COL703 Assignment 2

Viraj Agashe

TOTAL POINTS

6.5 / 7

QUESTION 1

1 **0.75 / 1**

+ 0 pts Please see model solutions on Teams before you raise a regrade request.

+ 0 pts Incorrect/Unattempted

✓ **+ 0.25 pts** *Part (a)*

✓ **+ 0.25 pts** *Part (b)*

✓ **+ 0.5 pts** *Part (c) (formula + proof)*

- 0.25 *Point adjustment*

F_3 after Skolemization and conversion to CNF is not ~ B(a), it is forall x, B(x).

QUESTION 2

2 **1 / 1**

✓ **+ 1 pts** *Correct*

+ 0.5 pts Partially Correct

+ 0 pts Incorrect/Unattempted

It should be $$\land$$ instead of $$\lor$$ in case-2, not deducting marks

QUESTION 3

3 **1 / 1**

✓ **+ 0.5 pts** *Correct formulas*

✓ **+ 0.5 pts** *Correct first order resolution*

QUESTION 4

4 **1 / 1**

✓ **- 0 pts** *Correct*

- 1 pts Incorrect/Unattempted

- 0.5 pts Only skolemised/no ground resolution

QUESTION 5

5 **1 / 1**

✓ **+ 1 pts** *Correct*

+ 0.5 pts Partially correct

+ 0 pts Incorrect/Unattempted

QUESTION 6

2 pts

6.1 **0.75 / 1**

- 0 pts Correct

✓ **- 0.25 pts** *Satisfiability of $$F_{n}$$ not proven*

- 1 pts Incorrect

- 1 pts Incomplete

Incomplete proof of at least n elements and satisfiability.

6.2 **1 / 1**

✓ **- 0 pts** *Correct*

- 1 pts Incorrect Model

- 0.5 pts Incorrect proof

- 0.5 pts Incomplete proof

- 1 pts Incomplete

Problem 1
Formalise the following as sentences of first order logic. Use B(x) for "x is a barber", and S(x, y) for "x shaves y".
(a) Every barber shaves all persons who do not shave themselves.
(b) No barber shaves any person who shaves himself.
Convert your answers to Skolem form and use ground resolution to show that (c), given below, is a consequence of (a) and (b).
(c) There are no barbers.

Solution:
(a) $F_1 = \forall x \forall y\, (B(x) \to (\neg S(y,y) \to S(x,y)))$
(b) $F_2 = \forall x \forall y\, (B(x) \to (S(y,y) \to \neg S(x,y)))$
To perform ground resolution, we must convert the matrices of these formulas in CNF. We obtain for the first formula:

$$F_1 = \forall x \forall y\, (\neg B(x) \vee S(y,y) \vee S(x,y))$$

Similarly for the first formula:

$$F_2 = \forall x \forall y\, (\neg B(x) \vee \neg S(y,y) \vee \neg S(x,y))$$

(c) $F_3 = \neg \exists x B(x)$, which after Skolemization and conversion to CNF is simply $\neg B(a)$.
Now, we need to prove that $F_1, F_2 \models F_3$, or $F_1 \wedge F_2 \wedge \neg F_3$ is unsatisfiable. To prove this, let us apply ground resolution. Choosing $x = a$ and $y = a$ in the ground instances of x and y in F_1 and F_2, we get:

1.	$\neg B(a) \vee S(a,a) \vee S(a,a)$	premise
2.	$\neg B(a) \vee \neg S(a,a) \vee \neg S(a,a)$	premise
3.	$B(a)$	premise
4.	$S(a,a)$	resolve(1,3)
5.	$\neg S(a,a)$	resolve(2,3)
6.	$\square$	resolve(5,6)

Problem 2
Fix a signature σ. Consider a relation $\sim$ on σ-assignments that satisfies the following two properties:

- If $A \sim B$ then for every atomic formula F we have $A \models F$ iff $B \models F$.
- If $A \sim B$ then for each variable x we have:
(i) for each $a \in U_A$ there exists $b \in U_B$ such that $A[x \mapsto a] \sim B[x \mapsto b]$
(ii) for all $b \in U_B$ there exists $a \in U_A$ such that $A[x \mapsto a] \sim B[x \mapsto b]$

Prove that if $A \sim B$ then for any formula F, $A \models F$ iff $B \models F$. You may assume that F is built from atomic formulas using the connectives $\vee$ and $\neg$, and the quantifier $\exists$.

Solution: Consider a formula F from the given connectives. We will prove the claim by induction on the depth of the formula, which we define as the number of logical connectives and quantifiers present in the formula.
Base Case: If F has zero depth, it contains zero connectives and quantifiers, it is an atomic formula by definition, for which we already know that $A \models F$ iff $B \models F$.
Inductive Step: Suppose the claim is true for all formulas with depth $< n$. Consider a formula with n depth. Then, structurally, it could be of one of the following forms:

1 **0.75 / 1**

+ 0 pts Please see model solutions on Teams before you raise a regrade request.

