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Problem 1
Let us introduce a new connective $\leftrightarrow$ which should abbreviate $\phi \to \psi \ \wedge \ \psi \to \phi$. Design introduction and elimination rules (like the ones we had in natural deduction) for $\leftrightarrow$ and show that they are derived rules.

Solution: The introduction and elimination rules for double implication can be designed as follows:

Introduction Rule:

$$\frac{\begin{array}{cc}[\phi] & [\psi] \\ \psi & \phi\end{array}}{\phi \leftrightarrow \psi} \leftrightarrow_i$$

Elimination Rules

$$\frac{\phi \leftrightarrow \psi \quad \phi}{\psi} \leftrightarrow_{e_1}$$

$$\frac{\phi \leftrightarrow \psi \quad \psi}{\phi} \leftrightarrow_{e_2}$$

$$\frac{\phi \leftrightarrow \psi \quad \neg\phi}{\neg\psi} \leftrightarrow_{e_3}$$

$$\frac{\phi \leftrightarrow \psi \quad \neg\psi}{\neg\phi} \leftrightarrow_{e_4}$$

We will now prove that these rules are derived rules. First, we prove that $\leftrightarrow_i$ is derived.

1.	ϕ	assumption
2.	ψ	premise
3.	$\phi \to \psi$	$\to_i$ 1-2
4.	ψ	assumption
5.	ϕ	premise
6.	$\psi \to \phi$	$\to_i$ 4-5
7.	$(\phi \to \psi) \wedge (\psi \to \phi)$	$\wedge_i$ 3, 6

Since $\phi \leftrightarrow \psi$ abbreviates $(\phi \to \psi) \wedge (\psi \to \phi)$, therefore, $\leftrightarrow_i$ is derived. Now, we will prove that $\leftrightarrow_{e_1}$ and $\leftrightarrow_{e_3}$ are also derived rules. The other two follow without loss of generality. First: $\leftrightarrow_{e_1}$:

1.	$(\phi \to \psi) \wedge (\psi \to \phi)$	premise
2.	ϕ	premise
3.	$\phi \to \psi$	$\wedge_{e_1}$ 1
4.	ψ	MP(3,2)

Now, $\leftrightarrow_{e_3}$:

1.	$(\phi \to \psi) \wedge (\psi \to \phi)$	premise
2.	$\neg\phi$	premise
3.	$\phi \to \psi$	$\wedge_{e_1}$ 1
4.	$\neg\psi$	MT(3,2)

We can get the other two rules by using $\wedge_{e_2}$ instead of $\wedge_{e_1}$ in the above proofs. Hence proved that the rules for $\leftrightarrow$ are derived.

Problem 2
Prove the validity of the following sequents using natural deduction proof rules:
(a) $(p \to r) \land (q \to r) \vdash p \land q \to r$
(b) $p \to q \land r \vdash (p \to q) \land (p \to r)$

Solution:
(a)

1.	$(p \to r) \land (q \to r)$	premise
2.	$(p \to r)$	$\land_{e_1} 1$
3.	$p \land q$	assumption
4.	p	$\land_{e_1}$ 3
5.	r	MP(2,4)
6.	$p \land q \to r$	$\to_i$ 3-5

(b)

1.	$p \to q \land r$	premise
2.	p	assumption
3.	$q \land r$	MP(1,2)
4.	q	$\land_{e_1}$ 3
5.	r	$\land_{e_2}$ 3
6.	$p \to q$	$\to_i$ 2-4
7.	$p \to r$	$\to_i$ 2-5
8.	$(p \to q) \land (p \to r)$	$\land_i$ 6, 7

Problem 3
An adequate set of connectives for propositional logic is a set such that for every formula of propositional logic there is an equivalent formula with only connectives from that set. For example, $\{\neg, \lor\}$ is adequate. Is $\{\neg, \leftrightarrow\}$ adequate? Justify your answer.

Solution: We will approach this problem as follows: We will show that any boolean formula build using $\{\neg, \leftrightarrow\}$ will have a certain property which a general formula in propositional logic will not have. This will prove the inadequacy of this set of connectives. We make the following claim.

