

# walkr

by David Kane, Andy Yao

**Abstract** The **walkr** package samples points using random walks from the intersection of the  $N$  simplex with  $M$  hyperplanes. Mathematically, the sampling space is all vectors  $x$  that satisfy  $Ax = b$ ,  $\sum x = 1$ , and  $x_i \geq 0$ . The sampling algorithms implemented are hit-and-run and Dikin walk, both of which are MCMC (Monte-Carlo Markov Chain) random walks. **walkr** also provide tools to examine and visualize the convergence properties of the random walks.

## Introduction

Consider all possible vectors  $x$  that satisfy the underdetermined matrix equation  $Ax = b$ , such that every component of  $x$  is  $\geq 0$  and sum to 1. How do we generate a diverse sample of such  $x$ 's? The **walkr** package uses MCMC (Monte-Carlo Markov Chain) random walks to generate such a sample.

**walkr** contains two MCMC random walks. Our first random walk is hit-and-run. Hit-and-run is a widely used MCMC sampling method that guarantees uniform sampling asymptotically, but mixes slower and slower as the dimensions of  $A$  increase. Our second random walk is the Dikin Walk. Dikin Walk generates a nearly uniform sample and exhibits much stronger mixing.

**walkr** also provides statistical diagnostics of the mixing and convergence properties of a MCMC random walk.

## Mathematical Background of Sampling Space

In this section, we go through the mathematical background of the space from which we are sampling – the intersection of the  $N$ -simplex and hyperplanes. The reader does not need to read this section in order to use our package or understand the sampling algorithms. However, this section should provide the reader a better sense of what the sample space is geometrically and mathematically.

1)  $Ax = b$  represent a system of  $M$  linear equations with  $N$  variables ( $M \ll N$ ). Hence,  $A$  is a  $M \times N$  matrix ( $M$  variables and  $N$  constraints),  $x$  is a  $N \times 1$  vector, and  $b$  is a  $M \times 1$  vector. Every row in  $A$  represents a hyperplane in  $\mathbb{R}^N$ .

2) The  $N$ -dimensional unit simplex (N-Simplex) is described by:

$$\begin{aligned} x_1 + x_2 + x_3 + \dots + x_N &= 1 \\ x_i &\geq 0 \end{aligned}$$

### Sampling space: simple 3D case

Let's begin with the simplest case – one linear constraint in 3 dimensional space.

$$x_1 + x_3 = 0.5$$

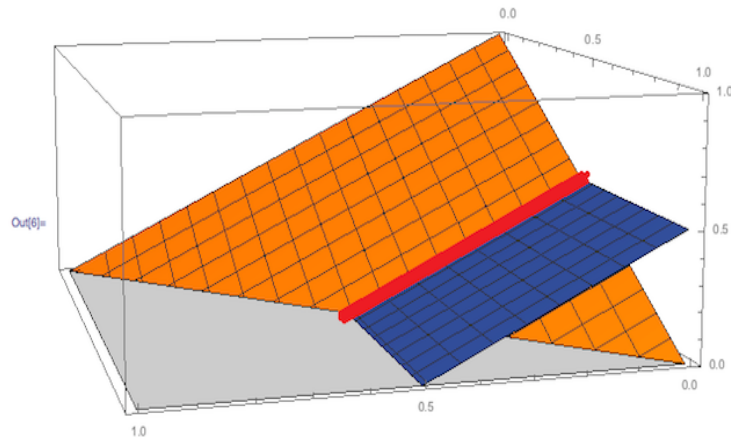
We can express this in terms of matrix equation  $Ax = b$ , where:

$$A = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \quad b = 0.5, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In addition, we require the solution space to be intersected with the 3D simplex:

$$\begin{aligned} \sum x_i &= 1 \\ x_i &\geq 0 \end{aligned}$$

In the following graph, we draw the intersection of the two. The orange equilateral triangle represents the 3D simplex, and the blue rectangle represents the plane  $w_1 + w_3 = 0.5$ . The intersection of the hyperplane (blue) with the simplex (orange) is the red line segment, which is our sampling space.