+ 0 pts Incorrect/Unattempted

✓ **+ 0.25 pts** *Part (a)*

✓ **+ 0.25 pts** *Part (b)*

✓ **+ 0.5 pts** *Part (c) (formula + proof)*

- 0.25 *Point adjustment*

F_3 after Skolemization and conversion to CNF is not ~ B(a), it is forall x, B(x).

Problem 1
Formalise the following as sentences of first order logic. Use B(x) for "x is a barber", and S(x, y) for "x shaves y".
(a) Every barber shaves all persons who do not shave themselves.
(b) No barber shaves any person who shaves himself.
Convert your answers to Skolem form and use ground resolution to show that (c), given below, is a consequence of (a) and (b).
(c) There are no barbers.

Solution:
(a) $F_1 = \forall x \forall y\,(B(x) \to (\neg S(y,y) \to S(x,y)))$
(b) $F_2 = \forall x \forall y\,(B(x) \to (S(y,y) \to \neg S(x,y)))$
To perform ground resolution, we must convert the matrices of these formulas in CNF. We obtain for the first formula:

$$F_1 = \forall x \forall y\,(\neg B(x) \vee S(y,y) \vee S(x,y))$$

Similarly for the first formula:

$$F_2 = \forall x \forall y\,(\neg B(x) \vee \neg S(y,y) \vee \neg S(x,y))$$

(c) $F_3 = \neg \exists x B(x)$, which after Skolemization and conversion to CNF is simply $\neg B(a)$.
Now, we need to prove that $F_1, F_2 \models F_3$, or $F_1 \wedge F_2 \wedge \neg F_3$ is unsatisfiable. To prove this, let us apply ground resolution. Choosing $x = a$ and $y = a$ in the ground instances of x and y in F_1 and F_2, we get:

1.	$\neg B(a) \vee S(a,a) \vee S(a,a)$	premise
2.	$\neg B(a) \vee \neg S(a,a) \vee \neg S(a,a)$	premise
3.	$B(a)$	premise
4.	$S(a,a)$	resolve(1,3)
5.	$\neg S(a,a)$	resolve(2,3)
6.	$\square$	resolve(5,6)

Problem 2
Fix a signature σ. Consider a relation $\sim$ on σ-assignments that satisfies the following two properties:

- If $A \sim B$ then for every atomic formula F we have $A \models F$ iff $B \models F$.
- If $A \sim B$ then for each variable x we have:
  (i) for each $a \in U_A$ there exists $b \in U_B$ such that $A[x \mapsto a] \sim B[x \mapsto b]$
  (ii) for all $b \in U_B$ there exists $a \in U_A$ such that $A[x \mapsto a] \sim B[x \mapsto b]$

Prove that if $A \sim B$ then for any formula F, $A \models F$ iff $B \models F$. You may assume that F is built from atomic formulas using the connectives $\vee$ and $\neg$, and the quantifier $\exists$.

Solution: Consider a formula F from the given connectives. We will prove the claim by induction on the depth of the formula, which we define as the number of logical connectives and quantifiers present in the formula.
Base Case: If F has zero depth, it contains zero connectives and quantifiers, it is an atomic formula by definition, for which we already know that $A \models F$ iff $B \models F$.
Inductive Step: Suppose the claim is true for all formulas with depth $< n$. Consider a formula with n depth. Then, structurally, it could be of one of the following forms:

- **Case 1:** $F = \neg G$. Then, since G has $n-1$ depth, we have by the inductive hypothesis that since $A \sim B$, $A \models G$ iff $B \models G$. Therefore, $A \not\models G$ iff $B \not\models G$. Furthermore, we know that by the definition of the satisfaction relation, $A \not\models \phi$ iff $A \models \neg\phi$. Therefore, $A \models \neg G$ iff $B \models \neg G$, which is simply $A \models F$ iff $B \models F$, as required.

- **Case 2:** $F = G \vee H$. Then, since G and H have $\leq n-1$ depth, we have by the inductive hypothesis that since $A \sim B$, $A \models G$ iff $B \models G$ and $A \models H$ iff $B \models H$.

  Further, $A \models F \iff A \models G \vee H \iff A \models G$ or $A \models H \iff B \models G$ or $B \models G$ (by inductive hypothesis) $\iff B \models G \vee H \iff B \models F$, as required.

- **Case 3:** $F = \exists x G$.
  We know that $A \models \exists x G$ iff $\exists a \in U_A$ such that $A[x \mapsto a] \models G$. Further, we know that for each $a \in U_A$, exists $b \in U_B$ such that $A[x \mapsto a] \sim B[x \mapsto b]$.

  Therefore, given $A \sim B$, we get, $A \models F \iff A \models \exists x G \iff \exists a \in U_A$ s.t. $A[x \mapsto a] \models G \iff \exists b \in U_B$ s.t. $B[x \mapsto b] \models G$. For this last step, using the inductive hypothesis, we know that since G has $\leq n-1$ depth, and $A[x \mapsto a] \sim B[x \mapsto b]$ we have $A[x \mapsto a] \models \phi$ iff $B[x \mapsto b] \models \phi$.

  It now follows that $\exists b \in U_B$ s.t. $B[x \mapsto b] \models G \iff B \models \exists x G \iff B \models F$.

