Department of Mathematics
MAL 250/MTL 106 (Probability and Stochastic Processes)
Tutorial Sheet No. 2

1. Suppose a jar has n chips numbered $1, 2, 3, \ldots, n$. A person draws a chip, returns it, draws another, returns it, and so on until he gets a chip which has been drawn before and then stops. Let X be the number of drawings required for this purpose. Find the probability distribution of random variable X.

$1)\ 0 \quad 2)\ \frac{1}{n^2} \quad 3)\ \frac{n-1}{n} \times \frac{2}{n} \quad 4)\ \frac{n-1}{n} \times \frac{n-2}{n} \times \frac{3}{n}$

2. Do the following functions define distribution functions.

(a) $F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le \frac{1}{2} \\ 1, & x > \frac{1}{2} \end{cases}$ (b) $F(x) = (\frac{1}{\pi})\tan^{-1}\text{x}, -\infty < x < \infty$ (c) $F(x) = \begin{cases} 1 - e^{-x}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$

not right cont. $RHL = \frac{1}{2} \Rightarrow X$

3. Consider the random variable X that represents the number of people who are hospitalized or die in a single head-on collision on the road in front of a particular spot in a year. The distribution of such random variables are typically obtained from historical data. Without getting into the statistical aspects involved, let us suppose that the cumulative distribution function of X is as follows:

x	0	1	2	3	4	5	6	7	8	9	10
$F(x)$	0.250	0.546	0.898	0.932	0.955	0.972	0.981	0.989	0.995	0.998	1.000

Find (a) $P(X = 10)$ (b) $P(X \le 5/X > 2)$.

a) 0.002 b) $\frac{0.074}{0.102}$

4. Let $B(k, n, p) = P(X \le k)$ denote the cdf of a binomial random variable X with parameters n, p. Show that $B(k, n, p) = 1 - B(n - k - 1, n, 1 - p)$.

5. For what values of α, p does the following function represent a probability mass function $p_X(x) = \alpha p^x, x = 0, 1, 2, \ldots$. Prove that the random variable having such a probability mass function satisfies the following no memory property $P(X > a + s/X > a) = P(X \ge s)$.

6. Let X be a random variable such that $P(X = 2) = \frac{1}{4}$ and its distribution function is given by

$$F_X(x) = \begin{cases} 0, & x < -3 \\ \alpha(x+3), & -3 \le x < 2 \\ \frac{3}{4}, & 2 \le x < 4 \\ \beta x^2, & 4 \le x < 8/\sqrt{3} \\ 1, & x \ge 8/\sqrt{3} \end{cases}.$$

$16\beta = \frac{3}{4} \Rightarrow \beta = \frac{3}{64}$

$5\alpha + \frac{1}{4} = \frac{3}{4} \Rightarrow \alpha = \frac{1}{10}$

Find α, β if 2 is the only jump discontinuity of F. Compute $P(X < 3/X \ge 2)$. $\frac{1/4}{1/2} = \frac{1}{2}$

7. Consider a random experiment of choosing a point in the annular disc of inner radius r_1 and outer radius r_2 $(r_1 < r_2)$. Let X be the distance of chosen point from the center of annular disc. Find the pdf of X.

8. Let X be an absolutely continuous random variable with density function f. Prove that the random variables X and $-X$ have the same distribution function if and only if $f(x) = f(-x)$ for all $x \in \Re$

9. The number of weekly breakdown of a computer is a random variable having a Poisson distribution with $\alpha = .03$.
(a) What is the probability that computer will have even number of breakdown during the given week?.
(b) If the computer is run for 10 consecutive weeks what is the probability that (i) atleast two weeks have no breakdown (ii) 10th is the first week to have a breakdown?.

10. Accidents in Delhi roads involving Blueline buses obey Poisson process with 9 per month of 30 days. In a randomly chosen month of 30 days.
(a) What is the probability that there are exactly 4 accidents in the first 15 days?
(b) Given that exactly 4 accidents occurred in the first 15 days, what is the probability that all the four occurred in the last 7 days out of these 15 days?

7) $\frac{2\pi x\,dx}{\pi(R^2 - r^2)} = f\,dx$. $\frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

8) $F_X(x) = F_{-X}(x)$ iff $f(x) = f(-x)$

$P(X \le x) = P(-X \le x)$

$P(X \le x) = P(X \ge -x) = 1 - P(X < -x) \Rightarrow F_X(x) = 1 - F_X(-x) \Rightarrow f(x) = f(-x)$

11. In a torture test a light switch is turned on and off until it fails. If the probability that the switch will fail any time it is turned 'on' or 'off' is 0.001, what is probability that the switch will fail after it has been turned on or off 1200 times?.

