MTL103 Minor 1

Viraj Agashe

TOTAL POINTS

19 / 25

QUESTION 1

1 Q1 **4.5 / 5**

+ **+ 0 pts** Incorrect/not attempted
+ ✓ **+ 5 pts** *Correct*
+ **+ 1 pts** Some positive approach towards correct solution
+ **+ 2 pts** For every point $$x$$, lies on the line segment between $$x*$$ and global min, $$f(x)\leq f(x\ast)$$.
+ **+ 1 pts** For some x on the line segment between $$x*$$ and global min, $$x\in N_\epsilon (x\ast)$$ but no proof.
+ **+ 2 pts** For some x on the line segment between $$x*$$ and global min, $$x\in N_\epsilon (x\ast)$$ with partially correct proof.
+ **+ 2 pts** Correct for $$n=1$$. that is $$R^n=R$$
+ **+ 0.5 pts** Some correct approach for $$n=1$$.
+ **- 0.5** *Point adjustment*

QUESTION 2

2 Q2 **5 / 10**

- **- 0 pts** Correct
- ✓ **- 10 pts** *unattempted/incorrect*
- **- 5 pts** if/only if part not proven or incomplete
- **+ 5** *Point adjustment*

QUESTION 3

3 Q3 **4 / 4**

- ✓ **- 0 pts** *Correct*
- **- 1 pts** Did not show feasible solution
- **- 1.5 pts** Did not show basic solution
- **- 1.5 pts** Did not show non-degenerate solution
- **- 4 pts** Incorrect

QUESTION 4

4 Q4 **3 / 3**

- ✓ **- 0 pts** *Correct*
- **- 3 pts** Wrong graph.
- **- 0.5 pts** Reason for degeneracy is not given.
- **- 0.5 pts** Feasible region is not marked.
- **- 0.5 pts** One point is wrong.
- **- 1 pts** Two points are wrong.
- **- 1.5 pts** Three points are wrong.
- **- 2 pts** Four points are wrong.
- **- 0.5 pts** Wrong bfs points are mentioned.

QUESTION 5

5 Q5 **2.5 / 3**

- ✓ **- 0 pts** *Correct*
- **- 0.5 pts** Definition of $$x_{ijg}$$ where $$i \in I, j \in J, g \in G$$, $$I,J,G$$ are sets of neighbourhoods, schools and grades
- **- 0.5 pts** Optimization Objective (min Total distance travelled by students)
- **- 0.5 pts** Non negativity constraints on $$x_{ijg}$$

✓ **- 0.5 pts** *Integer constraint on $$x_{ijg}$$*

- 0.5 pts Capacity constraint on school capacity $$C_{jg}$$

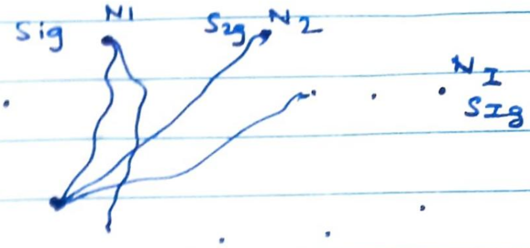
- 0.5 pts Assignment constraint on student population $$S_{ig}$$

- 3 pts Incorrect/Not attempted

MTL103 Minor 1.

Qs. Let us define

$x_{ijg} \equiv$ No. of students in neighbourhood i going to school j in grade g



Then we have the constraints:

$$x_{ijg} \geq 0 \quad \forall\, i,j,g \quad \text{—— ①}$$

$$\sum_{j \in J} x_{ijg} = S_{ig} \quad \forall\, i,g \quad \text{—— ②}$$

(Assign ALL students in neighbourhood i grade g to a school)

$$\sum_{i \in I} x_{ijg} \leq C_{jg} \quad \forall\, j,g \quad \text{—— ③}$$

(Should be less than or equal to capacity of schools)

We want to minimize:

$$f = \sum_{G}\sum_{J}\sum_{I} x_{ijg} \cdot d_{ij}$$

$\therefore$ The LP is:

$$\min \sum_{G}\sum_{J}\sum_{I} x_{ijg} \cdot d_{ij}$$

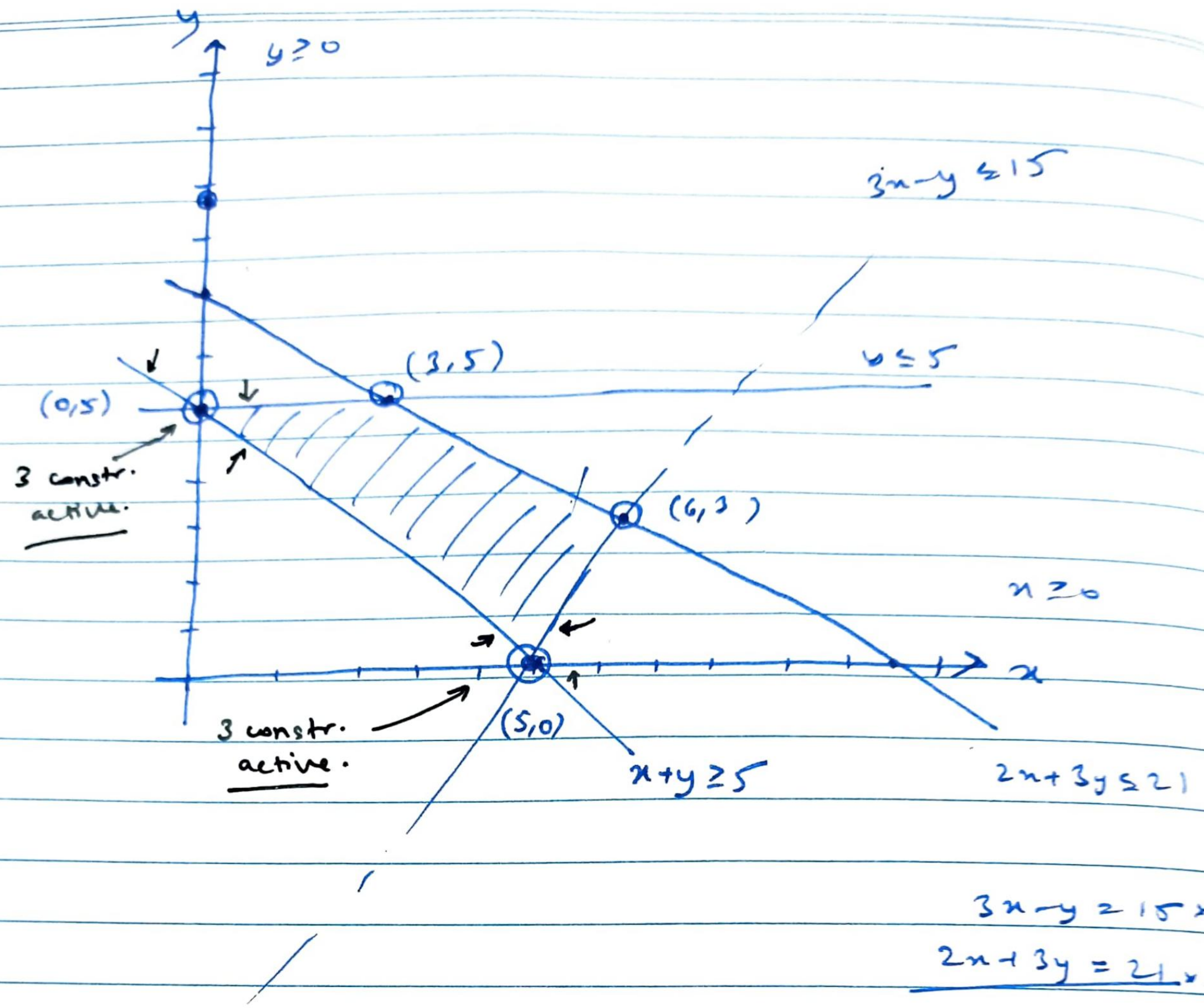
s.t.:

$$\sum_{j \in J} x_{ijg} = S_{ig} \quad \forall\, i,g$$

$$\sum x_{ijg} \leq C_{jg} \quad \forall\, j,g$$

$$x_{ijg} \geq 0 \quad \forall\, i,j,g$$

Q4.



$3x - y = 15 \times 2$

$2x + 3y = 21 \times 3$

$6x + 9y = 63$

$-6x \overset{(+)}{-} 2y \overset{(-)}{=} 30$

$11y = 33$

$y = 3$

$2x + \cancel{9} = \cancel{21}\,12$

The feasible region is as drawn above.

Note that since it is non-empty,

corner point ≡ BFS.

The corner points are,

(0,5), (3,5), (6,3), (5,0)

Note that at (0,5) and (5,0), there are 3 constraints active, while the dimensionality of the space is 2.

∴ These are degenerate.

DEGENERATE : (0,5), (5,0)

NON-DEGENERATE : (3,5), (6,3)

Q3. $P = \{ x \in \mathbb{R}^n \mid Ax \le b, \ x \ge 0 \}$

$P' = \{ (x, z) \in \mathbb{R}^{n+m} \mid Ax + z = b, \ x \ge 0, \ z \ge 0 \}$

Given x^* in BFS of P

Note that,

$Ax^* + b - Ax^* = b$. Further, $x^* \ge 0$ since $x^* \in P$

and $Ax^* \le b \Rightarrow b - Ax^* \ge 0$.

$\therefore$ $(x^*, b - Ax^*)$ is a feasible solution.

Suppose $A = m \times n$. Since x^* is a non-degenerate BFS,

it has exactly m non-zero entries in its vector representation.

Now, consider the solution

$(x^*, b - Ax^*)$.

x^* is a BFS $\Rightarrow \exists$ n L.I. active constraints at x^*.

Notice that since x^* has m non-zero entries, $\therefore$ $n-m$

entries are zero $\Rightarrow$ $n-m$ constraints of the form $x_i \ge 0$ are

active.

