frac{y}{z}}\ln x$, $\frac{\partial u}{\partial z}=-\frac{y}{z^2}x^{\frac{y}{z}}\ln x$.

(8) $\frac{\partial u}{\partial x}=\frac{z(x-y)^{z-1}}{1+(x-y)^{2z}}$,

$\frac{\partial u}{\partial y}=-\frac{z(x-y)^{z-1}}{1+(x-y)^{2z}}$,

$\frac{\partial u}{\partial z}=\frac{(x-y)^z\ln(x-y)}{1+(x-y)^{2z}}$.

2. 设 $T=2\pi\sqrt{\frac{l}{g}}$，求证 $l\frac{\partial T}{\partial l}+g\frac{\partial T}{\partial g}=0$.

证　因为　$\frac{\partial T}{\partial l}=2\pi\cdot\frac{1}{2\sqrt{\frac{l}{g}}}\cdot\frac{1}{g}=\frac{\pi}{\sqrt{gl}}$,

$$\frac{\partial T}{\partial g}=2\pi\cdot\frac{1}{2\sqrt{\frac{l}{g}}}\cdot\left(-\frac{l}{g^2}\right)=-\frac{\pi}{g}\sqrt{\frac{l}{g}},$$

所以
$$l\frac{\partial T}{\partial l}+g\frac{\partial T}{\partial g}=\pi\sqrt{\frac{l}{g}}-\pi\sqrt{\frac{l}{g}}=0.$$

3. 设 $z=e^{-\left(\frac{1}{x}+\frac{1}{y}\right)}$,求证 $x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=2z$.

证 因为
$$\frac{\partial z}{\partial x}=\frac{1}{x^2}e^{-\left(\frac{1}{x}+\frac{1}{y}\right)},\quad \frac{\partial z}{\partial y}=\frac{1}{y^2}e^{-\left(\frac{1}{x}+\frac{1}{y}\right)},$$

所以
$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=2e^{-\left(\frac{1}{x}+\frac{1}{y}\right)}=2z.$$

4. 设 $f(x,y)=x+(y-1)\arcsin\sqrt{\frac{x}{y}}$,求 $f_x(x,1)$.

解
$$f_x(x,y)=1+\frac{y-1}{\sqrt{1-\frac{x}{y}}}\cdot\frac{1}{2\sqrt{\frac{x}{y}}}\cdot\frac{1}{y},$$
$$f_x(x,1)=1.$$

5. 曲线 $\begin{cases}z=\dfrac{x^2+y^2}{4},\\ y=4\end{cases}$ 在点(2,4,5)处的切线对于 x 轴的倾角是多少?

解 设 $z=f(x,y)$. 按偏导数的几何意义,$f_x(2,4)$ 就是曲线在点(2,4,5)处的切线对于 x 轴的斜率,而 $f_x(2,4)=\left.\frac{1}{2}x\right|_{x=2}=1$,即 $k=\tan\alpha=1$,于是倾角 $\alpha=\frac{\pi}{4}$.

6. 求下列函数的 $\frac{\partial^2 z}{\partial x^2}$,$\frac{\partial^2 z}{\partial y^2}$ 和 $\frac{\partial^2 z}{\partial x\partial y}$:

(1) $z=x^4+y^4-4x^2y^2$;　　(2) $z=\arctan\frac{y}{x}$;

(3) $z=y^x$.

解 (1)
$$\frac{\partial z}{\partial x}=4x^3-8xy^2,\quad \frac{\partial^2 z}{\partial x^2}=12x^2-8y^2,$$
$$\frac{\partial z}{\partial y}=4y^3-8x^2y,\quad \frac{\partial^2 z}{\partial y^2}=12y^2-8x^2,$$
$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}(4x^3-8xy^2)=-16xy.$$

(2)
$$\frac{\partial z}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2},\quad \frac{\partial^2 z}{\partial x^2}=\frac{2xy}{(x^2+y^2)^2}.$$

隐函数求导公式得

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}=-\frac{1}{J}\begin{vmatrix}1 & -u\cos v\\ 0 & -u\sin v\end{vmatrix}$$

$$=\frac{\sin v}{e^{u}(\sin v-\cos v)+1},$$

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}=-\frac{1}{J}\begin{vmatrix}0 & -u\cos v\\ 1 & -u\sin v\end{vmatrix}$$

$$=\frac{-\cos v}{e^{u}(\sin v-\cos v)+1},$$

$$\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}=-\frac{1}{J}\begin{vmatrix}-e^{u}-\sin v & 1\\ -e^{u}+\cos v & 0\end{vmatrix}$$

