Problem Sheet 4 Solutions

Ques.1

a) $x(\tau_2 - \tau_1 - t)$
b) $x(-(-t - \tau_2) - \tau_1)$
c) Trapezoid with base $3T/2$.

Ques.2

$$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3]$$

$$h[n] = 2\delta[n+1] + 2\delta[n-1]$$

$$x[n] = \{\underset{\uparrow}{1}, 2, 0, -1\}$$

$$h[n] = \{2, \underset{\uparrow}{0}, 2\}$$

a) $y_1[n] = x[n] * h[n]$

$$y_1[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y_1[n] = \sum_{k=0}^{3} x[k]h[n-k]$$

For

$$y_1[-2] = \sum_{k=0}^{3} x[k]h[-2-k] = 0$$

$$y_1[-1] = \sum_{k=0}^{3} x[k]h[-1-k] = 2$$

$$y_1[0] = \sum_{k=0}^{3} x[k]h[-k] = 4$$

$$y_1[1] = \sum_{k=0}^{3} x[k]h[1-k] = 2$$

$$y_1[2] = \sum_{k=0}^{3} x[k]h[2-k] = 2$$

$$y_1[3] = \sum_{k=0}^{3} x[k]h[3-k] = 0$$

$$y_1[4] = \sum_{k=0}^{3} x[k]h[4-k] = -2$$

		x[n]			
		1	2	0	-1
h[n]	2	2	4	0	-2
	0	0	0	0	0

2 2 4 0 -2

$$y_1[n] = \{2,\underset{\uparrow}{4},2,2,0,-2\}$$

b) $y_2[n] = x[n+2] * h[n]$

$$y_2[n] = x[n+2] * h[n]$$
$$y_2[n] = x[n] * \delta[n+2] * h[n]$$
$$y_2[n] = x[n] * h[n] * \delta[n+2]$$
$$y_2[n] = y_1[n] * \delta[n+2]$$
$$y_2[n] = y_1[n+2]$$

c) $y_3[n] = x[n] * h[n+2]$

$$y_3[n] = x[n] * h[n+2]$$
$$y_3[n] = x[n] * h[n] * \delta[n+2]$$
$$y_3[n] = y_1[n] * \delta[n+2]$$
$$y_3[n] = y_1[n+2]$$

Ques.3

a) False To prove $x[n] * \{h[n]g[n]\} = \{x[n] * h[n]\}g[n]$

LHS=$x[n] * \{h[n]g[n]\}$

$$= \sum x[k]h[n-k]g[h-k]$$

RHS=$\{x[n] * h[n]\}g[n]$

$$= \sum x[k]h[n-k]\}g[n]$$

LHS $\neq$ RHS

Counter example assume $g[n] = \delta[n]$

b) True $y(t) = x(t) * h(t)$ then $y(2t) = 2x(t) * h(2t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$
$$y(2t) = \int_{-\infty}^{\infty} x(\tau)h(2t-\tau)d\tau$$

Put $\acute{\tau} = \tau/2 \quad d\acute{\tau} = d\tau/2$

$$y(2t) = 2\int_{-\infty}^{\infty} x(2\acute{\tau})h(2t-2\acute{\tau})d\acute{\tau}$$
$$y(2t) = 2x(t) * h(2t)$$

In general prove that

$$y(at) = |a|x(at) * h(at)$$

c) True

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$
$$y(-t) = \int_{-\infty}^{\infty} x(\tau)h(-t-\tau)d\tau$$

Assume $-\tau = \tau$

$$y(-t) = \int_{-\infty}^{\infty} x(-\tau)h(-t+\tau)d\tau$$

$$y(-t) = \int_{-\infty}^{\infty} x(-\tau)h(-(t-\tau)d\tau$$

But $x(-\tau) = x(\tau)$ & $h[-(t-\tau)] = h(t-\tau)$

$$y(-t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = y(t)$$

Hence $y(t)$ is even signal.

