```
MODULE PieceInclusionProof
EXTENDS Integers, TLC, Sequences, FiniteSets

SIZE ≜ 256   Must be a power of 2. TODO: Make CONSTANT and add ASSUME .
HEIGHT ≜ 9   TODO: Calculate this from SIZE, as log2(SIZE) + 1.

 --algorithm PieceInclusionProof

variables
    HashCounter = −1;
    HashRecord = ⟨⟩;

macro RepCompress(a, b, height, var)begin
    if (⟨a, b, height⟩ ∈ DOMAIN HashRecord) then
        var := HashRecord[⟨a, b, height⟩];
     else
        HashCounter := HashCounter − 1;
        HashRecord := (⟨a, b, height⟩ :> HashCounter) @@ HashRecord;
        var := HashCounter
    end if ;
end macro ;

process test_hash = "test hash"
variables h1, h2, h3;

begin
    L1:
        RepCompress(1, 2, 0, h1);
    L2:
        RepCompress(1, 2, 0, h2);
    L3:
        RepCompress(2, 1, 0, h3);

        assert h1 = h2;
        assert h1 ≠ h3;
end process ;

fair process merkle_tree = "merkle tree"
variables h,
    input, row, rowSize, nextRow, index, proof_element, root, challenge,
    cursor_index, cursor_row, cursor_element, proof_index, challenge_path_acc, place_acc,
    rows = ⟨⟩     ;
    height = −1;

    proof_path = ⟨⟩;
    proof_elements = ⟨⟩;
begin
    BuildTree:
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49          input := [i ∈ 1 .. SIZE ↦ i] ;
50          row := input ;
51          rows := ⟨⟩    ;
52      RowLoop:
53          height := height + 1 ;
54          rows := Append(rows, row) ;
55           It would be nice to make this assert an invariant, but how do we make an invariant
56           over a process variable?
57          assert height > 1 ⇒ Len(rows[height − 1]) = 2 ∗ Len(rows[height]) ;

59          nextRow := ⟨⟩ ;
60          index := 1 ;
61          rowSize := Cardinality(DOMAIN row) ;
62          if rowSize > 1 then
63              HashRow:
64                  RepCompress(row[index], row[index + 1], height, h) ;
65                  nextRow := Append(nextRow, h) ;
66              Advance: index := index + 2 ;
67              if index < Cardinality(DOMAIN row) then
68                 goto HashRow ;
69               else
70                  row := nextRow ;
71              end if ;
72              Repeat:  goto RowLoop ;
73           else
74              assert Len(rows) = HEIGHT ;
75          end if ;
76      Proofs:
77          challenge := 1 ;
78      MakeProof:
79          cursor_index := challenge ;
80          cursor_row := 1 ;
81          cursor_element := rows[cursor_row][cursor_index] ;
82          proof_path := ⟨⟩ ;
83          proof_elements := ⟨⟩ ;
84      S1:
85          if cursor_index%2 = 1 then
86              proof_path := Append(proof_path, FALSE) ;
87              proof_element := rows[cursor_row][cursor_index + 1] ;
88              RepCompress(rows[cursor_row][cursor_index],
89                          proof_element,
90                          cursor_row − 1,
91                          cursor_element) ;
92           else
93              proof_path := Append(proof_path, TRUE) ;
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94              proof_element := rows[cursor_row][cursor_index − 1] ;
95              RepCompress(proof_element,
96                          rows[cursor_row][cursor_index],
97                          cursor_row − 1,
98                          cursor_element) ;
99          end if ;

101         proof_elements := Append(proof_elements, proof_element) ;
102     ProofLoop:
103         cursor_row := cursor_row + 1 ;
104         cursor_index := (cursor_index + 1) ÷ 2 ;
105         if cursor_row < Len(rows) then
106             goto S1 ;
107         end if ;
108     FinishProof:
109         root := rows[Len(rows)][1] ;

111     CheckProof:
112         proof_index := 1 ;
113         height := 0 ;
114         cursor_index := challenge ;
115         cursor_element := rows[height + 1][cursor_index] ;
116         challenge_path_acc := 0 ;
117         place_acc := 1 ;
118     ProofCheckLoop:
119         if proof_path[proof_index] then
120             RepCompress(proof_elements[proof_index],
121                         cursor_element,
122                         height,
123                         cursor_element) ;
124             challenge_path_acc := challenge_path_acc + place_acc ;
125          else
126            RepCompress(cursor_element,
127                        proof_elements[proof_index],
128                        height,
129                        cursor_element) ;
130         end if ;
131         place_acc := place_acc ∗ 2 ;

133         proof_index := proof_index + 1 ;
134         height := height + 1 ;
135         if height < Len(proof_elements) then
136             goto ProofCheckLoop ;
137         end if ;

139     CheckRoot:
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        assert cursor_element = root ;
        assert challenge_path_acc = challenge − 1 ;   challenges are 1-indexed because TLA+.