$$=\frac{\cos v-e^{u}}{u[e^{u}(\sin v-\cos v)+1]},$$

$$\frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}=-\frac{1}{J}\begin{vmatrix}-e^{u}-\sin v & 0\\ -e^{u}+\cos v & 1\end{vmatrix}$$

$$=\frac{\sin v+e^{u}}{u[e^{u}(\sin v-\cos v)+1]}.$$

11. 设 $y=f(x,t)$，而 $t=t(x,y)$ 是由方程 $F(x,y,t)=0$ 所确定的函数，其中 f,F 都具有一阶连续偏导数. 试证明

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\dfrac{\partial f}{\partial x}\dfrac{\partial F}{\partial t}-\dfrac{\partial f}{\partial t}\dfrac{\partial F}{\partial x}}{\dfrac{\partial f}{\partial t}\dfrac{\partial F}{\partial y}+\dfrac{\partial F}{\partial t}}.$$

证法一　由方程组 $\begin{cases}y=f(x,t),\\ F(x,y,t)=0\end{cases}$ 可确定两个一元隐函数 $y=y(x)$，$t=t(x)$. 分别在两个方程两端对 x 求导可得

$$\begin{cases}\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial t}\cdot\dfrac{\mathrm{d}t}{\mathrm{d}x},\\[2ex] \dfrac{\partial F}{\partial x}+\dfrac{\partial F}{\partial y}\cdot\dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{\partial F}{\partial t}\cdot\dfrac{\mathrm{d}t}{\mathrm{d}x}=0.\end{cases}$$

移项得

$$\begin{cases}\dfrac{\mathrm{d}y}{\mathrm{d}x}-\dfrac{\partial f}{\partial t}\cdot\dfrac{\mathrm{d}t}{\mathrm{d}x}=\dfrac{\partial f}{\partial x},\\[2ex] \dfrac{\partial F}{\partial y}\cdot\dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{\partial F}{\partial t}\cdot\dfrac{\mathrm{d}t}{\mathrm{d}x}=-\dfrac{\partial F}{\partial x}.\end{cases}$$

当 $D=\begin{vmatrix} 1 & -\frac{\partial f}{\partial t} \\ \frac{\partial F}{\partial y} & \frac{\partial F}{\partial t}\end{vmatrix}=\frac{\partial F}{\partial t}+\frac{\partial f}{\partial t}\cdot\frac{\partial F}{\partial y}\neq 0$ 时,解方程组得

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{D}\cdot\begin{vmatrix} \frac{\partial f}{\partial x} & -\frac{\partial f}{\partial t} \\ -\frac{\partial F}{\partial x} & \frac{\partial F}{\partial t}\end{vmatrix}=\frac{\frac{\partial f}{\partial x}\cdot\frac{\partial F}{\partial t}-\frac{\partial f}{\partial t}\cdot\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial t}+\frac{\partial f}{\partial t}\cdot\frac{\partial F}{\partial y}}.$$

证法二　分别在 $y=f(x,t)$ 及 $F(x,y,t)=0$ 两端求全微分,得

$$\begin{cases}\mathrm{d}y=f_x\mathrm{d}x+f_t\mathrm{d}t, & (1)\\ F_x\mathrm{d}x+F_y\mathrm{d}y+F_t\mathrm{d}t=0. & (2)\end{cases}$$

由(2),得

$$F_t\mathrm{d}t=-(F_x\mathrm{d}x+F_y\mathrm{d}y). \tag{3}$$

将 F_t 乘(1)式两端,并以(3)式代入,得

$$F_t\mathrm{d}y=f_xF_t\mathrm{d}x-f_t(F_x\mathrm{d}x+F_y\mathrm{d}y),$$

即

$$(F_t+f_tF_y)\mathrm{d}y=(f_xF_t-f_tF_x)\mathrm{d}x.$$

故当 $F_t+f_tF_y\neq 0$ 时,有

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{f_xF_t-f_tF_x}{F_t+f_tF_y}.$$

习题 9－6　多元函数微分学的几何应用

1. 设 $\boldsymbol{f}(t)=f_1(t)\boldsymbol{i}+f_2(t)\boldsymbol{j}+f_3(t)\boldsymbol{k}$, $\boldsymbol{g}(t)=g_1(t)\boldsymbol{i}+g_2(t)\boldsymbol{j}+g_3(t)\boldsymbol{k}$, $\lim\limits_{t\to t_0}\boldsymbol{f}(t)=\boldsymbol{u}$, $\lim\limits_{t\to t_0}\boldsymbol{g}(t)=\boldsymbol{v}$, 证明 $\lim\limits_{t\to t_0}[\boldsymbol{f}(t)\times\boldsymbol{g}(t)]=\boldsymbol{u}\times\boldsymbol{v}$.

