COL751 - Lecture 18

1 Edmond's Result on Edge-Disjoint Reachability Trees (aka Edge-Disjoint Arborescences)

Theorem 1 *Let $G = (V, E)$ be a digraph and let $s \in V$. Then G has k edge-disjoint reachability trees rooted at s if and only if* $\text{MAX-FLOW}(s, v, G) \geqslant k$, *for each $v \in V$.*

If G has k edge-disjoint reachability trees rooted at s then clearly for each $v \neq s$, the (s, v)-max-flow in G is at least k.

We establish the converse via a proof given by Lovasz.

Definition 1 *We say a tree T rooted at s (with one or more vertices) is* **nice** *if for each vertex $v \in V$, we have* $\text{MAX-FLOW}(s, v, G - T) \geqslant k - 1$.

Lemma 1 *A tree T is nice iff for any set $X \subsetneq V$ containing s, size of cut (X, X^c) in $G - T$ is at least $k - 1$.*

Note that it suffices to provide construction of a *nice* tree T rooted at s of size n. We will compute tree T in $n - 1$ steps, choosing one edge each time. In the beginning T is intialized as singleton vertex s.

Definition 2 *A set $X \subsetneq V$ is said to be* **critical** *if*

- *it contains source s, and*
- *size of cut (X, X^c) in graph $(G - T)$ is $k - 1$.*

Such a set is referred as *critical* because while growing tree T we cannot choose another edge from the cut (X, X^c).

Lemma 2 *For any two critical sets $X, Y \subsetneq V$ satisfying $X \cup Y \neq V$, the union $X \cup Y$ is critical as well.*

Proof: By Submodularity of cuts,

$$\delta(X) + \delta(Y) \geqslant \delta(X \cap Y) + \delta(X \cup Y). \tag{1}$$

We have $\delta(X) = \delta(Y) = k - 1$ and by Lemma 1, $\delta(X \cap Y),\ \delta(X \cup Y) \geqslant k - 1$. This implies $\delta(X \cap Y),\ \delta(X \cup Y)$ must be exactly $k - 1$. □

Now let us consider the scenario when T is a nice tree of size less than n, and let $\mathcal{R}$ be vertices in T reachable from s.

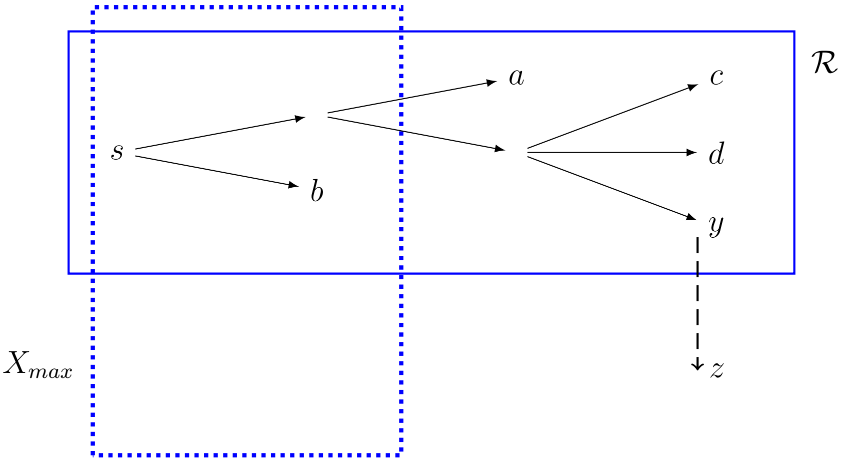

Figure 1: Depiction of sets X_{max} and $\mathcal{R}$ in Lemma 3.

Lemma 3 *Among all the critical sets whose union with $\mathcal{R}$ is not V, let X_{max} be a critical set having maximum cardinality. Then following holds.*

- *There exists an edge $(y, z) \in E$ satisfying $y \in \mathcal{R} \setminus X_{max}$ and $z \in (\mathcal{R} \cup X_{max})^c$.*
- *$T \cup (y, z)$ is a nice tree.*

Proof: Observe that $\delta(\mathcal{R} \cup X_{max}) \geqslant k$ as the corresponding cut does not contain any edges of T, and $\delta(X_{max}) = k - 1$. This implies that there must exist an edge, say (y, z), lying in the set $(\mathcal{R} \setminus X_{max}) \times (\mathcal{R} \cup X_{max})^c$.

Now let us assume on the contrary that $T \cup (y, z)$ is not a nice tree. Then there exists a critical set Y in graph $(G - T)$ satisfying $(y, z) \in (Y, Y^c)$-cut. Since X_{max}, Y are critical sets not containing z, by Lemma 2, we have $X_{max} \cup Y$ is also critical, violating the definition of X_{max}. Therefore, $T \cup (y, z)$ must be a nice tree. $\square$

Lemma 3 shows that any nice tree T with less than n nodes can be extended by adding an edge $e = (y, z)$ to obtain another nice tree. This is however an existential result. We can compute the edge by iterating over all possible edges e in $G \setminus T$ and checking if $T \cup e$ is a nice tree.

Homework Design an $O(m^2k^2)$ time algorithm to compute k edges disjoint reachability trees rooted at a vertex s in a digraph G satisfying $\text{MAX-FLOW}(s, v, G) \geqslant k$, for $v \in V$.