This file contains sketch of solutions discussed in lecture 13

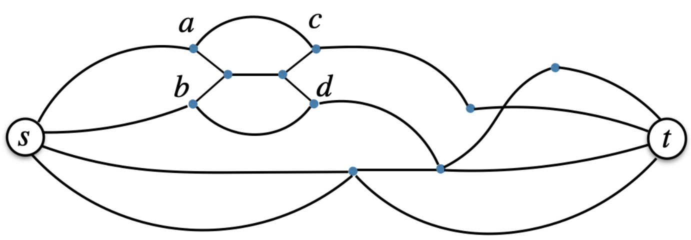
COL751: Quiz-1 Name:
Maximum marks: 10 Entry number: 2 0

Question 1 Let $G = (V, E)$ be a graph on n vertices, and $(x_1, y_1), \ldots, (x_n, y_n)$ be n vertex pairs. Present an algorithm to compute a subgraph $H = (V, E_H \subseteq E)$ with $O(n^{1.5} \log n)$ edges such that $distance(x_i, y_i, G) = distance(x_i, y_i, H)$, for $i \in [1, n]$. [3 marks]

Question 2 Let G be an undirected unit-capacited graph on n vertices and G_f be residual graph with respect to some (s, t)-max-flow.

A. Prove that the number of directed edges entering SCC of s is same as (s, t)-max-flow value. [2 marks]

B. Explain what are (i) SCCs of G_f, (ii) intra-cluster edges, (iii) the DAG G_f^{scc}, if G is as shown below. [1 marks]



Question 3 Let $G = (V, E)$ be an n vertex weighted undirected graph and $Z \subseteq V$ be a set of size k.

A. Explain how can you compute a $Z \times Z$ distance oracle with stretch 5, $O\big((nk)^{2/3} \log n\big)$ space, and $O(1)$ query time. [3 marks]

B. Let $H_Z = (Z, E_Z)$ be a complete graph such that $wt(x, y, H_Z) = distance(x, y, G)$, for each $x, y \in Z$. Explain how can you use this new graph along with result of (i) as black-box to improve space of $Z \times Z$ oracle to $O(n)$ when $k = n^{2/3}$. [1 marks]