d) $y(t) = x(t) * h(t)$

$$E_V\{y(t)\} = x(t) * E_V\{h(t)\} + E_V\{x(t)\} * h(t)$$

If this is true,

$$x(t) * \left[\frac{h(t)}{2}\right] + \frac{[x(-t) * h(-t)]}{2}$$
$$= \frac{[x(t) * h(t)]}{2} + x(t) * \left[\frac{h(-t)}{2}\right] + x(t) * \left[\frac{h(t)}{2}\right] + \frac{[x(-t) * h(t)]}{2}$$

This is equivalent to condition:

$E_v\{x(t)\} * h(t) - O_d\{x(t)\} * h(-t) = 0.$

If we have a purely even signal, this reduces to $E_v\{x(t)\} * h(t) = 0$, which is definitely not true.

Ques.4

$\widetilde{x_1}(t)$ & $\widetilde{x_2}(t)$ is periodic $T = T_0$

a) $\tilde{y}(t) = \widetilde{x_1}(t) \otimes \widetilde{x_2}(t)$

$$\tilde{y}(t) = \int_0^{T_0} \widetilde{x_1}(\tau)\, \widetilde{x_2}(t-\tau)d\tau$$

If $\tilde{y}(t)$ is periodic with $T = T_0$, then $\tilde{y}(t) = \tilde{y}(t+T_0)$.

$$\tilde{y}(t+T_0) = \int_0^{T_0} \widetilde{x_1}(\tau)\widetilde{x_2}(t+T_0-\tau) = \int_0^{T_0} \widetilde{x_1}(\tau)\widetilde{x_2}(t-\tau) = \tilde{y}(t)$$

Hence proved.

b) $\widetilde{y_a}(t) = \int_a^{a+T_0} \widetilde{x_1}(\tau)\, \widetilde{x_2}(t-\tau)\; d\tau$

Let $a = kT_0 + b \qquad 0 \le b \le T_0$

$$\widetilde{y_a}(t) = \int_{kT_0+b}^{kT_0+b+T_0} \widetilde{x_1}(\tau)\, \widetilde{x_2}(t-\tau)\; d\tau$$

where k is integer.

Let $\lambda + kT_0 = \tau$, then

$$\widetilde{y_a}(t) = \int_b^{b+T_0} \widetilde{x_1}(\lambda + kT_0)\, \widetilde{x_2}(t-\lambda-kT_0)\; d\tau = \int_b^{b+T_0} \widetilde{x_1}(\tau)\, \widetilde{x_2}(t-\tau)\; d\tau$$

$$\widetilde{y_a}(t) = \int_b^{T_0} \widetilde{x_1}(\tau)\, \widetilde{x_2}(t-\tau)\; d\tau + \widetilde{y_a}(t) + \int_{T_0}^{b+T_0} \widetilde{x_1}(\tau)\, \widetilde{x_2}(t-\tau)\; d\tau$$

Substitute $\lambda + T_0 = \tau$ in second term,

$$\int_{T_0}^{b+T_0} \widetilde{x_1}(\tau)\,\widetilde{x_2}(t-\tau)\ d\tau = \int_0^b \widetilde{x_1}(\tau)\,\widetilde{x_2}(t-\tau)\ d\tau$$

$$\widetilde{y_a}(t) = \int_b^{T_0} \widetilde{x_1}(\tau)\,\widetilde{x_2}(t-\tau)\ d\tau + \int_0^b \widetilde{x_1}(\tau)\widetilde{x_2}(t-\tau)\ d\tau$$

$$\widetilde{y_a}(t) = \int_0^{T_0} \widetilde{x_1}(\tau)\,\widetilde{x_2}(t-\tau)\ d\tau$$

$$\widetilde{y_a}(t) = \tilde{y}(t)$$

Hence proved.

