COL202: Discrete Math	Tutorial 9
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The solutions for the ($\star$) marked problems must be submitted on Gradescope by 11:59 am on 15th November.
The ♦ marked problems will be discussed in the tutorial.

In this tutorial, we will discuss **undirected** graphs. Throughout this tutorial, n represents the number of vertices, and m the number of edges. Unless specified otherwise, the graphs have no self-loops or multi-edges. We start with a summary of the definitions and results discussed in class.

Graph Isomorphism. Given two undirected graphs $G = (V, E)$ and $G' = (V', E')$, we say that G and G' are isomorphic if there exists a bijection $f : V \to V'$ such that $\{u, v\} \in E$ if and only if $\{f(u), f(v)\} \in E'$.
We discussed a few different attempts for efficiently testing whether two graphs are isomorphic or not. Currently, the best known algorithm for testing whether two graphs are isomorphic takes $n^{\log n}$ time.

Paths, Cycles, Walks and Tours. A path from a vertex u to a vertex v does not repeat vertices. A cycle is a path that starts and ends at the same vertex. A walk from u to v can repeat vertices and edges. A tour is a walk that does not repeat edges (can repeat vertices). The length of a path/cycle/walk/tour is the number of edges in the path/cycle/walk/tour.

Theorem 1. *Let* $\mathbf{A}$ *denote the adjacency matrix of a graph* G*. For any* $k \in \mathbb{N}$*,* $\mathbf{A}^k[u, v]$ *denotes the number of walks from* u *to* v *of length exactly* k*.*

For any two vertices $u, v \in V$, let

$$\mathsf{dist}_G(u, v) = \begin{cases} \text{length of the shortest path from } u \text{ to } v \text{ in } G \\ \infty \text{ if there exists no path from } u \text{ to } v \text{ in } G \end{cases}$$

Note that

$$\mathsf{dist}_G(u, v) = \begin{cases} \text{the smallest } k > 0 \text{ such that } \mathbf{A}^k[u, v] > 0 \\ \infty \text{ if } \mathbf{A}^k[u, v] = 0 \text{ for all } k \leq n \end{cases}$$

Eulerian Tour. The first important class of results that we saw is related to a famous puzzle from the 1700s called the Konigsberg bridge puzzle. The key takeaways from this part are the proof techniques for characterizing certain families of graphs.

Definition 1. *Let* $G = (V, E)$ *be an undirected graph. An Eulerian tour in* G *is a sequence of vertices* $(v_0, v_1, \ldots, v_k)$ *such that*

- *every consecutive pair form an edge -* $\{v_{i-1}, v_i\} \in E$ *for all* $i \leq k$
- *every edge is visited exactly once: for all* $\{u, v\} \in E$*, there exists a unique index* $j \leq k$ *such that* $((u = v_{j-1}$ *and* $v = v_j)$ *or* $(u = v_j$ *and* $v = v_{j-1}))$*.*

A closed Eulerian tour is an Eulerian tour that starts and ends at the same vertex. A graph G is said to be Eulerian if it has a closed Eulerian tour.

Theorem 2. *An undirected graph $G = (V, E)$ is Eulerian (i.e. has a closed Eulerian tour) if and only if every vertex in V has even degree and G is connected.*

One direction is easy (if the graph is Eulerian, then all vertices must have even degree). The other direction requires some work. We saw three different approaches for proving this result. Please try these approaches by yourself when you are going through this material.

1. induction on the number of vertices (we did not discuss this in detail)

2. approach via cycle partitioning - first show that if all vertices have even degree, then the edge set can be partitioned into cycles. Then 'stitch' together all these cycles to form a closed Eulerian tour.

3. the longest tour approach - consider the longest tour in a graph where every vertex has even degree. First show that this tour starts and ends at the same vertex. Next, show that every vertex is present in this tour. Finally, argue that every edge is present in this tour (and hence it is a closed Eulerian tour).

Hamiltonian Cycles. After Eulerian tours, we discussed a similar-looking, but very different class of graphs called Hamiltonian graphs, defined below.

Definition 2. *Let $G = (V, E)$ be an undirected graph. A Hamiltonian cycle in G is a cycle that visits each vertex exactly once (that is, the length of the cycle is n). Any graph that contains a Hamiltonian cycle is called a Hamiltonian graph.*

Unlike Eulerian graphs, we do not have any efficient tests to determine whether a graph is Hamiltonian or not. However, certain families of graphs are guaranteed to have a Hamiltonian cycle.

