COL 351: Analysis and Design of Algorithms

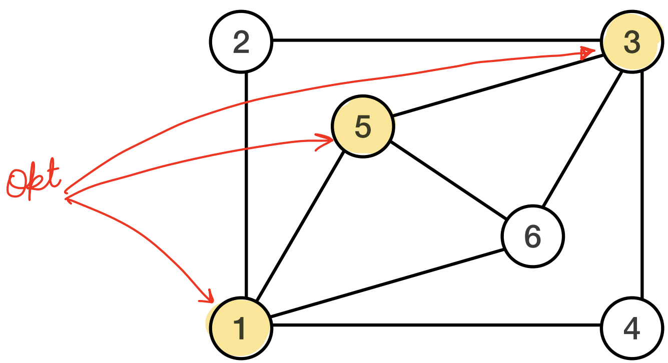
Lecture 3

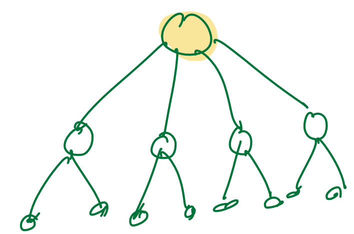
Facebook Network

Given: A Facebook network $G = (V, E)$ with n users.

Question: What is the minimum number of accounts that when deactivated results in zero friends of all remaining users.

Example:





counter-example

Ques: Can greedily removing vertices of maxdegree gives optimal ?

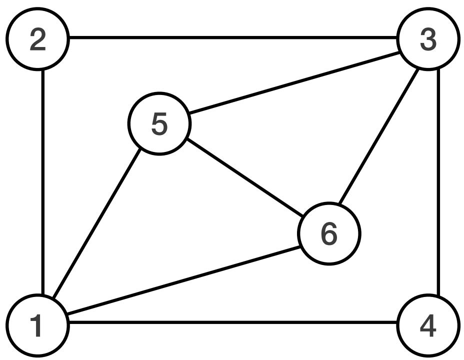
No

Vertex Cover

Given: A graph $G = (V, E)$ with n vertices.

Def: A subset $S \subseteq V$ such that for each $(a, b) \in E$, **at least** one end-point of (a, b) lies in S.

Example:



Can we find optimal solution efficiently?

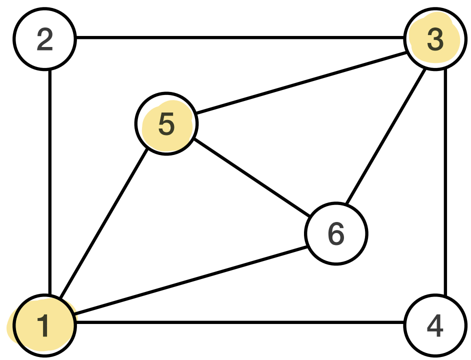
No, vertex cover is NP-hard problem and a poly-time sol^n is very-unlikely.

Optimization question: Find a vertex-cover of minimum possible size.

Approximate Vertex Cover

Given: A graph $G = (V, E)$ with n vertices.

Def: A subset $S \subseteq V$ such that for each $(a, b) \in E$, **at least** one end-point of (a, b) lies in S.



opt sol^n = {1, 3, 5}

Approximation Problem: Find a solution S satisfying …

$$|VC_{opt}| \leq |S| \leq \text{constant} * |VC_{opt}|$$

Greedy Approach-I

Repeatedly pick an uncovered edge, and add one of its end-points in the solution set.

Initialise $S := \phi$.