    IncrementChallenge:
        challenge := challenge + 1 ;
        if challenge ≤ Len(input) then
            goto MakeProof ;
        end if ;
end process ;

end algorithm ;

 BEGIN TRANSLATION
CONSTANT defaultInitValue
VARIABLES HashCounter, HashRecord, pc, h1, h2, h3, h, input, row, rowSize,
          nextRow, index, proof_element, root, challenge, cursor_index,
          cursor_row, cursor_element, proof_index, challenge_path_acc,
          place_acc, rows, height, proof_path, proof_elements

vars ≜ ⟨HashCounter, HashRecord, pc, h1, h2, h3, h, input, row, rowSize,
        nextRow, index, proof_element, root, challenge, cursor_index,
        cursor_row, cursor_element, proof_index, challenge_path_acc,
        place_acc, rows, height, proof_path, proof_elements⟩

ProcSet ≜ {"test hash"} ∪ {"merkle tree"}

Init ≜  Global variables
        ∧ HashCounter = −1
        ∧ HashRecord = ⟨⟩
        Process test_hash
        ∧ h1 = defaultInitValue
        ∧ h2 = defaultInitValue
        ∧ h3 = defaultInitValue
        Process merkle_tree
        ∧ h = defaultInitValue
        ∧ input = defaultInitValue
        ∧ row = defaultInitValue
        ∧ rowSize = defaultInitValue
        ∧ nextRow = defaultInitValue
        ∧ index = defaultInitValue
        ∧ proof_element = defaultInitValue
        ∧ root = defaultInitValue
        ∧ challenge = defaultInitValue
        ∧ cursor_index = defaultInitValue
        ∧ cursor_row = defaultInitValue
        ∧ cursor_element = defaultInitValue
        ∧ proof_index = defaultInitValue
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187         ∧ challenge_path_acc = defaultInitValue
188         ∧ place_acc = defaultInitValue
189         ∧ rows = ⟨⟩
190         ∧ height = −1
191         ∧ proof_path = ⟨⟩
192         ∧ proof_elements = ⟨⟩
193         ∧ pc = [self ∈ ProcSet ↦ CASE self = "test hash" → "L1"
194                                      □  self = "merkle tree" → "BuildTree"]

196 L1 ≜  ∧ pc["test hash"] = "L1"
197         ∧ IF (⟨1, 2, 0⟩ ∈ DOMAIN HashRecord)
198               THEN ∧ h1′ = HashRecord[⟨1, 2, 0⟩]
199                    ∧ UNCHANGED ⟨HashCounter, HashRecord⟩
200               ELSE ∧ HashCounter′ = HashCounter − 1
201                    ∧ HashRecord′ = (⟨1, 2, 0⟩ :> HashCounter′) @@ HashRecord
202                    ∧ h1′ = HashCounter′
203         ∧ pc′ = [pc EXCEPT ![“test hash”] = “L2”]
204         ∧ UNCHANGED ⟨h2, h3, h, input, row, rowSize, nextRow, index,
205                      proof_element, root, challenge, cursor_index, cursor_row,
206                      cursor_element, proof_index, challenge_path_acc,
207                      place_acc, rows, height, proof_path, proof_elements⟩

209 L2 ≜  ∧ pc["test hash"] = "L2"
210         ∧ IF (⟨1, 2, 0⟩ ∈ DOMAIN HashRecord)
