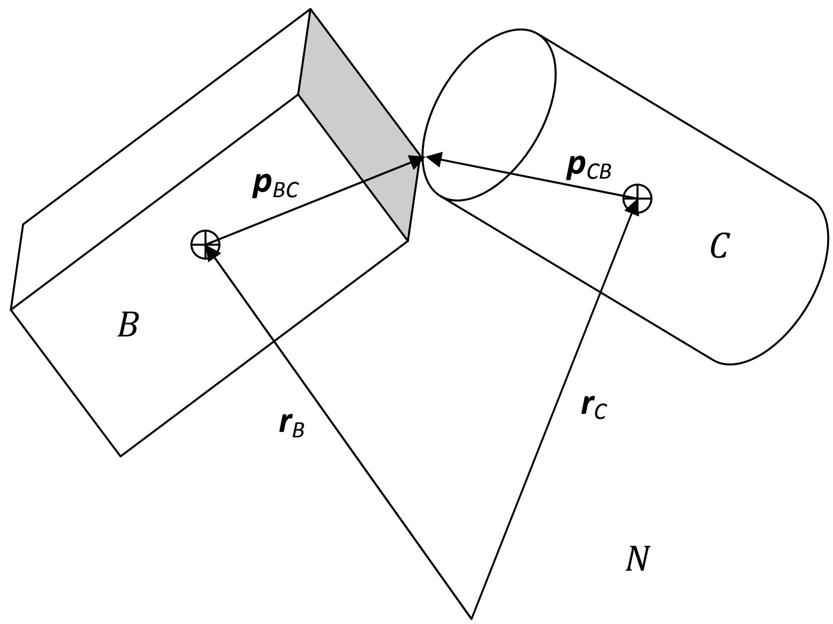
Ball joint (spherical joint) constraint expression: a point fixed on body *B* must be at the same location in inertial space as a point fixed on body *C*:



In vector notation, this is $\boldsymbol{r}_B + \boldsymbol{p}_{BC} = \boldsymbol{r}_C + \boldsymbol{p}_{CB}$ or $(\boldsymbol{r}_B + \boldsymbol{p}_{BC}) - (\boldsymbol{r}_C + \boldsymbol{p}_{CB}) = 0$. For the Udwadia-Kalaba form of the constraint, we need to take two derivatives of this expression. The first derivative (in the *N* frame) is:

$$\frac{{}^N d}{dt}\big((\boldsymbol{r}_B + \boldsymbol{p}_{BC}) - (\boldsymbol{r}_C + \boldsymbol{p}_{CB}) = 0\big)$$
$$\frac{{}^N d}{dt}(\boldsymbol{r}_B) + \frac{{}^B d}{dt}\boldsymbol{p}_{BC} + \boldsymbol{\omega}^{B/N} \times \boldsymbol{p}_{BC} \; - \frac{{}^N d}{dt}\boldsymbol{r}_C - \frac{{}^C d}{dt}\boldsymbol{p}_{CB} - \boldsymbol{\omega}^{C/N} \times \boldsymbol{p}_{CB} = 0$$

Since $\boldsymbol{p}_{BC}$ and $\boldsymbol{p}_{CB}$ are fixed in the frames B and C, respectively,

$$\frac{{}^N d}{dt}(\boldsymbol{r}_B) - \frac{{}^N d}{dt}\boldsymbol{r}_C + \boldsymbol{\omega}^{B/N} \times \boldsymbol{p}_{BC} \; - \boldsymbol{\omega}^{C/N} \times \boldsymbol{p}_{CB} = 0$$

