COL 751 : Practice Sheet - 1

1. Coin Tosses
You are given a biased coin whose heads probability is p, and you keep tossing it until you get heads. Let X be the number of coin tosses needed. What is the expected value of X?

2. Number of Empty Bins
We throw n balls randomly, uniformly, and independently into k bins. Use Linearity of Expectation to compute the expected number of empty bins.

3. Red and Blue balls
There is a bag containing n red balls and k blue balls. We take out balls one by one, uniformly randomly and throw them away. What is the expected number of blue balls left after all the red balls have been taken out?

4. Expected Sum Without Replacement
An urn contains n balls numbered $1, \dots, n$. We remove k balls at random (without replacement) and add up their numbers. Find the expected value of sum.

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6. Noodle Problem
There are n noodles in a plate. We repeat the following n times: Pick two free ends randomly uniformly and tie them. What is the expected number of loops at the end?

7. Matching
Consider a bipartite graph with two sets of nodes, A and B of size n. Each node in set A is connected to each node in set B with probability p independently. Let X be the number of edges in the maximum matching of the graph. Argue that $E(X)$ is at least np.

8. Graph Cuts
Let $G = (V, E)$ be an undirected graph with m edges. Prove that there exists a partition (X, Y) of vertices of G so that the number of edges across the (X, Y) partition/cut is at least $m/2$.
Hint: Take a random partition and compute expected number of edges in it.

9. Magnet Blocks
A total of n bar magnets are placed end to end in a line with random independent orientations. Adjacent like poles repel, ends with opposite polarities join to form blocks. Find the expected number of blocks of joined magnets.

10. Hitting Sets
Let U be a universe of size n, and $\mathcal{S} = (S_1 \ldots, S_m)$ be a family of subsets of U, each of size at least d. A set R is called a hitting set for family $\mathcal{S}$ if $S_i \cap R$ is non-empty, for $1 \leqslant i \leqslant m$.

Let $R = \{r_1, \ldots, r_k\}$ be a random set such that each element r_i is a uniformly random vertex from universe, chosen independent of other elements, where $k = \frac{n \cdot \ln(mn)}{d}$. Prove that R is a hitting set for family $\mathcal{S}$ with probability at least $(1 - 1/n)$.

11. Approximate Distance Preserver
Consider the following algorithm for computing an approximate distance preserver with an additive stretch.

Algorithm 1: Approximate-Distance-Preserver(G)

$R \leftarrow$ A random set obtained by adding each vertex $r \in V$ in R with probability $\frac{1}{\sqrt{n}}$;
$H \leftarrow$ Union of edges of BFS trees rooted at vertices of R;
foreach $v \in V$ **do**
    $L(v) \leftarrow$ Neighbors of v arranged in arbitrary order;
    $A(v) \leftarrow$ Vertices in $L(v)$ appearing before first occurrence of an element of R (if any);
    Add edges in $v \times A(v)$ to H;

Prove the following claims:

(a) H preserves all-pairs distances up to additive stretch +2.

(b) The expected size of $A(v)$ is $\sqrt{n}$, for each $v \in V$.

(c) The expected number of edges in H is $O(n\sqrt{n})$.

(d) With probability $(1 - 1/n^5)$, the number of edges in H is at most $O(n\sqrt{n} \log n)$.