211               THEN ∧ h2′ = HashRecord[⟨1, 2, 0⟩]
212                    ∧ UNCHANGED ⟨HashCounter, HashRecord⟩
213               ELSE ∧ HashCounter′ = HashCounter − 1
214                    ∧ HashRecord′ = (⟨1, 2, 0⟩ :> HashCounter′) @@ HashRecord
215                    ∧ h2′ = HashCounter′
216         ∧ pc′ = [pc EXCEPT ![“test hash”] = “L3”]
217         ∧ UNCHANGED ⟨h1, h3, h, input, row, rowSize, nextRow, index,
218                      proof_element, root, challenge, cursor_index, cursor_row,
219                      cursor_element, proof_index, challenge_path_acc,
220                      place_acc, rows, height, proof_path, proof_elements⟩

222 L3 ≜  ∧ pc["test hash"] = "L3"
223         ∧ IF (⟨2, 1, 0⟩ ∈ DOMAIN HashRecord)
224               THEN ∧ h3′ = HashRecord[⟨2, 1, 0⟩]
225                    ∧ UNCHANGED ⟨HashCounter, HashRecord⟩
226               ELSE ∧ HashCounter′ = HashCounter − 1
227                    ∧ HashRecord′ = (⟨2, 1, 0⟩ :> HashCounter′) @@ HashRecord
228                    ∧ h3′ = HashCounter′
229         ∧ Assert(h1 = h2, “Failure of assertion at line 34, column 9.”)
230         ∧ Assert(h1 ≠ h3′, “Failure of assertion at line 35, column 9.”)
231         ∧ pc′ = [pc EXCEPT ![“test hash”] = “Done”]
232         ∧ UNCHANGED ⟨h1, h2, h, input, row, rowSize, nextRow, index,
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                    proof_element, root, challenge, cursor_index, cursor_row,
                    cursor_element, proof_index, challenge_path_acc,
                    place_acc, rows, height, proof_path, proof_elements⟩

test_hash ≜ L1 ∨ L2 ∨ L3

BuildTree ≜ ∧ pc["merkle tree"] = "BuildTree"
            ∧ input' = [i ∈ 1 .. SIZE ↦ i]
            ∧ row' = input'
            ∧ rows' = ⟨⟩
            ∧ pc' = [pc EXCEPT !["merkle tree"] = "RowLoop"]
            ∧ UNCHANGED ⟨HashCounter, HashRecord, h1, h2, h3, h, rowSize,
                         nextRow, index, proof_element, root, challenge,
                         cursor_index, cursor_row, cursor_element,
                         proof_index, challenge_path_acc, place_acc,
                         height, proof_path, proof_elements⟩

RowLoop ≜ ∧ pc["merkle tree"] = "RowLoop"
          ∧ height' = height + 1
          ∧ rows' = Append(rows, row)
          ∧ Assert(height' > 1 ⇒ Len(rows'[height' − 1]) = 2 ∗ Len(rows'[height']),
                   "Failure of assertion at line 57, column 9.")
          ∧ nextRow' = ⟨⟩
          ∧ index' = 1
          ∧ rowSize' = Cardinality(DOMAIN row)
          ∧ IF rowSize' > 1
              THEN ∧ pc' = [pc EXCEPT !["merkle tree"] = "HashRow"]
              ELSE ∧ Assert(Len(rows') = HEIGHT,
                            "Failure of assertion at line 74, column 13.")
