# ELL205 Signals and Systems
# Mid-Term Examination

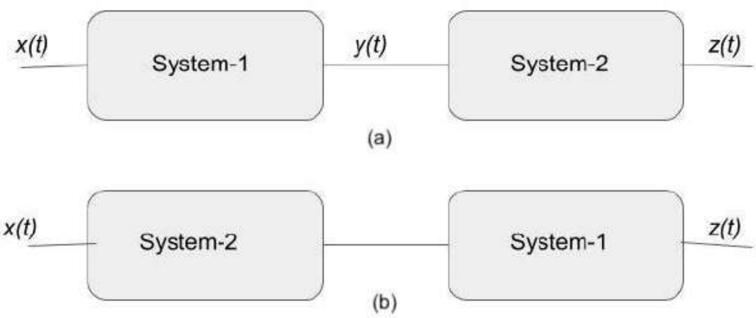
**Semester I, 2023-2024**

**60 marks, 120 minutes**

*NOTE:*

1) *Please work out your answers and then write them neatly in the answer script. The logic used should be clear. Your answers should be short and to the point. Simply state the property that you are using. Using short-hand notation is fine - eg.* $x(t) \leftrightarrow a_k \Rightarrow x(t-1) \leftrightarrow a_k e^{-j2\pi k/T}$ *(Shift property)*
2) *Your answers should be legible, else they will not be evaluated.*
3) *Answers not accompanied by clear justification will not receive any credit.*
4) *You need not answer questions in order, but the question and part numbers should be very clearly indicated in the left side of the page.*

1) (**17 marks**) Consider a linear time-invariant system (system-1) with impulse response $h(t) = \exp(-t)u(t)$, input $x(t)$, and output $y(t)$. Let $z(t) = \sum_{n=-\infty}^{\infty} y(t-nT)$ denote the output of a system with input $y(t)$ (system-2) connected in cascade to system-1 as shown in Fig. (a) below.




a) (**4 marks**) (i) Is system-2 linear? (ii) Is it time-invariant? (iii) Is it stable? (iv) Is it invertible?
ANS: The system is linear. $x(t-\tau)$ for $\tau$ a multiple of $T$ produces the same output - hence not time-invariant. If $x(t) = u(t)$ and therefore bounded to unity, output is infinite - hence not stable. No. $x(t) = \delta(t)$ and $x(t) = \delta(t-T)$ produce the same output.

b) (**2 marks**) Express $z(t)$ of Fig (a) in terms of $x_p(t) = \sum_{n=-\infty}^{\infty} x(t+nT)$ and $h(t)$. Is $z(t)$ periodic?

ANS: Note that $h_p(t) = \sum_{l=-\infty}^{\infty} h(t-lT)$ is periodic with time-period $T$. Now:

$$\begin{aligned}
y(t) &= \int_{\infty}^{\infty} x(t)h(t-\tau)d\tau \\
z(t) &= \sum_{l=-\infty}^{\infty} y(t-lT) = \sum_{l=-\infty}^{\infty} \int_{\infty}^{\infty} x(\tau)h(t-\tau-lT)d\tau \\
&= \int_{\infty}^{\infty} x(\tau) \sum_{l=-\infty}^{\infty} h(t-\tau-lT) = \int_{\infty}^{\infty} x(\tau)h_p(t-\tau)d\tau
\end{aligned}$$

Clearly since $h_p(t)$ is periodic, $z(t+T) = \int_{\infty}^{\infty} x(\tau)h_p(t+T-\tau)d\tau = \int_{\infty}^{\infty} x(\tau)h_p(t-\tau)d\tau = z(t)$. $z(t)$ is clearly periodic with time-period $T$.

c) (**2 marks**) Let $h_p(t) = \sum_{l=-\infty}^{\infty} h(t-lT)$. Can you express $z(t)$ in terms of $x(t)$ and $h_p(t)$? Can system-1 and system-2 be interchanged to obtain $z(t)$ as depicted in Fig (b)?

