| COL202: Discrete Math | Tutorial 1 |
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The solutions for the ($\star$) marked problems must be submitted at the start of the tutorial. The ($\blacklozenge$) marked problems will be discussed in the tutorial (if time permits, you can also discuss the other problems with the instructor/TAs).

**Notations.** Let $\mathbb{N}$ denote the set of natural numbers $\{1, 2, 3, \ldots\}$. Let $\mathbb{Z}$ denote the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

**Definition 1** (Injective, Surjective and Bijective Functions)**.** *A function $f : \mathcal{A} \to \mathcal{B}$ is*

- *injective if, for all distinct $a, a' \in \mathcal{A}$, $f(a) \neq f(a')$.*
- *surjective if, for all $b \in \mathcal{B}$, there exists an $a \in \mathcal{A}$ such that $f(a) = b$.*
- *bijective if it is both injective and surjective.*

**Definition 2** (Countably infinite sets)**.** *A set $\mathcal{A}$ is countably infinite if there exists a bijection between $\mathcal{A}$ and $\mathbb{N}$.*

**Theorem 1** (Schröder-Bernstein)**.** *Let $\mathcal{A}$, $\mathcal{B}$ be two infinite sets. Suppose there exists an injective function $f : \mathcal{A} \to \mathcal{B}$ and an injective function $g : \mathcal{B} \to \mathcal{A}$. Then there exists a bijection between $\mathcal{A}$ and $\mathcal{B}$.*

*Similarly, if there exists a surjective functions $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{A}$, then there exists a bijection between $\mathcal{A}$ and $\mathcal{B}$.*

## 1 Problems: Infinite Sets, Enumeration

1.1. ($\star$) Let $\mathcal{A}$ be any infinite set (you should not assume that $\mathcal{A}$ is countable; it can be the set of real numbers).

1. Consider the set of all functions $\mathcal{F} = \{f : \mathcal{A} \to \{0, 1\}\}$.
   (a) Show that there exists a surjective function $g : \mathcal{F} \to \mathcal{A}$.
   (b) Show that there does not exist a surjective function $h : \mathcal{A} \to \mathcal{F}$.
2. Let $\mathcal{B}$ be some finite set, and let $\mathcal{P} = \{f : \mathcal{B} \to \mathcal{A}\}$. Is there a bijection between $\mathcal{A}$ and $\mathcal{P}$?

1.2. ($\blacklozenge$ Infinite Bit Strings) In class, we showed that there exists no surjective function from $\mathbb{N}$ to the set of all infinite bit strings. However, we did not formally define what an infinite bit string is. Can we 'combine' two infinite bit strings? Can we make infinite copies of a finite string to produce an infinite string?

In this problem, we will formally define an infinite bit string as a function $f : \mathbb{N} \to \{0, 1\}$. Note that the bit at position $i \in \mathbb{N}$ is $f(i)$. We will now define a few sets using infinite bit strings:

- Let $\mathcal{S} = \{f : \mathbb{N} \to \{0, 1\}\}$. This captures the set of all infinite bit strings.

- Let $\mathcal{T} = \{f : \mathbb{N} \to \{0,1\}$ s.t. $\exists n_0 \in \mathbb{N}$ such that $f(j) = 0$ for all $j > n_0\}$. Informally, this is the set of all infinite bit strings that have only finitely many ones.
- Let $\mathcal{W} = \{f : \mathbb{N} \to \{0,1\}$ s.t. $\forall n_0 \in \mathbb{N}, \exists n_1 > n_0$ s.t. $f(n_1) = f(n_1 + 1) = 1\}$. This is the set of all strings that have infinitely many occurrences of 11.

Note that you need to be careful when you 'combine' two infinite bit strings. For instance, suppose we have infinite strings $u$ and $v$, represented using functions $f_u$ and $f_v$. To define the combination of $u$ and $v$, you need to define the corresponding function $f : \mathbb{N} \to \{0,1\}$.

1. Show that there exists a bijection between $\mathcal{S}$ and $\mathcal{W}$.
2. Show that there exists a bijection between $\mathcal{T}$ and $\mathbb{N}$.

1.3. (A New Sport at Olympics 2036)[1] The following 'blindfolded shooting event' has been proposed for Olympics 2036. In this event, the participant is blindfolded. The referee picks an integer $x \in \mathbb{Z}$, which denotes the initial position of the target on the X axis. The referee also picks a velocity $v \in \mathbb{N}$. At time $t = 0$, the target starts moving forward, along the X axis, with velocity $v$. The objective of the participant is to shoot the target **in a finite number of shots**. What should be the participant's strategy? Note that the participant does not know the values $x$ or $v$, it only knows that $x \in \mathbb{Z}$, $v \in \mathbb{N}$, and the time instant $t = 0$ (when the target starts moving).

## 2 Problems: Well Ordering Principle

2.1. ($\star$) Suppose there exists a predicate $P : \mathbb{N} \to \{T, F\}$ such that

- (base case) $P(1) = T$
- (induction step) for all $i \in \mathbb{N}$, $P(i) \implies P(i+1)$.

Then, the *Principle of Mathematical Induction* (PMI) states that $P(n)$ holds for all $n \in \mathbb{N}$.

Prove the Principle of Mathematical Induction, assuming the Well Ordering Principle.

2.2. ($\blacklozenge$) Prove that for any natural number $n$ greater than 50, there exist ~~natural~~ whole numbers $x, y, z$ such that $n = 6 \cdot x + 14 \cdot y + 21 \cdot z$.

## 3 Problems: Logic

3.1. ($\star$) Prove that the following Boolean formula is a tautology.

$$[p \implies (q \implies r)] \iff [(p \land q) \implies r]$$

3.2. ($\blacklozenge$) Using verbal reasoning, discuss why the following are always true (irrespective of the predicate $P$):

- $\forall x \in S : [P(x) \implies (\exists y \in S : P(y))]$

[1]Many thanks to Pravar Kataria for proposing this problem.

- $\exists x \in S : [P(x) \implies (\forall y \in S : P(y))]$

3.3. (Fun with nested quantifiers) A function $f : \mathbb{N} \to [0,1]$ is said to be a negligible function if for all $c \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $f(n) < n^{-c}$. Prove that if $\mu_1$ and $\mu_2$ are negligible functions, then so are $\mu(n) = \mu_1(n) + \mu_2(n)$ and $\mu(n) = \mu_1(n) \cdot n^{100}$.

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This is version 1.0 of the tutorial sheet. Let me know if something is unclear. In case of any doubt or for help regarding writing proofs, feel free to contact the instructor or TAs.


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