                   ∧ pc' = [pc EXCEPT !["merkle tree"] = "Proofs"]
          ∧ UNCHANGED ⟨HashCounter, HashRecord, h1, h2, h3, h, input, row,
                       proof_element, root, challenge, cursor_index,
                       cursor_row, cursor_element, proof_index,
                       challenge_path_acc, place_acc, proof_path,
                       proof_elements⟩

HashRow ≜ ∧ pc["merkle tree"] = "HashRow"
          ∧ IF (⟨(row[index]), (row[index + 1]), height⟩ ∈ DOMAIN HashRecord)
              THEN ∧ h' = HashRecord[⟨(row[index]), (row[index + 1]), height⟩]
                   ∧ UNCHANGED ⟨HashCounter, HashRecord⟩
              ELSE ∧ HashCounter' = HashCounter − 1
                   ∧ HashRecord' = (⟨(row[index]), (row[index + 1]), height⟩ :> HashCounter') @@ Ha
                   ∧ h' = HashCounter'
          ∧ nextRow' = Append(nextRow, h')
          ∧ pc' = [pc EXCEPT !["merkle tree"] = "Advance"]
          ∧ UNCHANGED ⟨h1, h2, h3, input, row, rowSize, index,
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                        proof_element, root, challenge, cursor_index,
                        cursor_row, cursor_element, proof_index,
                        challenge_path_acc, place_acc, rows, height,
                        proof_path, proof_elements⟩

Advance ≜ ∧ pc["merkle tree"] = "Advance"
          ∧ index' = index + 2
          ∧ IF index' < Cardinality(DOMAIN row)
                THEN ∧ pc' = [pc EXCEPT !["merkle tree"] = "HashRow"]
                     ∧ row' = row
                ELSE ∧ row' = nextRow
                     ∧ pc' = [pc EXCEPT !["merkle tree"] = "Repeat"]
          ∧ UNCHANGED ⟨HashCounter, HashRecord, h1, h2, h3, h, input,
                       rowSize, nextRow, proof_element, root, challenge,
                       cursor_index, cursor_row, cursor_element,
                       proof_index, challenge_path_acc, place_acc, rows,
                       height, proof_path, proof_elements⟩

Repeat ≜ ∧ pc["merkle tree"] = "Repeat"
         ∧ pc' = [pc EXCEPT !["merkle tree"] = "RowLoop"]
         ∧ UNCHANGED ⟨HashCounter, HashRecord, h1, h2, h3, h, input, row,
                      rowSize, nextRow, index, proof_element, root,
                      challenge, cursor_index, cursor_row, cursor_element,
                      proof_index, challenge_path_acc, place_acc, rows,
                      height, proof_path, proof_elements⟩

Proofs ≜ ∧ pc["merkle tree"] = "Proofs"
         ∧ challenge' = 1
         ∧ pc' = [pc EXCEPT !["merkle tree"] = "MakeProof"]
         ∧ UNCHANGED ⟨HashCounter, HashRecord, h1, h2, h3, h, input, row,
                      rowSize, nextRow, index, proof_element, root,
                      cursor_index, cursor_row, cursor_element,
                      proof_index, challenge_path_acc, place_acc, rows,
                      height, proof_path, proof_elements⟩

MakeProof ≜ ∧ pc["merkle tree"] = "MakeProof"
            ∧ cursor_index' = challenge
            ∧ cursor_row' = 1
            ∧ cursor_element' = rows[cursor_row'][cursor_index']
            ∧ proof_path' = ⟨⟩
            ∧ proof_elements' = ⟨⟩
            ∧ pc' = [pc EXCEPT !["merkle tree"] = "S1"]
            ∧ UNCHANGED ⟨HashCounter, HashRecord, h1, h2, h3, h, input,
                         row, rowSize, nextRow, index, proof_element, root,
                         challenge, proof_index, challenge_path_acc,
                         place_acc, rows, height⟩
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S1 ≜ ∧ pc["merkle tree"] = "S1"
     ∧ IF cursor_index%2 = 1
          THEN ∧ proof_path' = Append(proof_path, FALSE)
               ∧ proof_element' = rows[cursor_row][cursor_index + 1]
