Tutorial-4 MTL103

1. If x' is any feasible solution of (P) and w' is feasible for (D) such that $w'(b–Ax') = 0$ and $(w'A–c)x' = 0$, then show that x' is optimal for (P) and w' is optimal for (D).

   (P) $\max z = cx$ subject to $Ax \leq b$ and $x \geq 0$

   (D) $\min y = bw$ subject to $Aw \geq c$ and $w \geq 0$

2. Write the dual. Then find the primal optimal solution from the optimal solution of the dual.

$$\begin{gathered} \min\ 3x_1 - 5x_2 - x_3 + 2x_4 - 4x_5 \\ \text{subject to:} \\ x_1 + x_2 + x_3 + 3x_4 + x_5 \leq 6 \\ -x_1 - x_2 + 2x_3 + x_4 - x_5 \geq 3 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{gathered}$$

3. Use duality to show that the following LP has an optimal solution.

$$\begin{gathered} \min\ 2x_1 - x_2 \\ \text{subject to:} \\ 2x_1 - x_2 - x_3 \geq 3 \\ x_1 - x_2 + x_3 \geq 2 \\ x_1, x_2, x_3 \geq 0 \end{gathered}$$

4. Use complementary slackness theorem to verify that $(n, 0, 0, \ldots, 0)$ is an optimal solution of LPP

$$\begin{gathered} \min\ \textstyle\sum_{j=1}^{n} jx_j \\ \text{subject to:} \\ \textstyle\sum_{j=1}^{i} x_j \geq i, \text{ where } i = 1, 2, 3....n \\ x_j \geq 0, \forall j \end{gathered}$$

5. Are the following statements true? Give reasons for your answer.

   - The primal LP (P) and its dual LP (D), both cannot have unbounded solution.
   - The primal LP (P) and its dual LP (D), both cannot be infeasible.
   - The dual(dual(dual)) of a LPP is the primal LPP.
   - If the primal LP (P) has a unique optimal solution and the dual LP (D) is feasible, then (D) also has a unique optimal solution.

6. Solve the following by dual simplex algorithm:

$$\begin{gathered} \min\ 80x_1 + 60x_2 + 80x_3 \\ \text{subject to:} \\ x_1 + 2x_2 + 3x_3 \geq 4 \\ 2x_1 + 3x_3 \geq 3 \\ 2x_1 + 2x_2 + x_3 \geq 4 \\ 4x_1 + x_2 + x_3 \geq 6 \\ x_1, x_2, x_3 \geq 0 \end{gathered}$$

7. Consider the primal problem

$$\begin{gathered}\min\ c^T x \\ \text{subject to: } Ax \geq 0 \\ x \geq 0\end{gathered}$$

Form the dual problem and convert it into an equivalent minimization problem. Derive a set of conditions on the matrix A and vectors b and c under which the dual is identical to the primal.

8. Let A be a given matrix. Show that exactly one of the following alternatives must hold -

   (a) There exists some $x \neq 0$ such that Ax = 0, x$\geq$ 0.

   (b) There exists some p such that $p^T A > 0^T$.

9. Consider the following linear programming problem of minimizing $c^T x$ subject to $Ax = b, x \geq 0$. Let x^* be an optimal solution, assumed to exist, and let p^* be the optimal solution to the dual.

   (a) Let $\bar{x}$ be the optimal solution to the primal, when c is replaced by some $\bar{c}$. Show that $(\bar{c}-c)^T(\bar{x}-x^*) \leq 0$.

   (b) Let the cost vector be fixed at c, but suppose that we now change b to $\bar{b}$, and let $\bar{x}$ be a corresponding optimal solution to the primal. Prove that $(p^*)^T(\bar{b}-b) \leq c^T(\bar{x}-x^*)$.

10. Let A be a symmetric square matrix. Consider the linear optimization problem

$$\begin{gathered}\min\ c^T x \\ \text{subject to:} \\ Ax \geq 0 \\ x \geq 0\end{gathered}$$

Prove that if x^* satisfies $Ax^* = c$ and $x^* \geq 0$, then x^* is an optimal solution.

11. Find the minimum and maximum of $f(x, y, z) = 3x^2 + y$ subject to the constraints $4x - 3y = 9$ and $x^2 + z^2 = 9$. Formulate the lagrangian dual and solve using KKT conditions.