Theorem 3 (Dirac, 1952)**.** *Let $G = (V, E)$ be a graph with $n \geq 3$ vertices, where every vertex has degree at least $n/2$. Then G has a Hamiltonian cycle.*

Note that this theorem is tight - there exist graphs where every vertex has degree at least $n/2 - 1$, but the graph is not Hamiltonian. For proving Dirac's theorem, we first argued that if every vertex has degree at least $n/2$, then the graph is connected. Next, we looked at the longest path in this graph. Suppose this path is of length k. We argued that its endpoints have all their neighbors on this path (otherwise we have a longer path). Next, we showed a cycle of length $k + 1$ (here we used the fact that this path's endpoints have degree at least $n/2$). Finally, we argued that the path must contain all vertices (otherwise we can construct a longer path). This gives us a Hamiltonian cycle of length n. See Exercise 2 for an extension of this result.

Graph connectivity when edges/vertices are removed. Even if a graph is connected, the connectivity may be lost if we remove one or more vertices/edges. Graphs where connectivity is preserved even after removal of vertices/edges are called 2-vertex/edge-connected graphs, defined below.

Definition 3 (Bridges and articulation points in a graph). *Let $G = (V, E)$ be an undirected graph. A bridge in G is an edge e whose removal increases the number of connected components in G. An articulation point is a vertex $v \in V$ whose removal increases the number of connected components in G.*

Definition 4 (2-Vertex/Edge Connectivity). *Let $G = (V, E)$ be an undirected graph on $n \geq 3$ vertices. We say that G is 2-vertex-connected (resp. 2-edge-connected) if for every $v \in V$ (resp. every $e \in E$), the graph $G \setminus \{v\}$ obtained after removing v (resp. the graph $G \setminus \{e\}$ obtained after removing e) is connected.*

In class, we proved that a graph is 2-vertex-connected if and only if, for any pair of vertices u, v, there is a cycle containing u and v. In other words, there exist two paths from u to v that don't share any vertices (other than the endpoints u and v). These are called vertex-disjoint paths, which are formally defined below.

Definition 5 (Vertex/Edge Disjoint Paths). *Let $G = (V, E)$ be any undirected graph, and let u, v be two vertices in V. Let $P_1 = (u = v_0, v_1, \ldots, v_{k-1}, v_k = v)$ and $P_2 = (u = w_0, w_1, \ldots, w_{\ell-1}, w_\ell = v)$ be two paths from u to v. We say that P_1 and P_2 are vertex disjoint if the internal vertices of P_1, P_2 are disjoint (that is, $\{v_1, \ldots, v_{k-1}\} \cap \{w_1, \ldots, w_{\ell-1}\} = \varnothing$). Similarly, we say that P_1 and P_2 are edge-disjoint if the set of edges in the paths P_1 and P_2 are disjoint (that is, $\{\{v_{i-1}, v_i\}\}_{i \leq k} \cap \left\{\{w_{j-1}, w_j\}_{j \leq \ell}\right\} = \varnothing$).*

One direction is easy (if every pair of vertices lies on some cycle, then the graph is 2-vertex-connected). For the other direction, we need to perform induction on the distance from u to v in G.

Incremental characterization of 2-vertex-connected graphs. Finally, we discussed in class how to incrementally build 2-vertex-connected graphs. For this, we need to define some operations on graphs.

Definition 6 (Operations on Graphs). *Let $G = (V, E)$ be any undirected graph. We define the following operations on G (which result in a new undirected graph).*

- *Vertex removal: for any $u \in V$, the graph $G - u = (V', E')$ where $V' = V \setminus \{u\}$ and $E' = E \setminus \{\{x, u\}$ s.t. $\{x, u\} \in E\}$.*
- *Edge addition: for any u, v such that $\{u, v\} \notin E$, consider $G + \{u, v\} = (V, E \cup \{u, v\})$*
- *Edge removal: for any u, v such that $\{u, v\} \in E$. consider $G - \{u, v\} = (V, E \setminus \{\{u, v\}\})$.*
- *Edge contraction: for any u, v such that $e = \{u, v\} \in E$, let G/e denote the graph obtained by 'contracting' the edge e (that is, merging the two end-points). More formally, let $V' = (V \setminus \{u, v\}) \cup \{v_e\}$,*[1]

  $E' = (E \setminus \{$*all edges adjacent to either* u *or* $v\}) \cup \{\{x, v_e\} :$ x *is neighbor of* u *or* $v\}$.[2]

  Then $G/e = (V', E')$.

[1] Here v_e is a new vertex not present in V.

[2] If both u and v are connected to some vertex w, we will connect w to the new vertex v_e only once.

- *Edge splitting: for any $e = \{u, v\} \in E$, consider $V' = V \cup \{v_e\}$ [3],*

$$E' = (E \setminus \{e\}) \cup \{\{u, v_e\}, \{v, v_e\}\}.$$

Then $G\%e = (V', E')$.

We discussed that any 2-vertex-connected graph can be obtained by starting with a 3-cycle, and applying edge-additions and edge-splitting. Similarly, we can obtain a 2-vertex-connected graph by first starting with a cycle, and then incrementally adding either edges, or adding 'ears'. An 'ear' added to a graph $G = (V, E)$ is a new graph $G' = (V', E')$, where $V' = V \cup \{v_1, \dots, v_k\}$, and $E' = E \cup \{\{u, v_1\}, \{v_1, v_2\}, \dots, \{v_{k-1}, v_k\}, \{v_k, v\}\}$ for some distinct vertices $u, v \in V$.

1 Tutorial Submission Problem

1.1. Let $G = (V, E)$ be a graph with $n \geq 3$ vertices. Prove that if G is 2-**edge**-connected, then for every $u, v \in V$, there exist two **edge**-disjoint paths from u to v.

2 Graph Theory

2.1. (♦, [MN09], Pg. 136, Ex. 8) For a graph G, let $L(G)$ denote the so-called line graph of G, given by

$$L(G) = \left(E, \left\{\{e, e'\} : e, e' \in E(G), e \cap e' \neq \emptyset\right\}\right).$$

Prove or disprove: G has a closed Eulerian tour **iff** $L(G)$ has a Hamiltonian cycle.

> **Solution:** One direction directly follows from the definitions. If G has a closed Eulerian tour, then there is a Hamiltonian Cycle in $L(G)$.
>
> The other direction does not hold : consider a graph G on vertices $V = \{1, 2, 3, 4\}$, with edge set $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{1, 3\}\}$. This graph does not have a closed Eulerian tour since two vertices have odd degree. However, the line-graph $L(G)$ has a Hamiltonian cycle.

2.2. Prove Ore's Theorem: Let $G = (V, E)$ be any graph on $n \geq 3$ vertices such that for all non-adjacent vertices[4] u, v in V, $\deg(u) + \deg(v) \geq n$. Then G has a Hamiltonian cycle.

> **Solution:** Proof is similar to Dirac's proof that we did in class.

2.3. Prove or disprove: let $G = (V, E)$ be a 2-vertex-connected graph. For every distinct pair of vertices $u, v \in V$, for every path P from u to v, there exists another path Q from u to v such that P and Q are vertex-disjoint.

[3] Here v_e is a new vertex that is not present in V.

[4] Two vertices u, v are said to be non-adjacent if $\{u, v\} \notin E$

Solution: The above statement is not true. Consider a graph on four vertices $\{1,2,3,4\}$. The edge set is $E = \{\{1,2\},\{2,3\},\{3,4\},\{1,3\},\{2,4\}\}$. Now, consider the path $P = (1,2,3,4)$. There is no other path Q from 1 to 4 such that P and Q are vertex-disjoint.

2.4. (♦ [Die12], Ch. 3, Ex. 9) Let G be a 2-vertex-connected graph with at least 4 vertices. Prove that for any edge $e \in E$, either $G - e$ or G/e is 2-vertex-connected.

Solution: Suppose G is a 2-vertex-connected graph, and let e be any edge such that $G - e$ is not 2-vertex-connected. We need to show that G/e is 2-vertex-connected. The solution relies on the following observations (which need a formal proof; you should be able to prove these formally).

Observation 1. *If G is 2-vertex-connected and G has at least 4 vertices, then G has a cycle of length at least 4.*

Since our graph G is 2-vertex-connected and has at least 4 vertices, we can assume it has a cycle of length at least 4. Consider an ear decomposition $(G_0, G_1, \ldots, G_i = G)$ of G starting with a cycle of length at least 4. If $G-e$ is 2-vertex-connected, then we are done. Therefore, let us consider the case where $G-e$ is not 2-vertex-connected. Let k be the first index where G_k contains the edge e.

