Name: ______________________ **Entry number:** ______________

COL352-Holi2024 Quiz 1 Duration: 1 hour

1. (10 points) For a string $x \in \{0,1\}^*$, let $\overline{x}$ denote the bit-wise complement of x (e.g. if $x = 10110$, then $\overline{x} = 01001$). Prove that if $L \subseteq \{0,1\}^*$ is a regular language, then $L' = \{x \in \{0,1\} \mid x\overline{x} \in L\}$ is also a regular language.

For $K \subseteq \{0,1\}^*$, let $\overline{K} = \{\overline{x} \mid x \in K\}$.

Claim: K regular $\Rightarrow \overline{K}$ regular.

Proof: Suppose $K = \mathcal{L}(D)$ for DFA D. Change the 0-transitions of D to 1-transitions, and vice versa, to get a DFA that recognises $\overline{K}$. ∎

Let $D = (Q, \Sigma, \delta, q_0, A)$ be a DFA recognising L. Let $L_{ij} = \{x \mid \delta(i, x) = j\}$ for $i, j \in Q$.

Claim: $L' = \bigcup_{q \in Q} \bigcup_{t \in A} \left(L_{q_0 q} \cap \overline{L_{qt}}\right)$

Proof: $x \in L' \Rightarrow x\overline{x} \in L$. Let $q = \delta(q_0, x)$, $t = \delta(q_0, x\overline{x}) = \delta(q, \overline{x})$. Then $x \in L_{q_0 q}$, $\overline{x} \in L_{qt}$, and $t \in A$. This implies $x \in \overline{L_{qt}}$ $\therefore x \in RHS$.

$x \in RHS \Rightarrow \exists\, q \in Q, t \in A$ such that $x \in L_{q_0 q} \cap \overline{L_{qt}}$.
This implies $x \in L_{q_0 q}$ and $\overline{x} \in L_{qt}$. $\therefore \delta(q_0, x) = q$, $\delta(q, \overline{x}) = t$.
Thus, $\delta(q_0, x\overline{x}) = t \in A$ $\therefore x\overline{x} \in L$ $\therefore x \in L'$. ∎

Since Q (and therefore, A) is finite, L' is regular because Reg is closed under finite boolean operations.

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2. (10 points) For a DFA $D = (Q, \Sigma, \delta, q_0, A)$, let the language $\mathcal{L}_{\text{all}}(D)$ be defined as $\mathcal{L}_{\text{all}}(D) = \{x \in \Sigma^* \mid \text{the run of } D \text{ on } x \text{ visits all states of } D \text{ at last once}\}$. Prove that for every DFA D, the language $\mathcal{L}_{\text{all}}(D)$ is necessarily regular.

Let $L_q = \{x \in \Sigma^* \mid$ the run of D on x visits q at least once$\}$.

Claim: L_q is regular.

Proof: Change the DFA so that q is the only accepting state, and all transitions going out of q come back to q. This DFA recognises L_q.

Now $\mathcal{L}_{all}(D) = \bigcap_{q \in Q} L_q$.

Since Q is finite, each L_q is regular, and Reg is closed under finite boolean operations, $\mathcal{L}_{all}(D)$ is regular.

Alternate answer: The following DFA recognises $\mathcal{L}_{all}(D)$

State space: $Q \times 2^Q$

Transition function δ_{all}:

$$\delta_{all}((q, S), a) = (\delta(q,a), S \cup \{\delta(q,a)\})$$

Initial state: $(q_0, \{q_0\})$

Accepting states: $\{(q, Q) \mid q \in Q\}$

$\delta_{all}((q_0, \{q_0\}), x) = (\delta(q_0, x), S)$, where S is the set of states of D visited in the run of D on x.