COL202 Major

Aaveg Jain

TOTAL POINTS

31 / 40

QUESTION 1

1 Problem 1 **5 / 5**

✓ **+ 1 pts** *Calculate the number of ways in which $$k$$ fixed points can be chosen out of $$n$$ points.*

✓ **+ 1 pts** *State that the left out $$(n-k)$$ elements must not be fixed.*

✓ **+ 2 pts** *Calculate the number of derangements of those $$(n-k)$$ elements.*

✓ **+ 1 pts** *Combine the two findings to reach the conclusion*

- 1 pts Proof writing guidelines not followed.

+ 0 pts Incorrect / Not attempted

QUESTION 2

Problem 2 5 pts

2.1 Problem 2.1 **2 / 2**

✓ **+ 2 pts** *All Correct*

+ 0 pts Incorrect/Unattempted

+ 1 pts Series form of Exponential generating function

+ 1 pts Closed form of Exponential generating function

2.2 Problem 2.2 **3 / 3**

✓ **+ 0.25 pts** *EGF for oddness*

✓ **+ 0.75 pts** *EGF for one partition even + one partition odd*

✓ **+ 0.25 pts** *Short explanation for the above EGF*

✓ **+ 0.75 pts** *EGF for both partitions even*

✓ **+ 0.25 pts** *Short explanation for the above EGF*

✓ **+ 0.25 pts** *Final EGF*

✓ **+ 0.5 pts** *Explicit formula for $$p_n$$*

+ 1 pts Overcounting/undercounting the partitions

+ 0 pts Incorrect/Unattempted

QUESTION 3

Problem 3 8 pts

3.1 Problem 3.1 **3 / 3**

Prove that every pair of this poset has a meet and a join, thereby concluding that it is a lattice.

✓ **+ 0.75 pts** *Give expressions for the meet and join of any two arbitrary elements of the poset.*

✓ **+ 0.75 pts** *Prove that the stated meet and join are actually the meet and join of the two elements.*

Prove that every subset of this lattice has a meet and a join, thereby concluding that the lattice is a complete lattice.

✓ **+ 0.75 pts** *Give expressions for the meet and join of any arbitrary subset of the lattice.*

✓ **+ 0.75 pts** *Prove that the stated meet and join are actually the meet and join of the subset.*

- **1 pts** Proof writing guidelines not followed.

+ **0 pts** Not attempted / Incorrect.

3.2 Problem 3.2 **5 / 5**

✓ **+ 0.5 pts** *Mention the method of proof*

✓ **+ 1 pts** *The minimum value that x can take is (1,1,...,1) and the maximum value that x can take is (n,n,...,n)*

✓ **+ 1 pts** *The least change in the value of x can be in one coordinate value*

✓ **+ 1.5 pts** *As f(x) is monotonic, the value of f(x) differs from the previous f(x) at one position and is 1 more than the value at that position in x*

✓ **+ 1 pts** *Conclusion that the loop can run for a maximum of n*k times*

+ **0 pts** Incorrect

QUESTION 4

Problem 4 11 pts

4.1 Problem 4.1 **0 / 3**

- **0 pts** Correct

✓ **- 1 pts** *Did not argue when 3SAT is unsatisfiable.*

- **2 pts** Showed an un-satisfiable 3SAT but did not argue about its construction.

- **1 pts** Did not show an example for un-satisfiable 3SAT

✓ **- 2 pts** *Did not argue about the construction of the equation and did not show an example for an un-satisfiable 3SAT..*

- **3 pts** Incorrectly argued about construction of 3SAT

There will not be 6C3 x 2^3 = 160 total clauses

4.2 Problem 4.2 **0 / 6**

+ **1.5 pts** Observing that probability of a clause to be true is 7/8

+ **2 pts** Showing expectation of these will be 7m/8

+ **1.5 pts** Arguing > 0 probability for R.V. to be greater than it's expectation

+ **1 pts** Argument about existence of Ceiling

- **1 pts** Not following proof guidelines

✓ **+ 0 pts** *Incorrect/ Not attempted*

4.3 Problem 4.3 **2 / 2**

✓ **+ 1 pts** *If less than 8 clauses, then all the clauses will be true (using Problem 4.2)*

✓ **+ 0.5 pts** *Give examples for m = 7, 6 etc*

✓ **+ 0.5 pts** *Conclusion*

+ **0 pts** incorrect

QUESTION 5

5 Problem 5 **5 / 5**

+ **5 pts** Correct

+ **2 pts** p1: Formally write $$f$$ and prove surjection or injection from Σ* to $$\mathcal{N}$$

+ **2 pts** p2: Formally write $$g$$ and prove surjection or injection from $$\mathcal{N}$$ to Σ*

+ **1 pts** p3: Use Schroder Bernstein theorem to prove the cardinality of both sets

✓ **+ 2 pts** *P1: Formally write the bijection$$f$$ from Σ* to $$\mathcal{N}$$*

✓ **+ 1.5 pts** *P2: Argue that the above function is one-one*

✓ **+ 1.5 pts** *P3: Argue that the above function is onto*

+ 1.5 pts P2: Write the inverse $$g$$ of $$f$$

+ 1.5 pts P3: Argue that the above function $$g$$ is indeed inverse of $$f$$

+ 0 pts No solution / incomplete solution

- 0.5 pts P4: For not following the guidelines of proof

QUESTION 6

6 Problem 6 **6 / 6**

✓ **- 0 pts** *Correct*

- 6 pts Not attempted or nothing substantial written.

- 5 pts Wrong or missing idea or proof details

- 5 pts Induction on trees cannot be done by adding a node/edge to create a larger tree. This requires a proof that all possible trees of this size can be generated this way. That proof will further bring you back to working with the tree that results from removal, so the argument is circular. This point has been made in class multiple times.

- 1 pts Induction variable not clearly and/or separately specified

- 1 pts Missed discussing the case where the walk begins at the removed vertex.

- 1 pts Missed discussing the case where the walk doesn't begin at the removed vertex.

- 4 pts Right direction but incorrect/incomplete arguments.

- 1 pts Wrong way of writing the induction hypothesis or missing induction hypothesis

- 0.5 pts Missed $$\forall v \in V$$ in the statement.

COL202: Discrete Mathematical Structures. I semester, 2022-23.
Major exam.
, Maximum Marks: .

Name (In CAPITAL letters as on Gradescope)	I	Ent. No.
AAVEG JAIN		2021CS10073

Important: If you write outside the box we may not grade it.

Problem 1 (5 marks) Given a permutation π of the set $[n]$, we say that i is a fixed point of π if $\pi(i) = i$. Count the number of permutations of $[n]$ with exactly k fixed points where $0 \leq k \leq n$. Please argue your answer. An answer without an argument will get 0.

Claim - no. of such perm = $\binom{n}{k} D_{n-k}$; D_n denotes the ~~...~~ no. of derangements of a set of n elements

no. of ways of choosing the k fixed points = no. of k-sized subsets of $[n]$ = $\binom{n}{k}$

no. of ways of ~~...~~ ~~...~~ assigning $\pi(i)$ to these values = 1 (all of them are fixed pt.s).

no. of ways of assigning $\pi(i)$ to remaining $n-k$ el. s.t. $\pi(i) \neq i$ $\forall$ such i, $= D_{n-k}$ (from def^n of D_n as stated above; no. of el. to be deranged $= n-k$)

thus by generalized prod. rule no. of permutat^{n}s $= \binom{n}{k} \times 1 \times D_{n-k} = \binom{n}{k} D_{n-k}$.

(also, $D_n = \sum_{k=0}^{n} \frac{n!(-1)^k}{k!}$ for $\forall n \geq 1$ using princ. of inclusion & exclusion as stated in textbook)

Problem 2.1 (2 marks) We say a sequence $\{a_n\}_{n\geq 0}$ captures a property if $a_i = 1$ iff i has that property, e.g.. if the property is evenness then the sequence will be $1, 0, 1, \ldots$. Write the exponential generating function the property "evenness."

