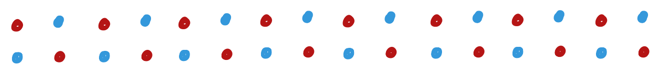
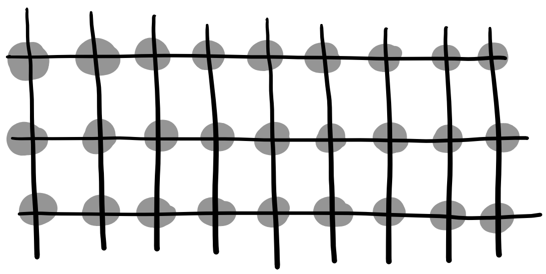
3.1: Hint: consider the pairs $\{1,2\}, \{3,4\}, \ldots, \{2n-1, 2n\}$.

3.2: If we consider only two horizontal lines, then we wont get a monochromatic rectangle.

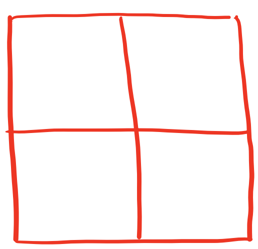


Hint: Consider three horizontal lines, and 9 vertical lines.



what can you say about the color pattern for the marked intersection points? Use this to conclude that there exists a monochromatic rectangle.

3.3: Hint: Consider four sub-squares inside the unit square. By PHP, two of the points are inside one sub-square.



3.4 Hint: the same conclusion would hold if we replace 9 with 10, 8, 6 or 4.

$f(x) = 11$ for 4 distinct integers.

$$f'(x) = f(x) - 11 = (x-\theta_1)(x-\theta_2)\cdots(x-\theta_4)\,g(x)$$
for $\theta_1, \dots, \theta_4 \in \mathbb{Z}$,
$g(x) \in \mathbb{Z}[x]$.

Suppose $\exists\ \theta \in \mathbb{Z}$ s.t. $f(\theta) = 9$
$\Rightarrow f'(\theta) = -2$

$$(\theta-\theta_1)(\theta-\theta_2)\cdots(\theta-\theta_4)\,g(\theta) = -2$$
for $\theta_1, \dots, \theta_4 \in \mathbb{Z}$,
$g(x) \in \mathbb{Z}[x]$.

-2 has 4 possible factors: $1, 2, -1, -2$.
$\Rightarrow (\theta-\theta_1), (\theta-\theta_2), \dots, (\theta-\theta_4), g(\theta) \in \{1, 2, -1, -2\}$

Moreover, exactly one of $\{(\theta-\theta_1), \dots, (\theta-\theta_4), g(\theta)\}$ can be ± 2. The rest must be ± 1.

$\Rightarrow$ At least three out of $\{(\theta-\theta_1), \dots, (\theta-\theta_4)\}$ must be either 1 or -1.

As a result, $\exists\ i \neq j$ s.t. $\theta - \theta_i = \theta - \theta_j$.

Contradiction since we assumed all θ_i are distinct. ∎

3.5 : Hint : Consider a related problem. For any n, there exists a multiple of n of the form $111\ldots1100\ldots0$.

Here 2 or 5 can divide n.

Proof of Easy Version :

Consider the set of numbers
$$S = \{1, 11, 111, \ldots, \underbrace{111\ldots11}_{n+1 \text{ ones}}\}$$

$|S| = n+1$. By PHP, $\exists$ two numbers $a, b \in S$, $a < b$, s.t. $a \bmod n = b \bmod n$.

Observation : $b - a$ is of the form $11\ldots100\ldots0$.

n divides $(b-a)$.

Proof of 3.4 : By the same argument as the easier version, $\exists\, k \in \mathbb{N}$ s.t.
$$n \cdot k = 11\cdots100\cdots0 = (11\cdots1) \times 10^c$$
for some $c \geq 0$.

Since 2 and 5 dont divide n,
$\gcd(n, 10^c) = 1$.

$\therefore$ n divides $11\ldots1$.

$\blacksquare$

3.6 : Hint : Take any two points, there exists a 'great circle' that passes through the two points.

great circle divides the sphere into two hemispheres.

Proof: Take 2 out of the 5 points, there exists a great circle through the two points.

Out of the remaining 3 points, two of them are in the same hemisphere.

Therefore, four of five points are in the same closed hemisphere.

This exercise is an illustration of how PHP can be used for geometric problems. This problem uses a non-trivial fact about spheres (great circle through any two points on a sphere). I will not expect you to know such facts for the quizzes / exams.

3.7 : Let $N = 10^n$. We need to show that $\exists\, k \in \mathbb{N}$ s.t. $F_k \bmod N = 0$.

Failed Attempt : We need to show that $\exists\, k \in \mathbb{N}$ s.t. $F_k \bmod N = 0$.

A natural first attempt is to consider $(F_1 \bmod N,\ F_2 \bmod N,\ \dots,\ F_{N+1} \bmod N)$

There are $N+1$ numbers, hence two of them must be equal. But it is not clear how to argue that $0 \in \{F_1 \bmod N, \dots, F_{N+1} \bmod N\}$.

Consider $(F_i \bmod N,\ F_{i+1} \bmod N)$.
There are only N^2 possible values.

Therefore, $\exists\ j_1 < j_2 \leq N^2+1$ s.t.

$$(F_{j_1} \bmod N,\ F_{j_1+1} \bmod N) = (F_{j_2} \bmod N,\ F_{j_2+1} \bmod N)$$

Let j_1^* be the smallest index ≥ 1 s.t. $\exists\ j_2 > j_1^*$ s.t. $(F_{j_1^*} \bmod N,\ F_{j_1^*+1} \bmod N) =$
$(F_{j_2} \bmod N,\ F_{j_2+1} \bmod N)$

Claim: $j_1^* = 1$.

Proof: Suppose $j_1^* > 1$.

Consider $(F_{j_1^*-1} \bmod N, \; F_{j_1^*} \bmod N)$.

$$F_{j_1^*-1} \bmod N = (F_{j_1^*+1} - F_{j_1^*}) \bmod N$$

$$= (F_{j_1^*+1} \bmod N - F_{j_1^*} \bmod N) \bmod N$$

$$= (F_{j_2+1} \bmod N - F_{j_2} \bmod N) \bmod N$$

$$= F_{j_2-1} \bmod N$$

Hence j_1^* is not the smallest index ≥ 1 s.t. $\exists\, j_2 > j_1^*$ s.t. $(F_{j_1^*} \bmod N, \; F_{j_1^*+1} \bmod N) = (F_{j_2} \bmod N, \; F_{j_2+1} \bmod N)$ ∎

As a result, $\exists\, j_2$ s.t. $F_{j_2} \bmod N = F_{j_2+1} \bmod N = 1$.

$\Rightarrow F_{j_2-1} \bmod N = 0$. ∎