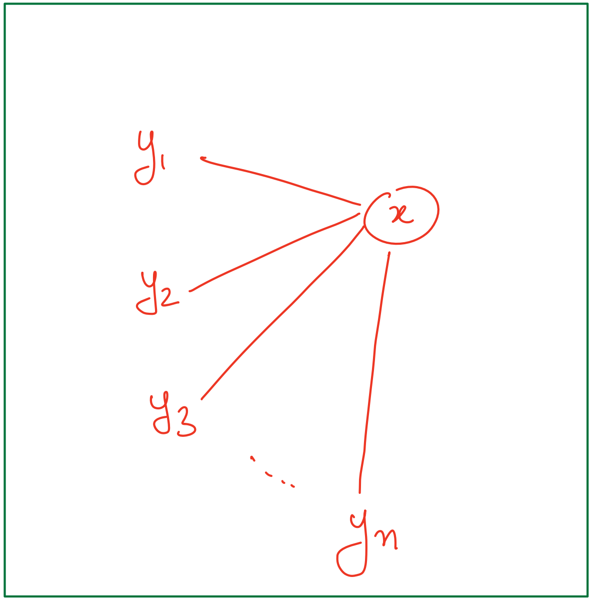
While there is an uncovered edge :

— Pick an uncovered edge (x, y).

— Add x ***or*** y to S.

— Mark all edges incident to “new" vertex added to S as covered.

Return S.



$|opt| = 1$

$|greedy| = n-1$

$OPT = \{x\}$

$greedy = \{y_1 \cdots y_n\}$

Approx-factor $= (n-1)$

Greedy Approach-II

Repeatedly choose vertices incident to the largest number of *currently* uncovered edges.

Initialise $S := \phi$.

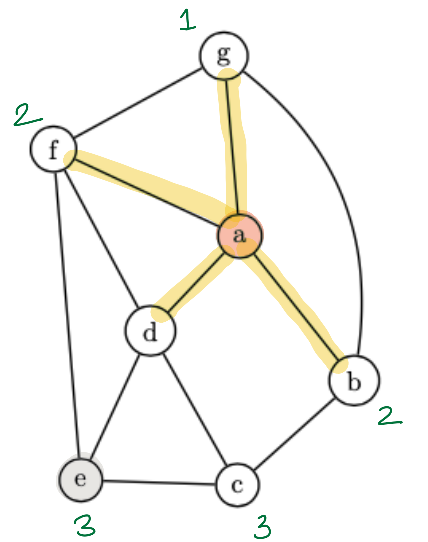
While there exists an uncovered edge :

— Pick a vertex x incident to maximum number of "uncovered edges".

— Add x to S.

— Mark all edges incident to x as covered.

Return S.



$S = \{a, e, \ldots\}$

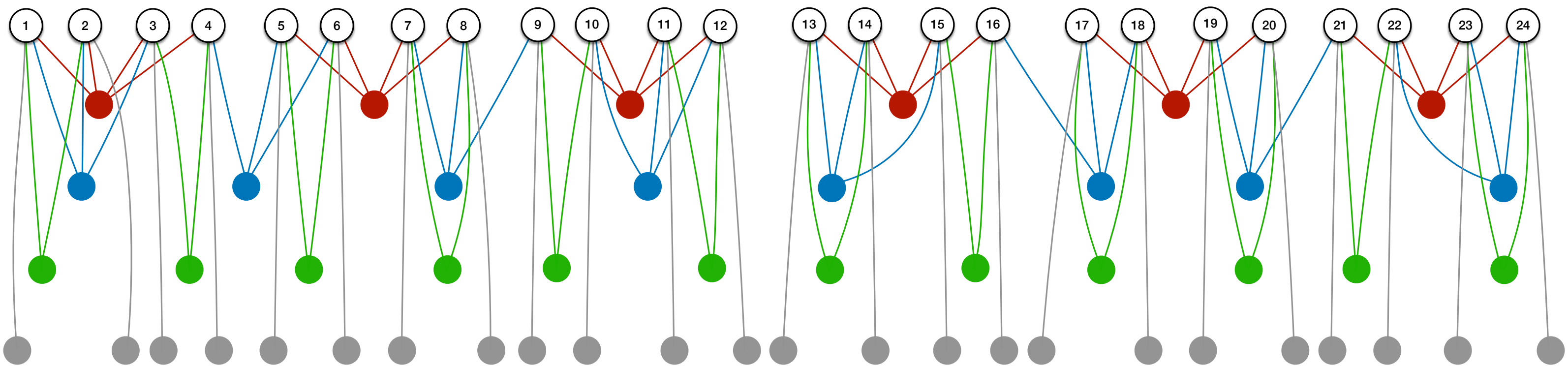
We will prove in lecture 3 that approximation is $O(\log n)$

Greedy Approach-II

⊛ $|\text{Input}| = O(\ell n \cdot \log n)$

⊛ approx-factor $= O(\log n)$

Repeatedly choose vertices incident to the largest number of *currently* uncovered edges.