**Figure 1:** The sampling space is the line segment (red), which is the intersection of the 3D simplex (orange) and a hyperplane (blue)

### Matrix Representation of Multiple Hyperplanes

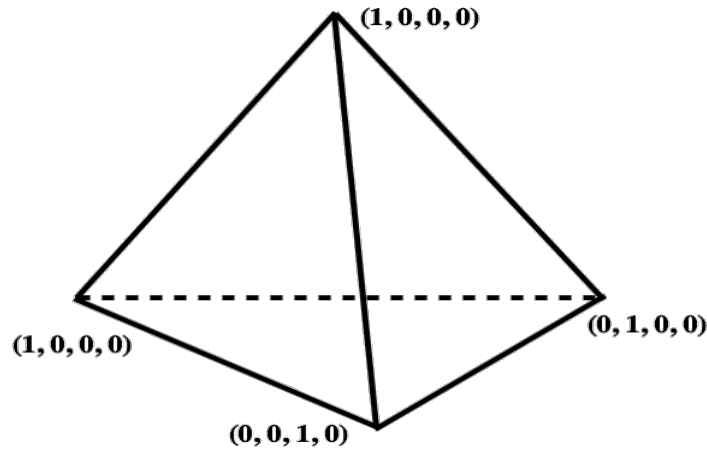
Every hyperplane is described by one linear equation. Thus, a system of linear equations is the intersection of hyperplanes. In general, if we have  $M$  linear equations and  $N$  variables, then  $Ax = b$  would look like:

$$A_{M \times N} = \underbrace{\begin{bmatrix} \dots \end{bmatrix}}_{N \text{ columns (variables)}} \Bigg\} M \text{ rows (constraints)}$$

Again,  $b$  is a  $M \times 1$  vector, and  $x$  is a  $N \times 1$  vector.

### 4D space

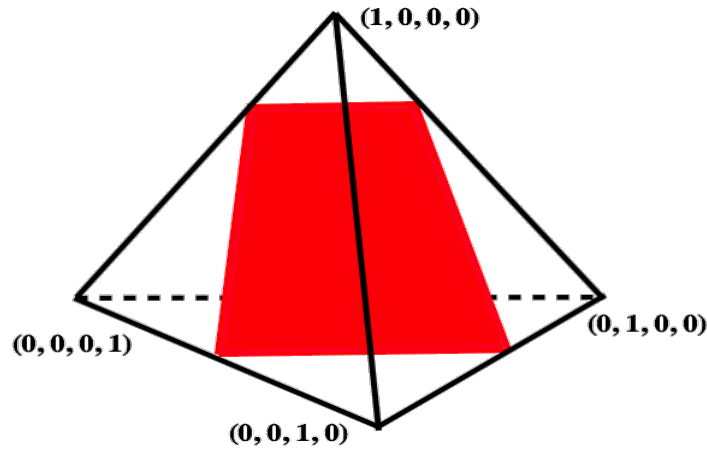
Just like how the 3D simplex is a 2D surface living in 3D space, the 4D simplex (i.e.  $x_1 + x_2 + x_3 + x_4 = 1$ ,  $x_i \geq 0$ ) could be viewed as a 3D object. Specifically, the 4D simplex is the following tetradhedron when viewed from 3D space, with vertices  $(1, 0, 0, 0)$ ,  $(0, 1, 0, 0)$ ,  $(0, 0, 1, 0)$ , and  $(0, 0, 0, 1)$ .



**Figure 2:** 4D Simplex Can Live in 3D Space

Now imagine the intersection of the 4D simplex with one hyperplane in 4D (1 equation, or 1 row in  $Ax = b$ ). For a specific  $A$  and  $b$ , we demonstrate the intersection in the figure below. The resulting shape is a trapezoid in 4D space.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 22 & 2 & 2 & 37 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 16 \end{bmatrix}$$



**Figure 3:** Intersection of a 4D Simplex and a Hyperplane

In higher dimensions, the same logic applies. Each row in  $Ax = b$  is a hyperplane living in  $\mathbb{R}^N$  (given  $N$  variables). Thus, geometrically, our sampling space is: **the intersection of hyperplanes with the  $N$ -simplex**.

#### From $Ax = b$ and the $N$ -simplex to $Ax \leq b$

Our sampling space is bounded (i.e. has finite volume in  $\mathbb{R}^N$ ). More formally, our sampling space is known as a **convex-polytope** in  $\mathbb{R}^N$ . Convex-polytopes are commonly described in the literature by a generic  $Ax \leq b$ . Here, we present a simple linear transformation which transforms the intersection of  $Ax = b$  and the  $N$ -simplex to the form  $Ax \leq b$ .

