Department of Mathematics
Tutorial Sheet No. 7
MAL 250/MTL 106 (Probability and Stochastic Processes)

1. Two communication satellites are placed in orbit. The lifetime of the the satellite is exponential distribution with mean $\frac{1}{\mu}$. If one fails its replacement is sent up. The time necessary to prepare and send up a replacement is exponential distribution with mean $\frac{1}{\lambda}$. Let $X(t)$ = the number of satellites in the orbit at time t. Assume $\{X(t), t \geq 0\}$ is a Markov process with state space $\{0, 1, 2\}$. Show that the infinitesimal generator matrix is given by

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda + \mu) & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix}$$

Write down the Kolmogorov forward and backward equations for the above process.

2. Find the limiting probabilities of the Markov chain in Problem no. 1. For what proportion of time in a year will the communication system be out of action with no satellites in operation. ?

3. In a parking lot with N (+ve integer) spaces the incoming traffic is according to a Poisson process with rate λ, but only as long as empty spaces are available. The occupancy times have an exponential distribution with mean $1/\mu$. Without loss of generality, assume that the system is modeled as a birth and death process. Let $X(t)$ be the number of occupied parking spaces at time t. Write the generator matrix Q. Write the forward Kolmogorov equations for the Markov process $\{X(t), t \geq 0\}$. Derive the equilibrium probability distribution of the process.

4. Consider the recent IIT Delhi Open House program. Assume that students from various schools arrive at the reception at the instants of a Poisson process with rate 2. At the reception main door, two program representatives separately explain the program to anybody entering the hall. Each explanation takes a time which is exponential distributed with parameter 1, and is independent of other explanations. After the explanation, the students enters the hall. If both representatives are busy the student goes directly into the hall. Let $X(t)$ be the number of busy representatives at time t. Without loss of generality, assume that the system is modeled as a birth and death process. Write the generator matrix Q. Write the forward Kolmogorov equations for the Markov process $\{X(t), t \geq 0\}$. Derive the equilibrium probability distribution of the process.

5. Consider a service station with two identical computers and two technicians. Assume that when both computers are in good condition, most of the work load is on one computer, exposed to a failure rate $\lambda = 1$, while the other computer's failure rate is $\lambda = 0.5$. Further assume that, if one of the computer fails, the other one takes the full load, thus exposed to a failure rate $\lambda = 2$. Among the technicians, one is with repair rate $\mu = 2$ while the second is with repair rate $\mu = 1$. If both work simultaneously on the same computer, the total repair rate is $\mu = 2.5$. Note that, at any given moment, they work so that repair rate is maximized.

   (a) Determine the infinitesimal generator matrix Q.

   (b) Draw the state transition diagram of the system.

   (c) Determine the steady state probabilities and the system availability.

6. Consider a taxi station where taxis and customers arrive independently in accordance with Poisson processes with respective rates of one and two per minute. A taxi will wait no matter how many other taxis are in the system. Moreover, an arriving customer that does not find a taxi also will wait no matter how many other customers are in the system. Note that a taxi can accommodate ONLY ONE customer by first come first service basis. Define

$$X(t) = \begin{cases} -n & \text{if } n \text{ number of taxis waiting for customers at time } t \\ n & \text{if } n \text{ number of customers waiting for taxis at time } t \end{cases}.$$

   (a) Write the generator matrix Q or draw the state transition diagram for the process $\{X(t), t \geq 0\}$.

(b) Write the forward Kolmogorov equations for the Markov process $\{X(t), t \geq 0\}$.

(c) Is a unique equilibrium probability distribution of the process exist? Justify your answer.

7. The birth and death process $\{X(t), t \geq 0\}$ is said to be a pure death process if λ_i=0 for all i. Suppose $\mu_i = i\mu$, $i = 1, 2, 3, \ldots$ and initially $X_0 = n$. Show that $X(t)$ has $B(n, p)$ distribution with $p = e^{-\mu t}$.

8. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Let X_1 be the epoch of the first arrival and X_n be the interarrival time between the $n-1$th and nth arrival. Let S_{n-1} be the epoch of the $n-1$th arrival.

   (a) Show that $P(X_1 > x) = e^{-\lambda x}$.

   (b) Show that $P(X_n > x \mid S_{n-1} = \tau) = e^{-\lambda x}$.

9. A cable car starts off with n riders. The times between successive stops of the car are independent exponential distributed random variables, each with rate λ. At each stop, one rider gets off. This takes no time and no additional riders get on. Let $X(t)$ denote the number of riders present in car at time t. Write down the Kolmogorov forward equations for the process $\{X(t), t \geq 0\}$. Find the mean and variance of the number of riders present in car at any time t.

10. A rural telephone switch has C circuits available to carry C calls. A new call is blocked if all circuits are busy. Suppose calls have duration which has exponential distribution with mean $\frac{1}{\mu}$ and inter-arrival time of calls is exponential distribution with mean $\frac{1}{\lambda}$. Assume calls arrive independently and are served independently. Model this process as a birth and death process and write the forward Kolmogorov equation for this process. Also find the probability that a call is blocked when the system in steady state.

11. The arrival of large jobs at a server forms a Poisson process with rate two per hour. The service times of such jobs are exponential distributed with mean 20 min. Only four jobs can be accommodated in the system at a time. Assuming that the fraction of computing power utilized by smaller jobs is negligible, determine the probability that a large job will be turned away because of lack of storage space. Also find the mean number of large jobs in the system at steady state.

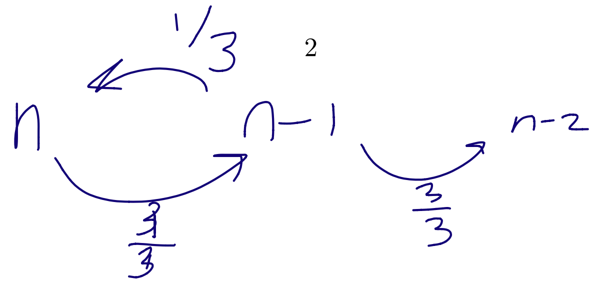
12. Suppose that you arrive at a single-teller bank to find seven other customers in the bank, one being served (First Come First service basis) and the other six waiting in line. You join the end of the line. Assume that, service times are independent and exponential distributed with rate μ. Model this situation as a birth and death process.

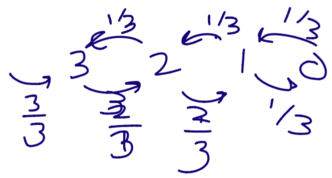
   (a) What is the distribution of time spend by you in the bank?

   (b) What is the expected amount of time you will spend in the bank?

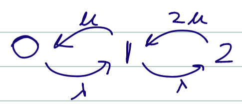
13. A digital camera needs three batteries to run. You buy a pack of 6 batteries, install three of these batteries into the camera. Whenever a battery is drained, you immediately replace the drained battery with the one new battery from the available stock. Assume that each battery lasts for an amount of time that is exponential distributed with mean $1/\mu$, independent of all other batteries. Eventually camera stops running, only two batteries will be left out in the camera that are not drained. Find the expected time that your camera will be able to run with the pack of batteries bought.

14. Consider New Delhi International Airport. Suppose that, it has three runway. Airplanes have been found to arrive at the rate of 20 per hour. It is estimated that each landing takes 3 minutes. Assume that a Poisson process for arrivals and an exponential distribution for landing times. Without loss of generality, assume that the system is modeled as a birth and death process. What is the steady state probability that the no waiting time to land? What is the expected number of airplanes waiting to land? Find the expected waiting time to land?





