# Intermediate Microeconomic Theory

Tools and Step-by-Step Examples

## Chapter 8:
## Cost Minimization

# Outline

- Isocost Lines
- Cost-Minimization Problem
- Input Demands
- Cost Functions
- Type of Costs
- Average and Marginal Cost
- Economies of Scale, Scope, and Experience
- Appendix. Cost-Minimization Problem–A Lagrangian Analysis

# Isocost Lines

# Isocost Lines

- An isocost line is the set of input combinations that yield the same total cost for the firm.

  That is, the combinations of $L$ and $K$ for which

$$TC = wL + rK,$$

  where $w > 0$ is the price of every unit of labor (wage per hour);

  $r > 0$ is the cost of each unit of capital (interest rate);

  $TC$ is a given total cost that the firm incurs.

# Isocost Lines

- This figure depicts the isocost line $TC = wl + rK$ or after solving for $K$, $K = \frac{TC}{r} - \frac{w}{r}L$.

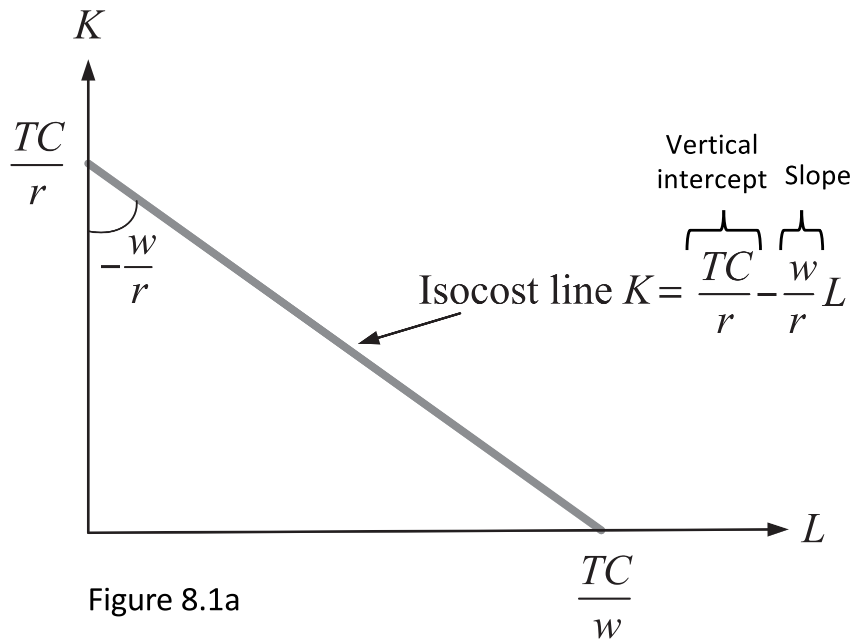


Figure 8.1a

- The firm faces a linear isocost regardless of its production function $q = f(L, K)$, because the isocost line is just a sum of costs.

# Isocost Lines

- An increase in $TC$ produces an increase in both the vertical $\frac{TC}{r}$ and horizontal $\frac{TC}{w}$ intercept, without altering the slope $\frac{w}{r}$.
  - It produces a parallel upward shift in the isocost line.
  - As the firm can incur in larger cost, it can choose among higher input combinations.

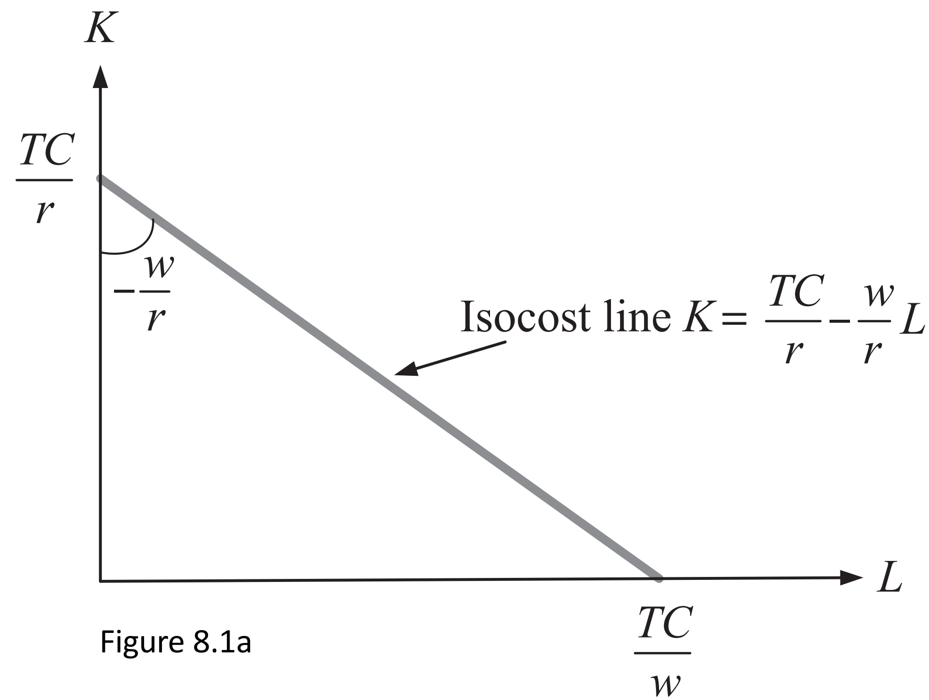


Figure 8.1a

# Isocost Lines

- If wages $w$ increase, the vertical intercept $\frac{TC}{r}$ is not affected, but the absolute value of the slope $\left|\frac{w}{r}\right|$ increases.
  - The isocost becomes steeper.
  - The firm can afford to hire fewer workers as their wages increase.

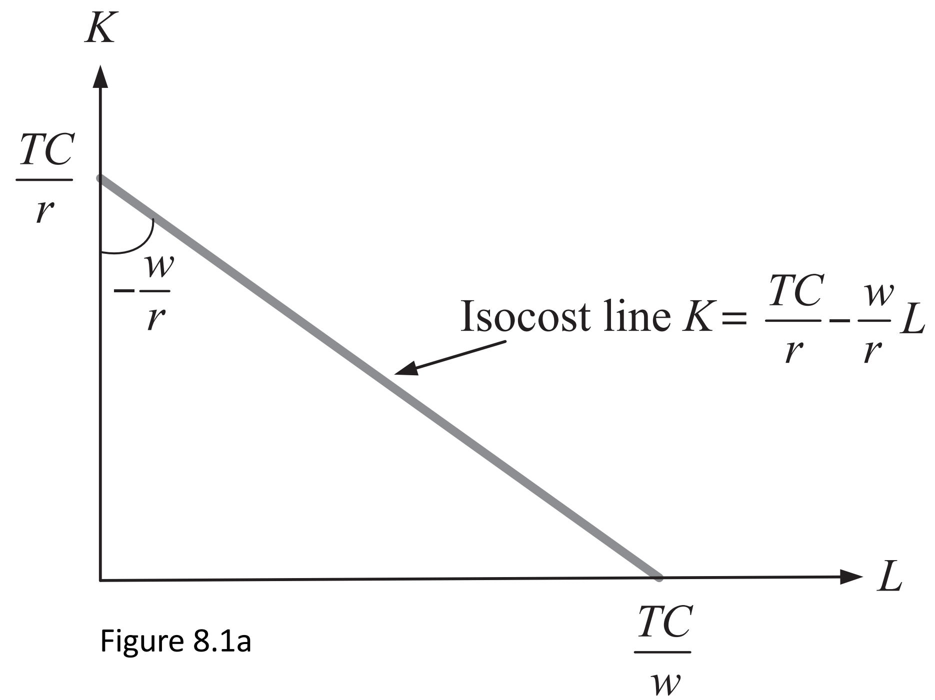


Figure 8.1a

# Isocost Lines

- If the interest rate $r$ increases, the vertical intercept $\frac{TC}{r}$ decreases, and the absolute value of the slope $\left|\frac{w}{r}\right|$ decreases.
  - The isocost becomes flatter.
  - The firm can afford fewer units of capital as its price increases.

