MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Time allowed: 100 minutes **End Term Examination** **Max. Marks: 40**

Name: Entry Number: Signature:

Multiple Selection Questions: **Section 1** **($4 \times 1 = 4$ marks)**

Each of the following questions 1 to 4 has four options out of which **one option is correct.** Write A, B, C or D which corresponds to the correct option. **1 mark** is awarded if the correct answer is written, **0 mark** for no answer or any incorrect answer.

1. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. r.v.s with $P(X_1 = 1) = p = 1 - P(X_1 = -1)$. Define $S_n = \sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$. When $p > \frac{1}{2}$, which one of the following statements is TRUE?
(A) $P\left(\lim_{n\to\infty} \frac{S_n}{n} = E(X_1)\right) = 1$. (B) $P\left(\lim_{n\to\infty} S_n = \infty\right) = 0$.
(C) $P\left(\lim_{n\to\infty} \frac{S_n}{n} = E(X_1)\right) = 0$. (D) $P\left(\lim_{n\to\infty} \frac{S_n}{n} = E(X_1)\right) = 0.5$. Answer:

2. In the following, which one is not the stochastic proc ocess (B) Kolmogorov process (C) Ito process (D) Yule process Answer:

3. Consider the DTMC $\{X_n, n = 0, 1, \ldots\}$ with two states $S = \{1, 2\}$ that has the transition probability matrix $P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$. The mean recurrence time for the state 1 is
(A) 1 (B) 2 (C) 0 (D) ∞ Answer:

4. Two types of jobs, labeled 1 and 2, arrive to a processor according to independent Poisson processes with rates λ_1 and λ_2. Let $X(t)$ denote the type of the last job arrival before time t. Which one of the following statements is TRUE?
(A) $\{X(t), t \geq 0\}$ is a CTMC and is an ergodic. (B) $\{X(t), t \geq 0\}$ is a CTMC and but not ergodic. (C) $\{X(t), t \geq 0\}$ is not a CTMC. (D) $\{X(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$. Answer:

——————————Space for Rough Work ——————————

Short Answer Type Questions: **Section 2** **($3 \times 2 = 6$ marks)**

Each of the following questions 5 to 7 has four options out of which **more than one options can be correct.** Write A, B, C or D which corresponds to the correct option for the first correct answer followed by space and the next correct answer and so on. **2 marks** is awarded if all correct answers are written, **0 mark** for no answer or partial correct answers or any incorrect answer.

5. Which of the following statements are TRUE? (A) A Markov process with discrete state space is known as a Markov chain. (B) A stochastic process with independent increments satisfies the Markov property. (C) Any stochastic process which is strict sense stationary and has a finite variance is always wide-sense stationary. (D) If a Gaussian process $\{X_t, t \geq 0\}$ is wide-sense stationary, then it is strict-sense stationary. Answer:

6. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. Bernoulli distributed r.v.s with $P(X_1 = 1) = p = 1 - P(X_1 = 0)$. Define $S_n = \sum_{i=1}^{n} X_i$ be the number of successes in n trials. Which of the following statements are TRUE? (A) For fixed $n = 2, 3, \ldots$, S_n follows Binomial distribution with parameters p and n. (B) $\{S_n, n = 1, 2, \ldots\}$ is a DTMC and is a strict-sence tationary process. (C) $\{S_n, n = 1, 2, \ldots\}$ is a DTMC and but is not a strict-sence tationary process. (D) $\{S_n, n = 1, 2, \ldots\}$ is a DTMC and is a wide-sence tationary process. Answer:

7. Consider an $M/M/1$ queueing system with mean service time as $\frac{1}{\mu}$ and customers arrival following Poisson process with rate λ, where $\lambda < \mu$. Which of the following statements are NOT TRUE? (A) Steady-state probabilities follows modified geometric distribution. (B) Response time by the customer follows exponential distribution. (C) This is not a stable system. (D) Mean waiting time by the customer is $\frac{1}{\mu - \lambda}$. Answer:

Space for Rough Work

Short Answer Type Questions: **Section 3** ($5 \times 2 = 10$ **marks**)

Write the answer upto 4 decimal places (D) or in fraction (F) or the expression (E) for the following questions 8 to 12. **2 marks** are awarded if answer is correct, and **0 mark** for no answer or partial correct answer or an incorrect answer.

$\Phi(2.33) = 0.99$

8. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with mean μ and variance σ^2. Let $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find a value of n (positive integer) such that there is at least a 99% chance that the $\overline{X}$ will be within 2 standard deviations of the μ. Answer:

9. Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables each having normal distribution with mean 0 and variance σ^2. Consider the stochastic process $\{S_n, n = 1, 2, \ldots\}$ where $S_n = exp(\sum_{i=1}^{n} X_i)$. Find $E[S_n]$ for all n.

   Answer (E):

10. Let $\{X_n, n = 0, 1, \ldots\}$ be an DTMC with state space $S = \{0, 1, \ldots\}$ and TPM $P := [p_{ij}]$, $i, j \in S$. For $i \in S$, suppose $0 < p_{ii} < 1$. Let τ_i be the r.v., known as sojourn time, denoting time taken for a change from state i into another state. Find the PMF of the r.v. τ_i.

    Answer (E):

11. Consider a pure birth process $\{X(t), t \geq 0\}$ with birth rates $\lambda_0 = \lambda; \lambda_i = i\lambda, i = 1, 2, \ldots$ for some $\lambda > 0$. Find $P(X(t) = j \mid X(0) = 1)$.

    Answer (E):

12. Consider $M/M/1/N$ queueing system. Let π_i be the steady-state probability that i customers in the system. Then, the effective arrival rate for this system is given by

    Answer (E):

Space for Rough Work

Subjective Type Questions: **Section 4** $(5+6+4+5=20$ **marks)**

Write the answer in the two pages provided for the questions 13 to 16. **Full marks** are awarded if all the steps are correct, and **partial marks** for an incorrect answer with wrong steps.

13. (a) State Central Limit Theorem.

    (b) Let $Y = e^X$ where $X \sim Exp(3)$. Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. r.v.s with same the distribution as Y. Let $Z_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Derive the approximate distribution of the r.v. Z_n when n is large? (2 + 3 marks)

Solution:

14. Let X_n denote the number of visitors at IIT Delhi website at time n. Assume that, $X_0 = 0$. At each time period, each visitor at the site independently leaves with probability p, and the number of new visitors that enter the site has a Poisson distribution with mean λ (independent of everything else).

(a) Show that $\{X_n, n = 0, 1, \ldots\}$ is a DTMC and determine its one-step transition probability matrix (TPM).

(b) Find a condition on p and λ under which the chain is ergodic

(c) Find the stationary distribution of ergodic $\{X_n, n = 0, 1, \ldots\}$ (2 + 2 + 2 marks)

Solution:

15. (a) Define Poisson process.

(b) Let $\{N(t), t \geq 0\}$ be a Poisson process where $N(t)$ denotes the number of arrival of events upto and including time t. Let T_1 be the time of first arrival of an event. For $0 \leq s \leq t$, find $P(T_1 \leq s \mid N(t) = 1)$. (2 + 2 marks)

Solution:

16. (a) Describe $M/M/\infty$ system.

(b) Derive the steady-state probabilities of the system size.

(c) Find the mean response time (spent time) by the customer in this system.

(2 + 2 + 1 marks)

Solution:

Extra Page