Hints and solutions to problem sheet-11

Section B

Q.1 Region of Convergence

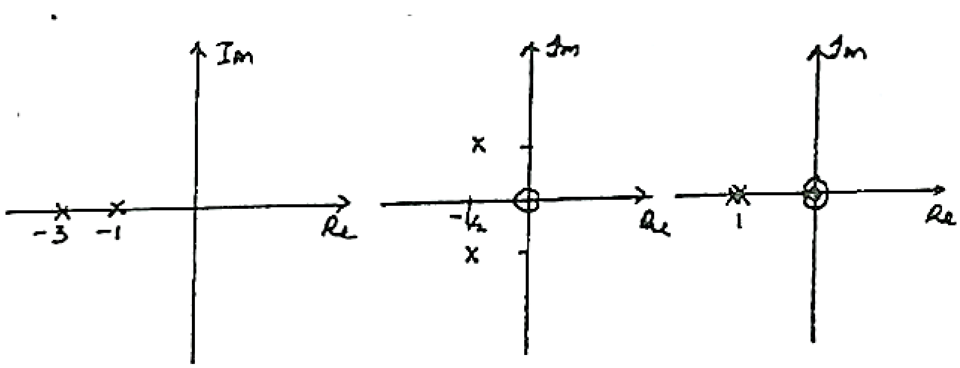
We know that.

$$g(t) = e^{2t}x(t) \overset{L}{\leftrightarrow} G(s) = X(s-2)$$

The ROC of $G(s)$ is the ROC of $X(s)$ shifted to the right by condition 2. We are also given that $X(s)$ has exactly $two\ poles$, located at $s = -1$, and $s = -3$. Since $G(s) = X(s-2), G(s)$ also has exactly $two\ poles$, located at $s = -1+2 = 1$ and $s = -3+2 = -1$. Since we are given $G(j\omega)$ exists, we may infer that the j −axis lies in the ROC of $G(s)$. Given this fact and the locations of the $poles$, we may conclude that $g(t)$ is a two-sided sequence. Obviously $x(t) = e^{-2t}g(t)$ will also be two sided.

Q.2 Laplace transforms of basic filters

The pole-zero plots for each of the three Laplace Transforms is as shown in Figure below.



a) We know that the magnitude of the Fourier transform may be expressed as

$$|H_1(j\omega)| = \frac{1}{(length\ of\ vector\ from\ \omega - axis\ to\ pole(-1))(length\ of\ vector\ from\ \omega\ to\ axis\ to\ pole(-2))}$$

We see that the right-hand side of the above expression is maximum for $\omega = 0$ and decreases as become increasingly more positive or more negative. Therefore $|H_1(j\omega\ )|$ is approximately low pass.

b) We know that the magnitude of the Fourier transform may be expressed as

$$|H_2(j\omega)| = \frac{(length\ of\ vector\ from\ \omega - axis\ to\ zero\ (0))}{\left(length\ of\ vector\ from\ \omega - axis\ to\ pole\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\right)\left(length\ of\ vector\ from\ \omega\ to\ axis\ to\ pole\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\right)}$$

We see that the right-hand side of the above expression is zero for $\omega = 0$. It first increases as $|\omega|$ increases until $|\omega|$ reaches ½. Then it starts decreasing as $|\omega|$ increases even further. Therefore, it is approximately band pass.

c) We know that the magnitude of the Fourier transform may be expressed,

$$|H_3(j\omega)| = \frac{(length\ of\ vector\ from\ \omega - axis\ to\ zero(0))^2}{(length\ of\ vector\ from\ \omega - axis\ to\ pole(-1))(length\ of\ vector\ from\ \omega\ to\ axis\ to\ pole(-1))}$$

As $|\omega|$ increases, $|H_3(j\omega)|$ decreases towards a value of 1 (because all the vector lengths become almost identical, and the ratio becomes 1). Therefore $|H_3(j\omega)|$ is approximately high pass.

Q.3 Properties of Laplace transforms

a) If $X(s)$ has only $one\ pole$, then $x(t)$ would be of the form Ae^{-at}. Clearly such a signal violates condition 2. Therefore, this statement is inconsistent with the given information. Therefore, this statement is inconsistent with the given information.