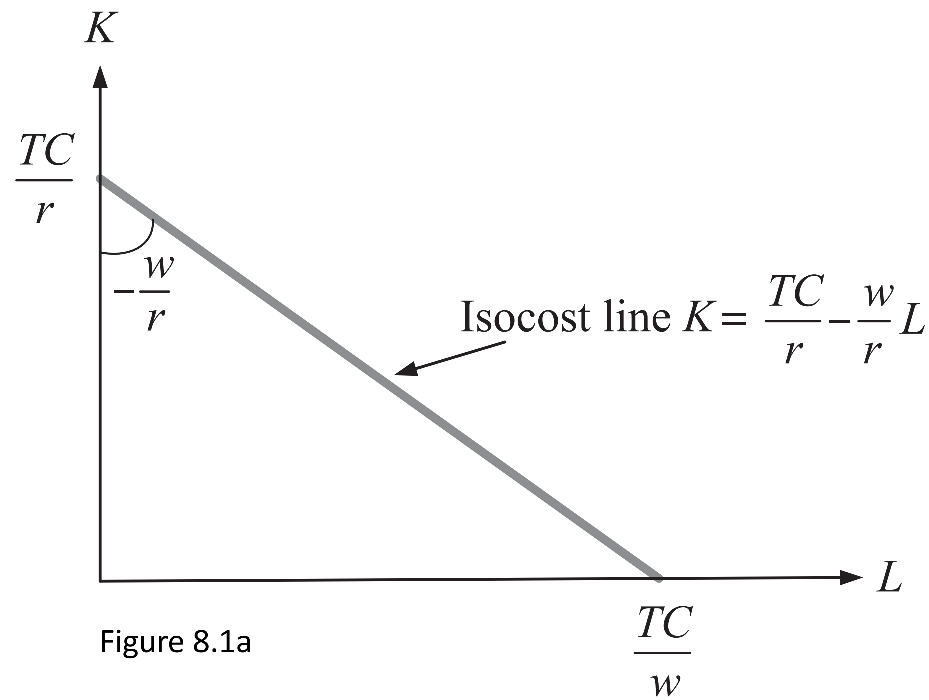


Figure 8.1a

# Isocost Lines

- *Example 8.1: A particular isocost.*
  - Consider a firm facing $w = \$10$, $r = \$15$, and incurring $TC = \$200$.
  - Its isocost line would be $200 = 10L + 15K$, or after solving for $K$, $K = \frac{200}{15} - \frac{10}{15}L$.

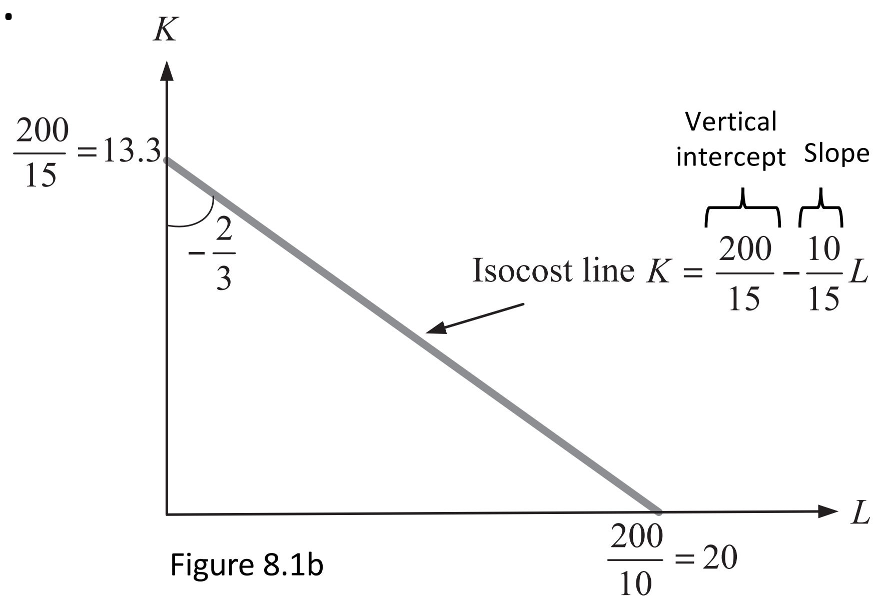


Figure 8.1b

# Cost-Minimization Problem

# Cost-Minimization Problem

- We combine the isoquant and the isocost to determine how many units of labor and capital the firm optimally hires.
- This figure depicts an isoquant line where the firm produces 100 units of inputs, with a set of isocosts each with a $TC$.

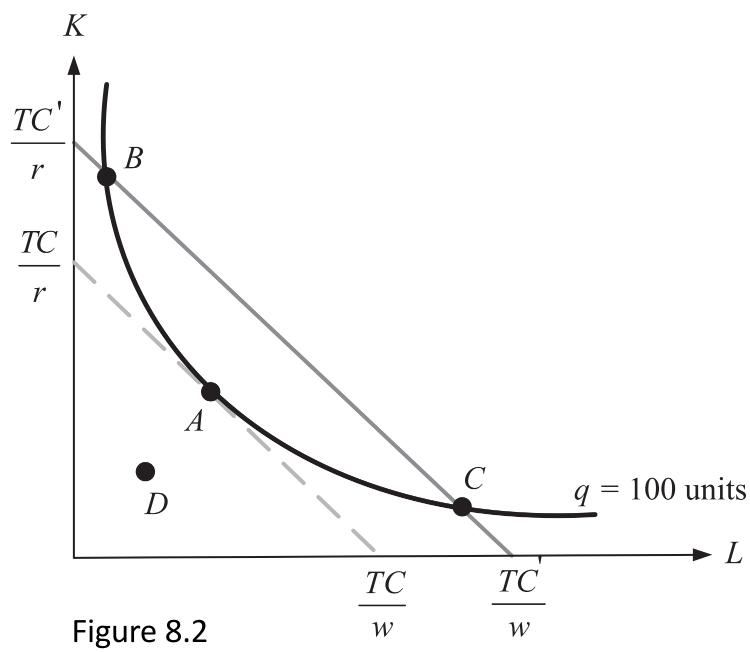


Figure 8.2

# Cost-Minimization Problem

- The cost-minimization problem (CMP) can be represented as

$$\min_{L,K} TC = wL + rK$$

$$\text{subject to } 100 = f(L, K).$$

- The problem ask the firm:

*Choose the input combination that minimizes your total cost TC, reaching an output level of 100 units.*

# Cost-Minimization Problem

- The CMP entails pushing the isocost inward, and reach the isoquant where $q = 100$.
  - Points $B$ or $C$ cannot be cost minimizing because, while the firm reaches $q = 100$, it does at a cost that could be reduced.
  - At point $A$, the firm minimizes its total cost and reaches $q = 100$.
  - At point $D$, with cheaper combinations of inputs, the firm does not reach the target $q = 100$.

