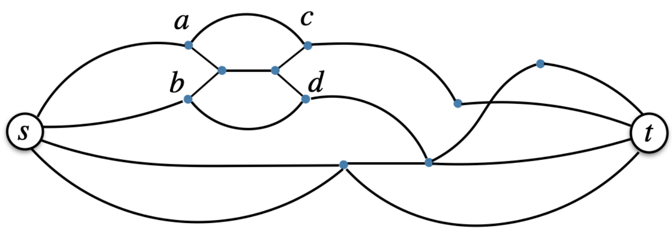
COL 751 : Practice Sheet - 3

1. SCCs of residual graph
Compute (s, t)-max-flow of the graph shown below and argue that its residual graph has exactly one non-singleton strongly connected component. Can you argue the same without actually computing the max-flow?



2. Nearest min-cut
Let $G = (V, E)$ be an undirected connected graph, $s \in V$ be a source vertex, and $t \in V$ be a sink vertex. An (s, t)-min-cut (A^*, B^*) of graph G is said to be the **nearest (s, t)-min-cut** if for each (s, t)-min-cut (A, B), we have $A^* \subseteq A$.

(i) For any (s, t)-min-cut (A, B) argue that $SCC(s, G_f) \subseteq A$, where G_f is residual graph corresponding to some max-flow f.

(ii) Show how can you use (i) to prove that there is a unique nearest (s, t)-min-cut in G.

(iii) Present an algorithm that given a max-flow f, computes nearest min-cut in $O(m + n)$ time.

3. Submodularity of cuts
Let $G = (V, E)$ be an undirected connected graph, $s \in V$ be a source vertex, $t \in V$ be a sink vertex, and G_f be residual graph corresponding to some max-flow f. Further, for any $x \in V$, let $R_{out}(x, G_f)$ be the set of vertices reachable from x in graph G_f.

(i) For any (s, t)-cut (A, B) in graph G prove that (A, B) is an (s, t)-min-cut if and only if $A = \cup_{x \in A} R_{out}(x, G_f)$.

(ii) Use result of (i) to prove that if (A_1, B_1) and (A_2, B_2) are two (s, t)-min-cuts, then $(A_1 \cap A_2, B_1 \cup B_2)$ is also an (s, t)-min-cut.

4. Min-cut oracle
Let $G = (V, E)$ be an undirected connected graph, $s \in V$ be a source vertex, $t \in V$ be a sink vertex, and λ be value of (s, t)-min-cut in G.

(i) Design an $O(n)$ sized oracle that given any query edge $e \in E$ answers whether or not e belongs to some (s, t)-min-cut in $O(1)$ time.

(ii) Design an $O(n\lambda)$ sized oracle that given any set $\mathcal{E} \subseteq E$ answers whether or not $\mathcal{E}$ belongs to some (s,t)-min-cut in $O(|\mathcal{E}|^2)$ time.

5. Menger's theorem
Prove using Max-Flow Min-Cut theorem that for any directed graph G with a source s and destination t, following holds:

(a) The size of a minimum (s,t) vertex-cut in G is equal to the maximum number of internally (s,t)-vertex-disjoint paths.

(b) The size of a minimum (s,t) edge-cut in G is equal to the maximum number of (s,t)-edge-disjoint paths.

Hint: Every vertex v in G can be split into a directed edge $(v_{in} \to v_{out})$.

6. k-vertex-connected subgraphs
Let $G = (V,E)$ be a k-vertex-connected graph, that is, it satisfies that for every $S \subseteq V$ of size at most $k-1$ the graph $G - S$ is connected.

Argue that the subgraph H obtained by recursively computing k spanning forests $T_1, \ldots, T_k$ (i.e. T_i is spanning forest of $G - (T_1 \cup \cdots \cup T_{i-1})$), and merging them may not be k-vertex-connected.

7. Deck of cards
A standard deck of cards, dealt into 13 piles of 4 cards each. Show that it is possible to select one card from each pile so that the selected cards contain exactly one card of each rank (Ace, 2, 3, ..., Queen, King).

8. Subgraph of $K_{n,n}$
Using Hall's theorem argue that any subgraph of $K_{n,n}$ with n^2-n+1 edges has a perfect matching.

9. Rooks on chess board
In a $2n \times 2n$ chess board, there are n rooks in each row as well as each column. Show that there exist a set S of $2n$ rooks such that no two rooks in set S lie in the same row or the same column.

10. Variant of Hall's theorem
Given a bipartite graph $G = (X,Y,E)$, the deficiency of G w.r.t. X is the maximum, over all subsets W of X, of the difference $|W| - |N(W)|$.

Using Hall's theorem argue that if the deficiency of a bipartite graph G is d, then G admits a matching of size at least $|X| - d$.