$\therefore$ Remaining m active constraints are from

$\boxed{Ax \le b}$

$\therefore$ x^* satisfies $Ax^* = b$. $\Rightarrow$ $b - Ax^* = 0$

$\therefore$ For the solution $(x^*, \underbrace{b - Ax^*}_{0})$

we have exactly ~~m~~

m- non zero entries.

$\therefore$ n active constraints from $x \ge 0, z \ge 0$,

m constraints from $Ax + z = b$

$\Rightarrow$ $m+n$ active constraints at $(x^*, \cancel{Ax^*}\ b - Ax^*)$

$\therefore$ Solution is basic. Since x has m non-zero entries, so does

$(x^*, b - Ax^*) \Rightarrow (x^*, b - Ax^*)$ is not degenerate.

Therefore we have proved that

$$y = (x^*, b - Ax^*)$$

is a basic, feasible, non-degenerate solution.

Q1. $f : \mathbb{R}^n \to \mathbb{R}$, $S \subset \mathbb{R}^n$.

Note that if f is convex over $\mathbb{R}^n$, it is convex over S.

Consider any point $x \in S$. Then we have that,

~~$f(x) \sim f(x^*)$~~

$$f(\lambda x^* + (1-\lambda)x) \leq \lambda f(x^*) + (1-\lambda) f(x) \quad \forall \lambda \in (0,1).$$

Now, we know $\exists\, \epsilon > 0$ s.t. $f(x^*) \leq f(x) \;\; \forall \; \|x - x^*\| \leq \epsilon$.

~~We want λ s.t. $x \neq x^*$~~

~~For $\|\lambda x^* + (1-\lambda)x\| \leq \epsilon$~~

~~$\Rightarrow \lambda x^* + (1-\lambda)x$~~

~~We~~

We claim $\exists\, \lambda$ s.t.

$$\| x^* - (\lambda x^* + (1-\lambda) x^*) \| \leq \epsilon$$

$$\Rightarrow \| (1-\lambda)(x^* - x) \| \leq \epsilon$$

$$1 - \lambda \leq \frac{\epsilon}{\|x^* - x\|}$$

$$\Rightarrow \underline{\lambda \geq 1 - \frac{\epsilon}{\|x^* - x\|}}$$

$\therefore$ If we pick $\lambda \geq 1 - \dfrac{\epsilon}{\|x^* - x\|}$ for any point x

we will have that $\|x^* - \underline{x}\| \leq \epsilon$

$\Rightarrow$ for this λ,

$$f(\lambda x^* + (1-\lambda)x) \geq f(x^*)$$

So we get, for our choice of λ (>0)

$$f(x^*) \leq f(\lambda x^* + (1-\lambda)x) \leq \lambda f(x^*) + (1-\lambda) f(x)$$

$$\Rightarrow \quad (1-\lambda) f(x^*) \leq (1-\lambda) f(x)$$

Since $\lambda \in (0,1)$, $1-\lambda > 0$

$$\Rightarrow \boxed{f(x^*) \leq f(x) \quad \forall x \in S}$$

Min $C^T d$

$(x_1, \ldots x_m, \underbrace{0, \ldots 0}_{n-m})$

$m \times d$

Q2. [$\Rightarrow$] Let x be a feasible solution.

We know for a standard LP,

the optimal solution must be basic feasible solution.

$\therefore$ At least $m-n$ variables are zero.

$\therefore |K| \geq m-n.$

Consider the problem,

$$\min \; C^T d$$

$$\underbrace{Ad = 0}_{m \text{ constraints.}}, \quad d_i \geq 0, \; i \in K.$$

Note that at least $m-n$ constraints of the form $d_i \geq 0$ will be active. Let

$I = \{i \mid d_i \neq 0\}$. then we have,

$$Ad = \sum_{j=1}^{n} A_j d_j$$

$$= \cancel{\sum_{j \in K} A_j d_j} + \sum_{j \in I} A_j d_i$$

$$= \sum_{j \in I} A_j d_i = 0$$

We know that the columns of A corresponding to the indices where $d_i \neq 0 \Rightarrow$ cols. of A corr. to indices where $x_i \neq 0 \Rightarrow A_j$'s are linearly independent

$$\Rightarrow d_i = 0 \;\; \forall i \in I.$$

$$\therefore d = (0, 0, \ldots \ldots 0)$$

$$\therefore C^T d = 0.$$

[$\Leftarrow$] Consider the LP problem,

$$Ad = 0.$$
$$d_i \geq 0, i \in K.$$

Then we have as earlier,

$$\sum_{j \in I} A_j d_j = 0.$$

Now consider the indices set I. We have for $i \in I$, $x_i \neq 0$.

~~For the optimal cost~~

If the optimal cost $c^T d = 0 \Rightarrow d = 0$

$\therefore$ A_j's are L.I.

$\Rightarrow$ Cols in A are LI.

$\Rightarrow$ x is BFS $\therefore$ ~~for~~ optimal.