Department of Mathematics
MAL 250/MTL 106 (Probability and Stochastic Processes)
Tutorial Sheet No. 3

1. X has a uniform distribution over the set of integers $\{-n, -(n-1), \ldots, -1, 0, 1, \ldots, (n-1), n\}$. Find the distribution of (i) $|X|$ (ii) X^2 (iii) $1/1+|X|$.

   i) $\frac{1}{2n+1}$ $n=0$; $\frac{2}{2n+1}$ ow   same

2. If X has $N(\mu, \sigma^2)$, find the distribution of $Y = a + bX$, and $Z = \left(\frac{X-\mu}{\sigma}\right)^2$.

3. Let X be uniformly distributed random variable on the interval $(0, 1)$. Define $Y = a + (b-a)X$, $a < b$. Find the distribution of Y.

4. Let X be a random variable with pdf $f(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$, $x > 0$ where $\theta > 0$ and $\alpha > 0$. Find the distribution of random variable $Y = \ln\left(\frac{X}{\theta}\right)$.

5. Let X be an random variable having an exponential distribution with parameter $\lambda > 0$. Let $Y = \left(X - \frac{1}{\lambda}\right)^2$. Find the pdf of Y.

6. Let X be the life length of an electron tube and suppose that X may be represented as a continuous random variable which is exponentially distributed with parameter λ. Let $p_j = P(j \leq X < j+1)$. Show that p_j is of the form $(1-\alpha)\alpha^j$ and determine α.

   $P_j = \int_j^{j+1} \lambda e^{-\lambda x} dx = \frac{\lambda}{-\lambda}\left(e^{-\lambda(j+1)} - e^{-\lambda j}\right) = e^{-\lambda j} - e^{-\lambda j - \lambda} = (e^{-\lambda})^j(1-e^{-\lambda})$

7. Consider the marks of MAL 250 examination. Suppose that marks are distributed normally with mean 76 and standard deviation 15. 15% of the best students obtained A as grade and 10% of the worst students fail in the course. (a) Find the minimum mark to obtain A as a grade. (b) Find the minimum mark to pass the course.

   a) $\phi\left(\frac{100-76}{15}\right) - \phi\left(\frac{x-76}{15}\right) = 0.15$

   $0.9452 - 0.15 = \phi\left(\frac{x-76}{15}\right)$

   $0.825 = \frac{x-76}{15}$

   $\Rightarrow x = 88.375$

   b) $\phi\left(\frac{c-76}{15}\right) - \phi\left(\frac{0-76}{15}\right) = 0.1 = \frac{c-76}{15} = -1.285 \Rightarrow c = 57.872$

8. Consider a nonlinear amplifier whose input X and output Y are related by its transfer characteristic

$$Y = \begin{cases} X^{\frac{1}{2}}, & X > 0 \\ -|X|^{\frac{1}{2}}, & X < 0 \end{cases}$$

   Find pdf of Y if X has $N(0, 1)$ distribution.

9. A point X is chosen at random in the interval $[-1, 3]$. Find the pdf of $Y = X^2$.

   $\frac{d}{dy}\int_0^{\sqrt{y}} \frac{1}{4}\, dx = \frac{1}{4} \times \frac{1}{2\sqrt{y}}$

10. Let X be a random variable with uniform distribution in the interval $(-\pi/2, \pi/2)$. Define

$$Z = \begin{cases} -1 & X \leq -\pi/4 \\ \tan(X) & -\pi/4 < X < \pi/4 \\ 1 & X \geq \pi/4. \end{cases}$$

   Find the distribution of the random variable Z.

   $z = -1$ $1/4$

   $z = 1$ $1/4$

   $f_z = \begin{cases} 0 & z < -1 \\ \frac{1}{\pi}\left(\frac{1}{1+z^2}\right) & -1 < z < 1 \\ 0 & z > 1 \end{cases}$

   $f_x = \frac{1}{\pi}$

   $\int_{-\pi/4}^{\tan^{-1} z} \frac{1}{\pi}\, dx \Rightarrow \frac{1}{\pi}\frac{1}{(1+z^2)}$

11. Find the probability distribution of a binomial random variable X with parameter n, p, truncated to the right at $X = r, r > 0$.

12. Find pdf of a doubly truncated normal $N(\mu, \sigma^2)$ random variable, truncated to the left at $X = \alpha$ and to the right at $X = \beta$.