Quiz

Q1 Pairs $(x_1, y_1) \dots (x_n, y_n)$

Short & long path

Hitting set

works for directed, weighted

Q2 # of edges entering SCC(s) = (s,t)-max flow value

Pf 1

A = vertices reachable from s in G_f

$B = V \setminus A$

Claim 1 : (A,B) is (s,t) - min-cut

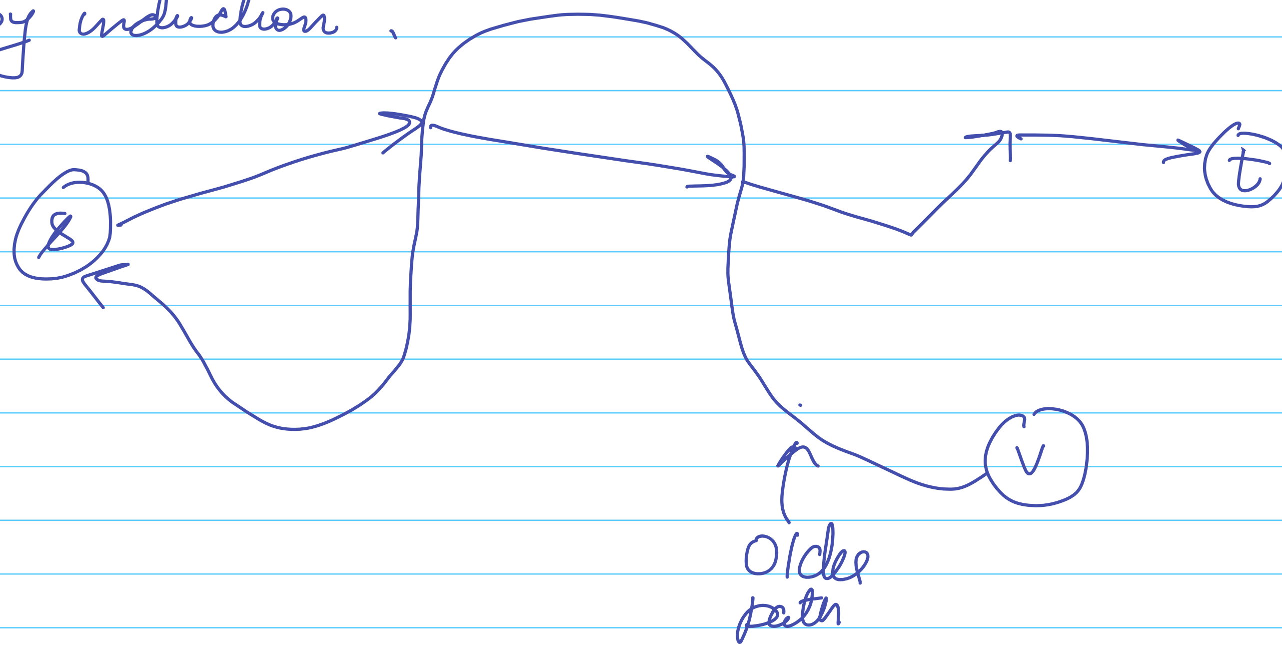
all edges from B to A

Claim 2 : A = SCC(s)

Implication of claim 2 : # of edges entering SCC(s) = (s,t)-max flow value

Proof of Claim 2

By induction.



Pf 2: Edges entering SCC(s) form anti-chain.

It is maximal due to following claim.

Claim: For any edge $e = (x, y)$ in G_f^{SCC}

there is an in-edge of 's' reachable from y.

Proof: $y \leftarrow x$

All edges other than s have out-edge in

Keep following them.

Q3: $Z \times Z$ oracle $\quad (nk)^{2/3}$ space

(a)

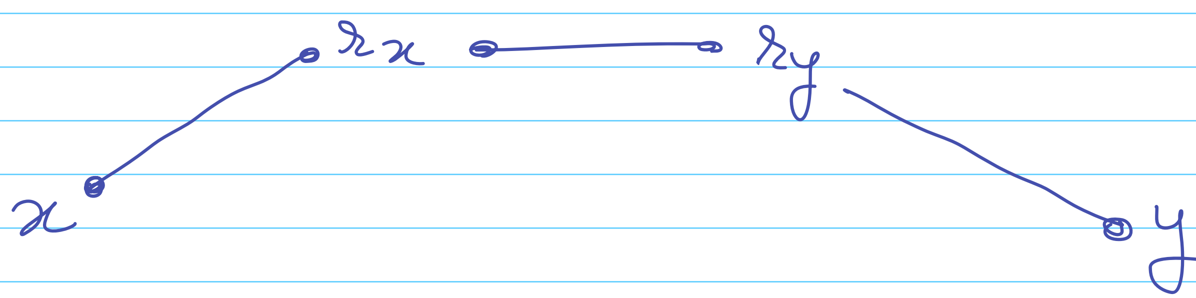
$k = |Z|$

stretch $= 5$

$\forall\, v \in Z$, store distance in $B(v, L \log n)$

$R \leftarrow$ rndm set of size $\frac{n}{L}$

$$\text{Storage} = k L \log n + \frac{n^2}{L^2} \qquad \left(L = \frac{n^{2/3}}{k^{1/3}} \right)$$



Stretch proof: See class Notes

(b) $k^{4/3} \log n \Rightarrow$ for $k \le n^{2/3} \log n$ $\quad O(n)$ space

COL 751 : Practice Sheet - 2

Note: Problems marked as $*$ are optional and much harder than other problems.

1. Distance Oracles of stretch 3
Argue that for any vertex x in an unweighted graph G (given as adjacency list representation), $BFS(x, r)$ can be computed in $O(r^2)$ time. Use this to prove that for unweighted graphs there exists a construction of distance oracle of multiplicative stretch 3 that takes $O(m\sqrt{n}\log n + n^2)$ pre-processing time.

2. Girth Conjecture
The Girth Conjecture by Erdos states that for every $k \geqslant 1$ and for sufficiently large n, there are n-vertex graphs with girth $2k + 2$ and $\Omega(n^{1+1/k})$ edges. Prove that if Girth Conjecture holds true then the greedy construction of $(2k - 1)$ spanner presented in Lecture 4 is of optimal size (up to constant factors).