Ques.5

a) $\varphi[n] = \left(\frac{1}{2}\right)^n u[n]$

$$\frac{1}{2}\varphi[n-1] = \left(\frac{1}{2}\right)^n\ u[n-1]$$

$$\varphi[n] - \frac{1}{2}\ \varphi[n-1] = \left(\frac{1}{2}\right)^n\ \{u[n] - u[n-1]\}$$

$$\left(\frac{1}{2}\right)^n\ \{u[n] - u[n-1]\} = \begin{cases} 0\ for\ n \neq 0 \\ 1\ for\ n = 0 \end{cases}$$

$$\left(\frac{1}{2}\right)^n\ \{u[n] - u[n-1]\} = \delta[n]$$

So $$\delta[n] = \varphi[n] - \left(\tfrac{1}{2}\right)\varphi[n-1]$$

In general

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\{\varphi[n-k] - \frac{1}{2}\varphi[n-k-1]\}$$

Thus, $x[n] = x[n] * \varphi[n] - \frac{1}{2}x[n] * \varphi[n-1] = x[n] * \varphi[n] * \delta[n] - \frac{1}{2}x[n] * \varphi[n] * \delta[n-1] = \left(x[n] * \delta[n] - \frac{1}{2}x[n] * \delta[n-1]\right) * \varphi[n] = \left(x[n] - \frac{1}{2}x[n-1]\right) * \varphi[n]$

$$= \textstyle\sum_{k=-\infty}^{\infty} \left(x[k] - \frac{1}{2}x[k-1]\right)\ \varphi[n-k].$$

Thus, $x[n] = \sum_{k=-\infty}^{\infty} a_k\varphi[n-k]$ where $a_k = x[k] - \frac{1}{2}x[k-1]$.

b)

$y[n] = \sum_{k=-\infty}^{\infty} a_k r[n-k]$, where $a_k = x[k] - \frac{1}{2}x[k-1]$.

c) $y[n] = x[n] * h[n]$

$$\delta[n] = \varphi[n] - \frac{1}{2}\varphi[n-1]$$
$$h[n] = r[n] - \frac{1}{2}r[n-1]$$
$$= \Psi[n] * r[n]$$

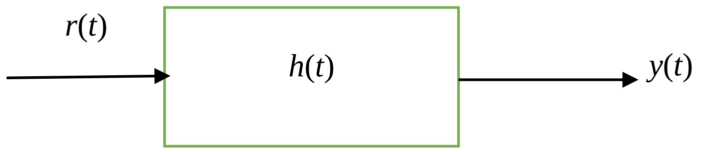
where $\Psi[n] = \delta[n] - \frac{1}{2}\delta[n-1]$
$$y[n] = x[n] * \Psi[n] * r[n]$$

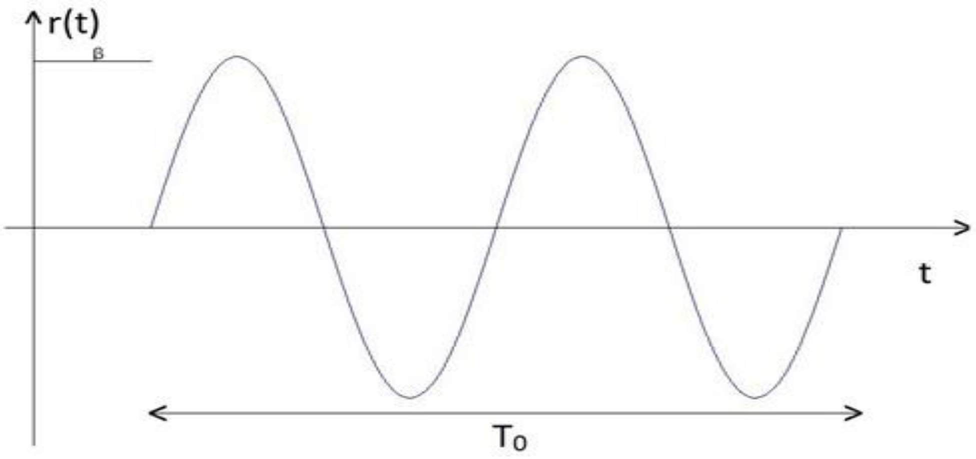
d) Already shown that
$$h[n] = r[n] - \frac{1}{2}r[n-1]$$
$$\Psi[n] * \varphi[n] = \left[\delta[n] - \frac{1}{2}\delta[n-1]\right] * \varphi[n] = \varphi[n] - \frac{1}{2}\varphi[n-1] = \delta[n].$$