1)



$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & -\lambda & \lambda & 0 \\ 1 & \mu & -(\mu+\lambda) & \lambda \\ 2 & 0 & 2\mu & -2\mu \end{array}$$

$$p'(t) = p(t) \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -(\mu+\lambda) & \lambda \\ 0 & 2\mu & 2\mu \end{bmatrix}$$

2)

$$0 = -\lambda x + \mu y$$
$$0 = \lambda x - (\mu+\lambda) y + 2\mu z$$
$$0 = \lambda y - 2\mu z$$
$$y = \frac{2\mu z}{\lambda} = \frac{\lambda x}{\mu}$$

$$y\left(1 + \frac{\lambda}{2\mu} + \frac{\mu}{\lambda}\right) = 1 \qquad [x \quad y \quad z]$$

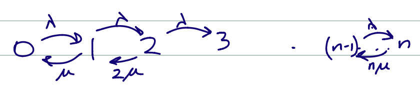
$$\Rightarrow y = \frac{2\mu\lambda}{2\mu\lambda + \lambda^2 + 2\mu^2}$$

$$x = \frac{2\mu^2}{2\mu\lambda + \lambda^2 + 2\mu^2}$$

$$z = \frac{\lambda^2}{2\mu\lambda + \lambda^2 + 2\mu^2}$$

$$\frac{2\mu^2}{2\mu\lambda + \lambda^2 + 2\mu^2}$$

3)



$$Q = \begin{array}{c|cccccc} & 0 & 1 & & & & \\ \hline 0 & -\lambda & \lambda & 0 & 0 & & \\ 1 & \mu & -(\mu+\lambda) & \lambda & & & \\ 2 & 0 & 2\mu & -(2\mu+\lambda) & \lambda & & \\ & 0 & 0 & 3\mu & -(3\mu+\lambda) & \lambda & 0 \end{array}$$

$$-\lambda\pi_0 + \mu\pi_1 = 0 \quad \Rightarrow \quad \mu\pi_1 = \lambda\pi_0 \quad \Rightarrow \quad \pi_1 = \frac{\lambda}{\mu}\pi_0$$
$$\lambda\pi_0 - (\mu+\lambda)\pi_1 + 2\mu\pi_2 = 0 \quad \Rightarrow \quad 2\mu\pi_2 = \lambda\pi_1 \quad \Rightarrow \quad \pi_2 = \frac{\lambda^2}{2!\mu^2}\pi_0$$
$$\lambda\pi_1 - (2\mu+\lambda)\pi_2 + 3\mu\pi_3 = 0 \quad \Rightarrow \quad 3\mu\pi_3 = \lambda\pi_2 \quad \Rightarrow \quad \pi_3 = \frac{\lambda^3}{3!\mu^3}\pi_0$$

$$\vdots$$

$$n\mu\pi_n = \lambda\pi_{n-1}$$

$$\Rightarrow \pi_0\left[1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2!\mu^2} + \frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3 + \cdots\right] = 1$$

4) $\lambda = 2$
$\mu = 1$

$$0 \underset{1}{\overset{2}{\rightleftarrows}} 1 \underset{2}{\overset{2}{\rightleftarrows}} 2$$

$$Q = \begin{bmatrix} -2 & 2 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix}$$

$$-2x + y = 0$$
$$2x - 3y + 2z = 0$$
$$2y - 2z = 0 \quad \Rightarrow \quad 2x = y = z$$
$$\Rightarrow \quad x = \frac{1}{5}, \quad y, z = \frac{2}{5}$$

5) $\lambda = 1$

$$0 \underset{2}{\overset{3}{\rightleftarrows}} 1 \underset{1.5}{\overset{2.5}{\rightleftarrows}} 2$$

$$Q = \begin{bmatrix} -3 & 3 & 0 \\ 2 & -4.5 & 2.5 \\ 0 & 1.5 & -1.5 \end{bmatrix}$$

$$-3x + 2y = 0$$
$$3x - 4.5y + 1.5z = 0$$
$$2.5y - 1.5z = 0$$
$$\frac{3x}{2} = y = \frac{3z}{5}$$
$$y\left(1 + \frac{2}{3} + \frac{5}{3}\right) = 1$$
$$\Rightarrow y = \frac{3}{10}$$
$$z = \frac{1}{2} \qquad x = \frac{2}{10}$$