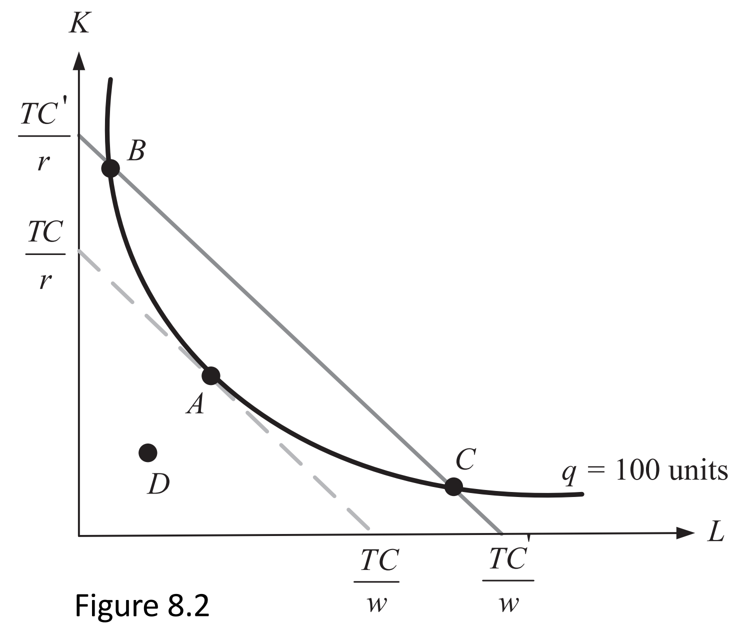


Figure 8.2

# Cost-Minimization Problem

- Combinations of labor and capital minimizing the firm's cost require that the firm's isoquant is tangent to its isocost
- This tangency condition implies that the slope of the isoquant (MRTS) and isocost coincide,

$$\frac{MP_L}{MP_K} = \frac{w}{r},$$

Or after cross-multiplying

$$\frac{MP_L}{w} = \frac{MP_K}{r}.$$

# Cost-Minimization Problem

- The condition $\frac{MP_L}{w} = \frac{MP_K}{r}$ states that when minimizing its cost, the firm rearranges inputs until the point where marginal product per \$ spent on additional units of labor coincide with that of capital
  - *Bang for the buck* must be the same across all inputs.

- If $\frac{MP_L}{w} > \frac{MP_K}{r}$, the firm could decrease its total costs by acquiring fewer units of capital, and using the savings to hire more workers who provide a higher marginal product per \$.

# Cost-Minimization Problem

- Tool 8.1. *Procedure to solve the Cost-Minimization Problem (CMP):*
  1. Set the tangency condition $\frac{MP_L}{MP_K} = \frac{w}{r}$. Cross-multiply and simplify.
  2. If the expression for the tangency condition:
     a. Contains both unknowns ($L$ and $K$), solve for $K$, and insert the result into the firm's output target $q = f(L, K)$.
     b. Contains only one unknown ($L$ or $K$), solve for that unknown, and insert the result into the firm's output target $q = f(L, K)$.

# Cost-Minimization Problem

- Tool 8.1. *Procedure to solve the Cost-Minimization Problem (CMP)* (cont.):

    2. If the expression for the tangency condition:

        c. Contains no input $L$ or $K$, compare $\frac{MP_L}{w}$ against $\frac{MP_K}{r}$.

        - If $\frac{MP_L}{w} > \frac{MP_K}{r}$, set $K = 0$ in the output target and solve for $L$.
        - If $\frac{MP_L}{w} < \frac{MP_K}{r}$, set $L = 0$ in the output target and solve for $K$.

# Cost-Minimization Problem

- Tool 8.1. *Procedure to solve the Cost Maximization Problem (CMP)* (cont.):

   3. If in step 2, one of inputs is negative (e.g., $L = -2$), then set the amount of that input equal to $0$ on the firm's output target (e.g., $q = a0 + bK$), and solve for the remaining input.
   4. If the values for all the unknowns $L$ and $K$ have not been found yet, use the tangency conditions from step 1 to find the remaining unknown.

# Cost-Minimization Problem

- *Example 8.2*: *CMP with Cobb-Douglas production functions*.
  - Consider a firm with Cobb-Douglas production function
    $$q = L^{1/2}K^{1/2},$$
    seeking to reach $q = 100$ and facing $w = \$40$, and $r = \$10$.
  - *Step 1*. Set the tangency condition, $\frac{MP_L}{MP_K} = \frac{w}{r}$,
    $$\frac{\frac{1}{2}L^{-1/2}K^{1/2}}{\frac{1}{2}L^{1/2}K^{-1/2}} = \frac{40}{10} \quad \Longrightarrow \frac{K}{L} = 4.$$
  - Solving for $K$, $K = 4L$.

    This result contains both inputs $K$ and $L$, so we move to step 2a.

# Cost-Minimization Problem

- *Example 8.2* (continued):
  - *Step 2a*. Inserting $K = 4L$ into the output target, $q = 100$, $100 = L^{1/2}K^{1/2}$,
  $$100 = L^{1/2}\underbrace{(4L)}_{K}{}^{1/2}.$$
  Rearranging and solving for $L$,
  $$100 = (4)^{1/2}L,$$
  $$L = \frac{100}{(4)^{1/2}} = \frac{100}{2} = 50 \text{ workers.}$$
  Because the firm hires a positive number of workers, we move to step 4.
  - *Step 4*. Plugging $L = 50$ into the tangency condition $K = 4L$, we find $K = 4{\times}50 = 200$ units of capital.

# Cost-Minimization Problem

- *Example 8.2* (continued):
  - *Summary*. The cost-minimizing input combination is
  $$(L, K) = (50{,}200).$$
  The firm uses more capital than labor because labor is four times as expensive as capital, while their marginal productivities are symmetric.

# Cost-Minimization Problem

- *Example 8.3*: *CMP with linear production functions*.
  - Consider a firm linear production function

    $$q = 2L + 8K,$$

    seeking to reach $q = 100$ and facing $w = \$40$, and $r = \$10$.
  - *Step 1*. Set the tangency condition, $\frac{MP_L}{MP_K} = \frac{w}{r}$,

    $$\frac{2}{8} = \frac{40}{10},$$

    which cannot hold because each side corresponds to a different number!

    As this result contains neither $K$ nor $L$, we move to step 2c.

# Cost-Minimization Problem

- *Example 8.3* (continued):
  - *Step 2c*. We obtained $\frac{2}{8} < \frac{40}{10}$, which entails $\frac{MP_L}{MP_K} < \frac{w}{r}$, or

$$\frac{MP_L}{w} < \frac{MP_K}{r}.$$

    The firm increases its purchases of capital as much as possible, leading to a corner solution where the firm only purchases capital but no labor ($L = 0$).

# Cost-Minimization Problem

- *Example 8.3* (continued):
  - *Step 4*. Inserting $L = 0$ into the output target of the firm, $100 = 2L + 8K$, and solving for $K$,

    $$100 = (2 \times 0) + 8K \qquad \Longrightarrow K = \frac{100}{8} = 12.5 \text{ units.}$$

  - *Summary*. The cost minimizing input combinations is $(L, K) = (0, 12.5)$.