~~...~~ from sequence capturing prop., $a_{2n} = 1$, $\forall n \geq 0$ and $a_{2n+1} = 0$ $\forall n \geq 0$ $\Rightarrow \hat{E}(x)$ = egf of evenness

$= \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} = \sum_{n\geq 0} \frac{a_{2n} x^{2n}}{(2n)!} + \sum_{n\geq 0} \frac{a_{2n+1} x^{2n+1}}{(2n+1)!} = \sum_{n\geq 0} \frac{x^{2n}}{(2n)!}$

$= \frac{e^x + e^{-x}}{2}$; using the formal series $e^x = \sum_{n\geq 0} \frac{x^n}{n!}$

Problem 2.2 (5 marks) Using the answer of Problem 2.1 write the egf for p_n = the number of ways of partitioning a set into two parts such that one of the parts is even in size. Find an explicit formula for p_n. No egf or no use of Problem 2.1 $\Rightarrow$ 0 marks.

(we have considered the partitions to be unordered i.e $\{1,2,3\},\{4,5,6\}$ is same as $\{4,5,6\},\{1,2,3\}$)

observe - $P_n = \dfrac{\sum_{k=0}^{n} \binom{n}{k} e_k e_{n-k}}{2!} + \sum_{k=0}^{n} \binom{n}{k} e_k O_{n-k}$, where

$e_k = 1$ iff k even, $O_k = 1$ iff k odd. since a partition of two even sets will repeat twice, we divide by $2!$

egf of $\Rightarrow$ $\dfrac{P_n}{n!} = \dfrac{1}{2!}\sum_{k=0}^{n} \dfrac{e_k}{k!}\dfrac{e_{n-k}}{(n-k)!} + \sum_{k=0}^{n} \dfrac{e_k}{k!}\dfrac{O_{n-k}}{(n-k)!}$

$\Rightarrow \hat{P}(x) = \frac{1}{2}\hat{E}(x)\hat{E}(x) + \hat{E}(x)\hat{O}(x)$; $\hat{E}(x) \to$ egf of e_k, $\hat{O}(x) \to$ egf of $O_k = \frac{e^x - e^{-x}}{2}$

$\hat{P}(x) \to$ egf of P_n can found in lect. paras

$= \dfrac{e^x + e^{-x}}{2}\left(\dfrac{1}{2}\cdot\dfrac{e^x+e^{-x}}{2} + \dfrac{e^x - e^{-x}}{2}\right)$

$= \dfrac{e^x+e^{-x}}{2}\left(\dfrac{e^x + e^{-x} + 2e^{+x} - 2e^{-x}}{4}\right) = \dfrac{3e^x - e^{-x}}{4} \times \dfrac{e^x + e^{-x}}{2}$

$= \dfrac{3e^{2x} - 1 + 3 - e^{-2x}}{8} = \dfrac{3e^{2x} - e^{-2x} + 2}{8}$; thus

$\dfrac{P_n}{n!} = [x^n]$ in $\hat{P}(x) = \dfrac{1}{8}\left[\dfrac{3\,2^n}{n!} - \dfrac{(-2)^n}{n!}\right] \Rightarrow \boxed{P_n = \dfrac{3\cdot 2^n - (-2)^n}{8} \quad n\geq 1}$

Problem 3.1 (3 marks) Given the set $[n]^k$, partial order $\preceq$ is defined as follows: $(x_1,\ldots,x_k) \preceq (y_1,\ldots,y_k)$ if $x_i \leq y_i$ for all $i \in [k]$. Argue that $([n]^k, \preceq)$ is a complete lattice. Recall $[n] = \{1,\ldots,n\}$.

