COL751 - Lecture 15

Recall the definition of Gomory Hu trees introduced in the previous lecture.

Definition 1 *(Lec 14) A tree $T = (V, E^*, c^*)$ is said to be a Gomory-Hu tree for a graph $G = (V, E, c)$ if it satisfies that for any distinct $x, y \in V$:*

1. $\lambda_{x,y,G} = \lambda_{x,y,T} = \min_{e \in P_{x,y}} c^*(e)$.
2. *If $e = \arg\min_{e \in P_{x,y}} c^*(e)$, then $T - e$ corresponds to an (x, y)-min-cut in G.*

1 Some Fundamental Properties of Cuts

Property 1 *(Lec 14) For any sequence of $k \geqslant 2$ distinct vertices $(x = x_1, x_2, \ldots, x_k = y)$, we have*

$$\lambda_{x,y,G} \geqslant \min_{i<k} \lambda_{x_i,x_{i+1},G}.$$

As an implication of Lemma 2, we prove the following alternate characterization of Gomory-Hu Trees.

Proposition 1 *(Alternate Characterization) A tree $T = (V, E^*, c^*)$ is a Gomory-Hu tree for $G = (V, E, c)$ iff the conditions 1 and 2 stated in Definition 1 hold for $(n-1)$ vertex pairs that corresponds to endpoints of edges in E^*.*

Proof: Let us suppose that conditions 1 and 2 stated in Definition 1 holds for pairs that correspond to endpoints of edges in E^*. Consider any distinct $x, y \in V$ that are not adjacent in T. Let $P_{x,y} = (x = x_1, x_2, \ldots, x_k = y)$ be unique path from x to y in tree T, and let $e = (x_i, x_{i+1})$ be edge of least weight on $P_{x,y}$.

By Lemma 2, $\lambda_{x,y,G} \geqslant \lambda_{x_i,x_{i+1},G} = c^*(e)$. Now, observe that $T - e$ is an (x, y)-cut and (x_i, x_{i+1})-min-cut, due to which it follows that $\lambda_{x,y,G} \leqslant \lambda_{x_i,x_{i+1},G}$.

This proves that $\lambda_{x,y,G} = c^*(e)$, and thus $T - e$ corresponds to an (x, y)-min-cut. $\square$

Property 2 *(Lec 14) Let $s, t \in V$ be distinct vertices and (A, A^c) be an (s, t)-min-cut in G. Then, for any two distinct vertices $x, y \in A$ there is a (x, y)-min-cut (B, B^c) in G such that either $B \subseteq A$ or $B^c \subseteq A$.*

(In other words, the (x, y)-min-cut is unaffected on considering A^c as a supernode.)

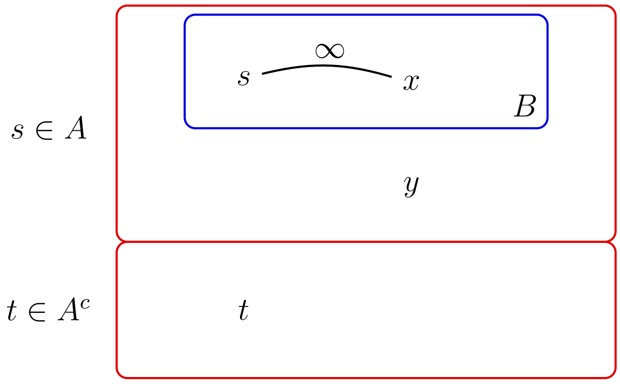

Figure 1: Illustration of cuts involved in Property 3.

Property 3 *Let $s, t \in V$ be distinct vertices and (A, A^c) be an (s,t)-min-cut in G. Let x, y be two distinct vertices in A and (B, B^c) be an (x,y)-min-cut in G such that $B \subseteq A$. If $s \in B$ then (A, A^c) is also an (y,t)-min-cut in G.*

Proof: Let us suppose the conditions stated in claim including the condition $s \in B$ holds true. Observe that we can add an edge of infinite capacity connecting s and x as they are not separated by any min-cut.

So, we have

$$\begin{aligned} \lambda_{y,t} &\geqslant \min(\lambda_{y,x}, \lambda_{x,s}, \lambda_{s,t}) && (by\ chain\ rule) \\ &= \min(\lambda_{x,y}, \lambda_{s,t}) && (due\ to\ edge\ of\ infinite\ weight) \\ &= \lambda_{s,t} && (as\ s\ and\ t\ are\ separated\ by\ (x,y)-min-cut) \end{aligned}$$

This along with the fact that y and t are separated by an (s,t)-min-cut implies that $\lambda_{y,t} = \lambda_{s,t}$. $\square$

2 Algorithm

Before understanding algorithm we explain notion of partial Gomory Hu Tree.

Definition 2 *A tree $T = (V^*, E^*, c^*)$ is said to be a* **Partial Gomory-Hu tree** *for a graph $G = (V, E, c)$ if V^* forms a partition of V and for any adjacent nodes X, Y in T^* there exists $(x,y) \in X \times Y$ satisfying:*

1. *$\lambda_{x,y,G} = \lambda_{X,Y,T} = c^*(X,Y)$.*
2. *$T - (X,Y)$ corresponds to an (x,y)-min-cut in G.*

Description of the algorithm: In our algorithm, we start with trivial solution for a partial Gomory-Hu tree $\mathcal{T} = (\{V\}, \emptyset)$ and recursively split non-singleton nodes until we get a tree with n nodes.