The second derivative (in the *N* frame) is:

$$\frac{{}^N d^2}{dt^2}\boldsymbol{r}_B - \frac{{}^N d^2}{dt^2}\boldsymbol{r}_C + \frac{{}^N d}{dt}\big(\boldsymbol{\omega}^{B/N} \times \boldsymbol{p}_{BC}\big) \; - \frac{{}^N d}{dt}\big(\boldsymbol{\omega}^{C/N} \times \boldsymbol{p}_{CB}\big) = 0$$
$$\frac{{}^N d^2}{dt^2}\boldsymbol{r}_B - \frac{{}^N d^2}{dt^2}\boldsymbol{r}_C + \frac{{}^N d}{dt}\big(\boldsymbol{\omega}^{B/N}\big) \times \boldsymbol{p}_{BC} + \boldsymbol{\omega}^{B/N} \times \big(\boldsymbol{\omega}^{B/N} \times \boldsymbol{p}_{BC}\big) - \frac{{}^N d}{dt}\big(\boldsymbol{\omega}^{C/N}\big) \times \boldsymbol{p}_{CB} - \boldsymbol{\omega}^{C/N} \times \big(\boldsymbol{\omega}^{C/N} \times \boldsymbol{p}_{CB}\big)$$
$$= 0$$

Now we need to get things in the form $A\ddot{x} = b$. The QuIRK state vector consists of the center-of-mass position for each body and the quaternion rotation from the body to inertial frame.

Exploiting the relations $\frac{{}^N d}{dt}\big(\boldsymbol{\omega}^{B/N}\big) = \frac{{}^B d}{dt}\big(\boldsymbol{\omega}^{B/N}\big)$ and $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$,

$$\frac{{}^{N}d^2}{dt^2}\boldsymbol{r}_B - \frac{{}^{N}d^2}{dt^2}\boldsymbol{r}_C - \boldsymbol{p}_{BC} \times \frac{{}^{B}d}{dt}\left(\boldsymbol{\omega}^{B/N}\right) + \boldsymbol{p}_{CB} \times \frac{{}^{C}d}{dt}\left(\boldsymbol{\omega}^{C/N}\right) + \boldsymbol{\omega}^{B/N} \times \left(\boldsymbol{\omega}^{B/N} \times \boldsymbol{p}_{BC}\right) - \boldsymbol{\omega}^{C/N} \times \left(\boldsymbol{\omega}^{C/N} \times \boldsymbol{p}_{CB}\right) = 0$$

Now we cast every vector onto the N basis vectors:

$${}^{N}\ddot{r}_B - {}^{N}\ddot{r}_C - \left({}^{N}Q^{B}\,{}^{B}p_{BC}\right)^{\times}\,{}^{N}Q^{B}\,{}^{B}\dot{\omega}^{B/N} + \left({}^{N}Q^{C}\,{}^{C}p_{CB}\right)^{\times}\,{}^{N}Q^{C}\,{}^{C}\dot{\omega}^{C/N} + \left(\left({}^{N}Q^{B}\,{}^{B}\omega^{B/N}\right)^{\times}\right)^{2}\,{}^{N}Q^{B}\,{}^{B}p_{BC} - \left(\left({}^{N}Q^{C}\,{}^{C}\omega^{C/N}\right)^{\times}\right)^{2}\,{}^{N}Q^{C}\,{}^{C}p_{CB} = 0$$

A superscript $^{\times}$ represents the skew-symmetric cross-product matrix form of a vector, and ${}^{N}Q^{B}$ is the direction-cosine matrix that transforms a vector from the B basis to the N basis coordinates.

Now we insert the identities ${}^{B}\omega^{B/N} = T_B\dot{q}_B$ and ${}^{C}\omega^{C/N} = T_C\dot{q}_C$ along with their derivatives:

$${}^{B}\omega^{B/N} = \underbrace{2\left[q_4 I - q_{123}^{\times} \quad -q_{123}\right]}_{T_B}\dot{q}_B$$

$${}^{N}\ddot{r}_B - {}^{N}\ddot{r}_C - \left({}^{N}Q^{B}\,{}^{B}p_{BC}\right)^{\times}\,{}^{N}Q^{B}\left(\dot{T}_B\dot{q}_B + T_B\ddot{q}_B\right) + \left({}^{N}Q^{C}\,{}^{C}p_{CB}\right)^{\times}\,{}^{N}Q^{C}\left(\dot{T}_C\dot{q}_C + T_C\ddot{q}_C\right) + \left(\left({}^{N}Q^{B}T_B\dot{q}_B\right)^{\times}\right)^{2}\,{}^{N}Q^{B}\,{}^{B}p_{BC} - \left(\left({}^{N}Q^{C}T_C\dot{q}_C\right)^{\times}\right)^{2}\,{}^{N}Q^{C}\,{}^{C}p_{CB} = 0$$

