Recurrence Relations

Lecture#7

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Introduction

- Here, we will discuss how recursive techniques can derive sequences and be used for solving counting problems.

- The procedure for finding the terms of a sequence in a recursive manner is called **recurrence relation**.

- We study the theory of linear recurrence relations and their solutions.

- Finally, we introduce **generating functions** for solving recurrence relations.

Definition

- A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous terms (Expressing F_n as some combination of F_i with $i < n$).

- **Example** –

Fibonacci series – $F_n = F_{n-1} + F_{n-2}$; $F_1 = F_2 = 1$

Tower of Hanoi – $F_n = 2 F_{n-1} + 1$; $F_1 = 1$

Linear Recurrence Relations

- A linear recurrence relation with constant coefficients is of the form:

  $$C_0a_n + C_1a_{n-1} + C_2a_{n-2} + \ldots\ldots\ldots + C_ka_{n-k} = f(n)$$

  where C_i is a constant and $C_k \neq 0$

- It is of **order k** because a_n can be represented in terms of previous k elements of the sequence, so the order of a recurrence relation is **difference of highest and lowest subscript of a_i** .

Degree of Recurrence Relation

- The **degree** of a recurrence relation is defined as the highest power of **a_i's.**

- A recurrence relation of degree 1 is called as **linear recurrence relation**, otherwise non-linear recurrence relations.

- In linear recurrence relation, every subscripted term **a_i**'s occurs to first power, i.e., it does not contain terms like **a_n^2 , a_n^3, $a_n.a_{n-1}$** etc.

Homogeneous Recurrence Relation

- $C_0a_n + C_1a_{n-1} + C_2a_{n-2} + \ldots\ldots\ldots + C_ka_{n-k} = f(n)$

  If f(n) is identically zero, it is called as **homogeneous** recurrence relation, otherwise non- homogeneous recurrence relation.

- $a_n = 2a_{n-1} + 3a_{n-2}$ is a linear homogeneous recurrence relation of order 2 and degree 1.

- $a_n = a_{n-1} + n + 5$ is a linear non-homogeneous recurrence relation of order 1 and degree 1.

Methods of solving RR

- Iteration
- Characteristic Roots
- Generating Functions

Iteration Method

- Solve the recurrence relation $\mathbf{a_n = a_{n-1} + 2,}$ $n \geq 2$ subject to initial condition $a_1 = 3$.
- Given $a_n = a_{n-1} + 2$ ....(1)

  then $a_{n-1} = a_{n-2} + 2$

  from(1) $a_n = (a_{n-2} + 2) + 2 = a_{n-2} + 2*2$ .....(2)

  also $a_{n-2} = a_{n-3} + 2$

  from(2) $a_n = (a_{n-3} + 2) + 2*2 = a_{n-3} + 3*2$

  In general, $a_n = a_{n-k} + k*2$

  put k = n-1; $a_n = a_1 + (n - 1) * 2 = 3 + 2n – 2$

therefore, $\mathbf{a_n = 2n + 1}$

Characteristic Roots Method

- In this method, we assume that solution to homogeneous recurrence relation $\mathbf{C_0a_n + C_1a_{n-1} + C_2a_{n-2} + \ldots\ldots\ldots + C_ka_{n-k} = 0}$ is of the form $\mathbf{a_n = Ar^n}$.

  Then, putting this in given recurrence relation

  $C_0 Ar^n + C_1 Ar^{n-1} + C_2 Ar^{n-2} + \ldots\ldots\ldots + C_k Ar^{n-k} = 0$

  or, $Ar^{n-k} [C_0 r^k + C_1 r^{k-1} + C_2 r^{k-2} + \ldots\ldots\ldots + C_k] = 0$

  or, $\mathbf{C_0 r^k + C_1 r^{k-1} + C_2 r^{k-2} + \ldots\ldots\ldots + C_k = 0}$

  This is called as characteristic equation of the recurrence relation and solutions to this are called as characteristic roots.

Characteristic Roots Method

- A characteristic equation of degree k has k characteristic roots.

- If these roots (e.g. r_1, r_2, r_3, ……, r_k) are all distinct and real, then general form of the solutions for homogeneous recurrence relation is:

  $$a_n = A_1\, r_1^{\,n} + A_2\, r_2^{\,n} + A_3\, r_3^{\,n} + \ldots\ldots\ldots + A_k\, r_k^{\,n}$$

  where A_1, A_2, A_3, ……., A_k are constants which may be chosen to satisfy any initial conditions.

