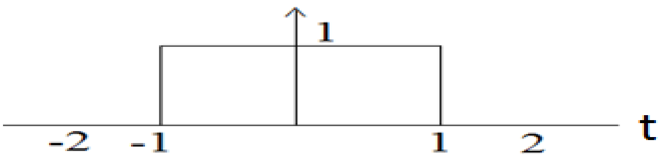
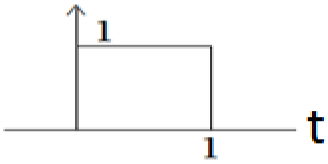
Problem sheet 6 Solution

Q1.

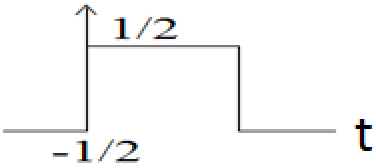


$$a_k = \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi}$$



$$b_k = \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi} e^{jk\frac{2\pi}{4}(-1)}$$

$$b_k = \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi} e^{-jk\frac{\pi}{2}}$$



$$g_k = \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi} e^{-jk\frac{\pi}{2}} \qquad k \neq 0$$

$$= 0 \qquad k = 0$$

$$\frac{dx(t)}{dt} = g(t)$$

$$jwkx_k \leftrightarrow g_k$$

$$x_k \leftrightarrow \frac{\sin\left(\frac{k\pi}{2}\right)}{jwkk\pi} e^{-jK\frac{\pi}{2}} \qquad k \neq 0$$

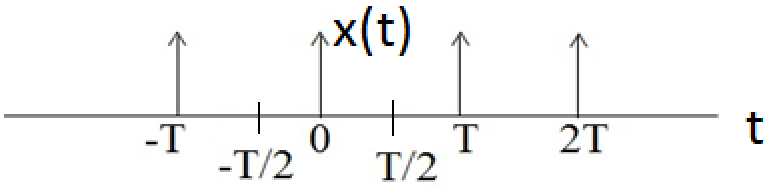
$$0 \qquad k = 0$$

$$x_k \leftrightarrow \frac{2\sin\left(\frac{k\pi}{2}\right)}{jk^2\pi^2} e^{-jk\frac{\pi}{2}} \qquad k \neq 0$$

$$0 \qquad k = 0$$

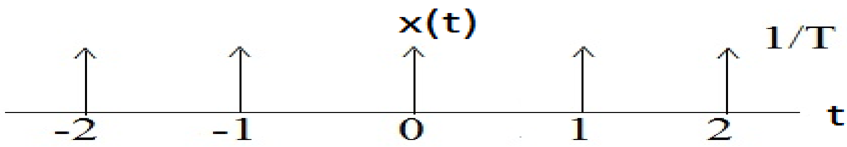
Q2.

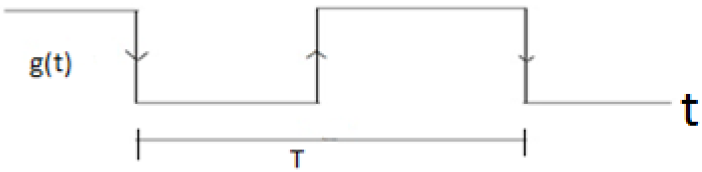
$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

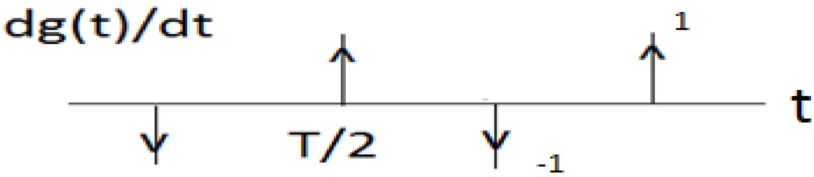


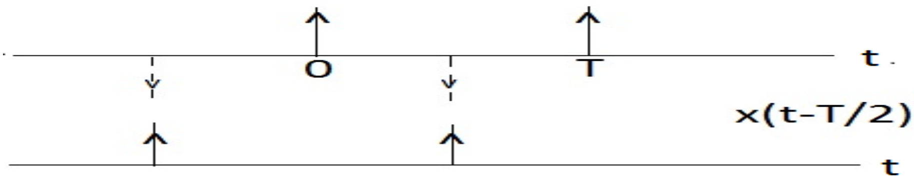
$$a_k = 1/T \int_{-T/2}^{T/2} \delta(t) e^{-jw_0 kt}\, dt$$

$$a_k = 1/T$$









$$\frac{dg(t)}{dt} = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$= x(t) - x(t - \frac{T}{2})$$