Claim 1. *Any boolean formula ϕ using both and only the atoms p, q and connectives from the set $\{\neg, \leftrightarrow\}$ satisfies that the truth table for formula ϕ has even number of rows with truth value T.*

Proof. We will prove this by induction on the length of the formula ϕ. Note that the minimum length of a formula involving two connectives p, q is 3, i.e. the formula $p \leftrightarrow q$. It is simple to observe that the truth table has two rows with the truth value T. Therefore the before case is proven.

In the inductive step, suppose all formulas with length $\leq n$ have an even number of rows with truth value T. Consider a formula ϕ of length n. Since we know that we only have the connectives $\{\neg, \leftrightarrow\}$, the formula ϕ can either be:

1. $\phi = \neg\psi$: By the inductive hypothesis, we know ψ has an even number of rows with truth value T. Since the number of rows in the truth table is also even, we must know that ψ has an even number of rows with truth value F. By semantics of $\neg$, we know that the number of rows with truth value T for $\neg\psi$ is the number of rows with truth value F for ψ, which is even.

2. $\phi = \psi \leftrightarrow \xi$: By the inductive hypothesis, we know ψ and ξ have an even number of rows with truth value T. It is useful to recall that there are *only 4 rows* in the truth table of ϕ since there are only two atoms p, q. Further, by semantics of $\leftrightarrow$, ϕ will be T wherever ψ and ξ are *both T* or *both F*. Now there are a few cases:

   - **Case I:** Both ψ and ξ have 0 or 4 rows with T. Then ϕ has T in all four rows.
   - **Case I:** ψ has 0 rows with T and ξ has 4 rows with T. Then ϕ has T in zero rows. The symmetric case follows.
   - **Case III:** ψ has 2 rows with T and ξ has 0 with T. Then ϕ has T in the 2 rows in which both ψ and ξ are F. The symmetric case follows.
   - **Case IV:** ψ has 2 rows with T and ξ has 4 with T. Then ϕ has T in the 2 rows in which both ψ and ξ are T. The symmetric case follows.
   - **Case V:** Both ψ and ξ have 2 rows each with T, and the two rows are exactly the same or totally disjoint. In this case, it is easy to see that ϕ will have 2 or 0 rows with T respectively (using the semantics of $\leftrightarrow$).
   - **Case VI:** Both ψ and ξ have 2 rows each with T, and if there is exactly one row in common where both have T, i.e. $\phi = T, \xi = T$. Then, there *must* exist a row in where both have a F. This is because, two of the other rows could possibly look like $\phi = T, \xi = F$ and $\phi = F, \xi = T$. However, the 4th row cannot contain any more Ts (since both ψ and ξ have only two rows with T). So, there are two rows, one in which both ψ and ξ are T, and one in which both are F. So, by semantics of $\leftrightarrow$ there are exactly 2 rows in the truth table of ϕ which are T.

Thus, the claim is proven. Further, note that formulas of propositonal logic such as $p \vee q$ and $p \wedge q$ have T in an odd number of rows. Thus, these cannot be represented using the connectives $\{\neg, \leftrightarrow\}$ and thus they are inadequate.

Problem 4
Show that the following sequents are not valid by finding a valuation in which the truth values of the formulas to the left of $\vdash$ are T and the truth value of the formula to the right of $\vdash$ is F.
(a) $\neg r \to (p \vee q), r \wedge \neg q \vdash r \to q$
(b) $p \to (q \to r) \vdash p \to (r \to q)$

Solution:
(a) The valuation is: $p = F, q = F, r = T$. Then, $\neg r \to (p \vee q) = F \to F \vee F = T$, and $r \wedge \neg q = T \wedge T = T$. For the RHS, $r \to q = T \to F = F$.
(b) The valuation is: $p = T, q = F, r = T$. Then, the LHS is $p \to (q \to r) = T \to (F \to T) = T \to T = T$, and the RHS is $p \to (r \to q) = T \to (T \to F) = T \to F = F$.

Problem 5
Let X be a set of propositional logic formulas. X is said to be a finitely satisfiable set (FSS) if every $Y \subseteq_{fin} X$ is satisfiable. Equivalently, X is an FSS if there is no finite subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of X such that $\neg(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n)$ is valid. (Note that if X is an FSS we are not promised a single valuation v which satisfies every finite subset of X. Each finite subset could be satisfied by a different valuation.)
Show that:
(a) Every FSS can be extended to a maximal FSS.
(b) If X is a maximal FSS then for every formula $\alpha, \alpha \in X$ iff $\neg\alpha \notin X$.
(c) If X is a maximal FSS then for all formulas α, β, $\alpha \vee \beta \in X$ iff $\alpha \in X \vee \beta \in X$.
(d) Every maximal FSS X generates a valuation v_X such that for every formula α, $v_X \models \alpha$ iff $\alpha \in X$.