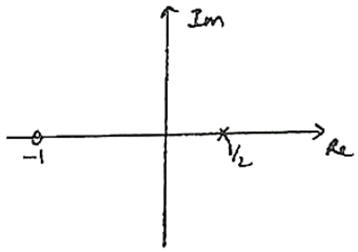
b) If $X(s)$ has only $two\ poles$, then $x(t)$ would be of the form $Ae^{-at}sin(\ \omega_0 t)$. Clearly such a signal could be made to satisfy all three conditions (Example, $\omega_0 = 80\pi, a = 19200$). Therefore, this statement is consistent with the given condition.

c) If $X(s)$ has $more\ than\ two\ poles$ (say 4 $poles$), then $x(t)$ could be assumed to be of the form $Ae^{-at}sin(\omega_0 t) + Be^{-bt}sin(\omega_0 t)$. Clearly such a signal could still be made to satisfy all three conditions. Therefore, this statement is consistent with the given information.

Q.4 Inverse System

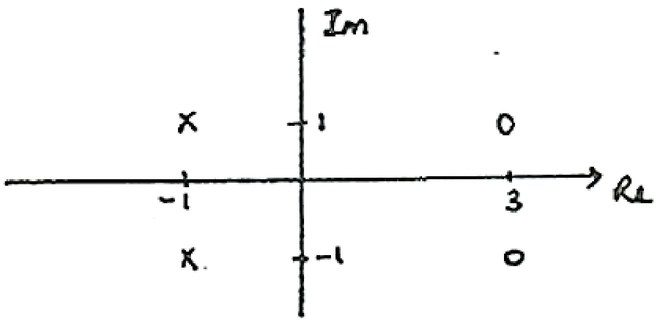
a) $H_1(s) = \frac{1}{H(s)}$.

b) From the above relationship it is clear that the $poles$ of the inverse system will be the $zeros$ of original system. Also, the $zeros$ of the inverse system will be the $poles$ of the original system. Therefore, the pole-zero-plot for $H_1(s)$ is given below,



Q.5 Properties of Laplace Transforms

Since $h(t)$ is real, its $poles$ and $zeros$ must occur in complex conjugate pairs. Therefore, the known $poles$ and $zeros$ of $H(s)$ are shown in Figure below.



Since $H(s)$ has exactly 2 $zeros$ at $infinity$, $H(s)$ has at least two more unknown finite $poles$. In case $H(s)$ has more than 4 $poles$, then it will have $a\ zero$ at some location for every additional $pole$. Furthermore, since $h(t)$ is causal and stable, all $poles$ of $H(s)$ must lie in the left half of the $s - plane$ and the ROC must include the $j\omega - axis$.

a) True. Consider

$$g(t) = h(t)e^{-3t} \overset{L}{\leftrightarrow} G(s) = H(s+3)$$

The ROC of $G(s)$ will be the ROC of $H(s)$ shifted by 3 to the left. Clearly this ROC will still include the $j\omega - axis$. Therefore, $g(t)$ has to be stable.

b) Insufficient information. As mentioned earlier, $H(s)$ has some unknown $poles$. So, we do not know which the rightmost $pole$ as in $H(s)$. Therefore, we cannot determine what exactly ROC is.

c) True. Since $H(s)$ is rational, $H(s)$ may be expressed as a ratio of two polynomials in s. Furthermore, since $h(t)$ is $real$, the coefficients of these polynomials will be $real$. That is,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{P(s)}{Q(s)}$$

Here, $P(s)$ and $Q(s)$ are polynomials in s. The differential equation relating $x(t)$ and $y(t)$ is obtained by taking the inverse Laplace transform of $Y(s)Q(s) = X(s)P(s)$. Clearly, this differential equation has to have only real coefficients.

d) False, we are given that $H(s)$ has 2 $zeros$ at $s = \infty$. Therefore, $\lim_{n\to\infty} H(s) = 0$.

e) True. The reasoning is at the beginning of the problem.

f) Insufficient information. $H(s)$ may have other $zeros$. So, reasoning at the beginning of the problem.

g) False. We know that $e^{3t}sin(t) = \left(\frac{1}{2j}\right)e^{(3+j)t} - \left(\frac{1}{2j}\right)e^{(3-j)t}$. Both $e^{(3+j)t}$ and $e^{(3-j)t}$ are eigen function of the LTI system. Therefore, the response of the system to these exponentials is $H(3+j)e^{(3+j)t}$ and $H(3-j)e^{(3-j)t}$, respectively. Since $H(s)$ has $zeros$ at $3+j$ and $3-j$, we know that the output of the system to the exponentials has to be zero. Hence, the response of the system to $e^{3t}sin(t)$ has to be zero.

Q.6 Laplace Transform for polynomial function

Since $x(t)$ has an impulse at $t = 0$, the numerator of $X(s)$ must be of the same/larger degree than the denominator polynomial of $X(s)$. This implies that $X(s)$ has at least 4 $zeros$.