Department of Mathematics

MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Major (I Semester 2016 - 2017)

Time allowed: 2 hours **Max. Marks: 50**

1. (a) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{A_n\}$ be a nondecreasing sequence of elements in $\mathcal{F}$. Prove that

$$P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

$C_i = A_{i+1} - A_i$ - 1

1/2 $C_i \cap C_j = \phi,\ i \neq j$ $P(\lim \cup C_i) = \sum P(C_i)$ 1/2 $= \lim P(A_n)$ 2

(4 marks)

(b) Let X and Y be independent random variables with $X \sim B(3, \frac{1}{3})$ and $Y \sim B(2, \frac{1}{2})$. Find $P(X = Y)$. $= P\{X=0, Y=0\} + P\{X=1, Y=1\} + P\{X=2, Y=2\}$ - 1 $= \sum P(X=i)P(Y=i) = 19/54$ - 1 (3 marks)

2. Let X and Y be two random variables such that $\rho(X, Y) = \frac{1}{2}$, $Var(X) = 1$ and $Var(Y) = 4$. Compute $Var(X - 3Y)$. $cov(X,Y) = 1$ 2 (4 marks)

$Var(X-3Y) = Var(X) + 9Var(Y) - 6cov(X,Y)$ - 1 $= 31$ - 1.

3. Let $\{X_n\}$ be a sequence of independent random variables defined by

$$P\{X_n = 0\} = 1 - \frac{1}{n}, \quad \text{and} \quad P\{X_n = 1\} = \frac{1}{n}, \quad n = 1, 2, \ldots.$$

Verify the following statements:

(a) $X_n \xrightarrow{p} 0$ (b) $X_n \xrightarrow{d} 0$ (c) $X_n \xrightarrow{a.s} 0.$ (2 + 2 + 2 marks)

4. Consider the random telegraph signal, denoted by $X(t)$, jumps between two states, 0 and 1, according to the following rules. At time $t = 0$, the signal $X(t)$ start with equal probability for the two states, i.e., $P(X(0) = 0) = P(X(0) = 1) = 1/2$, and let the switching times be decided by a Poisson process $\{Y(t), t \geq 0\}$ with parameter λ independent of $X(0)$. At time t, the signal

$$X(t) = \frac{1}{2}\left(1 - (-1)^{X(0)+Y(t)}\right), t > 0.$$

$E(X(t)) = \frac{1}{2}$ 1/2

$E(X^2(t)) = 1/2 < \infty$ 1/2

$cov(X(t), X(s)) = \frac{1}{4} e^{-2\lambda|t-s|}$ - 2.

Is $\{X(t),\ t \geq 0\}$ covariance/wide sense stationary? Justify your answer. (5 marks)

5. Suppose there are three white and three black balls in two urns (labeled 1, 2) distributed so that each urn contains three balls. We say that the system is in state $i \in \{0, 1, 2, 3\}$, if there are i white balls in Urn 1. At each stage one ball is drawn at random from each urn and interchanged. Let X_n denote the state of the system after the nth draw. Prove that $\{X_n, n = 0, 1, \ldots\}$ is a discrete time Markov chain. Write the one-step probability transition matrix or draw the state transition diagram for this Markov chain. (4 + 2 marks)

$X_{n+1} = X_n + Z_n$, $Z_n \in \{0, \pm 1\}$, $P_{i,i} = 2\left(\frac{i}{3}\right)\left(\frac{3-i}{3}\right)$, $P_{i,i+1} = \left(\frac{3-i}{3}\right)^2$, $P_{i,i-1} = \left(\frac{i}{3}\right)^2$

6. Consider a branching process, denoted by Galton-Watson process, that model a population in which each individual in generation n produces some random number of individuals in generation $n + 1$, according, in the simplest case, to a fixed probability distribution that does not vary from individual to individual. That is, the first generation of individuals is the collection of off-springs of a given individual. The next generation is formed by the off-springs of these individuals. Let X_n denote the number of individuals of the nth generation, starting with $X_0 = 1$ individual (the size of the zeroth generation). Let Y_i (or

3 (a) $\lim_{n\to\infty} P\{|X_n - X| > \epsilon\} = 0$ - 1

$\lim_{n\to\infty} P\{|X_n| > \epsilon\} = 0$ - 1

(b) $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ - 1

$\lim_{n\to\infty} P\{X_n \leq x\} = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$ - 1

(c) $P\{\lim_{n\to\infty} X_n = X\} = 1$ - 1

$P\{\lim_{n\to\infty} X_n = 0\} = 1$ - 1

$Y_{i,n}$) be the number of offspring of the ith individual of the nth generation. Suppose that, $\{Y_i, i = 1, 2, \ldots\}$ are non-negative integer valued i.i.d. random variables with probability mass function $p_j = P(Y_i = j), j = 0, 1, \ldots$ and independent of the size of the generation. Then

(1) - $\{0\}$ - absorbing state

(P) - $\{1, 2, \ldots\}$ - transient states

P.T.

$P(X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \ldots X_0) = P(X_{n+1}=j \mid X_n=i)$ — (3)

$$X_n = \sum_{i=1}^{X_{n-1}} Y_i, \quad n = 1, 2, \ldots$$

and $\{X_n, n = 0, 1, \ldots\}$ is a discrete time Markov chain. Classify the states of the chain as transient, +ve recurrent or null recurrent. (3+2) (5 marks)

7. Consider the recent IIT Delhi Open House program. Assume that students from various schools arrive at the reception at the instants of a Poisson process with rate 4 per minute. At the reception main door, two program representatives separately explain the program to any student entering Dogra hall. Each explanation takes a time which is exponential distribution with parameter 2 per minute, and is independent of other explanations. After the explanation, the students enters Dogra hall. If both representatives are busy the student goes directly into Dogra hall. Let $X(t)$ be the number of busy representatives at time t. Without loss of generality, assume that the system $\{X(t), t \geq 0\}$ is modeled as a birth and death process. Write the generator matrix Q or draw the state transition diagram. Write the forward Kolmogorov equations for the Markov process $\{X(t), t \geq 0\}$. Derive the equilibrium probability distribution of the process.

$\pi'(t) = \pi(t)Q$ -(2) $\quad \pi = \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$ -(2) -(2) (2 + 2 + 2 marks)

8. Suppose that you arrive at a single-teller bank to find ten other customers in the bank, one being served (First Come First service basis) and the other ~~six~~ nine waiting in line. You join the end of the line. Assume that, service times are independent and exponential distributed with parameter μ. Model this situation as a birth and death process.

   (a) What is the distribution of waiting time before your own service in the bank? = Sum of 10 independent exponential dist = Gamma(10, μ)

   (b) What is the expected amount of time you will spend in the bank?

   $= \frac{10}{\mu} + \frac{1}{\mu} = \frac{11}{\mu}$ (3 + 2 marks)

9. A toll bridge with 10 booths at the entrance can be modeled as a 10 server Markovian queueing system with infinite capacity. Assume that the vehicle arrival follows a Poisson process with parameter 8 per minute and the service times are independent exponential distributed random variables with mean 1 minute.

   (a) Draw the state transition diagram for a birth and death process for the system.

   (b) Find the limiting state probabilities.

   (c) If 2 more booths are installed, i.e, total 12 booths, what is the maximum arrival rate such that the limiting state probabilities exist?

(a) (2 + 2 + 2 marks)

(b) π_n from $M/M/10/\infty$

(c) $\lambda < 12$