Problem sheet–4

1. Convolution of continuous signals

Determine the convolution of the following two signals.

a) $x(-t-\tau_1)$ and $\delta(t-\tau_2)$
b) $x(-t-\tau_1)$ and $\delta(-t-\tau_2)$
c) $rect(t/T)$ and $rect\left(\frac{2t}{T}\right)$

2. Convolution of Discrete time signals

Let x[n] = δ[n] + 2δ[n- 1] - δ[n - 3] and h[n] = 2δ[n + 1] + 2δ[n - 1]. Compute and plot each of the following convolutions:

a) $y_1[n] = x[n] * h[n]$
b) $y_2[n] = x[n+2] * h[n]$
c) $y_3[n] = x[n] * h[n+2]$

3. Properties of Convolution integral

Determine if each of the following statements is true in general. Provide proofs for those that you think are true and counter examples for those that you think are false.

a) $x[n] * \{h[n]g[n]\} = \{x[n] * h[n]\}g[n]$
b) If $y(t) = x(t) * h(t)$, then $y(2t) = 2x(2t) * h(2t)$.
c) If x(t) and h(t) are odd signals, then $y(t) = x(t) * h(t)$ is an even signal.
d) If $y(t) = x(t) * h(t)$, then $Ev\{y(t)\} = x(t) * Ev\{h(t)\} + Ev\{x(t)\} * h(t)$, where E_v stands for even part of the signal.

4. Periodic Convolution (Convolution of periodic signals)

Let $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ be two periodic signals with a common period T_o. It is not too difficult to check that the convolution of $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ does not converge. However, it is sometimes useful to consider a form of convolution for such signals that is referred to as periodic convolution. Specifically, we define the periodic convolution of $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ as

$$\tilde{y}(t) = \int_0^{T_0} \tilde{x}_1(\tau)\tilde{x}_2(t-\tau)d\tau = \tilde{x}_1(t) \circledast \tilde{x}_2(t)$$

Note that we are integrating over exactly one period.

a) Show that $\tilde{y}(t)$ is periodic with period T_0.
b) Consider the signal

$$\tilde{y}_a(t) = \int_a^{a+T_0} \tilde{x}_1(\tau)\tilde{x}_2(t-\tau)d\tau$$

where a is any arbitrary real number. Show that

$$\tilde{y}_a(t) = \tilde{y}(t)$$

Hint: Write $a = kT_0 - b$, where $0 \le b < T_0$.

5. Signal representation

Consider the following set of signals:

$$\varphi[n] = (\tfrac{1}{2})^n\, u[n],$$

$$\varphi_k[n] = \varphi[n-k],\ k = 0, 1, \pm 2, \pm 3, \ldots.$$

a) Show that an arbitrary signal can be represented in the form

$$x[n] = \sum_{k=-\infty}^{+\infty} a_k\, \varphi[n-k]$$

by determining an explicit expression for the coefficient a_k in terms of the values of the signal x[n].

b) Let r[n] be the response of an LTI system to the input x[n] = φ[n]. Find an expression for the response y[n] to an arbitrary input x[n] in terms of r[n] and x[n].

c) Show that y[n] can be written as

$$y[n] = \psi[n] * x[n] * r[n]$$

by finding the signal $\boldsymbol{\psi}$[n].

d) Use the result of part (c) to express the impulse response of the system in terms of r[n]. Also, show that

$$\psi[n] * \varphi[n] = \delta[n]$$

6. Distance approximation using Radar

a) In this question we introduce the problem of measuring the radar range to an object by transmitting a radio frequency (RF) pulse and determining the round trip time delay for the echo of the pulse to return to the radar. In this question we identify the LTI system describing the propagation of the pulse. Let the transmitted RF pulse be given by

$$x(t) = \begin{cases} sin(w_c t), 0 \le t \le T \\ 0, \quad otherwise \end{cases}$$

Suppose we transmit an impulse from the radar to determine the impulse response of the round trip propagation to the target. The impulse is delayed in time and attenuated in amplitude which results in the impulse response

$$h(t) = \alpha\, \delta(t - \beta)$$

where α represents the attenuation factor and β is the round trip time delay. Use the convolution of x(t) with h(t) to verify this result.