13. State True or False with valid reasons for the following statements.

    (a) Let X be a discrete random variable with taking values $\frac{3^k}{2^k}$, $k = 0, 1, \ldots$ and such that $P(X = \frac{3^k}{2^k}) = \frac{1}{2^{k+1}}$. $Var(X)$ exists. F

    (b) The MGF of a discrete random variable Y is given by $M_Y(t) = \frac{1}{10}e^{-3t} + \frac{1}{5}e^{-t} + \frac{2}{5} + \frac{3}{10}e^{2t}$. T

    (c) If the characteristic function of a random variable W is $\varphi_W(t) = e^{4t}$, then $P(1 < W \leq 5) = \frac{1}{4}$. F

a) $E(x) = \sum \frac{1}{2}\left(\frac{3}{4}\right)^k$

$E(x^2) = \sum \frac{1}{2}\left(\frac{9}{4}\right)^k \times \frac{1}{2^k}$

$= \sum \frac{1}{2}\left(\frac{9}{8}\right)^k$ does not exist.

14. Prove that for any random variable X, $E[X^2] \geq [E[X]]^2$. Discuss the nature of X when one have equality?

15. Suppose that two teams are plying a series of games, each of which is independently won by team A with probability 0.5 and by team B with probability 0.5. The winner of the series is the first team to win four games. Find the expected number of games that are played. $4 \times (0.5)^4 \times 2 + 5 \times (0.5)^5 \times 2 \times {}^4C_3 + 6 \times \ldots$

16. Let X be a random variable having a Poisson distribution with parameter λ. Prove that, for $n = 1, 2, \ldots$

$$E[X^n] = \lambda E[(X+1)^{n-1}].$$

17. A certain alloy is formed by combining the melted mixture of two metals. The resulting alloy contains a certain percent of lead, say X, which may be considered as a random variable. Suppose that X has the following pdf

$$f(x) = \begin{cases} 6 * 10^{-6} x(100 - x) & 0 \leq x \leq 100 \\ 0 & \text{otherwise} \end{cases}.$$

Suppose that P, the net profit realised in selling this alloy per pound, is the following function of the percent content is lead: $P = C_1 + C_2 X$. Compute the expected profit (per pound). Also find the variance of the profit P.

18. Let X be a random variable having a binomial distribution with parameters n and p. Prove that

$$E\left(\frac{1}{X+1}\right) = \frac{1 - (1-p)^{n+1}}{(n+1)p}.$$

19. Let X be a continuous random variable with CDF $F_X(x)$. Define $Y = F_X(X)$.

    (a) Find the distribution of Y.

    (b) Find the variance of Y, if it exist?

20. Suppose that X is a continuous random variable having the following pdf:

$$f(x) = \begin{cases} \frac{e^x}{2}, & x \leq 0 \\ \frac{e^{-x}}{2}, & x > 0. \end{cases}$$

Let $Y = |X|$. Obtain $E(Y)$ and $Var(Y)$.

21. The mgf of a r.v. X is given by $M_X(t) = exp(\mu(e^t - 1))$.
(a) What is the distribution of X? (b) Find $P(\mu - 2\sigma < X < \mu + 2\sigma)$, given $\mu = 4$.

22. Let X be exponentially distributed random variable with parameter $\lambda > 0$.

    (a) Find $P\left(| X - 1 |> 1 \mid X > 1\right)$

    (b) Explain whether there exists a random variable $Y = g(X)$ such that the cumulative distribution function of Y has uncountably many discontinuity points. Justify your answer.

23. The moment generating function of a discrete random variable X is given by $M_X(t) = \frac{1}{6} + \frac{1}{2}e^{-t} + \frac{1}{3}e^t$. If μ is the mean and σ^2 is the variance of this random variable, find $P(\mu - \sigma < X < \mu + \sigma)$.

24. Using MGF, find the limit of Binomial distribution with parameters n and p as $n \to \infty$ such that $np = \lambda$ so that $p \to 0$.

25. Assume that, taxis are waiting in a queue for passengers to come. Passengers for these taxis arrive according to a Poisson process with an average of 60 passengers per hour. A taxi departs as soon as two passengers have been collected or 3 minutes have expired since the first passenger has got in the taxi. Suppose you get in the taxi as first passenger. What is the distribution of your waiting time for the departure? Also, find its variance.

24) $MGF(B(n,p)) = (pe^t + 1 - p)^n$
$MGF(P(\lambda)) = e^{\lambda(e^t - 1)}$

$$e^{\frac{\ln(pe^t+1-p)}{p/\lambda}}$$

$$e^{\frac{e^t-1}{(pe^t+1-p)^{\frac{1}{\lambda}}}} \qquad a = e^{\ln a}$$

$$= e^{\lambda(e^t-1)}$$

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Answer for Selected Problems

1. (i) $$P[\mid X \mid = k] = \begin{cases} \frac{1}{2n+1}, & k = 0 \\ \frac{2}{2n+1}, & k = \pm 1, \pm 2, \ldots, \pm n \\ 0, & \text{otherwise} \end{cases}$$

   (ii) $$P[X^2 = k] = \begin{cases} \frac{1}{2n+1}, & k = 0 \\ \frac{2}{2n+1}, & k = 1^2, 2^2, \ldots, n^2 \\ 0, & \text{otherwise} \end{cases}$$

   (iii) $$P[\tfrac{1}{|X|+1} = k] = \begin{cases} \frac{1}{2n+1}, & k = 1 \\ \frac{2}{2n+1}, & k = \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}$$

2. (i) $f_Y(y) = \frac{1}{|b|\sqrt{2}\pi\sigma} e^{\frac{-1}{2}\left(\frac{y-(a+\mu b)}{b\sigma}\right)^2}, y \epsilon R$

   (ii) $$f_Z(z) = \begin{cases} \frac{1}{2} \times \frac{1}{\sqrt{2\pi z}} e^{\frac{-z}{2}}, & z > 0 \\ 0, & \text{otherwise} \end{cases}$$

4. $f_Y(y) \; = \; \frac{\alpha e^y}{(e^y+1)^{\alpha+1}}, \qquad -\infty < y < \infty$

5. $f_Y(y) \; = \; \frac{\lambda e^{-\lambda(\sqrt{y}+\frac{1}{\lambda})}}{2\sqrt{y}}, \qquad y > 0$

7. (a) 91.6% (b)56.8%

8. $f_Y(y) = \sqrt{\frac{2}{\pi}} \mid y \mid e^{-\frac{1}{2}y^4}, \qquad -\infty < y < \infty$

9. $$f_Y(y) = \begin{cases} \frac{1}{4\sqrt{y}}, & 0 < y < 1 \\ \frac{1}{8\sqrt{y}}, & 1 < y < 9 \\ 0, & \text{otherwise} \end{cases}$$

10. Z has mixed type distribution where p.m.f. is given by

$$P[Z = z] = \begin{cases} \frac{1}{4}, & z = -1, 1 \\ 0, & \text{otherwise} \end{cases}$$

and density function given by

$$f_Z(z) = \begin{cases} \frac{1}{(1+z^2)}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

11. $P[X = x] = \dfrac{\binom{n}{x} p^x q^{n-x}}{\sum_{i=0}^{r-1} \binom{n}{i} p^i q^{n-i}}, \quad x = 0, 1, \ldots, r-1$

12. $f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma(\phi(\beta)-\phi(\alpha))} exp(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2), & \alpha < x < \beta \\ 0, & \text{otherwise} \end{cases}$

13. (a)False
    (b)True
    (c)False

14. If $E(X^2) = (E(X))^2$, then X is a degenerate random variable taking a fixed value with probability 1.

15. $X =$ No. of games played, $P(X = k) = p_k(> 0), k = 4, 5, 6, 7$ $E(X) = \frac{93}{16}$

17. $E(P) = C_1 + 50C_2 \qquad Var(P) = 500C_2^2$

19. (a)$Y \sim U(0, 1)$ (b)$\frac{1}{12}$

20. $E(Y) = 1, \qquad Var(Y) = 1 \;\; where \;\; Y =| X |$

21. (a) $X \sim P(\mu)$ (b) $\sum_{k=1}^{7} \frac{e^{-\mu}\mu^k}{k!}; \;\; \mu = 4$

22. (a) $e^{-\lambda}$

23. $P(-1.062 < X < 0.73) = \frac{2}{3}$

25. $X =$Waiting time

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad P(X = 3) = e^{-3\lambda}$$

2) $f_x = \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2 \cdot \frac{1}{2}}$

$Y = a + bX$

$\sigma_y = b\sigma_x$

$\mu_y = a + b\mu_x$

$\frac{1}{b\sqrt{2\pi}\sigma} e^{-\left(\frac{y-(a+b\mu)}{b\sigma}\right)^2 \cdot \frac{1}{2}}$

$Z = \left(\frac{X-\mu}{\sigma}\right)^2$

$X < \sigma\sqrt{Z} + \mu$

$F_Z(z) = \int_0^{\sigma\sqrt{z}+\mu} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2} dx$

$f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\sigma\sqrt{z}-\mu+\mu}{\sigma}\right)^2} \left[\frac{\sigma}{2\sqrt{z}}\right]$

3) $Y = a + (b-a)X$

↳ Uniformly dist over a & b

4) $f(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$

$F(y) \int_0^{\theta e^y} \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} dx$

$\Rightarrow f_y(y) = \frac{\alpha\theta^\alpha}{(\theta e^y + \theta)^{\alpha+1}} \theta e^y$

$= \frac{\alpha e^y}{(e^y+1)^{\alpha+1}}$

5) X:

$f(x) = \lambda e^{-\lambda x}$

$Y = \left(X - \frac{1}{\lambda}\right)^2$

$F_y = \int_0^{\sqrt{y}+1/\lambda} \lambda e^{-\lambda x} dx$ $\Rightarrow f_y = \frac{1}{2}\lambda e^{-\lambda(\sqrt{y} + 1/\lambda)} \frac{1}{\sqrt{y}}$

$\cancel{F_y = \frac{\lambda}{-\lambda}\left(-1 + e^{-\lambda(y^2+\frac{1}{\lambda})}\right)}$

$\cancel{f_y = \frac{d}{dy}\left(1 - e^{-\lambda(y^2+\frac{1}{\lambda})}\right)}$

$\cancel{= +e^{-\lambda(y^2+\frac{1}{\lambda})}(+2\lambda y)}$

$\cancel{= 2\lambda y\, e^{-\lambda(y^2+\frac{1}{\lambda})}}$

12) $Y = X / X \in [a, b]$

$$\begin{cases} 0 & y < a \\ P(x \le y / X \in [a,b]) & a \le y < b \\ 1 & b \le y \end{cases}$$

$$P(X \le y \ / X \in [a,b]) = \frac{P(X \le y,\ a \le X \le b)}{P(a \le X \le b)}$$

$$= \frac{P(a < X \le y)}{P(a \le X \le b)}$$

$$= \frac{\phi(\frac{y-\mu}{\sigma}) - \phi(\frac{a-\mu}{\sigma})}{\phi(\frac{b-\mu}{\sigma}) - \phi(\frac{a-\mu}{\sigma})}$$

$$\Rightarrow f_y = \begin{cases} 0 \\ \frac{1}{D} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{y-\mu}{\sigma})^2} \\ 0 \end{cases}$$

16) $X \sim P(\lambda)$

$p(x) = \frac{e^{-\lambda} \lambda^x}{x!}$

~~$E[X^n] = \lambda E[(x+1)^{n-1}]$~~

~~for $n = 1$, $E(x) = \lambda$~~

~~assume $E[X^n] = \lambda E[(X+1)^{n-1}]$~~

~~T.P. $E[X^{n+1}] = \lambda E[(X+1)^n]$~~

$$\sum \frac{x^n e^{-\lambda} \lambda^x}{x!}$$

$$= \lambda \sum \frac{x^{n-1}}{x-1!} e^{-\lambda} \lambda^{x-1}$$

$$= \lambda \sum \frac{(x+1)^{n-1} e^{-\lambda} \lambda^x}{x!}$$

$$= \lambda E((x+1)^{n-1})$$

17) $f(x) = \begin{cases} 6 \times 10^{-6} x (100 - x) & 0 \le x \le 100 \\ 0 & ow \end{cases}$

$$E(x) = \int_0^{100} 6 \times 10^{-6} x^2 (100 - x)\, dx$$

$$= 6 \times 10^{-6} \left(100 \times \frac{10^6}{3} - \frac{10^8}{4}\right)$$

$$= 6\times10^{-6} \times \frac{10^8}{12}$$

$$= 50$$

$$E(X^2) = \int_0^{100} 6\times10^{-6}\, x^3(100-x)\,dx$$

$$= 6\times10^{-6}\left[100\times\frac{10^8}{4} - \frac{10^{10}}{5}\right)$$

$$= 6\times 10^4 \times \frac{1}{20}$$

$$= 3000$$

$$E_p[X] = c_1 + c_2 50$$

$$E_p[X^2] = c_1^2 + c_2^2 3000 + 100c_1c_2$$

$$\sigma^2 = 500c_2^2$$

18) $X \sim B(n,p)$

$$E\left(\frac{1}{X+1}\right) = \sum_{k=0}^{n} \frac{1}{k+1}\, {}^nC_k\, p^k (1-p)^{n-k}$$

$$= \frac{1}{p(n+1)}\sum_{k=0}^{n} \frac{n+1!}{(k+1)!\,(n-k)!}\, p^{k+1}(1-p)^{n-k}$$

$$= \frac{1}{p(n+1)}\left[(p+1-p)^{n+1} - (1-p)^{n+1}\right]$$

$$= \frac{1-(1-p)^{n+1}}{p(n+1)}$$

21) $M_X(t) = e^{\mu(e^t-1)}$

$\Rightarrow X \sim P(\mu)$

$$P(x=X) = \frac{e^{-\mu}\mu^x}{x!}$$

$$E(X) = \mu$$

$$E(X^2) = \mu\, E[(X+1)]$$

$$= \mu[\mu+1] = \mu^2+\mu$$

$\Rightarrow \sigma^2 = \mu.$

$\Rightarrow \sigma = 2$

$\mu = 4$

$$P(0 < X < 8)$$

$$= \sum_{i=1}^{7} \frac{e^{-4}4^i}{\underline{|i}}$$

25) $\lambda = \frac{60}{60}$

$3 \text{ minute} = t$

$\Rightarrow$ exp. dist $\Rightarrow 3e^{-3\lambda}$