第四章

不定积分

习题 4－1　不定积分的概念与性质

1. 利用导数验证下列等式：

(1) $\int \frac{1}{\sqrt{x^2+1}}dx = \ln(x+\sqrt{x^2+1})+C$；

(2) $\int \frac{1}{x^2\sqrt{x^2-1}}dx = \frac{\sqrt{x^2-1}}{x}+C$；

(3) $\int \frac{2x}{(x^2+1)(x+1)^2}dx = \arctan x+\frac{1}{x+1}+C$；

(4) $\int \sec x dx = \ln|\tan x+\sec x|+C$；

(5) $\int x\cos x dx = x\sin x+\cos x+C$；

(6) $\int e^x \sin x dx = \frac{1}{2}e^x(\sin x-\cos x)+C$.

解　(1) $\frac{d}{dx}[\ln(x+\sqrt{x^2+1})+C] = \frac{1}{x+\sqrt{x^2+1}}\cdot\left(1+\frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}$.

(2) $\frac{d}{dx}\left(\frac{\sqrt{x^2-1}}{x}+C\right) = \frac{\frac{x}{\sqrt{x^2-1}}\cdot x-\sqrt{x^2-1}}{x^2} = \frac{1}{x^2\sqrt{x^2-1}}$.

(3) $\frac{d}{dx}\left(\arctan x+\frac{1}{x+1}+C\right) = \frac{1}{x^2+1}-\frac{1}{(x+1)^2} = \frac{2x}{(x^2+1)(x+1)^2}$.

(4) $\frac{d}{dx}(\ln|\tan x+\sec x|+C) = \frac{1}{\tan x+\sec x}\cdot(\sec^2 x+\sec x\tan x) = \sec x$.

(5) $\frac{d}{dx}(x\sin x+\cos x+C) = \sin x+x\cos x-\sin x = x\cos x$.

(6) $\frac{d}{dx}\left[\frac{1}{2}e^x(\sin x-\cos x)+C\right] = \frac{1}{2}e^x(\sin x-\cos x)+\frac{1}{2}e^x(\cos x+\sin x)$

$= e^x\sin x$.

2. 求下列不定积分：

(1) $\int \frac{dx}{x^2}$；　　(2) $\int x\sqrt{x}dx$；

(37) $\int \frac{\cot x}{1+\sin x}dx = \int \frac{\cos x}{\sin x(1+\sin x)}dx = \int\left(\frac{1}{\sin x} - \frac{1}{1+\sin x}\right)d(\sin x)$

$$= \ln\left|\frac{\sin x}{1+\sin x}\right| + C.$$

(38) $\int \frac{dx}{\sin^3 x\cos x} = -\int \cot x\sec^2 x d(\cot x) \xlongequal{u=\cot x} -\int u\left(1+\frac{1}{u^2}\right)du$

$$= -\frac{u^2}{2} - \ln|u| + C = -\frac{\cot^2 x}{2} - \ln|\cot x| + C.$$

(39) $\int \frac{dx}{(2+\cos x)\sin x} = \int \frac{d(\cos x)}{(2+\cos x)(\cos^2 x-1)}$

$$\xlongequal{u=\cos x} \int \frac{du}{(2+u)(u^2-1)}$$

$$= \int\left[\frac{1}{6(u-1)} - \frac{1}{2(u+1)} + \frac{1}{3(u+2)}\right]du$$

$$= \frac{1}{6}\ln|u-1| - \frac{1}{2}\ln|u+1| + \frac{1}{3}\ln|u+2| + C$$

$$= \frac{1}{6}\ln(1-\cos x) - \frac{1}{2}\ln(1+\cos x) + \frac{1}{3}\ln(2+\cos x) + C.$$

(40) 解法一

$$\int \frac{\sin x\cos x}{\sin x+\cos x}dx = \int \frac{\frac{1}{2}(\sin x+\cos x)^2 - \frac{1}{2}}{\sin x+\cos x}dx$$

$$= \frac{1}{2}\int(\sin x+\cos x)dx - \frac{1}{2}\int\frac{1}{\sin x+\cos x}dx$$

$$= \frac{1}{2}(-\cos x+\sin x) - \frac{1}{2}\int\frac{1}{\sin x+\cos x}dx,$$

令 $u=\tan\frac{x}{2}$,则 $\sin x = \frac{2u}{1+u^2}$,$\cos x = \frac{1-u^2}{1+u^2}$,$dx = \frac{2}{1+u^2}du$,故有

$$\int\frac{1}{\sin x+\cos x}dx = \int\frac{2}{2u+1-u^2}du = -\int\frac{2}{(u-1)^2-(\sqrt{2})^2}du$$

$$= -\frac{1}{\sqrt{2}}\int\frac{1}{u-1-\sqrt{2}}du + \frac{1}{\sqrt{2}}\int\frac{1}{u-1+\sqrt{2}}du$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{u-1+\sqrt{2}}{u-1-\sqrt{2}}\right| + C',$$

因此有

$$\int\frac{\sin x\cos x}{\sin x+\cos x}dx = \frac{1}{2}(\sin x-\cos x) - \frac{1}{2\sqrt{2}}\ln\left|\frac{\tan\frac{x}{2}-1+\sqrt{2}}{\tan\frac{x}{2}-1-\sqrt{2}}\right| + C.$$

解法二

$$\int \frac{\sin x\cos x}{\sin x+\cos x}\mathrm{d}x = \int \frac{\sin x\cos x}{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}\mathrm{d}x \xlongequal{u=x+\frac{\pi}{4}} \int \frac{2\sin^2 u-1}{2\sqrt{2}\sin u}\mathrm{d}u$$

$$= \frac{1}{\sqrt{2}}\int \sin u\mathrm{d}u - \frac{1}{2\sqrt{2}}\int \csc u\mathrm{d}u$$

$$= -\frac{\cos\left(x+\frac{\pi}{4}\right)}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\ln\left|\csc\left(x+\frac{\pi}{4}\right)-\cot\left(x+\frac{\pi}{4}\right)\right| + C.$$