First, note that the equality part of the simplex constraint ( $\sum x = 1$ ) could be added as an extra row in  $Ax = b$

$$A = \begin{bmatrix} & & \dots & & \\ & & \dots & & \\ & & \dots & & \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} \dots \\ \dots \\ \dots \\ 1 \end{bmatrix}$$

Second, to find the complete solution to the new  $Ax = b$  (i.e. the set of all possible  $x$ 's that satisfy  $Ax = b$ ), we must find the Null Space of  $A$  (all  $x$ 's that satisfy  $Ax = 0$ ), then add on any particular solution to  $Ax = b$  (This procedure can be found in any Linear Algebra textbook).

Mathematically, if the original  $A$  was  $M \times N$ , then after adding on the extra row from the simplex, the basis vectors, each with  $N$  components, which span the Null Space of our new  $A$  will be:

$$\left\{ v_1, v_2, v_3, \dots, v_{N-(M+1)} \right\}$$

Using any particular solution,  $v_{\text{particular}}$ , the complete solution to the new  $Ax = b$  will be

$$\left\{ v_{\text{particular}} + \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \dots + \alpha_{N-(M+1)} v_{N-(M+1)} \mid \alpha_i \in \mathbb{R} \right\}$$

Lastly, we tag on the  $x_i \geq 0$  constraints, and with some algebraic manipulations:

$$v_{\text{particular}} + \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \dots + \alpha_{N-(M+1)} v_{N-(M+1)} \geq \begin{bmatrix} 0 \\ 0 \\ \dots \\ \dots \\ \dots \\ 0 \end{bmatrix}$$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \dots + \alpha_{N-(M+1)} v_{N-(M+1)} \geq -v_{\text{particular}}$$

$$V\alpha \geq -v_{\text{particular}}, \quad \text{where: } V = [v_1 \ v_2 \ \dots \ v_{N-(M+1)}], \quad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_{N-(M+1)} \end{bmatrix}$$

And finally, we arrive at the form  $Ax \leq b$ .

$$-V\alpha \leq v_{\text{particular}}$$

We have performed a **transformation** from "x-space" (coordinates described by  $x_1, x_2, \dots$ ) to "α-space" (coordinates described by  $\alpha_1, \alpha_2, \dots$ ). The geometric object described is still the same convex polytope. In fact, **walkr** internally performs this transformation, samples the  $\alpha$ 's, maps them back to "x-space", and then returns the sampled points.

The user need not be concerned with this transformation affecting the uniformity or mixing properties of our MCMC sampling algorithms. This is because the transformation above is an affine transformation, which preserves uniformity. Simply put, sampling in either space is equivalent.

Having understood that the intersection of  $Ax = b$  with the unit-simplex is a convex polytope, we are ready to dive into the core of **walkr** – MCMC random walks.

## Random Walk: How to pick starting points?

MCMC random walks need a starting point,  $x_0$ , in the interior of the convex polytope. `walkr` generates such starting points using linear programming. Specifically, the `lse1` function of `limSolve` finds  $x$  which:

$$\begin{aligned} &\text{minimizes} && |Cx - d|^2 \\ &\text{subject to} && Ax \leq b \end{aligned}$$

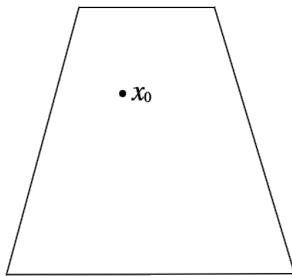
Thus, we randomly generate  $C$  and  $d$  obtaining  $x$  which satisfy  $Ax \leq b$ . We discovered that the  $x$ 's generated this way fall randomly on the boundaries of our convex polytope, due to the minimizing property of linear programming. Thus, we repeat this for say, 10 times, and then take an average of the  $x$ 's generated. This averaged point is  $x_0$ , our starting point.