3. Distance Oracles with stretch less than 3
Show that any distance oracle with multiplicative stretch strictly less than 3 for unweighted graphs takes $\Omega(n^2)$ space. (Hint: Argue that you can use such a distance oracle as black-box to identify edges of a bipartite graph $G = (A, B, E)$ satisfying $|A| = |B| = n$.)

4. Subset Distance Oracles
Let $G = (V, E)$ be an n vertex unweighted graph and $Z \subseteq V$ be a set of size k. Argue that there exists a $Z \times Z$ distance oracle with stretch 5 that takes $O\big((nk)^{2/3}\log n\big)$ space. Can you improve space further to $O(n \log n)$ for $k \leqslant n^{3/4}$? (Hint: Think simple!)

5. Distance Spanner for vertex pairs in $Z \times V$
Let $G = (V, E)$ be an n vertex unweighted graph and $Z \subseteq V$ be a set of size k. Argue that there exists a $Z \times V$ distance spanner with stretch 3 that takes $O\big(n\sqrt{k}\log n\big)$ space.

6. Distributed Systems
Let $G = (V, E)$ be a large network comprising of n nodes, where each node is associated with a local computer possessing a storage capacity of $O(\sqrt{n}\log n)$. Show that it is possible to store partial information about G in the local computers so that for any $x, y \in V$ a 3-approximation to (x, y) distance can be computed solely based on local information stored at nodes x and y.

7. Distance Oracles with (3, 2) stretch
A distance oracle is said to have (α, β) stretch if for every $x, y \in V$ it satisfies

$$dist(x, y) \;\leqslant\; \widehat{d}(x, y) \;\leqslant\; \alpha \cdot dist(x, y) + \beta.$$

Show that for n vertex unweighted graphs you can compute a distance oracle of stretch $(3, 2)$ that takes $O(n^2 \log^2 n)$ pre-processing time.

8. Distance Spanners for Directed graphs

Show that there exists n vertex digraphs for which any finite stretch distance spanner takes $\Omega(n^2)$ space.

9. Diameter Preservers

Show that for any n vertex strongly connected directed graph $G = (V, E)$ we can compute a subgraph $H = (V, E_H)$ with $O(n^{1.5} \log n)$ edges satisfying $diam(H) \leqslant \lceil 1.5\, diam(G) \rceil$.

Hint: Use the idea of hitting vertices of high (i.e. $\geqslant \sqrt{n}$) in-degree/out-degree.

10. Additive Spanners for Weighted graphs

Show that for any n vertex weighted graph G with edge weights in range $[1, W]$, it is possible to compute a $+2W$ additive spanner in $\widetilde{O}(n^2)$ time. Further, show that there exists weighted graphs with edge weights in range $[1, W]$, for which any $+W$ approximate distance oracle takes $\Omega(n^2)$ space.

11. Approximate Distance Matrix (*)

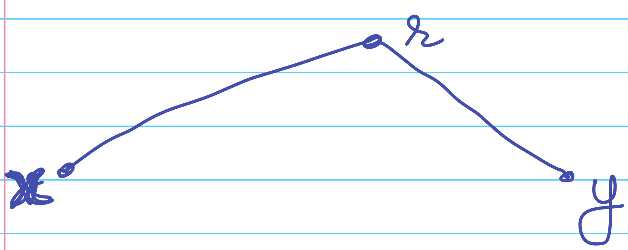
An approximate distance matrix $\hat{M}$ is said to have (α, β) stretch if for every $x, y \in V$ it satisfies

$$dist(x, y) \leqslant \hat{M}[x, y] \leqslant \alpha \cdot dist(x, y) + \beta.$$

Show that for any n vertex unweighted graph it is possible to compute in $\widetilde{O}(n^2)$ time an approximate distance matrix of stretch $(2, c)$, for some large enough constant c.

Q6 Each $x \in V$ stores distances in $B(x, \sqrt{n})$ locally.

and also stores distance (x, z), $\forall z \in R$.

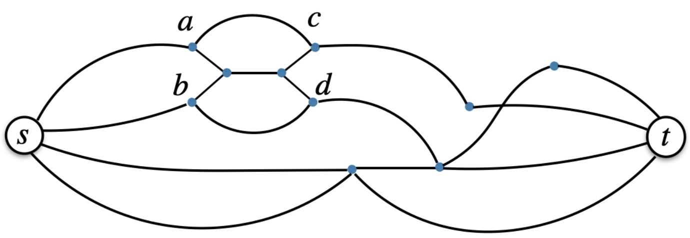


Q8 $K_{n,n}$ subgraph — needs to store information for all edges.