From these facts, conclude that:

(e) Any FSS X is simultaneously satisfiable (that is, for any FSS X, there exists v_X such that $v_X \models X$)
(f) For all X and all α, $X \models \alpha$ iff there exists $Y \subseteq_{fin} X$ such that $Y \models \alpha$.

Solution:
(a) Let X be an arbitrary FSS. Let $q_0, q_1, q_2, \ldots$ be an enumeration of Φ, as defined in class. We define an infinite sequence of sets $X_0, X_1, X_2, \ldots$ as follows: $X_0 = X$, and for $i \geq 0$,

$$X_{i+1} = \begin{cases} X_i \cup \{q_i\} & \text{if } X_i \cup \{q_i\} \text{ is an FSS} \\ X_i & \text{otherwise} \end{cases}$$

We observe that each set in this sequence of sets is an FSS, by construction. Now let us define $W = \bigcup_{i \geq 0} X_i$. We claim that W is a maximal FSS extending X. We prove this as follows:

- If W is not an FSS, then there is a subset $Z \subseteq_{\text{fin}} W$ which is not satisfiable. Let $|Z| = n$ and let $Z = \{p_1, p_2, \ldots, p_n\}$. We can write Z as $\{q_{i_1}, q_{i_2}, \ldots, q_{i_n}\}$ where the indices correspond to our enumeration of Φ. Then we see that $Z \subseteq_{\text{fin}} X_{j+1}$ in the sequence $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq W$, where $j = \max(i_1, i_2, \ldots, i_n)$. This implies that X_{j+1} is not an FSS, which violates our construction. Therefore W is an FSS.

- Now we show that W is maximal. Suppose it is not, i.e. $W \cup \{p\}$ is an FSS for some formula $p \notin W$. Let $p = q_j$ in our enumeration of Φ. Since $q_j \notin W, q_j$ was not added at step $j+1$ in our construction. This means that $X_j \cup \{q_j\}$ is not an FSS. In other words, there exists $Z \subseteq_{\text{fin}} X_j$ such that $Z \cup \{q_j\}$ is not an FSS, or exists a finite subset which is unsatisfiable. Since $X_j \subseteq W$, we must have $Z \subseteq_{fin} W$ as well, which contradicts the assumption that $W \cup \{q_j\}$ is an FSS, since there exists a finite unsatisfiable subset. So, W must be maximal. ■

(b)
[$\Longleftarrow$] Note that $\{\alpha, \neg\alpha\}$ is not satisfiable. Therefore, $\{\alpha, \neg\alpha\} \nsubseteq X$. Therefore, $\alpha \in X$ only if $\neg\alpha \notin X$.

[$\Longrightarrow$] Suppose that a maximal FSS X does not contain either of α and $\neg\alpha$. Therefore, $X \cup \{\alpha\}$ and $X \cup \{\neg\alpha\}$ both have finite subsets which are unsatisfiable, and therefore inconsistent. Let these finite subsets be named P and Q respectively. Note that P must contain α, since if not, then $P \setminus \{\alpha\} \subseteq_{fin} X$ itself is unsatisfiable, which contradicts the fact that X is an FSS. Symmetrically, Q must contain $\neg\alpha$.

Let $P \setminus \{\alpha\} = \{\beta_1, \beta_2, \ldots, \beta_n\}$ and $Q \setminus \{\neg\alpha\} = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$. Let $p = \beta_1 \wedge\ \beta_2 \wedge \ldots \wedge \beta_n$ and $q = \gamma_1 \wedge \gamma_2 \wedge \ldots \wedge \gamma_m$. Then, we have $\vdash \neg(\alpha \wedge p)$ and $\vdash \neg(\neg\alpha \wedge q)$. This is equivalent to $\vdash \neg\alpha \vee \neg p$ and $\vdash \neg\neg\alpha \vee \neg q$. Using the fact that $\neg a \vee b \equiv a \to b$ and $\neg\neg a \equiv a$, we get $\vdash \alpha \to \neg p$ and $\vdash \neg\alpha \to \neg q$.