ANS: Note that $x_p(t) = \sum_{l=-\infty}^{\infty} x(t-lT)$ is periodic with time-period $T$. Now:

$$\begin{aligned} y(t) &= \int_{\infty}^{\infty} h(t)x(t-\tau)d\tau \\ z(t) &= \sum_{l=-\infty}^{\infty} y(t-lT) = \sum_{l=-\infty}^{\infty} \int_{\infty}^{\infty} h(\tau)x(t-\tau-lT)d\tau \\ &= \int_{\infty}^{\infty} h(\tau) \sum_{l=-\infty}^{\infty} x(t-\tau-lT) = \int_{\infty}^{\infty} h(\tau)x_p(t-\tau)d\tau \end{aligned}$$

Clearly since $x_p(t)$ is periodic, $z(t+T) = \int_{\infty}^{\infty} h(\tau)x_p(t+T-\tau)d\tau = \int_{\infty}^{\infty} h(\tau)x_p(t-\tau)d\tau = z(t)$. $z(t)$ is clearly periodic with time-period $T$.

d) (**4 marks**) Express $z(t)$ in terms of $x_p(t)$ and $h_p(t)$.

ANS: Start with $z(t) = \int_{\infty}^{\infty} h(\tau)x_p(t-\tau)d\tau$. Using the fact that $x_p(t)$ is periodic modulo $T$, we can break up the summation over $\tau$ as follows. Write $\tau = \tau^{'} + mT$ where $m$ is an integer with range of $\tau^{'}$ begin 0 to $T$ only. Then

$$z(t) = \int_0^T h(\tau)x_p(t-\tau)d\tau$$

e) (**2 marks**) What are the complex Fourier series coefficients of $h_p(t)$?

ANS: $h_p(t) = \sum_{l=-\infty}^{\infty} h(t-lT) = \sum_{l=-\infty}^{\infty} \exp\left(-(t-lT)\right) u(t-lT)$. Consider $h_p(t)$ for $0 \le t \le T$ only. Then the range of $l$ is limited to $-\infty$ to 0 so that: $h_p(t) = \sum_{l=-\infty}^{0} \exp\left(-(t-lT)\right) = \exp(-t)\sum_{l=0}^{\infty} \exp\left(-lT\right) = \frac{\exp(-t)}{1-e^{-T}}$. Clearly

$$a_k = \frac{1}{T}\int_0^T h_p(t)e^{-j2\pi kt/T}dt = \frac{1}{T}\frac{1}{1+j2\pi k/T}$$

f) (**3 marks**) Evaluate the following summation using the properties of Fourier series: $\sum_{k=-\infty}^{\infty} \frac{1}{\left|1+j\frac{2\pi k}{T}\right|^2}$.

ANS: Since $h_p(t) \leftrightarrow a_k = \frac{1}{T}\frac{1}{1+j2\pi k/T}$, it follows from the linearity property of fourier coefficients that: $Th_p(t) = h_p^{\prime}(t) \leftrightarrow a_k^{\prime} = \frac{1}{1+j2\pi k/T}$. Hence, it follows from Parseval's relation that:

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \frac{1}{\left|1+j\frac{2\pi k}{T}\right|^2} = \sum_{k=-\infty}^{\infty} |a_k^{\prime}|^2 &= \frac{1}{T}\int_0^T |h_p^{\prime}(t)|^2 dt \\ &= \frac{1}{T}\int_0^T |Th_p(t)|^2 dt \\ &= \frac{T^2}{T(1-e^{-T})^2}\int_0^T e^{-2t}dt \\ &= \frac{T}{(1-e^{-T})^2}\frac{1-e^{-2T}}{2} \\ &= \frac{T(1-e^{-2T})}{2(1-e^{-T})^2} \end{aligned}$$

