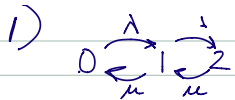


Department of Mathematics
MAL 250/MTL 106 (Probability and Stochastic Processes)
Tutorial Sheet No. 8

1. Consider a $M/M/1$ queuing model. Find the waiting time distribution for any customer in this model. Deduce the mean waiting time from the above distribution.
2. Patients visit a doctor in accordance with a Poisson process at the rate of 8 per hour, and the time doctor takes to examine any patient is exponential with mean 6 minutes. All arriving patients attended by the doctor.
 - (a) Find the probability that the patient has to wait on arrival.
 - (b) Find the expected total time spent (including the service time) by any patient who visits the doctor.
3. Consider a multiplexer that collects traffic formed by messages arriving according to exponentially distributed interarrival times. The multiplexer is formed by a buffer and a transmission line. Assume that, the transmission time of a message is exponentially distributed with the mean value 10 ms. From measurements on the state of the buffer, we know that the idle buffer probability is 0.8.
 - (a) The underlying queueing model is
 - (b) The mean delay (waiting time) for the message is
4. Consider that two identical $M/M/1$ queuing systems with the same rates λ, μ are in operation side by side (with separate queues) in a premises. Find the distribution of the total number, N , in the two systems taken together in long-run.
5. Prove that mean time spent in an $M/M/1$ system having arrival rate λ and service rate 2μ is less than the mean time spent in an $M/M/2$ system with arrival rate λ and each service rate μ .
6. Ms. H. R. Cutt runs a one-person, unisex hair salon. She does not make appointments, but runs the salon on a first-come, first-served basis. She finds that she is extremely busy on Saturday mornings, so she is considering hiring a part-time assistant and even possibly moving to a larger building. Having obtained a master's degree in operations research (OR) prior to embarking upon her career, she elects to analyze the situation carefully before making a decision. She thus keeps careful records for a succession of Saturday mornings and finds that customers seem to arrive according to a Poisson process with a mean arrival rate of 5/hr. Because of her excellent reputation, customers were always willing to wait. The data further showed that customer processing time (aggregated female and male) was exponentially distributed with an average of 10 min. Cutt interested to calculate the following measures:
 - (a) What is the average number of customers in the shop?
 - (b) What is the average number of customers waiting for a haircut?
 - (c) What is the percentage of time an arrival can walk right in without having to wait at all?
 - (d) If waiting room has only four seats at present, what is the probability that a customer, upon arrival, will not able to find a seat and have to stand.
7. City Hospital's eye clinic offers free vision tests every Wednesday evening. There are three ophthalmologists on duty. A test takes, on the average, 20 min, and the actual time is found to be approximately exponentially distributed around this average. Clients arrive according to a Poisson process with a mean of 6/hr, and patients are taken on a first-come, first-served basis. The hospital planners are interested in knowing:
 - (a) What is the average number of people waiting?
 - (b) What is the average amount of time a patient spends at the clinic?
8. Consider New Delhi International Airport. Suppose that, it has three runway. Airplanes have been found to arrive at the rate of 20 per hour. It is estimated that each landing takes 3 minutes. Assume that a Poisson process for arrivals and an exponential distribution for landing times. Without loss of generality, assume that the system is modeled as a birth and death process. What is the steady state probability that the no waiting time to land? What is the expected number of airplanes waiting to land? Find the expected waiting time to land?