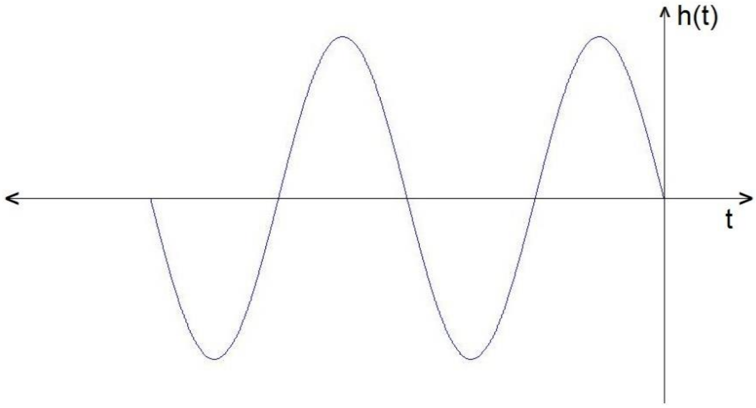
Ques.6

a) Received signal is delayed and attenuated.
So, $r(t) = \alpha \sin(\omega_c t - \beta)$
where α is attenuating factor and β is delay
$$x(t) * h(t) = x(t) * \alpha\delta(t-\beta)$$
$$= \alpha x(t-\beta)$$
$$= \alpha \sin(\omega_c t - \beta)$$
Hence $h(t) = \alpha\delta(t-\beta)$ is the correct impulse response.

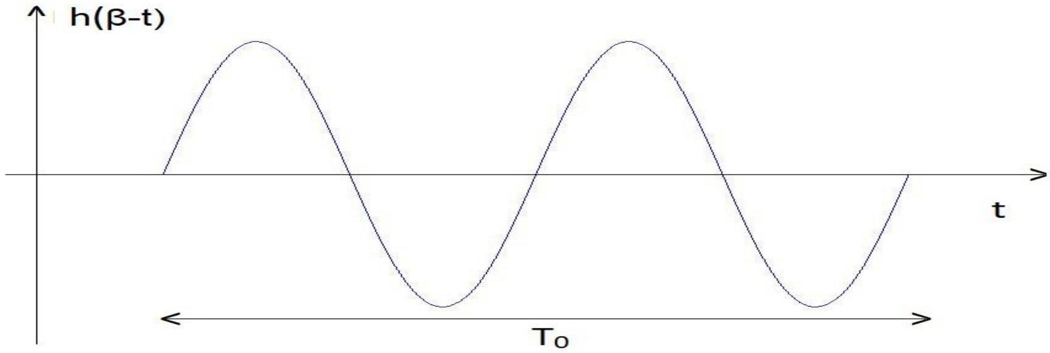
b) Matched filter is commonly used for detecting signals in the presence of noise. Matched filter response is the time-reversed version of the transmitted signal. Let us investigate what happens when the reflected signal passes through a matched filter.





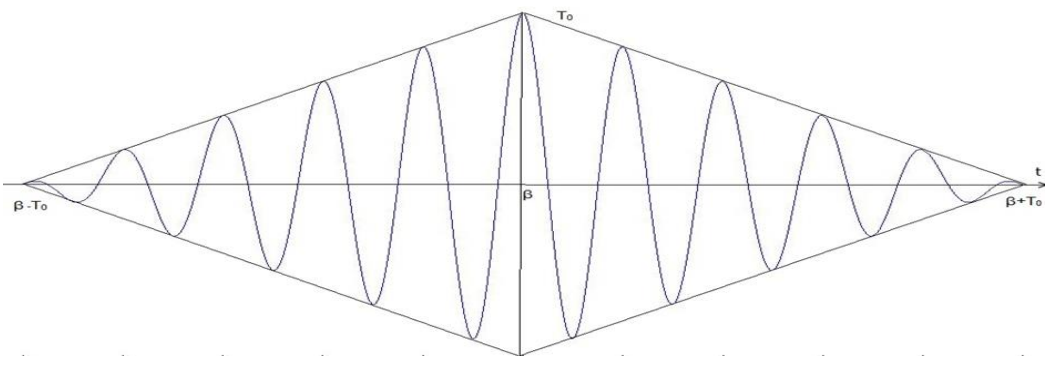


For convolution, we take the reversed version of the impulse response and shift is through various t.



