Name: ____________________ Roll No: ______________

(COL 202) Discrete Mathematics 3 November, 2023

Quiz 2

Duration: 45 minutes **(12 marks)**

- Be clear in your writing. If you use a statement proved in class or in the problem set, then write down the entire statement before using it.
- **You will not get a new sheet, so make sure you are certain when you write something.** Make a judicious decision of which tool(s) to use to get a clean and short answer that fits in the space.

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1. **(4 points)** The *complement* of a graph G is a graph $\overline{G}$ on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. Prove that at least one among G and $\overline{G}$ is *connected*.

   Suppose G is disconnected. We want to show that $\overline{G}$ is connected. So suppose v and w are vertices. If vw is not an edge in G, then it is an edge in $\overline{G}$, and so we have a path from v to w in $\overline{G}$. On the other hand, if vw is an edge in G, then this means v and w are in the same component of G. Since G is disconnected, we can find a vertex u in a different component, so that neither uv nor uw are edges of G. Then vuw is a parth from v to w in $\overline{G}$. This shows that any two vertices in $\overline{G}$ have a path (in fact a path of length one or two) between them in $\overline{G}$, so $\overline{G}$ is connected.

2. **(4 points)** Prove that a planar bipartite graph on n nodes has at most $2n-4$ edges.

   From Euler's formula, we have $v-e+f=2$. Since there are no cycles of length 3, every face has degree 4 or greater. From the handshake lemma, we then have

   $$4f \leq \sum_{f \in F} \deg(f) = 2e$$

   Substituting, we have

   $$2 = v - e + f \leq v - e + \frac{e}{2}$$

   which is the required result.

3. **(4 points)** If C is a cycle, and e is an edge connecting two non-adjacent nodes of C, then we call e a *chord* of C. Prove that if every node of a graph G has degree at least 3, then G contains a cycle with a chord.

   Suppose $u_0 u_1 \dots u_k$ is the longest path in $G = (V, E)$. Since $deg(u_0) \geq 3$, there exist two neighbors w, v of u_0 not equal to u_1. Since $u_0 u_1 \dots u_k$ is the longest path, we must have that $w, v \in \{u_2, u_3, \dots, u_k\}$. Suppose $w = u_i, v = u_j$ with $i \leq j$. Then $u_0 u_1 \dots u_j u_0$ is a circle with chord $u_0 u_i$.