Intermediate Microeconomic Theory

Tools and Step-by-Step Examples

Production Functions

Outline

- Production Function
- Marginal and Average Product
- Relationship between AP_L and MP_L
- Isoquants
- Marginal Rate of Technical Substitution
- Special Types of Production Functions
- Returns to Scale
- Technological Progress
- Appendix A. MRT as the Ratio of Marginal Products
- Appendix B. Elasticity of Substitution

Production Function

Production Function

- Inputs are factors of production, such as labor, capital, land, and any other element that the firm can transform into units of output.
- The production function represents how a certain amount of inputs is transformed into an amount of output q.
- *Example*:

$$q = f(L, K)$$

This production function describes how specific amounts of labor L and capital K are transformed into an amount of output q.

Production Function

- *Example 7.1: Examples of production functions.*
  - The Cobb-Douglas function is
  $$q = AK^{\alpha}L^{\beta},$$
  where A is a positive parameter, and $\alpha, \beta \in (0,1)$.
  - Consider $A = 3$, $\alpha = \beta = \frac{1}{2}$, $K = 4$ machines and $L = 9$ workers.
  - The maximum output the firm can generate is
  $$q = 3\times4^{1/2}\times9^{1/2} = 18 \text{ units.}$$

Production Function

- *Example 7.1* (continued):
  - If instead the firm produces only 14 units, it is not efficiently managing its available inputs.
    - The firm's efficiency is
    $$\frac{14}{18} = 0.77.$$
    - Alternatively, the firm has an inefficiency level of
    $$1 - 0.77 = 0.23.$$

Production Function

- *Example 7.1* (continued):
  - Other types of production functions are:

    (1) $q = aK + bL$, where a, b are positive parameters, and K, L enter linearly.

    (2) $q = Amin\{aK, bL\}$, where A, a, b are positive parameters, and K and L must be used in a certain proportion.

    (3) $q = AK^{\alpha} + bL$, where A, a, b are positive parameters, and one input (in this case L) enters linearly and the other input enters nonlinearly.

Marginal and Average Product

Average Product

- The average product is the total units of output per unit of input.
  - The average product of labor is $AP_L = \frac{q}{L}$.
  - The average product of capital is $AP_K = \frac{q}{K}$.
- *Example*:
  - If a firm produces $q = 100$ units of output, and hire $L = 4$ workers, its average product per worker is

$$AP_L = \frac{100}{4} = 25 \text{ units.}$$

  - Every worker produces *on average* 25 units ("labor productivity").

Average Product

- Consider Production function $q = 100\sqrt{L}$:
  - At A,
    - $L_A = 4.$
    - $q_A = 200.$
    - $AP_{L_A} = \frac{200}{4} = 50$ units.
  - At B,
    - $L_B = 16.$
    - $q_B = 400.$
    - $AP_{L_B} = \frac{400}{16} = 25$ units.

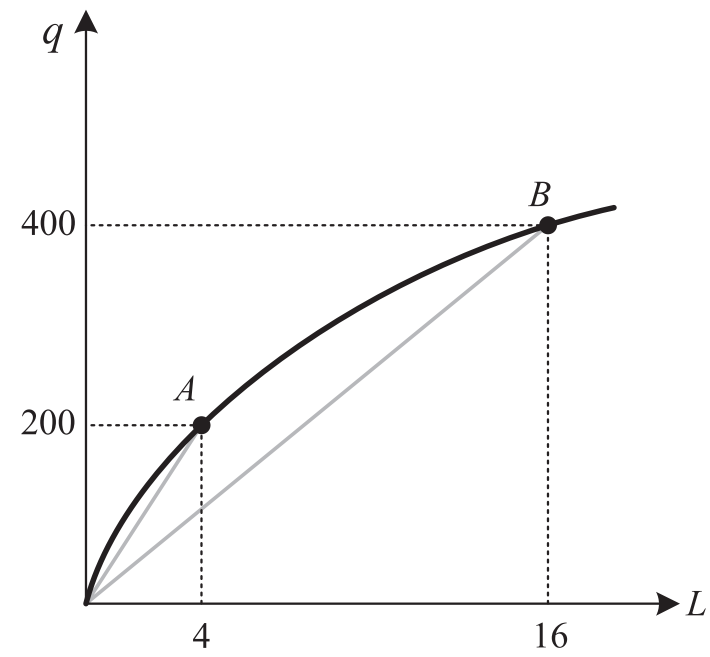


Figure 7.1

Average Product

- *Example 7.2* (continued):
  - Consider production function $q = 5L^{1/2} + 3L - 6$.
  - The average product of labor is

$$AP_L = \frac{q}{L} = \frac{5L^{1/2} + 3L - 6}{L}$$
$$= 5L^{1/2-1} + 3 - \frac{6}{L}$$
$$= \frac{5}{L^{1/2}} + 3 - \frac{6}{L}.$$

  - As L increases, AP_L increases if $\frac{\partial AP_L}{\partial L} \geq 0$,

$$-\frac{5}{2L^{\frac{3}{2}}} + \frac{6}{L^2} \geq 0,$$

Average Product

- *Example 7.2* (continued):

$$-\frac{6}{L^2} \geq \frac{5}{2L^{3/2}},$$

$$L^{3/2-2} \geq \frac{5}{12} \Longrightarrow \frac{12}{5} \geq L^{\frac{1}{2}}$$