# Input Demands

# Input Demands

- We now use the previous analysis in a more general setting, where input prices ($w$ and $r$) and output target $q$ are not concrete numbers but parameters.
- It allows us to find labor and capital demands and do comparative statics.

# Input Demands

- *Example 8.4*: *Finding input demands with Cobb-Douglas production function*.
  - Consider a firm with Cobb-Douglas production function
    $$q = L^{1/2}K^{1/2},$$
    seeking to reach $q$, and facing input prices $w$ and $r$.
  - *Step 1*. Set the tangency condition, $\frac{MP_L}{MP_K} = \frac{w}{r}$,
    $$\frac{\frac{1}{2}L^{-1/2}K^{1/2}}{\frac{1}{2}L^{1/2}K^{-1/2}} = \frac{w}{r} \qquad \Longrightarrow \frac{K}{L} = \frac{w}{r}$$
  - Solving for $K$, $K = \frac{w}{r}L$.

    This result contains both $K$ and $L$, so we move to step 2a.

# Input Demands

- *Example 8.4* (continued):
  - *Step 2a*. Inserting $K = \frac{w}{r}L$ into the output target, $q = L^{1/2}K^{1/2}$,

$$q = L^{1/2}\underbrace{\left(\frac{w}{r}L\right)}_{K}^{1/2}.$$

Rearranging, and solving for $L$,

$$q = \left(\frac{w}{r}\right)^{1/2} L,$$

$$L = \frac{q}{\left(\frac{w}{r}\right)^{1/2}} = \frac{q\sqrt{r}}{\sqrt{r}}.$$

# Input Demands

- *Example 8.4* (continued):
  - *Step 4*. Plugging labor demand $L = \frac{q\sqrt{r}}{\sqrt{w}}$ into the tangency condition $K = \frac{w}{r}L$, we find that capital demand is

$$K = \frac{w}{r}\frac{q\sqrt{r}}{\sqrt{w}} = \frac{q\sqrt{w}}{\sqrt{r}}.$$

  - If we evaluate labor and capital input demands at the parameter values in example 8.3, with $q = 100$ units, $w = \$40$, and $r = \$10$, we obtain the same results,

$$L = \frac{100\sqrt{10}}{\sqrt{40}} = 50 \text{ workers},$$

$$K = \frac{100\sqrt{40}}{\sqrt{10}} = 200 \text{ units of capital}.$$

# Input Demands

- Comparative statics with input demands from example 8.4. (with Cobb-Douglas production function):
  - Labor demand, $L = \frac{q\sqrt{r}}{\sqrt{w}}$:
    - *Increasing in* $q$. As the firm seeks to produce more units, it needs to hire more workers.
    - *Decreasing in* $w$. As it faces higher salaries, it responds hiring less workers.
    - *Increasing in* $r$. As capital becomes more expensive, labor becomes relatively more attractive, and the firm responds hiring more workers.
  - Capital demand, $K = \frac{q\sqrt{w}}{\sqrt{r}}$:
    - Increasing in $q$, decreasing in $r$, but increasing in $w$.

# Input Demands

- *Example 8.5*: *Finding input demands with a linear production function*.
  - Consider a firm linear production function
$$q = 2L + 8K,$$
    seeking to reach $q$, and facing input prices $w$ and $r$.
  - *Step 1*. Set the tangency condition, $\frac{MP_L}{MP_K} = \frac{w}{r}$,
$$\frac{2}{8} = \frac{w}{r}.$$
    As this result contains neither $K$ nor $L$, we move to step 2c.

# Input Demands

- *Example 8.5* (continued):
  - *Step 2c*. Comparing the marginal product per \$ across inputs,

$$\frac{MP_L}{w} < \frac{MP_K}{r} \text{ if } \frac{2}{8} < \frac{w}{r},$$
$$\frac{1}{4} < \frac{w}{r},$$

which induces the firm to hire no workers ($L = 0$).

Otherwise, the marginal product per \$ spent on labor is now higher than that on capital, entailing that the firm hires no capital ($K = 0$).

# Input Demands

- *Example 8.5* (continued):
  - *Step 4*: If $\frac{1}{4} < \frac{w}{r}$, $L = 0$.

    The demand for capital is found inserting $L = 0$ into output target $q = 2L + 8K$ and solving for $K$,

    $$q = (2\times 0) + 8K,$$

    $$K = \frac{q}{8},$$

    which is increasing in $q$.

# Input Demands

- *Example 8.5* (continued):
  - *Step 4* (cont.):
    - If $\frac{1}{4} > \frac{w}{r}$, $K = 0$. The demand for labor is found inserting $K = 0$ into output target $q = 2L + 8K$ and solving for $L$,

$q = 2L + (8 \times 0) \quad \Longrightarrow L = \frac{q}{2}$, which is also increasing in $q$.

# Input Demands

- Comparative statics with input demands from example 8.5. (with linear production function):
  - Labor and capital demands, $L = \frac{q}{2}$ and $K = \frac{q}{8}$, are increasing in the output $q$ the firm seeks to produce.
  - An increase in salary $w$ does not affect any of the input demands, except in one scenario:
    - When $\frac{1}{4} > \frac{w}{r}$, the firm produces using $(L, K) = \left(\frac{q}{2}, 0\right)$; but if $w$ increases enough to yield $\frac{1}{4} < \frac{w}{r}$, the firm changes its input usage to $(L, K) = \left(0, \frac{q}{8}\right)$.

# Input Demand–Responses

- ***Response to changes in its own price.***
  - The demand for an input is decreasing in its own price → the input demand has a negative slope.

