Problem sheet – 7

1. Fourier Transform Synthesis

Using the Fourier Transform Synthesis equation (given below), determine the inverse Fourier transform of the below signals-

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dw$$

a) $X_1(jw) = 2\pi\delta(w) + \pi\delta(w - 4\pi) + \pi\delta(w + 4\pi)$

b) $X_2(jw) = \begin{cases} 2, 0 \le w \le 2 \\ -2, -2 \le w < 0 \\ 0 \quad , |w| > 2 \end{cases}$

2. Fourier Transform using properties of Fourier Transform

a) Using Fourier Transform tables determine the Fourier transform of following signals

$$x(t) = t\left(\frac{\sin t}{\pi t}\right)^2$$

b) Use Parseval's relation and the result of the previous part to determine the numerical value of

$$A = \int_{-\infty}^{\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt$$

3. Time scaling in convolution using Fourier Transform

Given the relationship

$$y(t) = x(t) * h(t)$$

and

$$g(t) = x(3t) * h(3t)$$

and given that x(t) has Fourier transform X(jw) and h(t) has Fourier transform H(jw), use Fourier transform properties to show that g(t) has the form

$$g(t) = Ay(Bt)$$

Determine the values of A and B.

4. Determination of x(t) from its Fourier Transform properties

Consider a signal x(t) with Fourier transform X(jw). Suppose we are given the following facts:

a) x(t) is real and non negative

b) $\mathcal{F}^{-1}\{(1 + jw)X(jw)\} = Ae^{-2t}u(t)$, where A is independent of t.

c) $\int_{-\infty}^{\infty} |X(jw)|^2 dw = 2\pi$

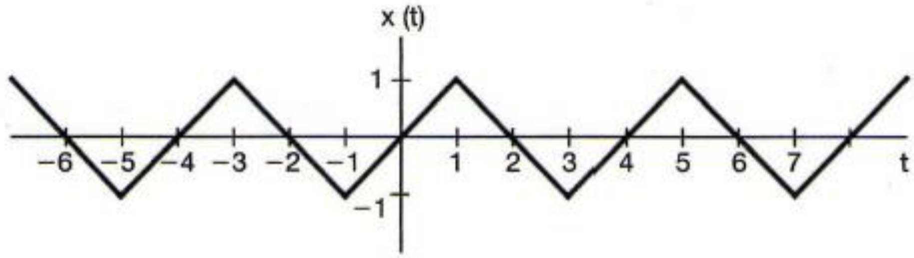
Determine a closed-form expression for x(t).

5. **Fourier Transform Properties and corresponding time domain signals**

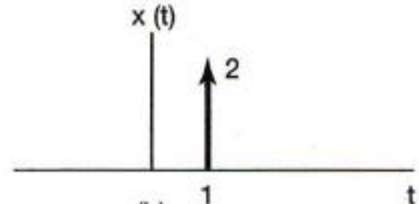
a) Determine which, if any, of the real signals depicted in Figure below have Fourier transforms that satisfy each of the following conditions:

1. Re{X(jw)} = 0
2. Im{X(jw)} = 0
3. There exists a real a such that $e^{j\alpha w}X(jw)$ is real
4. $\int_{-\infty}^{\infty} X(jw)dw = 0$
5. $\int_{-\infty}^{\infty} wX(jw)dw = 0$
6. X(jw) is periodic

b) Construct a signal that has properties (1), (4), and (5) and does not have the others.



(a)



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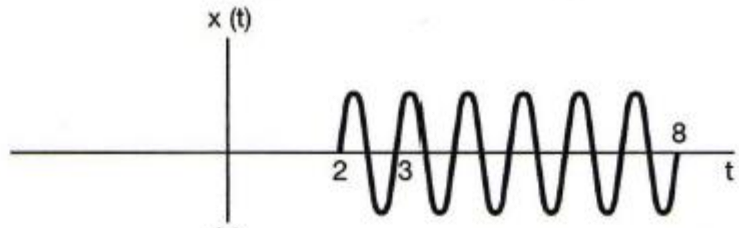

