MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 5 (Moments)

1. Let X_1 and X_2 be independent exponential distributed random variables with parameters 5 and 4 respectively. Define $X_{(1)} = \min\{X_1, X_2\}$ and $X_{(2)} = \max\{X_1, X_2\}$.

   (a) Find $Var(X_{(1)})$? (b) Find the distribution of $X_{(1)}$? (c) Find $E(X_{(2)})$?

2. Let X and Y be two non-negative continuous random variables having respective CDFs F_X and F_Y. Suppose that for some constants a & $b > 0$, $F_X(x) = F_Y\left(\frac{x-a}{b}\right)$. Determine $E(X)$ in terms of $E(Y)$.

3. Let X be a random variable having an exponential distribution with parameter $\frac{1}{2}$. Let Z be a random variable having a normal distribution with mean 0 and variance 1. Assume that, X and Z are independent random variables. (a) Find the pdf of $T = \frac{Z}{\sqrt{\frac{X}{2}}}$. (b) Compute $E(T)$ and $Var(T)$.

4. Let X and Y be two identically distributed random variables with Var(X) and Var(Y) exist. Prove or disprove that Var $\left(\frac{X+Y}{2}\right) \leq$ Var (X).

5. Let X and Y be i.i.d. random variables each having a $\mathcal{N}(0,1)$. Calculate $E[(X+Y)^4/(X-Y)]$.

6. Let $X_1, \ldots, X_5$ be a random sample from $N(0, \sigma^2)$. Find a constant c such that $Y = \dfrac{c(X_1 - X_2)}{\sqrt{X_3^2 + X_4^2 + X_5^2}}$ has a t-distribution. Also, find $E(Y)$.

7. Consider the metro train arrives at the station near your home every quarter hour starting at 5:00 AM. You walk into the station every morning between 7:10 and 7:30 AM, with the time in this interval being a uniform random variable, that is $\mathcal{U}([7:10, 7:30])$.

   (a) Find the distribution of time you have to wait for the first train to arrive?

   (b) Also, find its mean waiting time?

8. Let X and Y be iid random variables each having uniform distribution (2,3). Find $E\left(\frac{X}{Y}\right)$?

9. Let X and Y be two random variables such that $\rho(X,Y) = \frac{1}{2}$, $Var(X) = 1$ and $Var(Y) = 4$. Compute $Var(X - 3Y)$.

10. Let $X_1, X_2, \ldots, X_n$ be iid random variables with $E(X_1) = \mu$ and $Var(X_1) = \sigma^2$. Define $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$. Find (a) $Var(\overline{X})$ (b) $E[S^2]$.

11. Pick the point (X,Y) uniformly in the triangle $\{(x,y) \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x\}$. Calculate $E[(X-Y)^2/X]$.

12. Find $E(Y/x)$ where (X,Y) is jointly distributed with joint pdf $f(x,y) = \begin{cases} \frac{y}{(1+x)^4} e^{-\frac{y}{1+x}}, & x, y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$

13. Let X have a beta distribution i.e., its pdf is $f_X(x) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}$, $0 < x < 1$ and Y given $X = x$ has binomial distribution with parameters (n, x). Find regression of X on Y. Is regression linear?

14. Let $X \sim EXP(\lambda)$. Find $E[X/X > y]$ and $E[X - y/X > y]$.

15. Consider n independent trials, where each trial results in outcome i with probability $p_i = 1/3, i = 1, 2, 3$. Let X_i denote the number of trials that result in outcome i amongst these n trials. Find the distribution of X_2. Find the conditional expectation of X_1 given $X_2 > 0$. Also determine cov $(X_1, X_2 \mid X_2 \leq 1)$.

16. (a) Show that $cov(X,Y) = cov(X, E(Y|\, X))$.

    (b) Suppose that, for constants a and b, $E(Y|\, X) = a + bX$. Show that $b = cov(X,Y)/Var(X)$.

17. Let X be a random variable which is uniformly distributed over the interval $(0, 1)$. Let Y be chosen from interval $(0, X]$ according to the pdf $f(y/x) = \begin{cases} 1/x, & 0 < y \leq x \\ 0, & \text{otherwise.} \end{cases}$ Find $E(Y^k/X)$ and $E(Y^k)$ for any fixed positive integer k.