9. Consider a telephone switching system consisting of n trunks with an infinite caller population. We assume that an arrival call is lost if all trunks are busy.
- Find the expression for π_n the steady state probability that n trunks are busy.
 - This above system must design the number of trunks, n , in order to guarantee a blocking probability lower than or equal to 2%. The following data are available: The arrival stream is Poisson with rate 6 per minute. Each call has a duration modeled by an exponentially distributed variable with mean value of 3 minutes. The value of n is
10. A service center consists of two servers, each working at an exponential rate of two services per hour. If customers arrive at a Poisson rate of three per hour, then, assuming a system capacity of at most three customers.
- What fraction of potential customers enter the system?
 - What would the value of part (a) be if there was only a single server, and his rate was twice as fast (that is $\mu = 4$) ?
11. Consider an automobile emission inspection station with three inspection stalls, each with room for only one car. It is reasonable to assume that cars wait in such a way that when a stall becomes vacant, the car at the head of the line pulls up to it. The station can accommodate at most four cars waiting (seven in the station) at one time. The arrival pattern is Poisson with a mean of one car every minute during the peak periods. The service time is exponential with mean 6 min. The chief inspector wishes to know the average number in the system during peak periods, the average wait (including service), and the expected number per hour that cannot enter the station because of full capacity.
12. In a parking lot with N spaces the incoming traffic is according to Poisson process with rate λ , but only as long as empty spaces are available. The occupancy times have an exponential distribution with mean $1/\mu$. Let $X(t)$ be the number of occupied parking spaces at time t .
- Determine rate matrix Λ and the forward Kolmogorov equations for the Markov process $\{X(t), t \geq 0\}$.
 - Determine the limiting equilibrium probability distribution of the process.
13. A toll bridge with 10 booths at the entrance can be modeled as a 10 server Markovian queueing system with infinite capacity. Assume that the vehicle arrival follows a Poisson process with parameter 8 per minute and the service times are independent exponential distributed random variables with mean 1 minute.
- Draw the state transition diagram for a birth and death process for the system.
 - Find the limiting state probabilities.
 - If 2 more booths are installed, i.e, total 12 booths, what is the maximum arrival rate such that the limiting state probabilities exist?
14. Consider the central library of IIT Delhi where there are 4 terminals. These terminals can be used to obtain information about the available literature in the library. If all terminals are occupied when someone wants information, then that person will not wait but leave immediately (to look for the required information somewhere else). A user session on a terminal takes exponential distributed time with average 2.5 minutes. Since the number of potential users is large, it is reasonable to assume that users arrive according to a Poisson process. On average 25 users arrive per hour.
- Determine the steady state probability that i terminals are occupied, $i = 0, 1, 2, 3, 4$.
 - What is the mean time spent in the system by any person?
 - How many terminals are required such that at most 5% of the arriving users find all terminals occupied?



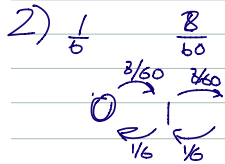
$$p = \frac{\lambda}{\mu}$$

$$\pi_n = (1-p)p^n$$

$$n \rightarrow \int_0^{\infty} (1-p)p^n \frac{\mu^n x^{n-1} e^{-\lambda x}}{\Gamma(n)} dx$$

$$E(N) = E(K) - \frac{1}{\mu} \quad \lambda E(K) = E(N) = \frac{p}{1-p} \Rightarrow E(K) = \frac{1}{\mu - \lambda}$$

$$= \frac{1}{\mu - \lambda} - \frac{1}{\mu}$$



$$p = \frac{4}{5}$$

$$\pi_n = \frac{1}{5} \times \left(\frac{4}{5}\right)^n$$

a) $\frac{1}{5}$

b) $\frac{8}{60} E(K) = E(N)$

$$= \frac{1}{\frac{1}{5} - \frac{8}{60}}$$

$$\Rightarrow E(K) =$$

3) $0.8 = 1-p$

$$p = 0.2$$

$$M/M/1/\infty$$

$$\mu = \frac{1}{10 \text{ ms}} = 0.1$$

$$\lambda = 0.02 / \text{ms}$$

$$E(N) = E(K) - \frac{1}{\mu}$$

$$= \frac{1}{\mu - \lambda} - \frac{1}{\mu}$$

$$= \frac{1}{0.08} - \frac{1}{0.1} \text{ ms}$$

$$=$$

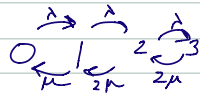
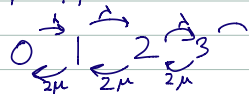
4) $(1-p)p^n$

$$\lambda = 1-p$$

$$n = 2 - 2p$$

$$(2-2p)(1+p)^n$$

5) M|M/1



6) $\lambda = \frac{5}{60}$

$\mu = 1/10$ $\rho = 5/6$

a) 5

b) $\frac{25}{360} = \frac{25}{6}$

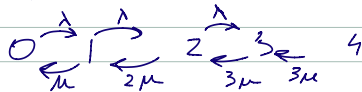
c) $\frac{1}{6}$

d) $1 - (\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4)$

7) $\mu = \frac{1}{20}$

$\lambda = \frac{1}{10}$

M/M/3



$$\pi_0 = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3}{6\mu^3} \left[1 + \frac{\lambda}{3\mu} + \left(\frac{\lambda}{3\mu}\right)^2 + \dots \right]}$$

$$= \frac{1}{1 + \rho + \frac{\rho^2}{2} + \frac{\rho^3}{2}(1-\rho)} = \frac{1}{X}$$