Claim- consider any set $S \neq \phi$, ~~let~~ $S = \{V_1, V_2, \ldots V_n\}$ $n \geq 1$, $V_i \in [n]^k$ $\forall i$

we claim that $\wedge S$ and $\vee S$ exist in $[n]^k$ and are ~~$\wedge S$ ...~~ $\alpha_i = \min(V_{1i}, V_{2i}, \ldots V_{ni})$; V_{ij} denotes j-th el. of V_i ;

and $\beta_i = \max(V_{1i}, V_{2i}, \ldots V_{ni})$; and u_i denotes i-th el. of $u \in [n]^k$.

observe that since $\leq$ is a total order on $[n]$, thus max. and min. of any set of el. of $[n]$ exist.

Pf - $\alpha_i \leq V_{ji}$ $\forall 1\leq j \leq n, 1 \leq i \leq k$ thus $\forall 1 \leq j \leq n$: $\alpha \leq V_j$ -(1)

also, consider any $x \in [n]^k$ s.t $\forall 1\leq j \leq n$: $x \leq V_j$

then $x_i \leq V_{ji}$ $\forall 1 \leq i \leq k, 1 \leq j \leq n$. $\Rightarrow x_i \leq \alpha_i \forall 1 \leq i \leq k$ $\Rightarrow x \leq \alpha$ -(2)

as $\min(V_{1i}, V_{2i}, \ldots V_{ni})$ is glb of the $(V_{1i}, V_{2i}, \ldots V_{ni})$ ($\leq$ is a total order)

hence from (1), (2) we conclude α is glb of S and exists in $[n]^k$

Same procedure can be foll. for $\vee S$ i.e $\forall 1 \leq j \leq n$: $V_j \leq \beta$ and if $y \in [n]^k$ is s.t $y \geq V_j$ $\forall 1 \leq j \leq n \Rightarrow y \geq \beta$, thus $\beta = \vee S$, hence $\vee S$ also exists in $[n]^k$ for all non empty sets $S \subseteq [n]^k$, thus from def of complete lattice $[n]^k$ is a complete lattice.

Name AAVEG JAIN	II	Ent. No. 2021CS10073

Problem 3.2 (5 marks) Let us suppose we are given a monotonic function $f : [n]^k \to [n]^k$, i.e.. $x \preceq y \Rightarrow f(x) \preceq f(y)$. We run the program given on the right. ($a \leftarrow b$ means "set the value of variable a to b".)

```
1: x ← (1, ..., 1)
2: repeat
3:     t ← x
4:     x ← f(x)
5: until x = t
6: Return x
```

Prove that the loop of lines (2)-(5) will execute at most kn times.

Pf- By contradict^n. Assume the converse i.e the program run atleast $kn+1$ times. We now show a contrad^n.

observe - in each iterat^n, we check if $x = f(x)$. If it is, then loop breaks, otherwise same loop is run on $f(x)$ and so on. Also, $(1, 1, \ldots 1) = x_0 \leq x \ \forall\, x \in [n]^k$ ($<$ is strict order corr. to given partial order).

Also, since program runs atleast $kn+1$ times, in each iterat^n we must have $f(x) \neq x$ for 1 through kn iterat^{n}s. Also, since x_0 is least el. we have $x_0 < f(x_0)$ (as, if $f(x_0) = x_0$, loop breaks)

By ind^n we show $f^{i}(x_0) < f^{i+1}(x_0)$ ($0 \leq i \leq kn-1$) (define $f^{(0)}(x_0) = x_0$) for $i=0$ its true (Base case) Assume $P(i)$: $f^{(i)}x_0 < f^{i+1}(x_0)$ ($\underbrace{f(f(\ldots f(x_0)))}_{i\text{-times}} = f^i(x_0)$) then from monotonicity of f and the fact that $f(x) \neq x$ for 1 thr. kn iterat^{n}s, we conclude $f^{i+1}(x_0) < f^{i+2}(x_0)$ i.e $P(i+1)$.

thus $f^{(i)}(x_0) < f^{(i+1)}(x_0)$ $\forall\, 0 \leq i \leq kn-1$

thus we h. $x_0 < f(x_0) \ldots < f^{kn-1}(x_0) < f^{kn}(x_0)$.

thus we $(kn+1)$ distinct values; but $kn+1 > |[n^k]| = nk$

thus we arrive at a contradict^n

hence by showing a contradict^n we showed that loop cannot run atleast $kn+1$ times i.e it runs $\leq kn$ times.

