Problem 1
Let $F_1 = (V, E_1)$ and $F_2 = (V, E_2)$ be any two acyclic graphs (a.k.a. forests) on the same vertex set V such that $|E_1| < |E_2|$. Prove that there exists an edge $e \in E_2 \setminus E_1$ such that $F = (V, E_1 \cup \{e\})$ is also an acyclic graph.

Solution: We first prove the following lemma:

Lemma: The number of edges in an acyclic graph $G = (V, E)$ is equal to $n - c$, where n is number of vertices and c is number of connected components.

Proof: Since the graph is acyclic, each connected component T_i is also acyclic. Let the total number of vertices in G be n, the total number of edges be e, and the number of vertices and edges in T_i be n_i and e_i respectively. Therefore, we have:

$$e_i = n_i - 1$$

$$\sum_{i=1}^{c} e_i = \sum_{i=1}^{c} (n_i - 1)$$

$$\sum_{i=1}^{c} e_i = \sum_{i=1}^{c} n_i - \sum_{i=1}^{c} 1$$

$$\boxed{e = n - c}$$

Now we proceed to prove the claim.

Proof by contradiction: Let the connected components of F_1 be $T_1, T_2, ..., T_{c_1}$. Suppose there is no edge $e \in E_2 \setminus E_1$ such that $F = (V, E_1 \cup \{e\})$ is an acyclic graph. $\Rightarrow$ There exists no edge e in $E_2 \setminus E_1$ which connects vertices from 2 different connected components of E_1, since otherwise, $F = (V, E_1 \cup \{e\})$ will be acyclic.

Note that since $e \notin E_1$ and $e \notin E_2 \backslash E_1$, therefore $e \notin E_2$. In other words, if V_i represents the vertices and W_i represents the edges of connected component T_i of F_1 respectively, there exists no edge $e \in E_2$ such that $e \notin W_i$ $\forall 1 \leq i \leq c_1$

So we must have, each connected component T_i of F_1 corresponds to at least 1 connected component in F_2, i.e. Number of connected components c_2 in $F_2 \geq$ number of connected components c_1 in F_1

Now by the previous lemma, we have $|E_1| = n - c_1$ and $|E_2| = n - c_2$. Therefore,

$$|E_2| - |E_1| = c_1 - c_2 \leq 0$$

This a contradiction to the fact that $|E_2| > |E_1|$.
Hence, proved by contradiction. ■

Problem 2
Prove that every connected graph $G = (V, E)$ has a closed walk which traverses every edge in E exactly twice. Hence, prove that every connected graph $G = (V, E)$ has a closed walk of length $2|V| - 2$ which visits every vertex in V at least once.

Solution: **(a)** *Claim:* Every connected graph $G = (V, E)$ with $|E| = n$ has a closed walk which traverses each edge in E exactly twice.

We will prove the above claim by strong induction on n.

Base Case: $n = 0$. Since the graph G is connected it must have one vertex (say v), in which case the required closed walk is simply v.

Induction Hypothesis: Let the claim be true for all $1 \leq k < n$.

Induction Step: Let us choose any edge $e = uv \in E$ of G. Consider the graph $G' = (V, E\backslash\{e\})$. Note that there are two possibilities:

1. G' is disconnected
2. G' is connected.

In both cases, we can construct the required walk.

Case 1: G' is disconnected. The graph is split into two connected components T_1 and T_2, whose vertex sets are V_1 and V_2 and edge sets are E_1 and E_2 respectively. We have, $u \in V_1$ and $v \in V_2$. Note that since $|E_1|, |E_2| < n$, by the IH there exist closed walks W_1 and W_2 of T_1 and T_2 respectively which traverse each edge of those components exactly twice.

So we construct the required walk as follows: Starting from u, follow the walk W_1 and return to u, then travel to v from u, follow the walk W_2 and return to v, then return back to u. All edges of G have been traversed twice.

Case 2: G' is connected. By the induction hypothesis, G' has a closed walk W which traverses each edge twice. Construct the required walk as follows: Starting from u, follow the closed walk W back to u. After this, travel from u to v and back. Thus, all edges of G have been traversed twice.

