Lecture 22 : Graphs

directed — undirected

directed graph $G = (V, E)$ where $E \subseteq V \times V$

Example :

$V = \{1, 2, \ldots, 5\}$

$E = \{(1,2), (2,1), (2,3), (4,3), (4,2), (4,5), (5,4), (5,1)\}$

Two popular representations :

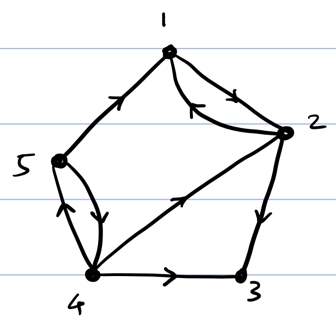
Adjacency List

1 - 2
2 - 1, 3
3 -
4 - 2, 3, 5
5 - 1, 4

Adjacency Matrix

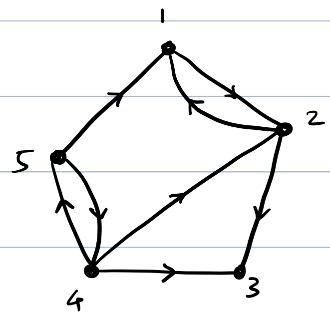
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

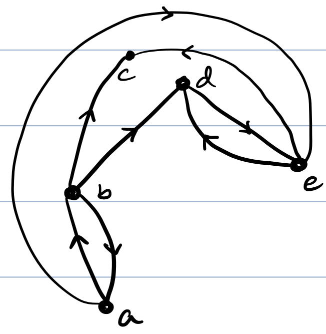
Both these representations contain the same info.
However, as we will see, the adjacency matrix rep. allows us to use tools from linear algebra.



How a graph is drawn is not important for now.

Graph Isomorphism : what's in the name ?





$G = (V, E)$ $\qquad\qquad$ $G' = (V', E')$

Are these two graphs the same ?

Literally speaking, no. In one case, $V = \{1, \ldots, 5\}$, and in another $V' = \{a, b, c, d, e\}$.

However, note that there exists a bijection $f : V \to V'$ s.t.
$(x, y) \in E$ if and only if $(f(x), f(y)) \in E'$.

Such graphs are said to be isomorphic.

Def : $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there exists a bijection $f : V \to V'$ s.t. $(x, y) \in E$ iff $(f(x), f(y)) \in E'$.

How to test if two graphs are isomorphic?

Naive algorithm : $n!\,\mathrm{poly}(n)$ time

Best known algorithm : $n^{(\log n)^2}$

Proving Isomorphism / Non Isomorphism :

[This is not part of the syllabus]

Powerful Server — Can perform superpolynomial computations

Weak Client — Can perform poly. time computations

Given : graphs (G_1, G_2)

How can the server prove to the client that G_1 and G_2 are not isomorphic?

Goal : interactive proofs (like a viva). Client asks the server a few questions. Server sends answers. If graphs are not isomorphic, then client accepts w.p. 1. If graphs are isomorphic, even a cheating server should not succeed in convincing the client (except with small probability)

Walks, Paths, Cycles and Distances :

$G = (V, E)$

[can be directed or undirected]

Def: walk $w = (v_1, v_2, \ldots, v_k)$ s.t. $(v_i, v_{i+1}) \in E$ for all $i \in \{1, \ldots, k-1\}$

path p from (u, v) is a sequence of vertices $(v_0 = u, v_1, \ldots, v_k = v)$ s.t. $(v_i, v_{i+1}) \in E$ for all $i \in \{1, \ldots, k-1\}$ and all v_i s are distinct. Length of path is k.

cycle $(v_0 = u, v_1, \ldots, v_k = u)$ where $(v_i, v_{i+1}) \in E$ for all $i \in \{1, \ldots, k-1\}$

Given $u, v \in V$, distance from u to v in G is the length of the shortest path from u to v (if such a path exists), and ∞ otherwise.

Thm: Let $G = (V, E)$ be a directed graph with adj. matrix A. For any $k \geq 0$, for any $u, v \in E$,
$A^k[u,v]$ = the number of walks from u to v of length exactly k.

Corollary: $A^n[u,v] > 0$ iff there exists a path from u to v.

