Department of Mathematics
Tutorial Sheet No. 5
MTL 106 (Introduction to Probability and Stochastic Processes)

1. Find $E(Y/x)$ where (X, Y) is jointly distributed with density

$$f(x,y) = \begin{cases} \frac{y}{(1+x)^4} e^{-\frac{y}{1+x}}, & x, y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

2. Let X have a beta distribution i.e. its pdf is

$$f_X(x) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} \ , \ 0 < x < 1$$

and Y given X = x has binomial distribution with parameters (n, x). Find $E(X/y)$.

3. For each fixed $\lambda > 0$, let X be a Poisson distributed random variable with parameter λ. Suppose λ itself is a random variable following a gamma distribution with pdf

$$f(\lambda) = \begin{cases} \frac{1}{\Gamma(n)} \lambda^{n-1} e^{-\lambda}, & \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$$

where n is a fixed positive constant. Find the pmf of the random variable X

4. Consider trinomial trials, where each trial independently results in outcome i with probability 1/3. With X_i equal to the number of trials that result in outcome i, find $E(X_1/X_2 > 0)$.

5. (a) Show that $cov(X, Y) = cov(X, E(Y|\ X))$.

   (b) Suppose that, for constants a and b, $E(Y|\ X) = a + bX$. Show that $b = cov(X, Y)/Var(X)$.

6. Let X be a random variable which is uniformly distributed over the interval $(0, 1)$. Let Y be chosen from interval $(0, X]$ according to the pdf

$$f(y/x) = \begin{cases} 1/x, & 0 < y \leq x \\ 0, & \text{otherwise.} \end{cases}$$

Find $E(Y^k/X)$ and $E(Y^k)$ for any fixed positive integer k.

7. Suppose that a signal X, standard normal distributed, is transmitted over a noisy channel so that the received measurement is $Y = X + W$, where W follows normal distribution with mean 0 and variance σ^2 is independent of X. Find $f_{X/y}(x/y)$ and $E(X \mid Y = y)$.

8. A real function g(x) is non-negative and satisfies the inequality $g(x) \geq b > 0$ for all $x \geq a$. Prove that for a random variable X if $E(g(X))$ exists then $P(X \geq a) \leq \frac{E(g(X))}{b}$.

9. Let X have a Poisson distribution with mean $\lambda \geq 0$, an integer. Show that $P(0 < X < 2(\lambda + 1)) \geq \frac{\lambda}{\lambda+1}$.

10. Does the random variable X exist for which $P[\mu - 2\sigma \leq X \leq \mu + 2\sigma] = 0.6$? Justify your answer.

11. The number of pages N in a fax transmission has geometric distribution with mean 4. The number of bits k in a fax page also has geometric distribution with mean 10^5 bits independent of any other page and the number of pages. Find the probability distribution of total number of bits in fax transmission.

12. Consider Bacteria reproduction by cell division. In any time t, a bacterium will either die (with probability 0.25), stay the same (with probability 0.25), or split into 2 parts (with probability 0.5). Assume bacteria act independently and identically irrespective of the time. Write down the expression for the generating function of the distribution of the size of the population at time $t = n$. Given that there are 1000 bacteria in the population at time $t = 50$, what is the expected number of bacteria at time $t = 51$.

13. Let N be a positive integer random variable and $X_1, X_2, \ldots$ be a sequence of iid random variables. N is independent of X_i's. Find the moment generating function (MGF) of $S_N = X_1 + X_2 + \ldots + X_N$, the random sum in terms of MGF of $X_i's$ and N. Also show that:
(a) $E[S_N] = E[N]E[X]$ (b) $Var[S_N] = E[N]Var[X] + [E[X]]^2 Var[N]$.

14. If $E[Y/X] = 1$, show that $Var[XY] \geq Var[X]$.

15. Suppose you participate in a chess tournament in which you play until you lose a game. Suppose you are a very average player, each game is equally likely to be a win, a loss or a tie. You collect 2 points for each win, 1 point for each tie and 0 points for each loss. The outcome of each game is independent of the outcome of every other game. Let X_i be the number of points you earn for game i and let Y equal the total number of points earned in the tournament. Find the moment generating function $M_Y(t)$ and hence compute $E(Y)$.

16. Let (X, Y) be two-dimensional random variable with joint pdf is given by

$$f(x,y) = \begin{cases} e^{-y}, & 0 < x < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

   (a) Find the conditional distribution of Y given $X = x$.
   (b) Find the regression of Y on X.
   (c) Show that variance of Y for give $X = x$ does not involve x.

17. Let $X \sim$ Bin (n, p). Use the CLT to find n such that: $P[X > n/2] \leq 1 - \alpha$. Calculate the value of n when α= 0.90 and p =0.45.

18. Suppose that 30 electronic devices say $D_1, D_2, \ldots, D_{30}$ are used in the following manner. As soon as D_1 fails, D_2 becomes operative. When D_2 fails, D_3 becomes operative etc. Assume that the time to failure of D_i is an exponentially distributed random variable with parameter $= 0.1(hour)^{-1}$. Let T be the total time of operation of the 30 devices. What is the probability that T exceeds 350 hours?

19. Let $X_1, X_2, \ldots, X_n$ be independent and $ln(X_i)$ has normal distribution $N(2i, 1)$, $i = 1, 2, \ldots, n$. Let $W = X_1^{\alpha} X_2^{2\alpha} \ldots X_n^{n\alpha}$, $\alpha > 0$ where α is any constant. Determine $E(W), Var(W)$ and the pdf of W.

20. For each $n \geq 1$, let X_n be an uniformly distributed random variable over set $\left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\right\}$. Prove that X_n convergence to $U[0,1]$ in distribution.

21. Suppose that X_i , $i = 1, 2, \ldots, 30$ are independent random variables each having a Poisson distribution with parameter 0.01. Let $S = X_1 + X_2 + \ldots + X_{30}$.
(a) Using central limit theorem evaluate $P(S \geq 3)$.
(b) Compare the answer in (a) with exact value of this probability.

22. Consider polling of n voters and record the fraction S_n of those polled who are in favour of a particular candidate. If p is the fraction of the entire voter population that supports this candidate, then $S_n = \frac{X_1 + X_2 + \ldots + X_n}{n}$, where X_i are independent Bernoulli distributed random variables with parameter p. How many voters should be sampled so that we wish our estimate S_n to be within 0.01 of p with probability at least 0.95?

23. Let $(\Omega, \Im, P) = ([0,1], \mathcal{B}(\mathbb{R}) \cap [0,1], \mathcal{U}([0,1]))$. Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of random variables with $X_n \stackrel{d}{=} \mathcal{U}\left(\left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}\right]\right)$. Prove or disprove that $X_n \xrightarrow{d} X$ with $X = \frac{1}{2}$.

24. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with mean 1 and variance 1600, and assume that these variables are non-negative. Let $Y = \sum_{k=1}^{100} X_k$.

   (a) What does Markov's inequality tell you about the probability $P(Y \geq 900)$.
   (b) Use the central limit theorem to approximate the probability $P(Y \geq 900)$. Final answer can be in terms of $\Phi(z)$ where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$.