2301 COL 202 Minor

Talegaonkar(29)

TOTAL POINTS

48.5 / 50

QUESTION 1

1 Choose the correct answers 12 / 12

✓ **- 0 pts** *All parts correct*

- 3 pts Part (a) incorrect

- 3 pts Part (b) incorrect

- 3 pts Part (c) incorrect

- 3 pts Part (d) incorrect

QUESTION 2

2 Brief justification 11 / 12

- 4 pts (a) part incorrect/unattempted

- 4 pts (b) part incorrect/unattempted

- 4 pts (c) part incorrect/unattempted

- 2 pts Partial marks for part (a)

✓ **- 1 pts** *Mostly correct part (a) [If both -1 and -2 are given, it was intentional, and not a mistake]*

- 2 pts Partial for part (b)

- 0 pts All parts correct

QUESTION 3

3 Counting 6 / 6

✓ **+ 1.5 pts** *part a Correct*

✓ **+ 1.5 pts** *part b Correct*

✓ **+ 1.5 pts** *part c Correct*

✓ **- 6 pts** *Normalize*

✓ **+ 1.5 pts** *part d Correct*

- 0 pts Click here to replace this description.

- 0 pts Click here to replace this description.

- 0 pts Click here to replace this description.

- 0 pts Click here to replace this description.

QUESTION 4

4 Chessboard 5 / 5

✓ **+ 1.5 pts** *Invariant used is correct*

✓ **+ 1 pts** *Proved that reversing colors in a row changes the total number of white squares (or black squares) by even number.*

✓ **+ 1 pts** *Proved that reversing colors in a column changes the total number of white squares (or black squares) by even number.*

✓ **+ 1 pts** *Proved that reversing colors in a 2x2 square changes the total number of white squares (or black squares) by even number.*

✓ **+ 0.5 pts** *Correct Conclusion*

+ 0 pts Incorrect

QUESTION 5

5 Bijection 5 / 5

✓ **- 0 pts** *Correct*

- 5 pts Incorrect/ unattempted

- 5 pts Proof that a bijection exists is given without giving a specific bijective function.

- 3 pts Partially correct

- 4 pts The given function is not surjective

QUESTION 6

6 Countable 5 / 5

+ 2.5 pts Partially correct

+ 4 pts Correct with incomplete explanation

✓ **+ 5 pts** *Completely correct solution*

+ 0 pts Incorrect

QUESTION 7

7 Closed Form 4.5 / 5

- 0 pts Correct

+ 0.5 pts Definition of $$S_m$$

- 5 pts No marks

✓ **- 0.5 pts** *Minor mistakes, but correct result*

- 1 pts Informally added or subtracted infinite sums without proving convergence. A formal solution would have involved limits

- 1 pts Incorrect answer but correct method

1 What does this even mean? This is extremely informal. The correct method is to define $$S_m$$ as the sum of $$T_n$$ for $$n=1, 2, \ldots m$$.

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(COL 202) Discrete Mathematics 13 September, 2023

Minor 1

Duration: 120 minutes **(50 points)**

Beware: Be clear in your writing. If you use a statement proved in class or in the problem set, then write down the entire statement before using it. You will not get a new sheet, so make sure you are certain when you write something (maybe use a dark pencil). Make a judicious decision of which tool(s) to use to get a clean and short answer that fits in the space. If you cheat, you will surely get an F in this course.

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1. ($4 \times 3 = 12$ **points)** In this question, each sub-question will have zero or more correct answers. You are to circle each correct answer and leave uncircled each incorrect answer. Each problem is worth 3 points and you get points if and only if you circle all of the correct answers and none of the wrong ones. There are no partial points.

   (a) Let w, b and n be propositions where w is "I walk to work", g is "I work in Gurugram", n is "I work at night". The sentence "When I work nights and I work in Gurugram, I don't walk to work" could be written using propositions and logical connectives as:

   (1) $(n \wedge g) \implies \neg w$ (2) $(n \vee g) \iff n$ (3) $n \implies \neg(w \wedge g)$ (4) $\neg(w \wedge g) \vee n$

   (b) Identify the *tautologies* among the following:

   (1) $(a \implies b) \iff (\neg a \implies \neg b)$ (2) $(a \implies b) \iff (\neg b \implies \neg a)$
   (3) $(a \implies b) \implies a$ (4) $(a \wedge b \wedge c) \iff (b \wedge c \wedge a)$

   (c) Identify those formulae which are *satisfiable*.

