COL751 - Lecture 12

1 Separating Set Family (Puzzle)

Let U be a universe of size N. We consider the problem of computing a set family $\mathcal{F} = (S_1, \ldots, S_r)$ of subsets of U such that for any distinct vertices $x, y \in U$, there is a set S_i that contains x but not y.

A natural choice for computing set S_i's is by picking elements from U randomly. We will see that this solution in-fact turns good for us. The algorithm to compute family $\mathcal{F}$ is presented below.

```
Let r = 12 log_e N;
for i = 1 to r do
  | Let S_i be uniformly random subset of U obtained by picking elements w.p. 1/2;
end
Return F = (S_1, ..., S_r);
```

Algorithm 1: Construction of 'Separating Set Family'

Lemma 1 *With high probability, for any $x, y \in U$, there exists an $i \in [1, r]$ such that S_i contains x but not y.*

Proof: Consider any distinct $x, y \in U$. For any set $S_i \in \mathcal{F}$, we have

$$Prob(x \in S_i,\ y \notin S_i) \ = \ \frac{1}{4}.$$

Since the sets $S_1, \ldots, S_r$ are computed independently, we have

$$\begin{aligned}
Prob(\forall i,\ S_i \text{ does not separate } x \text{ from } y) \ &= \ Prob(\nexists i \text{ that satisfy } x \in S_i,\ y \notin S_i) \\
&= \ \prod_{1 \leqslant i \leqslant r} \Big(1 - Prob(x \in S_i,\ y \notin S_i)\Big) \\
&= \ \left(1 - \frac{1}{4}\right)^{12 \log_e N} \\
&\leqslant \ \frac{1}{N^3}.
\end{aligned}$$

Thus, by union bound,

$$Prob(\exists\ x, y \in U \text{ satisfying } x, y \text{ are not separated by any } S_i) \ \leqslant \ \sum_{x \neq y \in U} \frac{1}{N^3} \ \leqslant \ \frac{1}{N}.$$

This proves that $\mathcal{F} = (S_1, \ldots, S_r)$ is a separating set family for pairs in U with probability at least $1 - 1/N$. $\square$

2 Bi-connectivity Certificate via Separating Set Family

Let $G = (V, E)$ be a 2-edge-connected undirected graph on n vertices. We will next see how to use Lemma 1 to construct a subgraph H of G that preserves 2-edge connectedness.

Remark By Max-Flow Min-Cut Theorem, for any pair $x, y \in V$, there exists k edge disjoint paths between x and y in G if and only if there is no (x, y) min-cut of size $k-1$ in G. In other words, on removal of any $k-1$ edges, the vertices x and y are still connected in G. We will use this equivalent definition to compute our certificate H.

Let $r = 12 \log_e m$;
for $i = 1$ *to* r **do**
  Let S_i be a uniformly random subset of E obtained by picking edges w.p. $\frac{1}{2}$;
  Let T_i be a spanning tree/forest of graph $G_i = (V, S_i)$;
end
Return $H = \big(V, \cup_{i=1}^{r} E(T_i)\big)$;

Algorithm 2: An alternate construction of 2-edge connectivity certificate

Lemma 2 *With probability $1 - 1/m$, for any edge $e = (x, y) \in E$ and any $F \subseteq E \setminus e$ of size 1, there exists an $i \in [1, r]$ that satisfy $e \in S_i$ and $F \cap S_i = \emptyset$.*

Proof: The proof directly follows by Lemma 1. $\square$

Lemma 3 *With high probability, the graph H is 2-edge connected.*

Proof: Consider any $x, y \in V$ and any subset $F \subseteq E(G)$ of size 1. In order to prove our claim it suffices to argue that there is a path from x to y in $H - F$.

Let P be a path from x to y in graph $G - F$. Such a path exists as G is 2-edge-connected. Now consider any edge $e = (a, b) \in P$.

Observe that w.p. $1 - 1/m$, $(S_1, \ldots, S_r)$ is a separating family for $(\{e\}, F)$. So, there is some S_i that contains e, but not F. This implies the endpoints of e (i.e. a and b) are connected by a path in graph T_i that does not contain edges in F. (Why?). Hence, we conclude that a, b are connected in $H - F$.