$$\left(\frac{12}{5}\right)^2 \geq \left(L^{1/2}\right)^2$$

$$L \leq \frac{144}{25} \cong 5.76 \text{ workers}$$

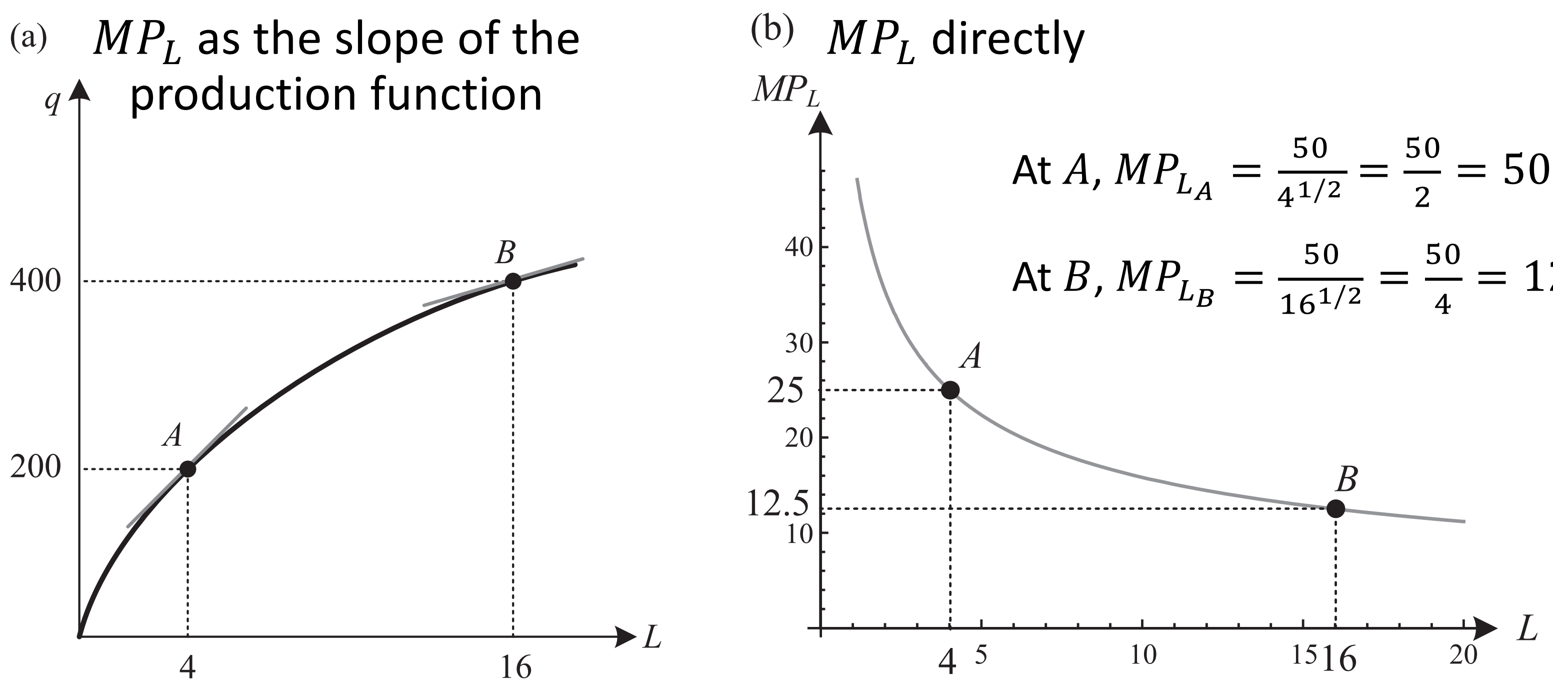
  - AP_L increases (decreases) in L for all $L \leq 5.76$ ($L > 5.76$).
  - AP_L reaches it maximum when $L = 5.76$ workers.

Marginal Product

- The marginal product is the rate at which total output increases as the firm uses an additional unit of either input.
  - The marginal product of labor is $MP_L = \frac{\Delta q}{\Delta L}$ when labor is discrete or $\frac{\partial q}{\partial L}$ when it is continuous.
  - The marginal product of capital is $MP_K = \frac{\Delta q}{\Delta K}$ when capital is discrete or $\frac{\partial q}{\partial K}$ when it is continuous.
- Graphically, the marginal product of an input can be interpreted as as the slope of the function when we marginally increase the amount of that input.

Marginal Product

- Consider production function $q = 100\sqrt{L}$.



At A, $MP_{L_A} = \frac{50}{4^{1/2}} = \frac{50}{2} = 50$ units.

At B, $MP_{L_B} = \frac{50}{16^{1/2}} = \frac{50}{4} = 12.5$ units.

Figure 7.2

MP_L is diminishing.

Marginal Product

- *Example 7.3: Finding marginal product.*
  - Consider the same production function $q = 5L^{1/2} + 3L - 6.$
  - The marginal product of labor is
$$MP_L = \frac{\partial q}{\partial L} = 5\frac{1}{2}L^{1/2-1} + 3 = \frac{5}{2L^{1/2}} + 3.$$
  - As *L* increases, MP_L decreases because
$$\frac{\partial MP_L}{\partial L} = -\frac{5}{4L^{\frac{3}{2}}} < 0, \text{ where } L > 0.$$
    - Additional workers bring more production to the firm, but a decreasing rate.

Relationship between AP_L and MP_L

Relationship between AP_L and MP_L

- The AP_L and MP_L exhibit interesting relationships:
  1. When the AP_L curve is increasing, MP_L lies above AP_L;
  2. When the AP_L curve is decreasing, MP_L lies below AP_L;
  3. When the AP curve reaches max, MP curve crosses AP .
- *Example*: Consider grades in a class.
  - You take a midterm exam. A few days later, the instructor informs how your average grade for the course is affected by the midterm:
    - the midterm *increases* your average if the midterm grade is higher than your previous average ($MP > AP$); or
    - the midterm *decreases* your average if the midterm grade is lower than your previous average ($MP < AP$); or
    - The midterm *does not affect* your average if the midterm grade coincides with your previous average ($MP = AP$).

Relationship between AP_L and MP_L

- The MP_L curve crosses the AP_L at the maximum point (the peak) of the AP_L curve.
- Consider production function $q = f(L)$.
  - The average product per worker is $AP_L = \frac{q}{L} = \frac{f(L)}{L}$.
  - To find the number of workers, L, at which AP_L reaches its maximum,

$$\frac{\partial AP_L}{\partial L} = \frac{f'(L)L - 1f(L)}{L^2} = 0,$$

    where we have used the quotient rule.
  - As $f'(L)$ is the marginal product of labor, $MP_L = \frac{\partial q}{\partial L}$,

$$\frac{MP_L L - 1f(L)}{L^2} = \frac{MP_L}{L} - \frac{f(L)}{L^2} = 0.$$

Relationship between AP_L and MP_L

- Multiplying both sides by L,

$$L\frac{MP_L}{L} - L\frac{f(L)}{L^2} = 0 \Longrightarrow MP_L - \frac{f(L)}{L} = 0.$$

- Note that $\frac{f(L)}{L} = AP_L$, then

$$MP_L = AP_L.$$

- This equation tells that, at the maximum of the AP_L curve, the MP_L curve crosses the AP_L curve.

