Department of Mathematics
Tutorial Sheet No. 6
MTL 106 (Introduction to Probability and Stochastic Processes)

1. Let $X(t) = A_0 + A_1 t + A_2 t^2$, where $A_i's$ are uncorrelated random variables with mean 0 and variance 1. Find the mean function and covariance function of $X(t)$.

2. In a communication system, the carrier signal at the receiver is modeled by $Y(t) = X(t)\cos(2\pi wt + \Theta)$ where $\{X(t), t \geq 0\}$ is a zero-mean and wide sense stationary process, Θ is a uniform distributed random variable with interval $(-\pi, \pi)$ and w is a positive constant. Assume that, Θ is independent of the process $\{X(t), t \geq 0\}$. Is $\{Y(t), t \geq 0\}$ wide sense stationary? Justify your answer.

3. Consider the random telegraph signal, denoted by $X(t)$, jumps between two states, 0 and 1, according to the following rules. At time $t = 0$, the signal $X(t)$ start with equal probability for the two states, i.e., $P(X(0) = 0) = P(X(0) = 1) = 1/2$, and let the switching times be decided by a Poisson process $\{Y(t), t \geq 0\}$ with parameter λ independent of $X(0)$. At time t, the signal

$$X(t) = \frac{1}{2}\left(1 - (-1)^{X(0)+Y(t)}\right), t > 0.$$

   Is $\{X(t),\ t \geq 0\}$ covariance/wide sense stationary?

4. Let A be a positive random variable that is independent of a strictly stationary random process $\{X(t), t \geq 0\}$. Show that $Y(t) = AX(t)$ is also strictly stationary random process.

5. The owner of a local one-chair barber shop is thinking of expanding because there seem to be too many people waiting. Observations indicate that in the time required to cut one person's hair there may be 0, 1 and 2 arrivals with probability 0.3, 0.4 and 0.3 respectively. The shop has a fixed capacity of six people whose hair is being cut. Let $X(t)$ be the number of people in the shop at any time t and $X_n = X(t_n^+)$ be the number of people in the shop after the time instant of completion of the nth person's hair cut. Prove that $\{X_n, n = 1, 2, \ldots\}$ is a Markov chain assuming i.i.d arrivals. Find its one step transition probability matrix.

6. Let X_0 be an integer-valued random variable, $P(X_0 = 0) = 1$, that is independent of the i.i.d. sequence $Z_1, Z_2, \ldots$, where $P(Z_n = 1) = p$, $P(Z_n = -1) = q$, and $P(Z_n = 0) = 1 - (p + q)$. Let $X_n = \max(0, X_{n-1} + Z_n), n = 1, 2, \ldots$. Prove that $\{X_n, n = 0, 1, \ldots\}$ is a discrete time Markov chain. Write the one-step transition probability matrix or draw the state transition diagram for this Markov chain.

7. Suppose that a machine can be in two states: 0 = working and 1 = out of order on a day. The probability that a machine is working on a particular day depends on the state of the machine during two previous days. Specifically assume that $P(X(n+1) = 0/X(n-1) = j, X(n) = k) = q_{jk} \quad j, k = 0, 1$ where $X(n)$ is the state of the machine on day n.

   (a) Show that $\{X(n), n = 1, 2, \ldots\}$ is not a discrete time Markov chain.

   (b) Define a new state space for the problem by taking the pairs (j, k) where j and k are 0 or 1. We say that machine is in state (j, k) on day n if the machine is in state j on day $(n - 1)$ and in state k on day n. Show that with this changed state space the system is a discrete time Markov chain.

   (c) Suppose the machine was working on Monday and Tuesday. What is the probability that it will be working on Thursday?

8. The transition probability matrix of a discrete time Markov chain $\{X_n, n = 0, 1, \ldots\}$ having three states 1, 2 and 3 is $P = \begin{pmatrix} 0.3 & 0.4 & 0.3 \\ 0.6 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.1 \end{pmatrix}$ and the initial distribution is $\pi = (0.7, 0.2, 0.1)$

   (a) Compute $P(X_2 = 3)$. (b) Compute $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$.