Observation 2. *Either $k = 0$, or G_k was obtained from G_{k-1} using ear addition.*

Finally, we consider the 'edge-contracted' graphs from k to i, and argue that (a) we eventually get G/e, and (b) all intermediate steps are either edge or ear additions.

Observation 3. *For all $j < k$, let $\hat{G}_j = G_j$. For all $j \geq k$, let $\hat{G}_j = G_j/e$. Then, $\hat{G}_0$ is a cycle, and for all $j \leq i$, $\hat{G}_j$ can be obtained from $\hat{G}_{j-1}$ using either edge or ear additions.*

Note that the proof of this observation uses the fact that $G - e$ is not 2-vertex-connected, make sure you are using it.

Since $\hat{G}_0$ is 2-vertex-connected, and ear and edge additions preserve 2-vertex-connectivity, the final graph in this sequence ($\hat{G}_i$) is also 2-vertex-connected.

2.5. ([Wes01], Ex. 4.2.14) Let $G = (V, E)$ be an undirected graph. A (u, v) necklace in a graph is a sequence of cycles $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots, G_k = (V_k, E_k)$ such that

- each G_i is a cycle in G (that is, $V_i \subseteq V$, $E_i \subseteq E$ for all i),
- the consecutive cycles share exactly one vertex (that is, $|V_i \cap V_{i-1}| = 1$ for all $2 \leq i \leq k$),
- the non-consecutive cycles are disjoint (that is, for all i, j such that $i < j - 1$, $|V_i \cap V_j| = 0$),

- $u \in V_1$ and $v \in V_k$.

Prove that G is 2-edge-connected if and only if for all $u, v \in G$, there is a (u, v) necklace in G.

Solution:

In class, we proved that a graph is 2-vertex-connected if and only if, for every pair of vertices $\{u, v\}$, there exist two vertex-disjoint paths from u to v. Similarly, one can show that a graph G is 2-edge-connected if and only if, for every pair of vertices $\{u, v\}$, there exist two edge-disjoint paths from u to v. We will use this for showing that a graph is 2-edge-connected if and only if it contains, for every pair $\{u, v\}$, a (u, v)-necklace.

- If there exists a (u, v)-necklace for every pair of vertices $\{u, v\}$: a (u, v)-necklace gives two edge-disjoint paths from u to v. As a result, since for every pair $\{u, v\}$, there are two edge-disjoint paths from u to v, the graph is 2-edge-disconnected.
- If G is 2-edge-connected: TODO

2.6. (♦) Prove that any n vertex graph G has at most $n - 1$ bridges.

Solution: The removal of every bridge divides the graph into two connected components. At max there can be n connected components.

3 Some practice problems for Quiz 4

The syllabus for Quiz 4 is Lectures 22-25. Here are a couple of sample Quiz 4 papers.

3.1 Paper 1

3.1. (15 marks) Construct a graph $G = (V, E)$ on n vertices that is 2-vertex-connected, has $\Theta(n^2)$ edges, but does not have a Hamiltonian cycle. Discuss informally why the graph does not have a Hamiltonian cycle.

3.2. (25 marks) Prove that in any 2-vertex-connected graph, for any two edges $e_1, e_2 \in E$, there exists a cycle in G containing both e_1 and e_2.

3.2 Paper 2

3.1. (20 marks) We say that two graphs have the same degree sequence if both graphs have the same number of vertices, and if we list the degrees of the vertices in each graph (in non-decreasing order), then both degree sequences are identical.

Construct two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that both graphs have a closed Eulerian tour, and both have an identical degree sequence, but the two graphs are not isomorphic.

3.2. (20 marks) Prove that every connected graph on $n \geq 3$ vertices has two vertices u, v such that $G - u$ and $G - v$ are both connected graphs.

References

[Die12] Reinhard Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.

[MN09] Jirí Matousek and Jaroslav Nesetril. *Invitation to Discrete Mathematics (2. ed.)*. Oxford University Press, 2009.

[Wes01] D.B. West. *Introduction to Graph Theory.* Featured Titles for Graph Theory. Prentice Hall, 2001.