MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 4 (Random Vector)

1. Let X and Y be independent random variables. The range of X is $\{1, 3, 4\}$ and the range of Y is $\{1, 2\}$. Partial information on the probability mass function is as follows: $p_X(3) = 0.50$; $p_Y(2) = 0.60$; $p_{X,Y}(4, 2) = 0.18$.
   (a) Determine p_X, p_Y and $p_{X,Y}$ completely. (b) Determine $P(|X - Y| \geq 2)$.

2. Let (X, Y) be a two-dimensional discrete type random variables with joint pmf $p(x, y) = cxy$ for $x = 1, 2, 3$ and $y = 1, 2, 3$ and equals zero otherwise. (a) Find c. (b) Find $P(1 \leq X \leq 2, Y \leq 2)$. (c) $P(Y = 3)$.

3. Show that $F(x, y) = \begin{cases} 0, & x < 0, y < 0 \text{ or } x + y < 1 \\ 1, & \text{otherwise} \end{cases}$ is not a distribution function.

4. Suppose that (X, Y) has joint pdf $f_{XY}(x, y) = \begin{cases} kx(x - y), & 0 < x < 2, -x < y < x \\ 0, & \text{otherwise} \end{cases}$. Find k. Evaluate $P(X < 1/Y = \frac{1}{2})$ and $P(Y < \frac{3}{2}/X = 1)$.

5. Let (X_1, X_2) be a random vector with the joint pdf $f(x_1, x_2) = \begin{cases} Kx_1x_2, & 0 < x_1 < 1, 0 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}$

   (a) Find the value of K? (b) Define $Y_1 = X_1^2$ and $Y_2 = X_1X_2$. Find the joint pdf of (Y_1, Y_2)

6. Consider a transmitter sends out either a 0 with probability p, or a 1 with probability $(1 - p)$, independently of earlier transmissions. Assume that the number of transmissions within a given time interval is Poisson distributed with parameter λ. Find the distribution of number of 1's transmitted in that same time interval?

7. Let $X_1, X_2, \ldots, X_5$ be iid random variables each having uniform distributions in the interval $(0, 1)$.

   (a) Find the probability that $\min(X_1, X_2, \ldots, X_5)$ lies between $(1/4, 3/4)$.
   (b) Find the probability that X_1 is the minimum and X_5 is the maximum among these random variables?

8. Let X_1, X_2 and X_3 be iid random variables each having the probability density function

$$f(x) = \begin{cases} e^{-x}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

   Find the joint distribution of (Y_1, Y_2, Y_3) where $Y_1 = X_1 + X_2 + X_3$, $Y_2 = \frac{X_1+X_2}{X_1+X_2+X_3}$ and $Y_3 = \frac{X_1}{X_1+X_2}$. Are Y_1 and Y_2 independent random variables? Justify your answer.

9. Let X, Y, Z be independent exponential distributed random variables with parameters λ, μ, ν respectively. Find $P(X < Y < Z)$.

10. Amit and Supriya work independently on a problem in Tutorial Sheet 5 of Probability and Stochastic Processes course. The time for Amit to complete the problem is exponential distributed with mean 5 minutes. The time for Supriya to complete the problem is exponential distributed with mean 4 minutes. Given that Amit requires more than 1 minutes, what is the probability that he finishes the problem before Supriya?

11. Let X and Y be independent random variables with $X \sim B(2, \frac{1}{3})$ and $Y \sim B(3, \frac{1}{2})$. Find $P(X = Y)$.

12. Evaluate all possible marginal and conditional distributions if (X, Y) has the following joint probability mass function
    (a) $P(X = j, Y = k) = q^{k-j}p^j, j = 1, 2, \ldots$ and $k = j + 1, j + 2, \ldots$ $q = 1 - p$
    (b) $P(X = j, Y = k) = \frac{15!}{j!k!(15-j-k)!}(\frac{1}{2})^j(\frac{1}{3})^k(\frac{1}{6})^{15-j-k}$
    for all admissible non negative integral values of j and k.

13. Find k, if the joint probability density of (X_1, X_2) is $f_{X_1X_2}(x_1, x_2) = \begin{cases} ke^{-3x_1-4x_2}, & x_1 > 0, x_2 > 0 \\ 0, & \text{otherwise} \end{cases}$. Also find the probability that the value of X_1 falls between 0 and 1 while X_2 falls between 0 and 2.

14. Suppose that (X, Y) has joint pdf $f(x, y) = \begin{cases} 1, & 0 < x, y < 1 \\ 0, & \text{otherwise} \end{cases}$. Find the pdf of $X + Y$.

15. Romeo and Juliet have a date at a given time, and each, independently, will be late by an amount of time, denoted by X and Y respectively, that is exponentially distributed with parameter λ. Find the pdf of, $X - Y$, the difference between their times of arrival?

16. Let X, Y and Z be independent and identically distributed random variables each having a uniform distribution over the interval $[0, 1]$. Find the joint density function of (V, W) where $V = XY$ and $W = Z^2$.

