Name:

COL202: Quiz-4

Maximum marks: 40 Kerberos id:

Instructions.

1. For each problem, you will receive 10% marks for leaving it blank. However, there will be penalty for submitting bogus proofs.

2. Please write your proofs clearly (marks will be deducted for skipping steps).

3. Clearly mark whether you have attempted the problem or not. In case there are multiple versions in a particular problem, clearly specify which version you have attempted.

   If nothing is marked, we will assume that you have not attempted the problem.

Question 1: Prefix-free Codes (20 marks) The probability of any event can be at most 1. This simple fact can be used to prove interesting combinatorial results.

Let $n \in \mathbb{N}$. An n-prefix-free set $\mathcal{C}$ is a set of strings of length at most n, such that for all distinct strings $x, y \in \mathcal{C}$, x is not a prefix of y. For instance, if $n = 3$, then $\mathcal{C}_1 = \{0, 10, 11\}$, $\mathcal{C}_2 = \{1, 01, 001, 000\}$ and $\mathcal{C}_3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$ are three such prefix-free sets.
Take any n-prefix-free set $\mathcal{C}$. Let N_i denote the number of strings in $\mathcal{C}$ of length i. Prove that

$$\sum_{i=1}^{n} \frac{N_i}{2^i} \leq 1.$$

Question 2 (20 marks). Let (Ω, p) be a probability distribution. Let $X_1, X_2, \ldots, X_n$ be independent random variables such that $X_i : \Omega \to [2n]$, $\mathbb{E}[X_i] = n$ and $\mathsf{Var}[X_i] = n/4$ for all i. Consider the random variable $X = \sum_{i=1}^{n} X_i$. Show that $\Pr[X > 3n^2/2] \leq \cancel{2^{-n}}$. 2^{-cn} for some const. c

(*Hint: If* $X > 3n^2/2$, *then the number of indices* $i \in [n]$ *such that* $X_i \geq 5n/4$ *must be at least* $n/8$. *If you are using this hint, then you must prove this too.*)

Easier version (12 marks) : Show that $\Pr[X > 3n^2/2] \leq 1/n$.