COL202: Discrete Math	Tutorial 9
Instructor: Venkata Koppula	Release date: 2nd November

The solutions for the ($\star$) marked problems must be submitted on Gradescope by 11:59 am on 8th November.
The $\blacklozenge$ marked problems will be discussed in the tutorial.

In this tutorial, we will discuss graphs. Throughout this tutorial, n represents the number of vertices, and m the number of edges. Unless specified otherwise, the graphs have no self-loops or multi-edges.

1 Tutorial Submission Problem

1.1. ($\star$) Let $G = (V, E)$ be an undirected graph, and let $\mathbf{A}$ denote the corresponding adjacency matrix. Prove that for all integers $k > 0$, and for any $u, v \in V$, $\mathbf{A}^k[u, v]$ is equal to the number of walks from u to v of length exactly k.

2 The Probabilistic Method (on Graphs)

The probabilistic method is extensively used for proving graph-theoretic properties. In fact, one of the first applications of the probabilistic method was for proving a lower bound on the Ramsey number, which you saw in Tutorial 7 (exercise 2.6). Here, we will see a few more applications of the probabilistic method (especially in the context of graphs).

2.1. ($\blacklozenge$: [AS92], Theorem 1.2.2) Let $G = (V, E)$ be an undirected graph where every vertex has degree at least δ. A dominating set in a graph is a subset of vertices $U \subseteq V$ such that every vertex in $V \setminus U$ has at least one neighbor in U. Clearly, if we take the entire vertex set V, then this is a dominating set. How small can the dominating set be? Show that there exists a dominating set of size at most $n(1 + \ln(\delta + 1))/(\delta + 1)$.

> **Solution:** Hint: pick a random subset $U \subseteq V$ (from some distribution). Construct a dominating set using U.

2.2. ([AS92], Ex 4) Let $G = (V, E)$ be a graph where every vertex has degree $\delta > 10$. Show that V can be partitioned into two disjoint subsets A, B such that $|A| \le O(n \ln \delta / \delta)$ and every vertex in B has at least one neighbor in A and at least one neighbor in B.

2.3. Let $G = (V, E)$ be an undirected graph where every vertex has degree exactly d. An independent set $S \subseteq V$ is a subset of vertices such that for every $a, b \in S$, $\{a, b\} \notin E$. Show that there exists an independent set of size at least $n/2d$.

3 General Properties of Graphs

3.1. (♦) Matrix multiplication can be performed in time $o(n^3)$. Use this fact to give an $o(n^3)$ time algorithm for the following problem: given an undirected graph G, check if there exist three vertices $a, b, c \in V$ such that $\{a, b\}$, $\{b, c\}$ and $\{c, a\}$ are all edges in E. The naive algorithm (checking all triplets) takes time $O(n^3)$.

Solution:

3.2. Consider an undirected graph $G = (V, E)$ where every vertex has degree at least d. Prove that G has a path of length at least d.

3.3. (♦ - [MN09], pg 130, Exercise 16) Consider any connected, undirected graph $G = (V, E)$ where, for any pair of distinct vertices u, v, either u and v have no common neighbors, or have exactly 2 common neighbors. Prove that all vertices of G have the same degree.

(Harder version) : Suppose you are given that for any pair of distinct vertices u, v, either u and v have no common neighbors, or have exactly 5 common neighbors. Prove that all vertices have the same degree.

Solution:

References

[AS92] Noga Alon and Joel Spencer. *The Probabilistic Method.* John Wiley, 1992.

[MN09] Jirí Matousek and Jaroslav Nesetril. *Invitation to Discrete Mathematics (2. ed.).* Oxford University Press, 2009.