$$kjw_0 g_k = x_k - x_k e^{-jw_0 kT/2}$$

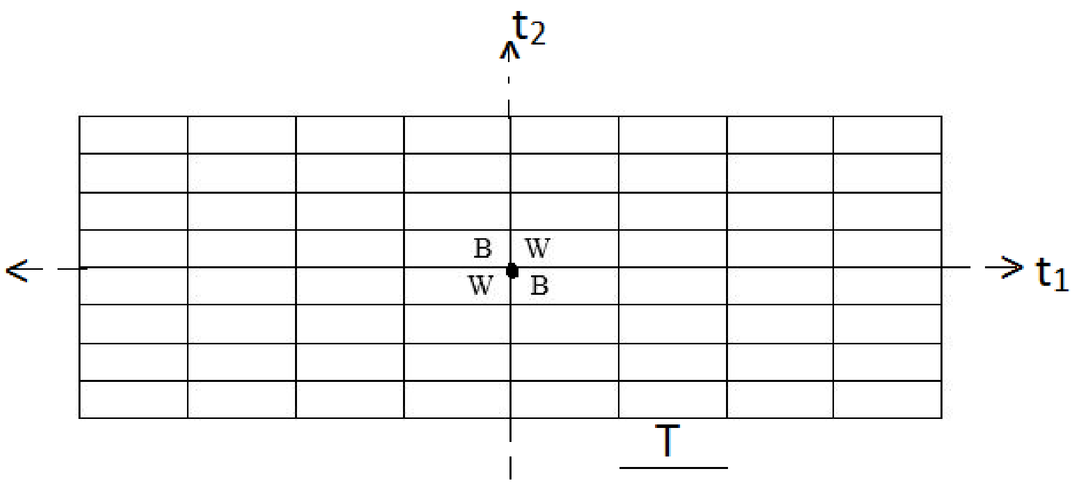
$$= x_k(1 - e^{-jk\pi})$$

$$g_k = \frac{1}{Tkjw_0}(1 - e^{-jk\pi})$$

$$g_k = \frac{1}{2\pi jk} e^{-jk\pi/2}(e^{jk\pi/2} - e^{-jk\pi/2})$$

$$g_k = \frac{\sin(\frac{k\pi}{2})}{\pi k} e^{-jk\pi/2}$$

Q3.



We assume that the chess board is of infinite size, so that we can have Fourier series.

$$x(t_1, t_2) = \sum_n \sum_m a_{mn} e^{+jmw_x t_1} \; e^{+jnw_y t_2}$$

$$a_{mn} = \left(\frac{1}{T_1 T_2}\right) \int_{T_1} \int_{T_2} x(t_1, t_2) e^{-jmw_x t_1} \; e^{-jnw_y t_2} \, dt_1 dt_2$$

$$T_1 = 2, \quad T_2 = 2, \qquad w_x = \pi, \quad w_y = \pi$$

$$x(t_1, t_2) = u(t_1)u(-t_2) + u(-t_1)u(t_2)$$

$a_{mn} = \left(\frac{1}{mn\pi^2}\right)$ If m & n are both odd.

=0 Otherwise

$x(t_1, t_2) = \sum_m \sum_n 2/mn\pi^2 \, e^{-jm\pi \, t_1} \; e^{-jn\pi t_2}$ For m and n odd

= 0 otherwise

Q4.

a) *x(t)* is real and odd
Hence, a_k are imaginary and odd & $a_0 = 0$
b) Period is T=2

$$w_0 = \frac{2\pi}{2} = \pi$$

c)

$$x(t) = a_1 e^{jw_1 t} + a_{-1} e^{-jw_1 t}$$

d)

$$|a_1|^2 + |a_{-1}|^2 = 1$$

$$|a_1|^2 = \frac{1}{2}$$

$$a_1 = \frac{1j}{\sqrt{2}}, \frac{1(-j)}{\sqrt{2}}$$

$$a_1 = \frac{j}{\sqrt{2}}, \frac{(-j)}{\sqrt{2}}$$

$$x(t) = \pm[\frac{j}{\sqrt{2}} e^{j\pi t} + \frac{(-j)}{\sqrt{2}} e^{-j\pi t}]$$

$$x(t) = \pm \frac{j}{\sqrt{2}} [e^{j\pi t} - e^{-j\pi t}]$$

$$x(t) = \pm \frac{j}{\sqrt{2}} \ 2j\left[e^{j\pi t} - e^{-j\pi t}\right] \ / \ 2j$$

$$x(t) = \pm\sqrt{2} \sin(\pi t)$$

Q5.

a) Pairs (a) and (b) are orthogonal. Pairs (c) and (d) are not orthogonal.
b) Orthogonal but not orthonormal $A_m = 1/w_0$
c) Orthonormal
d) We have

$$\int_{t_0}^{t_0+T} e^{jmw_0 t} e^{-jnw_0 t} \, dt = \frac{e^{j(m-n)w_0 t_0}\left(e^{j(m-n)2\pi} - 1\right)}{(m-n)w_0}$$

$$= \begin{cases} 0 \ \ when\, m \neq n \\ jT \ \ when\, m = n \end{cases}$$

Therefore, function are orthogonal but not orthonormal.