COL 751 : Practice Sheet - 3

1. SCCs of residual graph

Compute (s,t)-max-flow of the graph shown below and argue that its residual graph has exactly one non-singleton strongly connected component. Can you argue the same without actually computing the max-flow?



2. Nearest min-cut

Let $G = (V, E)$ be an undirected connected graph, $s \in V$ be a source vertex, and $t \in V$ be a sink vertex. An (s,t)-min-cut (A^*, B^*) of graph G is said to be the **nearest (s,t)-min-cut** if for each (s,t)-min-cut (A, B), we have $A^* \subseteq A$.

(i) For any (s,t)-min-cut (A, B) argue that $SCC(s, G_f) \subseteq A$, where G_f is residual graph corresponding to some max-flow f.

(ii) Show how can you use (i) to prove that there is a unique nearest (s,t)-min-cut in G.

(iii) Present an algorithm that given a max-flow f, computes nearest min-cut in $O(m+n)$ time.

3. Submodularity of cuts

Let $G = (V, E)$ be an undirected connected graph, $s \in V$ be a source vertex, $t \in V$ be a sink vertex, and G_f be residual graph corresponding to some max-flow f. Further, for any $x \in V$, let $R_{out}(x, G_f)$ be the set of vertices reachable from x in graph G_f.

(i) For any (s,t)-cut (A, B) in graph G prove that (A, B) is an (s,t)-min-cut if and only if $A = \cup_{x \in A} R_{out}(x, G_f)$.

(ii) Use result of (i) to prove that if (A_1, B_1) and (A_2, B_2) are two (s,t)-min-cuts, then $(A_1 \cap A_2, B_1 \cup B_2)$ is also an (s,t)-min-cut.

4. Min-cut oracle

Let $G = (V, E)$ be an undirected connected graph, $s \in V$ be a source vertex, $t \in V$ be a sink vertex, and λ be value of (s,t)-min-cut in G.

(i) Design an $O(n)$ sized oracle that given any query edge $e \in E$ answers whether or not e belongs to some (s,t)-min-cut in $O(1)$ time.

(ii) Design an $O(n\lambda)$ sized oracle that given any set $\mathcal{E} \subseteq E$ answers whether or not $\mathcal{E}$ belongs to some (s,t)-min-cut in $O(|\mathcal{E}|^2)$ time.

5. Menger's theorem

Prove using Max-Flow Min-Cut theorem that for any directed graph G with a source s and destination t, following holds:

(a) The size of a minimum (s,t) vertex-cut in G is equal to the maximum number of internally (s,t)-vertex-disjoint paths.

(b) The size of a minimum (s,t) edge-cut in G is equal to the maximum number of (s,t)-edge-disjoint paths.

Hint: Every vertex v in G can be split into a directed edge $(v_{in} \rightarrow v_{out})$.

6. k-vertex-connected subgraphs

Let $G = (V,E)$ be a k-vertex-connected graph, that is, it satisfies that for every $S \subseteq V$ of size at most $k-1$ the graph $G - S$ is connected.

Argue that the subgraph H obtained by recursively computing k spanning forests $T_1, \ldots, T_k$ (i.e. T_i is spanning forest of $G - (T_1 \cup \cdots \cup T_{i-1})$), and merging them may not be k-vertex-connected.

7. Deck of cards

A standard deck of cards, dealt into 13 piles of 4 cards each. Show that it is possible to select one card from each pile so that the selected cards contain exactly one card of each rank (Ace, 2, 3, ..., Queen, King).

8. Subgraph of $K_{n,n}$

Using Hall's theorem argue that any subgraph of $K_{n,n}$ with $n^2 - n + 1$ edges has a perfect matching.

9. Rooks on chess board

In a $2n \times 2n$ chess board, there are n rooks in each row as well as each column. Show that there exist a set S of $2n$ rooks such that no two rooks in set S lie in the same row or the same column.