2) (**5 marks**) You are given a linear time-invariant system with input $x[n]$ and output $y[n]$ related by

$$y[n] = \sum_{k=n-5}^{n+5} \left(\frac{1}{2}\right)^{|k|} x[k].$$

a) (**2 marks**) Determine its impulse response.
ANS: We know that impulse response is response of the system at unit impulse input. So,

$$h[n] \quad = \quad \sum_{k=n-5}^{n+5} \left(\frac{1}{2}\right)^{|k|} \delta[k]$$

The above equation is zero everywhere except the points where one of the numbers between $n-5$ & $n+5$ is zero, where the above expression evaluates to 1. Hence,

$$h[n] = \begin{cases} 1 & -5 \leq n \leq 5 \\ 0 & otherwise \end{cases}$$

b) (**3 marks**) Determine its output to $x[n] = \delta[n+1] + \delta[n] + \delta[n-1]$.
ANS: As the given system is an LTI system, for $x[n] = \delta[n+1] + \delta[n] + \delta[n-1]$, $y[n] = h[n+1] + h[n] + h[n-1]$
Now, we can calculate y[n] either by explicitly adding all the 3 components or by plotting the graph for all 3 and adding them graphically. In either case, the final result we get is

$$y[n] = \begin{cases} 3 & -4 \leq n \leq 4 \\ 2 & n = -5, 5 \\ 1 & n = -6, 6 \\ 0 & otherwise \end{cases}$$

3) (**8 marks**) Consider a periodic signal $x(t)$ with complex Fourier series coefficients $a_k$. You are given that $a_k = 0$ except in the range $0 \leq k \leq N-1$.

a) (**1 mark**) Is $x(t)$ real or complex?
Since $a_k = a^*_{-k}$ for real signals, it is clear that the periodic signal is complex.

b) (**7 marks**) Let $b_k = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{N-1} a_m \delta[k-m-lN]$. Determine the periodic signal $y(t)$ with Fourier series coefficients $b_k$.
This is an extension of a problem from the textbook, which was covered in the tutorials - 3.23 d). 3.41 also covers similar ideas. Here, the FS coefficients are $....a_0, a_1, \ldots, a_{N-1}, a_0, a_1, a_2....$ You can see that $b_k = a_0\delta[n-lN] + a_1\delta[n-1-lN] + .... + a_{N-1}\delta[n-N+1-lN]$

Approach 1 : Make use of knowledge of the FS of the impulse train $\sum_{n=-\infty}^{\infty} \delta(t-nT) \longleftrightarrow a_k = 1/T$.

$$x(t) = \sum_{k=-\infty}^{\infty} b_k e^{j2\pi kt/T} = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \sum_{m=0}^{N-1} a_m \delta[k-m-lN] e^{j2\pi kt/T}$$

We can now use the sampling property of the delta function. Looking at $\delta[k-m-lN]e^{j2\pi kt/T}$ as a function of $k$, and using the sampling property of the delta function, we have:

$$\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \sum_{m=0}^{N-1} a_m \delta[k-m-lN] e^{j2\pi(m+lN))t/T} = \underbrace{\sum_{m=0}^{N-1} a_m e^{j2\pi mt/T}}_{x(t)} \sum_{l=-\infty}^{\infty} e^{j2\pi lNt/T}$$

Since $\sum_{n=-\infty}^{\infty} \delta(t-nT) \longleftrightarrow c_k = 1/T$, it implies that $T \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{k=-\infty}^{\infty} e^{j2\pi kt/T}$. Using this, we have:

$$y(t) = \underbrace{\sum_{m=0}^{N-1} a_m e^{j2\pi mt/T}}_{x(t)} \underbrace{0um_{l=-\infty}^{\infty} e^{j2\pi lNt/T}}_{T\sum_n \delta(t-nT/N)} = Tx(t) \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{nT}{N}\right) = T \sum_{n=-\infty}^{\infty} x\left(\frac{nT}{N}\right) \delta\left(t - \frac{nT}{N}\right)$$

This is not the only way to approach this problem. This is a generalization of a tutorial problem.