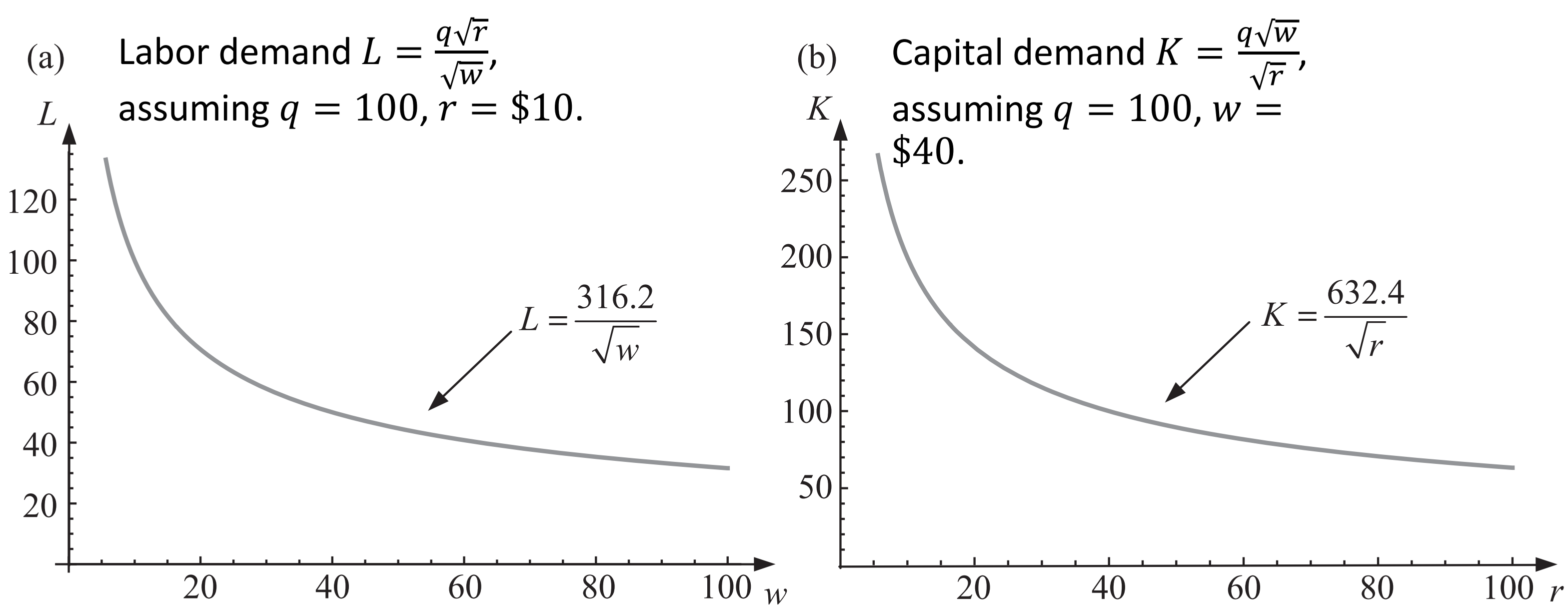


Figure 8.3

# Input Demand–Responses

- *Response to changes in its own price*.
  - The sensitivity of input demand to variations in its price is measured using its price elasticity,
$$\varepsilon_{L,w} = \frac{\%\Delta L}{\%\Delta w} = \frac{\frac{\Delta L}{L}}{\frac{\Delta w}{w}} = \frac{\Delta L}{\Delta w}\frac{w}{L},$$
or, if the change in salary $w$ is infinitely small, $\varepsilon_{L,w} = \frac{\partial L}{\partial w}\frac{w}{L}$, where $\frac{\partial L}{\partial w}$ represents the slope of the labor demand curve.
    - If salaries $w$ increase by $1\%$, the firm would reduce the number of workers its hires by $\varepsilon_{L,W}\%$.
  - Similarly, the elasticity of capital with respect to its price $r$ is $\varepsilon_{K,r} = \frac{\partial K}{\partial r}\frac{r}{K}$.

# Input Demand–Responses

- *Response to changes in its own price.*
  - In the case of the fixed-proportion production function, input demand becomes vertical, as the firm does not change its input combination when input prices change.
    - The slope of labor demand is $\frac{\partial L}{\partial w} = -\infty$, yielding $\varepsilon_{L,W} = -\infty$.
  - In the case of a linear production function, its input demand is flat.
    - The slope of labor demand is $\frac{\partial L}{\partial w} = 0$, yielding $\varepsilon_{L,W} = 0$.

# Input Demand–Responses

- *Response to changes in the price of the other input.*
  - The demand for an input increases as we increase the price of the other input, shifting upwards.
    - As labor becomes more expensive (higher w), capital becomes more attractive.
    - In example 8.4,
      - the demand for capital $K = \frac{q\sqrt{w}}{\sqrt{r}}$ increases in salaries, $w$;
      - the demand for labor $K = \frac{q\sqrt{r}}{\sqrt{r}}$ increases in the price of capital $r$.
      - Graphically, the demand function for labor (capital) would shift outwards as the price of the other input, capital (labor), becomes more expensive.

# Input Demand–Responses

- *Response to changes output*.
  - When the firm increases the demand for inputs to produce more units of $q$, such input is *normal*.
  - When the firm's input demands decrease in $q$, the input is *inferior*.
  - *Example*: A firm with different types of labor:
    - Chief executive officers, midlevel managers, sellers, accountants, secretaries, information technology personnel, and janitors.
    - While it may initially hire more workers in all categories as it increases in output, it might sign software contracts when output is large enough, and as a result firing some secretaries which would become inferior inputs.

# Cost Functions

# Cost Functions

- Total cost. The expenditures that a firm incurs when hiring the optimal amounts of labor and capital identified by its labor and capital demand,

$$TC = wL^{*} + rK^{*}.$$

# Cost Functions

- *Example 8.6: Finding TC in the Cobb-Douglas case.*
  - Labor and capital demands found in example 8.4 were $L = \frac{q\sqrt{r}}{\sqrt{w}}$ and $K = \frac{q\sqrt{w}}{\sqrt{r}}$.
  - Total cost is

$$TC = w \overbrace{\frac{q\sqrt{r}}{\sqrt{w}}}^{L} + r \overbrace{\frac{q\sqrt{w}}{\sqrt{r}}}^{K}$$

$$= qw^{1/2}r^{1/2} + qr^{1/2}w^{1/2}$$

$$= 2q\sqrt{rw}.$$

# Cost Functions

- *Example 8.6* (continued):
  - Total cost

$$TC = 2q\sqrt{rw}$$

  increases as $q$, $r$, and $w$ increase.

  - If $w = \$40$, $r = \$10$ and $q = 100$, total cost simplifies to

$$TC = 2\times 100\sqrt{10\times 40} = \$4{,}000.$$

# Cost Functions

- *Example 8.7: Finding TC in linear production case.*
  - Labor and capital demands found in example 8.5 were

| | When $\frac{1}{4} < \frac{w}{r} \Longrightarrow r < 4w$ | When $\frac{1}{4} < \frac{w}{r} \Longrightarrow r > 4w$ |
|---|---|---|
| | $L = 0$ and $K = \frac{q}{8}$ | $L = \frac{q}{2}$ and $K = 0$ |
| • Total cost is | $TC = w0 + r\frac{q}{8} = r\frac{q}{8}$ | $TC = w\frac{q}{2} + r0 = w\frac{q}{8}$ |
| | Increasing in $q$ and r.<br>Independent of $w$. | Increasing in $q$ and $w$.<br>Independent of r. |

  - If $w$ increases enough, the condition $r > 4w$ can revert to $r < 4w$.

# Sunk vs. nonsunk costs

- Sunk costs. Costs that cannot be recovered, even if the firm chooses to shut down its operations.
  - *Example:* The rental a firm pays for the building it uses, if the lease prohibits subletting.
- Nonsunk costs. Costs that can be sold back if the firm were to shut down its operations (recovering a portion of the cost).
  - *Example:* Most of raw materials.

# Long-run vs. short-run costs

- In the long run, the firm have enough time to vary the amount of all inputs as much as necessary.
- In the short run, the amount of at least one input is considered to be fixed.
- *Example*: Faculty positions at universities.
  - Acquiring a new computer (a form of capital) can be done in few hours.
  - Hiring a new professor would require a long process (4-5 months if not longer: posting ads, interviews of candidates, fly-outs, offer to selected candidate, and negotiation.
- Short-run costs are higher (or equal, but never lower) than long-run costs.

# Long-run vs. short-run costs

- *Example 8.8: Comparing long- and short-run costs.*
  - Consider a firm with Cobb-Douglas production function $q = L^{1/2}K^{1/2}$.
  - Capital in the short run is fixed at $\bar{K} = 150$ units.
  - We find the cost-minimizing units of labor inserting $\bar{K} = 150$ into the firm's production function and solving for $L$,

$$q = L^{1/2}150^{1/2},$$

$$(q)^2 = \left(L^{1/2}150^{1/2}\right)^2,$$

$$q^2 = 150L \quad \Rightarrow L = \frac{q^2}{150}.$$

which increases in $q$.