6) $\cancel{\mu} = 1$
$\lambda = 2$

$$-n \qquad \xrightarrow{2} -1 \underset{1}{\overset{2}{\rightleftarrows}} 0 \underset{1}{\overset{2}{\rightleftarrows}} 1 \underset{1}{\overset{2}{\rightleftarrows}} 2 \qquad \underset{1}{\overset{2}{\leftleftarrows}} n$$

7)

$$0 \xleftarrow{\mu} 1 \xleftarrow{2\mu} 2 \qquad \xleftarrow{3\mu} n$$

$$Q = \begin{bmatrix} 0 & 0 & & & \\ \mu & -\mu & & & \\ & 2\mu & -2\mu & & \\ & & & \ddots & \\ & & & n\mu & -n\mu \end{bmatrix}$$

$P'(t) = P(t)Q$

$x_0'(t) = \mu x_1(t)$

$x_1'(t) = -\mu x_1(t) + 2\mu x_2(t)$

$x_2'(t) = -2\mu x_2(t) + 3\mu x_3(t)$

$\vdots$

$x_{n-1}'(t) = -(n-1)x_{n-1}(t) + n\mu x_n(t)$

$x_n'(t) = -n\mu x_n(t)$

$\Rightarrow x_n(t) = e^{-n\mu t}$

$x_{n-1}'(t) = -(n-1)\mu x_{n-1}(t) + n\mu e^{-n\mu t}$

$x_{n-1}'(t) + (n-1)x_{n-1}(t)$

$=$

$\frac{d}{dt}\left(x_{n-1} e^{(n-1)\mu t}\right) = \int n\mu e^{-\mu t} dt$ use PMI in this

$x_{n-1} = e^{-(n-1)\mu t} \, n\left(e^{-\mu t}\right)$

$= n\left(1 - e^{-\mu t}\right) e^{-(n-1)\mu t}$

8)

$P(X_n > x / S_{n-1} = \tau)$

$= P(S_n - S_{n-1} > x / S_{n-1} = \tau)$

$= P(X > x / S_{n-1} = \tau)$

$= P(X > x)$

$= e^{-\lambda x}$

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Answer for Selected Problems

Note: $\rho = \frac{\lambda}{\mu}$ everywhere

1 $P'(t) = P(t)Q$ $K.F.E$ with $P(t) = (P_0(t), P_1(t), P_2(t))$ and
$P'(t) = QP(t)$ $K.B.E$ with $P(t) = (P_0(t), P_1(t), P_2(t))^T$
where $P_i(t) = P[X(t) = i]$.

2 a)$\pi_0 = \frac{1}{1+\rho+\frac{\rho^2}{2}}, \pi_1 = \rho\pi_0, \pi_2 = \frac{\rho^2}{2}\pi_0$ b)π_0.

3. $S = \{0, 1, 2, \ldots, N\}$, $P'(t) = P(t)Q$
$K.F.E$ where $P(t) = (P_0(t), P_1(t), P_2(t), \ldots, P_N(t))$

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & \ldots & 0 & 0 \\ \mu & -(\mu+\lambda) & \lambda & & \ldots & 0 \\ 0 & 2\mu & -(2\mu+\lambda) & \lambda & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & N\mu & -N\mu \end{pmatrix}$$

and $\pi_k = \frac{\rho^k}{k!}\pi_0$, $1 \le k \le N$ with $\pi_0 = \frac{1}{1+\sum_{i=1}^{N}\frac{\rho^i}{i!}}$.

4. $S = \{0, 1, 2\}$ $P'(t) = P(t)Q$ $K.F.E$ where $P(t) = (P_0(t), P_1(t), P_2(t))$

$$Q = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{pmatrix}.$$

Then, $\pi_0 = \frac{1}{5}$, $\pi_1 = \frac{2}{5}$, $\pi_2 = \frac{2}{5}$.

