Q1 Well-behaved numbers

4 Points

Call a complex number x *well-behaved* if there exists a natural number d and there exist $d+1$ integers, $a_0, a_1, \ldots, a_d$, not all zero, such that $\sum_{i=0}^{d} a_i x^i = 0$. Let $W \subseteq \mathbb{C}$ denote the set of all well-behaved complex numbers.

(To get some intuition, observe that every rational number is well-behaved, and so are some irrational real numbers like $2^{1/4}$, and some non-real complex numbers like $-(1/2) + (\sqrt{3}/2)\iota$, etc.)

Is W countable? Prove your answer. You may use the fact that a nonzero polynomial of degree d has at most d roots.

Claim: W is countable.

Proof:
Consider the equation

$$a_o + a_1 x + a_2 x^2 + ... a_d x^d = 0$$

Let A_d denote the set of roots of the above equation. Notice that the above equation is a non-zero polynomial of degree d, so it must have at most d roots. Let us denote the set of roots of the above equation by A_d. Then we have, $|A_d| \leq d$.

Every well-behaved complex number must satisfy at least one such equation by definition. Therefore, we may write the set of well behaved complex numbers as,

$$W = \bigcup_{n=1}^{\infty} A_d$$

Now we know by the result proved in class, a countably infinite union of countable sets is itself countable. Here, A_d are finite for all d and since degrees are natural numbers (which are countable), thus it is a countably infinite union.

Hence, W is countable. ■

Q2 Non-conflicting transportation

6 Points

Call two walks in a graph *conflicting* if there exists an edge in the graph traversed by both of them (possibly in opposite directions). Given a connected graph G and a subset T of its vertices with $|T|$ even, we call a set R of $|T|/2$ paths a *transportation of* T *in* G if the set of endpoints of paths in R is exactly the set T. We call such a transportation R a *non-conflicting transportation of* T *in* G if no two (different) paths in R are conflicting.

Prove or disprove the following statement. For every connected graph G and every even-sized subset T of its vertices, there exists a non-conflicting transportation of T in G.

Claim: Consider the statement,
$P(n)$: For every connected graph G of size m and every even-sized subset T of its vertices of size n, there exists a non-conflicting transportation of T in G.
We will attempt to prove using induction on n.

Base Case: Let $n = 2$. In that case, only one path exists from the two vertices to one another. No two different paths conflict and the claim is hence true.

Let $P(n)$ be true for some even n. Then, consider $P(n + 2)$.
Consider any two connected vertices v1, v2 (i.e. edge exists between v1 and v2) and consider the remaining n vertices.

By induction hypothesis there exist n/2 non conflicting paths amongst the other n vertices. Thus we simply append the path v1v2 to the set of non conflicting paths and we are done.