# Quiz-3

**Q3 :** G has exactly one odd length cycle, say C.

**Goal :** Find max matchg in $O(mn)$ time.

**Sol$^n$ :** Use Edmonds Algo.

Number of odd cycle contractions $= 1$

So, time to find aug path will be $O(m)$ only.

**ALTERNATE**

Find the cycle C in $O(m)$ time.

Let H be obtained from G by removng a vertex of C.

$|M_H| \geq |M_G| - 1$ as one vertex removed.

So first find a matchg of H in $O(m\sqrt{n})$ time,

then add back removed vertex.

Finally find an aug path in $O(m)$ time.

# Quiz-3

**Q4** $G = (V, E)$ $\quad L_G$ = Count of vertices free under some max matching

Find size of

$$F_G = \{ S \subsetneq V \mid S \text{ is free under some max matching} \}$$

in $\text{poly}(n) \cdot L_G^{\,def(G)}$ time

**Sol$^n$:** Let $Y$ = Vertices free under some max matching

Time to find $Y$ = $n \cdot$ TIME (MAX-MATCHING)

Possible choices for $S \leq {}^{|Y|}C_{def(G)} \leq |Y|^{def(G)}$

Time to verify any given $S$ = poly$(n)$

# PS 4

## Q4: Given: $G$ is $2k$-edge-conn

Using MINOR Q3A we can find a subgraph $H$ of $G$ s.t.

- $H$ is $2k$-edge-conn
- $H$ has $O(nk)$ edges
- working space $= O(nk^2 \log n)$

**Lemma:** Let $H$ be $2k$-edge-con with $O(nk)$ edges, then we can find an orientation of $H$ that is strongly $k$-edge-conn in $O(nk)$ sp.

**Proof:** We need to "IMPLICITLY" store the sequence
$$H_0 \to H_1 \to H_2 \to \cdots \to H_r \equiv H$$
and then add directions

Space needed to store each o/p is $O(1)$ for edge add or $O(\deg)$ for vertex add

So, total space needed to find seq $D_0 \to D_1 \to \cdots \to D_r$ is $O(nk)$.

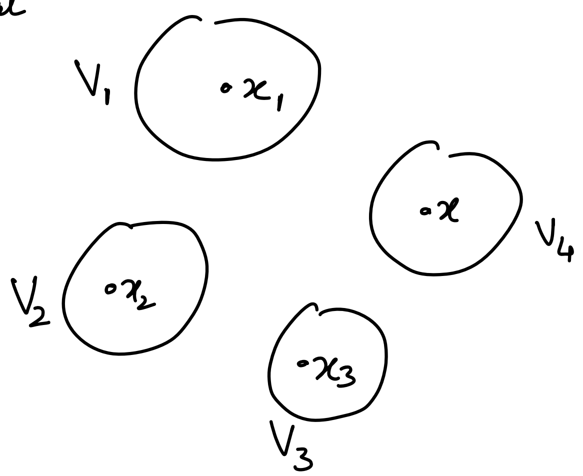
**Lemma:** Let $H$ be a directed graph with $O(nk)$ edges such that $\text{max-flow}(s, v, H) \geq k \quad \forall v \in V(H)$. Then in

$O(nk)$ space we can find a tree rooted at "$s$" such that max-flow$(s, v, H-T) \geq k-1$ $\forall v \in V(H)$.

**Proof:** Recall we defined a tree $T$ as NICE if $\forall X \subseteq V$ containing $s$, size of $(X, X^c)$ cut in $G-T$ is $\geq k-1$. Also any NICE tree of SIZE $\not\leq n$ can be extended. So, we can just try greedy approach. Space used remains $O(nk)$.

**Ques 6** Let us first consider Multiway Cut Prob (where $k = |X|$)

Goal




Let $A$ = opt sol$^n$, and

$A_i = (V_i, V_i^c)$

**Observe:** $wt(A) = \sum_{i=1}^{k} wt(A_i)$

**Assume:** $wt(A_1) \leq \cdots \leq wt(A_k)$

Let $B_i$ = min-cut$(x_i, X - x_i)$, then $wt(B_i) \leq wt(A_i)$

Our sol$^n$ "C" comprises of $k-1$ lightest cuts from $B_1 \ldots B_k$.

