1. **[6 marks]** Consider the following first-order logic formula:

$$\exists x \; (P(x) \rightarrow \forall y \; P(y))$$

If we consider the universe as the students in COL703, and interpret $P(x)$ as '*x likes Kohli*', then the statement above means that *there is someone in the class who if (s)he likes Kohli, then everyone in the class likes Kohli*.

If the above formula is valid, you should prove it. However, if the formula is not valid, then you should give an interpretation that makes this formula *false* (you may use or adapt the interpretation given above if you wish to).

Ans: We prove that the formula is valid, by showing the unsatisfiability of the negation of the formula. The negated formula can be written as:

$$\begin{gathered} \neg\exists x \; (P(x) \rightarrow \forall y \; P(y)) \\ \forall x \; \neg(P(x) \rightarrow \forall y \; P(y)) \\ \forall x \; (P(x) \wedge \neg\forall y \; P(y)) \\ \forall x \; (P(x) \wedge \exists y \; \neg P(y)) \\ \forall x \; P(x) \wedge \exists y \; \neg P(y)) \end{gathered}$$

We can skolemize the second conjunct to get $\neg P(a)$, for some constant symbol 'a'. Now, unsatisfiability can be proved using ground resolution, by producing the ground instance $P(a)$ from the first conjunct, and resolving it with $\neg P(a)$, the skolemized form of the second conjunct.

For the given interpretation, the formula simply says that either everyone likes Kohli or there is someone who doesn't like Kohli.

2. **[9 marks]** Let S be a set of clauses – $\{C_1, \ldots, C_m\}$, where each C_i is a disjunction of n_i literals – $\{l_{i1}, \ldots, l_{in_i}\}$. Let us define S^* to be the set of clauses

$$\bigcup_{i=1}^{m} \; \bigcup_{1 \leq j < k \leq n_i} \{\{l_{ij}, l_{ik}\}\}$$

Prove that S is renamable Horn if and only if S^* is satisfiable.

Recall that a *renamable Horn formula* is a CNF formula that can be turned into a Horn formula by negating (all occurrences of) some of its variables. For example,

$$(p_1 \vee \neg p_2 \vee \neg p_3) \wedge (p_2 \vee p_3) \wedge (\neg p_1)$$

is renamable Horn because it can be turned into a Horn formula by negating p_1 and p_2.

Ans: ($\Leftarrow$) Suppose S^* is satisfiable, and let M be a model for S^* (i.e., think of M as the set of all the literals that are set to *true*). Let A be the set of positive literals in M. Consider the renaming that negates (all occurrences of) all those variables in S that are also in A. If a renamed clause has two positive literals, then it means that none of these two (the original literals before renaming) were in M. But S^* has a clause containing exactly these two literals (S^* has one such clause for every pair of literals appearing in a clause, by definition). If none of the two literals were in M, then how could M be a model for S^*!

($\Rightarrow$) Suppose S is renamable Horn. Let A be the set of variables whose (every) occurrence needs to be negated, for the renaming. Let M be the set of literals consisting of all the propositional variables in A and the complements of all the propositional variables occurring in S but not in A. We claim that M is a model for S^*. Consider any clause $\{l_{ij}, l_{ik}\}$ ($i \in [1, m], 1 \leq j < k \leq n_i$) of S^*. If l_{ij}, l_{ik} are both positive, then the renaming A (and therefore, M) must contain one of them. If l_{ij}, l_{ik} are both negative, then one of the $\neg l_{ij}, \neg l_{ik}$ must not be in A (otherwise, both would be complemented and both would become positive!). This means that one of l_{ij}, l_{ik} must be in M. Lastly if one of l_{ij}, l_{ik} is positive and the other one is negative – say l_{ij} is positive and l_{ik} is negative, then A must complement l_{ij} if it complements l_{ik}. Therefore, $l_{ij} \in A$ if $\neg l_{ik} \in A$. Clearly, either $l_{ij} \in M$ (because it is in A), or $\neg l_{ik} \notin A$ in which case $l_{ik} \in M$.

3. **[3.5+3.5+3 = 10 marks]** An *input* resolution refutation of a set of clauses (let us restrict ourselves to propositional logic), Γ, is defined to be a refutation of Γ in which every resolution inference has at least one of its hypotheses in Γ. Consider the following claims:

   (a) Input resolution is at least as powerful as unit resolution. That is, if unit resolution can refute a set of clauses, then input refutation can also refute that set.