   (1) $(a \vee b) \wedge (a \vee \neg b) \wedge (\neg a \vee b) \wedge (\neg a \vee \neg b)$ (2) $(a \wedge b) \wedge (a \wedge \neg b)$
   (3) $(a \implies b) \implies (\neg b \implies \neg a)$ (4) $(a \wedge b) \implies (a \wedge \neg b)$

   (d) For countably infinite sets A and B, $A \cap B$ can be

   (1) Countably infinite (2) Uncountable (3) Finite (4) Empty

2. ($3 \times 4 = 12$ **points)** Answer the following questions with a brief justification.

   (a) Arrange the following functions in a sequence $f_1, f_2, \ldots, f_7$ so that $f_i = O(f_{i-1})$. Additionally, if $f_i = \Theta(f_{i+1})$, indicate that: $n \log n$, $(\log\log n)^{\log n}$, $(\log n)^{\log\log n}$, $n \cdot 2^{\sqrt{\log n}}$, $(\log n)^{\log\log n}$, $n^{1+\frac{1}{\log n}}$, n^2. Assume that all the logarithms are to the base 2.

Sequence $= (\log\log n)^{\log n}, n^2, n.2^{\sqrt{\log n}}, n\log n, n^{1+\frac{1}{\log n}}, (\log n)^{\log\log n}, (\log n)^{\log\log n}$

To determine this sequence, we can take logarithm on all terms and then compare them accordingly.

   (b) How many different ways can you choose 18 muffins from a choice of apple, blueberry, chocolate-chip and date muffins, if there are 9 apple, 8 blueberry, 6 chocolate chip, but an unlimited number of date muffins.

$x_1 + x_2 + x_3 + x_4 = 18$, $x_1 \leq 9$, $x_2 \leq 8$, $x_3 \leq 6$.

We can use Principle of Inclusion - Exclusion.

Ways = Total no. of ways − Ways with $x_1 \geq 10$ − Ways with $x_2 \geq 9$ − Ways with $x_3 \geq 7$

+ Ways with $x_1 \geq 10$ & $x_2 \geq 9$ +

$= \binom{21}{3} - \binom{11}{3} - \binom{12}{3} - \binom{14}{3} + \binom{5}{3} + \binom{4}{3}$

Alternate method for (c):

Let $x_1 = -2 + x_0$, where $x_0 \geqslant 0$.

$\therefore x_1 + x_2 + x_3 + x_4 = -2 + x_0 + x_2 + x_3 + x_4 = 10$

$x_0 + x_2 + x_3 + x_4 = 12$, $x_0 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0, x_4 \geqslant 0$

For each value of x_0, there is only one value of x_1.

$\therefore$ No. of soln. $= \binom{12+3}{3} = \binom{15}{3}$

(c) Count the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 10, x_1 \geq -2, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$.

No. of solutions = Solutions with $x_1 \geqslant 0$ + Soln. with $x_1 = -1$ + Soln. with $x_1 = -2$

Solutions with $x_1 \geqslant 0 = \binom{10+3}{3} = \binom{13}{3}$

Soln. with $x_1 = -1 = \binom{11+2}{2}$, Soln. with $x_1 = -2 = \binom{12+2}{2}$

$\therefore$ Total no. of soln. $= \binom{13}{3} + \binom{13}{2} + \binom{14}{2} = \binom{14}{3} + \binom{14}{2} = \binom{15}{3}$

3. **(6 points)** How many 6-character passwords can be made using only the characters from the set $\{A, B, C, D, E, F, 1, 2, 3, 4\}$ if

(a) The password must contain at least one letter and at least one digit (repeats allowed).

(b) The password contains four letters and two digits (in any order and repeats allowed).

(c) No character is used more than once.

(d) No two letters are adjacent, no two digits are adjacent, and no character is used more than once. Briefly explain your answers for each of the cases.

