1.

The set of polynomials with integer coefficients can be injectively mapped to the set of finite length sequences of integers, which is a countable set. Therefore, the set of polynomials with integer coefficients is countable.

Each such polynomial p has a finite set of roots, say R_p. The set of well-behaved numbers is, by definition, given by W = union over all p of R_p. Thus, W is the union of a countable collection of finite (and therefore, countable) sets. Therefore, W is countable.

2.

Solution 1

Define a set A as,

A = { {$w_1, \ldots, w_{T/2}$} | $w_1, \ldots, w_{T/2}$ are walks in G such that the set of their endpoints are exactly the set T }.

Since graph G is connected, A is a non-empty set.

Define the set B as,

B = { $\text{len}(w_1) + \ldots + \text{len}(w_{T/2})$ | {$w_1, \ldots, w_{T/2}$} belongs to A }.

Similarly, B is a non-empty subset of natural numbers.

By WOP, B has a minimum element say m. Let W = { $w_1, \ldots w_{T/2}$ } be the set of walks in A with total length m.

Claim: $w_1, \ldots w_{T/2}$ all are paths in G and no two of them are conflicting.

Proof. Suppose w_c is not a path. Then $w_c = v_1, \ldots v_k$ such that $v_i = v_j$ for some I < j.

Now we can replace w_c by $w'_c = v_1, \ldots v_i, v_{j+1}, \ldots v_k$ in W to get another member W' of A. Clearly $\text{len}(w'_c) < \text{len}(w_c)$, so the total length of walks in W' is less than m, which is a contradiction.

Now suppose there exists a conflicting edge between two walks w_i and w_j and let that edge be (u, v).

$w_i = a_1, a_2, \ldots u, v, \ldots a_p$; $w_j = b_1, b_2, \ldots, b_l, u, v, b_{l+2}, \ldots b_q$.

Replace w_i and w_j by the walks $w'_i = a_1, a_2, \ldots u, b_l, \ldots, b_2, b_1$ and

$w'_j = a_p, \ldots, v, b_{l+2}, \ldots b_q$ to get another member W' of A. $\text{len}(w'_i) + \text{len}(w'_j) = \text{len}(w_i) + \text{len}(w_j) - 2$, so the total length of walks in W' is less than m, which is a contradiction.

Solution 2

Since the graph G is connected, it has a spanning tree, say H. Since the edges in H are a subset of G, if we prove the claim for H, it directly follows for the graph G.

So, we prove that there exists a non-conflicting transportation in a tree H for any subset T of vertices where |T| is even.

Proof by induction of number of vertices in H,

Base case: |V| = 2, then we can pick the single edge between these 2 vertices if T=V, and the empty set of paths If T is empty.

Induction Hypothesis: Assume there exists a non-conflicting transportation #vertices < n.

Inductive step: Consider a tree H with n vertices, and a subset T of vertices. Let a be a leaf node of this tree which is connected to vertex b.

Case 1: a does not belong to T. In this case we can simply remove a from the graph and get the result using IH.

Case 2: Both a, b belong to T. In this case we include the edge (a, b) to the transportation, remove the vertex a from graph and a,b from T. Now we can complete the argument using IH.

Case 3: a belongs to T, b does not belong to T. In this case we remove the vertex a from the graph, and replace a by b to T. From IH we get a non-conflicting transportation. Consider a path which has an end point as b, extend this path by appending edge (a, b) to it. This creates a non-conflicting transportation for original tree.