COL751 - Lecture 10

Definition 1 *A matching in an undirected graph G refers to a set of edges $\mathcal{E}$ where no two edges in $\mathcal{E}$ share a common vertex.*

Definition 2 *A vertex cover in an undirected graph G is a set of vertices V_C such that every edge in G is connected to at least one vertex in V_C.*

The lemma below implies that in any graph G size of maximum matching is at most the size of minimum vertex cover.

Lemma 3 *For any matching $\mathcal{E}$ and any vertex cover V_C, we have $|\mathcal{E}| \leqslant |V_C|$.*

Proof: Each edge within the matching $\mathcal{E}$ corresponds to a distinct vertex in the vertex cover V_C implying $|\mathcal{E}| \leqslant |V_C|$. $\square$

1 König-Egervary Theorem

Theorem 4 (König-Egervary Theorem) *In a bipartite graph the maximum matching size is same as the size of minimum vertex cover.*

Let $G = (X, Y, E)$ be a bipartite graph satisfying $|X| = n$ and $|Y| = k$. We compute a directed graph $H = (V_H, E_H, c)$ on $n + k + 2$ vertices where $V_H = X \cup Y \cup \{s, t\}$.

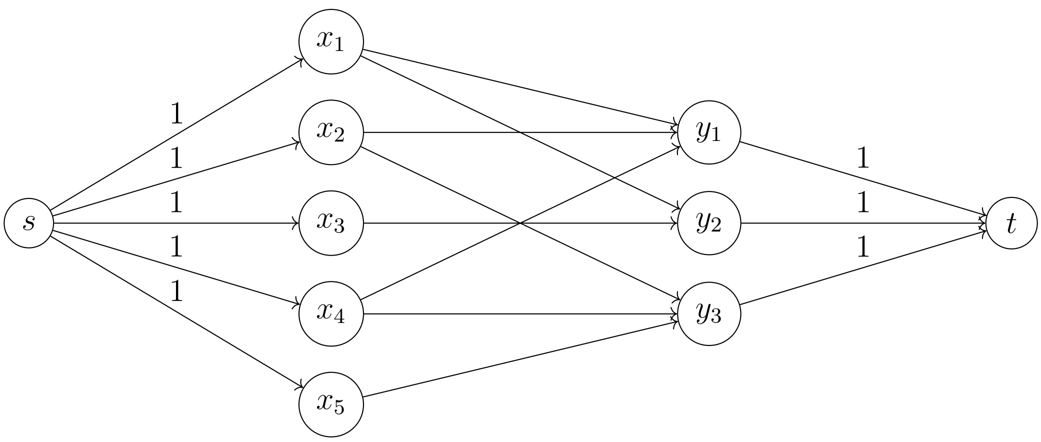

Figure 1 – A digraph H computed from bipartite graph $G = (X, Y, E)$

The edges in H, and corresponding capacities are as follows.

- For $i = 1$ to n: add edge (s, x_i) of capacity 1.
- For $j = 1$ to k: add edge (y_j, t) of capacity 1.

- For $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k$ satisfying $(x_i, y_j) \in E$: add directed edge (x_i, y_j) of capacity ∞.

Lemma 5 *The size of maximum matching in G is same as the value of (s,t)-max-flow.*

Proof: Take any integral maximum (s,t) flow f. The edges (x_i, y_j) in G carrying non-zero flow in H corresponds to a matching as at most one outgoing edge of x_j (resp. incoming edge of y_j) can carry a flow of value one or more.

Next let M be a matching in G. We can find a flow of value $|M|$ in graph H as follows: (1) for each matched x_i pass a unit flow along (s, x_i); (2) for each matched y_j pass a unit flow along (y_j, t); (3) for each $(x_i, y_j) \in M$ pass a unit flow along (x_i, y_j) in H. This corresponds to a valid flow of value $|M|$. This 1-1 correspondence proves our claim. $\square$

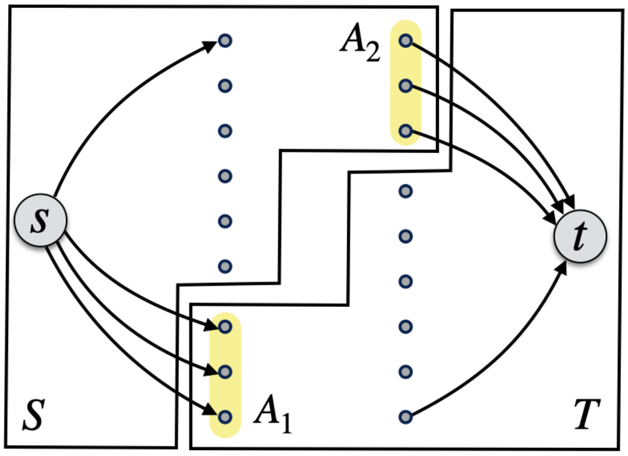

Figure 2 – Depiction of sets $A_1 = X \cap T$ and $A_2 = Y \cap S$.