Further, we have that $\vdash (\alpha \to \beta) \to ((\delta \to \gamma) \to ((\alpha \vee \delta) \to (\beta \vee \gamma)))$ (proved in class). Substituting $\alpha = \alpha, \delta = \neg\alpha$, $\beta = \neg p$ and $\gamma = \neg q$ we get $\vdash (\alpha \vee \neg\alpha) \to (\neg p \vee \neg q)$. Since $\vdash \alpha \vee \neg\alpha$, we get $\vdash \neg p \vee \neg q$. By rewriting $\vee$ in terms of $\wedge$, we can get $\neg(p \wedge q)$.

However, this implies that $(P \setminus \{\alpha\} \cup Q \setminus \{\neg\alpha\}) \subseteq_{\text{fin}} X$ is inconsistent, and therefore unsatisfiable (since $p \wedge q$ represents the formula for $P \setminus \{\alpha\} \cup Q \setminus \{\neg\alpha\}$). And since both $P \setminus \{\alpha\} \subseteq X$ and $Q \setminus \{\neg\alpha\} \subseteq X$ and so $P \setminus \{\alpha\} \cup Q \setminus \{\neg\alpha\} \subseteq X$. But this is a contradiction to the fact that X is an FSS. Therefore, X must contain atleast one of $\{\alpha, \neg\alpha\}$. ■

(c)
[$\Longrightarrow$] Suppose that $\alpha \in X \vee \beta \in X$ but $\alpha \vee \beta \notin X$. Since X is maximal, there exists a finite subset of $X \cup \{(\alpha \vee \beta)\}$ that is not satisfiable. However, note that if $\alpha \in X$. This means $X \cup \{\alpha\}$ is satisfiable, and so $X \cup \{(\alpha \vee \beta)\}$ should also be satisfiable. A similar argument for β. Therefore, if $\alpha \in X \vee \beta \in X$ then $\alpha \vee \beta \in X$

[$\Longleftarrow$] Suppose a maximal FSS X contains $\alpha \vee \beta$ but does not contain either of α and β. Therefore, $X \cup \{\alpha\}$ and $X \cup \{\beta\}$ both have finite subsets which are unsatisfiable, and therefore inconsistent. Let these finite subsets be named P and Q respectively. Note that P must contain α, since if not, then $P \setminus \{\alpha\} \subseteq_{fin} X$

itself is unsatisfiable, which contradicts the fact that X is an FSS. Symmetrically, Q must contain β.

Let $P\backslash\{\alpha\} = \{\xi_1, \xi_2, \ldots, \xi_n\}$ and $Q\backslash\{\beta\} = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$. Let $p = \xi_1 \wedge \xi_2 \wedge \ldots \wedge \xi_n$ and $q = \gamma_1 \wedge \gamma_2 \wedge \ldots \wedge \gamma_m$. Then, we have $\vdash \neg(\alpha \wedge p)$ and $\vdash \neg(\beta \wedge q)$. This is equivalent to $\vdash \neg\alpha \vee \neg p$ and $\vdash \neg\beta \vee \neg q$. Using the fact that $\neg a \vee b \equiv a \rightarrow b$, we get $\vdash \alpha \rightarrow \neg p$ and $\vdash \beta \rightarrow \neg q$.

Further, we have that $\vdash (\alpha \rightarrow \xi) \rightarrow ((\delta \rightarrow \gamma) \rightarrow ((\alpha \vee \delta) \rightarrow (\xi \vee \gamma)))$ (proved in class). Substituting $\alpha = \alpha, \delta = \beta$, $\xi = \neg p$ and $\gamma = \neg q$ we get $\vdash (\alpha \vee \beta) \rightarrow (\neg p \vee \neg q)$. By deduction theorem, we may also write $(\alpha \vee \beta) \vdash (\neg p \vee \neg q)$, or $(\alpha \vee \beta) \wedge (p \wedge q)$ is inconsistent, and therefore unsatisfiable.