Hence by PMI, the claim is true, and there exists a closed walk which traverses each edge of a connected graph twice. ■

(b) For every connected graph $G = (V, E)$ $\exists$ a tree T on V which is a subgraph of G, also called the spanning tree of the graph (as discussed in the class).
We know that the total number of edges in a tree on V $= |V| - 1$.
We have already proved that every connected graph graph has a closed walk that traverse every edge exactly twice. Therefore, our tree T also has a closed walk that traverses every edge exactly twice, with the length of this walk $= 2(|V| - 1)$

Suppose this closed walk does not visit a vertex v. In that case, it will not be able to traverse any edge connected to v . But this walk traverses every edge exactly twice, hence it must visit every vertex V at least once .Thus, $\exists$ a closed walk in G(V,E) of length $2|V| - 2$ that visits every vertex at least once. ■

Problem 3
A graph is said to be *regular* if the degrees of all its vertices are the same, A matching M in a graph is said to be a *perfect matching* if for every vertex v, M contains an edge incident on v (that is, all vertices are matched by M). Prove that the edge set of every bipartite regular graph can be partitioned into perfect matchings. (Convince yourself that the claim is not true if the graph is (i) regular but not bipartite, even if it has an even number of vertices (ii) bipartite but not regular, even if the graph is connected, both sides of its bipartition contain an equal number of vertices, and the degree of every

vertex is at least d, where d is as large a constant as you want.)

Solution: Suppose the degree of each vertex in the given bipartite regular graph $G = (V, E)$ is d. Since the graph is bipartite, we partition the vertex set V into two disjoint sets L and R such that each edge has one vertex incident in L and R respectively.

Claim: $|L| = |R|$

Proof: Suppose $|L| = n_1$ and $|R| = n_2$. Therefore, the total number of edges incident on any $u \in L$ is $n_1 d$ whereas the total number of edges incident on any $v \in R$ will be $n_2 d$.
By definition, all edges of a bipartite graph have one vertex incident in L and R, therefore, $n_1 = n_2$.

Now we prove the following lemma:
Lemma: A regular bipartite graph has a perfect matching.

Proof by contradiction:
Let no such perfect matching exist. Note that if there exists a matching which matches all $u \in L$, it must be a perfect matching since $|L| = |R|$. Therefore, no such matching exists which matches all of L.

According to Hall's theorem, in a bipartite graph, if there does not exist a matching which matches all of L in a bipartite graph then $\exists$ a set S $\subseteq$ L, such that $|S| > |N(S)|$, where $N(S) = \{v \in R | \exists\ u \in L$ such that uv is an edge$\}$

Suppose $|S| = x$ and $|N(S)| = y$, with $x > y$. Since G is regular, every very vertex of S has a degree d. Thus, the total number of edges that will connect $u \in S$ and $v \in N(S)$ will be xd. Thus, the total number of edges with a vertex incident in $N(S)$ is at least xd. However, since $|N(S)| = y$, the total number of edges incident on vertices in $N(S)$ is yd. So we have, $yd \geq xd$, which contradicts that $y < x$. Hence there must exist a perfect matching.

Finally, consider the following claim:

Claim: The edge set of every bipartite regular graph can be partitioned into perfect matchings.

Proof by induction: We rephrase the claim as follows: The edge set of every bipartite regular graph with degree of each vertex d can be partitioned into perfect matchings. We will induct on d.

Base Case: $d = 1$. By the above lemma, there exists a perfect matching of G. Since there is only one possible vertex each vertex of L can map to, the edge set E itself is the perfect matching.

Induction Hypothesis: Let the claim be true for the degree of each vertex being $d-1$, for some $2 \leq d \leq |L|$.

Induction Step: Consider the claim for the degree of each vertex being d. By the above lemma, there exists a perfect matching M of G. Let the edges involved in M be E^*. Now consider the graph on the same vertex set and edge set $E \backslash E^*$. Under this edge set, the degree of each vertex is $d - 1$ and the graph is still bipartite. By the induction hypothesis, this graph can be partitioned into perfect matchings. Let the partition be P. Therefore, $P' = P \cup \{E^*\}$ forms the required partition of the edge set.