# Long-run vs. short-run costs

- *Example 8.8* (continued):
  - In this context, the short-run total cost becomes
  $$STC = wL^* + r\bar{K} = w\frac{q^2}{150} + r150.$$
  - Considering the same input prices as in example 8.4., $w = \$40$ and $r = \$10$,
  $$STC = \$1{,}500 + \frac{4}{15}q^2,$$
  which lies above the long-run total cost in example 8.6, $TC = 40q$.

# Long-run vs. short-run costs

- *Example 8.8* (continued):
  - If $q = 150$ units,
    $STC = \$7{,}500$,
    $TC = \$6{,}000$.
    $STC(150) > TC(150)$
  - If $q = 75$ units,
    $STC(75) = TC(75)$

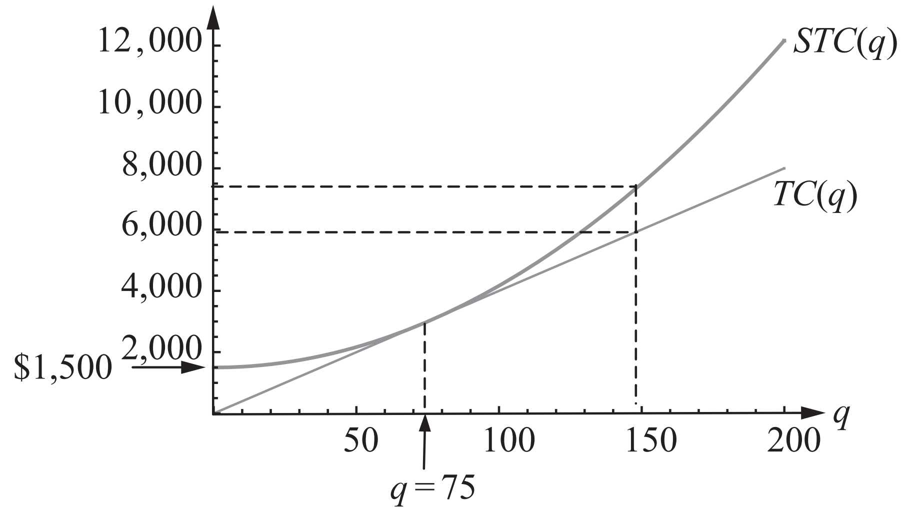

Figure 8.4

To produce $q = 75$, the firm has $K = \frac{q\sqrt{w}}{\sqrt{r}} = \frac{75\sqrt{40}}{\sqrt{10}} = 150$, which coincides with the fixed amount of capital $\bar{K} = 150$ in the short run.

For all other $q \neq 75$, the fixed amount of capital $\bar{K} = 150$, $STC(q) = TC(q)$.

# Average and Marginal Cost

# Average and Marginal Cost

- Average cost (AC). The total cost that the firm incurs per unit of output,

$$AC = \frac{TC}{q}.$$

  - *Example:* If $TC = \$1{,}000$ and $q = 20$ monitors, $AC = \frac{1000}{20} = \$50$ per monitor.

- Marginal cost (MC). The rate at which total costs increases as the firm produces 1 more unit,

$$AC = \frac{\partial TC}{\partial q}.$$

# Average and Marginal Cost

- Graphically, $MC$ measures the slope of the $TC$ curve:
  - When $TC \uparrow$, its slope must be positive → $MC$ is positive.
  - When $TC \downarrow$, its slope must be negative → $MC$ is negative.
- The $AC$ and $MC$ curves exhibit a similar relationship than the relationship between average and marginal product, $AP$ and $MP$.
- The $MC$ curve crosses the $AC$ curve at it minimum.

# Average and Marginal Cost

- *Example 8.9: Finding average and marginal cost.*
  - Consider a firm with Cobb-Douglas function in example 8.4,
  $$q = L^{1/2}K^{1/2}$$
  where $TC = 40q$.
  - The firm's average cost and marginal cost are
  $$AC = \frac{40q}{q} = 40,$$
  $$MC = \frac{\partial(40q)}{\partial q} = 40.$$
  - Hence, $AC$ and $MC$ curves are both constant, and $AC = MC$.
  - Graphically they are depicted by a horizontal line at height $\$40$.

# Average and Marginal Cost

- *Example 8.9:* (continued):

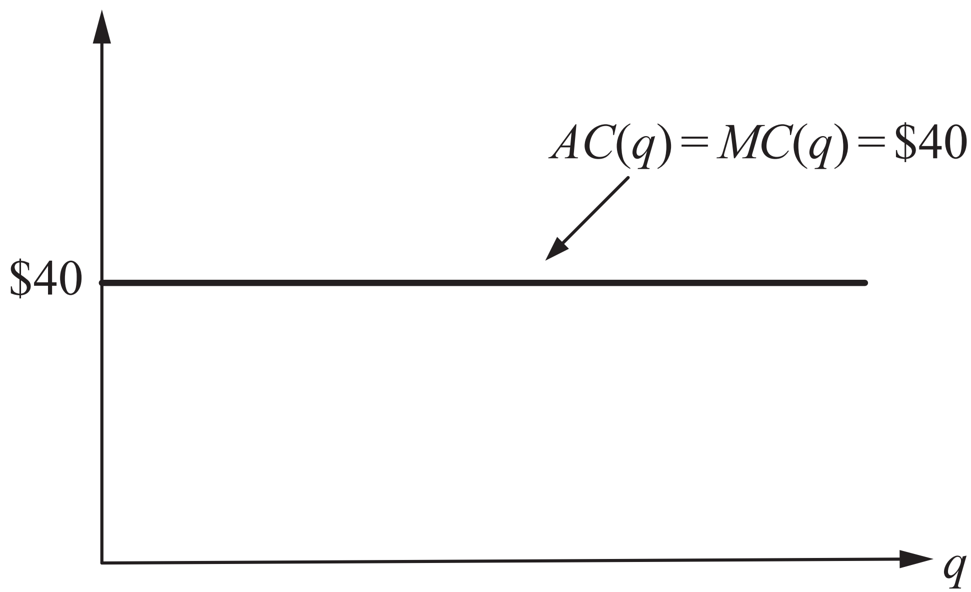


Figure 8.5

# Average and Marginal Cost

- *Example 8.9* (continued):
  - In the case of the linear production function in example 8.7,
    - $TC = r\frac{q}{8}$ when $r < 4w$ (i.e., labor is expensive relative to capital).
    - $TC = w\frac{q}{2}$ when $r > 4w$ (i.e., labor is relatively cheap).
  - In this context,
    - When $r < 4w$, $AC = \frac{r\frac{q}{8}}{q} = \frac{r}{8}$ and $MC = \frac{\partial\left(r\frac{q}{8}\right)}{\partial q} = \frac{r}{8}$.
    - When $r > 4w$, $AC = \frac{w\frac{q}{2}}{q} = \frac{w}{2}$ and $MC = \frac{\partial\left(w\frac{q}{2}\right)}{\partial q} = \frac{w}{2}$.
  - Hence, $AC$ and $MC$ are constant in $q$, and $AC = MC$.
  - Graphically, $AC$ and $MC$ overlap, being a flat line.

