Tutorial - 4

(1) P: max $z = cx$

$Ax \leq b$

$x \geq 0$

D: min $y = bw$

$A^T w \geq c$

$w \geq 0$

$w' b = w' A x'$

and $w' A x' = c x'$

This gives $w' b = c x'$

To prove x' is optimal for P.

Let x_1 be feasible sol^n

then by weak duality we have

$$c^T x_1 \leq b^T w' = c^T x'$$

$\Rightarrow c^T x_1 \leq c^T x' \quad \forall x$ feasible

$\Rightarrow x'$ is optimal for P

$||^{ly}$ w' is optimal for D.

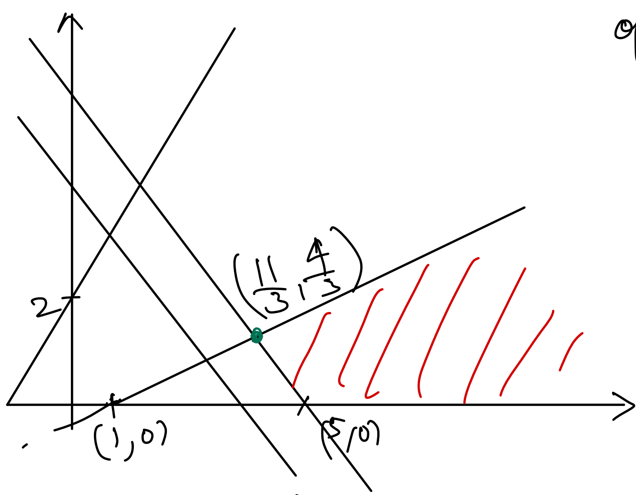
Q2

The dual of P:

$$\max \quad -6y_1 + 3y_2$$

$$-y_1 - y_2 \le 3$$
$$-y_1 - y_2 \le -5$$
$$-y_1 + 2y_2 \le -1$$
$$-3y_1 + y_2 \le 2$$
$$-y_1 - y_2 \le -4.$$
$$y_1 \ge 0, \; y_2 \ge 0$$

Solving using graphical method



optimal value is attained at $\left(\frac{11}{3}, \frac{4}{3}\right)$

max value = -18

By strong duality, we have optimal value = -18 for P.

Q3 The dual of the given LPP

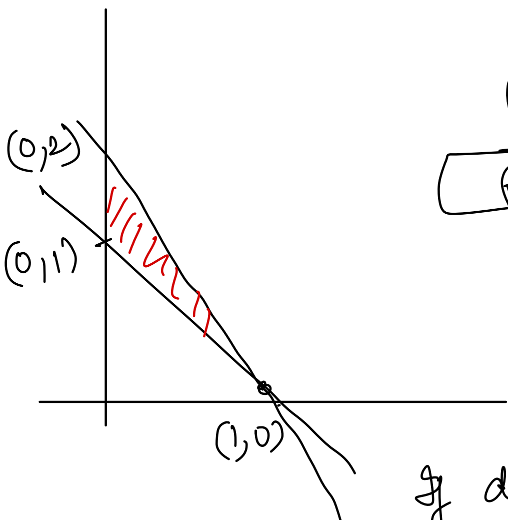
$$\max Z = 3y_1 + 2y_2$$

s.t. $\quad 2y_1 + y_2 \leq 2$

$\quad -y_1 - y_2 \leq -1$

$\quad y_1, y_2 \geq 0.$

Using graphical method:



Feasible	Z
(0, 1)	2
$\boxed{(0, 2)}$	$\boxed{4}$
(1, 0)	3

If dual has optimal, primal has optimal

Q4 Primal problem

$$\min \sum_{j=1}^{n} j x_j$$

$$\sum_{j=1}^{i} x_j \geq i \qquad \forall i = 1, 2 \ldots n$$

$$x_j \geq 0 \quad \forall j$$

$$A = \begin{bmatrix} 1 & & & 0 \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n \times n}$$

$$b' = [1 \; 2 \; 3 \; \ldots \; n] = c$$

the primal can be written as:

$$\min c^T x$$
$$Ax \geq b \qquad - (P)$$
$$x \geq 0, \; x \in R^n$$

Dual:

$$\max b^T y$$
$$A^T y \leq c \qquad - (D)$$
$$y \geq 0, \; y \in R^n$$

Now, To verify

$x^* = (n, 0, 0 \cdots 0)$ is optimal for P

Let's say $y^* = (y_1^*, y_2^* \cdots y_n^*)$ are optimal for (D)

Observe that

only n^{th} constraint have 0 slack, rest all the constraints have non zero slack