Hence, we have proved that if for σ-structures A, B, if $A \sim B$ then $A \models F \iff B \models F$ ■.

Problem 3
Express the following by formulas of first-order logic:
(a) Any person is happy if all their children are rich.
(b) All graduates are rich.
(c) Someone is a graduate if they are a child of a graduate.
(d) All graduates are happy.
Use first-order resolution to show that (d) is entailed by (a), (b), and (c). Indicate the substitutions in each resolution step.

Solution:
(a) $F_1 = \forall x\, (\forall y\, (C(x,y) \wedge R(y)) \to H(x))$
(b) $F_2 = \forall x\; G(x) \to R(x)$
(c) $F_3 = \forall x \forall y\; C(x,y) \wedge G(x) \to G(y)$
(b) $F_4 = \forall x\; G(x) \to H(x)$
Now we will apply first order resolution to try to show that (d) is entailed by (a), (b), (c). First, we convert the formulas to CNF. We obtain:

1. $C(x,y) \vee H(x)$
2. $\neg R(y) \vee H(x)$
3. $\neg G(z) \vee R(z)$
4. $\neg C(u,v) \vee \neg G(u) \vee G(v)$
5. $G(w)$
6. $\neg H(w)$

Now, performing resolution:

2 **1 / 1**

✓ **+ 1 pts** *Correct*

+ 0.5 pts Partially Correct

+ 0 pts Incorrect/Unattempted

It should be $\land$ instead of $\lor$ in case-2, not deducting marks

- **Case 1:** $F = \neg G$. Then, since G has $n-1$ depth, we have by the inductive hypothesis that since $A \sim B$, $A \models G$ iff $B \models G$. Therefore, $A \not\models G$ iff $B \not\models G$. Furthermore, we know that by the definition of the satisfaction relation, $A \not\models \phi$ iff $A \models \neg\phi$. Therefore, $A \models \neg G$ iff $B \models \neg G$, which is simply $A \models F$ iff $B \models F$, as required.

- **Case 2:** $F = G \vee H$. Then, since G and H have $\leq n-1$ depth, we have by the inductive hypothesis that since $A \sim B$, $A \models G$ iff $B \models G$ and $A \models H$ iff $B \models H$.

  Further, $A \models F \iff A \models G \vee H \iff A \models G$ or $A \models H \iff B \models G$ or $B \models G$ (by inductive hypothesis) $\iff B \models G \vee H \iff B \models F$, as required.

- **Case 3:** $F = \exists x G$.
  We know that $A \models \exists x G$ iff $\exists a \in U_A$ such that $A[x \mapsto a] \models G$. Further, we know that for each $a \in U_A$, exists $b \in U_B$ such that $A[x \mapsto a] \sim B[x \mapsto b]$.

  Therefore, given $A \sim B$, we get, $A \models F \iff A \models \exists x G \iff \exists a \in U_A$ s.t. $A[x \mapsto a] \models G \iff \exists b \in U_B$ s.t. $B[x \mapsto b] \models G$. For this last step, using the inductive hypothesis, we know that since G has $\leq n-1$ depth, and $A[x \mapsto a] \sim B[x \mapsto b]$ we have $A[x \mapsto a] \models \phi$ iff $B[x \mapsto b] \models \phi$.

  It now follows that $\exists b \in U_B$ s.t. $B[x \mapsto b] \models G \iff B \models \exists x G \iff B \models F$.

Hence, we have proved that if for σ-structures A, B, if $A \sim B$ then $A \models F \iff B \models F$ $\blacksquare$.

Problem 3
Express the following by formulas of first-order logic:
(a) Any person is happy if all their children are rich.
(b) All graduates are rich.
(c) Someone is a graduate if they are a child of a graduate.
(d) All graduates are happy.
Use first-order resolution to show that (d) is entailed by (a), (b), and (c). Indicate the substitutions in each resolution step.

Solution:
(a) $F_1 = \forall x\, (\forall y\, (C(x,y) \wedge R(y)) \rightarrow H(x))$
(b) $F_2 = \forall x\; G(x) \rightarrow R(x)$
(c) $F_3 = \forall x \forall y\; C(x,y) \wedge G(x) \rightarrow G(y)$
(b) $F_4 = \forall x\; G(x) \rightarrow H(x)$
Now we will apply first order resolution to try to show that (d) is entailed by (a), (b), (c). First, we convert the formulas to CNF. We obtain:

1. $C(x,y) \vee H(x)$
2. $\neg R(y) \vee H(x)$
3. $\neg G(z) \vee R(z)$
4. $\neg C(u,v) \vee \neg G(u) \vee G(v)$
5. $G(w)$
6. $\neg H(w)$

Now, performing resolution:

1.	$C(x, y) \lor H(x)$	premise
2.	$\neg R(y) \lor H(x)$	premise
3.	$\neg G(z) \lor R(z)$	premise
4.	$\neg C(u, v) \lor \neg G(u) \lor G(v)$	premise
5.	$G(w)$	premise
6.	$\neg H(w)$	premise
7.	$\neg C(w, v) \lor G(v)$	resolve(4,5) [w/u]
8.	$\neg C(w, v) \lor R(v)$	resolve (7,3) [v/u]
9.	$R(v) \lor H(w)$	resolve(1,8) [w/x][v/y]
10.	$H(w)$	resolve(2,9) [v/y] [w/x]
11.	$\square$	resolve(9,6)