# Output Elasticity to Total Cost

- The marginal cost $MC = \frac{\partial TC}{\partial q}$ measures how much total cost increases if the firm increases its output by 1 units.
- However, this measure is not unit-free.
- Consider a firm producing computer monitors in the US, and another firm producing cars in Germany.
  - The MC from the first firm would be in $/monitor.
  - The MC form the second firm would be in €/car.
- We can apply the definition of elasticity to obtain a unit-free measure of how total cost changes in output.

# Output Elasticity to Total Cost

- Output elasticity of total cost is

$$\varepsilon_{TC,q} = \frac{\%\Delta TC}{\%\Delta q} = \frac{\frac{\Delta TC}{TC}}{\frac{\Delta q}{q}} = \frac{\Delta TC}{\Delta q}\frac{q}{TC},$$

or $\varepsilon_{TC,q} = \frac{\partial TC}{\partial q}\frac{q}{TC}$ when the change $q$ is small.

- Because $MC = \frac{\partial TC}{\partial q}$, this elasticity can be rewritten as

$$\varepsilon_{TC,q} = MC\,\frac{q}{TC}.$$

- As $AC = \frac{TC}{q}$, its inverse is $\frac{1}{AC} = \frac{q}{TC}$,

$$\varepsilon_{TC,q} = MC\,\frac{1}{AC} \quad \Rightarrow \varepsilon_{TC,q} = \frac{MC}{AC}.$$

# Output Elasticity to Total Cost

- When $MC > AC$, $\varepsilon_{TC,q} = \frac{MC}{AC}$ satisfies $\varepsilon_{TC,q} > 1$.
  - Total costs increase *more* than proportionally to $1\%$ increase in output.
- When $MC < AC$, $\varepsilon_{TC,q} = \frac{MC}{AC}$ satisfies $\varepsilon_{TC,q} < 1$.
  - Total costs increase *less* than proportionally to $1\%$ increase in output.
- When $MC = AC$, $\varepsilon_{TC,q} = \frac{MC}{AC}$ satisfies $\varepsilon_{TC,q} = 1$.
  - Total costs responds *proportionally* to $1\%$ increase in output.

# Output Elasticity to Total Cost

- *Example 8.10: Output elasticity in the Cobb-Douglas case.*
  - Consider the firm with Cobb-Douglas function in example 8.4,
    $$q = L^{1/2}K^{1/2}$$
    where $TC = 40q$.
  - The total elasticity is
    $$\varepsilon_{TC,q} = \frac{\partial TC}{\partial q}\frac{q}{TC} = 40\,\frac{q}{40q} = 1.$$
    - If the firm increases its output by $1\%$, its total costs also increase by $1\%$.

# Average and Marginal Cost

- *Example 8.10* (continued):
  - In the firm has the linear production function in example 8.7,
    - $TC = r\frac{q}{8}$ when $r < 4w$.
    - $TC = w\frac{q}{2}$ when $r > 4w$.
  - When $4r < w$, output elasticity becomes

$$\varepsilon_{TC,q} = \frac{\partial TC}{\partial q}\frac{q}{TC} = \frac{r}{8}\frac{q}{r\frac{q}{8}} = \frac{q}{r}.$$

    - If the firm seeks to produce $1\%$ more units of output, its total cost increase by $\frac{q}{r}\%$.
  - When $r > 4w$, $\varepsilon_{TC,q} = \frac{q}{W}$.

# Economies of Scale, Scope, and Experience

# Economies of Scale

- A firm experiences economies of scale when its average cost, $AC$, decreases in output $q$.
  - *Examples*:
    - Task specialization.
    - Large capital investments spreaded over large output levels.
- A firm suffers from diseconomies of scale when its average cost, $AC$, increases in output $q$.
  - *Example*: Managerial diseconomies.

# Economies of Scale

- *Example 8.11: Testing for economies of scale.*
  - Consider a firm with $TC = a + bq + cq^2$, where $a, b, c \geq 0$. The average cost is

$$AC = \frac{TC}{q} = \frac{a + bq + cq^2}{q} = \frac{a}{q} + b + cq.$$

  - This expression reaches its minimum at

$$\frac{\partial AC}{\partial q} = 0,$$

$$-\frac{a}{q^2} + c = 0,$$

$$q = \left(\frac{a}{c}\right)^{1/2}.$$

# Economies of Scale

- *Example 8.11* (continued):

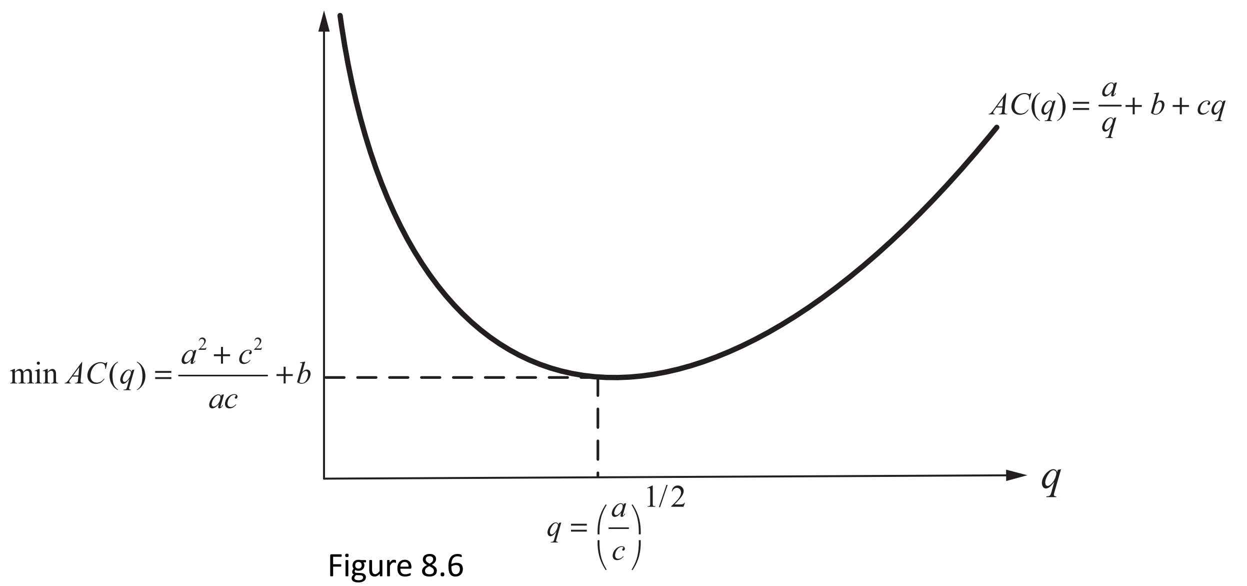


Figure 8.6

- For $q < \left(\frac{a}{c}\right)^{1/2}$, $AC$ curve is decreasing → economies of scale.
- For $q > \left(\frac{a}{c}\right)^{1/2}$, $AC$ curve is increasing → diseconomies of scale.