17. The random variable X represents the amplitude of cosine wave; Y represents the amplitude of a sine wave. Both are independent and uniformly distributed over the interval (0,1). Let R represent the amplitude of their resultant, i.e., $R^2 = X^2 + Y^2$ and θ represent the phase angle of the resultant, i.e., $\theta = \tan^{-1}(\text{Y/X})$. Find the joint and marginal pdfs of θ and R.

18. Let X and Y be continuous random variables having joint distribution which is uniform over the square which has corners at $(2, 2), (-2, 2), (-2, -2)$ and $(2, -2)$. Determine $P(|Y| > |X| + 1)$.

19. Suppose that, we choose a point (X, Y) uniformly at random in E where $E = \{(x, y) \mid |x| + |y| \leq 1\}$. Find the joint pdf of (X, Y).

20. Let X and Y be independent rvs with X follows $Exp(1)$ and Y follows $U[0, 1]$. Let $Z = \max\{X, Y\}$. Find $P(Z \leq z)$.

21. Mr. Ram and Mr. Ramesh agree to meet between 5:00 PM and 6:00 PM, with the understanding that each will wait no longer than 20 minutes for the other (and if other does not show up they will leave). Suppose that each of them arrive at a time distributed uniformly at random in this time interval, independent of the other. What is the probability that they will meet?

22. Let (X, Y, Z, U) be a 4-dim rv with joint pdf $f_{X,Y,Z,U}(x, y, z, u) = \begin{cases} e^{-u}, & 0 < x < y < z < u < \infty \\ 0, & \text{otherwise} \end{cases}$. Find the marginal pdf of X.

23. Let A, B and C be independent random variables each with uniform distributed on interval $(0, 1)$. What is the probability that $Ax^2 + Bx + C = 0$ has real roots?

24. Aditya and Aayush work independently on a problem in Tutorial Sheet 4 of Probability and Stochastic Processes course. The time for Aditya to complete the problem is exponential distributed with mean 5 minutes. The time for Aayush to complete the problem is exponential distributed with mean 3 minutes.

    (a) What is the probability that Aditya finishes the problem before Aayush?

    (b) Given that Aditya requires more than 1 minutes, what is the probability that he finishes the problem before Aayush?

    (c) What is the probability that one of them finishes the problem a minute or more before the other one?

25. Let A, B and C be independent random variables. Suppose that A and C are uniformly distributed on $[0, 1]$. Further, B is uniformly distributed on $[0, 2]$. What is the probability that $Ax^2 + Bx + C = 0$ has real roots?

26. Prove that the correlation coefficient between any two random variables X and Y lies in the interval $[-1, 1]$.

27. For each fixed $\lambda > 0$, let X be a Poisson distributed random variable with parameter λ. Suppose λ itself is a random variable following a gamma distribution with pdf

$$f(\lambda) = \begin{cases} \frac{1}{\Gamma(n)} \lambda^{n-1} e^{-\lambda}, & \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$$

where n is a fixed positive constant. Find the pmf of the random variable X.

28. The number of pages N in a fax transmission has geometric distribution with mean 4. The number of bits k in a fax page also has geometric distribution with mean 10^5 bits independent of any other page and the number of pages. Find the probability distribution of total number of bits in fax transmission.

29. It is known that a IIT bus will arrive at random at Nilgiri hostel bus stop sometime between 8:30 A.M. and 8:45 A.M. Rahul decides that he will go at random to this location between these two times and will wait at most 5 minutes for the bus. If he misses it, he will take the cycle rickshaw. What is the probability that he will take the cycle rickshaw?

30. Let (X, Y) be a two-dimensional continuous type random variables with joint pdf

$$f(x, y) = xe^{-x(y+1)} \quad \text{for} \quad x, y > 0 \quad \text{and} \quad f(x, y) = 0 \quad \text{otherwise.}$$

Define $Z = XY$. Then, find $P(Z \leq z)$?

31. Let (X, Y) ba a two-dimensional random variable with joint pdf

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}Q(x,y)}, \quad -\infty < x, y < \infty$$

where $\sigma_1, \sigma_2 > 0, \mid \rho \mid < 1$ and

$$Q(x, y) = \frac{1}{(1-\rho^2)} \left[\left(\frac{x-\mu_1}{\sigma_1} \right)^2 - 2\rho \left(\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} \right) + \left(\frac{y-\mu_2}{\sigma_2} \right)^2 \right]$$

Find the conditional pdf $f_{Y/X}(y/x)$.

32. Let X_0 be an integer-valued random variable, $P(X_0 = 0) = 1$, that is independent of the i.i.d. sequence $Z_1, Z_2, \ldots$, where Z_n can take values in the set $\{-1, 1\}$ such that $P(Z_n = -1) = \frac{2}{5}, P(Z_n = 1) = \frac{3}{5}$. Let $X_n = X_{n-1} + Z_n,\ n = 1, 2, \ldots$.

    (a) Find $P(X_3 = 1)$?

    (b) Find the value of $P(X_5 = -1 \mid X_2 = 0)$?