5. a) $S = \{0, 1, 2\}$, $Q = \begin{pmatrix} -3 & 3 & 0 \\ 2 & -4.5 & 2.5 \\ 0 & 1.5 & -1.5 \end{pmatrix}$. Then, $\pi = (\frac{1}{5}, \frac{3}{10}, \frac{1}{2})$
c) $1 - \pi_0 = \pi_1 + \pi_2 = \frac{4}{5}$.

6. $S = \{0, \pm1, \pm2, \ldots\}$, $P'(t) = P(t)Q$
$K.F.E$ where $P(t) = (P_0(t), P_1(t), P_2(t), \ldots, P_N(t))$
$\pi_n = (1-\rho)\rho^n$, $n = 1, 2, 3, \ldots$ where $\rho = \frac{1}{2}$

9. As in Q7. $X(t) \sim B(n, p)$ where $p = e^{-\mu t}$ mean=np, Var=npq where $q = 1 - p$.

10. $X(t)$ = number of calls in the system at time t. $S = \{0, 1, 2, \ldots, C\}$. Then, blocking probability is given as

$$\pi_C = \frac{\frac{\rho^C}{C!}}{1+\displaystyle\sum_{i=1}^{C}\frac{\rho^i}{i!}}.$$

11. $X(t)$ = number of large jobs in the system at time t with $S = \{0, 1, 2, 3, 4\}$.
$\pi = (\frac{243}{473}, \frac{2}{3}\times\frac{243}{473}, \frac{2}{9}\times\frac{243}{473}, \frac{4}{81}\times\frac{243}{473}, \frac{2}{3^5}\times\frac{243}{473})$
blocking probability= π_4.

12. a) $gamma(8, \lambda)$ b) $\frac{8}{\lambda}$.

13. $X(t)$ = number of non drained batteries at time t with $S = \{6, 5, 4, 3, 2\}$. Then, mean = $\frac{4}{3\mu}$.

14. $X(t)$ =number of busy runways at time t with $S = \{0, 1, 2, 3\}$. $\lambda = 20hour^{-1}$ $\mu = 20hour^{-1}$
a)$1 - \pi_3$,b)$L_W = \displaystyle\sum_{n=4}^{\infty}(n-3)\pi_n = 0.04535$, c)$T_W = L_W/\lambda = 0.0022725$

9)

$0 \xleftarrow{\lambda} \cdots \quad n-1 \xleftarrow{\lambda} n$

$$Q = \begin{bmatrix} 0 & 0 & & & \\ \lambda & -\lambda & & & \\ 0 & \lambda & -\lambda & & \\ 0 & 0 & \lambda & -\lambda & \\ & & & & \\ & & & \lambda & -\lambda \end{bmatrix}$$

$p'(t) = p(t)\,Q(t)$

$x_0'(t) = \lambda x_1(t)$

$x_1'(t) = -\lambda x_1(t) + \lambda(x_2(t))$

$x_2'(t) = -\lambda x_2(t) + \lambda x_3(t)$

$\vdots$

$x_{n-1}'(t) = -\lambda x_{n-1}(t) + \lambda x_n(t)$

$x_n'(t) = -\lambda x_n(t)$

$\Rightarrow x_n = e^{-\lambda t}$

$x_{n-1} e^{\lambda t} = \int \lambda \, dt$

$x_{n-1} = \lambda t e^{-\lambda t}$

$x_{n-2} = \frac{\lambda^2 t^2}{2} e^{-\lambda t}$

$n e^{-\lambda t} + (n-1)\lambda t e^{-\lambda t} + (n-2)\frac{\lambda^2 t^2}{2!} e^{-\lambda t}$

$e^{-\lambda t}\left[n + (n-1)\lambda t + (n-2)\frac{\lambda^2 t^2}{2!} + \cdots \right]$

14) $q_{i,i+1} = \lambda_i$

$q_{i,i-1} = \mu_i$

$\pi_i = \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} \pi_0$