Solve the recurrence relation $a_n = a_{n-1} + 2\, a_{n-2}$, $n \geq 2$ with the initial conditions $a_0 = 0$, $a_1 = 1$.

Given $a_n = a_{n-1} + 2\, a_{n-2}$

i.e. $a_n - a_{n-1} - 2\, a_{n-2} = 0$

Then the characteristic eqn. is **$r^2 - r - 2 = 0$**

or, $(r - 2)(r + 1) = 0$

or, $r = 2, -1$

Therefore, the general solution is

$a_n = A_1 (2)^n + A_2 (-1)^n$

Given $a_0 = 0$ then $A_1 + A_2 = 0$ .....(1)

and $a_1 = 1$ then $2A_1 - A_2 = 1$ .....(2)

Solving (1) and (2) we get $A_1 = ⅓$ and $A_2 = -⅓$

Hence the solution to the given homogeneous RR is:

$a_n = ⅓ (2)^n - ⅓ (-1)^n$

Solve the recurrence relation $a_n - 4a_{n-1} + 4 a_{n-2} = 0$, $n \geq 2$ with the initial conditions $a_0 = a_1 = 1$.

Given $a_n - 4a_{n-1} + 4 a_{n-2} = 0$

Then the characteristic eqn. is **$r^2 - 4r + 4 = 0$**

or, $(r - 2)(r - 2) = 0$

or, $r = 2, 2$

Therefore, the general solution is

$a_n = A_1 (2)^n + n A_2 (2)^n$

Given $a_0 = 1$ then $A_1 = 1$ .....(1)

and $a_1 = 1$ then $2A_1 + 2A_2 = 1$ .....(2)

Solving (1) and (2) we get $A_1 = 1$ and $A_2 = -½$

Hence the solution to the given homogeneous RR is:

$a_n = (2)^n - ½ n (2)^n = (1 - ½ n) 2^n$

Solve the homogeneous recurrence relation
$a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$

The characteristic eqn. is

$$\mathbf{r^3 - 8r^2 + 21r - 18 = 0}$$

or, $r^3 - 2r^2 - 6r^2 + 12r + 9r - 18 = 0$

or, $r^2(r - 2) - 6r(r - 2) + 9(r - 2) = 0$

or, $(r - 2)(r^2 - 6r + 9) = 0$

or, $(r - 2)(r - 3)^2 = 0$

So, $r = 2, 3, 3$

Hence the solution to the homogeneous RR is:

$$\mathbf{a_n = A_1(2)^n + A_2(3)^n + nA_3(3)^n}$$

Non Homogeneous Recurrence Relations

- $C_0a_n + C_1a_{n-1} + C_2a_{n-2} + \ldots\ldots\ldots + C_ka_{n-k} = f(n)$

  If $f(n) \neq 0$, then it a non-homogeneous linear recurrence relation of order k.

- Its solution $\mathbf{a_n}$ consists of two parts:

- [ ] Homogeneous solution $\mathbf{a_n^{(h)}}$ by keeping $f(n) = 0$.
- [ ] Particular solution $\mathbf{a_n^{(p)}}$ by keeping $f(n)$ on the right hand side.

- The required general solution is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

Case I. When $f(n) = k$, a constant

- Solve $\mathbf{a_{n+2} - 5a_{n+1} + 6a_n = 2}$, with the initial conditions $a_0 = 1$ and $a_1 = -1$.

The characteristic equation for homogeneous part is: $r^2 - 5r + 6 = 0$

or, $r = 2, 3$

So, the homogeneous solution is: $\mathbf{a_n^{(h)} = C_1 2^n + C_2 3^n}$

Let particular solution is of the form $a_n = P$.

Then, $P - 5P + 6P = 2$ or, $P = 1$

So, the particular solution is: $\mathbf{a_n^{(p)} = 1}$

Hence, the general solution is given as,

$a_n = a_n^{(h)} + a_n^{(p)}$ So, $\mathbf{a_n = C_1 2^n + C_2 3^n + 1}$ .....(1)

Case I. When f(n) = k, a constant

- $a_n = C_1 2^n + C_2 3^n + 1$ .....(1)

Now using initial conditions $a_0 = 1$ and $a_1 = -1$, we can evaluate constant terms C_1 and C_2.