证 $\lim\limits_{t\to t_0}[\boldsymbol{f}(t)\times\boldsymbol{g}(t)]=\lim\limits_{t\to t_0}\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ f_1(t) & f_2(t) & f_3(t)\\ g_1(t) & g_2(t) & g_3(t)\end{vmatrix}$

$$=\lim_{t\to t_0}\big(f_2(t)g_3(t)-f_3(t)g_2(t),f_3(t)g_1(t)-f_1(t)g_3(t),f_1(t)g_2(t)-f_2(t)g_1(t)\big)$$

$$=\Big(\lim_{t\to t_0}[f_2(t)g_3(t)-f_3(t)g_2(t)],\lim_{t\to t_0}[f_3(t)g_1(t)-f_1(t)g_3(t)],\lim_{t\to t_0}[f_1(t)g_2(t)-f_2(t)g_1(t)]\Big)$$

$$=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \lim\limits_{t\to t_0}f_1(t) & \lim\limits_{t\to t_0}f_2(t) & \lim\limits_{t\to t_0}f_3(t)\\ \lim\limits_{t\to t_0}g_1(t) & \lim\limits_{t\to t_0}g_2(t) & \lim\limits_{t\to t_0}g_3(t)\end{vmatrix}=\boldsymbol{u}\times\boldsymbol{v}.$$

$$\begin{cases} L_x = \dfrac{\lambda}{3} + 2\mu x = 0, \\ L_y = \dfrac{\lambda}{4} + 2\mu y = 0, \\ L_z = 2z + \dfrac{\lambda}{5} = 0. \end{cases}$$

又由约束条件,有

$$\frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 1,$$
$$x^2 + y^2 = 1.$$

解此方程组,得 $x = \frac{4}{5}, y = \frac{3}{5}, z = \frac{35}{12}$. 于是,得可能的极值点 $M_0\left(\frac{4}{5}, \frac{3}{5}, \frac{35}{12}\right)$. 由问题本身可知,距离最短的点必定存在,因此 M_0 就是所求的点.

18. 在第一卦限内作椭球面 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ 的切平面,使该切平面与三坐标面所围成的四面体的体积最小. 求这切平面的切点,并求此最小体积.

解　设切点为 $M(x_0, y_0, z_0)$, $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$,

$$\boldsymbol{n} = (F_x, F_y, F_z) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right).$$

曲面在点 M 处的切平面方程为

$$\frac{x_0}{a^2}(x - x_0) + \frac{y_0}{b^2}(y - y_0) + \frac{z_0}{c^2}(z - z_0) = 0,$$

即

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1.$$

于是,切平面在三个坐标轴上的截距依次为 $\frac{a^2}{x_0}, \frac{b^2}{y_0}, \frac{c^2}{z_0}$, 切平面与三个坐标面所围成的四面体的体积为

$$V = \frac{1}{6} \cdot \frac{a^2 b^2 c^2}{x_0 y_0 z_0}.$$

在 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ 的条件下,求 V 的最小值,即求分母 xyz 的最大值. 作拉格朗日函数

$$L(x, y, z) = xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right).$$

令

$$\begin{cases} L_x = yz + \dfrac{2\lambda x}{a^2} = 0, & (1) \\ L_y = xz + \dfrac{2\lambda y}{b^2} = 0, & (2) \\ L_z = xy + \dfrac{2\lambda z}{c^2} = 0. & (3) \end{cases}$$

$(1)\cdot x+(2)\cdot y+(3)\cdot z$,并由约束条件$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$,得

$$\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}=\frac{1}{3},$$

从而

$$x=\frac{a}{\sqrt{3}},\quad y=\frac{b}{\sqrt{3}},\quad z=\frac{c}{\sqrt{3}}.$$

于是,得可能极值点 $M\left(\dfrac{a}{\sqrt{3}},\dfrac{b}{\sqrt{3}},\dfrac{c}{\sqrt{3}}\right)$. 由此问题的性质知,所求的切点为 $M\left(\dfrac{a}{\sqrt{3}},\dfrac{b}{\sqrt{3}},\dfrac{c}{\sqrt{3}}\right)$,四面体的最小体积为

$$V_{\min}=\frac{\sqrt{3}}{2}abc.$$

19. 某厂家生产的一种产品同时在两个市场销售,售价分别为 p_1 和 p_2,销售量分别为 q_1 和 q_2,需求函数分别为

$$q_1=24-0.2p_1,\quad q_2=10-0.05p_2,$$

总成本函数为

$$C=35+40(q_1+q_2).$$

试问:厂家如何确定两个市场的售价,能使其获得的总利润最大?最大总利润为多少?