Since we have derived $\square$, therefore it follows that $F_1 \land F_2 \land F_3 \land \neg F_4$ is unsatisfiable, and thus $F_1, F_2, F_3 \models F_4$, as required. ■

Problem 4
Let us consider the sentences that follow. Everyone who loves all animals is loved by someone. Anyone who kills an animal is loved by no one. Ramesh loves all animals. Either Ramesh or Curiosity killed the cat, who is named Molly. Did Curiosity kill Molly? Use resolution to answer this.

Solution:
Let us express the given statements as first order logic formulas. Let the predicate $A(x)$ represent "x is an animal", $C(x)$ represent "x is a cat", $L(x, y)$ express "x loves y" and $K(x, y)$ expresses "x kills y". Then in order of the given statements, we have:

1. $F_1 = \forall x \forall y A(y) \land L(x, y) \rightarrow \exists z L(z, x)$
2. $F_2 = \forall x \forall y A(y) \land K(x, y) \rightarrow \forall z \neg L(z, x)$
3. $F_3 = \forall x A(x) \rightarrow L(\text{Ramesh}, x)$
4. $F_4 = C(\text{Molly})$
5. $F_5 = K(\text{Ramesh}, \text{Molly}) \lor K(\text{Curiosity}, \text{Molly})$

Further, based on our real world knowledge of the fact that cats are animals, we add the clause, $F_6 = \forall x C(x) \rightarrow A(x)$. Finally, we add $F_7 = \neg K(Curiosity, Tuna)$, which is the negation of our query. Now, let us express convert to CNF form. The clauses we obtain are:

1. $A(f(x)) \lor L(g(x), x)$
2. $\neg L(x, f(x)) \lor L(g(x), x)$
3. $\neg L(y, x) \lor \neg A(z) \lor \neg K(x, z)$
4. $\neg A(x) \lor L(Ramesh, x)$
5. $K(Ramesh, Molly) \lor K(Curiosity, Molly)$
6. $C(Molly)$
7. $\neg C(x) \lor A(x)$
8. $\neg K(Curiosity, Molly)$

3 **1 / 1**

✓ **+ 0.5 pts** *Correct formulas*

✓ **+ 0.5 pts** *Correct first order resolution*

1.	$C(x,y) \vee H(x)$	premise
2.	$\neg R(y) \vee H(x)$	premise
3.	$\neg G(z) \vee R(z)$	premise
4.	$\neg C(u,v) \vee \neg G(u) \vee G(v)$	premise
5.	$G(w)$	premise
6.	$\neg H(w)$	premise
7.	$\neg C(w,v) \vee G(v)$	resolve(4,5) [w/u]
8.	$\neg C(w,v) \vee R(v)$	resolve (7,3) [v/u]
9.	$R(v) \vee H(w)$	resolve(1,8) [w/x][v/y]
10.	$H(w)$	resolve(2,9) [v/y] [w/x]
11.	$\square$	resolve(9,6)

Since we have derived $\square$, therefore it follows that $F_1 \wedge F_2 \wedge F_3 \wedge \neg F_4$ is unsatisfiable, and thus $F_1, F_2, F_3 \models F_4$, as required. ■

Problem 4
Let us consider the sentences that follow. Everyone who loves all animals is loved by someone. Anyone who kills an animal is loved by no one. Ramesh loves all animals. Either Ramesh or Curiosity killed the cat, who is named Molly. Did Curiosity kill Molly? Use resolution to answer this.

Solution:
Let us express the given statements as first order logic formulas. Let the predicate $A(x)$ represent "x is an animal", $C(x)$ represent "x is a cat", $L(x,y)$ express "x loves y" and $K(x,y)$ expresses "x kills y". Then in order of the given statements, we have:

1. $F_1 = \forall x \forall y A(y) \wedge L(x,y) \rightarrow \exists z L(z,x)$
2. $F_2 = \forall x \forall y A(y) \wedge K(x,y) \rightarrow \forall z \neg L(z,x)$
3. $F_3 = \forall x A(x) \rightarrow L(\text{Ramesh}, x)$
4. $F_4 = C(\text{Molly})$
5. $F_5 = K(\text{Ramesh}, \text{Molly}) \vee K(\text{Curiosity}, \text{Molly})$

Further, based on our real world knowledge of the fact that cats are animals, we add the clause, $F_6 = \forall x C(x) \rightarrow A(x)$. Finally, we add $F_7 = \neg K(Curiosity, Tuna)$, which is the negation of our query. Now, let us express convert to CNF form. The clauses we obtain are:

1. $A(f(x)) \vee L(g(x), x)$
2. $\neg L(x, f(x)) \vee L(g(x), x)$
3. $\neg L(y,x) \vee \neg A(z) \vee \neg K(x,z)$
4. $\neg A(x) \vee L(Ramesh, x)$
5. $K(Ramesh, Molly) \vee K(Curiosity, Molly)$
6. $C(Molly)$
7. $\neg C(x) \vee A(x)$
8. $\neg K(Curiosity, Molly)$

Now, we resolve:

1.	$A(f(x)) \vee L(g(x), x))$	premise
2.	$\neg L(x, f(x)) \vee L(g(x), x)$	premise
3.	$\neg L(y, x) \vee \neg A(z) \vee \neg K(x, z)$	premise
4.	$\neg A(u) \vee L(Ramesh, u)$	premise
5.	$K(Ramesh, Molly) \vee K(Curiosity, Molly)$	premise
6.	$C(Molly)$	premise
7.	$\neg C(w) \vee A(w)$	premise
8.	$\neg K(Curiosity, Molly)$	premise
9.	$A(Molly)$	resolve(6,7) [Molly/w]
10.	$K(Ramesh, Molly)$	resolve(5,8)
11.	$\neg L(y, x) \vee \neg K(x, Molly)$	resolve(3,9) [Molly/z]
12.	$\neg A(f(Ramesh)) \vee L(g(Ramesh), Ramesh)$	resolve(2,4) [Ramesh/x, f(Ramesh)/u]
13.	$\neg L(y, Ramesh)$	resolve(10, 11) [Ramesh/x]
14.	$L(g(Ramesh), Ramesh)$	resolve(1,12) [Ramesh/x]
15.	$\square$	resolve(12,13) [g(Ramesh)/y]

Therefore, we must have that F_7 is true, i.e. curiosity killed the cat.

Problem 5
Let $A(x_1, \cdots, x_n)$ be a formula with no quantifiers and no function symbols. Prove that $\forall x_1 \cdots \forall x_n A(x_1, \cdots, x_n)$ is satisfiable if and only if it is satisfiable in an interpretation with there being just one element in the universe

Solution:
[$\Longleftarrow$] If there is an interpretation with only one element in the universe under which $\forall x_1 \cdots \forall x_n A(x_1, \cdots, x_n)$ is satisfiable, then it is satisfiable.

[$\Longrightarrow$] Suppose $F = \forall x_1 \cdots \forall x_n A(x_1, \cdots, x_n)$ is satisfiable. Note that F is a closed formula in Skolem form, since A is free from any quantifiers. Then, by the Herbrand theorem, it has a Herbrand model $\mathcal{H}$. Since we consider constants to be function symbols with zero arity, A is free from constants as well. Consider any $a \in U_{\mathcal{H}}$. Then we have $\mathcal{H} \models A(a, \cdots a)$. Now, let us consider a sub-structure $\mathcal{H}'$ of $\mathcal{H}$ which has $U_{\mathcal{H}'} = \{a\}$. Then, since A is quantifier-free, we have $\mathcal{H}' \models A(a, \cdots a)$. Further, since $|U_{\mathcal{H}'}| = 1$, we may also write $\mathcal{H}' \models \forall x_1 \cdots \forall x_n A(x_1, \cdots x_n)$. Therefore, we have that F is satisfiable in an interpretation with there being just one element in the universe.

Problem 6
In this question, we work with first-order logic without equality.
(a) Consider a signature σ containing only a binary relation R. For each positive integer n show that there is a satisfiable σ-formula F_n such that every model A of F_n has at least n elements.
(b) Consider a signature σ containing only unary predicate symbols $P_1, \ldots, P_k$. Using the question 2 (above), or otherwise, show that any satisfiable σ-formula has a model where the universe has at most 2^k elements.

Solution:
(a) Consider the formula F_n to be defined as:

$$F_n = \forall x \neg R(x, x) \wedge (\forall x \forall y \forall z R(x, y) \wedge R(y, z) \rightarrow R(x, z)) \wedge \exists x_1 \exists x_2 \ldots \exists x_n R(x_1, x_2) \ldots R(x_{n-1}, x_n)$$

4 **1 / 1**

✓ **- 0 pts** *Correct*

- 1 pts Incorrect/Unattempted

- 0.5 pts Only skolemised/no ground resolution

Now, we resolve:

1.	$A(f(x)) \vee L(g(x), x))$	premise
2.	$\neg L(x, f(x)) \vee L(g(x), x)$	premise
3.	$\neg L(y, x) \vee \neg A(z) \vee \neg K(x, z)$	premise
4.	$\neg A(u) \vee L(Ramesh, u)$	premise
5.	$K(Ramesh, Molly) \vee K(Curiosity, Molly)$	premise
6.	$C(Molly)$	premise
7.	$\neg C(w) \vee A(w)$	premise
8.	$\neg K(Curiosity, Molly)$	premise
9.	$A(Molly)$	resolve(6,7) [Molly/w]
10.	$K(Ramesh, Molly)$	resolve(5,8)
11.	$\neg L(y, x) \vee \neg K(x, Molly)$	resolve(3,9) [Molly/z]
12.	$\neg A(f(Ramesh)) \vee L(g(Ramesh), Ramesh)$	resolve(2,4) [Ramesh/x, f(Ramesh)/u]
13.	$\neg L(y, Ramesh)$	resolve(10, 11) [Ramesh/x]
14.	$L(g(Ramesh), Ramesh)$	resolve(1,12) [Ramesh/x]
15.	$\square$	resolve(12,13) [g(Ramesh)/y]

Therefore, we must have that F_7 is true, i.e. curiosity killed the cat.

