1.

We can model the network of friends as an undirected graph, with a vertex corresponding to each user, and an edge (P1, P2) iff P1 and P2 are friends.

Since there is a path from every user to every other user, the graph is connected.

Since there are no cyclical paths, the graph is acyclic.

Suppose there are N users. The graph shall be a connected acyclic graph with N-1 edges. By handshake lemma, the sum of degrees of all vertices shall be 2(N-1)

Since the graph is connected, every vertex must have degree at least 1. There are no vertices of degree more than 2021, and one vertex for every degree from 2 to 2021. Let the number of nodes of degree 1 be k. The sum of degrees of all vertices = k + (2 + 3 + … + 2021). The number of vertices N shall be k + 2020.

Sum of degrees for vertices = 2(k+2020 - 1) = k+(2 +3 + … + 2021)

-> k = (2 + 3 + … + 2021) - 4038

-> N = (2 + 3 + … + 2021) - 2018

-> N = 2041212

2.

Let G be a bipartite graph with bipartition (L, R) defined as follows.

L={a_1,…,a_n} is a set of n vertices (where a_i represents the i'th column of A).

R={b_0,…,b_{n-1}} is a set of n vertices (where b_j represents the remainder j mod n).

Edge set E is defined as follows: For each i,j, {a_i,b_j} \in E if the i'th column of A contains a number that is j mod n.

We first claim that the above bipartite graph has a perfect matching using Hall's theorem.

Consider any subset T of L. We claim that |T| <= |N(T)|, where N(T) is the set of neighbours of vertices in T.

The columns indexed by elements of T contain k*|T| (distinct) numbers. All these numbers, mod n, have one of |N(T)| remainders. But for every j in {0,..,n-1}, exactly k elements of A are j mod n. Therefore, k*|T| <= k*|N(T)|. Thus |T| <= |N(T)|.

Thus, by Halls theorem, the graph contains a perfect matching, say M.

Construct a set S as follows. For each a_i, look at its partner under M, say b_j. Since {a_I,b_j} \in E, the i'th column of M is guaranteed to contain a number which is j mod n. Include one such number in S. Thus, we include one number from each column of A in S. Moreover, since each b_j is matched to exactly one a_i, we include in S exactly one number that is j mod n.

Thus, S is a nice set.