解法一　总收入函数为

$$R=p_1q_1+p_2q_2=24p_1-0.2p_1^2+10p_2-0.05p_2^2,$$

总利润函数为

$$L=R-C=32p_1-0.2p_1^2-0.05p_2^2+12p_2-1\,395.$$

由极值的必要条件,得方程组

$$\begin{cases} \dfrac{\partial L}{\partial p_1}=32-0.4p_1=0, \\ \dfrac{\partial L}{\partial p_2}=12-0.1p_2=0. \end{cases}$$

解此方程组,得 $p_1=80$,$p_2=120$.

由问题的实际意义可知,厂家获得总利润最大的市场售价必定存在,故当 $p_1=80,p_2=120$ 时,厂家所获得的总利润最大,其最大总利润为

$$L\Big|_{p_1=80,p_2=120}=605.$$

解法二　两个市场的价格函数分别为

$$p_1=120-5q_1,\quad p_2=200-20q_2,$$

总收入函数为

$$R=p_1q_1+p_2q_2=(120-5q_1)q_1+(200-20q_2)q_2,$$

总利润函数为

$$\begin{aligned}L=R-C&=(120-5q_1)q_1+(200-20q_2)q_2-[35+40(q_1+q_2)]\\&=80q_1-5q_1^2+160q_2-20q_2^2-35.\end{aligned}$$

由极值的必要条件,得方程组

$$\begin{cases}\dfrac{\partial L}{\partial q_1}=80-10q_1=0,\\[2mm]\dfrac{\partial L}{\partial q_2}=160-40q_2=0.\end{cases}$$

解此方程组得 $q_1=8,q_2=4$.

由问题的实际意义可知,当 $q_1=8,q_2=4$,即 $p_1=80,p_2=120$ 时,厂家所获得的总利润最大,其最大总利润为

$$L\Big|_{q_1=8,q_2=4}=605.$$

20. 设有一小山,取它的底面所在的平面为 xOy 坐标面,其底部所占的闭区域为 $D=\{(x,y)\mid x^2+y^2-xy\leqslant 75\}$,小山的高度函数为 $h=f(x,y)=75-x^2-y^2+xy$.

(1) 设 $M(x_0,y_0)\in D$,问 $f(x,y)$ 在该点沿平面上什么方向的方向导数最大?若记此方向导数的最大值为 $g(x_0,y_0)$,试写出 $g(x_0,y_0)$ 的表达式.

(2) 现欲利用此小山开展攀岩活动,为此需要在山脚找一上山坡度最大的点作为攀岩的起点,也就是说,要在 D 的边界线 $x^2+y^2-xy=75$ 上找出(1)中的 $g(x,y)$ 达到最大值的点. 试确定攀岩起点的位置.

解　(1) 由梯度与方向导数的关系知,$h=f(x,y)$ 在点 $M(x_0,y_0)$ 处沿梯度

$$\mathbf{grad}\,f(x_0,y_0)=(y_0-2x_0)\boldsymbol{i}+(x_0-2y_0)\boldsymbol{j}$$

方向的方向导数最大,方向导数的最大值为该梯度的模,所以

$$g(x_0,y_0)=\sqrt{(y_0-2x_0)^2+(x_0-2y_0)^2}=\sqrt{5x_0^2+5y_0^2-8x_0y_0}.$$

(2) 欲在 D 的边界上求 $g(x,y)$ 达到最大值的点,只需求 $F(x,y)=g^2(x,y)=5x^2+5y^2-8xy$ 达到最大值的点. 因此,作拉格朗日函数

$$L=5x^2+5y^2-8xy+\lambda(75-x^2-y^2+xy).$$

令

$$\begin{cases} L_x = 10x - 8y + \lambda(y - 2x) = 0, & (1) \\ L_y = 10y - 8x + \lambda(x - 2y) = 0. & (2) \end{cases}$$

又由约束条件，有

$$75 - x^2 - y^2 + xy = 0. \qquad (3)$$

(1) + (2)，得

$$(x + y)(2 - \lambda) = 0,$$

解得 $y = -x$ 或 $\lambda = 2$.

若 $\lambda = 2$，则由(1)得 $y = x$，再由(3)得 $x = y = \pm 5\sqrt{3}$.

若 $y = -x$，则由(3)得 $x = \pm 5$, $y = \mp 5$.

于是得到四个可能的极值点：

$M_1(5, -5)$，　$M_2(-5, 5)$，　$M_3(5\sqrt{3}, 5\sqrt{3})$，　$M_4(-5\sqrt{3}, -5\sqrt{3})$.

由于 $F(M_1) = F(M_2) = 450$，$F(M_3) = F(M_4) = 150$，故 $M_1(5, -5)$ 或 $M_2(-5, 5)$ 可作为攀岩的起点.