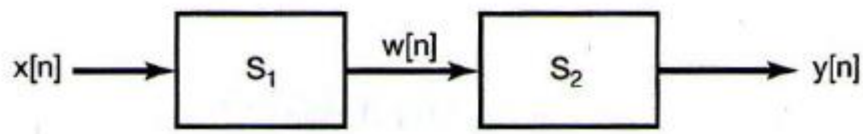
b) In the previous question, the target range is determined by estimating the time delay β from the received signal r(t). In principle this may be accomplished by measuring the onset time of the received pulse. However, in practice the received signal is contaminated with noise and maybe weak. For these reasons, the time delay is determined by passing the received signal through an LTI system commonly referred to as a matched filter. An important property of this system is that it optimally discriminates against certain types of noise in the received waveform. The impulse response of the matched filter is a reflected, or time reversed version of the transmitted signal x(t). That is h_m(t) = x(-t).

$$h_m(t) = \begin{cases} -sin(w_c t),\ -T \le t \le 0 \\ 0, \quad otherwise \end{cases}$$

The terminology "matched filter" refers to the fact that the impulse response of the radar receiver is "matched" to the transmitted signal. Estimate the time delay from the matched filter output.

7. Cascade of LTI systems

Consider the cascade of the following two systems S_1 and S_2, as depicted in figure below.



S_1: causal LTI

$$w[n] = \frac{1}{2} w[n-1] + x[n]$$

S_2: causal LTI

$$y[n] = \alpha y[n-1] + \beta w[n]$$

The difference equation relating x[n] and y[n] is:

$$y[n] = -\frac{1}{8} y[n-2] + \frac{3}{4} y[n-1] + x[n]$$

a) Determine α and β
b) Show the impulse response of the cascaded system.

8. Properties related to LTI systems

Determine if each of the following statements concerning LTI systems is true or false. Justify your answers.

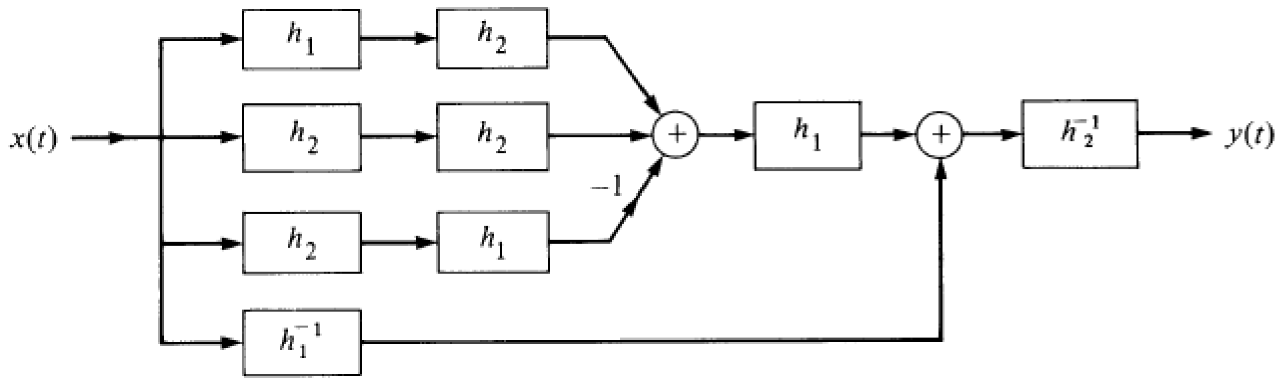
a) If h(t) is the impulse response of an LTI system and h(t) is periodic and nonzero, the system is unstable.
b) The inverse of a causal LTI system is always causal.
c) If $|h[n]| \leq K$ for each n, where K is a given number, then the LTI system with h[n] as its impulse response is stable.
d) If a discrete-time LTI system has an impulse response h[n] of finite duration, the system is stable.
e) If an LTI system is causal, it is stable.
f) The cascade of a non-causal LTI system with a causal one is necessarily non-causal.
g) A continuous-time LTI system is stable if and only if its step response s(t) is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |s(t)| dt < \infty$$

h) A discrete-time LTI system is causal if and only if its step response s[n] is zero for $n < 0$.

9. Evaluation of LTI system using block diagrams

Find the combined impulse response of the LTI system in the figure given below. Recall that $x(t) * h(t) * h^{-1}(t) = x(t)$.



