Department of Mathematics
Tutorial Sheet No. 6
MAL 250/MTL 106 (Probability and Stochastic Processes)

1. Let $X(t) = A_0 + A_1 t + A_2 t^2$, where $A_i's$ are uncorrelated random variables with mean 0 and variance 1. Find the mean function and covariance function of $X(t)$.

2. In a communication system, the carrier signal at the receiver is modeled by $X(t) = \cos(2\pi wt + \theta)$ where θ is a uniform distributed random variable with interval $(-\pi, \pi)$ and w is a positive constant. Is $\{X(t),\ t \geq 0\}$ covariance/wide sense stationary?

3. Consider the random telegraph signal, denoted by $X(t)$, jumps between two states, 0 and 1, according to the following rules. At time $t = 0$, the signal $X(t)$ start with equal probability for the two states, i.e., $P(X(0) = 0) = P(X(0) = 1) = 1/2$, and let the switching times be decided by a Poisson process $\{Y(t), t \geq 0\}$ with parameter λ independent of $X(0)$. At time t, the signal
$$X(t) = \frac{1}{2}\left(1 - (-1)^{X(0)+Y(t)}\right), t > 0.$$
Is $\{X(t),\ t \geq 0\}$ covariance/wide sense stationary?

4. Let A be a positive random variable that is independent of a strictly stationary random process $\{X(t), t \geq 0\}$. Show that $Y(t) = AX(t)$ is also strictly stationary random process.

5. Is the stochastic process $\{X(t), t \in T\}$ stationary, whose probability distribution under a certain condition given by
$$P\{X(t) = n\} = \begin{cases} \frac{(at)^{n-1}}{(1+at)^{n+1}} & n = 1, 2, \ldots \\ \frac{(at)}{(1+at)} & n = 0 \end{cases}$$

6. Let X_0 be an integer-valued random variable that is independent of the i.i.d. sequence $Z_1, Z_2, \ldots$, where $P(Z_n = 1) = p$, $P(Z_n = -1) = q$, and $P(Z_n = 0) = 1 - (p + q)$. Let $X_n = \max(0, X_{n-1} + Z_n), n = 1, 2, \ldots$. Prove that $\{X_n, n = 0, 1, \ldots\}$ is a discrete time Markov chain. Write the one-step transition probability matrix or draw the state transition diagram for this Markov chain.

7. Suppose that a machine can be in two states: 0 = working and 1 = out of order on a day. The probability that a machine is working on a particular day depends on the state of the machine during two previous days. Specifically assume that $P(X(n+1) = 0/X(n-1) = j, X(n) = k) = q_{jk} \quad j, k = 0, 1$ where $X(n)$ is the state of the machine on day n.
(i) Show that $\{X(n), n = 1, 2, \ldots\}$ is not a Markov chain.
(ii) Define a new state space for the problem by taking the pairs (j, k) where j and k are 0 or 1. We say that machine is in state (j, k) on day n if the machine is in state j on day $(n - 1)$ and in state k on day n. Show that with this changed state space the system is a Markov chain.
(iii) Suppose the machine was working on Monday and Tuesday. What is the probability that it will be working on Thursday?

8. The transition probability matrix of a discrete time Markov chain $\{X_n, n = 1, 2, \ldots\}$ having three states 1, 2 and 3 is $P = \begin{pmatrix} 0.3 & 0.4 & 0.3 \\ 0.6 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.1 \end{pmatrix}$ and the initial distribution is $\pi = (0.7, 0.2, 0.1)$
(a) Compute $P(X_2 = 3)$. (b) Compute $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$.

9. A particle starts at the origin and moves to and fro on a straight line. At any move it jumps either 1 unit to the right or 1 unit to the left each with probability 1/3 or 1/3 respectively. With probability 1/3, the particle may stay at the same position in any move. Model this process as a discrete time Markov chain and draw the state transition diagram for the chain. Classify the states of the chain as transient, +ve recurrent or null recurrent.

10. One way of spreading information on a network uses a rumor-spreading paradigm. Suppose that there are 5 hosts currently on the network. Initially, one host begins with a message. In every round, each host that has the message contacts another host chosen independently and uniformly at random from the other 4 hosts, and sends the message to the host. The process stops when all hosts has the message. Model this process as a discrete time Markov chain and find one step transition probability matrix for the chain. Classify the states of the chain as transient, positive recurrent or null recurrent.

11. For $j = 0, 1, \ldots$, let $P_{jj+2} = v_j$ and $P_{j0} = 1 - v_j$, define the transition probability matrix of Markov chain. Discuss the character of the states of this chain.

12. For a Markov chain $\{X_n, n = 1, 2, \ldots\}$ with state space $E = \{0, 1, 2, 3, 4\}$ and transition probability matrix P given below, classify the states of the chain. Also determine the closed communicating classes.

$$(a)\ P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \quad (b)\ P = \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$$

13. Show that if a Markov chain is irreducible and $P_{ii} > 0$ for some state i then the chain is aperiodic.

14. Let 0 be an absorbing state and for $j > 0$, $P_{jj} = p$, $P_{jj-1} = q$ where $p + q = 1$. Find $f_{j0}^{(n)}$, the probability that absorption takes place exactly at n^{th} step given initial state is j.