Relationship between AP_L and MP_L

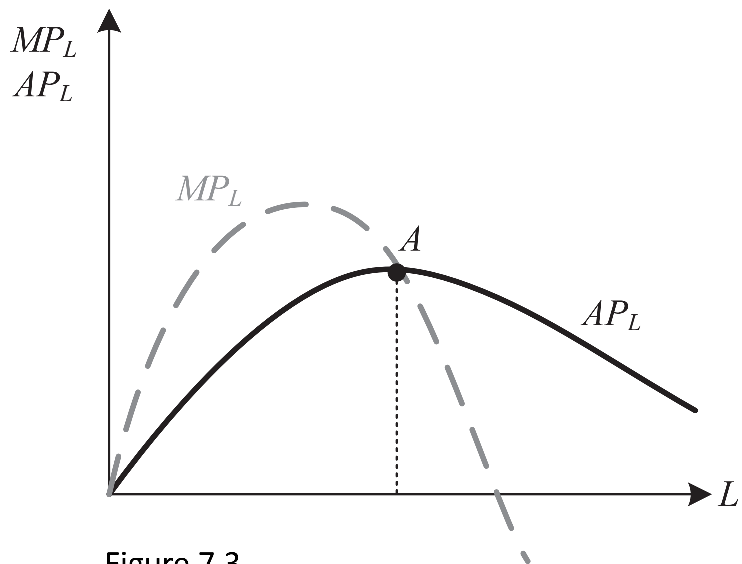


Figure 7.3

- When the AP_L curve is increasing, MP_L lies above AP_L;
- When the AP_L curve is decreasing, MP_L lies below AP_L;
- When the AP curve reaches the max, MP curve crosses AP .

Relationship between AP_L and MP_L

- *Example 7.4: Relationship between* AP_L and MP_L.
  - Consider the production function in examples 7.2 and 7.3,

$$q = 5L^{1/2} + 3L - 6.$$

  - From example 7.2,

$$AP_L = \frac{5}{L^{1/2}} + 3 - \frac{6}{L},$$

  which reaches its maximum at $L = \frac{144}{25} \cong 5.76$, where its height becomes,

$$AP_L = \frac{5}{(5.76)^{1/2}} + 3 - \frac{6}{5.6} \cong 4.04.$$

Relationship between AP_L and MP_L

- *Example 7.4* (continued):
  - If we evaluate the MP_L curve from example 7.3

$$MP_L = \frac{5}{2L^{1/2}} + 3$$

at the same $L = 5.76$, the height of the MP_L curve is

$$MP_L = \frac{5}{2(5.76)^{1/2}} + 3 \cong 4.04,$$

confirming that the MP_L crosses the AP_L at its maximum point.

Isoquants

Isoquants

- The isoquant curve represents combinations of labor and capital that yield the same amount of output.

  - At A, the firm uses an input combination intense in capital.
  - At B, it uses a labor-intense input combination, producing the same $q = 100$ than at A.
  - At C, the firm reaches a higher output, $q = 200$.

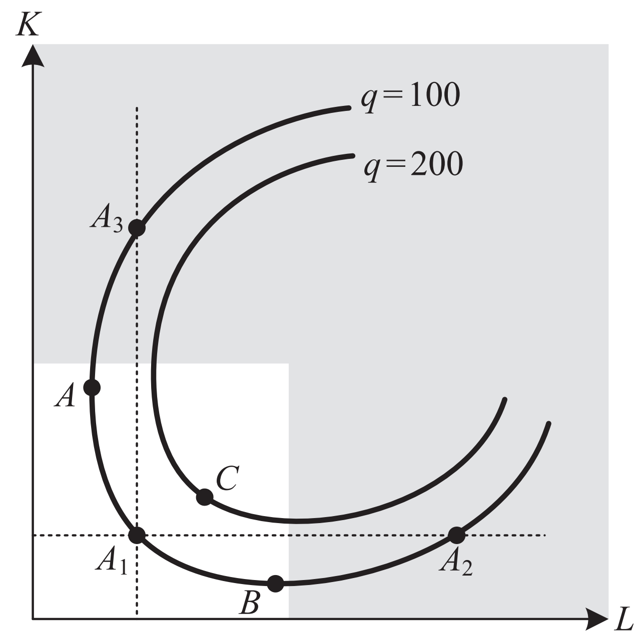


Figure 7.4

Isoquants

- The isoquant curve represents combinations of labor and capital that yield the same amount of output.

  - The shaded areas are unprofitable for the firm.
  - It would not choose A_2 or A_3 because it can reach the same output with less inputs at A_1.

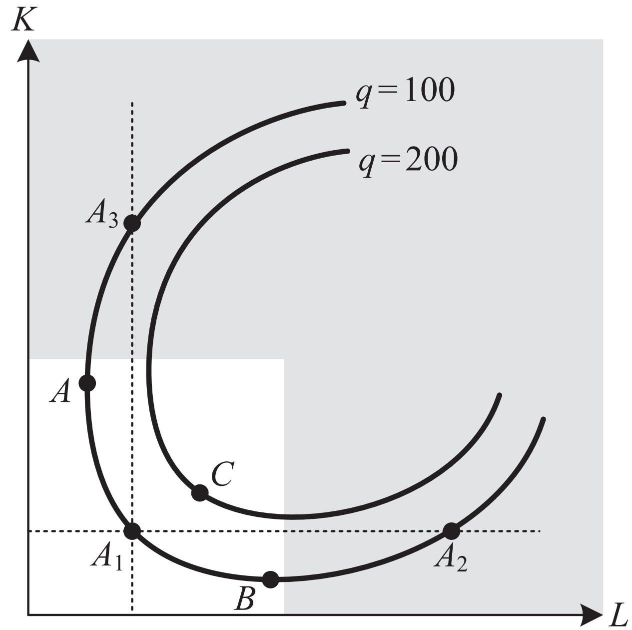


Figure 7.4

Isoquants

- *Example 7.5: Finding isoquant curves for a Cobb-Douglas production function*.
  - Consider a Cobb-Douglas production function $q = 5L^{1/2}K^{1/2}$.
  - To find the isoquant corresponding to $q = 100$ units:
    - Insert this output level into the production function,

$$100 = 5L^{1/2}K^{1/2},$$
$$20 = L^{1/2}K^{1/2}.$$

    - Solve for capital K,

$$20^2 = \left(L^{1/2}K^{1/2}\right)^2,$$
$$400 = LK \quad \Longrightarrow K = \frac{400}{L}.$$

