Discrete Mathematics
BCSC0010

Module 1

PERMUTATIONS

(Lecture16)

Introduction

- A **permutation of a set S is an ordered arrangement of the elements** of S.
- In other words, it is a sequence containing every element of S exactly once.
- **Example: Consider the set S = {1, 2, 3}**

The sequence (3; 1; 2) is one permutation of S.

There are 6 different permutations of S. They are:

(1; 2; 3) ; (1; 3; 2) ; (2; 1; 3) ; (2; 3; 1) ; (3; 1; 2) ; (3; 2; 1)

r-Permutation

- An r-**permutation of a set S, is an ordered arrangement (sequence) of** r distinct elements of S.

  (For this to be well-defined, r needs to be an integer with (0 ≤r ≤S).
- **Examples:**

  There is only one 0-permutation of any set: the empty sequence ().

  For the set S = **{1, 2, 3}** the sequence (3; 1) is a 2-permutation.

  (3; 2; 1) is both a permutation and 3-permutation of S (since |S| = 3).

  There are 6 different 2-permutations of S. They are:

  (1; 2) ; (1; 3) ; (2; 1) ; (2; 3) ; (3; 1) ; (3; 2)

r-Permutation

- We usually are interested in the number of such permutations without listing them. The number of permutations of *n objects taken r at a time will be denoted by*

$$P(n, r) \quad \text{(other texts may use } {}_nP_r,\ P_{n,r},\ \text{or } (n)_r\text{)}.$$

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Example

- Find the number *m of permutations of six objects, say, A, B, C, D, E, F, taken three at a time.*
- In other words, find the number of “three-letter words” using only the given six letters without repetition.

Example

Let us represent the general three-letter word by the following three positions:

$$\underline{\qquad},\ \underline{\qquad},\ \underline{\qquad}$$

The first letter can be chosen in 6 ways; following this the second letter can be chosen in 5 ways; and, finally, the third letter can be chosen in 4 ways. Write each number in its appropriate position as follows:

$$\underline{\ 6\ },\ \underline{\ 5\ },\ \underline{\ 4\ }$$

By the Product Rule there are $m = 6 \cdot 5 \cdot 4 = 120$ possible three-letter words without repetition from the six letters. Namely, there are 120 permutations of 6 objects taken 3 at a time. This agrees with the formula in Theorem 5.4:

$$P(6, 3) = 6 \cdot 5 \cdot 4 = 120$$

Permutations with Repetitions

Frequently we want to know the number of permutations of a multi set, that is, a set of objects some of which are alike. We will let

$$P(n;\ n_1,\ n_2,\ \ldots,\ n_r)$$

denote the number of permutations of *n objects of which n1 are alike, n2 are alike, . . ., nr are alike. The general* formula follows:

$$P(n;\ n_1,\ n_2,\ldots,\ n_r) = \frac{n!}{n_1!\,n_2!\ldots n_r!}$$

Example

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- We seek the number of permutations of 7 objects of which 3 are alike (the three *E's), and 2 are alike (the two N's).*

$$m = P(7;\, 3,\, 2) = \frac{7!}{3!2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 420$$

Ordered Samples

- Many problems are concerned with choosing an element from a set *S, say, with n elements. When we choose* one element after another, say, *r times, we call the choice an ordered sample of size r. We consider two cases.*

(1) Sampling with replacement

- Here the element is replaced in the set *S before the next element is chosen. Thus, each time there are n ways* to choose an element (repetitions are allowed).
- The Product rule tells us that the number of such samples is:

$$n \cdot n \cdot n \cdots n \cdot n (r \text{ factors}) = n^r$$

(2) Sampling without replacement

- Here the element is not replaced in the set *S before the next element is chosen. Thus, there is no repetition* in the ordered sample. Such a sample is simply an *r-permutation. Thus the number of such samples is*:

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Example :

- Three cards are chosen one after the other from a 52-card deck. Find the number *m of ways* this can be done:

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  (a) with replacement; (b) without replacement.

(a) Each card can be chosen in 52 ways. Thus $m = 52(52)(52) = 140\,608$.

(b) Here there is no replacement. Thus the first card can be chosen in 52 ways, the second in 51 ways, and the third in 50 ways. Therefore:

$m = P(52, 3) = 52(51)(50) = 132\,600$

Example :

- Find the number *n of distinct permutations that can be formed from all the letters of each word:*

(*a) THOSE; (b) UNUSUAL; (c) SOCIOLOGICAL.*

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(*a*) *THOSE;* (*b*) *UNUSUAL;* (*c*) *SOCIOLOGICAL.*

(*a*) $n = 5! = 120$, since there are 5 letters and no repetitions.

(*b*) $n = \frac{7!}{3!} = 840$, since there are 7 letters of which 3 are U and no other letter is repeated.

(*c*) $n = \frac{12!}{3!2!2!2!}$, since there are 12 letters of which 3 are O, 2 are C, 2 are I, and 2 are L. (We leave the answer using factorials, since the number is very large.)

Examples

- A class contains 8 students. Find the number *n of samples of size 3:*

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- A class contains 8 students. Find the number *n of samples of size 3: (a)With replacement; (b)Without replacement.*

(*a*) Each student in the ordered sample can be chosen in 8 ways; hence, there are

$$n = 8 \cdot 8 \cdot 8 = 8^3 = 512 \text{ samples of size 3 with replacement.}$$

(*b*) The first student in the sample can be chosen in 8 ways, the second in 7 ways, and the last in 6 ways. Thus, there are $n = 8 \cdot 7 \cdot 6 = 336$ samples of size 3 without replacement.

Examples

- Find the number *m of ways that 7 people can arrange themselves:*
  (*a) In a row of chairs; (b) Around a circular table.*

Examples

- Find the number *m of ways that 7 people can arrange themselves: (a) In a row of chairs; (b) Around a circular table.*

(*a*) Here $m = P(7, 7) = 7!$ ways.

(*b*) One person can sit at any place at the table. The other 6 people can arrange themselves in $6!$ ways around the table; that is $m = 6!$.

This is an example of a *circular permutation*. In general, n objects can be arranged in a circle in $(n - 1)!$ ways.

Next Topic

Combinations