COL 202 Discrete Mathematics **Diwali 2023**

Tutorial 1

1. [Submission Problem for Group 1] Prove or disprove the following:
   (a) Every natural number can be written as either the sum of two perfect squares or the difference of two perfect squares or both. (You may include 0^2 if needed.)
   (b) here exist irrational numbers x and y such that x^y is rational.
   (c) $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is irrational.

2. [Submission Problem for Group 2] Prove or disprove that the following pairs of propositions are logically equivalent:
   (a) $\neg(p \vee q \vee r) \vee s$ and $(\neg p \vee s) \wedge (\neg q \vee s) \wedge (\neg r \vee s)$
   (b) $(p \wedge q) \implies r$ and $(p \implies r) \vee (q \implies r)$
   (c) $(p \implies q) \implies r$ and $p \implies (q \implies r)$

3. [Submission Problem for Group 3] By $A \iff B$, we denote the pair of logical statements $A \implies B$ and $B \implies A$. Consider the following:
   (a) Suppose that
   $$a + b + c = d$$
   where a, b, c, d are nonnegative integers. Let P be the assertion that d is even. Let W be the assertion that exactly one among a, b, c are even, and let T be the assertion that all three are even. Prove by a detailed case analysis that
   $$P \iff [W \vee T]$$
   (b) Now suppose that
   $$w^2 + x^2 + y^2 = z^2$$
   where w, x, y, z are nonnegative integers. Let P be the assertion that z is even, and let R be the assertion that all three of w, x, y are even. Prove by a case analysis that $P \iff R$:

4. [Submission Problem for Group 4]
   (a) The Twin Prime conjecture is one of the most famous open problems in mathematics. It says that there are infinitely many pairs of "nearby" prime numbers, i.e., primes that differ by 2 (e.g., 3 and $5, 11$ and 13, and so on). Write the statement of the twin prime conjecture in formal notation. You may use quantifiers such as $\forall$ ("for all") and $\exists$ ("there exists"), and use the notation $\mathbb{N}$ for the set of natural numbers. You may also use the proposition $p(x)$ to denote "x is prime". Suppose, in the future, someone proves that the twin prime conjecture is false. What would be the correct statement of the result in that case? Write it in formal notation.

(b) Another well-known problem is Bertrand's postulate which says that for any natural number n, there is always a prime number between n and $2n$. Write the statement of Bertrand's postulate in formal notation.

(c) Here's another fact about primes: There are infinitely many prime numbers that do not have the digit 7 in their decimal expansion. Write this statement in formal notation.

5. [Bonus] Prove the following

 (a) There is an irrational number a such that $a^{\sqrt{3}}$ is rational.

 (b) $\log_4 6$ is irrational.

 (c) Let the coefficients of the polynomial

$$a_0 + a_1 x + \cdots + a_{m-1} x^{m-1} + x^m$$

 be integers. Then any real root of the polynomial is either integral or irrational.

 (d) Use part (c) to show that $\sqrt[m]{k}$ is irrational whenever k is not an m-th power of some integer.