Isoquants

- *Example 7.5* (continued):
  - Graphically, the isoquant $K = \frac{400}{L}$
    - is a curve approaching the vertical axis when L is close to zero (but it never crosses this axis);
    - and that approaches the horizontal axis when L is larger (without ever crossing).

Marginal Rate of Technical Substitution

Marginal Rate of Technical Substitution

- The slope of the isoquant answers the question:

How many units of capital must the firm give up to maintain its output level unaffected after hiring an extra worker?

- Marginal rate of technical substitution (MRTS). After increasing the quantity of labor by 1 unit, the MRTS measures the amount by which capital must be reduced so that output remains constant.

Marginal Rate of Technical Substitution

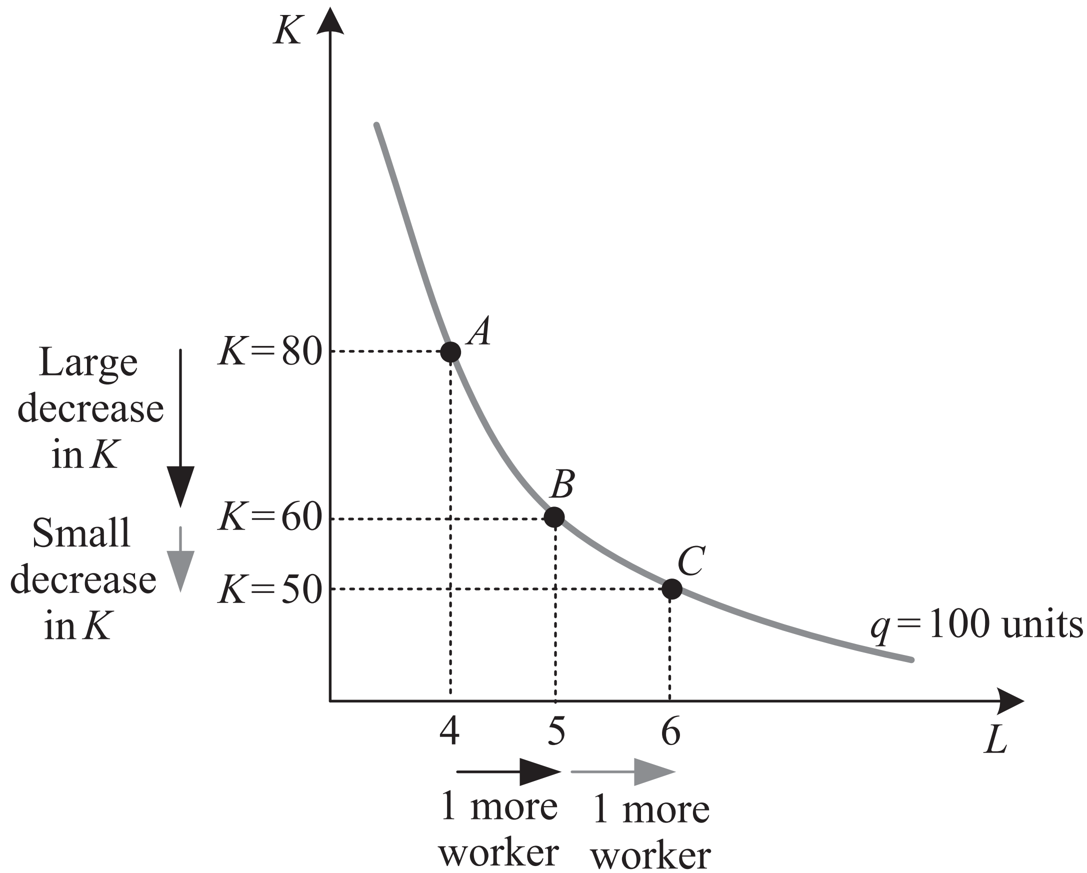


Figure 7.5

Marginal Rate of Technical Substitution

- Intuitively,
  - When capital is abundant, the firm is willing to give up many units of capital to hire one more worker.
  - When capital becomes more scarce, the firm is less willing to replace it with workers.

Marginal Rate of Technical Substitution

- *Example 7.6: Finding the MRTS of a Cobb-Douglas production function.*
  - Consider a firm with $q = 8L^{1/2}K^{1/2}$.
  - $MP_L = 8\frac{1}{2}L^{1/2-1}K^{1/2} = 4L^{-1/2}K^{1/2}$.
  - $MP_K = 8\frac{1}{2}L^{1/2}K^{1/2-1} = 4L^{1/2}K^{-1/2}$.
  - Hence, the MRTS is
  $$MRTS = \frac{MP_L}{MP_K} = \frac{4L^{-1/2}K^{1/2}}{4L^{1/2}K^{-1/2}} = \frac{K}{L},$$
  which is decreasing in L.
  - Graphically, the slope of the isoquant (MRTS) falls as we move rightward toward more units of L. That is, the isoquant is bowed in form the origin.

Marginal Rate of Technical Substitution

- *Example 7.7: Finding the MRTS of a linear production function.*
  - Consider a firm with $q = aL + bK$, where $a, b > 0$.
  - $MP_L = a$ and $MP_K = b$.
  - Hence, the MRTS is
  $$MRTS = \frac{MP_L}{MP_K} = \frac{a}{b},$$
  which is not a function of L or K. It is just a constant.
  - If $a = 6$ and $b = 3$, $MRTS = \frac{6}{3} = 2$. The slope of the isoquant would be -2 in all its points.
  - Graphically, the isoquant would be a straight line.