$\Rightarrow y_n^* \in \mathbb{R}$ and $y_i^* = 0 \;\forall i = 1, 2 \ldots, n-1$

also, as $x_1 \neq 0 \Rightarrow 1^{st}$ const^n of dual will have zero slack

$$\Rightarrow y_1^* + y_2^* \cdots y_n^* = 1$$

$$\Rightarrow y_n^* = 1$$

So the dual should have optimal sol^n

$$y_1^* = y_2^* \cdots y_{n-1}^* = 0 \text{ and } y_n = 1.$$

To check for feasibility for above y (Exercise)

Primal value at $x^* = n \times 1 + 2 \times 0 \cdots + n \times 0 = n$

Dual value at $y^* =: 1 \times 0 + 2 \times 0 \; \ldots \; n \times 1 = n$

By strong duality, x^* is optimal for P
and y^* is optimal for D

HP.

Q5 P: $\max z = c^T x$ D: $\min w = b^T y$

$Ax \le b$ $A^T y \ge c$

$x \ge 0$ $y \ge 0$

- P and D cannot have unbdd sol^n:

By weak duality, we have

$$z = c^T x \le b^T y = w \quad (\text{for } x \,\&\, w \text{ feasible})$$

If P is unbounded, we can push $z \uparrow$, without any limits. Hence, there are no feasible y for dual

||ly it can be proved for unbounded Sol^n for D.

- P and D, both can be feasible:

Counter example

$c = (1)$, $A = 0$

$b = (-1)$

both primal & dual are infeasible sets.

(Verify)

- The dual(dual(dual)) of an LPP is the primal

False

dual(dual) = primal. and dual(dual(dual)) = dual of LPP.

- Counter example: (Verify)

max: $x_1 + x_2$

$x_1 \leq 1$

$x_2 \leq 1$ $\rightarrow$ optimal $= (1,1)$

$x_1 + 2x_2 \leq 3$

$2x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$

dual: $y_1 + y_2 + 3y_3 + 3y_4$

$y_1 + 2y_3 + 2y_4 \geq 1$

$y_2 + 2y_3 + y_4 \geq 1$

$y \geq 0$.

$y = (1, 1, 0, 0)$

& $\left(0, 0, \frac{1}{3}, \frac{1}{3}\right)$

are sol^n.

Q6 min $80x_1 + 60x_2 + 80x_3$

S.t. $x_1 + 2x_2 + 3x_3 \geq 4$

$2x_1 + 3x_3 \geq 3$

$2x_1 + 2x_2 + x_3 \geq 4$

$4x_1 + x_2 + x_3 \geq 6$

$x_1, x_2, x_3 \geq 0$

Introducing slack variables:

max: $-80x_1 - 60x_2 - 80x_3 + 0S_1 + 0S_2 + 0S_3 + 0S_4$

$-x_1 - 2x_2 - 3x_3 + S_1 = -4$ 1

$-2x_1 - 3x_3 + S_2 = -3$

$-2x_1 - 2x_2 - x_3 + S_3 = -4$

$-4x_1 - x_2 - x_3 + S_4 = -6.$

$x_1, x_2, x_3, x_4, S_1, S_2, S_3, S_4 \geq 0$

Iter-1

		C_j	-80	-60	-80	0	0	0	0
B	C_B	x_B	$\boxed{x_1}$	x_2	x_3	S_1	S_2	S_3	S_4
S_1	0	-4	-1	-2	-3	1	0	0	0
S_2	0	-3	-2	0	3	0	1	0	0
S_3	0	-4	-2	-2	-1	0	0	1	0
$\boxed{S_4}$	0	-6	(-4)	-1	-1	0	0	0	1
$Z=0$		Z_j	0	0	0	0	0	0	0
		$Z_j - C_j$	80	60	80	0	0	0	0
Ratio $\frac{Z_j - C_j}{S_{4j}}$, $S_{4j} < 0$			-20	-60	-80	-	-	-	-

$R_4 = R_4 / -4$

$R_1 = R_1 + R_4$

$R_2 = R_2 + 2R_4$

$R_3 = R_3 + \cancel{2R_4}$

Leaving Basis variable: S_4, entering var: x_1.
pivot element = -4.