               ∧ IF (⟨(rows[cursor_row][cursor_index]), proof_element', (cursor_row − 1)⟩ ∈ DOMAIN H
                    THEN ∧ cursor_element' = HashRecord[⟨(rows[cursor_row][cursor_index]), proof_e
                         ∧ UNCHANGED ⟨HashCounter, HashRecord⟩
                    ELSE ∧ HashCounter' = HashCounter − 1
                         ∧ HashRecord' = (⟨(rows[cursor_row][cursor_index]), proof_element', (curso
                         ∧ cursor_element' = HashCounter'
          ELSE ∧ proof_path' = Append(proof_path, TRUE)
               ∧ proof_element' = rows[cursor_row][cursor_index − 1]
               ∧ IF (⟨proof_element', (rows[cursor_row][cursor_index]), (cursor_row − 1)⟩ ∈ DOMAIN H
                    THEN ∧ cursor_element' = HashRecord[⟨proof_element', (rows[cursor_row][cursor_
                         ∧ UNCHANGED ⟨HashCounter, HashRecord⟩
                    ELSE ∧ HashCounter' = HashCounter − 1
                         ∧ HashRecord' = (⟨proof_element', (rows[cursor_row][cursor_index]), (curso
                         ∧ cursor_element' = HashCounter'
     ∧ proof_elements' = Append(proof_elements, proof_element')
     ∧ pc' = [pc EXCEPT ![“merkle tree”] = “ProofLoop”]
     ∧ UNCHANGED ⟨h1, h2, h3, h, input, row, rowSize, nextRow, index, root,
                  challenge, cursor_index, cursor_row, proof_index,
                  challenge_path_acc, place_acc, rows, height⟩

ProofLoop ≜ ∧ pc["merkle tree"] = "ProofLoop"
            ∧ cursor_row' = cursor_row + 1
            ∧ cursor_index' = ((cursor_index + 1) ÷ 2)
            ∧ IF cursor_row' < Len(rows)
                 THEN ∧ pc' = [pc EXCEPT ![“merkle tree”] = “S1”]
                 ELSE ∧ pc' = [pc EXCEPT ![“merkle tree”] = “FinishProof”]
            ∧ UNCHANGED ⟨HashCounter, HashRecord, h1, h2, h3, h, input,
                         row, rowSize, nextRow, index, proof_element, root,
                         challenge, cursor_element, proof_index,
                         challenge_path_acc, place_acc, rows, height,
                         proof_path, proof_elements⟩

FinishProof ≜ ∧ pc["merkle tree"] = "FinishProof"
              ∧ root' = rows[Len(rows)][1]
              ∧ pc' = [pc EXCEPT ![“merkle tree”] = “CheckProof”]
              ∧ UNCHANGED ⟨HashCounter, HashRecord, h1, h2, h3, h, input,
                           row, rowSize, nextRow, index, proof_element,
                           challenge, cursor_index, cursor_row,
                           cursor_element, proof_index, challenge_path_acc,
                           place_acc, rows, height, proof_path,
                           proof_elements⟩
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CheckProof ≜ ∧ pc["merkle tree"] = "CheckProof"
             ∧ proof_index' = 1
             ∧ height' = 0
             ∧ cursor_index' = challenge
             ∧ cursor_element' = rows[height' + 1][cursor_index']
             ∧ challenge_path_acc' = 0
             ∧ place_acc' = 1
             ∧ pc' = [pc EXCEPT !["merkle tree"] = "ProofCheckLoop"]
             ∧ UNCHANGED ⟨HashCounter, HashRecord, h1, h2, h3, h, input,
                          row, rowSize, nextRow, index, proof_element,
                          root, challenge, cursor_row, rows, proof_path,
                          proof_elements⟩

ProofCheckLoop ≜ ∧ pc["merkle tree"] = "ProofCheckLoop"
                 ∧ IF proof_path[proof_index]
                     THEN ∧ IF (⟨(proof_elements[proof_index]), cursor_element, height⟩ ∈ DOMAIN Ha
                               THEN ∧ cursor_element' = HashRecord[⟨(proof_elements[proof_index]
                                    ∧ UNCHANGED ⟨HashCounter,
                                                 HashRecord⟩