Special Types of Production Functions

Linear Production Function

- The linear production function takes the form

$$q = aL + bK, \text{ where } a, b \text{ are positive parameters.}$$

- Solving for K, the isoquant of this production function is a straight line

$$K = \frac{q}{b} - \frac{a}{b}L,$$

where $\frac{q}{b}$ is the vertical intercept and $\frac{a}{b}$ denotes its negative slope.

- The slope (MRTS) is constant along all points of the isoquant because $\frac{a}{b}$ is not a function of L or K.
  - The firm can substitute units of capital and labor at the same rate regardless of the number of input that it employs.

Linear Production Function

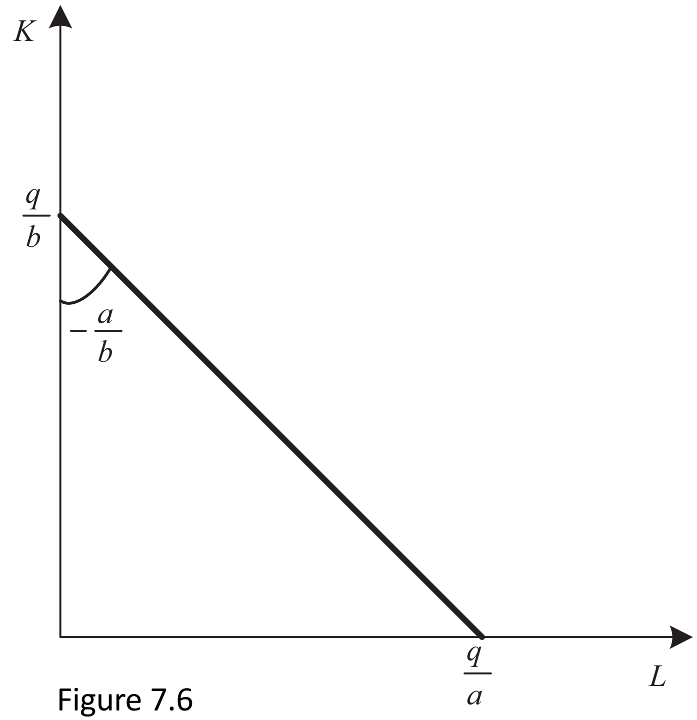


Figure 7.6

- Linear production functions can represent firms capable of substituting between inputs, such as two types of oils or computers.

Fixed-Proportions Production Function

- The firm cannot substitute between inputs and still maintain the same output level.
- Instead, the firm must use inputs in a fixed proportion.
- This production function takes the form

$$q = Amin\{aL, bK\}, \text{ where } A, a, b \text{ are positive.}$$

- *Example*: $q = \min\{2L, 3K\}$.
- An increase in one input without a proportional increase in the other input will not result in an increase in production.

Fixed-Proportions Production Function

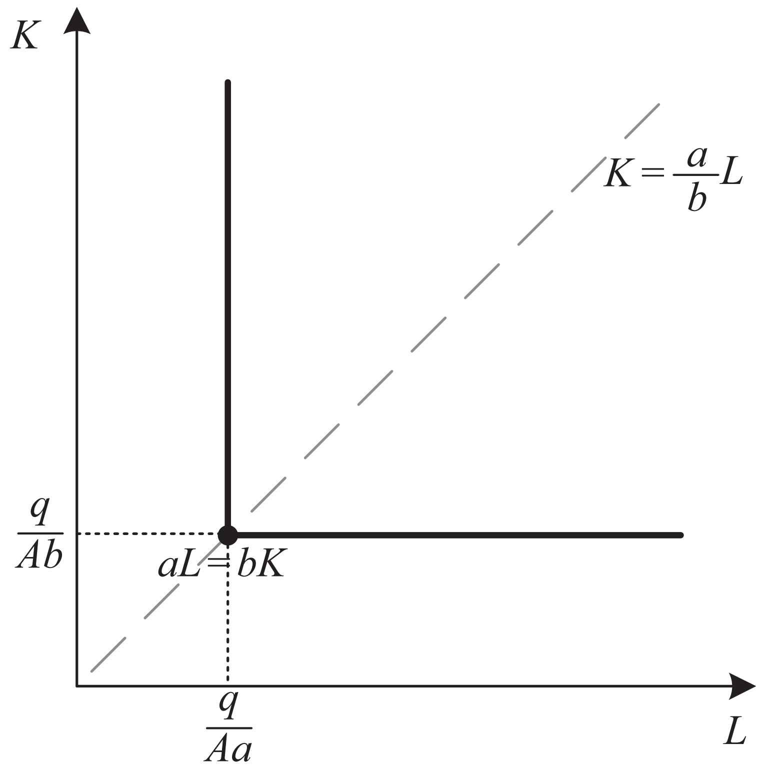


Figure 7.7

Fixed-Proportions Production Function

- Depending on the amounts of L and K, the firm faces either of the following cases:

  1. **If $\mathbf{min}\{\boldsymbol{aL}, \boldsymbol{bK}\} = \boldsymbol{aL}$** (when $aL < bK$), $q = AaL$.

     Solving for L, $L = \frac{q}{Aa}$, which is a vertical line.