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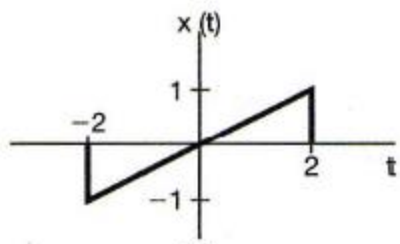

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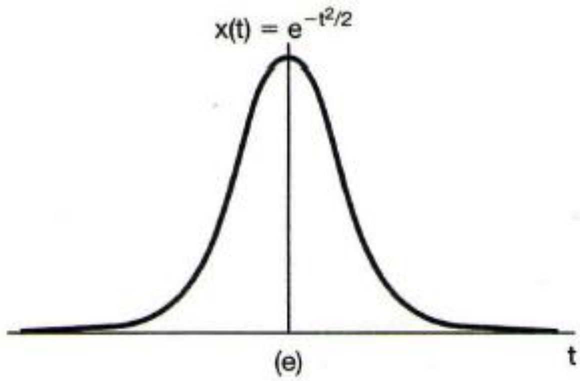
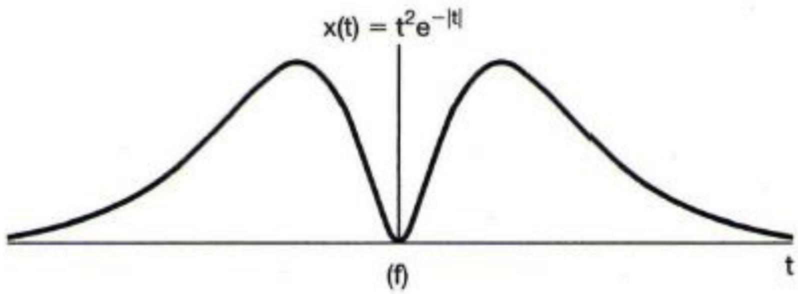


(e)



(f)

6. Multiplication property in Fourier Transform

Suppose g(t) = x(t) cos(t) and the Fourier transform of the g(t) is

$$G(jw) = \begin{cases} 1, & |w| \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

a) Determine x(t).

b) Specify the Fourier Transform X_1(jw) of a signal x_1(t) such that

$$g(t) = x_1(t)\cos\left(\frac{2}{3}t\right)$$

7. Hilbert Transform

a) Let us consider a system with a real and causal impulse response h(t) that does not have any singularities at t = 0. We know that either the real or the imaginary part of H(jw) completely determines H(jw). In this problem we derive an explicit relationship between H_R(jw) and H_I(jw), the real and imaginary parts of H(jw).

To begin, note that since h(t) is causal

$$h(t) = h(t)u(t),$$

except perhaps at t = 0. Now, since h(t) contains no singularities at t = 0, the Fourier transforms of both sides of eq. above must be identical. Use this fact, together with the multiplication property, to show that

$$H(jw) = \frac{1}{j\pi}\int_{-\infty}^{\infty} \frac{H(j\eta)}{w-\eta} d\eta$$

Use eq. above to determine an expression for H_R(jw) in terms of H_I(jw) and one for H_I(jw) in terms of H_R(jw).

b) The operation

$$y(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$$

is called the Hilbert transform. We have just seen that the real and imaginary parts of the transform of a real, causal impulse response h(t) can be determined from one another using the Hilbert transform

Now consider eq. above and regard y(t) as the output of an LTI system with input x(t). Show that the frequency response of this system is

$$H(jw) = \begin{cases} -j, & w > 0 \\ j, & w < 0 \end{cases}$$

c) What is the Hilbert transform of the signal x(t) = cos 3t?

8. Bandwidth and rise time of a system

Let H(jw) be the frequency response of a continuous-time LTI system, and suppose that H(jw) is real, even, and positive. Also, assume that

$$\max_{w} \{H(jw)\} = H(0)$$

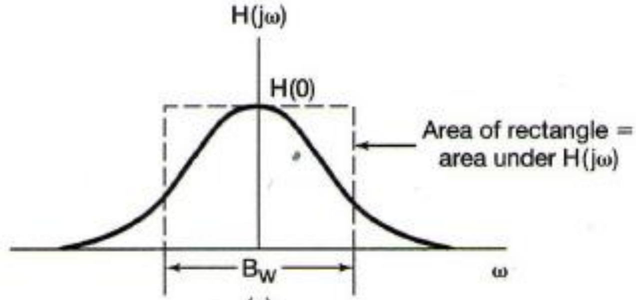
a) Show that
   i. The impulse response, h(t), is real
   ii. max |h(t)| = h(0).