10. Variant of Hall's theorem

Given a bipartite graph $G = (X,Y,E)$, the deficiency of G w.r.t. X is the maximum, over all subsets W of X, of the difference $|W| - |N(W)|$.

Using Hall's theorem argue that if the deficiency of a bipartite graph G is d, then G admits a matching of size at least $|X| - d$.

Q2 (i) For any min-cut, $SCC(s) \subseteq A$

Proof:

$s \in A$ and $SCC(s)$ can't contain cut edges.

(ii) By quiz-1 ques, $(SCC(s), V - SCC(s))$ is min-cut

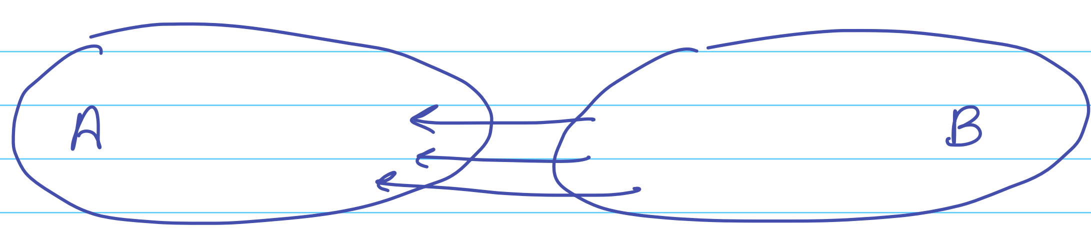
and $SCC(s) \subseteq A$ $\quad \Rightarrow (SCC(s), V - SCC(s))$ is NMC.

$\forall (A,B)$ min cuts.

(iii) Given f, NMC can be obtaind by findg $SCC(s)$

<u>Q 3.(i)</u> (A,B) is min-cut iff $A = \bigcup_{x \in A} Rout(x, G_f)$

Proof: Let $A = \bigcup_{x \in A} Rout(x, G_f)$

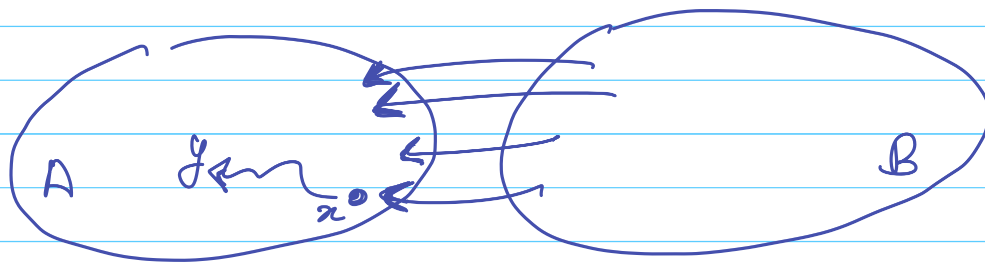


All edges in G_f from B to A

$\Rightarrow$ By Prop 3 of Lec 9, (A, B) is min-cut

Next assume (A,B) is min cut

$\Rightarrow$



all edges in G_f from B to A.

Suppose $y \in R_{out}(x, G_f)$ for some $x \in A$

Then clearly $y \in A$.

PROPERTY 3:

(*) Suppose (A,B) is (s,t)-min-cut

$\Rightarrow$ all edges in $A \times B$ are fully saturated

$\Rightarrow$ all edges in G_f from B to A.

(*) Suppose in G_f all edges from B to A.

$\Rightarrow$ $s \in A$ and $t \in B$

$\Rightarrow$ (A, B) is (s,t) cut in which all edges are saturated

$\Rightarrow$ (A, B) is (s,t) - min-cut

Q4

(i) Min-cut oracle.