     In this example, $q = \min\{2L, 3K\}$, if the firm produces $q = 100$, the vertical segment of the isoquant happens when

     $$2L < 3K \Longrightarrow \frac{2}{3}L < K,$$

     where the vertical line lies at

     $$L = \frac{q}{Aa} = \frac{100}{2} \text{ workers.}$$

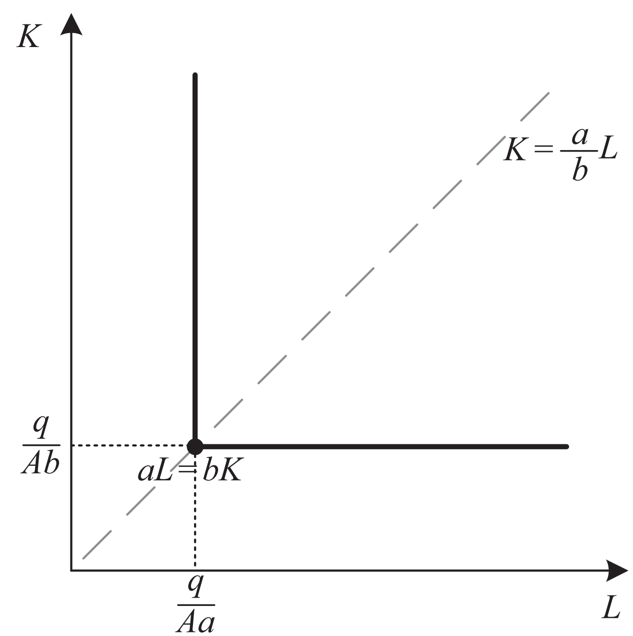


Figure 7.7

Fixed-Proportions Production Function

- Depending on the amounts of L and K, the firm faces either of the following cases:

2. **If $\min\{aL, bK\} = bK$** (when $aL > bK$), $q = AbK$.

   Solving for K, $K = \frac{q}{Ab}$, which is a horizontal line.

   In this example, $q = \min\{2L, 3K\}$, if the firm produces $q = 100$, the horizontal segment of the isoquant occurs when

   $$2L < 3K \Longrightarrow L > \frac{2}{3}K,$$

   where the horizontal line lies at

   $$K = \frac{q}{Ab} = \frac{100}{3} \text{ units of capital.}$$

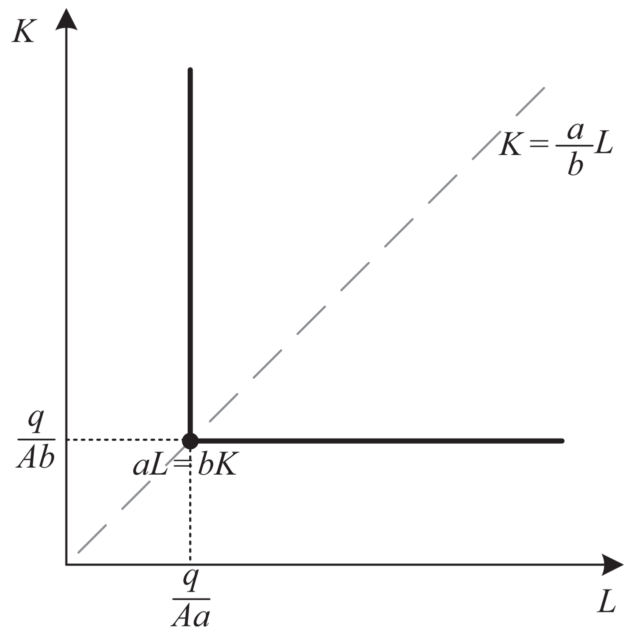


Figure 7.7

Fixed-Proportions Production Function

- Depending on the amounts of L and K, the firm faces either of the following cases:

  3. **If $\min\{aL, bK\}$ is either aL or bK** (when $aL = bK$), $q = AaL = AbK$.

     This occurs at the kink of the isoquant.

     Solving for K, yields a kink at $K = \frac{a}{b}L$.

     In this example, $q = \min\{2L, 3K\}$, the kink happens at
     $$K = \frac{2}{3}L.$$
     Graphically, the kinks of all isoquants are crossed by a ray from the origin with slope $\frac{2}{3}$.

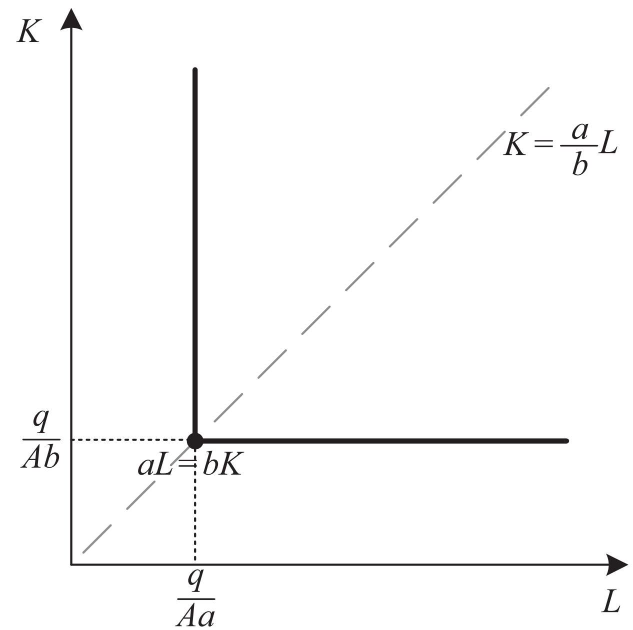


Figure 7.7

Fixed-Proportions Production Function

- The MRTS is not well defined because we can find infinitely many slopes for the isoquant at its kink.
- We can nonetheless say the slope of the isoquant is infinite in its vertical segment, and zero in its horizontal segment.
- This type of production function is common in firms that cannot easily substitute across inputs without altering total output.
- *Examples*:
    - Firms in the chemical industry.
    - Firms with a highly automated production process.

Cobb-Douglas Production Function

- The Cobb-Douglas production function takes the form

$$q = AL^{\alpha}K^{\beta} \text{ where } A, \alpha, \beta \text{ are positive.}$$

- *Example*: If $A = 1$ and $\alpha = \beta = 1/2$,

$$q = L^{1/2}K^{1/2}.$$

We find the isoquant

$$q^2 = \left(L^{1/2}K^{1/2}\right)^2,$$

$$q^2 = LK,$$

$$K = \frac{q^2}{L}.$$

Cobb-Douglas Production Function

The slope of the isoquant (MRTS) becomes

$$MRTS = \frac{MP_L}{MP_K}$$

$$= \frac{1/2L^{-1/2}K^{1/2}}{1/2L^{1/2}K^{-1/2}}$$

$$= \frac{K^{1/2+1/2}}{L^{1/2+1/2}}$$

$$= \frac{K}{L}.$$

Constant Elasticity of Substitution Production Function

- The constant elasticity of substitution (CES) production function takes the form

$$q = \left(aL^{\frac{\sigma-1}{\sigma}} + bK^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}},$$

where σ represents the elasticity of substitution.

