COL751 - Lecture 25

Let M be any maximum matching of G, $S = (s_1, \ldots, s_k)$ be the set of M-free vertices, and T be an M-*alternating tree* rooted at S. Let T_{final} be tree obtained after performing all valid odd length cycle contractions in G. Note that while constructing T_{final} we grow tree even after all valid cycle contractions. Let $V_{odd}(T_{final})$ and $V_{even}(T_{final})$ be respectively the vertices at odd and even depth in T_{final}. Further, let $V_{not}(T_{final}) = V_{not}(T)$ be those vertices in G that do not lie in T or T_{final}.

We present below some important lemmas and definitions.

Lemma 1. *Vertices in $V_{odd}(T_{final})$ are matched under every maximum matching.*

Proof: It was proved in Lecture 24 that the set corresponding to vertices in $V_{odd}(T_{final})$ is a Tutte-Berge maximizer. Now the vertices of any Tutte-Berge maximizer R are matched under every perfect matching. (Proof: Homework). This proves that vertices in $V_{odd}(T_{final})$ are matched under every maximum matching. $\square$

We next prove that vertice of G not lying in T are also matched under every optimal matching.

Lemma 2. *Vertices in $V_{not}(T_{final})$ are matched under every maximum matching.*

Proof: Let us assume on contrary that a vertex $v \in V_{not}(T_{final})$ that is free under another optimal matching M'. Consider the graph $H = (V, M \oplus M')$ which is union of vertex disjoint even-length paths and cycles.

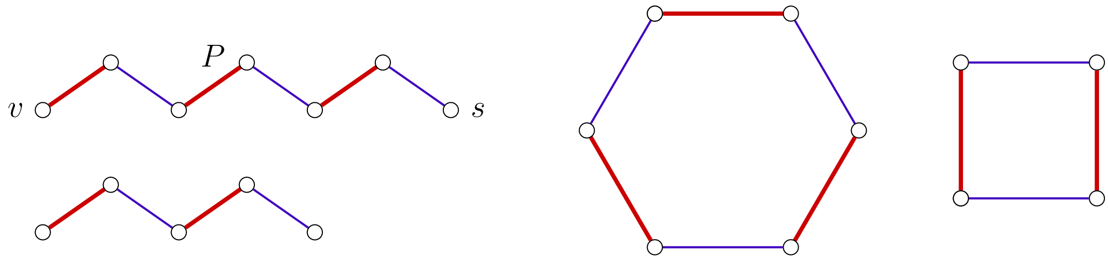

Figure 1: Depiction of paths and cycles in $M \oplus M'$. Edges in M are shown in red, and edges in alternate matching M' are shown in blue. The vertex v is M'-free.

Let P be the path in H containing v. The path P is an M-alternating path from v to an M-free vertex in G, say s. However, any path from an M-free vertex to v must contain two consecutive edges not lying in M. (Refer to the definition of an M-alternating tree in Lecture 23). This contradicts our assumption, thereby proving the claim. $\square$

Definition 1. *A graph H on n vertices is said to be factor-critical if every subgraph of H of $n-1$ vertices has a perfect matching.*

Lemma 3. *Vertices in $V_{even}(T)$ are unmatched under some maximum matching of G. Furthermore, each component of $V_{even}(T)$ is factor-critical.*

Proof: Consider the induced graph H corresponding to a supernode $h \in V_{even}(T)$. Recall that the supernode h is obtained by performing odd cycle contractions in H. It suffices to show that on removal of any vertex v from H, the resultant graph has a perfect matching. Let C be the first cycle contracted in H. The claim trivially holds if H was an odd-length cycle. By induction, we can assume claim holds for contracted graph H/C. Now let v be a vertex in H. We have following cases.

- $v \notin C$: In such a case a perfect matching of $H/C - v$ can be extended to obtain a perfect matching of $H - v$.
- $v \notin C$: Let v_C be supernode obtained after contracting cycle C in H. Then a perfect matching of $H/C - v_C$ can be extended to obtain a perfect matching of $H - v$.

This proves that vertices in $V_{even}(T)$ are free under some maximum matching of G, and the components of $V_{even}(T)$ are factor-critical. $\square$

Gallai–Edmonds decomposition

Gallai–Edmonds decomposition is a partition of the vertices of an undirected graph $G = (V, E)$ into three subsets which provides information on the structure of all maximum matchings in G.

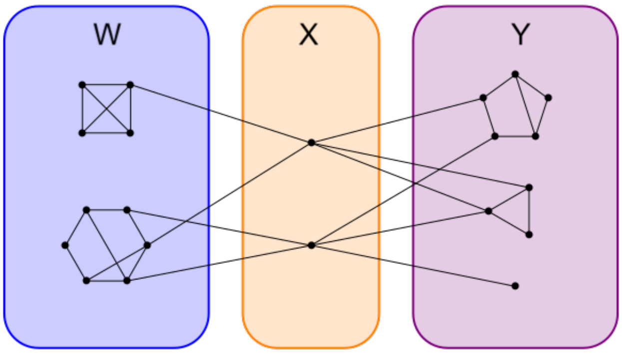

Figure 2: Gallai–Edmonds decomposition of an undirected graph G.

Theorem 1 (Gallai-Edmonds)**.** *The vertex set of any graph $G = (V, E)$ can be decomposed into sets W, X, Y such that the following holds:*

1. *Every maximum matching M_{opt} in G has the following structure:*
   - *M_{opt} restricted to $G[W]$ is a perfect matching.*
   - *M_{opt} restricted to each component of $G[Y]$ is a near-perfect matching.*
   - *Each node in X is connected to a component in $G[Y]$ using an M_{opt}-edge. Also, no two nodes in X are matched to vertices in the same component of $G[Y]$.*

2. *Components of induced graph $G[Y]$ are factor-critical graphs, and induced graph $G[W]$ has a perfect matching.*

3. *If $G[Y]$ has k components, then $def(G) = k - |X|$.*

Proof: Let Y be those vertices in G that unmatched under at least one maximum matching, X be the neighbors of Y, and W be $V \setminus (X \cup Y)$. By Lemma 1-3, it can be argued that following holds true (why?).

$$W = V_{not}(T), \quad X = V_{odd}(T), \quad \text{and} \quad Y = V_{even}(T).$$

Also Claim 1 and Claim 2 follows from Lemma 1-3. Finally, $def(G) = oc(G \setminus X) - |X|$ as X is a Tutte-Berge maximizer. Since $k = oc(G \setminus X)$, the last claim also holds true. □