Then,

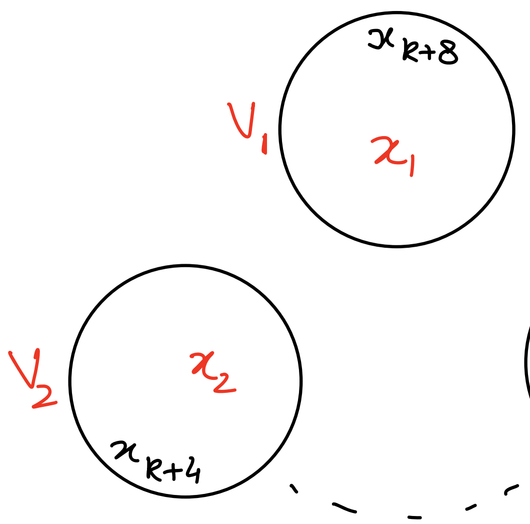
$$wt(C) \leq \sum_{i=1}^{k-1} wt(A_i) \leq \left(1 - \frac{1}{k}\right) \sum_{i=1}^{k} wt(A_i) = 2\left(1 - \frac{1}{k}\right) wt(A)$$

Now consider Stiener Cut Prob with $X = (x_1 \dots x_r)$, $k \leq r$.

Fix an opt sol$^n$ $A$ and let $V_1 \dots V_k$ be corresp partition, such that $V_i$ contains vertex $x_i \in X$.

Define cut $A_i = (V_i, V_i^c)$, and assume $wt(A_1) \leq \dots \leq wt(A_k)$

Note $wt(A) = \frac{1}{2}\left( \sum_{i=1}^{k} wt(A_i) \right)$



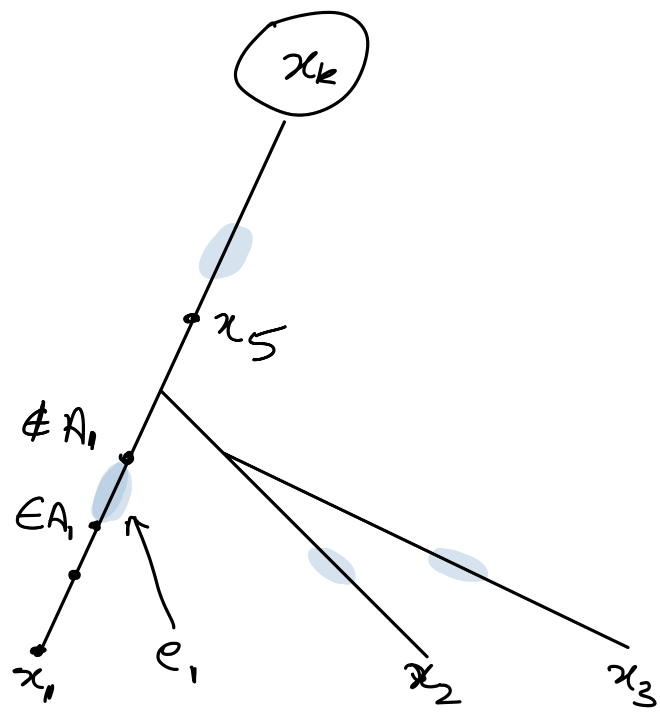

$(V_1 \dots V_k)$ = Partition of $V$

We root GHT at $x_k$

Let $e_i$ = First ancestor edge of $x_i$ not in $G[V_i]$

Then, by GHT def$^n$

$c(e_i) \leq c(A_i)$



Note that $\bigcup_{i=1}^{k-1} cut(e_i)$ also separate $x_1 \dots x_k$

Now let $f_1 \dots f_{k-1}$ be edges of our sol$^n$ in G HT.

So, $\bigcup_{i=1}^{k-1} cut(f_i)$ separate some $k$ vertices in X.