Problem 4 (3 + 6 + 2 = 11 marks) Variables $P_1, P_2, \ldots$ can take values T or F. We use the term *literal* to denote P_i or $\neg P_i$. A *disjunctive clause of size 3* is a term of the form $L_1 \vee L_2 \vee L_3$ where L, L_2, L_3 are literals involving 3 *distinct* variables. An expression of the form $\bigwedge_{i=1}^{m}(L_{i,1} \vee L_{i,2} \vee L_{i,3})$ is a *3-SAT expression*, where $L_{i,j}$s are all literals. A 3-SAT expression is *satisfiable* if there exists a truth value setting of $P_1, P_2, \ldots$ such that the expression evaluates to T.

Problem 4.1 (3 marks) Assume for now that no disjunctive clause of size 3 is repeated in any 3-SAT expression. Show by construction that there exists a 3-SAT expression that is *not* satisfiable.

consider six variables $P_1, P_2, P_3 \ldots P_5, P_6$. there are $\binom{6}{3} = 20$ 3-tuples of these variables. consider a 3-SAT expr.

$= \bigwedge_{i \neq j \neq k} (P_i \vee P_j \vee P_k) \wedge \left(\bigwedge_{i \neq j \neq k} (\bar{P_i} \vee \bar{P_j} \vee \bar{P_k}) \right)$ that

has overall ~~for~~ $20 + 20 = 40$ such terms. We show that in any setting of $P_1 \ldots P_6$, there exists 3 which are all T or all F. ~~Pf~~ – ~~by cases~~. $|T| + |F| = 6$ $|T| \to$ no. of variables which are T, sim. $|F|$

if $|T| = |F| = 3$, we are done. If not, then either $|T| > |F|$ or $|F| > |T|$; in either case we have atleast 3 var. with same truth value since $|T| > |F| \Rightarrow 2|T| > 6 \Rightarrow |T| > 3$ and vice versa.

thus ~~...~~ in the given 3-SAT expr, atleast one clause is ~~...~~ F and thus the expr. is not satisfiable.

Problem 4.2 (6 marks) Show that there exists a setting of the truth values of $P_1, P_2, \ldots$ such that at least $\lceil 7m/8 \rceil$ of the disjunctive clauses are T for any 3-SAT expression with m disjunctive clauses of size 3. Here too assume that all the clauses are distinct. (Hint: Use the Probabilistic Method).

con

Name AAVEG JAIN	III	Ent. No. 2021CS10073

Problem 4.3 (2 marks) Using the result of Problem 4.2 (even if you didn't solve it) argue that any 3-SAT expression that is not satisfiable must have at least 8 clauses.

from result - there exists one such setting where atmost $m - \lceil \frac{7m}{8} \rceil$ are F,

now, $m < 8$, $\lceil \frac{7m}{8} \rceil = m \Rightarrow$ atmost 0 clauses. i.e none are F

hence for all setting, all are T i.e it is satisfiable.

for $m \geq 8$, $m - \lceil \frac{7m}{8} \rceil$ is atleast 1, thus there exists a setting where atleast 1 clause is F, thus a setting which is not satisfiable must have atleast 8 clauses. (for non satisfiable, all such setting should have atleast 1 F)

Problem 5 (5 marks) Given a finite set of alphabets Σ prove that the set Σ^* of all finite strings with alphabets from Σ is countably infinite

let $\Sigma = \{a_1, a_2, \ldots a_n\}$; $n < \infty$ since Σ is finite.

consider a map $f: \mathbb{N} \to \Sigma^*$ s.t $f(0) = f(1) = \lambda$ (empty str.)