It is intuitively clear that the output will be maximum when $h(t)$ is reversed and shifted by β, because it leads to maximum overlap. Thus, by detecting the time at which output becomes maximum, we calculate β.

$y(t)$ can also be computed analytically.



Ques.7

a)

$$w[n] = \frac{1}{2}w[n-1] + x[n]$$

$$y[n] = \alpha y[n-1] + \beta w[n]$$

$$w[n] = \frac{1}{\beta}\{y[n] - \alpha y[n-1]\} \cdots\cdots (1)$$

$$\frac{1}{2}w[n-1] = \frac{1}{2\beta}\{y[n-1] - \alpha y[n-2]\} \cdots\cdots (2)$$

From (1) and (2)

$$w[n] - \frac{1}{2}w[n-1] = \frac{y[n]}{\beta} - \left(\alpha + \frac{1}{2}\right)\frac{y[n-1]}{\beta} + \frac{\alpha y[n-2]}{2\beta}$$

$$y[n] = \beta x[n] - \frac{\alpha}{2}y[n-2] + \left(\alpha + \frac{1}{2}\right)y[n-1]$$

Thus $\beta = 1$ and $\alpha = \frac{1}{4}$

b)

$$w[n] = \frac{1}{2}w[n-1] + x[n]$$

Evaluate the impulse response

$$w[0] = \frac{1}{2}w[-1] + \delta[0]$$

$w[-1] = 0$ {System is causal LTI}

$$w[0] = \delta[0] = 1$$

$$w[1] = \frac{1}{2}w[0] = \frac{1}{2}$$

$$w[2] = \frac{1}{4}w[0] = \frac{1}{4}$$

$$\vdots$$

Thus

$$w[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

Similarly

$$h_2[n] = \left(\frac{1}{4}\right)^n \; u[n]$$

$$h[n] = h_1[n] * h_2[n]$$

$$= \sum_{k=-\infty}^{k=\infty} \left(\frac{1}{2}\right)^k \; u[k] \left(\frac{1}{4}\right)^{n-k} \; u[n-k]$$

$$= \sum_{k=-\infty}^{k=\infty} \left(\frac{1}{2}\right)^k \; \left(\frac{1}{4}\right)^{n-k}$$

$$h[n] = \left(\frac{1}{2}\right)^{2n} \; [2^{n+1} - 1]u[n]$$

Ques.8

a) True

$$\int_{-\infty}^{\infty} |h(t)|dt = \sum_{k=-\infty}^{k=\infty} \int_{KT}^{KT+T} |h(t)|\, dt = \infty$$

b) False

$$h(t) = \delta(t-1)$$
$$h^{-1}(t) = \delta(t+1)$$

$h^{-1}(t)$ is non causal system.

c) False. Suppose $h[n] = u[n]$ then

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} 1 = \infty$$

d) False if $h[n] = \infty$ then the system is unstable.

e) False. $h(t) = u(t)$ then

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{0}^{\infty} dt = \infty$$

f) False

$$h_1(t) = \delta(t+1)$$
$$h_2(t) = \delta(t-1)$$
$$h(t) = h_1(t) * h_2(t) = \delta(t)$$

which is not causal.

g) False

$s(t)$ will always integrate to (+/-) infinity (i.e., $\int_{-\infty}^{\infty} s(t)dt = \infty$) whenever $h(t)$ has non-zero mean. For stability, $h(t)$ should be absolutely integrable and it can have non-zero mean. For example, $h(t) = rect(t)$ corresponds to a stable signal but its step-response will integrate to infinity.