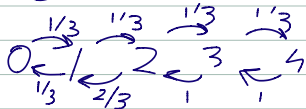
$$\pi_1 = \frac{\rho}{X}$$

$$\pi_2 = \frac{\frac{\rho^2}{2}}{X} \quad \frac{\rho^3}{2} \left[\left(\frac{\rho}{3}\right)^{n-3} \right]$$

$$\pi_n = \frac{\frac{\rho^3}{2} \left(\frac{\rho}{3}\right)^{n-3}}{X}$$

8) $\mu = \frac{1}{3}$

$\lambda = \frac{1}{3}$



$$\pi_0 = \frac{1}{1 + 1 + \frac{1}{2} + \frac{1}{6} \left[1 + \left(\frac{1}{3}\right) + \frac{1}{3^2} + \dots \right]}$$

$$= \frac{1}{2 + \frac{1}{2} + \frac{1}{4}} = \frac{4}{11}$$

$$\pi_1 = \frac{4}{11}$$

$$\pi_2 = \frac{2}{11}$$

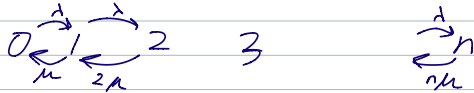
$$a) \left(\frac{10}{11} \right)$$

$$b) \sum_{n=3}^{\infty} (n-3) \pi_n = E(N_2)$$

c) W : Waiting time

$$\lambda E(W) = E(N_2)$$

9) M/M/n/n.



$$\pi_0 = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \dots + \frac{\lambda^n}{n!\mu^n} + \frac{\lambda^n}{n!\mu^n}$$

$$\pi_j = \frac{\frac{\lambda^j}{j!\mu^j}}{\sum_{n=0}^n \frac{\lambda^n}{n!\mu^n}}$$

$$\lambda = 6 \quad \mu = \frac{1}{3}$$

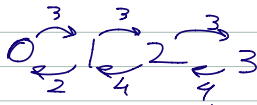
$$\frac{\frac{1}{10} 18^n}{\sum_{j=0}^n \frac{18^j}{j!}} < \frac{2}{100}$$

$$50 \times \left(\frac{18^n}{10} \right) < \sum_{j=0}^n \frac{18^j}{j!}$$

$$\frac{18^n}{10} < \frac{18^{18}}{18!}$$

$n > 18$
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10) $\lambda = 3$
 $\mu = 2$



$$a) \pi_0 = \frac{1}{1 + \frac{3}{2} + \frac{3 \times 3}{4} + \frac{3 \times 3 \times 3}{4 \times 4} + \frac{3 \times 3 \times 3 \times 3}{4 \times 4 \times 4 \times 4}}$$

$$\pi_1 = \pi_0 \frac{3}{2}$$

$$\pi_2 = \pi_0 \frac{3}{2} \frac{3}{4}$$

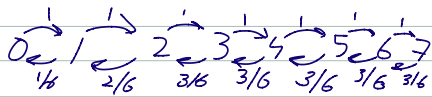
$$\pi_3 = 1 - \pi_0 - \pi_1 - \pi_2$$

$$a) 1 - \pi_3$$

$$11) M/M/3/4$$

$$\lambda = 1$$

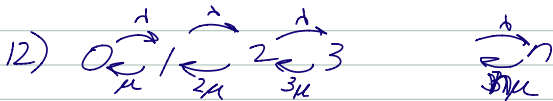
$$\mu = 1/6$$



$$a) \sum_{i=0}^7 \pi_i i = E(N)$$

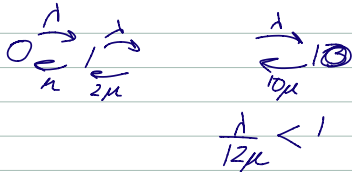
$$b) E(R) = \frac{E(N)}{\lambda}$$

$$c) \lambda \pi_n$$



$$13) \lambda = 8$$

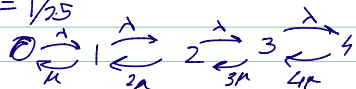
$$\mu = 1$$



$$14) M/M/4/4$$

$$\lambda = 25/60$$

$$\mu = 1/25$$



$$\pi_0 = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3}{6\mu^3} + \frac{\lambda^4}{24\mu^4}}$$

$$a) \pi_i = \frac{\lambda^i}{i! \mu^i} \pi_0 \quad E(N) = \sum_{i=0}^{\infty} \pi_i i$$

$$b) E(N) \times \frac{1}{\lambda_{eff}} \Rightarrow \lambda_{eff} = \lambda(1 - \pi_4)$$

$$c) \pi_n < \frac{5}{100}$$