COL 202 Discrete Mathematics **Diwali 2023**

Tutorial 2

1. [Submission Problem for Group 1] Prove the following by induction:

   (a) A graph (or a network) is a structure consisting of a set of objects (also known as vertices or nodes) some pairs of which are connected via edges. Assume that there are no self-edges (or loops).The degree of a vertex is the number of other vertices that share an edge with it. Say we are given a set of k colors. A graph is said to be k-colorable if each vertex can be assigned one of the k colors in a way that all neighboring vertices (i.e., any pair of vertices that share an edge) have different colors. (Some of the k colors may be left unused.) Prove that any graph with maximum degree d is $(d+1)$-colorable.

   (b) The number of subsets of an n-element set is 2^n

   (c) The number of ways of ranking n different objects is $n!$.

2. [Submission Problem for Group 2] The sequence of Fibonacci numbers $\{F_n\}_{n\in\mathbb{N}\cup\{0\}}$ is defined as follows: $F_0 = 0, F_1 = 1$, and $\forall n \geq 2, F_n = F_{n-1}+F_{n-2}$. Prove the following using induction.

   (a) The Fibonacci number F_{5k} is a multiple of 5, for all integers $k \geq 1$.

   (b) $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$

3. [Submission Problem for Group 3] Let $P(x), Q(x)$, and $R(x)$ be the statements "x is a clear explanation", "x is satisfactory", and "x is an excuse", respectively. Suppose that the domain for x consists of all English text. Express each of these statements using quantifiers, logical connectives, and $P(x), Q(x)$ and $R(x)$.

   (a) All clear explanations are satisfactory.

   (b) Some excuses are unsatisfactory.

   (c) Some excuses are not clear explanations.

   (d) Does (c) follow from (a) and (b)?

4. [Submission Problem for Group 4] For each of the following propositions, indicate which of these are false when the domain ranges over a) $\mathbb{Z}_{>0}$, b) $\mathbb{Z}$, c) $\mathbb{R}$

   (a) $\forall x \exists y : 2x - y = 0$.

   (b) $\forall x \exists y : x - 2y = 0$.

   (c) $\forall x, x < 10 \implies (\forall y, y < x \implies y < 9)$

   (d) $\forall x \exists y, [y > x \land \exists z, y + z = 100]$

5. [Bonus] Let $P(x, y)$ be a statement about the variables x and y. Consider the following two statements: $A := (\forall x)(\exists y)(P(x, y))$ and $B := (\exists y)(\forall x)(P(x, y))$. The universe is the set of integers.

(a) Prove: $(\forall P)(B \implies A)$ ("B always implies A" i.e., for all P , if B is true then A is true).

(b) Prove: $\neg(\forall P)(A \implies B)$ (i. e., A does not necessarily imply B). In other words, $(\exists P)(A \not\Longrightarrow B)$. To prove this, you need to construct a counterexample, i. e., a statement $P(x, y)$ such that the corresponding statement A is true but B is false. Make $P(x, y)$ as simple as possible.

6. [Bonus] Let r be a positive real number satisfying $r^2 = r + 1$. Using induction, show that for all $n \in \mathbb{N}, F_n \geq r^{n-2}$.

7. [Bonus] Problems 3.17, 3.18, 3.49, and 3.50 from `https://courses.csail.mit.edu/6.042/spring18/mcs.pdf`