e) We have

$$\int_{-T}^{T} x_e(t) x_0(t) \, dt = \frac{1}{4} \int_{-T}^{T} [x(t) + x(-t)][x(t) - x(-t)] dt$$

$$= \frac{1}{4}\int_{-T}^{T} x^2(t)dt - 1/4\int_{-T}^{T} x^2(-t)dt$$

$$= 0$$

f) $\int_a^b \frac{1}{\sqrt{A_k}} \frac{\emptyset_k(t)}{\sqrt{A_l}} \emptyset_l^*(t)dt = \frac{1}{\sqrt{A_k A_l}} \int_a^b \int_a^b \emptyset_k(t)\emptyset_l^*(t)\, dt$

$$= \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases}$$

Therefore, functions are orthonormal.

g) $\int_a^b |x(t)|^2\, dt = \int_a^b x(t)x^*(t)dt$

$$= \int_a^b \sum_i a_i \emptyset_i(t) \sum_j a_j \emptyset_j^*(t)\, dt$$

$$= \sum_i \sum_j a_i\, a_j^* \int_a^b \emptyset_i(t)\, \emptyset_j^*(t)dt$$

$$= \sum_i |a_i|^2$$

h) $y(T) = \int_{-\infty}^{\infty} h_i(T-\tau)\emptyset_j(\tau)d\tau$

Q6.

We can write signal x1(t) in terms of impulse and unit step function,

$x1\,(t)$ = δ(t +3)+u(t + 1) − u(t − 1)+ δ(t -2)

$x2\,(t)$ = u(t + 1.5) − u(t − 1.5)

$y(t) = x1(t) * x2(t)$

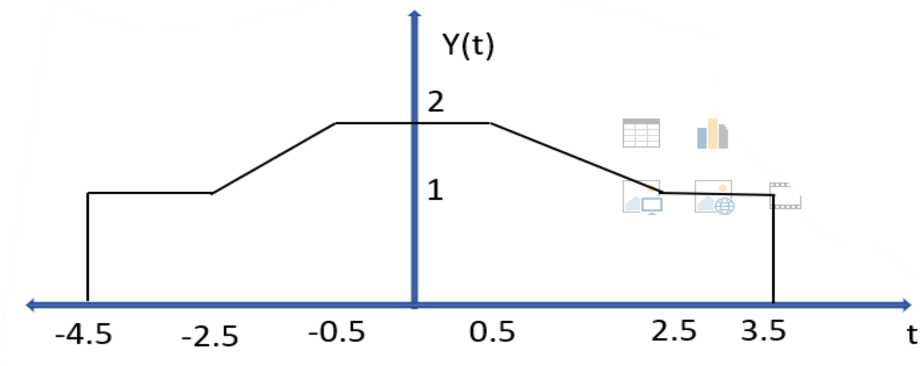
1) x2(t)* δ(t +3)= u(t + 4.5) − u(t + 1.5)

2) x2(t)* δ(t -2)=u(t -0.5) − u(t -3.5)

3) x2(t)*[u(t + 1) − u(t − 1)]=r(t+2.5)-r(t+0.5)-r(t-0.5)+r(t-2.5)

By adding results of 1,2 & 3 we can get expression for y(t)

Plot:



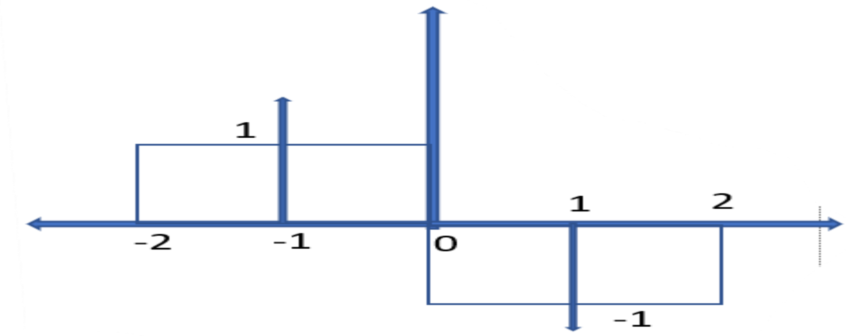
Q7.

As we can see, x(t) is a combination of trapezoid, rectangle & triangle and as we know fourier series is linear ,

By finding F.S. of individual we can find F.S. of x(t) by adding individual F.S.

Another way to do this question is , first we can differentiate x(t) .

after differentiation dx(t)/dt is combination of unit step & delta function.



a) As per above hint u can find fourier series of x(t).
b) $y(t) = x(t) * h(t)$

   If x(t) is a complex exponential

   Then $y(t) = H(jw)e^{jwt}$

   Where $H(jw) = \int_{-\infty}^{\infty} h(\tau)e^{-jw\tau}d\tau.$