12. The time to failure of certain units is exponentially distributed with parameter λ. At time $t = 0$, n identical units are put in operation. The units operate, so that failure of any unit is not affected by the behavior of the other units. For any $t > 0$, let N_t be the random variable whose value is the number of units still in operation time t. Find the distribution of the random variable N_t.

13. The life time (in hours) of a certain piece of equipment is a continuous random variable X, having pdf

$$f_X(x) = \begin{cases} \frac{xe^{-x/100}}{10^4}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases} .$$

If four pieces of this equipment are selected independently of each other from a lot, what is the probability that atleast two of them have life length more than 20 hours?.

14. Suppose that f and g are density function and that $0 < \lambda < 1$ is a constant. Is $\lambda f + (1 - \lambda)g$ a density function? Is fg a density function? Explain.

15. The probability of hitting an aircraft is 0.001 for each shot. Assume that the number of hits when n shots are fired is a random variable having a binomial distribution. How many shots should be fired so that the probability of hitting with two or more shots is above 0.95?

16. An airline knows that 5 percent of the people making reservation on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. Assume that passengers come to airport are independent with each other. What is the probability that there will be a seat available for every passenger who shows up?

17. Let X be a random variable with cumulative distribution function given by:

$$F_X(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{25}, & 1 \le x < 2 \\ \frac{x^2}{10}, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases} .$$

Determine the cumulative discrete distribution functions F_d and one continuous F_c and the constants α and β with $\alpha + \beta = 1$ such that: $F_X(x) = \alpha F_d(x) + \beta F_c(x)$.

18. Specifications for a certain job calls for washers with an inside diameter of $.300 \pm .005$ inch. (a) If the inside diameter of the washers follows normal distribution with parameter $\mu = .302$ inch and $\sigma = .003$ inch then what percentage of washers meet the specifications?. (b) If 10 such randomly chosen washers are inspected find the probability that not more than 2 would be outside the specifications.

19. Suppose that the life length of two electronic devices say D_1 and D_2 have normal distributions $N(40, 36)$ and $N(45, 9)$ respectively. If a device is to be used for 45 hours, which device would be preferred? If it is to be used for 42 hours which one should be preferred?

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Answer for selected Problems

1. $P[X = k] = (\frac{n-1}{n})(\frac{n-2}{n})\dots(\frac{n-(k-2)}{n})(\frac{k-1}{n})$; $k = 2, 3, \dots, n+1$

2. a)No b)No c)Yes

3. a) 0.002 b) 0.7255

5. $0 < \alpha < 1$, $0 < p < 1$

6. $\alpha = \frac{1}{10}$; $\beta = \frac{3}{64}$; $P(X < 3/X \geq 2) = \frac{1}{2}$

7. $f_X(x) = \begin{cases} \frac{2x}{r_2^2 - r_1^2} & r_1 \leq x \leq r_2 \\ 0 & \text{otherwise} \end{cases}$

9. a)$\sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^{2n}}{(2n)!}$ where $\lambda = 0.03$

   b) (i) $1-[\,{}^{10}C_0 p^0 (1-p)^{10} + {}^{10}C_1 p^1 (1-p)^9]$ (ii) $p^9(1-p)^1$ where $p = e^{-0.03}$

10. N_t: Number of accidents in Delhi roads in time $(0, t]$

    a)$P[N_{15} = 4] = \frac{e^{-\lambda t}(\lambda t)^4}{4!} = 0.1898$ where $\lambda = \frac{9}{30}$, $t = 15$
    b)$\frac{P[N_8=0, N_7=4]}{P[N_{15}=4]} = 0.0474$

11. $(1 - 0.001)^{1200}$

12. $P[N_t = k] = {}^nC_k (e^{-\lambda t})^k (1 - e^{-\lambda t})^{n-k}$ where $k = 0, 1, 2, \dots, n$

13. $1 - [\,{}^4C_1 p^1\,(1-p)^3 + {}^4C_0 (1-p)^4\,p^0]$, where $p = P[X > 20]$

15. $P[X \geq 2] = [1 - [(1-p)^n + {}^nC_1 p^1 (1-p)^{n-1}]] \geq 0.95$ where $p = 0.001$
    $n \simeq 4742$

16. $[1 - (0.95)^{52} - 52C_1(0.05)(0.95)^{51}]$

17. $F_X(x) = \alpha F_d(x) + (1-\alpha)F_c(x)$ where $\alpha = \frac{1}{2}$,

$$F_d(x) = \begin{cases} 0 & x < 1 \\ \frac{2}{25} & 1 \leq x < 2 \\ \frac{4}{5} & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases} \quad ; \quad F_c(x) = \begin{cases} 0 & 0 \leq x < 2 \\ \frac{(x^2-4)}{5} & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

18. a) 83.14% b) $p^{10} +^{10} C_1(1-p)p^9 +^{10} C_2(1-p)^2 p^8$ where $p = 0.8314$

19. $i)D_2$ $ii)D_2$

4)

$B(k, n, p)$

$$f(k) = \sum_{i=0}^{k} {}^nC_i \, p^i \, (1-p)^{n-i}$$

$$B(k, n, p) = 1 - B(n-k-1, n, 1-p)$$

$$\hookrightarrow = \sum_{i=0}^{n-k-1} {}^nC_i \, (1-p)^i \, p^{n-i}$$

$$= \sum_{j=k+1}^{n} {}^nC_j \, p^j \, (1-p)^{n-j}$$

5) $\alpha p^x \quad x = 0, 1, 2 \ldots$

$$\alpha \left(1 + p + p^2 \ldots \ldots p^\infty\right) = 1$$

$\Rightarrow \quad p < 1$

$$\alpha\left(\frac{1}{1-p}\right) = 1$$

$$\Rightarrow \quad \alpha = 1 - p$$

9) a) $\frac{e^{-\alpha}\alpha^2}{2!} + \frac{e^{-\alpha}\alpha^4}{4!} + \frac{e^{-\alpha}\alpha^6}{6!}$

b)

11) $(0.999)^{1200}\left(0.001 + 0.999 \times 0.001 + 0.999 \times 0.001^2 + \ldots \right)$

$$= (0.999)^{1200} \times \frac{0.001}{1 - 0.999}$$

12) $p(x) = \lambda e^{-\lambda x}$

$${}^nC_k \left[\lambda e^{-\lambda t}\right]^k \left[1 - \lambda e^{-\lambda t}\right]^{n-k}$$

13) $f_X(x) = \begin{cases} \frac{x e^{-x/100}}{10^4} & \infty > x > 0 \\ 0 & x \leq 0 \end{cases}$

$$\int_0^t \frac{x e^{-x/100}}{10^4} \, dx = \frac{-100 x}{10^4} e^{-x/100} \Big|_0^t + \frac{100}{10^4} \int_0^t e^{-x/100} \, dx$$

$$= \frac{-t}{100} e^{-t/100} + \frac{1}{\cancel{100}} \cancel{-100} \left(e^{-t/100} - 1\right)$$

$$= 1 - e^{-t/100} - \frac{t}{100} e^{-t/100}$$

$$= 1 - e^{-t/100}\left(1 + \frac{t}{100}\right)$$

$$= P(x > 20) = e^{-20/100}\left(1 + \frac{20}{100}\right) = k$$

$$\text{ans} = {}^4C_2 \, k^2 (1-k)^2 + {}^4C_3 \, k^3 (1-k) + {}^4C_4 \, k^4$$

14) f , g

$\lambda f + (1-\lambda) g$ ✓

fg ×

15) $1 - (0.999)^n - n(0.999)^{n-1}(0.001) \geq 0.95$

$0.05 \geq 0.999^{n-1}(1 + 0.001n)$

16) $1 - {}^{52}C_0\, 0.95^{52} - {}^{52}C_1\, 0.95^{51}\, 0.05$

17) $f_d = \frac{1}{25}$ $x=1$

$\frac{4}{10} - \frac{1}{25}$ $\frac{9}{25}$ $x=2$

$\frac{1}{10}$ $x=3$

$F_D = \begin{cases} 0 & x<1 \\ 2/25 & 1\leq x<2 \\ 4/5 & 2\leq x<3 \\ 1 & 3\leq x \end{cases}$

$\alpha = \frac{1}{2}$

$f_c = \begin{cases} 0 & x<2 \\ x/5 & 3>x>2 \end{cases}$

$\int_2^x x/5\, dx$ $\frac{x^2-2^2}{5\times 2}$

$= F_C = \begin{cases} 0 & x<2 \\ \frac{x^2-2^2}{5} & 2<x<3 \\ 1 & 3<x \end{cases}$

18)