Problem 5
Let $A(x_1, \cdots, x_n)$ be a formula with no quantifiers and no function symbols. Prove that $\forall x_1 \cdots \forall x_n A(x_1, \cdots, x_n)$ is satisfiable if and only if it is satisfiable in an interpretation with there being just one element in the universe

Solution:
[$\Longleftarrow$] If there is an interpretation with only one element in the universe under which $\forall x_1 \cdots \forall x_n A(x_1, \cdots, x_n)$ is satisfiable, then it is satisfiable.

[$\Longrightarrow$] Suppose $F = \forall x_1 \cdots \forall x_n A(x_1, \cdots, x_n)$ is satisfiable. Note that F is a closed formula in Skolem form, since A is free from any quantifiers. Then, by the Herbrand theorem, it has a Herbrand model $\mathcal{H}$. Since we consider constants to be function symbols with zero arity, A is free from constants as well. Consider any $a \in U_{\mathcal{H}}$. Then we have $\mathcal{H} \models A(a, \cdots a)$. Now, let us consider a sub-structure $\mathcal{H}'$ of $\mathcal{H}$ which has $U_{\mathcal{H}'} = \{a\}$. Then, since A is quantifier-free, we have $\mathcal{H}' \models A(a, \cdots a)$. Further, since $|U_{\mathcal{H}'}| = 1$, we may also write $\mathcal{H}' \models \forall x_1 \cdots \forall x_n A(x_1, \cdots x_n)$. Therefore, we have that F is satisfiable in an interpretation with there being just one element in the universe.

Problem 6
In this question, we work with first-order logic without equality.
(a) Consider a signature σ containing only a binary relation R. For each positive integer n show that there is a satisfiable σ-formula F_n such that every model A of F_n has at least n elements.
(b) Consider a signature σ containing only unary predicate symbols $P_1, \ldots, P_k$. Using the question 2 (above), or otherwise, show that any satisfiable σ-formula has a model where the universe has at most 2^k elements.

Solution:
(a) Consider the formula F_n to be defined as:

$$F_n = \forall x \neg R(x, x) \wedge (\forall x \forall y \forall z R(x, y) \wedge R(y, z) \rightarrow R(x, z)) \wedge \exists x_1 \exists x_2 \ldots \exists x_n R(x_1, x_2) \ldots R(x_{n-1}, x_n)$$

5 **1 / 1**

✓ **+ 1 pts** *Correct*

+ 0.5 pts Partially correct

+ 0 pts Incorrect/Unattempted

Now, we resolve:

1.	$A(f(x)) \vee L(g(x), x))$	premise
2.	$\neg L(x, f(x)) \vee L(g(x), x)$	premise
3.	$\neg L(y, x) \vee \neg A(z) \vee \neg K(x, z)$	premise
4.	$\neg A(u) \vee L(Ramesh, u)$	premise
5.	$K(Ramesh, Molly) \vee K(Curiosity, Molly)$	premise
6.	$C(Molly)$	premise
7.	$\neg C(w) \vee A(w)$	premise
8.	$\neg K(Curiosity, Molly)$	premise
9.	$A(Molly)$	resolve(6,7) [Molly/w]
10.	$K(Ramesh, Molly)$	resolve(5,8)
11.	$\neg L(y, x) \vee \neg K(x, Molly)$	resolve(3,9) [Molly/z]
12.	$\neg A(f(Ramesh)) \vee L(g(Ramesh), Ramesh)$	resolve(2,4) [Ramesh/x, f(Ramesh)/u]
13.	$\neg L(y, Ramesh)$	resolve(10, 11) [Ramesh/x]
14.	$L(g(Ramesh), Ramesh)$	resolve(1,12) [Ramesh/x]
15.	$\square$	resolve(12,13) [g(Ramesh)/y]

Therefore, we must have that F_7 is true, i.e. curiosity killed the cat.

Problem 5
Let $A(x_1, \cdots, x_n)$ be a formula with no quantifiers and no function symbols. Prove that $\forall x_1 \cdots \forall x_n A(x_1, \cdots, x_n)$ is satisfiable if and only if it is satisfiable in an interpretation with there being just one element in the universe

Solution:
[$\Longleftarrow$] If there is an interpretation with only one element in the universe under which $\forall x_1 \cdots \forall x_n A(x_1, \cdots, x_n)$ is satisfiable, then it is satisfiable.