## Random Walk: Hit-and-run

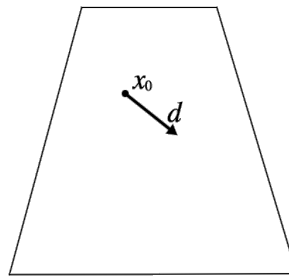
The hit-and-run algorithm is as follows:

1. Set starting point  $x_0$  as current point
2. Randomly generate a direction  $\vec{d}$ . If we are in  $N$  dimensions, then  $d$  will be a vector of  $N$  components. Specifically,  $d$  is a uniformly generated unit vector on the  $N$  dimensional unit-sphere
3. Find the chord  $S$  through  $x_0$  along the directions  $\vec{d}$  and  $-\vec{d}$ . We find end points  $s_1$  and  $s_2$  of the chord by going through the rows of  $Ax_0 \leq b$  one by one, setting the inequality to equality (so we hit the surface). Then, parametrize the chord along  $x_0$  by  $s_1 + t(s_2 - s_1)$ , where  $t \in [0, 1]$
4. Pick a random point  $x_1$  along the chord  $S$  by generating  $t$  from `Uniform[0,1]`
5. Set  $x_1$  as current point
6. Repeat algorithm until number of desired points sampled

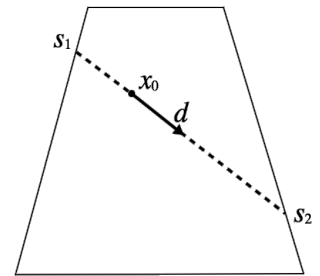
Here is a picture of the hit-and-run algorithm:



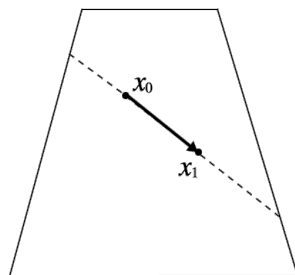
**Figure 4:** Pick  $x_0$



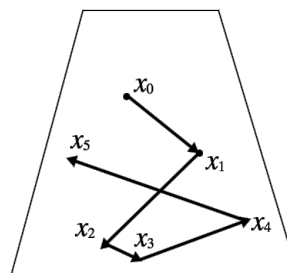
**Figure 5:** Random direction  $d$



**Figure 6:** Find endpoints of chord



**Figure 7:** move to random point  $x_1$  on the chord



**Figure 8:** repeat the algorithm

## Random Walk: Dikin Walk

### Preliminary Definitions

Recall, our sampling space is a convex polytope. We call this convex polytope  $K$ , which can be described in the form  $Ax \leq b$ .

For the definitions below, let  $a_i$  represent a row in  $A$ ,  $x_i$ ,  $b_i$  represent the  $i^{th}$  element of  $x$  and  $b$ . Also recall that  $A$  is a  $M \times N$  matrix.

### Log Barrier Function $\phi$ :

$$\phi(x) = \sum -\log(b_i - a_i^T x)$$

We can compute and simplify the Hessian of the Log Barrier:

### Hessian of Log Barrier $H_x$ :

$$H_x = \nabla^2 \phi(x) = \dots = A^T D^2 A \quad , \quad \text{where:}$$

$$D = \text{diag}\left(\frac{1}{b_i - a_i^T x}\right)$$

**Note:**  $H_x$  is a  $N \times N$  linear operator.  $D$  is a  $M \times M$  diagonal matrix.

### Definition - Dikin Ellipsoid $D_{x_0}^r$

$D_{x_0}^r$ , the Dikin Ellipsoid centered at  $x_0$  with radius  $r$  is defined as:

$$D_{x_0}^r = \{y \mid (y - x)^T H_{x_0} (y - x) \leq r^2\}$$

The shape of the Dikin Ellipsoid with radius  $r$  is a function of  $A$ ,  $b$ , and its center  $x_0$ . Thus, throughout the convex polytope  $K$ , the Dikin Ellipsoid changes shape depending on where the center  $x_0$  is.

### Algorithm Dikin

1. Begin with a point  $x_0 \in K$ . This starting point must be in the polytope.
2. Construct  $D_{x_0}$ , the Dikin Ellipsoid centered at  $x_0$
3. Pick a random point  $y$  from  $D_{x_0}$
4. If  $x_0 \notin D_y$ , then reject  $y$  (be careful, this condition is counter-intuitive)
5. If  $x_0 \in D_y$ , then accept  $y$  with probability  $\min(1, \sqrt{\frac{\det(H_y)}{\det(H_{x_0})}})$  (the big picture is that the ratio of the determinants are equal to the ratio of volumes of the ellipsoids centered at  $x_0$  and  $y$ . Thus, the geometric argument would be that this way the Dikin walk can avoid extreme corners of the region)
6. repeat until obtained number of desired points