  - When $\sigma \to \infty$, CES coincides with the linear production function, where the firm can substitute inputs.
  - When $\sigma = 0$, CES converges to the fixed-proportion production function.
  - When $\sigma = 1$, CES coincides with the Cobb-Douglas production function.

Returns to Scale

Returns to Scale

- Returns to scale. Consider that all inputs are increased by a common factor, $\lambda > 1$. Hence,L is increased to λL, and K is increased to λK. If the firm increases as follows:
  - $\lambda^a > \lambda$ (when $a > 1$), the firm exhibits *increasing* returns to scale.
  - $\lambda^a < \lambda$ (when $a < 1$), the firm exhibits *decreasing* returns to scale.
  - $\lambda^a = \lambda$ (when $a = 1$), the firm exhibits *constant* returns to scale.

Returns to Scale

- Consider a firm doubling the units of all inputs ($\lambda = 2$):
  - If output increases more than proportionally (more than double), we have increasing returns to scale.
  - If output increases less than proportionally (it falls short from doubling), we have decreasing returns to scale.
  - If output increases proportionally (exactly doubling), we have constant returns to scale.

Returns to Scale

- *Example 7.8: Testing for returns to scale*.
  - Consider a Cobb-Douglas production function
  $$q = AL^{\alpha}K^{\beta}.$$
  - If we increase all inputs by λ, (λL and λK), total output is now
  $$A(\lambda L)^{\alpha}(\lambda K)^{\beta} = A\lambda^{\alpha}L^{\alpha}\lambda^{\beta}K^{\beta},$$
  $$\lambda^{\alpha+\beta}\underbrace{\left(AL^{\alpha}K^{\beta}\right)}_{q} = \lambda^{\alpha+\beta}q.$$
  - Output increased by $\lambda^{\alpha+\beta}$, giving rise to 3 possible cases:
    - If $\alpha + \beta > 1$, increasing returns to scale.
    - If $\alpha + \beta < 1$, decreasing returns to scale.
    - If If $\alpha + \beta = 1$, constant returns to scale.

Returns to Scale

- *Example 7.8* (continued):
  - Consider a linear production function
$$q = aL + bK.$$
  - If we increase all inputs by λ, total output becomes
$$a(\lambda L) + b(\lambda K) = \lambda\underbrace{(aL + bK)}_{q} = \lambda q.$$
  - Output increased proportionally to inputs. The firm's production process exhibits constant returns to scale.

Returns to Scale

- *Example 7.8* (continued):
  - Consider a fixed-proportions production function
  $$q = Amin\{aL, bK\}.$$
  - If we increase all inputs by λ, total output becomes
  $$Amin\{a\lambda L, b\lambda K\} = \lambda \underbrace{Amin\{aL, bK\}}_{q} = \lambda q.$$
  - Output responds proportionally to a given increase in inputs. The firm's production process exhibits constant returns to scale.

Appendix A.
MRTS as the Ratio of Marginal Products

MRTS as Ratio of Marginal Products

- We show that the slope of the isoquant is measured by $\frac{MP_L}{MP_K}$.
- Consider a firm with production function $q = f(L, K)$.
- To evaluate the slope of the isoquant, we simultaneously increase labor (e.g., by 1 unit) and decrease capital.
- Hence, we totally differentiate the production function with respect to L and K

$$dq = \underbrace{\frac{\partial f(L,K)}{\partial L}}_{MP_L} dL + \underbrace{\frac{\partial f(L,K)}{\partial K}}_{MP_K} dK,$$

$$dq = MP_L dL + MP_K dK.$$

MRTS as Ratio of Marginal Products

- Because we are moving along different points of the firm's isoquant, the output level is the same, entailing $dq = 0$. Then,

$$dq = MP_L dL + MP_K dK,$$
$$0 = MP_L dL + MP_K dK,$$
$$MP_K dK = -MP_L dL.$$

- Because we are interested in the slope of the isoquant, we solve for $-\frac{dK}{dL}$, which reflects the rate at which the firm needs to decrease K if L increases by 1 unit,

$$\underbrace{-\frac{dK}{dL}}_{\text{Slope of isoquant}} = \underbrace{\frac{MP_L}{MP_K}}_{\text{Ratio of marginal products (MRTS)}}.$$

Appendix B.
Elasticity of Substitution

Elasticity of Substitution

- The elasticity of substitution is a common measure of how easy it is for a firm to substitute labor for capital,

$$\sigma = \frac{\%\Delta \frac{K}{L}}{\%\Delta MRTS} = \frac{\dfrac{\left(\frac{K}{L}\right)_B - \left(\frac{K}{L}\right)_A}{\left(\frac{K}{L}\right)_A}}{\dfrac{MRTS_B - MRTS_A}{MRTS_A}}.$$

- The elasticity of substitution tells us that, if the MRTS increases by 1%, the capital labor ratio that the firm uses, $\frac{K}{L}$, increases by $\sigma\%$.

Elasticity of Substitution

- Finding MRTS at two points to obtain σ.