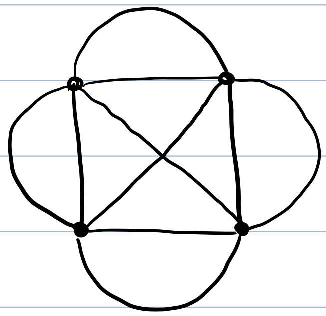
For $u \neq v$, $$\text{dist}_G(u,v) = \begin{cases} \infty & \text{if } A^n[u,v] = 0. \\ \min.\ k & \text{s.t. } A^k[u,v] > 0 \end{cases}$$

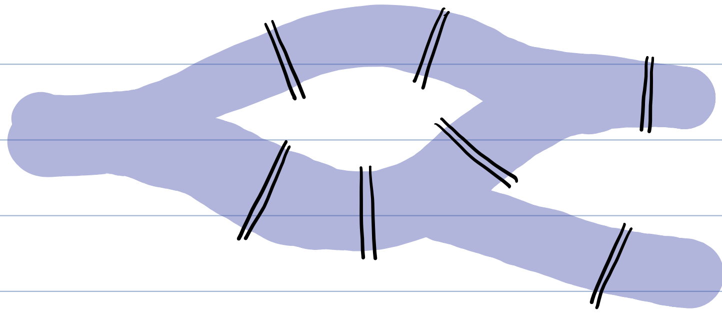
Proof of Theorem is using (regular) induction on k.
It is left as an exercise.

Errata: In class, I had mentioned that $(I + A)^k$ counts the number of walks of length at most k. This is not correct. It only tells you whether there exists a walk of length at most k. For any k, any $u, v \in V$,
$\exists$ walk of length at most k from u to v in G
$\iff (I + A)^k[u,v] > 0$.

Eulerian Graphs:

A puzzle: draw the following figure without lifting your pen, and without retracing any path





Can you start from some point, and come back to the same point after crossing each bridge exactly once?

This is widely believed to be the first theorem in graph theory. See 'Konigsberg Seven Bridges' problem.

We can cast the above problem in graph-theoretic language as follows: given an undirected graph $G = (V, E)$, does there exist a closed walk $(v_0, v_1, \ldots, v_k = v_0)$ s.t.

(a) $(v_i, v_{i+1}) \in E$ for all $i \in \{0, 1, \ldots, k\}$

(b) for every edge in E, there exists a unique i s.t. $e = (v_i, v_{i+1})$.

Such a closed walk is called an Eulerian walk, and the corresponding graphs are Eulerian graphs.

There is a very clean characterization of Eulerian graphs. Moreover, you dont need to look at 'global' properties of the graph, only 'local' ones related to the degree of each vertex.

Def: For any vertex $u \in V$, let

$N_G(u) = \{ v \in V \text{ s.t. } (u,v) \in E \}$ neighbourhood of u.

$\deg_G(u) = |N_G(u)|$ degree of u

Def: An undirected graph $G = (V, E)$ is connected if for every $u, v \in V$, $\mathrm{dist}_G(u,v)$ is finite.

Thm: An undirected connected graph $G = (V, E)$ is Eulerian if and only if every vertex has even degree.

Proof: One direction is easy.

Part 1: G is Eulerian

$\Rightarrow$ every vertex has even degree

Since G is Eulerian, there exists a closed walk $(v_0, v_1, \ldots, v_k = v_0)$ s.t.

- $\forall\, i \in \{0, \ldots, k-1\}$, $(v_i, v_{i+1}) \in E$.
- $\forall\, e \in E$, $\exists$ unique i s.t. $e = (v_i, v_{i+1})$

Take any vertex $v \in V$. For each edge $e = (u,v) \in E$, label it as 'incoming' if (u,v) appears in the closed walk, and label it as 'outgoing' if (v, u) appears in the closed walk.

Note that every edge incident on v is labeled as either 'incoming' or 'outgoing'. Moreover, there is a bijection between 'incoming' and 'outgoing' edges. Hence every vertex must have even degree.

<u>Part 2</u>: Connected graph with all vertices having even degree $\Rightarrow$ graph is Eulerian

To be contd...