Department of Mathematics
MTL 106 (Introduction to Probability and Stochastic Processes)
Tutorial Sheet No. 3

1. Let X be uniformly distributed random variable over the interval $[0, 10]$. Find the CDF of $Y = \max\{2, \min\{4, X\}\}$.

2. If X has $N(\mu, \sigma^2)$, find the distribution of $Y = a + bX$, and $Z = \left(\frac{X-\mu}{\sigma}\right)^2$.

3. Let X be uniformly distributed random variable on the interval $(0, 1)$. Define $Y = a + (b-a)X, \quad a < b$. Find the distribution of Y.

4. Let X be a random variable with pdf $f(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}, \quad x > 0$ where $\theta > 0$ and $\alpha > 0$. Find the distribution of random variable $Y = \ln\left(\frac{X}{\theta}\right)$.

5. Suppose that X is a continuous random variable with pdf $f_X(x) = e^{-x}$ for $x > 0$. Define $Y = \begin{cases} X, & X < 1 \\ \frac{1}{X}, & X \geq 1 \end{cases}$.

   (a) Discuss whether the distribution of Y is discrete or continuous or mixed type.

   (b) Determine the pmf/pdf as applicable to this case.

6. Let X be the life length of an electron tube and suppose that X may be represented as a continuous random variable which is exponentially distributed with parameter λ. Let $p_j = P(j \leq X < j+1)$. Show that p_j is of the form $(1-\alpha)\alpha^j$ and determine α.

7. Consider the marks of MTL 106 examination. Suppose that marks are distributed normally with mean 76 and standard deviation 15. 15% of the best students obtained A as grade and 10% of the worst students fail in the course. (a) Find the minimum mark to obtain A as a grade. (b) Find the minimum mark to pass the course.

8. Consider a nonlinear amplifier whose input X and output Y are related by its transfer characteristic
$$Y = \begin{cases} X^{\frac{1}{2}}, & X > 0 \\ -|X|^{\frac{1}{2}}, & X < 0 \end{cases}$$
   Find pdf of Y if X has $N(0, 1)$ distribution.

9. Let the phase X of a sine wave be uniformly distributed in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Define $Y = \sin X$. Find the distribution of Y.

10. Let X be a random variable with uniform distribution in the interval $(-\pi/2, \pi/2)$. Define
$$Z = \begin{cases} -1 & X \leq -\pi/4 \\ \tan(X) & -\pi/4 < X < \pi/4 \\ 1 & X \geq \pi/4. \end{cases}$$
    Find the distribution of the random variable Z.

11. Find the probability distribution of a binomial random variable X with parameter n, p, truncated to the right at $X = r, r > 0$.

12. Find pdf of a doubly truncated normal $N(\mu, \sigma^2)$ random variable, truncated to the left at $X = \alpha$ and to the right at $X = \beta$.

13. State True or False with valid reasons for the following statements.

    (a) Let X be a discrete random variable with taking values $\frac{3^k}{2^k},\ k = 0, 1, \ldots$ and such that $P(X = \frac{3^k}{2^k}) = \frac{1}{2^{k+1}}$. $Var(X)$ exists.

(b) The MGF of a discrete random variable Y is given by $M_Y(t) = \frac{1}{10}e^{-3t} + \frac{1}{5}e^{-t} + \frac{2}{5} + \frac{3}{10}e^{2t}$.

(c) If the characteristic function of a random variable W is $\varphi_W(t) = e^{4t}$, then $P(1 < W \leq 5) = \frac{1}{4}$.

14. Prove that for any random variable X, $E[X^2] \geq [E[X]]^2$. Discuss the nature of X when one have equality?

15. Suppose that two teams are plying a series of games, each of which is independently won by team A with probability 0.5 and by team B with probability 0.5. The winner of the series is the first team to win four games. Find the expected number of games that are played.

16. Let Φ be the characteristic function of a random variable X. Prove that $1- \mid \Phi(2u) \mid^2 \leq 4\left(1- \mid \Phi(u) \mid^2\right)$.

17. (a) Let X be a uniformly distributed random variable on the interval $[a, b]$ where $-\infty < a < b < \infty$. Find the distribution of the random variable $Y = \frac{X-\mu}{\sigma}$ where $\mu = E(X)$ and $\sigma^2 = Var(X)$. Also, find $P(-2 < Y < 2)$.

(b) Suppose Shimla's temperature is modeled as a random variable which follows normal distribution with mean 10 Celcius degrees and standard deviation 3 Celcius degrees. Find the mean if the temperature of Shimla were expressed in Fahreneit degrees.

18. Let X be a random variable having a binomial distribution with parameters n and p. Prove that

$$E\left(\frac{1}{X+1}\right) = \frac{1-(1-p)^{n+1}}{(n+1)p}.$$

19. Let X be a continuous random variable with CDF $F_X(x)$. Define $Y = F_X(X)$.

(a) Find the distribution of Y.

(b) Find the variance of Y, if it exist?

20. Consider a random variable X with $E(X) = 1$ and $E(X^2) = 1$.

(a) Find $E[(X - E(X))^4]$ if it exists.

(b) Find $P(-1/2 < X \leq 3)$ and $P(X = 0)$.

21. The mgf of a r.v. X is given by $M_X(t) = exp(\mu(e^t - 1))$.
(a) What is the distribution of X? (b) Find $P(\mu - 2\sigma < X < \mu + 2\sigma)$, given $\mu = 4$.

22. Let X be exponentially distributed random variable with parameter $\lambda > 0$.

(a) Find $P\left(\mid X - 1 \mid > 1 \mid X > 1\right)$

(b) Explain whether there exists a random variable $Y = g(X)$ such that the cumulative distribution function of Y has uncountably many discontinuity points. Justify your answer.

23. Let X be a random variable with Poisson distribution with parameter λ. Show that the characteristic function of X is $\varphi_X(t) = exp\left[\lambda(e^{it} - 1)\right]$. Hence, compute $E(X^2)$, $Var(X)$ and $E(X^3)$.

24. Let X be a random variable with $N(0, \sigma^2)$. Find the moment generating function for the random variable X. Deduce the moments of order n about zero for the random variable X from the above result.

25. The moment generating function of a discrete random variable X is given by $M_X(t) = \frac{1}{6} + \frac{1}{2}e^{-t} + \frac{1}{3}e^t$. If μ is the mean and σ^2 is the variance of this random variable, find $P(\mu - \sigma < X < \mu + \sigma)$.