Problem sheet–11

Section A

Laplace Transforms

1. **Prove properties in Table 9.1 (Properties of Laplace transforms)**

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

2. **Derive Laplace transforms of elementary functions given in Table 9.2 (Laplace transforms of elementary functions)**

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{R}e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{R}e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathcal{R}e\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathcal{R}e\{s\} > 0$
13	$[e^{-\alpha t}\cos \omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{R}e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin \omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{R}e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} > 0$

Section B

1. Region of Convergence

Let x(t) be a signal that has a rational Laplace transform with exactly two poles, located at s = -1 and s = -3. If $g(t) = e^{2t}x(t)$ and $G(jw)$ [the Fourier transform of g(t)] converges, determine whether x(t) is left sided, right sided, or two sided.

2. Laplace transforms of basic filters

Using geometric evaluation of the magnitude of the Fourier transform from the corresponding pole-zero plot, determine, for each of the following Laplace transforms, whether the magnitude of the corresponding Fourier transform is approximately low pass, high pass, or band pass:

a) $H_1(s) = \frac{1}{(s+1)(s+3)}, \quad Re\{s\} > -1$

b) $H_2(s) = \frac{s}{s^2+s+1}, \quad Re\{s\} > -\frac{1}{2}$

c) $H_3(s) = \frac{s^2}{s^2+2s+1}, \quad Re\{s\} > -1$

3. Properties of Laplace transforms

Suppose we are given the following three facts about the signal x(t):

1. x(t) = 0 for t < 0.
2. x(k/80) = 0 for k = 1, 2, 3, ...
3. x(l/160) = e^{-120}.

Let X(s) denote the Laplace transform of x(t), and determine which of the following statements is consistent with the given information about x(t):

(a) X(s) has only one pole in the finites-plane.

(b) X(s) has only two poles in the finites-plane.

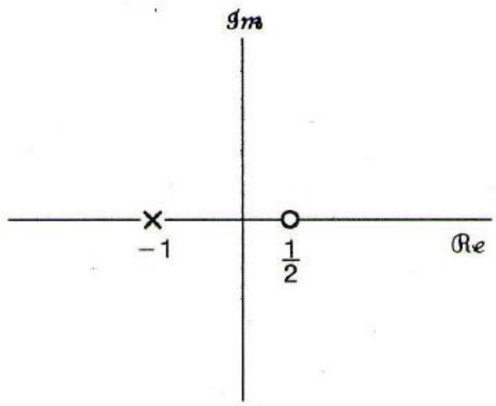
(c) X(s) has more than two poles in the finites-plane.

4. Inverse System

The inverse of an LTI system H(s) is defined as a system that, when cascaded with H(s), results in an overall transfer function of unity or, equivalently, an overall impulse response that is an impulse.

(a) If $H_1(s)$ denotes the transfer function of an inverse system for H(s), determine the general algebraic relationship between H(s) and $H_1(s)$.

(b) Shown in the figure below is the pole-zero plot for a stable, causal system H(s). Determine the pole-zero plot for the associated inverse system.



5. Properties of Laplace Transforms

Consider a stable and causal system with a real impulse response h(t) and system function H(s). It is known that H(s) is rational, one of its poles is at $-1 + j$, one of its zeros is at $3 + j$, and it has exactly two zeros at infinity. For each of the following statements, determine whether it is true, whether it is false, or whether there is insufficient information to determine the statement's truth.

(a) h(t)e^{-3t} is absolutely integrable.

(b) The ROC for H(s) is (Re{s} > - 1.

(c) The differential equation relating inputs x(t) and outputs y(t) for S may be written in a form having only real coefficients.

(d) $\lim_{s\to\infty} H(s) = 1$

(e) H(s) does not have fewer than four poles.

(f) $H(jw) = 0$ for at least one finite value of w.

(g) If the input to S is $e^{3t}\sin(t)$, the output is $e^{3t}\cos(t)$.

6. Laplace Transform for polynomial function

The Laplace transform X(s) of a signal x(t) has four poles and an unknown number of zeros. The signal x(t) is known to have an impulse at t = 0. Determine what information, if any, this provides about the number of zeros and their locations.