2201 - MTL106 : Quiz

Q1. $(\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow$ Probability space, $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$.

$\mathbb{Q} : \mathcal{F} \longrightarrow [0,1]$ defined by $\mathbb{Q}(A) = \mathbb{P}(A|B)$.

We know that $\mathbb{Q}$ is a probability measure over Ω with $\mathbb{Q}(B) = 1$.

Given: $C \in \mathcal{F}$ with $\mathbb{Q}(C) > 0$.

$$L.H.S = \mathbb{Q}(A|C) = \frac{\mathbb{Q}(A \cap C)}{\mathbb{Q}(C)} = \frac{\mathbb{P}(A \cap C | B)}{\mathbb{P}(C|B)}$$

$$= \frac{\mathbb{P}((A \cap C) \cap B)}{\mathbb{P}(B)} \cdot \frac{\mathbb{P}(B)}{\mathbb{P}(C \cap B)}$$

$$= \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(B \cap C)} = \mathbb{P}(A | B \cap C) = R.H.S$$

3-marks

Q2. $(\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow$ probability space, $E, F, G \in \mathcal{F}$ and E is independent of F and G.

- Given statement "E is independent of $F \cup G$" is false — $\frac{1}{2}$ mark.

- Counter example: Let $\Omega = \{ (i,j) : 1 \leq i,j \leq 6 \}$

  $\mathcal{F} = \mathcal{P}(\Omega)$, the power set of Ω.

  $\mathbb{P}(A) = \frac{|A|}{36}$ for any $A \subseteq \Omega$.

Consider the events

$E = \{(2,b) : 1 \le b \le 6\}$

$F = \{(a,5) : 1 \le a \le 6\}$

$G = \{(a,b) : a+b = 7,\ 1 \le a, b \le 6\}$.

Then $\mathbb{P}(E) = \mathbb{P}(F) = \mathbb{P}(G) = 1/6$.

$\mathbb{P}(E \cap F) = \mathbb{P}(E \cap G) = \frac{1}{36}$.

Hence E is independent of F and G.

Here $\mathbb{P}(F \cup G) = \frac{11}{36}$, $\quad E \cap (F \cup G) = \{(2,5)\}$

Now, $\mathbb{P}(E \cap (F \cup G)) = \frac{1}{36} \neq \mathbb{P}(E) \cdot \mathbb{P}(F \cup G)$

$= \frac{1}{6} \cdot \frac{11}{36}$

Hence E is NOT independent of $F \cup G$.

$3\frac{1}{2}$ marks.

<u>Q3.</u> $X \to$ random variable with $E[|X|] < +\infty$.

$\Phi_X(\cdot) \to$ characteristic fn of X.

$1 - \Phi_X(t) = 1 - \mathbb{E}[e^{itX}] = \mathbb{E}[1 - e^{itX}]$

~~$= 1 - E[\cos tX] - i\,\mathbb{E}[\sin tX]$~~

~~$\Rightarrow |1 - \Phi_X(t)| \le \int (1 - \ldots$~~

$\Rightarrow |1 - \Phi_X(t)| \le \mathbb{E}[|1 - e^{itX}|]$

$\le \mathbb{E}[\sqrt{2(1 - \cos tX)}]$.

$\to 1\frac{1}{2}$ marks

We know that for any x, $2(1 - \cos x) \le x^2$.

Thus, $|1 - \Phi_X(t)| \le E[\sqrt{t^2 X^2}] = t\,E[|X|]$ for $t > 0$. — o —

~~This completes the proof.~~

$1\frac{1}{2}$ marks

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Q4:- Let X be a random variable with pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty.$$

Let $h(x) = e^x$. Then h is strictly increasing and differentiable function on $\mathbb{R}$. Moreover $h^{-1}(y) = \ln y$ for $y > 0$ and

$$\frac{d}{dy} h^{-1}(y) = \frac{1}{y}.$$

$1\frac{1}{2}$

Hence
$$f_Y(y) = f_X(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right|$$

$$= \begin{cases} \frac{1}{y\sqrt{2\pi}} e^{-\frac{1}{2}(\ln y)^2}, & \text{if } y > 0 \\ 0, & \text{otherwise} \end{cases}$$

$1\frac{1}{2}$

where $Y = e^X$.

Let $m_X(t)$ be the moment generating function of X. Then $m_X(t) = \mathbb{E}[e^{tX}]$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{x^2}{2}} dx = e^{\frac{t^2}{2}}.$$

$$\Rightarrow \mathbb{E}[Y^t] = \mathbb{E}[e^{tX}] = e^{t^2/2}$$

$$\Rightarrow \mathbb{E}[Y] = e^{\frac{1}{2}}.$$

— 2 marks.