MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Time allowed: 100 minutes   Mid Term Examination   Max. Marks: 40

Name: Entry Number: Signature:

Multiple Selection Questions: **Section 1**   ($4 \times 1 = 4$ **marks**)

Each of the following questions 1 to 4 has four options out of which **one option is correct.** Write A, B, C or D which corresponds to the correct option. **1 mark** is awarded if the correct answer is written, **0 mark** for no answer or any incorrect answer.

1. Let $\Omega = \{0, 1, \ldots\}$. Let $\mathcal{F}$ be the largest σ-field on Ω. Define a probability on $(\Omega, \mathcal{F})$ by $P(\{n\}) = k2^{-n}$ where k is a constant. Find k? (A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{1}{4}$ (D) 1 Answer:

2. If $X \sim U(0,1)$. What is the distribution of $Y = -\ln X$?
(A) $Y \sim U(-1,0)$ (B) $Y \sim U(-1,1)$ (C) $Y \sim Exp(1)$ (D) $Y \sim N(0,1)$. Answer:

3. The MGF of a r.v. X is given by $M_X(t) = e^{\lambda(e^t-1)}$ with $0 < \lambda < 1$. What is the mean factorial of X, E(X!)?
(A) $e^{-\lambda}$ (B) $\lambda e^{-\lambda}$ (C) λ (D) $\frac{e^{-\lambda}}{1-\lambda}$ Answer:

4. The joint PDF of (X, Y) is given by: $f_{X,Y}(x,y) = \begin{cases} 2(1-x), & (x,y) \in R \\ 0, & \text{otherwise} \end{cases}$. The value of R is given by (A) $\{(x,y) \mid 0 < x < 1, 0 < y < 1\}$ (B) $\{(x,y) \mid 0 < x < 2, 0 < y < 2\}$ (C) $\{(x,y) \mid 0 < x < y < 1\}$ (D) $\{(x,y) \mid 0 < y < x < 1\}$ Answer:

———————Space for Rough Work ———————

Short Answer Type Questions: **Section 2** ($3 \times 2 = 6$ **marks**)

Each of the following questions 5 to 7 has four options out of which **more than one options can be correct**. Write A, B, C or D which corresponds to the correct option for the first correct answer followed by space and the next correct answer and so on. **2 marks** is awarded if all correct answers are written, **0 mark** for no answer or partial correct answers or any incorrect answer.

5. Let X be a continuous type r.v. with PDF $f(x) = \begin{cases} \alpha + \beta x^2, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$. If $E(X) = \frac{3}{5}$, which of the following statements are TRUE?
(A) $\alpha = \frac{3}{5}$ (B) $\alpha = \frac{6}{5}$ (C) $\beta = \frac{3}{5}$ (D) $\beta = \frac{6}{5}$ Answer:

6. Let X and Y be discrete type random variables with respective PMFs given by

$$p_X(x_1) = p_1,\ p_X(x_2) = 1 - p_1,\ p_Y(y_1) = p_2,\ p_Y(y_2) = 1 - p_2, 0 < p_1, p_2 < 1.$$

Which of the following statements are TRUE? (A) If X and Y are independent, then X and Y are uncorrelated. (B) If X and Y are uncorrelated, then X and Y are independent. (C) If X and Y are independent, then X and Y need not to be uncorrelated. (D) If X and Y are uncorrelated, then X and Y need not to be independent. Answer:

7. Let $X \sim B(n,p)$ and $Y \sim B(m,p)$, independent of X and $m \neq n$. Which of the following statements are TRUE? (A) $X - Y$ is negative with positive probability. (B) $X - Y$ is zero with positive probability. (C) $X + Y \sim B(n+m, p)$. (D) $X - Y$ is positve with positive probability. Answer:

Space for Rough Work

Short Answer Type Questions: **Section 3** ($5 \times 2 = 10$ **marks**)

Write the answer upto 4 decimal places (D) or in fraction (F) or the expression (E) for the following questions 8 to 12. **2 marks** are awarded if answer is correct, and **0 mark** for no answer or partial correct answer or an incorrect answer.

8. A box contains two unbiased coins and one two-headed coin. Suppose you pick a coin at random and toss it, and get heads. What is the probability that it is the two-headed coin?

   Answer (D/F):

9. Atul and Arun play a match consisting of a series of games, where Atul has probability p of winning each game. Assume that, each game is independent with each other games and $0 < p < 1$. Both Atul and Arun play with a 'win by two' rule, that is, the first player to win two games more than his opponent wins the match. What is the probability that Atul wins the match?

   Answer(E):

10. Let X denotes the day of the of the week, mapped so that Monday is 1, Tuesday is 2 etc. That is, X takes values 1, 2, ..., 7 with equal probabilities. Let Y denotes the next day after X, again represented as an integer between 1 and 7. Find $P(X < Y)$?

    Answer (D/F):

11. Consider that, two women are pregnant. Assume that, both with same due date and the two birth times are i.i.d. On a timeline, define time 0 to be the instant when the due date begins. Assume that, the time when the woman gives birth has a normal distribution, centered at 0 and with standard deviation 6 days. Let Z be the time of first of the two births (in days). What is the mean of Z? Answer(D/F/E):

12. Let N be the number of eggs lays by a chicken which follows Poisson distribution with parameter λ. Each egg hatches a chick with probability p, independently. Let X be the number which hatch. What is the covariance between N and X? Answer(E):

Subjective Type Questions: **Section 4** ($4 \times 5 = 20$ **marks**)

Write the answer in the two pages provided for the questions 13 to 16. **Full marks** are awarded if all the steps are correct, and **partial marks** for an incorrect answer with wrong steps.

13. (a) Write axiomatic definition of probability.

    (b) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{A_n\}$ be a nondecreasing sequence of elements in $\mathcal{F}$. Prove that

$$P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

14. Let X be a continuous type r.v. with CDF $F(x)$ and PDF $f(x)$.

(a) Suppose the mean of X exists, prove that

$$E(X) = -\int_{-\infty}^{0} F(x)dx + \int_{0}^{\infty} (1 - F(x))dx.$$

(b) Let $f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$. Find $E(X)$, if it exists. (3 + 2 marks)

Solution:

15. Let A, B and C be i.i.d. random variables each uniform distributed on $(0, 1)$. What is the probability that $Ax^2 + Bx + C = 0$ has real roots?

Solution:

16. Suppose X has Poisson distribution with parameter Λ, which is also a r.v. having exponential distribution with mean 1. Find conditional expectation $E(e^{-\Lambda} \mid X = 1)$. (5 marks)

Solution: