Name:

COL202: Quiz-3

Maximum marks: 40

Kerberos id:

**Instructions.**

1. For each problem, you will receive 10% marks for leaving it blank. However, there will be penalty for submitting bogus proofs.
2. Please write your proofs clearly (marks will be deducted for skipping steps).
3. Clearly mark whether you have attempted the problem or not (put a tick next to "Attempted" in case you have attempted one or more parts of the problem).

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**Question 1: $n$ sticks in a line (10 marks).** There are $n$ sticks of distinct heights, arranged in a line in a uniformly random order. Suppose we look from the left. Let $X$ denote the number of sticks visible. Compute $\mathbb{E}[X]$.

**Question 2: Random Permutations (10 marks)** Let $\sigma$ be a uniformly random permutation over $\{1, 2, \ldots, n\}$. Let $A = (\sigma(1), \sigma(2), \ldots, \sigma(n))$. We are interested in the length of the longest increasing subsequence in $A$. For instance, if $\sigma(1) = 3$, $\sigma(2) = 5$, $\sigma(3) = 4$, $\sigma(4) = 1$, $\sigma(5) = 2$, then $A = (3, 5, 4, 1, 2)$ and the length of the longest increasing subsequence in $A$ is 2.

Let $X$ be a random variable denoting the length of the longest increasing subsequence. Prove that there exists a constant $c > 0$ such that $\mathbb{E}[X] \geq c\sqrt{n}$.

**ATTEMPTED**

**NOT ATTEMPTED**

**Question 3: Independent Events (20 marks).** Let $(\Omega, p)$ be any discrete probability distribution. Two events $A \subseteq \Omega$ and $B \subseteq \Omega$ are said to be independent if

$$\Pr\left[A \cap B\right] = \Pr[A] \cdot \Pr[B].$$

When we consider $n$ events, there are two ways to define independence:

- **(mutual independence)** $n$ events $A_1, \ldots, A_n$ are said to be mutually independent if for any subset $I \subseteq \{1, 2, \ldots, n\}$,

$$\Pr\left[\bigcap_{i \in I} A_i\right] = \prod_{i \in I} \Pr[A_i].$$

- **(pairwise independence)** $n$ events $A_1, \ldots, A_n$ are said to be pairwise independent if for any $1 \leq i < j \leq n$,

$$\Pr\left[A_i \cap A_j\right] = \Pr[A_i] \cdot \Pr[A_j].$$

1. (2 marks) Suppose we sample a uniformly random permutation $\sigma$ over $\{1, 2, \ldots, n\}$. Let $A_1$ denote the event that $\sigma(1) = 1$, and $A_2$ the event that $\sigma(2) = 2$. Are $A_1$ and $A_2$ independent events? Give a one-line justification for your answer.

2. (3 marks) Let $\Omega = \{0, 1\}^3$. Define a probability distribution over $\Omega$, and three events $A_1, A_2, A_3$ such that they are pairwise independent but not mutually independent.

3. (3 marks) Let $A \subseteq \Omega$ and $B \subseteq \Omega$ be two events, and let $\overline{B} = \Omega \setminus B$. Prove that if $A$ and $B$ are independent events, then

$$\Pr\left[\, A \cap \overline{B} \,\right] = \Pr[A] \cdot \Pr\left[\, \overline{B} \,\right].$$

4. (2 marks) Let $A_1, \ldots, A_n$ be mutually independent events over sample space $\Omega$, and let $I \subseteq \{1, 2, \ldots, n\}$ be any subset. Give an expression for the following probability in terms of $\{\Pr[A_i]\}$:

$$\Pr\left[\left(\bigcap_{i \in I} \overline{A_i}\right) \cap \left(\bigcap_{i \notin I} A_i\right)\right].$$

You only need to state the final expression (don't need to prove it).

5. (10 marks) Let $(\Omega, p)$ be a discrete probability distribution such that $\Omega$ is finite. Let $A_1, A_2, \ldots, A_n$ be mutually independent events such that $0 < \Pr[A_i] < 1$ for all $i \in [n]$. Prove that $|\Omega| \geq 2^n$.