<u>CLAIM</u>: For $i \leq k-1$, we have

$$c(f_i) \leq \max\left(c(e_1) \dots c(e_i)\right) \leq wt(A_i)$$

Bcoz $e_1 \dots e_i$ is some partition of $(x_1, x_2 \dots x_i, x_k)$

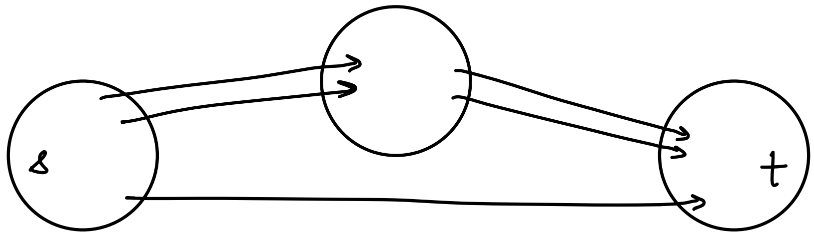
bcoz $wt(A_1) \leq \dots \leq wt(A_k)$ and $c(e_i) \leq wt(A_i)$

and $(f_1 \dots f_i)$ is greedy partition obtained by choosing edges of least wt.

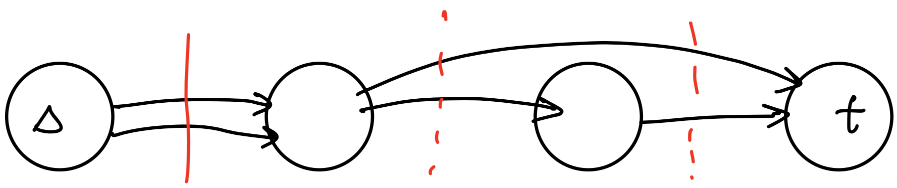
Thus, $wt\left(\bigcup_{i=1}^{k-1} cut(f_i)\right) \leq \sum_{i=1}^{k-1} wt(A_i) \leq 2\left(1-\frac{1}{k}\right) wt(A).$

**Q9:** Compute SCCs of residual graph $G_f$ where $f = (s,t)$-max-flow.

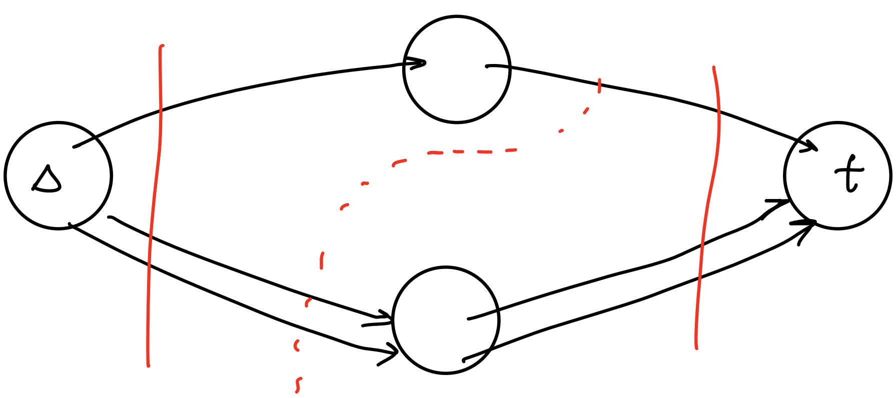
There are at most two distinct $(s,t)$-cuts $\iff$ $G_f^{scc}$ has at most three supernodes




