MTL103
Tutorial 2

1. Graph the convex hull of points (0, 5), (3, 5), (6, 3), (5, 0), (3, 3), (2.5, 2.5). Which of these points are extreme points of the hull? Express the non-extreme point (among given points), if any, as a convex combination of the extreme points.

2. Express the point $x = (0, 1)$ as a convex combination of the extreme points of the set $\{(x_1, x_2)^T : x_1 - x_2 \leq 3, 2x_1 + x_2 \leq 4, x_1 + 3 \geq 0\}$

3. Show that a hyperplane $H = \{x \mid Ax = b\}$ and a half-space $H^+ = \{x \mid Ax \geq k\}$ are convex sets.

4. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a convex function and let c be some constant. Show that the set $S = \{x \in \mathbb{R}^n \mid f(x) \leq c\}$ is convex.

5. Find the number of degenerate and non-degenerate basic feasible solutions for the system graphically $2x + 3y \leq 21,\ 3x - y \leq 15,\ x + y \geq 5,\ y \leq 5,\ x, y \geq 0.$

6. Find all basic solutions of the following systems and classify them as degenerate/non-degerate
   (a) $x_1 + 2x_2 + 3x_3 + 4x_4 = 7$
   $2x_1 + x_2 + x_3 + 2x_4 = 3$
   (b) $8x_1 + 6x_2 + 12x_3 + x_4 + x_5 = 6$
   $9x_1 + x_2 + 2x_3 + 6x_4 + 10x_5 = 11$

7. Consider the following system:

$$x_1 + 2x_2 + x_3 \leq 3$$
$$-2x_1 + 2x_2 + 2x_3 \leq 3$$
$$x_1, x_2, x_3 \geq 0$$

   The point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is feasible. Verify whether it is basic.

8. Let P and Q be polyhedra in $\mathbb{R}^n$. Let $P + Q = \{x + y \mid x \in P, y \in Q\}$.
   (a) Show that $P + Q$ is a polyhedron.
   (b) Show that every extreme point of $P + Q$ is the sum of an extreme point of P and an extreme point of Q.

9. Let $A_1, \ldots A_n$ be a collection of vectors in $\mathbb{R}^m$. Let

$$C = \left\{ \sum_{i=1}^{n} \lambda_i A_i \mid \lambda_1, \ldots, \lambda_n \geq 0 \right\}$$

   Show that any element of C can be expressed in the form $\sum_{i=1}^{n} \lambda_i A_i$, with $\lambda_i \geq 0$, and with at most m of the coefficients A_i being nonzero.

10. Consider the standard form polyhedron $\{x \mid Ax = b, x \geq 0\}$, and assume that the rows of the matrix A are linearly independent. Suppose that two different bases lead to the same basic solution. Show that the basic solution is degenerate.

11. Consider the linear program: Minimize $c^T x$ subject to $Ax \leq b, x \geq 0$, where c is a nonzero vector. Suppose that the point x_0 is such that $Ax_0 < b$ and $x_0 > 0$. Show that x_0 cannot be an optimal solution.