COL202: Discrete Math	Tutorial 5
Instructor: Venkata Koppula	Release date: 20th September, 2024

The solutions for the ($\star$) marked problems must be submitted on Gradescope by 11:59am on the day of your tutorial.

References:

- Lecture notes L13, L14 and L15
- Structural Induction: Section 5.4 from here
- PHP: Section 15.8 from [LLM18]

1 Tutorial Submission Problem ($\star$)

This problem is taken from here (Problem 5.88).
The set $S \subset \{[,]\}^*$ of all strings with balanced square brackets is defined recursively as follows:

- (Base case): The empty string $\epsilon \in S$.
- (Recursive Step):
  1. If $x \in S$, then $[x] \in S$.
  2. If $x_1, x_2 \in S$, then $x_1 x_2 \in S$ (here, x_1 and x_2 are concatenated).

Prove that for any $x \in S$, any prefix y of x, the number of open square brackets (that is, the symbol '[') in y is at least as many as the number of closed square brackets (that is, the symbol ']') in y.

2 Problems - Structural Induction

2.1. Consider the set S defined recursively as follows:

Structural Induction for S subset of K
Strong induction for K subset of S
K -> set of+ve int multiple of 3

- Base case: $3 \in S$
- Recursive step: If $x \in S$ and $y \in S$, then $x + y \in S$

Prove that S is the set of all positive integers that are multiples of 3.

☆ 2.2. Let S be a set of strings of a's and b's recursively defined as follows:

- $a \in S,\ b \in S$
- If $u \in S$ and $v \in S$, then $uv \in S$

answered by GPT

Also, we recursively define an operation R on S as:

- $R(a) = a,\ R(b) = b$

- If $u \in S$, then $R(au) = R(u)a$ and $R(bu) = R(u)b$

Prove by structural induction that for all $u, v \in S$, $R(uv) = R(v)R(u)$.

2.3. This problem is taken from here (Problem 7)

Integer trees are a recursively defined data type. Every tree is either an empty tree, we denote it EmptyTree or a tuple of the form (l, T_1, T_2) where l is an integer and T_1 and T_2 are trees which we call the left and right subtrees respectively. Write an algorithm for finding the minimum integer in the tree. Assume for simplicity that all integers stored are non-negative. Prove the correctness of your algorithm using mathematical induction.

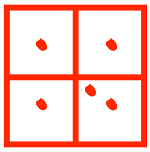
3 Problems - PHP

3.1. (Easy) Show that for any $n+1$ size subset of $\{1, 2, \ldots, 2n\}$ with distinct elements, there exist two numbers that are co-prime.

3.1) (1,2), (3,4), ...(2n-1,2n) each pair co-prime n-pairs

3.2. (Easy) Consider the Cartesian plane with just integer coordinates (that is, the set of all points $(x, y) \in \mathbb{Z} \times \mathbb{Z}$). Let each point be coloured either red or blue. Show that there exists a rectangle which has all the vertices of the same colour.

3.2) solution at end of pdf



3.3. (Easy) Given five points inside a unit square, prove that there must be two points whose distance is at most $\frac{1}{\sqrt{2}}$.

3.4. (♦) Let $f(x)$ be a polynomial with integer coefficients such that $f(x) = 11$ has at least 4 distinct integer roots. Prove that for all integers y, $f(y) \neq 9$.

3.5. (♦) Let n be any odd number such that 5 does not divide n. Prove that there exists a multiple of n that has only ones (in the decimal expansion). Example: if $n = 7$, then 7 divides 111111.

3.5) solution at end of pdf

3.6. (♦) Prove that for any set of 5 points on the surface of a sphere, there exists a closed hemisphere that contains at least 4 of the points.

3.7. Show that for any $n > 0$, there exists a Fibonacci number ending with n zeroes.

Additional Problems for Pigeonhole Principle: Problems 15.38-15.53 from [LLM18]

This is version 1.0 of the tutorial sheet. Let me know if something is unclear. In case of any doubt or for help regarding writing proofs, feel free to contact me or TAs.

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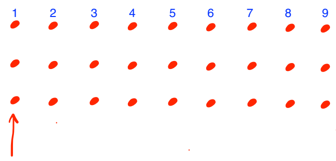
Q 3.2



2^3 combinations of R / B

2 colors 3 dots
so atleast two have same color

9 columns 8 combinations so atleast two columns have same combination

Q 3.5

Consider the series : 1, 11, 111, 1111, … n+1 numbers
There are atleast 2 numbers with same value of % n
Take those 2 numbers from the above and subtract to get Z = 111…1000...0 (a 1s b 0s)
Z = 11…1 x 10^b
Z % n = 0
((11…1 % n) x (10^b % n))%n = 0
but 10^b %n != 0
so, 11...1 % n = 0