If $G_f^{scc}$ has "$\geq 4$" supernodes then at least 3 cuts




← three



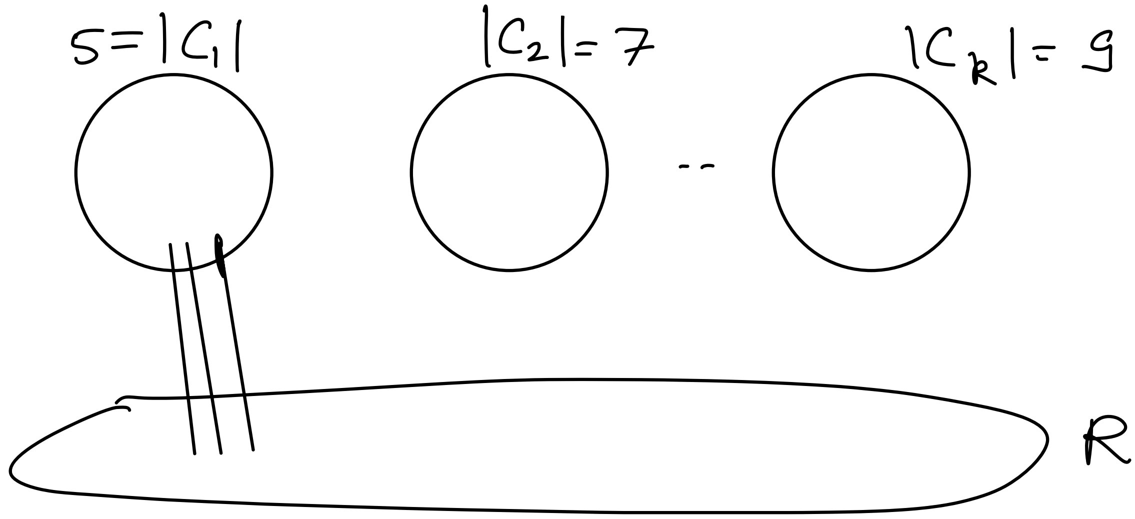

← four

PS 5

Q6 To Prove : Bridgeless cubic graph has perfect matching

By Tutte Berge Thm

$\exists R \subseteq V$ s.t. $def(G) = oc(G-R) - |R|$




CLAIM : For any odd component $C$ in $G-R$, no. of edges leaving $C$ is odd, but not one.

REASON : $\sum_{w \in C} deg(w) = 3|C|$ and edges with both endpoints in $C$ are counted twice

$\Rightarrow oc(G-R) \leq \dfrac{\text{No of edges leaving } R}{3} \leq \dfrac{3|R|}{3} = |R|$

$\Rightarrow def(G) = $ zero.

## Q3 (a) $P$ = A shortest aug-path wrt $M$

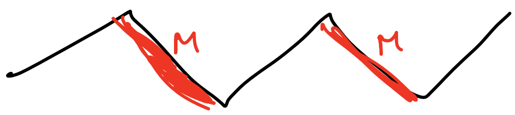
$Q$ = an aug path wrt $M \oplus P$

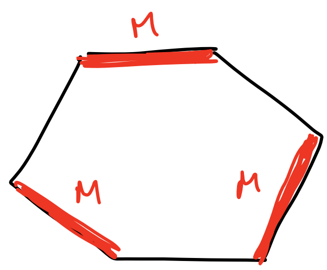
Then, $|Q| \geq |P| + 2|P \cap Q|$

Let $N = M \oplus P \oplus Q$

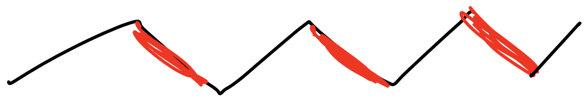
Then $\underline{\underline{M \oplus N}} = P \oplus Q \leftarrow$ size $|P| + |Q| - 2|P \cap Q|$

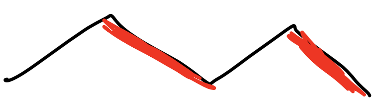
$P_1$




$P_2$





$P_1$ $P_2$ are $M$-aug path and $|P_1|, |P_2| \geq |P|$

$$|P| + |Q| - 2|P \cap Q| = |P \oplus Q| = |M \oplus N| \geq 2|P|$$

$$\Rightarrow |Q| \geq |P| + 2|P \cap Q|$$

(b) Let $\mathcal{C} = (P_1 \dots P_k)$ be collection of shortest M-aug path that is inclusion maximal.

Let $Q$ = Aug path wrt $M \oplus P_1 \oplus \dots \oplus P_k$.

Assume WLG $(Q \cap P_k)$ is not empty

and let $M_0 := M \oplus P_1 \oplus \dots \oplus P_{k-1}$

$Q$ is aug wrt $M_0 \oplus P_k$

$\Rightarrow |Q| \geq |P| + 2|Q \cap P_k| \gneq |P|$