Lecture Summary :

→ What is a proof using axiomatic system? Example using '$\alpha\beta\gamma$' puzzle'. We will prove formally using induction that $\alpha\beta \not\rightarrow \alpha\gamma$

→ Logic puzzle - Blue-eyed island. [Terence Tao's blog] we will see a formal proof using induction.

→ Definition of sets, basic properties

→ Definition of functions - injective, surjective, bijective

→ when are two infinite sets of same/different cardinality?

COL 202 JARGON & NOTATIONS :

SETS :

Def 1.1 : Set - unordered collection of distinct objects

Examples :

$$A = \{3, 7, 11\}$$: Explicit representation

$$= \{4n + 3 : n \in \{0, 1, 2\}\}$$: Implicit representation

Notation : $x \in S$ - x is in set S

$x \notin S$ - x is not in set S

Def 1.2 : Cardinality of S, denoted by $|S|$, is number of elements in S.

Examples of Infinite Sets :

$\mathbb{N}$: natural numbers $\{1, 2, 3, \ldots\}$

$\mathbb{Q}$: rational numbers

$\mathbb{R}$: real numbers

Notation : Empty set / null set - ϕ

Def 1.3 : subset - P is subset of Q
$P \subseteq Q$ if every element of P is element of Q

Q is superset of P.

Def 1.4 : power set of S, $\mathcal{P}(S)$: set of all subsets of S.

OPERATIONS ON SETS :

Union : $A \cup B$

By definition of the union operator, if $z \in A \cup B$, then $z \in A$ or $z \in B$. Similarly, if $z \in A$, then $z \in A \cup B$, for any B.

Intersection : $A \cap B$

By definition of the intersection operator, if $z \in A \cap B$, then $z \in A$ and $z \in B$. Similarly, if $z \in A$ and $z \in B$, then $z \in A \cap B$.

Set difference : $A \setminus B$ - all elements of A which are not in B

By definition of the set difference operator, if $z \in A \setminus B$, then $z \in A$ and $z \notin B$.

→ Union and intersection are associative

$$A \cup (B \cup C) = (A \cup B) \cup C$$

This is an axiom.

Thm 1.1 : $A \cup (B \cap C)$

$= (A \cup B) \cap (A \cup C)$

Distributive Property

$A \cap (B \cup C)$

$= (A \cap B) \cup (A \cap C)$

Pf of $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$:

we will show $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ (i)

and $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ (ii)

(i) Take any $z \in A \cup (B \cap C)$

By def of '$\cup$', z is in A, or z is in $B \cap C$ (or both)

if $z \in A$, then $z \in A \cup B$ and $z \in A \cup C$

By def. of '$\cup$'

if $z \in B \cap C$, then $z \in B$, $z \in C$

By def. of '$\cap$'

$\Rightarrow z \in A \cup B$ and $z \in A \cup C$ [By def. of '$\cup$']

(ii) Take any $z \in (A \cup B) \cap (A \cup C)$.

$z \in A \cup B$ and $z \in A \cup C$

By def. of '$\cap$'

If $z \in A$, then $z \in A \cup (B \cap C)$

By def. of '$\cup$'

If $z \notin A$, then $z \in B$ and $z \in C$

$z \in A \cup B$, and if $z \notin A$, then $z \in B$

Similarly, $z \in A \cup C$, and if $z \notin A$, then $z \in C$

$\Rightarrow z \in B \cap C$

By def of '$\cap$'

$\Rightarrow z \in A \cup (B \cap C)$

By def of '$\cup$'.

Thm 1.2: A - finite set

$A_1 \dots A_k$ are subsets of A s.t.

→ $\bigcup_{i=1}^{k} A_i = A$

→ $\forall\, i \neq j,\ A_i \cap A_j = \phi.$

Then $|A| = \sum_{i=1}^{k} |A_i|$

Proof idea: each element of A is counted exactly once in LHS and RHS.

