COL 751 : Practice Sheet - 2

Note: Problems marked as $*$ are optional and much harder than other problems.

1. Distance Oracles of stretch 3
Argue that for any vertex x in an unweighted graph G (given as adjacency list representation), $BFS(x, r)$ can be computed in $O(r^2)$ time. Use this to prove that for unweighted graphs there exists a construction of distance oracle of multiplicative stretch 3 that takes $O(m\sqrt{n}\log n + n^2)$ pre-processing time.

2. Girth Conjecture
The Girth Conjecture by Erdos states that for every $k \geqslant 1$ and for sufficiently large n, there are n-vertex graphs with girth $2k + 2$ and $\Omega(n^{1+1/k})$ edges. Prove that if Girth Conjecture holds true then the greedy construction of $(2k - 1)$ spanner presented in Lecture 4 is of optimal size (up to constant factors).

3. Distance Oracles with stretch less than 3
Show that any distance oracle with multiplicative stretch strictly less than 3 for unweighted graphs takes $\Omega(n^2)$ space. (Hint: Argue that you can use such a distance oracle as black-box to identify edges of a bipartite graph $G = (A, B, E)$ satisfying $|A| = |B| = n$.)

4. Subset Distance Oracles
Let $G = (V, E)$ be an n vertex unweighted graph and $Z \subseteq V$ be a set of size k. Argue that there exists a $Z \times Z$ distance oracle with stretch 5 that takes $O\big((nk)^{2/3}\log n\big)$ space. Can you improve space further to $O(n \log n)$ for $k \leqslant n^{3/4}$? (Hint: Think simple!)

5. Distance Spanner for vertex pairs in $Z \times V$
Let $G = (V, E)$ be an n vertex unweighted graph and $Z \subseteq V$ be a set of size k. Argue that there exists a $Z \times V$ distance spanner with stretch 3 that takes $O\big(n\sqrt{k}\log n\big)$ space.

6. Distributed Systems
Let $G = (V, E)$ be a large network comprising of n nodes, where each node is associated with a local computer possessing a storage capacity of $O(\sqrt{n}\log n)$. Show that it is possible to store partial information about G in the local computers so that for any $x, y \in V$ a 3-approximation to (x, y) distance can be computed solely based on local information stored at nodes x and y.

7. Distance Oracles with (3, 2) stretch
A distance oracle is said to have (α, β) stretch if for every $x, y \in V$ it satisfies

$$dist(x, y) \;\leqslant\; \widehat{d}(x, y) \;\leqslant\; \alpha \cdot dist(x, y) + \beta.$$

Show that for n vertex unweighted graphs you can compute a distance oracle of stretch $(3, 2)$ that takes $O(n^2 \log^2 n)$ pre-processing time.

8. Distance Spanners for Directed graphs

Show that there exists n vertex digraphs for which any finite stretch distance spanner takes $\Omega(n^2)$ space.

9. Diameter Preservers

Show that for any n vertex strongly connected directed graph $G = (V, E)$ we can compute a subgraph $H = (V, E_H)$ with $O(n^{1.5} \log n)$ edges satisfying $diam(H) \leqslant \lceil 1.5\, diam(G) \rceil$.

Hint: Use the idea of hitting vertices of high (i.e. $\geqslant \sqrt{n}$) in-degree/out-degree.

10. Additive Spanners for Weighted graphs

Show that for any n vertex weighted graph G with edge weights in range $[1, W]$, it is possible to compute a +$2W$ additive spanner in $\widetilde{O}(n^2)$ time. Further, show that there exists weighted graphs with edge weights in range $[1, W]$, for which any $+W$ approximate distance oracle takes $\Omega(n^2)$ space.

11. Approximate Distance Matrix (*)

An approximate distance matrix $\hat{M}$ is said to have (α, β) stretch if for every $x, y \in V$ it satisfies

$$dist(x, y) \;\leqslant\; \hat{M}[x, y] \;\leqslant\; \alpha \cdot dist(x, y) + \beta.$$

Show that for any n vertex unweighted graph it is possible to compute in $\widetilde{O}(n^2)$ time an approximate distance matrix of stretch $\big(2, c\big)$, for some large enough constant c.