[$\Longrightarrow$] Suppose $F = \forall x_1 \cdots \forall x_n A(x_1, \cdots, x_n)$ is satisfiable. Note that F is a closed formula in Skolem form, since A is free from any quantifiers. Then, by the Herbrand theorem, it has a Herbrand model $\mathcal{H}$. Since we consider constants to be function symbols with zero arity, A is free from constants as well. Consider any $a \in U_{\mathcal{H}}$. Then we have $\mathcal{H} \models A(a, \cdots a)$. Now, let us consider a sub-structure $\mathcal{H}'$ of $\mathcal{H}$ which has $U_{\mathcal{H}'} = \{a\}$. Then, since A is quantifier-free, we have $\mathcal{H}' \models A(a, \cdots a)$. Further, since $|U_{\mathcal{H}'}| = 1$, we may also write $\mathcal{H}' \models \forall x_1 \cdots \forall x_n A(x_1, \cdots x_n)$. Therefore, we have that F is satisfiable in an interpretation with there being just one element in the universe.

Problem 6
In this question, we work with first-order logic without equality.
(a) Consider a signature σ containing only a binary relation R. For each positive integer n show that there is a satisfiable σ-formula F_n such that every model A of F_n has at least n elements.
(b) Consider a signature σ containing only unary predicate symbols $P_1, \ldots, P_k$. Using the question 2 (above), or otherwise, show that any satisfiable σ-formula has a model where the universe has at most 2^k elements.

Solution:
(a) Consider the formula F_n to be defined as:

$$F_n = \forall x \neg R(x, x) \wedge (\forall x \forall y \forall z R(x, y) \wedge R(y, z) \rightarrow R(x, z)) \wedge \exists x_1 \exists x_2 \ldots \exists x_n R(x_1, x_2) \ldots R(x_{n-1}, x_n)$$

Using the above formula, we are ensured that any model of F_n requires at least n elements in the domain. The condition $\forall x \neg R(x,x)$ ensures that the relation R is not reflexive. Furthermore, the condition $(\forall x \forall y \forall z R(x,y) \wedge R(y,z) \rightarrow R(x,z))$ ensures that the relation R is transitive.

Claim 1. *If F_n is satisfiable, $x_i = x_j$ iff $i = j$.*

Proof. Suppose this is not true, i.e. $x_i = x_j$ for some $i < j$. We know that $R(x_i, x_{i+1}) \dots R(x_{j-1}, x_j)$ are all true. Therefore, by transitivity, $R(x_i, x_j)$ must be true. But since $x_i = x_j$, this violates the condition of irreflixibility. Therefore, $x_i = x_j$ only for $i = j$.

(b) Consider a formula F on the signature σ such that $A \models F$. We will use the results of question 2 and prove that there also exists an assignment B such that $A \sim B$, and prove that $|U_B| \leq 2^k$.

We define the σ-assignment B as follows:

- **Universe:** To define the universe U_B, let us first define the equivalence relation $\simeq$ over U_A. We consider $c_1 \simeq c_2$ if $\forall\ i\ P_i(c_1) = P_i(c_2)$. Note that the relation $\simeq$ induces at most 2^k equivalence classes on U_A, since each equivalence class C can be represented by k bits, with the i-th bit set to 1 if $P_i(c) = 1$ for $c \in C$, and there can be at most 2^k such bit sequences. Let $g : U_A \rightarrow U_A$ be a function which maps each element of U_A to a fixed element of its equivalence class. The range of the function g is therefore defined to be U_B. Note that $|U_B| \leq 2^k$.
- **Predicates:** Define $P_i^B = \{g(x) | x \in P_i^A\}$, i.e. we map all elements which were in P_i^A to their equivalence class representative.
- **Function Symbols:** For every function symbol f_A, we define f_B such that it maps the inputs $g(A[t_1]), g(A[t_2]), ..., g(A[t_k])$ to $g(f_A(A[t_1], ..., A[t_k]))$
- **Constants and Variable:** For any constant or variable $x \in U_B$, we have it to $B[x] = g(A[x])$.

We argue that any σ-assignments A and B constructed as above preserves satisfiability of F. First, we prove the following claim.

Claim 2. *For any term t, we have $B[t] = g(A[t])$*

Proof. We proceed by induction on the length of terms.

Base Case: If t is a constant symbol or a variable, we have by definition $B[x] = g(A[x])$.

Inductive Step: Consider any k-ary function symbol f. Therefore, we have:

$$B[t] = B[f(x_1, ...x_k)] = f_B(g(A[x_1]), ...g(A[x_k])) = g(f_A(A[x_1], ..., A[x_k])) = g(A[f(x_1, ..., x_k)]) = g(A[t])$$

Therefore, we have, for all terms t, $B[t] = g(A[t])$ are required. Now, we prove the main claim:

Claim 3. *For any σ-assignments A and B constructed as above, $A \models F$ iff $B \models F$*

Proof. We proceed by induction on the depth of the formula, which is defined as the number of quantifiers and logical connectives in the formula, as in Q2.

