MTL103: Optimization Methods and Applications	Date: April 13, 2023

Minor 2
(solution)

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Question 1. *Consider the following problem:*

minimize $15x_1^3 + 40x_2^2$
Subject to $2x_1 + x_2 \geq 30$
$5x_1 - 6x_2 \leq 17$
$x_1, x_2 \geq 0.$

a) Write down the corresponding Lagrange dual problem.
b) Write down the KKT optimality conditions for the problem.

Solution: (a) We define the Lagrangian $L(x, \lambda, \mu)$ associated with the given problem as

$$L(x, \lambda, \mu) = (15x_1^3 + 40x_2^2) + \lambda_1(-2x_1 - x_2 + 30) + \lambda_2(5x_1 - 6x_2 - 17) + \lambda_3(-x_1) + \lambda_4(-x_2)$$

So the Lagrangian dual function $g(\lambda, \mu)$ is given by

$$\begin{aligned} g(\lambda, \mu) &= \inf_x L(x, \lambda, \mu) \\ &= \inf_x \left[(15x_1^3 + 40x_2^2) + \lambda_1(-2x_1 - x_2 + 30) + \lambda_2(5x_1 - 6x_2 - 17) + \lambda_3(-x_1) + \lambda_4(-x_2)\right] \end{aligned}$$

Thus, the Lagrangian dual problem is

max $g(\lambda, \mu)$

Subject to $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$

(b) Let x^* and (λ^*, μ^*) be any primal and dual optimal points with zero duality gap.
Then the KKT optimality conditions for the problem are

$$\begin{aligned} 2x_1^* + x_2^* &\geq 30 \\ 5x_1^* - 6x_2^* &\leq 17 \\ x_1^*, x_2^* &\geq 0 \\ \lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^* &\geq 0 \\ \lambda_1^*(-2x_1^* - x_2^* + 30) &= 0 \\ \lambda_2^*(5x_1^* - 6x_2^* - 17) &= 0 \\ \lambda_3^*(-x_1^*) &= 0 \\ \lambda_4^*(-x_2^*) &= 0 \\ 45(x_1^*)^2 - 2\lambda_1^* + 5\lambda_2^* - \lambda_3^* &= 0 \\ 80x_2^* - \lambda_1^* - 6\lambda_2^* - \lambda_4^* &= 0 \end{aligned}$$

Question 2. *Consider a linear programming problem given in the standard form*

Minimize $c^T x$
Subject to $Ax = b$
$x \geq 0.$

Let x^* *be an optimal solution, assumed to exist, and let* p^* *be an optimal solution to the dual.*
a) Let $\overline{x}$ *an optimal solution to the primal , when c is replaced by some* $\overline{c}$*. Show that* $(\overline{c}-c)^T(\overline{x}-x^*) \leq 0$*.*
b) Let $\overline{x}$ *an optimal solution to the primal, when b is replaced by some* $\overline{b}$*. Show that* $(p^*)^T(\overline{b}-b) \leq c^T(\overline{x}-x^*)$*.*

Solution: (a) The given primal LPP is

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & \begin{array}{c} Ax = b \\ x \geq 0 \end{array} \end{aligned} \tag{1}$$

So the corresponding dual problem will be

$$\begin{aligned} \text{maximize} \quad & p^T b \\ \text{subject to} \quad & \begin{array}{c} p^T A \leq c^T \\ \text{p is unrestricted in sign} \end{array} \end{aligned} \tag{2}$$

The new primal, when c is replaced by some $\bar{c}$ will be,

$$\begin{aligned} \text{minimize} \quad & \bar{c}^T x \\ \text{subject to} \quad & \begin{array}{c} Ax = b \\ x \geq 0 \end{array} \end{aligned} \tag{3}$$

Given that $\bar{x}$ is an optimal solution to 3 and x^* was an optimal solution to 1. Notice that, the constraints of both 1 and 3 are the same. So, both $\bar{x}$ and x^* will be feasible for both primals 1 and 3.
Now, x^* is an optimal solution for 1 and $\bar{x}$ is a feasible solution to 1. Thus,

$$c^T x^* \leq c^T \bar{x}.$$

Similarly, x^* is a feasible solution for 3 and $\bar{x}$ is an optimal solution to 3. Thus,

$$\bar{c}^T \bar{x} \leq \bar{c}^T x^*.$$

Now, adding the above two equations, we get,

$$c^T x^* + \bar{c}^T \bar{x} \leq c^T \bar{x} + \bar{c}^T x^* \implies (\bar{c} - c)^T (\bar{x} - x^*) \leq 0$$

Hence the proof of part (a).

(b) Given that p^* is an optimal solution to the dual 2. Since x^* was an optimal solution to primal 1, by Strong duality theorem, we have,

$$c^T x^* = (p^*)^T b \tag{4}$$

When b is replaced by $\bar{b}$, the new primal will be,

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & \begin{array}{c} Ax = \bar{b} \\ x \geq 0 \end{array} \end{aligned} \tag{5}$$

So the corresponding dual problem will be,

$$\begin{array}{ll} \text{maximize} & p^T \bar{b} \\ \text{subject to} & \begin{array}{c} p^T A \leq c^T \\ \text{p is unrestricted in sign} \end{array} \end{array} \tag{6}$$

Notice that, constraints of 2 and 6 are same. Since p^* is an optimal solution to 2, p^* will be a feasible solution to 6. Also, given that $\bar{x}$ is an optimal solution to 5. Thus by the weak duality theorem, we get,

$$(p^*)^T \bar{b} \leq c^T \bar{x} \tag{7}$$

Now, from 4 and 7, we get,

$$(p^*)^T (\bar{b} - b) \leq c^T (\bar{x} - x^*).$$

Hence the proof of part (b).