and for $n > 1$; consider $n = p_1^{i_1} p_2^{i_2} \ldots p_k^{i_k}$ (Fund. thm. of Arith.) where $i_1, i_2 \ldots i_k$ are all non-zero (p_n denotes n^{th} prime)

then $f(n) = a_{i_1} a_{i_2} \ldots a_{i_k}$. Claim - f is a surj^n from $\mathbb{N}$ to Σ^*

f is total f^n trivially. consider any str s.

if s is empty, then $f(0) = s$ and hence $\exists n : f(n) = s$.

if s is not empty, then ~~$s = a_1 a_2 \ldots$ finite~~

$s = a_{n_1} a_{n_2} \ldots a_{n_s}$ (s is finite). consider $n = p_1^{n_1} p_2^{n_2} \ldots p_s^{n_s}$

then acc. to above rule, we have $n_1, n_2, \ldots n_s \neq 0$

and hence $f(n) = s$. thus $\forall s \in \Sigma^* : \exists n \in \mathbb{N} : f(n) = s$

and hence f is a surj^n. $\Rightarrow$ $\mathbb{N}$ surj Σ^* and

hence Σ^* is countable. Also, observe that Σ^* is

non finite, since $x_1 = a_1$, $x_2 = a_1 a_1$, $x_3 = a_1 a_1 a_1$ ...

is a non finite sequence of el. ($\{x_n\}$ bij $\mathbb{N}$)

thus Σ^* is countably infinite (countable $\Rightarrow$ count. ∞ or finite).

Problem 6 (6 marks) Given a tree $T = (V, E)$ prove by induction that for every $v \in V$ there is a walk that begins and ends at v and uses every edge *exactly* twice.

We prove by strong ind^n on the order of the tree.

Base case ($n = 2$): for T^2, let T^2 be $\{\{v_1, v_2\}, \{v_1, v_2\}\}$ $= e_1$ then the walk $v_1 e_1 v_1$ and $v_2 e_1 v_2$ satisfies our cond^n and hence $P(2)$ is T. ($P(n)$ defined below)

Ind^n predicate – $P(i)$: for every vertex V of a tree with $|T| = i$, a walk exists that begins and ends at V and uses each edge exactly twice

Ind^n hypothesis → Assume $P(j)$ $\forall j < i+1$. We now show $P(i+1)$

consider any vertex $v \in V(T)$. From connectivity of tree, $N(V) \neq \phi$; let $N(V) = \{v_1, v_2, \ldots v_k\}$ and consider the graph $G \setminus V$. Claim – $G \setminus V$ has exactly k components. Pf – two neighbours (x, y) of v cannot be in same comp. of $G - v$, as if they are, then consider the path P in $G - v$ b/w. them. Thus, in T, there exist two distinct paths (atleast) b/w these neighbours x, y → P and $x\,v\,y$. thus T has a cycle which is a contradict^n. Now consider the k comp. of $G - v$ → $C_1, C_2, \ldots C_k$ with C_i cont. v_i (C_i can contain only one neighbour as shown) then $\forall i$: C_i is a tree (conn. and acyclic) and thus there is a walk W_i which starts at v_i and ends at v_i. consider the walk $W = v\, e_1\, W_1\, e_1\, e_2\, W_2\, e_2 \ldots \ldots e_k W_k e_k\, v$ (e_i is edge b/w v and v_i)

Claim – W satisfies $P(i+1)$. consider any edge $E \in E(T)$. either E belongs to a comp. of $G-v$ in which case it occ. exactly twice from ind^n hypothesis, or it is a edge b/w v and v_j for some j. but for all e_j, e_j occurs exactly twice in W. [e_j can't be in any comp. of $G - v$). thus all edges occur exactly twice in W. also W begins and ends at V. Thus W satisfies $P(i+1)$ and hence $P(i)$ is T $\forall i \leq i$

Note – we haven't showed $P(1)$ since for this graph, no edges are present and hence $W = V$)