Problem sheet – 12

Section A

Z transforms

1. **Prove properties in Table 10.1(Properties of z-transforms)**

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$ $x_1[n]$ $x_2[n]$	$X(z)$ $X_1(z)$ $X_2(z)$	R R_1 R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	z_0R
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R
10.5.9		Initial Value Theorem If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z\to\infty} X(z)$		

2. **Derive z transforms of elementary functions given in Table 10.2 (z transforms of elementary functions)**

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Section B

1. Z-transform of stable system

We are given the following five facts about a discrete-time signal x[n] with z transform X(z):

a. x[n] is real and right-sided.
b. X(z) has exactly two poles.
c. X(z) has two zeros at the origin.
d. X(z) has a pole at $z = \frac{1}{2}e^{j\pi/3}$.
e. $X(1) = 8/3$

Determine X(z) and specify its region of convergence.

2. Elementary transformation in z-transform

Let x[n] be a discrete-time signal with z-transform X(z). For each of the following signals, determine the z-transform in terms of X(z):

a) $\Delta x[n]$, where Δ is the first difference operator defined by

$$\Delta x[n] = x[n] - x[n-1]$$

b) $x_1[n] = \begin{cases} x\left[\frac{n}{2}\right], n\ even \\ 0, \quad n\ odd \end{cases}$

c) $x_2[n] = x[2n]$

3. Z-transforms and stability criterion

Determine which of the following z-transforms could be the transfer function of a discrete-time linear system that is not necessarily stable, but for which the unit sample response is zero for n < 0. State your reasons clearly.

a) $\frac{(1-z^{-1})^2}{1-\frac{1}{2}z^{-1}}$

b) $\frac{(z-1)^2}{z-\frac{1}{2}}$

c) $\frac{\left(z-\frac{1}{4}\right)^5}{\left(z-\frac{1}{2}\right)^6}$

d) $\frac{\left(z-\frac{1}{4}\right)^6}{\left(z-\frac{1}{2}\right)^5}$

4. Solution of Difference Equation from z-transform

A sequence x[n] is the output of an LTI system whose input is s[n] . The system is described by the difference equation

$$x[n] = s[n] - e^{8\alpha}s[n-8],$$

where $0 < \alpha < 1$.

(a) Find the system function

$$H_1(z) = \frac{X(z)}{S(z)}$$

, and plot its poles and zeros in the z-plane. Indicate the region of convergence.

(b) We wish to recover s[n] from x[n] with an LTI system. Find the system function

$$H_2(z) = \frac{X(z)}{S(z)}$$

such that y[n] = s[n]. Find all possible regions of convergence for $H_2(z)$, and for each, tell whether or not the system is causal or stable.

(c) Find all possible choices for the unit impulse response $h_2[n]$ such that,

$$y[n] = h_2[n] * x[n] = s[n]$$