                               ELSE ∧ HashCounter' = HashCounter − 1
                                    ∧ HashRecord' = (⟨(proof_elements[proof_index]), cursor_eleme
                                    ∧ cursor_element' = HashCounter'
                          ∧ challenge_path_acc' = challenge_path_acc + place_acc
                     ELSE ∧ IF (⟨cursor_element, (proof_elements[proof_index]), height⟩ ∈ DOMAIN Ha
                               THEN ∧ cursor_element' = HashRecord[⟨cursor_element, (proof_eleme
                                    ∧ UNCHANGED ⟨HashCounter,
                                                 HashRecord⟩
                               ELSE ∧ HashCounter' = HashCounter − 1
                                    ∧ HashRecord' = (⟨cursor_element, (proof_elements[proof_index
                                    ∧ cursor_element' = HashCounter'
                          ∧ UNCHANGED challenge_path_acc
                 ∧ place_acc' = place_acc ∗ 2
                 ∧ proof_index' = proof_index + 1
                 ∧ height' = height + 1
                 ∧ IF height' < Len(proof_elements)
                     THEN ∧ pc' = [pc EXCEPT ![“merkle tree”] = “ProofCheckLoop”]
                     ELSE ∧ pc' = [pc EXCEPT ![“merkle tree”] = “CheckRoot”]
                 ∧ UNCHANGED ⟨h1, h2, h3, h, input, row, rowSize, nextRow,
                              index, proof_element, root, challenge,
                              cursor_index, cursor_row, rows, proof_path,
                              proof_elements⟩

CheckRoot ≜ ∧ pc["merkle tree"] = "CheckRoot"
            ∧ Assert(cursor_element = root,
                     "Failure of assertion at line 140, column 9.")
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                    ∧ Assert(challenge_path_acc = challenge − 1,
                             "Failure of assertion at line 141, column 9.")
                    ∧ pc′ = [pc EXCEPT ![“merkle tree”] = “IncrementChallenge”]
                    ∧ UNCHANGED ⟨HashCounter, HashRecord, h1, h2, h3, h, input,
                                 row, rowSize, nextRow, index, proof_element, root,
                                 challenge, cursor_index, cursor_row,
                                 cursor_element, proof_index, challenge_path_acc,
                                 place_acc, rows, height, proof_path,
                                 proof_elements⟩

IncrementChallenge ≜ ∧ pc[“merkle tree”] = “IncrementChallenge”
                     ∧ challenge′ = challenge + 1
                     ∧ IF challenge′ ≤ Len(input)
                           THEN ∧ pc′ = [pc EXCEPT ![“merkle tree”] = “MakeProof”]
                           ELSE ∧ pc′ = [pc EXCEPT ![“merkle tree”] = “Done”]
                     ∧ UNCHANGED ⟨HashCounter, HashRecord, h1, h2, h3, h,
                                  input, row, rowSize, nextRow, index,
                                  proof_element, root, cursor_index,
                                  cursor_row, cursor_element, proof_index,
                                  challenge_path_acc, place_acc, rows,
                                  height, proof_path, proof_elements⟩

merkle_tree ≜ BuildTree ∨ RowLoop ∨ HashRow ∨ Advance ∨ Repeat
                  ∨ Proofs ∨ MakeProof ∨ S1     ∨ ProofLoop ∨ FinishProof
                  ∨ CheckProof ∨ ProofCheckLoop ∨ CheckRoot
                  ∨ IncrementChallenge

Next ≜ test_hash ∨ merkle_tree
           ∨ Disjunct to prevent deadlock on termination
             ((∀ self ∈ ProcSet : pc[self] = “Done”) ∧ UNCHANGED vars)

Spec ≜ ∧ Init ∧ □[Next]_vars
       ∧ WF_vars(merkle_tree)

Termination ≜ ◇(∀ self ∈ ProcSet : pc[self] = “Done”)

END TRANSLATION
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