18. Suppose that a signal X, standard normal distributed, is transmitted over a noisy channel so that the received measurement is $Y = X + W$, where W follows normal distribution with mean 0 and variance σ^2 is independent of X. Find $f_{X/y}(x/y)$ and $E(X \mid Y = y)$.

19. Suppose X follows $Exp(1)$. Given $X = x$, Y is a uniform distributed rv in the interval $[0, x]$. Find the value of $E(Y)$.

20. Consider Bacteria reproduction by cell division. In any time t, a bacterium will either die (with probability 0.25), stay the same (with probability 0.25), or split into 2 parts (with probability 0.5). Assume bacteria act independently and identically irrespective of the time. Write down the expression for the generating function of the distribution of the size of the population at time $t = n$. Given that there are 1000 bacteria in the population at time $t = 50$, what is the expected number of bacteria at time $t = 51$.

21. Let N be a positive integer random variable and $X_1, X_2, \ldots$ be a sequence of iid random variables. N is independent of X_i's. Find the moment generating function (MGF) of $S_N = X_1 + X_2 + \ldots + X_N$, the random sum in terms of MGF of $X_i's$ and N. Also show that:
(a) $E[S_N] = E[N]E[X]$ (b) $Var[S_N] = E[N]Var[X] + [E[X]]^2 Var[N]$.

22. If $E[Y/X] = 1$, show that $Var[XY] \geq Var[X]$.

23. Suppose you participate in a chess tournament in which you play until you lose a game. Suppose you are a very average player, each game is equally likely to be a win, a loss or a tie. You collect 2 points for each win, 1 point for each tie and 0 points for each loss. The outcome of each game is independent of the outcome of every other game. Let X_i be the number of points you earn for game i and let Y equal the total number of points earned in the tournament. Find the moment generating function $M_Y(t)$ and hence compute $E(Y)$.

24. Let (X, Y) be two-dimensional random variable with joint pdf is given by $f(x, y) = \begin{cases} e^{-y}, & 0 < x < y < \infty \\ 0, & \text{otherwise} \end{cases}$

    (a) Find the conditional distribution of Y given $X = x$.
    (b) Find the regression of Y on X.
    (c) Show that variance of Y for give $X = x$ does not involve x.

25. Let $X_1, X_2, \ldots, X_n$ be independent and $ln(X_i)$ has normal distribution $N(2i, 1)$, $i = 1, 2, \ldots, n$. Let $W = X_1^{\alpha} X_2^{2\alpha} \ldots X_n^{n\alpha}$, $\alpha > 0$ where α is any constant. Determine $E(W), Var(W)$ and the pdf of W.

26. Let (X, Y) be a two-dimensional continuous type random variables. Assume that, $E(X), E(Y)$ and $E(XY)$ are exist. Suppose that, $E(X \mid Y = y)$ does not depend on y. Find $E(XY)$.

27. For each fixed $\lambda > 0$, let X be a Poisson distributed random variable with parameter λ. Suppose λ itself is a random variable following exponential distribution with parameter 1. Find the probability mass function of X.

28. Let X and Y be two discrete random variables with

$$P(X = x_1) = p_1, \quad P(X = x_2) = 1 - p_1, \quad 0 < p_1 < 1;$$

and

$$P(Y = y_1) = p_2, \quad P(Y = y_2) = 1 - p_2, \quad 0 < p_2 < 1.$$

If the correlation coefficient between X and Y is zero, check whether X and Y are independent random variables.

29. Suppose the length of a telephone conversation between two persons is a random variable X with cumulative distribution function

$$P(X \le t) = \begin{cases} 0, & -\infty < t < 0 \\ 1 - e^{-0.04t}, & 0 \le t < \infty \end{cases},$$

where the time is measured in minutes.

(a) Given that the conversation has been going on for 20 minutes, compute the probability that it continues for at least another 10 minutes.

(b) Show that, for any $t > 0$, $E(X/X > t) = t + 25$.

30. A real function g(x) is non-negative and satisfies the inequality $g(x) \ge b > 0$ for all $x \ge a$. Prove that for a random variable X if $E(g(X))$ exists then $P(X \ge a) \le \frac{E(g(X))}{b}$.

31. Let X have a Poisson distribution with mean $\lambda \ge 0$, an integer. Show that $P(0 < X < 2(\lambda + 1)) \ge \frac{\lambda}{\lambda+1}$.

32. Does the random variable X exist for which $P[\mu - 2\sigma \le X \le \mu + 2\sigma] = 0.6$? Justify your answer.