An edge $e = (x, y)$ lies in (s,t) - min-cut iff

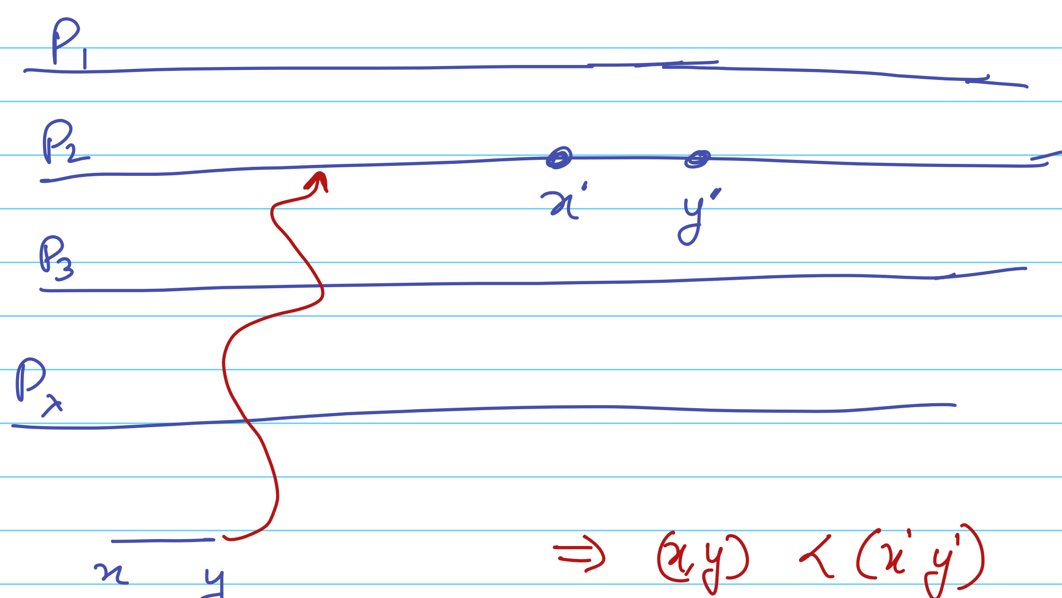
$$SCC(x) \neq SCC(y).$$

Store "representative vertex" in SCCs to check this in $O(1)$ time.

(ii) Check if $\mathcal{E}$ is contained in an (s,t) min cut in $O(|\mathcal{E}|^2)$ time.

— All edges must be inter cluster

— No two edges should form a chain

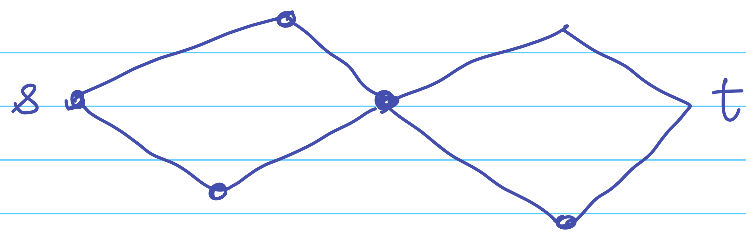


$\Rightarrow (x, y) \prec (x', y')$

Q5 Menger's Theorem for Directed graphs.

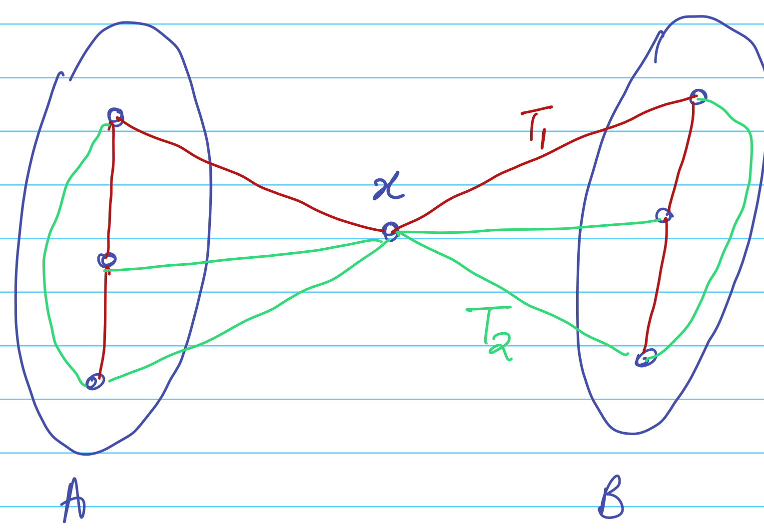
Each edge — ∞ capacity

Each v — $(v_{in} \longrightarrow v_{out})$ $\forall v \neq s, t$.



Q6 Counter example for $k=1$ (can be extended for general k)

Let $G = K_7$



For the choice of T_1 T_2 above graph $H = T_1 \cup T_2$ is not 2-vertex connected as $H \setminus x$ has 2 components.