Question 3. *Let us consider the following problem*

$$\begin{aligned}
&\textit{Maximize } v\\
&\textit{Subject to } Af + dv \le 0\\
&\qquad\qquad f \le b\\
&\qquad\qquad -f \le 0.
\end{aligned}$$

*where $A_{|V|\times|E|}$ is an incidence matrix of a graph, and the $d \in \mathbb{R}^{|V|}$. Consider this to be **D** in the primal-dual method. Formulate corresponding **DRP**.*

Solution: The dual problem (**D**) is

$$\begin{aligned}
&\text{Maximize } v\\
&\text{Subject to } Af + dv \le 0\\
&\qquad\qquad f \le b\\
&\qquad\qquad -f \le 0.
\end{aligned}$$

So the primal problem (**P**) is

$$\begin{aligned}
&\text{Minimize } \textstyle\sum_{j=1}^{|E|}(\lambda_j b_j)\\
&\text{Subject to } \mu_x - \mu_y + \lambda_e \ge 0, \text{ for all edge } e = (x, y) \in E\\
&\qquad\qquad \textstyle\sum_{i=1}^{|V|}(\mu_i d_i) = 1\\
&\qquad\qquad \mu \ge 0, \lambda \ge 0
\end{aligned}$$

Define, $Q_1 = \{i \mid \sum_{j=1}^{|E|}(a_{ij} f_j) + d_i v = 0\}$
$Q_2 = \{j \mid f_j = b_j\}$
$Q_3 = \{k \mid f_k = 0\}$

So the restricted primal problem (**RP**) is defined by

$$\begin{aligned}
&\text{Minimize } \textstyle\sum_{j=1}^{|E|} x_j^{(a)} + y^{(a)}\\
&\text{Subject to } \mu_x - \mu_y + \lambda_e + x_e^{(a)} \ge 0, \text{ for all edge } e = (x, y) \in E\\
&\qquad\qquad \textstyle\sum_{i=1}^{|V|}(\mu_i d_i) + y^{(a)} = 1\\
&\qquad\qquad x_e^{(a)} \ge 0, \text{ for all edge } e \in E\\
&\qquad\qquad y^{(a)} \ge 0\\
&\qquad\qquad \mu_i \ge 0, \text{ for all } i \in Q_1\\
&\qquad\qquad \lambda_j \ge 0, \text{ for all } j \in Q_2\\
&\qquad\qquad \mu_i = \lambda_j = 0, \text{ for all } i \notin Q_1 \text{ and } j \notin Q_2
\end{aligned}$$

Hence, the dual of the restricted primal (**DRP**) is defined by

$$\begin{aligned}
&\text{Maximize } v\\
&\text{Subject to } Af + dv \le 0\\
&\qquad\qquad f \le 1\\
&\qquad\qquad v \le 1\\
&\qquad\qquad f_j \le 0, \text{ for all } j \in Q_2\\
&\qquad\qquad f_k \ge 0, \text{ for all } k \in Q_3
\end{aligned}$$

Question 4. *For the following matrix game, formulate an appropriate LP and compute all mixed-strategy Nash equilibria.*

$$\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \\ 0 & 0 \end{bmatrix}$$

Solution: Pay off matrix for row player is A and for column player is $-A$.
Let strategy set for row player $R = \{r_1, r_2, r_3, r_4\}$ and for column player it is $C = \{c_1, c_2\}$.

LP formulation for row player (consider this as LP1 problem):

Maximize z
Subject to $z \leq 0 \cdot x_1 + \frac{1}{2} \cdot x_2 - \frac{1}{2} \cdot x_3 + 0 \cdot x_4$
$z \leq x_1 + 0 \cdot x_2 + x_3 + 0 \cdot x_4$
$x_1 + x_2 + x_3 + x_4 = 1$
$x_i \geq 0$ for all $i = 1, 2, 3, 4$

Similarly, for column player (consider this LP2 problem):

Minimize w
Subject to $w \geq 0 \cdot y_1 + y_2$
$w \geq \frac{1}{2} \cdot y_1 + 0 \cdot y_2$
$w \geq -\frac{1}{2} \cdot y_1 + y_2$
$w \geq 0 \cdot y_1 + 0 \cdot y_2$
$y_1 + y_2 = 1$
$y_i \geq 0$ for all $i = 1, 2$

Let x^* and y^* be the optimal solution of LP1 and LP2 respectively. So by the minmax theorem, we have that (x^*, y^*) is a mixed strategy Nash equilibrium. Note that LP1 and LP2 are dual for each other, and hence we can solve any one of them and use complimentary slackness to solve the other.
Now,

$$y_1 + y_2 = 1 \Rightarrow y_2 = 1 - y_1$$

From LP2, we get

Minimize w
Subject to $w \geq 1 - y_1$
$w \geq \frac{1}{2} y_1$
$w \geq -\frac{1}{2} y_1 + y_2$
$w \geq 1 - \frac{3}{2} y_1$
$1 \geq y_1 \geq 0$

Consider the above problem as LP3. Note that LP2 and LP3 are equivalent. Solving LP3 graphically gives,

$$y_1 = \frac{2}{3}, y_2 = \frac{1}{3} \text{ and } w = \frac{1}{3}$$

.
Let $y^* = (\frac{2}{3}, \frac{1}{3})$. Now we use the complementary slackness property to obtain optimal solution $x^* = (\frac{1}{3}, \frac{2}{3}, 0, 0)$ for LP1.