MA 201

PARTIAL DIFFERENTIAL EQUATIONS

TUTORIAL SHEET-6

$\boxed{\text{Fourier Series, Integral and Transform: Properties and applications}}$

$\boxed{\text{Laplace Transform: Properties and applications}}$

1. Given the Fourier series for the function $f(x) = x^4, \quad -\pi < x < \pi$, as
$$x^4 = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{8(-1)^n}{n^4}(\pi^2 n^2 - 6)\cos nx$$
find the Fourier series for $f(x) = x^5, \quad -\pi < x < \pi$.

2. Deduce the Fourier series for the function $f(x) = e^{ax}, \ -\pi < x < \pi, \ a$ a real number. Hence find the values of the following series:
$$\text{(a)} \ \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2}, \quad \text{(b)} \ \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}.$$

3. Given the half-range sine series
$$x(\pi - x) = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3}, \ 0 \le x \le \pi,$$
use Parseval's theorem to deduce the value of the series $\sum_{n=1}^{\infty} 1/(2n-1)^6$.

4. Find the Fourier integral representation of the following non-periodic function:
$$f(t) = \begin{cases} \sin t, & t^2 < \pi^2, \\ 0, & t^2 > \pi^2. \end{cases}, \quad f(t) = \begin{cases} 0, & -\infty < t < -1, \\ -1, & -1 < t < 0, \\ 1, & 0 < t < 1, \\ 0, & 1 < t < \infty. \end{cases}$$

5. Express
$$f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & x > \pi, \end{cases}$$
as Fourier sine integral and hence evaluate
$$\int_0^{\infty} \frac{1 - \cos(\pi\sigma)}{\sigma} \sin(\pi\sigma) d\sigma.$$

6. Find the Fourier transforms of
$$f(t) = \begin{cases} e^{-3t}, & t > 0, \\ e^{2t}, & t < 0. \end{cases}, \quad f(t) = \begin{cases} t^n e^{-2t}, & t > 0, \\ 0, & t < 0, \end{cases} \quad n > -1$$

7. Find the Fourier sine and cosine transforms of $f(t) = te^{-t}, t > 0$.

8. If $U(x,t)$ is the temperature at time t and α the thermal diffusivity of a semi-infinite metal bar, find the temperature distribution in the bar at any point at any subsequent time by solving the following boundary value problem using Fourier transform:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \alpha \frac{\partial^2 U}{\partial x^2}, \quad x > 0,\ t > 0 \\ \frac{\partial U}{\partial x}(0,t) &= 0, \quad U(x,0) = f(x) = \begin{cases} 0, & x < 0, \\ 1, & 0 < x < 1, \\ 0, & x > 1. \end{cases} . \end{aligned}$$

9. Find the Laplace transforms of

$$\text{(i) } te^{3t}\cos 4t, \text{(ii) } t\int_0^t e^{-3t}\sin 2t\, dt, \text{(iii) } \int_0^t \frac{e^{-3t}\sin 2t}{t}\, dt.$$

10. Find the Laplace transform of the following unit step functions:
(i) $2H(\sin \pi t) - 1$, (ii) $H(t^3 - 6t^2 + 11t - 6)$.

11. Find the inverse Laplace transforms:
(i) $\dfrac{2s+3}{s^2+4s+6}$, (ii) $\dfrac{2s^2-3s+5}{s^2(s^2+1)}$.

12. Using convolution theorem, find the following:
(i) $\mathcal{L}^{-1}\left\{\dfrac{s^2+4s+4}{(s^2+4s+13)^2}\right\}$, (ii) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2(s+2)^2}\right\}$.

13. Solve the following ODEs using Laplace transform for $t \in [0,\infty)$:
(i) $\ddot{y} + 2\dot{y} + 5y = e^{-t}\sin t,\ y(0) = 0,\ \dot{y}(0) = 1$, (ii) $t\ddot{y} + 2\dot{y} + ty = 0,\ y(0) = 1$.

14. Solve the following IBVP for one dimensional heat conduction equation for a rod of unit length and with unit diffusivity using Laplace transform:

$$\begin{aligned} U_t &= U_{xx},\ 0 < x < 1, t > 0, \\ U(x,0) &= 3\sin(2\pi x),\ 0 < x < 1, \\ U(0,t) &= 0 = U(1,t),\ t > 0. \end{aligned}$$