Base Case: For atomic formulas F, which are of the form $P_i(t)$, we have that $A \models P_i(t) \iff A[t] \in P_i^A \iff g(A[t]) \in P_i^B \iff B[t] \in P_i^B \iff B \models P_i(t)$

Inductive Step: Suppose the claim is true for formulas of depth $< n$. Now, of depth n can be constructed using a connective or a quantifier. We given the case of the existential quantifier: $A \models \exists x F$ iff $A[x \mapsto a] \models F$. Now note that we can construct $B[x \mapsto b]$ from $A[x \mapsto a]$ using the construction given above, with $b = g(a)$, and the depth of the formula F is less than n, therefore using the inductive hypothesis we get $B[x \mapsto b] \models F \iff B \models \exists x F$. We can prove the case of the for-all quantifier as well as logical

6.1 **0.75 / 1**

- **- 0 pts** Correct
- ✓ **- 0.25 pts** *Satisfiability of $$F_{n}$$ not proven*
- **- 1 pts** Incorrect
- **- 1 pts** Incomplete

Incomplete proof of at least n elements and satisfiability.

Using the above formula, we are ensured that any model of F_n requires at least n elements in the domain. The condition $\forall x \neg R(x,x)$ ensures that the relation R is not reflexive. Furthermore, the condition $(\forall x \forall y \forall z R(x,y) \wedge R(y,z) \rightarrow R(x,z))$ ensures that the relation R is transitive.

Claim 1. *If F_n is satisfiable, $x_i = x_j$ iff $i = j$.*

Proof. Suppose this is not true, i.e. $x_i = x_j$ for some $i < j$. We know that $R(x_i, x_{i+1}) \ldots R(x_{j-1}, x_j)$ are all true. Therefore, by transitivity, $R(x_i, x_j)$ must be true. But since $x_i = x_j$, this violates the condition of irreflixibility. Therefore, $x_i = x_j$ only for $i = j$.

(b) Consider a formula F on the signature σ such that $A \models F$. We will use the results of question 2 and prove that there also exists an assignment B such that $A \sim B$, and prove that $|U_B| \leq 2^k$.

We define the σ-assignment B as follows:

- **Universe:** To define the universe U_B, let us first define the equivalence relation $\simeq$ over U_A. We consider $c_1 \simeq c_2$ if $\forall\ i\ P_i(c_1) = P_i(c_2)$. Note that the relation $\simeq$ induces at most 2^k equivalence classes on U_A, since each equivalence class C can be represented by k bits, with the i-th bit set to 1 if $P_i(c) = 1$ for $c \in C$, and there can be at most 2^k such bit sequences. Let $g : U_A \rightarrow U_A$ be a function which maps each element of U_A to a fixed element of its equivalence class. The range of the function g is therefore defined to be U_B. Note that $|U_B| \leq 2^k$.
- **Predicates:** Define $P_i^B = \{g(x) | x \in P_i^A\}$, i.e. we map all elements which were in P_i^A to their equivalence class representative.
- **Function Symbols:** For every function symbol f_A, we define f_B such that it maps the inputs $g(A[t_1]), g(A[t_2]), ..., g(A[t_k])$ to $g(f_A(A[t_1], ..., A[t_k]))$
- **Constants and Variable:** For any constant or variable $x \in U_B$, we have it to $B[x] = g(A[x])$.

We argue that any σ-assignments A and B constructed as above preserves satisfiability of F. First, we prove the following claim.

Claim 2. *For any term t, we have $B[t] = g(A[t])$*

Proof. We proceed by induction on the length of terms.

Base Case: If t is a constant symbol or a variable, we have by definition $B[x] = g(A[x])$.

Inductive Step: Consider any k-ary function symbol f. Therefore, we have:

$$B[t] = B[f(x_1, ...x_k)] = f_B(g(A[x_1]), ...g(A[x_k])) = g(f_A(A[x_1], ..., A[x_k])) = g(A[f(x_1, ..., x_k)]) = g(A[t])$$

Therefore, we have, for all terms t, $B[t] = g(A[t])$ are required. Now, we prove the main claim:

Claim 3. *For any σ-assignments A and B constructed as above, $A \models F$ iff $B \models F$*

Proof. We proceed by induction on the depth of the formula, which is defined as the number of quantifiers and logical connectives in the formula, as in Q2.

Base Case: For atomic formulas F, which are of the form $P_i(t)$, we have that $A \models P_i(t) \iff A[t] \in P_i^A \iff g(A[t]) \in P_i^B \iff B[t] \in P_i^B \iff B \models P_i(t)$

Inductive Step: Suppose the claim is true for formulas of depth $< n$. Now, of depth n can be constructed using a connective or a quantifier. We given the case of the existential quantifier: $A \models \exists x F$ iff $A[x \mapsto a] \models F$. Now note that we can construct $B[x \mapsto b]$ from $A[x \mapsto a]$ using the construction given above, with $b = g(a)$, and the depth of the formula F is less than n, therefore using the inductive hypothesis we get $B[x \mapsto b] \models F \iff B \models \exists x F$. We can prove the case of the for-all quantifier as well as logical connectives similarly.

Therefore, we have proved that the σ-assignment B preserves satisfiability, and it has a universe of size $|U_B| \leq 2^k$, as required. ■.

6.2 **1 / 1**

✓ **- 0 pts** *Correct*

- 1 pts Incorrect Model

- 0.5 pts Incorrect proof

- 0.5 pts Incomplete proof

- 1 pts Incomplete