We can argue the same for each edge $e \in P$, thereby proving that x, y are connected in $H - F$. $\square$

Theorem 4 *For any 2-edge-connected undirected graph $G = (V, E)$ on n vertices, we can compute a sparse 2-edge-connectivity certificate $H = (V, E_H \subseteq E)$ with at most $O(n \log n)$ edges.*

3 k-connectivity Certificate via Separating Set Family

Let G be a k-edge-connected graph with n vertices and m edges. That is, for each pair $(x, y) \in V \times V$ of distinct vertices, there are k-edge disjoint paths between x and y in G. We will next see how to use randomisation to compute a sparse subgraph H of G which is k-edge-connected.

```
Let r = 4(k+1)(k-1) log_e m;
for i = 1 to r do
    Let S_i be a uniformly random subset of E obtained by picking edges w.p. 1/(k-1);
    Let T_i be a spanning tree/forest of graph G_i = (V, S_i);
end
Return H = (V, ∪_{i=1}^{r} E(T_i));
```

Algorithm 3: An alternate construction of k-edge connectivity certificate

Lemma 5 *With probability $1 - 1/m$, for any edge $e = (x, y) \in E$ and any $F \subseteq E \setminus e$ of size $k-1$, there exists an $i \in [1, r]$ that satisfy $e \in S_i$ and $F \cap S_i = \emptyset$.*

Proof: Consider any disjoint $e \in E$ and $F \subseteq E$ of size $k-1$. For any set $S_i \in \mathcal{F}$, we have

$$Prob(e \in S_i,\ F \cap S_i = \emptyset) \ = \ \frac{1}{k-1}\left(1 - \frac{1}{k-1}\right)^{k-1} \geqslant \frac{1}{4(k-1)}.$$

Since the sets $S_1, \ldots, S_r$ are computed independently, we have

$$\begin{aligned} Prob(\forall i,\ S_i \text{ does not separate } e \text{ from } F) &= Prob(\nexists i \text{ that satisfy } e \in S_i,\ F \cap S_i = \emptyset) \\ &= \prod_{1 \leqslant i \leqslant r} \Big(1 - Prob(e \in S_i,\ F \cap S_i = \emptyset)\Big) \\ &= \left(1 - \frac{1}{4(k-1)}\right)^{4(k+1)(k-1)\log_e m} \\ &\leqslant \frac{1}{m^{k+1}}. \end{aligned}$$

Thus, by union bound,

$$Prob(\exists\, (e, F) \text{ satisfying } e, F \text{ are not separated by any } S_i) \leqslant \sum_{(e,F) \in E \times E^{k-1}} \frac{1}{m^{k+1}} \leqslant \frac{1}{m}.$$

This proves that with probability at least $1 - 1/m$, $\mathcal{F} = (S_1, \ldots, S_r)$ is a separating set family for pairs $(e, F) \in E \times E^{k-1}$. $\square$

Lemma 6 *With high probability, the graph H is k-edge connected.*

Proof: Consider any $x, y \in V$ and any subset $F \subseteq E(G)$ of size $k-1$. In order to prove our claim it suffices to argue that there is a path from x to y in $H - F$.

Let P be a path from x to y in graph $G-F$. Such a path exists as G is k-edge-connected. Now consider any edge $e = (a, b) \in P$.

Observe that w.p. $1 - 1/m$, $(S_1, \ldots, S_r)$ is a separating family for $(\{e\}, F)$. So, there is some S_i that contains e, but not F. This implies the endpoints of e (i.e. a and b) are connected by a path in graph T_i that does not contain edges in F. (Why?). Hence, we conclude that a, b are connected in $H - F$.

We can argue the same for each edge $e \in P$, thereby proving that x, y are connected in $H - F$. $\square$

Theorem 7 *For any k-edge-connected undirected graph G on n vertices, we can compute a sparse k-edge-connectivity certificate H with just $O(nk^2 \log n)$ edges.*

Remark We had seen in Lecture 09, section 3 a very simple construction of k-edge-connectivity certificate with just $O(kn)$ edges. However, the older construction does not directly work for k-vertex-connectivity. We will see later that ideas presented in Lemma 2-6 are helpful in computing sparse k-vertex connectivity certificates as well.