MA201 Mathematics III
Solutions to Complex Analysis Tutorial 05

Application of Residues in Summing the Series

45. Suppose that f is analytic on the plane except for poles $w_1, w_2, \cdots, w_N$, none of which are integers, and suppose that $\lim_{z\to\infty} |zf(z)| = 0$.

    Then we have $\displaystyle\sum_{n=-\infty}^{\infty} f(n) = -\sum_{j=1}^{N} \text{Res}\,(f(z)\pi\cot(\pi z);\ w_j)$. Using it find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+a^2}$ where a is chosen such that none of the denominators vanish.

    Answer: Let $f(z) = \frac{1}{z^2+a^2}$ where a is not an integer.
    Let $g(z) = f(z)\pi\cot(\pi z)$. Then

    $$\sum_{n=-\infty}^{\infty} f(n) = \frac{1}{a^2} + 2\sum_{n=1}^{\infty} \frac{1}{n^2+a^2}\ .$$

    f has a simple pole at $z = \pm ia$. Then, the residue of g at $z = ia$ is equal to $\frac{\pi\cot(\pi ia)}{2ia} = \frac{(-1)\pi\coth(\pi a)}{2a}$ and the residue of g at $z = -ia$ is equal to $\frac{\pi\cot(\pi ia)}{2ia} = \frac{(-1)\pi\coth(\pi a)}{2a}$.

    $$\frac{1}{a^2} + 2\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = (-1)\left(\frac{(-1)\pi\coth(\pi a)}{a}\right)\ .$$

    This gives that

    $$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \left(\frac{1}{2}\right)\left(\frac{\pi\coth(\pi a)}{a} - \frac{1}{a^2}\right)\ .$$

Argument Principle and Rouche's Theorem

46. Let C denote the unit circle $|z| = 1$, described in the positive sense. Determine the change in the argument of $f(z)$ as z describes C once if $f(z) = (z^3+2)/z$.
    Answer:
    The function $f(z) = (z^3+2)/z$ has a simple pole at $z = 0$ and has no zeros in $|z| < 1$.

    $$\text{Change in the argument of } f = \Delta_C \arg\,(f(z)) = 2\pi(N-P)$$

    where N is the number of zeros and P is the number of poles of f inside C (counting to its multiplicities).
    So, $\Delta_C \arg\,(f(z)) = 2\pi(0-1) = -2\pi$.
    The image curve Γ winds around the origin once in the counterclockwise direction in the w-plane.

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47. Using Rouche's theorem, find the number of roots of the equation $z^9 - 2z^6 + z^2 - 8z - 2 = 0$ lying in $|z| < 1$.

Answer:
Rouche's Theorem: Suppose that (i) two functions f and g are analytic inside and on a simple closed contour C and (ii) $|g(z)| < |f(z)|$ at each point on the contour C. Then the function f and $f+g$ have the same number of zeros, counting multiplicities, inside the contour C.
Set $g(z) = z^9 - 2z^6 + z^2 - 2$, $f(z) = -8z$ and $P(z) = z^9 - 2z^6 + z^2 - 8z - 2$.
Observe that

$$\begin{aligned} |g(z)| = |z^9 - 2z^6 + z^2 - 2| &\leq |z|^9 + 2|z|^6 + |z|^2 + 2 \leq 6 \qquad \text{for } |z| = 1 \\ |f(z)| = |-8z| &= 8|z| = 8 \qquad \text{for } |z| = 1 \\ |g(z)| \leq 6 &< 8 = |f(z)| \qquad \text{on } |z| = 1 \end{aligned}$$

By the Rouche's theorem, the function f and $f + g \equiv P$ have same number of zeros inside $|z| = 1$. Since f has only a simple zero at $z = 0$ inside $|z| = 1$, the function $f + g \equiv P$ has only one zero inside $|z| = 1$. Therefore, the equation $P(z) = 0$ has only one root in $|z| < 1$.

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48. How many roots of the equation $z^4 - 5z + 1 = 0$ are situated in the domain $|z| < 1$? In the annulus $1 < |z| < 2$?
Answer:
In the domain $|z| < 1$:
Set $g(z) = z^4 + 1$, $f(z) = -5z$ and $P(z) = z^4 - 5z + 1$.
Observe that

$$\begin{aligned} |g(z)| = |z^4 + 1| &\leq |z|^4 + 1 \leq 2 \qquad \text{for } |z| = 1 \\ |f(z)| = |-5z| &= 5|z| = 5 \qquad \text{for } |z| = 1 \\ |g(z)| \leq 2 &< 5 = |f(z)| \qquad \text{on } |z| = 1 \end{aligned}$$

By the Rouche's theorem, the function f and $f + g \equiv P$ have same number of zeros inside $|z| = 1$. Since f has only a simple zero at $z = 0$ inside $|z| = 1$, the function $f + g \equiv P$ has only one zero inside $|z| = 1$. Therefore, the equation $P(z) = 0$ has only one root in $|z| < 1$.

In the domain $|z| < 2$:
Set $g(z) = -5z + 1$, $f(z) = z^4$ and $P(z) = z^4 - 5z + 1$.
Observe that

$$\begin{aligned} |g(z)| = |-5z + 1| &\leq 5|z| + 1 \leq 11 \qquad \text{for } |z| = 2 \\ |f(z)| = |z^4| &= |z|^4 = 16 \qquad \text{for } |z| = 2 \\ |g(z)| \leq 11 &< 16 = |f(z)| \qquad \text{on } |z| = 2 \end{aligned}$$

By the Rouche's theorem, the function f and $f + g \equiv P$ have same number of zeros inside $|z| = 2$. Since f has only a zero of order 4 at $z = 0$ inside $|z| = 2$, the function $f + g \equiv P$ has four zeros inside $|z| = 2$. Therefore, the equation $P(z) = 0$ has four roots in $|z| < 2$.

In the domain $1 < |z| < 2$:
The equation $P(z) = 0$ has 4 roots in $|z| < 2$ and it has 1 root in $|z| < 1$. Therefore, we conclude that the equation $P(z) = 0$ has 3 roots in the domain $1 < |z| < 2$.