However, this is exactly the formula for the set $(P\backslash\{\alpha\} \cup Q\backslash\{\neg\alpha\} \cup \{\alpha \vee \beta\})$, and since both $P\backslash\{\alpha\} \subseteq_{fin} XX$ and $Q \setminus \{\neg\alpha\} \subseteq_{fin} X$, and $\{\alpha \vee \beta\} \subseteq_{fin} X$, we have that $(P \setminus \{\alpha\} \cup Q \setminus \{\neg\alpha\} \cup \{\alpha \vee \beta\}) \subseteq_{fin} X$, and it is unsatisfiable, which is a contradiction. Therefore, X must contain atleast one of $\{\alpha, \beta\}$. ■

(d) Let us define the valuation v_X as the following. Suppose $\mathcal{P} = \{p_1, p_2 \ldots\}$ is the set of all atomic propositions. The valuation v_X has the following form:

$$p_i = \begin{cases} T & \text{if } p_i \in X \\ F & \text{otherwise} \end{cases}$$

We claim that v_X satisfies the property that $v_X \vDash \alpha$ iff $\alpha \in X$. We will prove this by induction on the length of the formula α.
Base case: α has a length of 1 , so it must be an atom, and the property follows by definition of v_X.
Inductive Step: Suppose the hypothesis is true for all formulas of length $< n$. Consider a formula α with $|\alpha| = n$. Then, there are only two structural possibilities for α:

- $\alpha = \neg\beta$: Note that $v_X \models \neg\beta$ iff $v_X \not\models \beta$. Further, by inductive hypothesis, this can occur iff $\beta \notin X$. Finally, $\beta \notin X$ iff $\beta \in X$ (by Part (b) above) and so, $v_X \models \neg\beta$ iff $\neg\beta \in X$

- $\alpha = \beta \vee \gamma$: Note that $v_X \models \beta \vee \gamma$ iff $v_X \models \beta \vee v_X \models \gamma$. Further, by inductive hypothesis, this can occur iff $\beta \in X \vee \gamma \in X$. Finally, $\beta \in X \vee \gamma \in X$ iff $\beta \vee \gamma \in X$ (by Part (c) above) and so, $v_X \models \beta \vee \gamma$ iff $\beta \vee \gamma \in X$.

(e) First, let us consider a maximal FSS X. Then, we claim that v_X as defined in Part (d) above simultaneously satisfies X. If this is not true, then there exists some $\beta \in X$ such that $v_X \not\models \beta$, however, by definition, this can occur iff $\beta \notin X$. Therefore, v_X simultaneously satisfies X.

So, for a general FSS Y, we can define the valuation $v_Y = v_{Y_{max}}$, where Y_{max} is the maximal FSS obtained by extending Y. Then v_Y simultanously satisfies Y, because it also simultaneously satisfies Y_{max}, and $Y \subseteq Y_{max}$.

(f)
[$\Longrightarrow$] If $Y \subseteq_{fin} X$ and $Y \models \alpha$ then $X \models \alpha$, since if a valuation v satisfies X then it also satisfies Y. Therefore, $Y \models \alpha \implies v \models \alpha$ for every valuation satisfying X. Therefore, $X \models \alpha$.

[$\Longleftarrow$] There are two cases we need to consider:

- X is an FSS: Let us consider the set $Z = X \cup \{\neg\alpha\}$. Suppose that Z is an FSS: Then, there must exist a valuation v which simultaneously satisfies Z, i.e. $v \models Z$. In other words, $v \models X$ and $v \models \neg\alpha$. However, since $X \models \alpha$, any valuation satisfying X must satisfy α, i.e. $v \models \alpha$. This is a contradiction, so Z must not be an FSS. Therefore, there must exist $W \subseteq_{fin} X \cup \{\neg\alpha\}$ such that W is not satisfiable, by definition of Finitely Satisfiable Sets. Further, W *must* contain $\neg\alpha$, else $Y \subseteq X$, and therefore X could not have been an FSS. So, we have that $W = Y \cup \{\neg\alpha\}$ is unsatisfiable, and therefore we get $Y \models \alpha$, as required.

- If X is not an FSS: By definition, $\exists Y \subseteq_{fin} X$ such that Y is not satisfiable, and therefore X is not satisfiable. Since no valuation satisfies Y, we can trivially say that $Y \models \alpha$.

Hence, we conclude that $X \models \alpha$ iff $\exists Y \subseteq_{fin} X$ such that $Y \models \alpha$. $\blacksquare$