10.Time domain analysis of a system

Determine the unit impulse response h(t) for a system specified by the equation:

$(D^2 + 3D + 2)y(t) = D\,x(t)$, where D stands for the differentiation operator.

11.Unit-step response

The unit-step response $s[n]$, that is response of the system to $u[n]$, of a discrete-time linear time-invariant (LTI) system is:

$$s[n] = e^{\beta n}u[n]$$

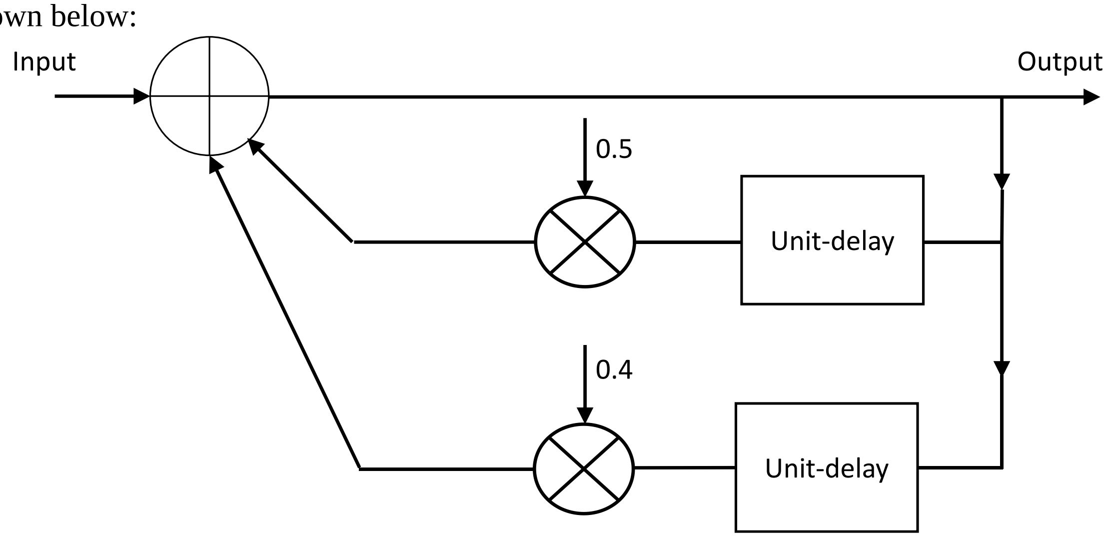
Find the output of the system if the input $x[n]$ is:

$$x[n] = e^{\alpha n}u[n] - e^{\alpha[n-1]}u[n-1],$$

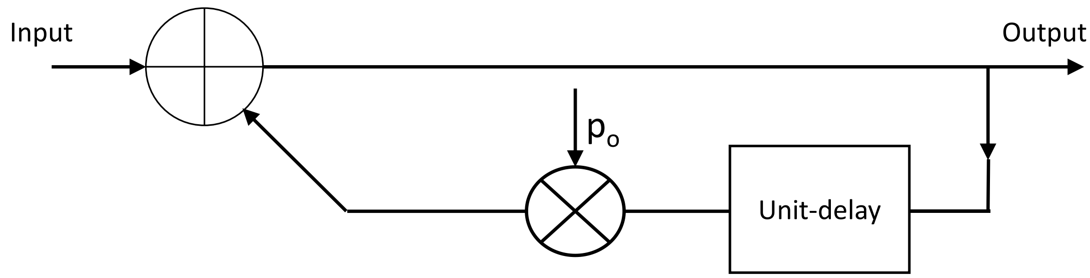
for $\propto, \beta < 0$.

12.Stability of the system

Consider a discrete-time LTI system implemented using unit-delays, multipliers and an adder as shown below:



Assuming that the above system starts at rest, we replace the above system with an equivalent discrete-time LTI system which is as shown below:



Find the value of p_o. Is the system stable? Justify.

13. Solving difference equation

Consider the causal LTI system described by the following difference equation:

$$y[n] = 2x[n] + 3x[n-1] - y[n-2]$$

If the impulse response $h[n]$ of the system is

$$h[n] = \{A\mathrm{j}^n + B(-\mathrm{j})^n\}u[n]$$

where $\mathrm{j} = \sqrt{-1}$. Find A and B.