Iter-2

B	C_B	x_B	C_j: x_1 (-80)	x_2 (-60)	x_3 (-80)	S_1 (0)	S_2 (0)	S_3 (0)	S_4 (0)
S_1	0	-2.5	0	-1.75	-2.75	1	0	0	-0.25
S_2	0	0	0	0.5	-2.5	0	1	0	-0.5
S_3	0	-1	0	-1.5	-0.5	0	0	1	-0.5
x_1	-80	1.5	1	0.25	0.25	0	0	0	-0.25
Z = -120		Z_j	-80	-20	-20	0	0	0	20
		$Z_j - C_j$	0	40	60	0	0	0	20
$\boxed{\text{Ratio } \frac{Z_j - C_j}{S_{4j}},\ S_{4j} < 0}$			-	-22.85	-21.81	-	-	-	-80

Leaving: S_1 and entering: x_3.
pivot element $\rightarrow$ 2.75

Further, steps can be performed similarly

After iter - 4,

$x_1 = 1.2308 \quad x_2 = 0.4615 \quad x_3 = 0.6154$

$\max -z = -175.3846.$

$\min z = 175.3846$

Q7

$$\min\ c^T x$$
$$\text{s.t. } Ax \geq b$$
$$x \geq 0$$

Dual:

$$\max\ b^T \cancel{x}y \qquad = \qquad \min\ -b^T y$$
$$A^T \cancel{x}y \leq c \qquad\qquad -A^T y \geq -c$$
$$y \geq 0 \qquad\qquad y \geq 0$$

For primal to be same as dual

we need to $\forall y$

$$-b^T y = c^T y \Rightarrow b = -c$$
$$-A^T y = Ay \Rightarrow A = -A^T$$

Q8. Consider the following pair of (P-D) problem

$$\min \ 0^T x \qquad\qquad \max \ e^T x$$
$$\text{s.t} \quad p^T A \geq e^T \quad -(P) \qquad\qquad Ax = 0 \quad -(D)$$
$$x \geq 0.$$

Suppose (b) holds

$p^T A > 0 \Rightarrow p^T A \geq c$

components of p can be scaled such that

$p^T A \geq e$ (equivalent, prove)

$\Rightarrow$ p is feasible with ~~optimal~~ cost 0.

By strong duality, the dual has optimal cost as well.

Suppose $\exists\, x \neq 0$, $Ax = 0$ and $x \neq 0$

Then x is feasible for (D) and

$e^T x > 0$ – a contradiction.

Thus only (b) holds.

Suppose (b) doesn't hold.

Then P is not feasible.

Then D can be either infeasible or unbounded.

Since 0 is always feasible for dual problem.

Thus dual problem is unbounded.

$e^T x$ is acheivable for some x, $Ax = 0$ and $x \geq 0$

$\Rightarrow \quad x \neq 0$

In other words, (a) holds.

Q9 : P: min $c^T x$ D: max $b^T y$

$Ax = b$ $A^T y \geq c$

$x \geq 0$ $y \in R$

x^* optimal for P and p^* optimal for D.

(a) $\tilde{x}$ optimal for (P), when c is replaced by $\tilde{c}$

$c^T x^* \leq c^T x \quad \forall x$ feasible.

$\Rightarrow c^T x^* \leq c^T \tilde{x}$

$||^{ly} \quad \tilde{c}^T \tilde{x} \leq \tilde{c}^T x^*$

$$(\tilde{c} - c)^T (\tilde{x} - x^*) = \underbrace{(\tilde{c}^T \tilde{x} - \tilde{c}^T x^*)}_{\leq 0} + \underbrace{(c^T x^* - c^T \tilde{x})}_{\leq 0.}$$

$$\Rightarrow (\tilde{c} - c)^T (\tilde{x} - x^*) \leq 0.$$

(b) $\tilde{x}$ optimal for $\tilde{b}$ and x^* optimal for b

To prove: $p^{*T} (\tilde{b} - b) \leq c^T (\tilde{x} - x^*)$

Let $\tilde{p}^*$ denote the optimal sol^n for modified dual (after $b \to \tilde{b}$)

by strong duality: $c^T \tilde{x} = \tilde{p}^{*T} \tilde{b}$

p^* feasible for modified dual:

$$p^{*T}\tilde{b} \le \tilde{p}^{*T}\tilde{b} = c^T\tilde{x}$$

$$p^{*T}\tilde{b} \le c^T\tilde{x}.$$

$$p^{*T}(\tilde{b}-b) - c^T(\tilde{x}-x^*)$$

$$= (p^{*T}\tilde{b} - c^T\tilde{x}) - (p^{*T}b - c^Tx^*)$$

$$\le 0.$$

Q10 P: $\min c^T x$

$Ax \geq c$

$x \geq 0$

D: $\max c^T y$

$A^T y \leq c$

$y \geq 0$

given: x^* such that $Ax^* = c$ and $x^* \geq 0$

Since A is symmetric, $A^T y \leq c \Rightarrow Ay \leq c$

Dual: $\max c^T y$

$Ay \leq c$

$y \geq 0$

If we set $y_1 = x^*$, we see that y_1 is feasible for D

Also primal objective: $c^T x^* = c^T y_1 =$ Dual objective

$\Rightarrow$ x^* is optimal for (P) and y_1 is ~~optimal for~~ (D)

(Duality theorem).