$|\text{opt-vertex-cover}| = 24 = \ell n$

greedy sol = Red vertices $\cup$ Blue vertices $\cup$ _ _.

= colored vertices.

deg = 4	Red — 6
deg = 3	Blue — 8
deg = 2	Green — 12
deg = 1	Grey — 24

$|\text{greedy sol}^n| = 24 + 12 + 8 + 6 =$

$= 24\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) \simeq \ell n \log n$

Greedy-III (and stupid) ?

Repeatedly pick an uncovered edge, and add both of its end-points in the solution set.

Initialise $S := \phi$.

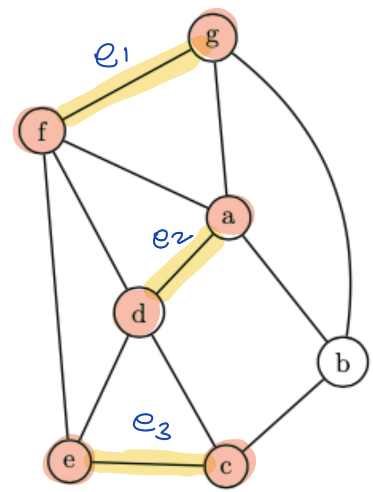
While there exists an uncovered edge :

— Pick an uncovered edge (x, y).

— Add x ***and*** y to S.

— Mark all edges incident to "new" vertices added to S as covered.

Return S.



Suppose edges & covered are : e_1 e_2 e_3

$|VC_{opt}| \geq 3$

This gives us 2-approximation.

Greedy-III

Repeatedly pick an uncovered edge, and add both of its end-points in the solution set.

Edges considered at this step don't share a vertex

Initialise $S := \phi$.

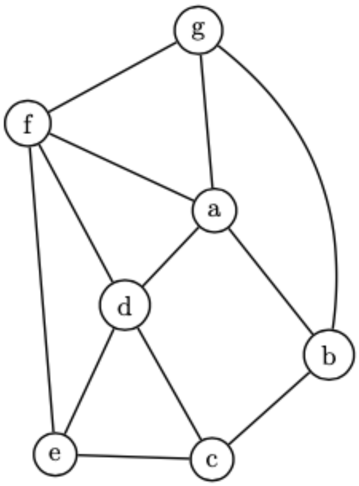
While there exists an uncovered edge :

— Pick an uncovered edge (x, y).

— Add x ***and*** y to S.

— Mark all edges incident to "new" vertices added to S as covered.

Return S.



Theorem:

If S is the output of the above algorithm, then $|S| \leqslant 2\ VC_{opt}$, where VC_{opt} is an optimal vertex cover.

Correctness

Theorem:

If S is the output of the algorithm, then $|S| \leqslant 2\ VC_{opt}$, where VC_{opt} is an optimal vertex cover.

Proof:

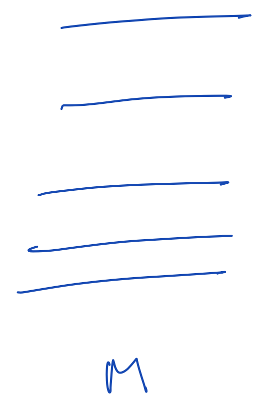
Claim (HW) : Edges considered do not share an ~~edge~~ vertex.