Finally, we rearrange terms so that those involving the second derivatives of states are isolated on one side of the expression:

$$\begin{aligned}
{}^{N}\ddot{r}_B - {}^{N}\ddot{r}_C - \left({}^{N}Q^{B}\,{}^{B}p_{BC}\right)^{\times}\,{}^{N}Q^{B}T_B\ddot{q}_B + \left({}^{N}Q^{C}\,{}^{C}p_{CB}\right)^{\times}\,{}^{N}Q^{C}T_C\ddot{q}_C
&= \left({}^{N}Q^{B}\,{}^{B}p_{BC}\right)^{\times}\,{}^{N}Q^{B}\dot{T}_B\dot{q}_B - \left({}^{N}Q^{C}\,{}^{C}p_{CB}\right)^{\times}\,{}^{N}Q^{C}\dot{T}_C\dot{q}_C - \left(\left({}^{N}Q^{B}T_B\dot{q}_B\right)^{\times}\right)^{2}\,{}^{N}Q^{B}\,{}^{B}p_{BC} \\
&\quad + \left(\left({}^{N}Q^{C}T_C\dot{q}_C\right)^{\times}\right)^{2}\,{}^{N}Q^{C}\,{}^{C}p_{CB}
\end{aligned}$$

$$\begin{aligned}
\left[I \quad -I \quad -\left({}^{N}Q^{B}\,{}^{B}p_{BC}\right)^{\times}\,{}^{N}Q^{B}T_B \quad \left({}^{N}Q^{C}\,{}^{C}p_{CB}\right)^{\times}\,{}^{N}Q^{C}T_C\right]\begin{bmatrix} {}^{N}\ddot{r}_B \\ {}^{N}\ddot{r}_C \\ \ddot{q}_B \\ \ddot{q}_C \end{bmatrix}
&= \left[\left({}^{N}Q^{B}\,{}^{B}p_{BC}\right)^{\times}\,{}^{N}Q^{B}\dot{T}_B\dot{q}_B - \left({}^{N}Q^{C}\,{}^{C}p_{CB}\right)^{\times}\,{}^{N}Q^{C}\dot{T}_C\dot{q}_C - \left(\left({}^{N}Q^{B}T_B\dot{q}_B\right)^{\times}\right)^{2}\,{}^{N}Q^{B}\,{}^{B}p_{BC}\right. \\
&\quad \left. + \left(\left({}^{N}Q^{C}T_C\dot{q}_C\right)^{\times}\right)^{2}\,{}^{N}Q^{C}\,{}^{C}p_{CB}\right]
\end{aligned}$$

Now we have the 3×14 and 3×1 matrices:

$$\begin{aligned}
A &= \left[I \quad -I \quad -\left({}^{N}Q^{B}\,{}^{B}p_{BC}\right)^{\times}\,{}^{N}Q^{B}T_B \quad \left({}^{N}Q^{C}\,{}^{C}p_{CB}\right)^{\times}\,{}^{N}Q^{C}T_C\right] \\
b &= \left[\left({}^{N}Q^{B}\,{}^{B}p_{BC}\right)^{\times}\,{}^{N}Q^{B}\dot{T}_B\dot{q}_B - \left({}^{N}Q^{C}\,{}^{C}p_{CB}\right)^{\times}\,{}^{N}Q^{C}\dot{T}_C\dot{q}_C - \left(\left({}^{N}Q^{B}T_B\dot{q}_B\right)^{\times}\right)^{2}\,{}^{N}Q^{B}\,{}^{B}p_{BC}\right. \\
&\quad \left. + \left(\left({}^{N}Q^{C}T_C\dot{q}_C\right)^{\times}\right)^{2}\,{}^{N}Q^{C}\,{}^{C}p_{CB}\right]
\end{aligned}$$