# Economies of Scale

- *Example 8.11* (continued):
  - The minimum of the $AC$ curve, $q = \left(\frac{a}{c}\right)^{1/2}$, could alternatively be found by using the property that the $MC$ and $AC$ cross each other at the minimum of t he $AC$ curve.
  - First, we find $MC$,
$$MC = \frac{\partial TC}{\partial q} = b + 2cq.$$
  - Second, $MC$ and $AV$ curves cross where $MC = AC$,
$$b + 2cq = \frac{a}{q} + b + cq.$$
  - Rearranging and solving for $q$,
$$\frac{a}{q} = cq \quad \Rightarrow q = \left(\frac{a}{c}\right)^{1/2}.$$

# Economies of Scale

- *Example 8.11* (continued):
  - Consider the firm's total cost function is
  $$TC = 10 + 2q + q^2,$$
  which implies that $a = 10$, $b = 2$, and $c = 1$.
  - The $AC$ curve becomes
  $$AC = \frac{10}{q} + 2 + q,$$
  which reaches its minimum at $q = \left(\frac{10}{1}\right)^{1/2} \cong 3.16$ units.
  - For al $q < 3.16$, the firm's $AC$ curve decreases in $q$, while for all $q > 3.16$ it increases in $q$.

# Economies of Scope

- Economies of scope. The situation where a firm incurs a lower total cost producing two different products than the total cost that two firms would incur producing each good separately,

$$TC(q_1, q_2) < TC(q_1, 0) + TC(0, q_2).$$

- Because often $TC(0,0) = 0$,

$$TC(q_1, q_2) < TC(q_1, 0) + TC(0, q_2) - \underbrace{TC(0,0)}_{\text{zero}}.$$

# Economies of Scope

- After rearranging,

$$TC(q_1, q_2) - TC(q_1, 0) < TC(0, q_2) - \mathrm{TC}(0,0).$$

  The increase in cost from starting to produce one good alone is larger than the additional costs of adding one more good to the firm's product line.

- *Example*: Television channels in a satellite network.

# Economies of Scope

- *Example 8.12: Economies of Scope.*
  - Consider a soda company producing 2 types of cola.
  - When the firm only produces regular cola (good 1),
  $$TC = (q_1, 0) = 3q_1 + 10.$$
  - When the firm only produces diet cola (good 2),
  $$TC(0, q_2) = 4q_2 + 10.$$

# Economies of Scope

- *Example 8.12* (continued):
  - When it simultaneously produces both types of colas,

$$TC(q_1, q_2) = (3 - \alpha)q_1 + (4 - \alpha)q_2 + (10 + \beta),$$

where $\alpha > 0$ indicates the cost savings effect that producing related products has on the unit cost of both regular and diet cola.

$\beta > 0$ represents the increased in fixed costs when producing 2 types of cola rather than 1.

# Economies of Scope

- *Example 8.12* (continued):
  - The firm exhibits economies of scope if

$$TC(q_1, q_2) < TC(q_1, 0) + TC(0, q_2),$$

$$(3 - \alpha)q_1 + (4 - \alpha)q_2 + (10 + \beta) < [3q_1 + 10] + [4q_2 + 10],$$

which simplifies to

$$\beta < 10 + \alpha(q_1 + q_2).$$

The firm benefits from economies of scope if the increase in the fixed costs from producing both goods (measured by $\beta$) is relatively lower than the cost-saving effect from producing both goods (measures by $\alpha$).

# Economies of Experience

- Economies of experience. The average variable cost (AVC) decreases during the firm's production history.
  - Often emerge because workers learn from previous periods to avoid product defect, because managers arrange workstations to improve work productivity, or achieve higher material yield.
- Economies of experience are expressed as

$$AVC(E) = \frac{A}{E^{\varepsilon}}.$$

where $A > 0$ denotes the AVC from the 1st unit;

$E = q_{t-1} + q_{t-2} \dots$, measures experience from production in previous periods;

$\varepsilon \in (0,1)$ represents experience elasticity.

# Economies of Experience

- Elasticity of experience $\varepsilon$ is

$$\varepsilon_{AVC,E} = \frac{\%\Delta AVC}{\%\Delta E}$$

$$= \frac{\frac{\Delta AVC}{AVC}}{\frac{\Delta E}{E}}$$

$$= \frac{\Delta AVC}{\Delta E}\frac{E}{AVC}.$$

- Or when the change in $E$ is relatively small,

$$\varepsilon_{AVC,E} = \frac{\partial AVC}{\partial E}\frac{E}{AVC}.$$

# Economies of Experience

- Because $AVC(E) = \frac{A}{E^{\varepsilon}}, \frac{\partial AVC}{\partial E} = -A\varepsilon E^{-(1+\varepsilon)}$.
- Then, experience elasticity becomes

$$\varepsilon_{AVC,E} = -A\varepsilon E^{-(1+\varepsilon)} \frac{E}{\frac{A}{E^{\varepsilon}}}.$$

- A $1\%$ increase in the firm's production experience, $E$, decreases its average variable costs by $\varepsilon\%$.

# Economies of Experience

- *Example 8.13: Slope of the experience curve.*
  - We can analyze the responsiveness of a firm's average costs ($AVC$) to its production experience by focusing on the slope of the experience curve,

$$\text{Slope of the experience curve} = \frac{AVC(2E)}{AVC(E)} = \frac{\frac{A}{(2E)^{\varepsilon}}}{\frac{A}{E^{\varepsilon}}} = \frac{E^{\varepsilon}}{2^{\varepsilon}E^{\varepsilon}} = \frac{1}{2^{\varepsilon}}.$$

  - This slope measures how much the $AVC$ decreases when cumulative output ($E$) doubles.
  - Because $\varepsilon \in (0,1)$, an increase in $\varepsilon$ entails a larger slope of the experience curve.

# Appendix. Cost-Minimization–A Lagrangian Analysis

# CMP–A Lagrangian Analysis

- The firm's cost-minimization problem is

$$\min_{L\geq 0, K\geq 0} TC = wL + rK$$

subject to $q = f(L, K)$.

- This is a constrained minimization problem, in which the constraint is given by the output target, $q = f(L, K)$.
- This problem has the Lagrangian function

$$\mathcal{L} = wL + rK + \lambda[q - f(L, K)].$$

where $\lambda$ denotes the Lagrange multiplier associated with the constraint.

# CMP–A Lagrangian Analysis

- Differentiating with respect to $L$,

$$w + \lambda \left[ -\underbrace{\frac{\partial f(L,K)}{\partial L}}_{MP_L} \right] = 0 \text{ or } \frac{w}{MP_L} = \lambda.$$

- Differentiating with respect to $K$,

$$r + \lambda \left[ -\underbrace{\frac{\partial f(L,K)}{\partial K}}_{MP_K} \right] = 0 \text{ or } \frac{r}{MP_K} = \lambda.$$

- Differentiating with respect to $\lambda$,

$$q - f(L,K) = 0 \text{ or } q = f(L,K).$$

which coincides with the constraint.

# CMP–A Lagrangian Analysis

- Because the results after differentiating with respect to $L$ and $K$ are both equal to $\lambda$,

$$\frac{w}{MP_L} = \frac{r}{MP_K}.$$

- After cross multiplying,

$$\frac{MP_L}{w} = \frac{MP_K}{r}.$$

When minimizing cost, the firm adjusts its inputs until it gets the same bang for the buck across all inputs.

- This result can be rewritten as $\frac{MP_L}{MP_K} = \frac{w}{r}$, which says that the firm hires inputs until the point in which the isoquant is tangent to the isocost.