COL202 Quiz 3

Aaveg Jain

TOTAL POINTS

5 / 5

QUESTION 1

1 Problem 1 **5 / 5**

✓ **+ 1 pts** ***Correct base case:** True for $$|X| = 1$$ as the graph has only one element.*

✓ **+ 1 pts** ***Correct induction hypothesis:** Every poset of size $$n$$ has a topological sorting.*

Correct inductive step: Using maximal/minimal element to go from poset of size $$n+1$$ to $$n$$.

+ 1 pts Arguing that the maximal/minimal element will be present as the poset is finite.

+ 2 pts Removing the maximal/minimal element from the poset of size $$n+1$$ to move to the inductive hypothesis defined for a poset of size $$n$$.

Correct inductive step: Go from a poset of size $$n$$ to a poset of size $$n+1$$.

✓ **+ 1 pts** *Proving that the new set is still a poset.*

✓ **+ 2 pts** *Proving $$P(n+1)$$ is true.*

+ 1 pts **Partially correct inductive step:** Going from a poset of size $$n$$ to one of size $$n+1$$ does not generalise for an arbitrary poset of size $$n+1$$.

- 0.75 pts Not following the guidelines of writing an induction proof or proofs in general.

Proof by Induction

1. State that the proof uses induction on $$|X|$$.
2. Define an appropriate predicate $$P$$.
3. Prove that $$P(0)$$ is true. This is called the base case
4. Prove that $$P(n)$$ implies $$P(n+1)$$ for every non-negative integer $$n$$. This is called the inductive step.
5. Invoke induction.

+ 0 pts Incorrect or no solution

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Name AAVEG JAIN	Ent. No. 2021CS10073

Important: Answer within the box. Anything written outside the box will be treated as rough work.

Problem 1

A *topological sorting* of a partially ordered set $(X, \preceq)$ is a bijection $f : X \to \{1, \ldots, |X|\}$ such that for all $x, y \in X$, $x \preceq y$ implies $f(x) \leq f(y)$. Prove that every finite poset $(X, \preceq)$ has a topological sorting. Your proof *must* proceed by induction on the size of X. Anything proved in class, in the tut sheets or the texts can be used without being reproved but please mention where you got that fact from.

Pf. by inductⁿ.
~~Proof~~ We prove the foll. predicate: $P(n)$: a poset $(X, \leq)$ with size $|X| = n$ has a topological sorting (t.s), $n \geq 1$.

Base case $n = 1$; ~~trivial~~, let $X = \{x_1\}$; set $f(x_1) = 1$ and we are done.

Indⁿ step: Indⁿ hypothesis: - assume $P(n)$ $n \geq 1$.
i.e $\exists$ a bijectⁿ $f_n : X \to \{1, 2, \ldots n\}$ s.t $\forall$ $x, y \in X$, $x \leq y \Rightarrow f(x) \leq f(y)$.
Consider now a set Y, $|Y| = n+1$; let $Y = \{x_1, x_2, \ldots x_n, x_{n+1}\}$ (all x_i are unique).
$\exists$ bijectⁿ $f : Y \setminus \{x_{n+1}\} \to \{1, 2, \ldots n\}$ which is a t.s by $P(n)$.
We now proceed by cases - Case 1 - x_{n+1} is a minimal el. in Y.
~~Set~~ define $g : Y \to \{1 \ldots n+1\}$; $g(x_{n+1}) = 1$, $g(x_i) = f(x_i) + 1$ $\forall$ $1 \leq i \leq n+1$
~~Second~~ ~~case~~ we now show g is a t.s. consider $x_i, x_j \in X$.
if $i, j > 1$, ~~then~~ and WLOG let $x_i \leq x_j$
$\Rightarrow f(x_i) \leq f(x_j) \Rightarrow g(x_i) \leq g(x_j)$. if $i = 1$ or $j = 1$, then from minimality $x_i \leq x_k$ $\forall$ $k = 2, \ldots n+1$ (WLOG let $j = 1$) and $g(x_i) = 1 \leq g(x_k)$ $\forall$ $k = 2, \ldots n+1$ ($g(x) \geq 1$ $\forall$ x). hence g is a t.s and $P(n+1)$ is T ($\forall$ x_i, x_j : $x_i \leq x_j \Rightarrow g(x_i) \leq g(x_j)$)
Case 2 - x_{n+1} is neither a minimal nor maximal el in Y.
By ~~lemma~~ result done in class, since Y is finite, x_{n+1} has an i.p and i.s (let it be x_i, x_j) i.e $x_i < x_{n+1} < x_j$. define $g : Y \to [n+1]$, $g(x_{n+1}) = f(x_j)$; $g(x_k) = f(x_k)$ $\forall$ $x_k \leq x_i$ and $g(x_k) = f(x_k) + 1$ $\forall$ $x_i \leq x_k$. if $s, t \neq 1$, then trivially $g(x_s) \leq g(x_t)$. ~~...~~ ~~if~~ similarly for all cases we show $x_i \leq x_j \Rightarrow g(x_i) \leq g(x_j)$. proved. ($s < i$, $t = n+1$ and $s(x)$;)
Case 3 - if x_{n+1} is a maximal el., then define $g : Y \to [n+1]$ $t = n+1$
$g(x_i) = f(x_i)$ $\forall$ $i \neq n+1$, $g(x_{n+1}) = n+1$. Similar to case 1 this case is proved.
Hence by ~~the~~ inductⁿ $P(n)$ is T $\forall$ $n \geq 1$.