9. Consider a time-homogeneous discrete time Markov chain $\{X_n, n = 0, 1, \ldots\}$ with state space $S = \{0, 1, 2, 3, 4\}$ and one-step transition probability matrix $P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.
   (a) Classify the states of the chain as transient, +ve recurrent or null recurrent.
   (b) When $P(X_0 = 2) = 1$, find the expected number of times the Markov chain visit state 1 before being absorbed.
   (c) When $P(X_0 = 1) = 1$, find the probability that the Markov chain absorbs in state 0.

10. Two gamblers, A and B, bet on successive independent tosses of an unbiased coin that lands heads up with probability p. If the coin turns up heads, gambler A wins a rupee from gambler B, and if the coin turns up tails, gambler B wins a rupee from gambler A. Thus the total number of rupees among the two gamblers stays fixed, say N. The game stops as soon as either gambler is ruined; i.e., is left with no money! Assume the initial fortune of gambler A is i. Let X_n be the amount of money gambler A has after the nth toss. If $X_n = 0$, then gambler A is ruined and the game stops. If $X_n = N$, then gambler B is ruined and the game stops. Otherwise the game continues. Prove that $\{X_n, n = 0, 1, \ldots\}$ is a discrete time Markov chain. Write the one-step transition probability matrix or draw the state transition diagram for this Markov chain.

11. For $j = 0, 1, \ldots$, let $P_{jj+2} = v_j$ and $P_{j0} = 1 - v_j$, define the transition probability matrix of Markov chain. Discuss the character of the states of this chain.

12. Show that if a Markov chain is irreducible and $P_{ii} > 0$ for some state i then the chain is aperiodic.

13. Let 0 be an absorbing state and for $j > 0$, $P_{jj} = p$, $P_{jj-1} = q$ where $p + q = 1$. Find $f_{j0}^{(n)}$, the probability that absorption takes place exactly at n^{th} step given initial state is j.

14. Consider a branching process, denoted by Galton-Watson process, that model a population in which each individual in generation n produces some random number of individuals in generation $n+1$, according, in the simplest case, to a fixed probability distribution that does not vary from individual to individual. That is, the first generation of individuals is the collection of off-springs of a given individual. The next generation is formed by the off-springs of these individuals. Let X_n denote the number of individuals of the nth generation, starting with $X_0 = 1$ individual (the size of the zeroth generation). Let Y_i (or $Y_{i,n}$) be the number of offspring of the ith individual of the nth generation. Suppose that, $\{Y_i, i = 1, 2, \ldots\}$ are non-negative integer valued i.i.d. random variables with probability mass function $p_j = P(Y_i = j), j = 0, 1, \ldots$ and independent of the size of the generation. Then
$$X_n = \sum_{i=1}^{X_{n-1}} Y_i, \quad n = 1, 2, \ldots$$
and $\{X_n, n = 0, 1, \ldots\}$ is a discrete time Markov chain. Classify the states of the chain.

15. Consider a DTMC on the non negative integers such that, starting from i, the chain goes to state $i+1$ with probability $p, 0 < p < 1$ and goes to state 0 with probability $1 - p$. Show that this DTMC has a unique steady state distribution π and then find π.

16. (a) Consider an aperiodic irreducible finite state space DTMC $\{X_n, n = 0, 1, \ldots\}$ with one step transition probability matrix $P = [P_{ij}], i, j \in S$ satisfying $\sum_j P_{ij} = \sum_i P_{ij} = 1$. Find the steady state distribution for this DTMC, if it exist.
   (b) Consider the simple random walk on a circle. Assume that K odd number of points labeled $0, 1, \ldots, K-1$ are arranged on a circle clockwise. From i, the walker moves to $i+1$ (with K identified with 0) with probability p $(0 < p < 1)$ and to $i - 1$ (with -1 identified with $K - 1$) with probability $1 - p$. Find the steady state distribution for this random walk, if it exist.