   (b) Unit resolution is at least as powerful as input resolution.

   (c) Input resolution is complete.

   For each of these claims, **state clearly** whether the claim is true or false, and justify your answer. Answers without a justification will not get any marks.

   Ans: (a) and (b) are True, (c) is False. We claim that input resolution and unit resolution are equally powerful (they can refute exactly the same set of clauses). We prove it below, but before that let us look at this counterexample for part (c). Consider the following clauses:

   $\{\{\neg a, \neg b\}, \{\neg a, b\}, \{a, \neg b\}, \{a, b\}\}$

   Input resolution (or unit resolution) cannot refute this set of clauses (clearly), but there is a simple resolution proof that can be obtained by resolving the first two clauses (to get $\neg a$), and then the last two clauses (to get a), and then the two resolvents (a and $\neg a$).

   Claim: A set S of clauses has a unit resolution proof if and only if it has an input resolution proof.

   Proof: Let P be the set of propositional atoms in S. We prove the claim by induction on the size of P.

   Base case: If P is empty, then S is either the empty set of clauses, or $S = \{\{\}\}$. In the first case, S is satisfiable, so it has neither a unit resolution proof nor an input resolution proof. In the second case, it has a trivial one-line unit and input resolution proof.

   Assume that the claim holds when P consists of i elements, for $1 \leq i \leq n$. Let us consider the case when P has $(n + 1)$ elements.

   Inductive step: ($\Rightarrow$) If S has a unit resolution proof, it must have a unit clause, say $\{l\}$. The literal l must either be a or $\neg a$ for some $a \in P$. Let us derive an S' by removing all clauses containing the literal l and by deleting $\neg l$ from the remaining clauses. If S has a unit resolution proof, S' will also have a unit resolution proof. (*Why? The only unit resolutions that are not possible any longer are the ones that involve $\{l\}$, but they aren't even required because $\neg l$ appears nowhere!*)

   But now since S' contains at most n atoms, (by induction hypothesis) S' must have an input resolution proof. Let us take this proof and construct an input resolution proof of S. Here is how we do this: Starting from the top, let us insert $\neg l$ back in all the clauses from where we had deleted them (recall the resolution completeness proof done in the class, which can also be found in the lecture notes by James Worrell). At the end, we will be left with an empty clause, or we may get $\{\neg l\}$ because of adding $\neg l$ back in the clauses from where we had deleted it. If we get an empty clause, we are done.

Otherwise, we can resolve $\{\neg l\}$ with $\{l\}$ (which we know is there in S). This (last step) is an input resolution, of course.

($\Leftarrow$) If S has an input resolution proof, then it must have a unit clause, say $\{l\}$. Construct an S' as follows: Initialise S' to S, take $\{l\}$, resolve it with every clause in S and add all the resolvents to S', and then remove every clause from S' that containts l or $\neg l$. Since S has an input resolution proof, S' will also have an input resolution proof. (*Why? Suppose the input resolution proof of S had a step where C_1 and C_2 where resolved to get R. If C_1 or C_2 contained $\neg l$, then replace R by the resolvent of R with C_1 or C_2. If C_1 or C_2 do not contain $\neg l$ but contain l, then l must appear in R and later be removed in a resolution step with an input clause C' (containing $\neg l$), to get R'. At that point we can directly replace R', with the resolvent of C' and $\{l\}$*).

Now, S' has n atoms or lesser. So, it must have a unit resolution proof. But every clause in S' is either already in S or is obtained by a unit resolution with $\{l\}$. Therefore, S has unit resolution proof.

4. **[3.5+3.5+3 = 10 marks]** Consider the predicate logic sentence $\phi = \phi_1 \wedge \phi_2 \wedge \phi_3$, where
$\phi_1 = \forall x\ (f(g(x)) = g(x))$
$\phi_2 = \forall x\ (g(f(x)) = f(x))$
$\phi_3 = \forall x\ \exists y\ ((x = g(y)) \vee (x = f(y)))$

(a) Let $\mathcal{H}$ be the Herbrand universe for the sentence ϕ. Let $M_{\mathcal{H}}$ be a model with the universe being $\mathcal{H}$, and with the obvious interpretations of functions f and g, i.e., if t is a term in $\mathcal{H}$, the terms $f(t)$ and $g(t)$ in $\mathcal{H}$ are defined to be the result of applying f and g respectively to t. If $M_{\mathcal{H}} \vDash \phi$, how many distinct elements can be present in the universe of $M_{\mathcal{H}}$.

