COL202: Discrete Mathematical Structures. I semester, 2022-23.
Amitabha Bagchi
Tutorial Sheet 6: Relations, functions and ordering relations.
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**Important:** The question marked with a ♠ is to be written on a sheet of paper and submitted to your TA within the first 10 minutes of the beginning of your tutorial session. Questions marked with a * are optional challenge problems and are not to be discussed in the tutorial.

**Problem 1 [1]**
Prove that for binary relations $\mathcal{R}, \mathcal{R}'$ from $A$ to $B$ and $\mathcal{S}, \mathcal{S}'$ from $B$ to $C$, if $\mathcal{R} \subseteq \mathcal{R}'$ and $\mathcal{S} \subseteq \mathcal{S}'$ then $\mathcal{R} \circ \mathcal{S} \subseteq \mathcal{R}' \circ \mathcal{S}'$.

**Problem 2 [1]**
Given $\mathcal{R} \subseteq A \times B$ and $\mathcal{S}, \mathcal{T} \subseteq B \times C$, prove or find an example that disproves

1. $\mathcal{R} \circ (\mathcal{S} \cup \mathcal{T}) = (\mathcal{R} \circ \mathcal{S}) \cup (\mathcal{R} \circ \mathcal{T})$
2. $\mathcal{R} \circ (\mathcal{S} \cap \mathcal{T}) = (\mathcal{R} \circ \mathcal{S}) \cap (\mathcal{R} \circ \mathcal{T})$
3. $\mathcal{R} \circ (\mathcal{S} \setminus \mathcal{T}) = (\mathcal{R} \circ \mathcal{S}) \setminus (\mathcal{R} \circ \mathcal{T})$

**Problem 3 [1]**
Show that a relation $\mathcal{R}$ on a set $A$ is

1. antisymmetric if and only if $\mathcal{R} \cap \mathcal{R}^{-1} \subseteq \mathcal{I}_A$.
2. transitive if and only $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$.
3. connected if and only if $(A \times A) \setminus \mathcal{I}_A \subseteq \mathcal{R} \cup \mathcal{R}^{-1}$.

**Problem 4 [1]**
Consider any preorder $\mathcal{R}$ on $A$. For each $a \in A$ let $[a]_\mathcal{R} = \{b \in A : a\mathcal{R}b \wedge b\mathcal{R}a\}$. Now let $B = \{[a]_\mathcal{R} : a \in A\}$. Define a relation $\mathcal{S} \subseteq B \times B$ as follows: $[a]_\mathcal{R}\mathcal{S}[b]_\mathcal{R}$ whenever $a\mathcal{R}b$. Show that $\mathcal{S}$ is a partial order.

**Problem 5**
Suppose we have a set $S$ and a partially ordered set $(T, \preceq_T)$, let $\mathcal{F}$ be the set of functions $f : S \to T$, i.e., all the functions from $S$ to $T$. We define a relation, $\preceq$, on $\mathcal{F}$ as follows: $f \preceq g$ if $f(x) \preceq_T g(x)$ for all $x \in S$. Show that $\preceq$ is a partial order on $\mathcal{F}$.

**Problem 6 ♠**
Given a set $X$, let $X_\preceq$ be the set of partial orders on $X$. For any two partial orders $\preceq_1, \preceq_2 \in X_\preceq$ we say that $\preceq_1 \trianglelefteq \preceq_2$ if $x_1 \preceq_1 x_2$ implies $x_1 \preceq_2 x_2$ for all $x_1, x_2 \in X$. Show that $(X_\preceq, \trianglelefteq)$ is a partially ordered set. Is it totally ordered?

**Problem 7**
For any $n > 0$, let $\mathbb{R}^{n\times n}$ be the set of $n \times n$ real matrices. We say an $A \in \mathbb{R}^{n\times n}$ is *positive semi-definite* if for every column vector $\boldsymbol{x} \in \mathbb{R}^{n\times 1}$, $\boldsymbol{x}^T A\boldsymbol{x} \geq 0$. Let $\mathcal{P}^n \subseteq \mathbb{R}^{n\times n}$ be the set of positive semi-definite $n \times n$ real matrices. We say that for $A, B \in \mathcal{P}^n$, $A \preceq B$ if $B - A$ is positive semidefinite. Prove that $\preceq$ defines a partial order on $\mathcal{P}^n$. Is $\preceq$ a total order?

**Problem 8**
Let $(S, \preceq_S)$ and $(T, \preceq_T)$ be two posets defined on disjoint sets $S, T$. The *linear sum* $S \oplus T$ of the two posets is $(S \cup T, \preceq)$ where for $x, y \in S \cup T$ we say $x \preceq y$ if either $x \preceq_S y$ or $x \preceq_T y$ or if $x \in S$ and $y \in T$. Show that $\preceq$ is a partial order on $S \cup T$. Draw an example of a graph for which a normal spanning tree's tree order can be represented as the linear sum of two tree orders.

**Problem 9**
Two partially ordered sets $(S, \preceq_S)$ and $(T, \preceq_T)$ are said to be *isomorphic* if there exists a bijection $f : S \to T$ such that $x \preceq_S y$ if and only if $f(x) \preceq_T f(y)$ for all $x, y \in S$. The function $f$ is called an *isomorphism.* Also a function $f : S \to T$ is said to be *increasing* if $x \preceq_S y$ implies $f(x) \preceq_T f(y)$ for all $x, y \in S$. A function $f : S \to T$ is said to be *strictly* increasing iff for $x \neq y$, $x \preceq_S y$ implies $f(x) \preceq_T f(y)$ and $f(x) \neq f(y)$ (this could also be denoted $f(x) \prec_T f(y)$).

Show by example that an increasing function need not be an isomorphism.

**Problem 10**
Suppose $(S, \preceq_S)$ and $(T, \preceq_T)$ are *isomorphic* and $f : S \to T$ is an isomorphism between them. Show that $f$ and $f^{-1}$ are both strictly increasing functions.

**Problem 11 * requires some knowledge of Linear Algebra**
For $i \in [n]$, let $\lambda_i : \mathcal{P}^n \to \mathbb{R}$ be the function mapping a matrix to its $i$ smallest eigenvalue. Is $\lambda_n$ an increasing function from $(\mathcal{P}^n, \preceq)$ to $(\mathbb{R}, \leq)$ where $\preceq$ and $\mathcal{P}^n$ are as defined in Problem 7? What about $\lambda_1$? What about $\lambda_i$ for $i \neq 1, n$?

# References

[1] S. Arun-Kumar, Lecture notes for *Introduction to Logic for Computer Science.*, IIT Delhi, 2002. http://www.cse.iitd.ernet.in/∼sak/courses/ilcs/logic.pdf