(a) No. of such passwords = Total no. of passwords − Passwords with no digits − Pwds. with no letters

$= 10^6 - 6^6 - 4^6$

(b) No. of passwords = (Ways of choosing 4 positions for letters) × (Ways to choose 4 letters) × (Ways to choose 2 digits)

$= \binom{6}{4} 6^4 4^2$

(c) No. of passwords = (Ways to choose 6 characters) × (No. of arrangements of 6 unique characters)

$= \binom{10}{6} \times 6! = {}^{10}P_6$

(d) $D_1 L_1 D_2 L_2 D_3 L_3$ or $L_1 D_1 L_2 D_2 L_3 D_3$ ← Only 2 configurations

No. of passwords = 2 × (Ways to choose and arrange 3 letters) × (Ways to choose and arrange 3 digits)

$= 2 \times \binom{6}{3} \times 3! \times \binom{4}{3} \times 3!$

2 configurations

As $P(x_0) = $ true, and if $q \longrightarrow r$, $P(q) \Rightarrow P(r)$, $P(x)$ is true for all states x attainable by machine.
In 63 w & 1 b, $P(x)$ is false.
$\therefore$ It is impossible to attain this state.

4. **(5 points)** An 8×8 chessboard is colored in the usual way, but that's boring, so you decide to fix this. You can take any row, column, or 2×2 square, and reverse the colors inside it, switching black to white and white to black. Prove that it's impossible to end up with 63 white squares and 1 black square.

We can solve this by creating a state machine S.
States $= (w, b)$, $w =$ no. of white squares, $b =$ no. of black squares.

Initial state$(x_0) = (32, 32)$.
Consider a preserved invariant P: $(w-b) \bmod 4 = 0$ for all states attainable by S.

Base case: $P(x_0) = $ true (as $0 \bmod 4 = 0$).

Transition 1: Flipping a row.
If a row has w white squares, it has $8-w$ black squares.
$\therefore$ Initial $w-b = w_0 + w - (8-w) = w_0 + 2w - 8$.
(w_0 for rest of chessboard)

Now, consider $(w_0 + 2w - 8) \bmod 4 = 0$ i.e. $P(\text{current state}) = $ true.
On flipping, $w - b = w_0 + 8 - 2w$
Now, $w_0 + 8 - 2w = w_0 + 2w - 8 + (16 - 4w) \Rightarrow (w_0 + 8 - 2w) \bmod 4 = 0$.
$\therefore$ $P(\text{next state}) = $ true.
$\therefore$ P is preserved for transition 1.

Transition 2: Flipping a column.
As a column also has w white squares & $8-w$ black squares, we can do the same thing as transition 1 & P is preserved.

Transition 3: Flipping a 2×2 square.
If $w_i = w$, $b_i = 4 - w$
If $P((w_i, b_i)) = $ true $\Rightarrow (w_0 + (2w - 4)) \bmod 4 = 0$.
Now, final $w - b = w_0 + 4 - 2w = w_0 + 2w - 4 + 8 - 4w$ (As $w_f = 4 - w$, $b_f = w$)
$= w_0 + 2w - 4 + 4\lambda$.
$\therefore$ $P((w_f, b_f)) = $ true $\Rightarrow$ P is preserved

5. **(5 points)** Recall that for $a, b \in \mathbb{R}$, $[a,b] = \{x \in \mathbb{R} \mid a < x < b\}$ and $(a,b) = \{x \in \mathbb{R} \mid a \le x \le b\}$. Find a bijection from $[0,1]$ to $(0,1)$.

Let $X = [0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ and

$Y = (0,1) = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$

If we show that X surj Y and Y surj X, we can conclude that X bij Y.

Now, as $X \subseteq Y$, Y surj X.

To show X surj Y,

we can define $f : X \to Y$ s.t.

$$f(x) = \begin{cases} 0, & \text{if } x = 1/3 \\ 1, & \text{if } x = 2/3 \\ 3x, & \text{if } 0 < x < 1/3 \\ x, & \text{otherwise} \end{cases}$$

The interval $[0, \frac{1}{3}]$ will cover all x in $[0,1]$, and 0 and 1 at the boundaries are covered by $\frac{1}{3}$ and $\frac{2}{3}$.

$\therefore$ X surj Y.

$\Rightarrow$ X bij Y.