Hence, by PMI the claim is true, and the edge set of every bipartite regular graph can be partitioned into perfect matchings. ■

Problem 4
Find an expression for the number of perfect matchings in a complete graph on n vertices, and prove

your answer. Your expression must involve only a constant number of applications of only the following mathematical operators: addition, subtraction, multiplication, division, exponentiation, and factorial.

Solution: Let $N(n)$ represent the number of perfect matchings in a complete graph on n vertices. We claim that when n is even,

$$N(n) = \prod_{i=2}^{n}(i-1)$$

and 0 when n is odd.

Proof: When n is odd, no perfect matching exists, because a perfect matching partitions all vertices into pairs of two containing distinct vertices, which is not possible if total vertices are odd

When n is even. Let us prove the claim by induction on n, the number of vertices of the graph. Consider the claim,

$$p(k) : N(2k) = \prod_{i=2}^{2k}(i-1)$$

Base Case: $k = 1$. If there are only 2 vertices, there can be only one perfect matching in which both are connected, therefore $N(2) = 1$ and $p(1)$ is true.

Induction Hypothesis: Let $P(n-1)$ be true for some $n \geq 1,\ n \in \mathbb{N}$.

Induction Step: Consider $P(n)$. Let $V = \{v_1, v_2, ...v_{2n}\}$ represent the set of vertices of the graph G. Pick any vertex, say v_1. There are $2n-1$ choices (the remaining vertices) for it's partner. Now for the remaining $2n-2$ vertices of the complete graph, by the induction hypothesis, there exist $\prod_{i=2}^{2(n-1)}(i-1)$ perfect matchings. Therefore by the multiplication principle, we must have,

$$N(2n) = (2n-1) * \prod_{i=2}^{2(n-1)}(i-1) = \prod_{i=2}^{2n}(i-1)$$

Therefore, we have by PMI, $p(k)$ is true $\forall k \in \mathbb{N}$ and $N(n) = \prod_{i=2}^{n}(i-1)$ ■.

Problem 5

Consider a Delhi Metro train consisting of n seats numbered $1, \ldots, n$ carrying m (distinct) passengers. The government rules for physical distancing prevent passengers from standing in the compartment during their journey. Moreover, they must leave a gap of at least two seats between themselves: if a seat k is occupied, seats $k-2$, $k-1$, $k+1$, $k+2$ must remain vacant. Find an expression for the number of ways in which the passengers can occupy seats while following the physical distancing norms, and prove your answer. Again, your expression must involve only a constant number of applications of only the following mathematical operators: addition, subtraction, multiplication, division, exponentiation, and factorial.

Solution: Let x_1 represent the number of empty seats before the seat occupied by person 1. Let $x_2, x_3, ...x_m$ represent the gap of seats between consecutive people, and x_{m+1} be the number of empty seats after the seat occupied by person m. Then note that we have,

$$x_1 + x_2 + ... + x_m + x_{m+1} + m = n$$

Further, $x_1 \geq 0, x_2 \geq 2, x_2 \geq 2, ...\ x_m \geq 2, x_m \geq 0$. By finding the number of non-negative integral solutions to the above equation under the given constraints, we can find the number of ways in which the passengers can occupy the seats (considering passengers to be identical, we will fix this later). Let the number of solutions be N.

Replace $t_i = x_i - 2,\ 2 \leq i \leq m$ and $t_i = x_i$ otherwise, in the above equation. Therefore, it reduces to,

$$t_1 + t_2 + ... + t_{m+1} \leq n - m - 2(m-1) \leq n - 3m + 2$$

Under the constraints, $t_i \geq 0, 1 \leq i \leq m+1$. Note that finding the number of solutions of this equation under the given constraints is equivalent to adding $(n - 3m + 2)$ identical balls to $m+1$ distinct bins. By the result proved in class, this is equal to,

$$N = \binom{n - 3m + 2 + m + 1 - 1}{m+1} = \binom{n - 2m + 2}{m+1}$$

Note that the value of N obtained assumes that all m passengers are identical. In every case, there are in fact $m!$ ways to arrange the passengers in the seats. Therefore, by the multiplication principle, we have the number of ways $N^* = Nm!$ given by,

$$\boxed{N^* = \binom{n - 2m + 2}{m+1} m!}$$