COL202: Discrete Math	**Tutorial 2**
Instructor: Venkata Koppula	Release date: 8nd August, 2024

The solutions for the ($\star$) marked problems must be submitted at the start of the tutorial. The ($\blacklozenge$) marked problems will be discussed in the tutorial (if time permits, you can also discuss the other problems with the instructor/TAs).

Notations. Let $\mathbb{N}$ denote the set of natural numbers $\{1, 2, 3, \ldots\}$. Let $\mathbb{Z}$ denote the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Definition 1 (Principle of Mathematical Induction). *Let $P : \mathbb{N} \to \{T, F\}$ be any predicate such that it satisfies the following two conditions:*

- $P(1) = T$
- *for all* $i \in \mathbb{N}$, $P(i) \implies P(i+1)$

Then, for all $n \in \mathbb{N}$, $P(n)$ *holds.*

Definition 2 (Strong Principle of Mathematical Induction). *Let $P : \mathbb{N} \to \{T, F\}$ be any predicate such that it satisfies the following two conditions:*

- $P(1) = T$
- *for all* $i \in \mathbb{N}$, $(P(1) \wedge P(2) \wedge \ldots \wedge P(i)) \implies P(i+1)$

Then, for all $n \in \mathbb{N}$, $P(n)$ *holds.*

1 Problems: PMI, Strong PMI

Clearly state whether you are using PMI or Strong PMI. If you are using any non-standard variant of PMI/Strong PMI, then you must first prove this variant (using Definitions 1, 2).

1.1. ($\star$) Let $P : \mathbb{N} \to \{T, F\}$, $S = \{n : P(n) = T\}$. Suppose S is non-empty. Prove that it has a minimal element, assuming PMI/Strong PMI.

1.2. ($\star$) Prove that for any natural numbers n, n_0, there exists non-negative integer q and $r \in \{0, 1, \ldots, n_0 - 1\}$ such that $n = q \cdot n_0 + r$.

1.3. ($\blacklozenge$) Consider the functions power and fpower on $\mathbb{N}$ defined below.

$$\mathsf{power}(x, n) = \left\{ \begin{array}{ll} 1 & \text{if } n = 0 \\ x \cdot \mathsf{power}(x, n-1) & \text{otherwise} \end{array} \right\}$$

$$\mathsf{fpower}(x, n) = \left\{ \begin{array}{ll} 1 & \text{if } n = 0 \\ \mathsf{fpower}(x^2, \lfloor n/2 \rfloor) & \text{if } n \bmod 2 = 0 \\ x \cdot \mathsf{fpower}(x^2, \lfloor n/2 \rfloor) & \text{if } n \bmod 2 = 1 \end{array} \right\}$$

Prove that for all naturals n and x, $\mathsf{power}(x, n) = \mathsf{fpower}(x, n)$

1.4. ($\blacklozenge$) Prove the following using induction:

$$\forall n \in \mathbb{N}, \quad \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{2n}}.$$

1.5. (Defining induction over $\mathbb{N} \times \mathbb{N}$) Let $P : \mathbb{N} \times \mathbb{N} \to \{T, F\}$ be some predicate s.t. :

- (base case) : $P(1,1) = T$.
- (induction step) : for all $(i,j) \in \mathbb{N} \times \mathbb{N}$, $P(i,j) = T$ implies $P(i+1,j) = T$ and $P(i,j+1) = T$.

1. Prove that for all $(i,j) \in \mathbb{N} \times \mathbb{N}$, $P(i,j) = T$.
2. Suppose, in the induction step, you were only given the following:
   - for all $(i,j) \in \mathbb{N} \times \mathbb{N}$, $P(i,j) = T$ implies that $P(i,j+1) = T$.

   How should you strengthen the base case in order to conclude that $P(i,j) = T$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$?

1.6. (Knight reachability problem) Consider an infinite chessboard, where each cell is indexed by $(x,y) \in \mathbb{N} \times \mathbb{N}$. There is a knight[1] at position $(1,1)$. Prove that for any $(x,y) \in \mathbb{N} \times \mathbb{N}$, the knight can reach (x,y) in finitely many steps.

Bound = 3(x+y)

2 Problems: Number Theory

The following is an optional problem, and will not be discussed in the tutorials (interested students can discuss this with me after class, or after tutorials)

2.1. Let $S = \{a + b\sqrt{-5} \ : \ a, b \in \mathbb{Z}\}$ be a subset of the complex numbers.

1. Is this set countably infinite or uncountably infinite?

For any two elements in this set, we can add and multiply them, the resulting number will also be in this set. For this set, we can define divisibility, and a notion that's similar to primality: we will call it irreducibility.

Definition 3. *Let $a, b \in S$. We say that a divides b if there exists $k \in S$ such that $b = a \cdot k$.*

Definition 4. *A number $z \in S$ is said to be irreducible if, for all $a, b \in S$, if $z = a \cdot b$, then either $a \in \{1, -1\}$ or $b \in \{1, -1\}$.*

2. Is the number 5 irreducible? Can you find some other number $k \in \mathbb{N}$ such that k is a prime number, but k is not irreducible?

We will now prove that the numbers 2 and 3 are irreducible. This might look surprising, since $2 \cdot 3 = (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5})$, but 2 or 3 do not divide $(1 + \sqrt{-5})$ or $(1 - \sqrt{-5})$. What property does the set of natural numbers have that this set S does not have?

The key trick for proving irreducibility is to look at the norm operator $\mathcal{N} : S \to \mathbb{N} \cup \{0\}$ defined as follows: for all $a, b \in \mathbb{Z}$, $\mathcal{N}(a + b\sqrt{-5}) = a^2 + 5b^2$.

[1] Starting from position $(a,b) \in \mathbb{N} \times \mathbb{N}$, the knight can go to one of the following eight positions (assuming the position is in $\mathbb{N} \times \mathbb{N}$): $(a+1, b+2)$, $(a+2, b+1)$, $(a-1, b+2)$, $(a-2, b+1)$, $(a+1, b-2)$, $(a+2, b-1)$, $(a-1, b-2)$ and $(a-2, b-1)$.

3. Prove that for any $x, y \in S$, $\mathcal{N}(x \cdot y) = \mathcal{N}(x) \cdot \mathcal{N}(y)$.
4. Can any element in S have norm equal to 2 or 3?
5. From here, conclude that 2 and 3 are irreducibles.

This is version 1.0 of the tutorial sheet. Let me know if something is unclear. In case of any doubt or for help regarding writing proofs, feel free to contact the instructor or TAs.

Venkata Koppula - kvenkata@iitd.ac.in
Ananya Mathur - cs5200416@iitd.ac.in
Anish Banerjee - cs1210134@cse.iitd.ac.in
Eshan Jain - cs5200424@cse.iitd.ac.in
Mihir Kaskhedikar - cs1210551@iitd.ac.in
Naman Nirwan - Naman.Nirwan.cs521@cse.iitd.ac.in
Pravar Kataria - Pravar.Kataria.cs121@cse.iitd.ac.in
Shashwat Agrawal - csz248012@cse.iitd.ac.in
Subhankar Jana - csz248009@iitd.ac.in