Def 1.5: Cartesian product of sets

$A \times B = \{(\underline{a, b}) : a \in A,\ b \in B\}$

ordering important

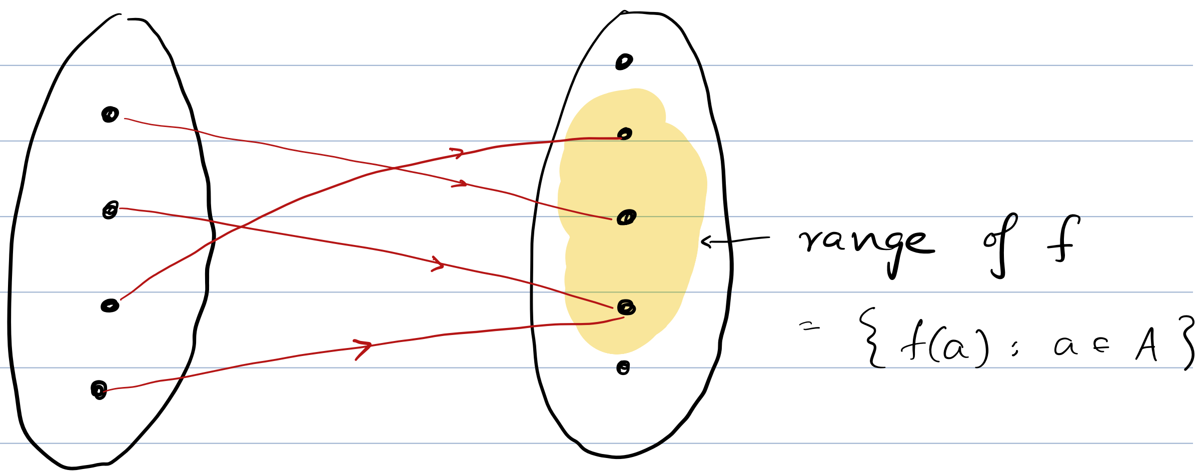
Example:

$\mathbb{N} \times \mathbb{N} = \mathbb{N}^2 = \{(a, b) : a \in \mathbb{N},\ b \in \mathbb{N}\}$

FUNCTIONS:

A, B : sets

$f : A \to B$ mapping from A to B.



domain : A co-domain : B

Def 1.6: $f : A \to B$. For any $b \in B$,

$$\text{preimage}(b) = \{a : f(a) = b\}$$

Thm 1.3: $f : A \to B$, A, B finite sets.

Then $$|A| = \sum_{b \in B} |\text{preimage}(b)|$$

Proof idea: for any $b \in B$, let A_b = preimage(b)

Since f is a function, $\forall\, b \neq b'$,

$$A_b \cap A_{b'} = \emptyset.$$

Moreover, since every element of A is mapped to some element of B,

$$A = \bigcup_{b \in B} A_b.$$

Using Thm 1.2, $|A| = \sum_{b \in B} |A_b|$

$$= \sum_{b \in B} |\text{preimage}(b)|$$

Def 1.7: $f: A \to B$ is injective if $\forall\, a \neq a'$, $f(a) \neq f(a')$

$f: A \to B$ is surjective if $\forall\, b \in B$, $\exists\, a \in A$ s.t. $f(a) = b$

$f: A \to B$ is bijective if it is both injective and surjective.

A, B : finite sets

Thm 1.4 : If $f : A \to B$ is injective, then $|A| \leq |B|$ (i)

If $f : A \to B$ is surjective, then $|A| \geq |B|$ (ii)

If $f : A \to B$ is bijective, then $|A| = |B|$. (iii)

Pf :

(i) $$|A| = \sum_{b \in B} |\text{preimage}(b)|$$

Since f is injective, for any $b \in B$, $|\text{preimage}(b)| \leq 1$

$$\therefore |A| \leq \sum_{b \in B} 1 = |B|$$

(ii) $$|A| = \sum_{b \in B} |\text{preimage}(b)|$$

Since f is surjective, for any $b \in B$, $|\text{preimage}(b)| \geq 1$.

$$\therefore \quad |A| \geq \sum_{b \in B} 1 = |B|$$

(iii) If f is injective and surjective, then $|A| \leq |B|$ and $|A| \geq |B|$.

$\Rightarrow |A| = |B|$ if $\exists$ bijection $f : A \to B$.

Using Thm 1.4, we can conclude that two finite sets A and B have same cardinality if $\exists$ bijection $f : A \to B$.

Def 1.8: Let A, B be two infinite sets. We say that A and B have same cardinality if $\exists$ bijective function $f : A \to B$.

Example: Consider $A = \mathbb{N}$,

$B = \{2, 4, 6, 8, \dots\}$

One might feel that $B \subseteq A$, and therefore B has smaller cardinality than A.

However, there is a bijection between A & B.

$f : A \to B$, $f(x) = 2x$ is a bijection. // In general, you need to prove that f is a bijection

Hence, as per def. 1.8, A and B have same cardinality.

Thm 1.5 : Let $A = \mathbb{N}$, $B = \mathbb{N} \times \mathbb{N}$.

A and B have same cardinality.