Put n = 0 and n = 1 in (1) we get,

$C_1 + C_2 + 1 = 1$ or, $C_1 + C_2 = 0$ ...(2)

$2C_1 + 3C_2 + 1 = -1$ or, $2C_1 + 3C_2 = -2$ ...(3)

By Solving (2) & (3), $C_1 = 2$, $C_2 = -2$

Therefore the general solution is given by,

$$\mathbf{a_n = 2*2^n - 2*3^n + 1}$$

Case II. When $f(n) = A\beta^n$, where β is not a characteristic root

- Solve the recurrence relation $\mathbf{a_n + 6a_{n-1} + 8a_{n-2} = 7*5^n}$

The characteristic equation for homogeneous part is:

$r^2 + 6r + 8 = 0$

or, $(r + 2)(r + 4) = 0$

or, $r = -2, -4$

So, the homogeneous solution is: $\mathbf{a_n^{(h)} = C_1(-2)^n + C_2(-4)^n}$

Case II. When $f(n) = A\beta^n$, where β is not a characteristic root

Let particular solution is of the form $a_n = C*5^n$.

Then, $C5^n + 6C5^{n-1} + 8C5^{n-2} = 7*5^n$

or, $C + (6/5)C + (8/25)C = 7$

or, $(63/25)C = 7$ or, $C = 175/63$

So, the particular solution is: $\mathbf{a_n^{(p)} = (175/63)5^n}$

Hence, the general solution is given as,

$a_n = a_n^{(h)} + a_n^{(p)}$ So, $\mathbf{a_n = C_1(-2)^n + C_2(-4)^n + (175/63)5^n}$

Case III. When $f(n) = A\beta^n$, where β is a characteristic root of multiplicity m

- Solve $a_n - 6a_{n-1} + 9a_{n-2} = 2*3^n$, with the initial conditions $a_0 = 1$ and $a_1 = -3$.

The characteristic equation for homogeneous part is: $r^2 - 6r + 9 = 0$

or, $r = 3, 3$

So, the homogeneous solution is $a_n^{(h)} = C_1 3^n + n\, C_2 3^n$

Let particular solution is of the form $a_n = Cn^2 3^n$.

Then, $Cn^2 3^n - 6C(n-1)^2 3^{n-1} + 9C(n-2)^2 3^{n-2} = 2*3^n$

or, $Cn^2 - 2C(n^2 - 2n + 1) + C(n^2 - 4n + 4) = 2$

or, $-2C + 4C = 2$ or, $C = 1$

So, the particular solution is $a_n^{(p)} = n^2 3^n$

Hence, the general solution is given as,

$a_n = a_n^{(h)} + a_n^{(p)}$ So, $a_n = C_1 3^n + nC_2 3^n + n^2 3^n$ .....(1)

Case III. When $f(n) = A\beta^n$, where β is a characteristic root of multiplicity m

- $a_n = C_1 3^n + nC_2 3^n + n^2 3^n$ .....(1)

  Now using initial conditions $a_0 = 1$ and $a_1 = -3$, we can evaluate constant terms C_1 and C_2.

  Put n = 0 and n = 1 in (1) we get,

  $C_1 = 1$ and $3C_1 + 3C_2 + 3 = -3$

  Solving these we get, $C_1 = 1$, $C_2 = -3$

  Therefore the solution is given by,

  $$\mathbf{a_n = 3^n - 3n3^n + n^2 3^n = (1 - 3n + n^2)\, 3^n}$$

Case IV. When f(n) is a polynomial

- **Solve the recurrence relation $y_{n+2} - y_{n+1} - 6y_n = n^2$.**

The characteristic equation for homogeneous part is:

$$r^2 - r - 6 = 0$$

or, $\quad r = -2, 3$

So, the homogeneous solution is:

$$y_n^{(h)} = C_1(-2)^n + C_2(3)^n$$

Let particular solution is of the form $y_n = C_1 n^2 + C_2 n + C_3$

Therefore,

$$C_1(n+2)^2 + C_2(n+2) + C_3 - [C_1(n+1)^2 + C_2(n+1) + C_3] - 6[C_1 n^2 + C_2 n + C_3] = n^2$$

Case IV. When f(n) is a polynomial

or, $(C_1 - C_1 - 6C_1)n^2 + (4C_1 + C_2 - 2C_1 - C_2 - 6C_2)n + (4C_1 + 2C_2 + C_3 - C_1 - C_2 - C_3 - 6C_3) = n^2 + 0.n + 0$

Comparing the coefficients of n^2, n and constant terms both sides,

$C_1 = -1/6, \quad C_2 = -1/18, \quad C_3 = -5/54$

So, $\mathbf{y_n^{(p)} = (-1/6)\, n^2 - (1/18)\, n - (5/54)}$

Hence, the general solution is given as,

$\mathbf{y_n = C_1\,(-2)^n + C_2\,(3)^n + (-1/6)\, n^2 - (1/18)\, n - (5/54)}$

Next Topic

- Generating Functions