Bijective map: $f : X \to Y$

$$f(x) = \begin{cases} 0, & \text{if } x = 1/3 \\ 1, & \text{if } x = 2/3 \\ x, & \text{if } x \neq \frac{1}{3^n} \text{ and } x \neq \frac{2}{3^n} \text{ for } n \in \mathbb{N} \\ 3x, & \text{if } x = \frac{1}{3^n} \text{ or } x = \frac{2}{3^n} \text{ for } n \in \mathbb{N}, n \geq 2 \end{cases}$$

f is a bijective map because:

1. $x = 0, 1$ & $x \neq \frac{1}{3^n}$ & $\frac{2}{3^n}$ are mapped in a one-one way.
2. Sequence $(\frac{1}{3^2}, \frac{1}{3^3}, \ldots)$ is mapped to sequence $(\frac{1}{3}, \frac{1}{3^2}, \ldots)$ & similarly for $2/3^n$, covering all numbers in both X and Y.

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6. (5 points) If $A = (a_0, a_1, \ldots)$ and $B = (b_0, b_1, \ldots)$ are countably infinite sets, Show that their product $A \times B$ is also a countable set by showing how to list the elements of $A \times B$.

$A \times B = \{(a_i, b_j) \mid i \in \mathbb{N} \cup \{0\}, j \in \mathbb{N} \cup \{0\}\}$

We can map $A \times B$ injectively to $\mathbb{N}$ to show that it is countable.

Consider $f : A \times B \longrightarrow \mathbb{N}$,

$f((a_i, b_j)) = 2^i 3^j, \quad \forall i, j \in \mathbb{N} \cup \{0\}$

Consider any two elements in $A \times B$,

say $X = (a_{i_1}, b_{j_1}), Y = (a_{i_2}, b_{j_2})$

Now, $(i_1, j_1) \neq (i_2, j_2)$ i.e: atleast one of $i_1 \neq i_2$ & $j_1 \neq j_2$ is true.

$\therefore f(X) = 2^{i_1} 3^{j_1}, \quad f(Y) = 2^{i_2} 3^{j_2}$.

Since atleast one of $i_1 \neq i_2$ and $j_1 \neq j_2$ is true,

atleast one of $2^{i_1} \neq 2^{i_2}$ and $3^{j_1} \neq 3^{j_2}$ is true

By Fundamental Theorem of Arithmetic, we can see that $f(X) \neq f(Y)$ for $X \neq Y$.

$\therefore f$ is an injective function.

$\Rightarrow A \times B$ is countable.

7. **(5 points)** Find a closed form for $S = \sum_{n=0}^{\infty} \frac{2n}{3^{n+1}}$

We can use the method of perturbations to find a closed form for S. $\left(T_n = \frac{2n}{3^{n+1}} \right)$

$$S = \frac{2\cdot 0}{3^1} + \frac{2\cdot 1}{3^2} + \frac{2\cdot 2}{3^3} + \cdots \lim_{n\to\infty} T_n$$

$$-\left(\frac{S}{3} = \frac{2\cdot 0}{3^2} + \frac{2\cdot 1}{3^3} + \cdots + \lim_{n\to\infty} \frac{T_n}{3} \right)$$

$$\frac{2S}{3} = \frac{2\cdot 0}{3^1} + \frac{2\cdot 1}{3^2} + \frac{2\cdot 1}{3^3} + \cdots + \lim_{n\to\infty} \frac{1}{3^{n+1}} - \lim_{n\to\infty} \frac{T_n}{3}$$

Now, $\lim_{n\to\infty} \frac{T_n}{3} = \lim_{n\to\infty} \frac{2n}{3^{n+2}} = 0$.

$$\frac{2S}{3} = 2\left(\frac{1}{3^2} + \frac{1}{3^3} + \cdots \right)$$

$$S = 3\left(\frac{1}{3^2} + \frac{1}{3^3} + \cdots \right) = 3 \cdot \frac{1}{3^2} \cdot \frac{1}{1-\frac{1}{3}} = \frac{1/3}{2/3} = \frac{1}{2}$$

$$\therefore \boxed{S = \frac{1}{2}}$$

Proof of $\lim_{n\to\infty} \frac{2n}{3^{n+2}} = 0$ can be given by L'Hopital rule.

$$\lim_{n\to\infty} \frac{2n}{3^{n+2}} = \lim_{n\to\infty} \frac{2}{3^{n+2} \ln 3} = 0$$

$\left(\lim_{n\to\infty} \frac{f(x)}{g(x)} = \lim_{n\to\infty} \frac{f'(x)}{g'(x)} \right.$ if they exist $\left. \right)$