Lemma 6 *Let (S,T) be an (s,t)-min-cut in H, and let $A_1 = X \cap T$ and $A_2 = Y \cap S$. Then $A_1 \cup A_2$ is a vertex cover of G. Moreover, capacity of (S,T)-cut is $|A_1| + |A_2|$.*

Proof: Observe that for each edge $(x, y) \in E \cap (X \times Y)$ at least one of following holds:

- $x \in A_1$, or
- $y \in A_2$.

This is because otherwise the (S,T)-cut contains an edge of capacity ∞, which is not possible. Thus $A_1 \cup A_2$ is a vertex cover of G.

The capacity of (S,T)-cut is $|A_1| + |A_2|$ as edges in H going from S to T lie in either $\{s\} \times A_1$ or $A_2 \times \{t\}$. $\square$

On combining Lemma 5 and Lemma 6, and using Max-Flow Min-Cut theorem, we get that $|A_1| + |A_2|$ is both the size of maximum matching as well as size of minimum vertex cover in G. This proves König-Egervary theorem.

2 Hall's Theorem

Theorem 7 (Hall's Theorem) *In a bipartite graph $G = (X, Y, E)$ satisfying that for every $W \subseteq X$ the size of W is at most the size of neighbor-set $N(W)$, there exists a matching in which every vertex of X is matched.*

Proof: The stated condition is clearly necessary. To prove it is sufficient, assume that $|N(W)| \geqslant |W|$, for all $W \subseteq X$. Consider the graph H constructed in the proof of König-Egervary theorem. By Lemma 5, it suffices to show that the value of (s,t)-max-flow (or, the size of (s,t)-min-cut) in H is $|X|$.

Assume on contrary there is an (s,t)-min-cut, say (S,T), of capacity strictly less than $|X|$. Let $A_1 = X \cap T$ and $A_2 = Y \cap S$. Then by Lemma 6, we have capacity of (S,T)-cut is $|A_1|+|A_2|$, which by our assumption is less than $|X|$. This implies $|A_2| < |X|-|A_1|$.

Let $W = X \cap S$. The set $N(W)$ is contained in A_2, as otherwise there would be an infinite-capacity edge crossing from S to T. Thus, $|N(W)| \leqslant |A_2| < |X| - |A_1| = |W|$, and we get a violation to Hall's criterion. $\square$

d-regular bipartite graph

As an application of Hall's Theorem we prove that every d-regular bipartite graph with equal partitions has a perfect matching.

Lemma 8 *For $d \geqslant 1$, every d-regular bipartite graph $G = (X, Y, E)$ has a perfect matching.*

Proof: Consider a set $W \subseteq X$. We will argue that $|N(W)| \geqslant |W|$. Observe that $\sum_{w \in W} |N(w)| = d|W|$. Now each vertex $y \in N(W)$ can have at most d neighbors, so by Pigeon hole principle, $|N(W)|$ is at least $d|W|/d = |W|$. This proves that Hall's criterion is satisfied, and thus G contains a matching in which each vertex of X (and similarly, each vertex of Y) is matched. $\square$

Latin matrix

A matrix M of size $(k \times n)$ with entries in the range $[1, n]$ satisfying $k \leqslant n$ is said to be Latin matrix if each row/column has distinct entries.

Lemma 9 *For every $k < n$, a latin matrix of size $k \times n$ can be extended to obtain a latin matrix of size $(k+1) \times n$.*

Proof: Let M be a latin matrix of size $k \times n$. Create a bipartite graph $G = (X, Y, E)$ where $X = (x_1, \ldots, x_n)$ corresponds to columns of M, and $Y = (y_1, y_2, \ldots, y_n)$ corresponds to possible deficient values. For $i, j \in [1, n]$, the edge (x_i, y_j) is added to G if and only if column C_i doesn't contain j. Now observe that G is $(n-k)$-regular (why?). Thus G contains a perfect matching. Given a perfect matching M, we append j to column C_i iff y_j is matched to x_i to obtain the latin matrix of size $(k+1) \times n$. $\square$