Let M = set of considered edges

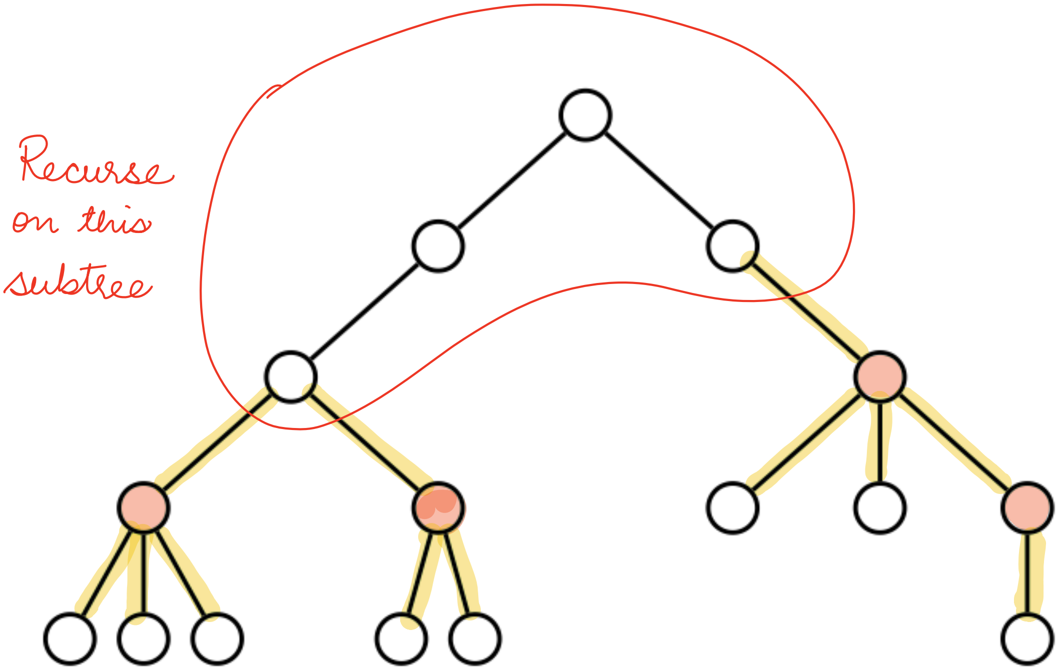
⊛ greedy sol^n size $= |S| = 2|M|$

⊛ $|VC_{opt}| \geqslant |M| = \frac{|S|}{2}$

Thus, we get $|S| \leq 2 \cdot |VC_{opt}|$



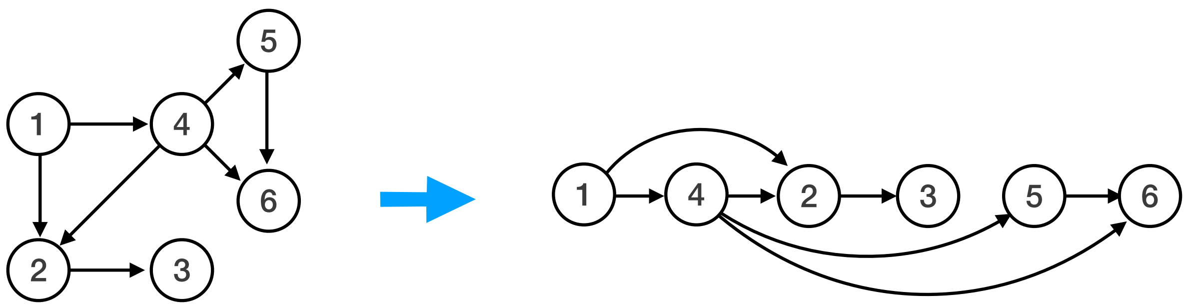
Vertex Cover in Trees/Forests



Theorem: We can compute a smallest vertex cover for tree on n vertices in $O(n)$ time

Topological Sort: Greedy Algorithm

GOAL : A linear ordering of vertices of DAG such that for every directed edge (u, v), u comes before v in the ordering.



H.W. 1 Prove that DAG contains a vertex of deg "0"

H.W. 2 Give $O(m+n)$ time algo to find topological ordering of DAG using only ARRAYS & DOUBLY-LINK-LIST.