Ans: Since there are no constant symbols in the formula, let us add a constant a in $\mathcal{H}$. This gives us two other elements: $f(a)$ and $g(a)$. We claim that (these) three elements is an upper bound. Why? Because $f(g(a))$ is same as $g(a)$ (from ϕ_1), and $g(f(a))$ is same as $f(a)$ (from ϕ_2). Further, $f(f(a))$ is same as $f(g(f(a)))$ (from ϕ_2), which is same as $g(f(a))$ (from ϕ_1), which is same as $f(a)$ (from ϕ_2). Similarly, $g(g(a))$ is same as $g(f(g(a)))$ (from ϕ_1), which is same as $f(g(a))$ (from ϕ_2), which is same as $g(a)$.
Now, consider ϕ_3 and suppose we are taking $x = a$. There must exist a $y \in \{a, f(a), g(a)\}$, such that $((a = g(y)) \vee (a = f(y)))$. Now, one of the following must hold:

- y is a, and $((a = g(a)) \vee (a = f(a)))$
- y is $f(a)$, and $((a = g(f(a))) \vee (a = f(f(a))))$, i.e., $(a = f(a))$
- y is $g(a)$, and $((a = g(g(a))) \vee (a = f(g(a))))$, i.e., $(a = g(a))$

Clearly, there can be at most two distinct elements: $\{f(a), g(a)\}$.
In fact, as we are going to prove below: ϕ_1, ϕ_2, and ϕ_3 let us derive that $(x = f(x))$ and $(x = g(x))$ for all x. One of them is proved in part (b), below, and the other proof is very similar. So, there can just be one element in $\mathcal{H}$. (But we will give full marks even to those who have argued that there cannot be more than two distinct elements.)

(b) Show using natural deduction for predicate logic that $\phi_1, \phi_2, \phi_3 \vdash \forall x\ (x = f(x))$.

Ans:

i.	$((x_0 = g(y_0)) \vee (x_0 = f(y_0)))$	x_0 arbitrary, and y_0 corresponding to x_0
ii.	$(x_0 = g(y_0))$	*assume, first disjunct in (i)*
iii.	$(f(x_0) = f(g(y_0)))$	applying f on equal terms from (ii)
iv.	$(f(x_0) = g(y_0))$	from ϕ_1 and (iii)
v.	$(f(x_0) = x_0)$	from (ii)

vi.	$\forall x\ (f(x) = x)$	since x_0 was arbitrary
vii.	$(x_0 = f(y_0))$	*assume, second disjunct in (i)*
viii.	$(f(x_0) = f(f(y_0)))$	applying f on equal terms from (vii)
ix.	$(f(x_0) = f(g(f(y_0))))$	replacing $f(y_0)$ by $g(f(y_0))$ in (viii), ϕ_2
x.	$(f(x_0) = g(f(y_0)))$	from ϕ_1 and (ix)
xi.	$(f(x_0) = f(y_0))$	from ϕ_2 and (x)
xii.	$(f(x_0) = x_0)$	from (vii)
xiii.	$\forall x\ (f(x) = x)$	since x_0 was arbitrary
xiv.	$\forall x\ (f(x) = x)$	$\vee$-elimination, (i), (ii)-(vi), (vii)-(xiii)

(c) Let $\phi_4 = \exists x\ \neg(f(x) = g(x))$. Show using Herbrand's theorem that $\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4$ is unsatisfiable.

Ans: We first prove that $\forall x\ (g(x) = x)$. From ϕ_3 we have that $((x_0 = g(y_0)) \vee (x_0 = f(y_0)))$. If we take the first disjunct, we get $g(x_0) = g(g(y_0)) = g(y_0) = x_0$ (and thus, $\forall x\ g(x) = x$). If we take the second disjunct, we get $g(x_0) = g(f(y_0)) = f(y_0) = x_0$ (and thus, $\forall x\ g(x) = x$). This is very similar to the proof above.

Clearly, $\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4$ can be proved unsatisfiable using ground resolution (after deriving $\forall x\ (g(x) = x)$ and $\forall x\ (f(x) = x)$).