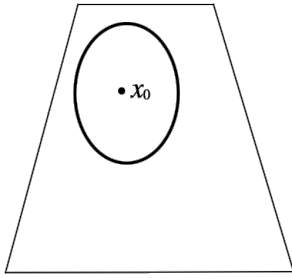


Figure 9: Step 1 and 2

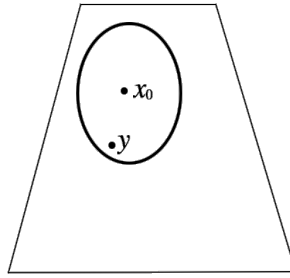


Figure 10: Step 3

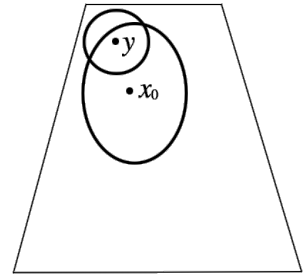


Figure 11: Step 4 Case I

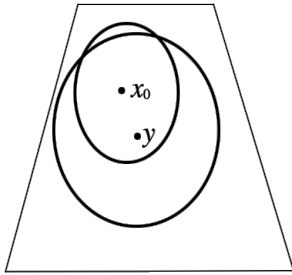


Figure 12: Step 4 Case II

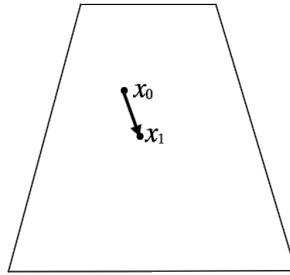


Figure 13: Step 5

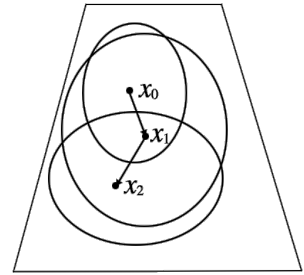


Figure 14: Step 6

### How to pick a random point uniformly from a Dikin Ellipsoid?

Let's say, we now have  $D_x^r$ , the Dikin Ellipsoid centered at  $x$  with radius  $r$ .

1. generate  $\zeta$  from the  $n$  dimensional Standard Gaussian (i.e. `zeta = rnorm(n,0,1)`)
2. normalize  $\zeta$  to be on the  $n$  dimensional ball with radius  $r$ , that is:  

$$\zeta = \langle x_1, x_2, \dots, x_n \rangle \rightarrow \langle \frac{rx_1}{\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}}, \frac{rx_2}{\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}}, \dots, \frac{rx_n}{\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}} \rangle$$
3. Solve for  $d$  in the matrix equation  $H_x d = A^T D \zeta$  (note, as long as  $x_0$  is not on the boundary of our polytope  $K$ ,  $H_x$  will be non-singular, thus,  $d$  will always be unique)
4.  $y = x_0 + d$  is our randomly sampled point from  $D_x^r$

### Important Theorem

In algorithm `Dikin`, what if the point  $y$  we accept is outside of our polytope  $K$ ? Luckily, there is no need to worry about that because of the following theorem:

**Theorem** – If  $x_0 \in K$ , then  $D_{x_0}^1 \subseteq K$ . That is, if our starting point  $x_0$  is in our polytope  $K$ , then the Dikin Ellipsoid centered at  $x_0$  with radius 1 will always be contained in  $K$ .

This is important because this way, we know for sure that if we set  $r = 1$ , then our algorithm will never sample points from outside the polytope  $K$ .

hhh

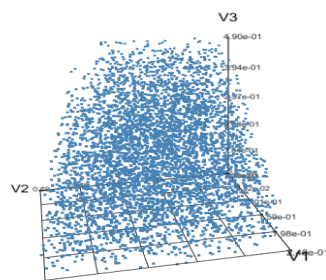
The shape of the Dikin Ellipsoid is a function of  $A$ ,  $b$ , and  $x_0$ . In other words, if we think in terms of running a MCMC chain within our polytope  $K$ , the Dikin Ellipsoid is able to reshape itself accordingly as it surveys through the polytope.

Although there are still two rejection components to the algorithm (see above), the rejection rate is much lower than expected because of this theorem.