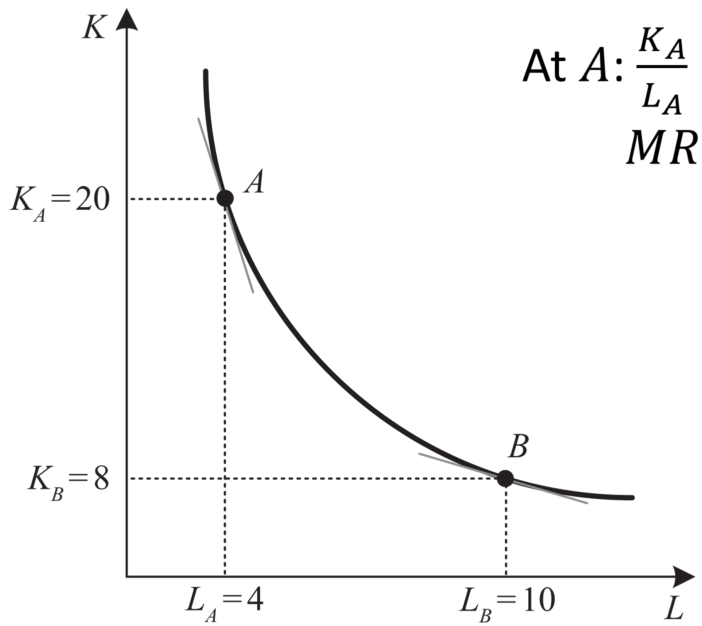


Figure 7.8

At A: $\frac{K_A}{L_A} = \frac{20}{4} = 5$, $\quad MRTS_A = 6.$

At B: $\frac{K_B}{L_B} = \frac{8}{10} = 0.8$, $\quad MRTS_B = 2.$

$$\sigma = \frac{\dfrac{\left(\frac{K}{L}\right)_B - \left(\frac{K}{L}\right)_A}{\left(\frac{K}{L}\right)_A}}{\dfrac{MRTS_B - MRTS_A}{MRTS_A}} = \frac{\dfrac{0.8 - 5}{5}}{\dfrac{2 - 6}{6}} = \frac{-0.84}{-\dfrac{2}{3}} = 1.26.$$

If the MRTS decreases by $2/3$ (about 66%), K/L decreases more than proportionally, by 84%.

Elasticity of Substitution

- *Linear production function*.
  - If the firm has a linear production function, $q = aL + bK$, its isoquants are straight lines.
  - $MRTS = \frac{a}{b}$ is constant along all the points of the isoquant.

$$\sigma = \frac{\frac{\left(\frac{K}{L}\right)_B - \left(\frac{K}{L}\right)_A}{\left(\frac{K}{L}\right)_A}}{0} = +\infty.$$

  - Regardless of % change in the K/L ratio, the elasticity of substitution is infinite.
    - The firm can substitute labor for capital very easily without altering its output.

Elasticity of Substitution

- *Fixed-proportions production function*.
  - Consider a firm with production function

$$q = Amin\{aL + bK\}.$$

  - $MRTS$ changes drastically as we move rightward, from the vertical to the horizontal segment of the L-shaped isoquant.

$$\sigma = \frac{\frac{\left(\frac{K}{L}\right)_B - \left(\frac{K}{L}\right)_A}{\left(\frac{K}{L}\right)_A}}{+\infty} = 0.$$

    - The firm cannot easily substitute units of labor for capital without affecting its output level.

Elasticity of Substitution

- *Cobb-Douglas production function*.
  - Consider a firm with production function $q = AL^{\alpha}K^{\beta}$.
  - First, rewrite the definition of the elasticity of substitution,

$$\sigma = \frac{\%\Delta \frac{K}{L}}{\%\Delta MRTS} = \frac{\frac{\Delta \frac{K}{L}}{\frac{K}{L}}}{\frac{\Delta MRTS}{MRT}} = \frac{\Delta \frac{K}{L}}{\Delta MRTS} \frac{MRTS}{\frac{K}{L}}.$$

  - Second, we find the MRTS,

$$MRTS = \frac{MP_L}{MP_K} = \frac{\alpha AL^{\alpha-1}K^{\beta}}{\beta AL^{\alpha}K^{\beta-1}} = \frac{\alpha}{\beta}\frac{K}{L}.$$

Elasticity of Substitution

- *Cobb-Douglas production function* (cont.).
  - Rearranging the expression of the MRTS yields the capital-labor ratio,

$$MRTS = \frac{\alpha}{\beta}\frac{K}{L},$$

$$MRTS\frac{\beta}{\alpha} = \frac{K}{L},$$

$$\Delta MRTS\frac{\beta}{\alpha} = \Delta\frac{K}{L},$$

$$\frac{\Delta\frac{K}{L}}{\Delta MRTS} = \frac{\beta}{\alpha}. \qquad (7.1)$$

Elasticity of Substitution

- *Cobb-Douglas production function* (cont.).
  - From the MRTS, $MRTS = \frac{\alpha}{\beta}\frac{K}{L}$, we also know

$$\frac{MRTS}{\frac{K}{L}} = \frac{\alpha}{\beta}. \tag{7.2}$$

  - Inserting (7.1) and (7.2) into the definition of elasticity of substitution,

$$\sigma = \frac{\Delta\frac{K}{L}}{\Delta MRTS}\frac{MRTS}{\frac{K}{L}} = \underbrace{\frac{\beta}{\alpha}}_{\text{From (7.1)}} \underbrace{\frac{\alpha}{\beta}}_{\text{From (7.2)}} = 1.$$

  - Therefore, the Cobb-Douglas production function has $\sigma = 1$, regardless of the value of parameters A, α, β.

Elasticity of Substitution

- *CES production function*.
  - Consider a firm with production function

$$q = \left(aL^{\frac{\sigma-1}{\sigma}} + bK^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}.$$

  - To find its elasticity of substitution, we first obtains its MRTS,

$$MRTS = \frac{MP_L}{MP_K} = \frac{aL^{-\frac{1}{\sigma}}}{bK^{-\frac{1}{\sigma}}} = \frac{a}{b}\left(\frac{K}{L}\right)^{\frac{1}{\sigma}},$$

$$ln(MRTS) = ln\frac{a}{b} + \frac{1}{\sigma}\ln\left(\frac{K}{L}\right),$$

$$\frac{1}{\sigma}ln\left(\frac{K}{L}\right) = ln(MRTS) - ln\frac{a}{b}.$$

Elasticity of Substitution

- *CES production function* (cont.)
  - Multiplying both sides by σ, yields

$$ln\left(\frac{K}{L}\right) = \sigma ln(MRTS) - \sigma ln\frac{a}{b}.$$

  - Therefore, the elasticity of substitution between labor and capital is the derivative of this expression with respect to $ln(MRTS)$,

$$\frac{\partial\, ln\left(\frac{K}{L}\right)}{\partial ln(MRTS)} = \sigma,$$

  which coincides with the term σ in the exponent of the CES function.