h) Of course, a causal system only respond after an input is applied. Thus,
$s[n] = 0$ For n < 0

Ques.9

$$h(t) = \{\{h_1 * h_2 + h_2 * h_2 - h_2 * h_1\} * h_1 + h_1^{-1}\} * h_2^{-1}$$

$$h(t) = \{h_1 + h_2 - h_3\} * h_1 + h_1^{-1} * h_2^{-1}$$

$$h(t) = h_2 * h_1 + h_1^{-1} * h_2^{-1}$$

Ques.10

$$(D^2 + 3D + 2)y(t) = Dx(t)$$

Solution using polynomials:

$$Y + 3AY + 2A^2Y = AX$$

$$\frac{Y}{X} = \frac{A}{1 + 3A + 2A^2} = \frac{A}{(1 + 2A)(1 + A)} = \frac{1}{1 + A} - \frac{1}{1 + 2A}$$

$$\frac{Y}{X} = \frac{1}{A}\left(\frac{A}{1 + A} - \frac{A}{1 + 2A}\right)$$

Thus,

$$h(t) = \frac{d}{dt}((e^{-t} - e^{-2t})u(t)) = (e^{-2t} - e^{-t})u(t).$$

Ques.11

$y[n] = (1 - R)^2\{e^{\beta n}u[n] * e^{\alpha n}u[n]\}$ where R is the unit-delay operator

$$y[n] = (1 - R)^2 \frac{\{e^{\beta(n+1)} - e^{\alpha(n+1)}\}}{e^{\beta} - e^{\alpha}} u[n]$$

$$y[n] = \frac{\{e^{\beta(n+1)} - e^{\alpha(n+1)}\}}{e^{\beta} - e^{\alpha}} u[n] - 2\frac{\{e^{\beta n} - e^{\alpha n}\}}{e^{\beta} - e^{\alpha}} u[n-1] + \frac{\{e^{\beta(n-1)} - e^{\alpha(n-1)}\}}{e^{\beta} - e^{\alpha}} u[n-2]$$

Ques.12

In the first system, $Y = X + 0.5RY + 0.4RY = X + 0.9RY$

In the second system, $Y = X + p_oRY$ Hence, $p_o = 0.9$

The impulse response of the system $h[n] = 0.9^n u[n]$, and hence $\sum_{n=0}^{+\infty}|0.9^n| = \frac{1}{1-0.9} = 10.$

The impulse response is absolutely summable and hence the system is stable.

Ques.13

Method 1:

$$Y = 2X + 3RX - R^2Y$$

$$\frac{Y}{X} = \frac{2 + 3R}{1 + R^2}$$

$$\frac{Y}{X} = \frac{1 + \frac{3}{2}\mathrm{j}}{1 + \mathrm{j}R} + \frac{1 - \frac{3}{2}\mathrm{j}}{1 - \mathrm{j}R}$$

$$h[n] = \left(1 + \frac{3}{2}\mathrm{j}\right)(-\mathrm{j})^n u[n] + \left(1 - \frac{3}{2}\mathrm{j}\right)(\mathrm{j})^n u[n]$$

Hence, $A = 1 - \frac{3}{2}\mathrm{j}$ and $B = 1 + \frac{3}{2}\mathrm{j}$

Method 2: Assuming $h[n] = (Az_1^n + Bz_2^n)u[n]$

Thus, $(Az_1^n + Bz_2^n)u[n] = 2\delta[n] + 3\delta[n-1] - (Az_1^{n-2} + Bz_2^{n-2})u[n-2]$

For $n = 0$, $\quad A + B = 2 \quad$ (Eq. 1)

For $n = 1$, $\quad Az_1 + Bz_2 = 3 \quad$ (Eq. 2)

Substituting $z_1 = \mathrm{j}$ and $z_2 = -\mathrm{j}$ in Eq. 1 and Eq. 2, we get $A = 1 - \frac{3}{2}\mathrm{j}$ and $B = 1 + \frac{3}{2}\mathrm{j}$