$P(|X| \geq a) \leq \frac{E(X)}{a}$

$P(|Y_n| \geq \varepsilon n^2) \leq \frac{E(Y_n)}{\varepsilon n^2}$

Department of Mathematics, IIT Delhi

2201-MTL106: Major Exam.

Time: 2 hours   Date: 20-11-2022   Total Marks: 45

Q.1) i) Let $\{Y_n : n \geq 1\}$ be a sequence of non-negative **i.i.d** random variables, defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with finite mean and variance. Show that $X_n := \frac{Y_n}{n^2}$ converges in probability. Explain whether X_n converges almost surely or not.

ii) Let $\{Z_n : n \geq 1\}$ be a sequence of random variables, defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, converges in distribution to a random variable Z with $Z = \mathtt{A}$ a.e. Then show that $\{Z_n\}$ converges to $\mathtt{A}$ in probability.

iii) Prove the following **Central limit theorem**: Let $\{X_n\}$ be a sequence of i.i.d. random variables with finite mean μ and variance σ^2 with $0 < \sigma^2 < +\infty$. Then $Y_n := \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}}$ converges in distribution to Y, where $\mathcal{L}(Y) = \mathcal{N}(0,1)$.

(2+2)+3+5 marks

Q.2) a) Let A be a random variable with zero mean and unit variance. Define

$$X(t) := A(-1)^t \quad t \in \mathbb{N}.$$

Explain whether $X(t)$ is covariance stationary or not.

b) Let $\{N(t) : t \geq 0\}$ be a Poisson process with intensity 3. Show that, for any $a > 0$, there holds

$$\frac{a^2}{3t + a^2} \leq \mathbb{P}(N(t) - 3t < a) \leq 1.$$

c) Let $\{X_n : n \geq 0\}$ be a DTMC with state space $\mathcal{S} = \{1, 2, \ldots, K\}$ and transition probability matrix $P = (p_{ij})$ where

$$p_{ij} = \begin{cases} \frac{1}{4}, & j = i+1 \\ \frac{3}{4}, & j = i-1 \end{cases} \quad 1 < i < K; \quad p_{ii} = 0, \ i \neq 1, K; \quad p_{12} = \frac{1}{4} = p_{KK}; \quad p_{11} = \frac{3}{4} = p_{K,K-1}.$$

Calculate $p_{12}^{(n)}$ for $n \to \infty$.

4+3+5 marks

Q.3) (a) Suppose $X_n \xrightarrow{2} X$. Prove or disprove

$$\mathrm{Var}(X_n) \to \mathrm{Var}(X).$$

$E(|X_n - X|^2) \to 0$

$E(X_n^2 + X^2 - 2XX_n) \to 0$

$\mathrm{Var}\,X_n + E^2(X_n) + \mathrm{Var}\,X + E^2(X)$

$-2(\mathrm{Cov}(X, X_n) - E(X)E(X_n))$

b) Let $\{X(t) : t \geq 0\}$ be a birth and death process with birth and death rate λ_n and μ_n respectively, where

$$\lambda_n = n + 2, \quad n \geq 0; \quad \mu_n = 2n, \quad n \geq 1.$$

i) Show that the second moment about zero $M_2(t)$ of $X(t)$ satisfies the differential equation:

$$M_2'(t) = 7M(t) - 2M_2(t) + 2\,,$$

$E(X^2(t))$

where $M(t)$ is the mean population size at time t.

ii) Show that $M_2(t)$ is given explicitly as

$$M_2(t) = (X(0))^2 e^{-2t} + 8(1 - e^{-2t}) + 7(X(0) - 2)(e^t - 1)e^{-2t}, \quad t \geq 0,$$

where $X(0)$ being the population size at $t = 0$.

3+(4+5) marks

Q.4) a) Let $\{X_n : n \geq 0\}$ be an irreducible Markov chain with one-step transition matrix P and state space S. Suppose there exists a vector $\Pi = (\Pi_i)_{i \in S}$ with $\Pi_i \geq 0, \quad i \in S$ such that $\Pi = \Pi P$. Show that, if $\Pi_i = 1$ for some i, then $0 < \Pi_j < +\infty$ for all $j \in S$.

b) The number of families migrating to an area follows a Poisson process with rate 4 per week. The number Y_i of the people in the i-th family has the distribution (independent)

$$\mathbb{P}(Y_i = 1) = \frac{1}{6} = \mathbb{P}(Y_i = 4), \quad \mathbb{P}(Y_i = 2) = \frac{1}{3} = \mathbb{P}(Y_i = 3)\,.$$

Find the variance of the total number of people migrating in 10 weeks.

c) Consider a CTMC with rate matrix $Q = \begin{pmatrix} -5 & 3 & 2 \\ 1 & -3 & 2 \\ 5 & 2 & -7 \end{pmatrix}$ and initial distribution $(0, 1, 0)$. Find $\mathbb{P}(\tau > 5)$ where τ denotes the first transition time of the Markov chain.

3+4+2 marks

Best of Luck!!!

$(\alpha \; \beta) \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = (\alpha \; \beta)$

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