1 Take any two vertices s, t in X.

2 Let $C_1, \ldots, C_k$ be connected-components in $\mathcal{T} - X$.

3 Let H be new graph obtained from G by contracting $C_1, \ldots, C_k$ into super-nodes.

4 Compute an (s,t)-min-cut, say (S_H, T_H), in H and let (S, T) be an (s,t)-cut in G obtained from (S_H, T_H) on uncontracting $C_1, \ldots, C_k$.

5 Split node X into two nodes $X_S = S \cap X$ and $X_T = T \cap X$, and for $j \in [1, k]$, connect C_j to X_S if $V(C_j) \subseteq S$ and X_T otherwise.

6 Set $c^*(X_S, X_T) = \lambda_{s,t,G}$.

7 Return $\mathcal{T}$.

Algorithm 1: Extend-Partial-Gomory-Hu-tree$(G, \mathcal{T}, X)$

The splitting process need computation of just one min-cut between an pair of nodes in a non-singleton set $X \in \mathcal{T}$. The pseudocode for the same is presented in Algorithm 1. Also see Figure 2.

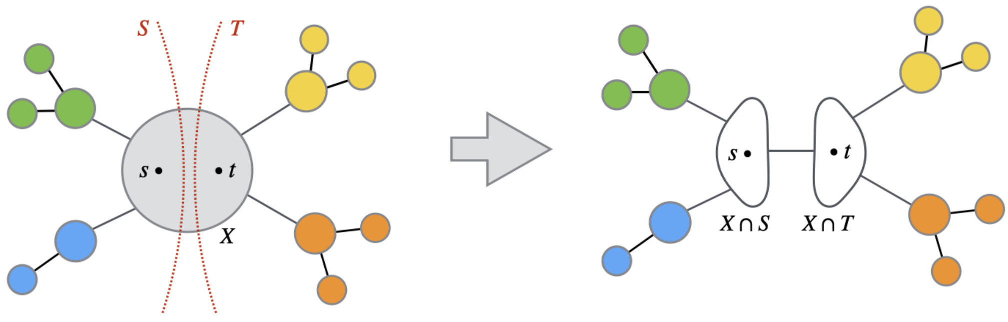

Figure 2: Splitting of a node X.

Lemma 1 *If $\mathcal{T}$ is a partial-Gomory-Hu-tree, then the cut (S, T) obtained in Step 3 of Algorithm 1 is an (s,t)-min-cut in G.*

Proof: Homework. Argue using Property 2 or Submodularity property from Lec 14. $\square$

Let $\mathcal{T}_1 = (\{V\}, \emptyset), \mathcal{T}_2, \ldots, \mathcal{T}_n$ be a sequence of trees obtained for G by recursively applying Algorithm 1. In order to prove correctness, we present the following lemma.

Lemma 2 *For $i \geqslant 3$, if tree $\mathcal{T}_{i-1}$ is a partial Gomory Hu tree, then so is tree $\mathcal{T}_i$.*

Proof: Let us suppose $\mathcal{T}_{i-1}$ is a partial Gomory Hu tree. Consider any two nodes X, Y that are adjacent in $\mathcal{T}_i$. We need to prove that there exists $x \in X$ and $y \in Y$ that satisfy

- $c^*(X, Y) = \lambda_{x,y,G}$.
- $\mathcal{T}_i - (X, Y)$ corresponds to an (x, y)-min-cut in G.

We have following three cases:

Case 1 X, Y were present in $\mathcal{T}_{i-1}$.
In this case the claim is obviously true.

Case 2 X, Y both appeared for first time in tree $\mathcal{T}_i$ due to splitting of a node in $\mathcal{T}_{i-1}$. In this case by Algorithm 1, the claim holds true.

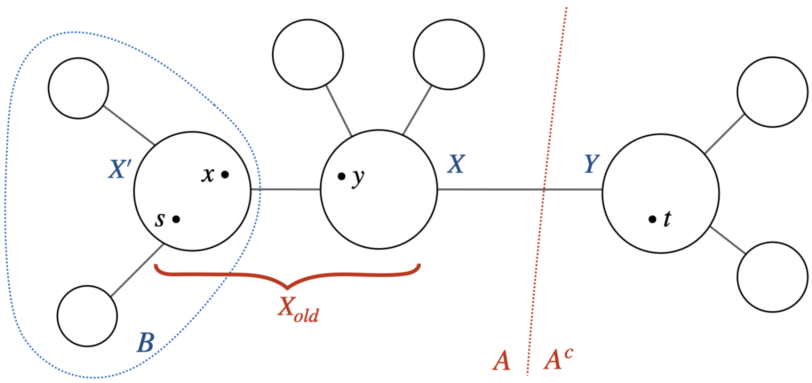

Figure 3: Illustration of Case 3(b).

Case 3 One of the nodes, say X, was created in round i by slitting an older node, say $X_{old} \in \mathcal{T}_{i-1}$, into X', X.

Let us suppose split occurred due to a vertex pair $(x, y) \in X' \times X$. At end of round $i-1$, by induction hypothesis we can assume there were nodes $(s, t) \in X_{old} \times Y$ satisfying the required claim. After splitting of X_{old}, either $s \in X'$ or $s \in X$.

- If $s \in X$, claim holds true.
- If $s \in X'$, then by Property 3, required vertex pair is $(y, t) \in X \times Y$.

This proves our claim. $\square$

Theorem 3 *For any n vertex undirected weighted graph $G = (V, E, c)$ one can construct a Gomory-Hu Tree in time $O\big(n \times \text{TIME}_{flow}(n)\big)$, where $\text{TIME}_{flow}(n)$ denotes the time to compute max-flow in an undirected graph on n vertices.*