15. Consider a branching process, denoted by Galton-Watson process, that model a population in which each individual in generation n produces some random number of individuals in generation $n+1$, according, in the simplest case, to a fixed probability distribution that does not vary from individual to individual. That is, the first generation of individuals is the collection of off-springs of a given individual. The next generation is formed by the off-springs of these individuals. Let X_n denote the number of individuals of the nth generation, starting with $X_0 = 1$ individual (the size of the zeroth generation). Let Y_i (or $Y_{i,n}$) be the number of offspring of the ith individual of the nth generation. Suppose that, $\{Y_i, i = 1, 2, \ldots\}$ are non-negative integer valued i.i.d. random variables with probability mass function $p_j = P(Y_i = j), j = 0, 1, \ldots$ and independent of the size of the generation. Then

$$X_n = \sum_{i=1}^{X_{n-1}} Y_i, \quad n = 1, 2, \ldots$$

and $\{X_n, n = 0, 1, \ldots\}$ is a discrete time Markov chain. Classify the states of the chain as transient, +ve recurrent or null recurrent.

16. The owner of a local one-chair barber shop is thinking of expanding because there seem to be too many people waiting. Observations indicate that in the time required to cut one person's hair there may be 0,1 and 2 arrivals with probability 0.3, 0.4 and 0.3 respectively. The shop has a fixed capacity of six people whose hair is being cut. Let X_n be the number of people in the shop at the completion of the nth person's hair cut. $\{X_n, n = 1, 2, \ldots\}$ is a Markov chain assuming i.i.d arrivals. Find its one step transition probability matrix. Determine the 'long run' proportion of time that the shop has six people in it; that it has 5 people in it.

17. Consider a DTMC on the non negative integers such that, starting from i, the chain goes to state $i + 1$ with probability $p, 0 < p < 1$ and goes to state 0 with probability $1 - p$.

    (a) Show that this DTMC is irreducible and recurrent.

    (b) Show that this DTMC has a unique steady state distribution π and then find π.

1) $E(X(t)) = E(A_0 + A_1 t + A_2 t^2)$
$= 0$

$E[(A_0 + A_1 t + A_2 t^2)(A_0 + A_1 s + A_2 s^2)]$

5 $P\{X(t)=n\} = \begin{cases} \dfrac{(at)^{n-1}}{(1+at)^{n+1}} & n=1,2,\dots \\ \dfrac{at}{1+at} & n=0 \end{cases}$

$$E(X(t)) = \sum_{n=1}^{\infty} \frac{n(at)^{n-1}}{(1+at)^{n+1}}$$

$$S = \frac{1}{1+at} + \frac{2at}{(1+at)^2} + \dots$$

$$\frac{Sat}{1+at} = \qquad + \frac{1at}{(1+at)^2} + \dots$$

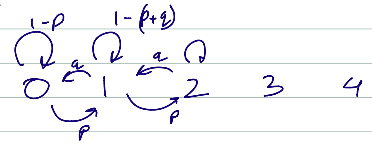
$$\frac{S}{1+at} = \frac{1}{1+at}\left(1-\frac{at}{1+at}\right)$$

$$\Rightarrow S = 1+at$$

6) $P(Z_n=1)=p$

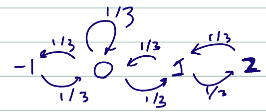
$-1 = q$

$0 = 1-p-q$

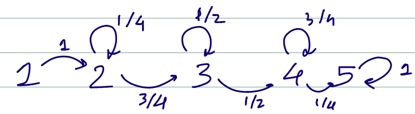


7) $P(X(n+1)=0 \mid X(n-1)=j, X(n)=k) = q_{jk}$

9)



10)



11) $P_{j,j+2} = v_j \qquad P_{j,0} = 1 - v_j$

$$\begin{bmatrix} 1-v_0 & 0 & v_0 & 0 & 0 & 0 \\ 1-v_1 & 0 & v_1 & 0 & 0 & \cdots \\ 1-v_2 & 0 & v_2 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & & & \\ 1-v_n & \cdot & v_n & \cdot & \cdot & \cdot \end{bmatrix}$$

12) a) $P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

3 is absorbing

b)

13) For each state, $d_i = 1$

$\Rightarrow$ It is aperiodic.

14) $P_{jj} = p$

$P_{j\,j+1} = q \qquad p + q = 1$

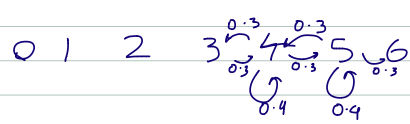
$f_{j0}^{(n)} = \begin{cases} 0 & n < j \\ {}^{n-1}C_{j-1}\, p^{n-j} q^{j} & \end{cases}$

15) $X_n = \sum_{i=1}^{X_{n-1}} Y_i$

$S = \{\underset{\text{abs}}{0}, \underbrace{1, 2, \ldots}_{\text{transient}}\}$.

16) $0, 1, 2$

$0.3, 0.4, 0.3$



	0	1	2	3	4	5	6
0	0.3	0.4	0.5	0	0	0	0
1	0.3	0.4	0.3	0	0	0	0
2	0	0.3	0.4	0.3	0	0	0
3	0	0	0.3	0.4	0.3	0	0
4	0	0	0	0.3	0.4	0.3	0
5	0	0	0	0	0.3	0.4	0
6	0	0	0	0	0	0.3	0.7

$\pi(P - I) = 0$

$[\pi]\Big[\quad\Big]$

17) $P_{i\,i+1} = p$

$P_{i0} = 1-p$

all states can be visited ⇒ irreducible

irreducible, MC → recurrent

$\pi(P - I) = 0$

$$\underset{1\times n}{\pi} \underset{n\times n}{\begin{bmatrix} p & p & & \\ 1-p & \cancel{0}\,1 & p & \\ 1-p & 0 & \cancel{0}\,1 & p \\ \vdots & & & \\ 1 & & & \end{bmatrix}} = \underset{1\times n}{0}$$

$\pi_0 p + (1-p)(1-\pi_0) = 0$

$\pi_0 p - \pi_1 = 0$

$\pi_1 p - \pi_2 = 0$