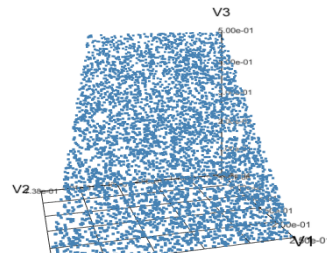
## Dikin versus Hitandrun

	hit-and-run	Dikin Walk
Uniform Sampling	Yes, needs $O(N^3)$ points, where $N$ is the dimension of the polytope	No, concentrates in the interior
Mixing	$O(\frac{N^2 R^2}{r^2})$ *, slows down substantially as dimension of polytope increases and polytope becomes "skinnier"	$O(MN)$ , where $A$ is $M \times N$ ; much stronger mixing.
Cost of One Step	$O(MN)$	$O(MN^2)$ , in practice, one step of Dikin is much more costly than hit-and-run
Rejection Sampling	No	Yes (see probability formula and $x \notin D_y$ ), but rejection rate not high

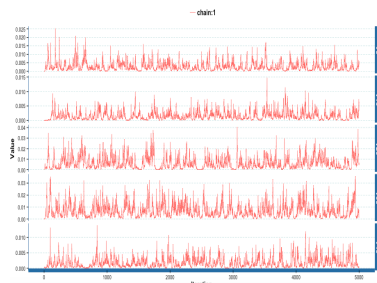
\* $R$  is the radius of the smallest ball that contains the polytope  $K$ .  $r$  is the radius of the largest ball that is contained within the polytope  $K$ . Thus,  $\frac{R}{r}$  increases as the polytope is "skinnier".



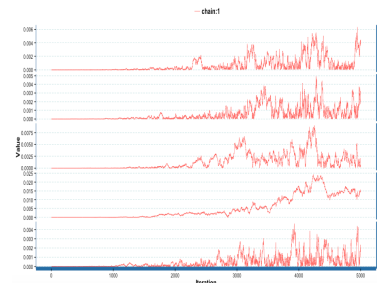
**Figure 15:** Dikin concentrates at the center



**Figure 16:** Hit-and-run samples the space thoroughly



**Figure 17:** Dikin mixes stronger



**Figure 18:** Hit-and-run mixes slower

## Using walkr to sample points

`walkr` has one main function `walkr` which makes it very easy for the user to sample points.

For example, define  $A$  and  $b$  as follows:

```
> A <- matrix(c(1, 0, 1, 0, 1), ncol = 3)
> b <- 0.5
```

Then, the sampling could be simply ran with `walkr`:

```
> ## n is the number of points sampled
> ## method is the sampling method
>
> hitandrun <- walkr(A = A, b = b, n = 1000, method = "hit-and-run")
> dikin <- walkr(A = A, b = b, n = 1000, method = "dikin")
```



```

> optimized_dikin <- walkr(A = A, b = b, n = 1000, method = "optimized-dikin")
> ## see some of the sampled points
>
> hitandrun[ , 10:15]
> dikin[ , 10:15]
> optimized_dikin[ , 10:15]

```

To see the difference in performance between Dikin in R and Optimized Dikin using Rcpp:

```

> A <- matrix(c(1,0,1,0,1), ncol = 5)
> b <- 0.5
> unoptimized <- function() {walkr(A = A, b = b, n = 5000, method = "dikin")}
> optimized <- function() {walkr(A = A, b = b, n = 5000, method = "optimized-dikin")}
> microbenchmark(
+   unoptimized(),
+   optimized(),
+   times = 20
+ )
>

```

## Using walkr to examine MCMC random walks

We could visualize/diagnose the MCMC chains by:

```

> vis_sampling(hitandrun, chains = 1)
> optimized_dikin(hitandrun, chains = 1)

```

## Conclusion

## Authors

David Kane  
 Managing Director  
 Hutchin Hill Capital  
 101 Federal Street, Boston, USA  
[dave.kane@gmail.com](mailto:dave.kane@gmail.com)

Andy Yao  
 Mathematics and Physics  
 Williams College  
 Williamstown, MA, USA  
[ay3@williams.edu](mailto:ay3@williams.edu)

## Bibliography

D. Bates and D. Eddelbuettel. Fast and elegant numerical linear algebra using the RcppEigen package. *Journal of Statistical Software*, 52(5):1–24, 2013. URL <http://www.jstatsoft.org/v52/i05/>. [p9]

R. Kannan and H. Narayanan. Random Walks on Polytopes and an Affine Interior Point Method for Linear Programming. *Mathematics of Operations Research*. [p9]

G. van Valkenhoef and T. Tervonen. *hitandrun: "Hit and Run" and "Shake and Bake" for Sampling Uniformly from Convex Shapes*. CRAN. [p9]

Kannan and Narayanan van Valkenhoef and Tervonen Bates and Eddelbuettel (2013)