Hint: If f (t, w) is a complex function of two variables, then

$$\left| \int_{-\infty}^{\infty} f(t,w) dw \right| \leq \int_{-\infty}^{\infty} |f(t,w)|\, dw$$

b) One important concept in system analysis is the bandwidth of an LTI system. There are many different mathematical ways in which to define bandwidth, but they are related to the qualitative and intuitive idea that a system with frequency response G(jw) essentially "stops" signals of the form e^{jwt} for values of w where G(jw) vanishes or is small and "passes" those complex exponentials in the band of frequency where G(jw) is not small. The width of this band is the bandwidth. These ideas will be made much clearer in further chapters but for now we will consider a special definition of bandwidth for those systems with frequency responses that have the properties specified previously for H(jw). Specifically, one definition of the bandwidth BW of such a system is the width of the rectangle of height H(j0) that has an area equal to the area under H(jw). This is illustrated in Figure below. Note that since H(j0) = max_wH(jw), the frequencies within the band indicated in the figure are those for which H(jw) is largest. The exact choice of the width in the figure is, of course, a bit arbitrary, but we have chosen one definition that allows us to compare different systems and to make precise a very important relationship between time and frequency. What is the bandwidth of the system with frequency response?

$$H(jw) = \begin{cases} 1, & |w| < W \\ 0, & |w| > W \end{cases}$$



(a)

c) Find an expression for the bandwidth Bw in terms of H(jw).

d) Let s(t) denote the step response of the system set out in part (a). An important measure of the speed of response of a system is the rise time, which, like the bandwidth, has a qualitative definition, leading to many possible mathematical definitions, one of which we will use. Intuitively, the rise time of a system is a measure of how fast the step response rises from zero to its final value,

$$s(\infty) = \lim_{t\to\infty} s(t)$$

Thus, the smaller the rise time, the faster is the response of the system. For the system under consideration in this problem, we will define the rise time as

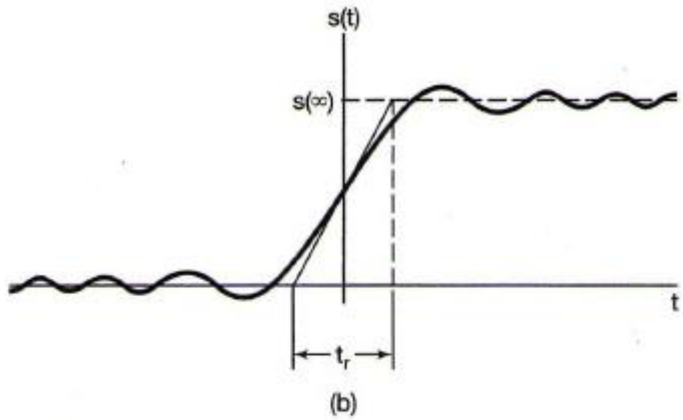
$$t_r = \frac{s(\infty)}{h(0)}$$

Since

$$s'(t) = h(t)$$

and also because of the property that h(0) = max_t h(t), t_r is the time it would take to go from zero to s(∞) while maintaining the maximum rate of change of s(t). This is illustrated in Figure below.

Find an expression for t_r in terms of H(jw).



e) Combine the results of parts (c) and (d) to show that

$$BW * t_r = 2\pi$$

Thus, we cannot independently specify both the rise time and the bandwidth of our system. For example, eq. above implies that, if we want a fast system (t_r small), the system must have a large bandwidth. This is a fundamental trade-off that is of central importance in many problems of system design.

9. Fourier Transform of Gaussian Pulse

Derive the Fourier Transform relations of the Gaussian pulse.