Department of Mathematics
Tutorial Sheet No. 7
MTL 106 (Introduction to Probability and Stochastic Processes)

1. Two communication satellites are placed in orbit. The lifetime of the the satellite is exponential distribution with mean $\frac{1}{\mu}$. If one fails its replacement is sent up. The time necessary to prepare and send up a replacement is exponential distribution with mean $\frac{1}{\lambda}$. Let $X(t)$ = the number of satellites in the orbit at time t. Assume $\{X(t), t \geq 0\}$ is a Markov process with state space $\{0, 1, 2\}$. Show that the infinitesimal generator matrix is given by

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda + \mu) & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix}$$

Write down the Kolmogorov forward and backward equations for the above process.

2. Find the limiting probabilities of the Markov chain in Problem no. 1. For what proportion of time in a year will the communication system be out of action with no satellites in operation.

3. Consider a machine repair model in which a job shop that consists of M machines and one serviceman. Suppose that the amount of time each machine runs before breaking down is exponentially distributed with mean $1/\lambda$, and suppose that the amount of time that it takes for the serviceman to fix a machine is exponentially distributed with mean $1/\mu$.
   (a) What is the average number of machines not in use?
   (b) What proportion of time is each machine in use?

4. Consider the recent IIT Delhi Open House program. Assume that students from various schools arrive at the reception at the instants of a Poisson process with rate 2. At the reception main door, two program representatives separately explain the program to anybody entering the hall. Each explanation takes a time which is exponential distributed with parameter 1, and is independent of other explanations. After the explanation, the students enters the hall. If both representatives are busy the student goes directly into the hall. Let $X(t)$ be the number of busy representatives at time t. Without loss of generality, assume that the system is modeled as a birth and death process. Write the generator matrix Q. Write the forward Kolmogorov equations for the Markov process $\{X(t), t \geq 0\}$. Derive the equilibrium probability distribution of the process.

5. Consider a service station with two identical computers and two technicians. Assume that when both computers are in good condition, most of the work load is on one computer, exposed to a failure rate $\lambda = 1$, while the other computer's failure rate is $\lambda = 0.5$. Further assume that, if one of the computer fails, the other one takes the full load, thus exposed to a failure rate $\lambda = 2$. Among the technicians, one is with repair rate $\mu = 2$ while the second is with repair rate $\mu = 1$. If both work simultaneously on the same computer, the total repair rate is $\mu = 2.5$. Note that, at any given moment, they work so that repair rate is maximized.

   (a) Determine the infinitesimal generator matrix Q.

   (b) Draw the state transition diagram of the system.

   (c) Determine the steady state probabilities and the system availability.

6. Consider a taxi station where taxis and customers arrive independently in accordance with Poisson processes with respective rates of one and two per minute. A taxi will wait no matter how many other taxis are in the system. Moreover, an arriving customer that does not find a taxi also will wait no matter how many other customers are in the system. Note that a taxi can accommodate ONLY ONE customer by first come first service basis. Define

$$X(t) = \begin{cases} -n & \text{if } n \text{ number of taxis waiting for customers at time } t \\ n & \text{if } n \text{ number of customers waiting for taxis at time } t \end{cases}.$$

   (a) Write the generator matrix Q or draw the state transition diagram for the process $\{X(t), t \geq 0\}$.

(b) Write the forward Kolmogorov equations for the Markov process $\{X(t), t \geq 0\}$.

(c) Is a unique equilibrium probability distribution of the process exist? Justify your answer.

7. The birth and death process $\{X(t), t \geq 0\}$ is said to be a pure death process if $\lambda_i=0$ for all i. Suppose $\mu_i = i\mu$, $i = 1, 2, 3, \ldots$ and initially $X_0 = n$. Show that $X(t)$ has $B(n, p)$ distribution with $p = e^{-\mu t}$.

8. (a) Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. For any $s, t \geq 0$, find
$P(N(t + s) - N(t) = k \mid N(u); 0 \leq u \leq t)$

(b) Let $\{N(t), t \geq 0\}$ be a Poisson process with rate 5. Compute
$P(N(2.5) = 15, N(3.7) = 21, N(4.3) = 21)$.

9. Consider the random telegraph signal, denoted by $X(t)$, jumps between two states, -1 and 1, according to the following rules. At time $t = 0$, the signal $X(t)$ start with equal probability for the two states, i.e., $P(X(0) = -1) = P(X(0) = 1) = 1/2$, and let the switching times be decided by a Poisson process $\{N(t), t \geq 0\}$ with parameter λ independent of $X(0)$. At time t, the signal

$$X(t) = X(0)(-1)^{N(t)}, t > 0.$$

Write the Kolmogorov forward equations for the continuous time Markov chain $\{X(t), t \geq 0\}$. Find the time-dependent probability distribution of $X(t)$ for any time t.

10. Let $\{X(t), t \geq 0\}$ be a pure birth process with $\lambda_n = n\lambda$, $n = 1, 2, \ldots, \lambda_0 = \lambda; \mu_n = 0$, $n = 1, 2, \ldots$. Find the conditional probability that $X(t) = n$ given that $X(0) = i(1 \leq i \leq n)$. Also, find the mean of this conditional distribution.

11. Consider a Poisson process with parameter λ. Let T_1 be the time of occurrence of the first event and let $N(T_1)$ denote the number of events occurred in the next T_1 units of time. Find the mean and variance of $N(T_1)T_1$.

12. Suppose that you arrive at a single-teller bank to find seven other customers in the bank, one being served (First Come First service basis) and the other six waiting in line. You join the end of the line. Assume that, service times are independent and exponential distributed with rate μ. Model this situation as a birth and death process.

(a) What is the distribution of time spend by you in the bank?

(b) What is the expected amount of time you will spend in the bank?

13. Let $X(t)$ be the number of bacteria in a colony at time t. Evolution of the population is described by the time that each of the individuals takes for division in two, independently of the other bacteria and the life time of each bacterium also independent. Assume that, time for division is exponential distribution with rate λ and life time of each bacterium is also exponential distribution with rate μ. Without loss of generality, assume that $\{X(t), t \geq 0\}$ is modeled as a birth and death process.

(a) Discuss the equilibrium probability distribution of the process.

(b) Prove that, $\frac{dM(t)}{dt} = (\lambda - \mu)M(t)$ where $M(t)$ denotes the mean number of bacteria at time t.

(c) Discuss $M(t)$ as $t \to \infty$.

14. Four workers share an office that contains four telephones. At any time, each worker is either working or on the phone. Each working period of worker i lasts for an exponentially distributed time with rate λ_i, and each on the phone period lasts for an exponentially distributed time with rate μ_i, $i = 1, 2, 3, 4$. Let $X_i(t)$ equal 1 if worker i is working at time t, and let it be 0 otherwise. Let $X(t) = (X_1(t), X_2(t), X_3(t), X_4(t))$.
(a) Argue that $\{X(t), t \geq 0\}$ is a continuous-time Markov chain and give its infinitesimal rates.
(b) What proportion of time are all workers working?