Approach 2 : Making use of the frequency shifting property $e^{jM\omega_0 t}x(t) \longleftrightarrow a_{k-M}$. We know

$$\begin{aligned} b_k &= \sum_{l=-\infty}^{\infty} \sum_{m=0}^{N-1} a_m \delta[k-m-lN] \\ &= \sum_{m=0}^{N-1} a_m \sum_{l=-\infty}^{\infty} \delta[k-m-lN] \end{aligned}$$

Here, $\sum_{l=-\infty}^{\infty} \delta[k-m-lN]$ is non-zero only when $k-m$ is an integral multiple of $N$. For a given $k$, the value of $m \in [0, N-1]$ such that $k-m$ is a multiple of $N$ is only when $m = k \mod (N)$. Hence, we get $b_k = a_{k \bmod (N)}$. Consider the frequency shifed signals $x_{lN}(t) = e^{jlN\omega_0 t}x(t) \longleftrightarrow a_{k-lN}$. It is clear from the Fourier coefficents $b_k$ that $y(t)$ is the sum of all frequency shifted signals, i.e. $y(t) = \sum_{l=-\infty}^{\infty} x_{lN}(t) = \sum_{l=-\infty}^{\infty} e^{jlN\omega_0 t}x(t) = x(t) \sum_{l=-\infty}^{\infty} e^{j2\pi lNt/T}$. This expression can be simplified to $T \sum_{n=-\infty}^{\infty} x\left(\frac{nT}{N}\right) \delta\left(t - \frac{nT}{N}\right)$ as done in Approach 1.

4) (**12 marks**) Consider a system with input x(t) and output y(t) described by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$$

Let $x(t)$ be a periodic signal of time-period $T$. Specifically $x(t)$ is defined over one time-period by

$$x(t) = \begin{cases} \frac{4t}{T} & 0 \le t \le T/4 \\ \frac{4}{3T}(T-t) & T/4 \le t \le T \end{cases}$$

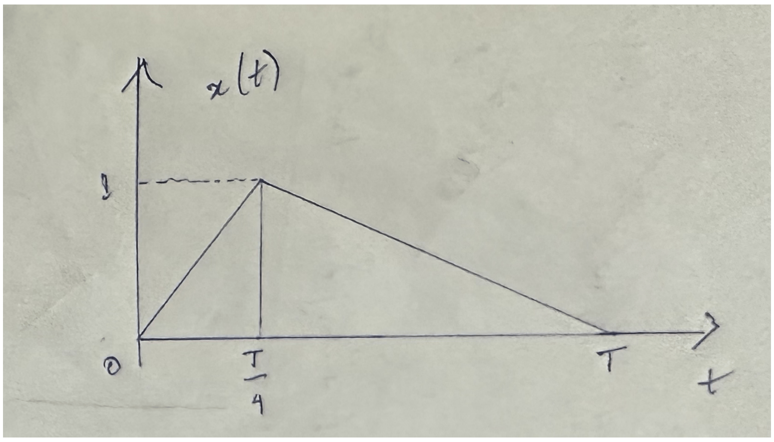
Find an expression for $y(t)$.
This linear constant coefficient differential equation obviously represents a linear time invariant system. With $e^{j\omega t}$ input, the output will be $H(j2\pi k/T)e^{j\omega t}$ where $H(j2\pi k/T)$ is a complex constant. We can simply use the super-position theorem. Write $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$. First find $a_k$. This can be done by differentiation and use of the properties of FS along with knowledge of the FS of the rectangular pulse train. Using $y(t) = H(j2\pi k/T)e^{j2\pi k/T}$ with $x(t) = e^{j2\pi kt/T}$, we can find $H(j2\pi k/T)$.
Given, x(t) is is a periodic signal, we can express it as a Fourier series.

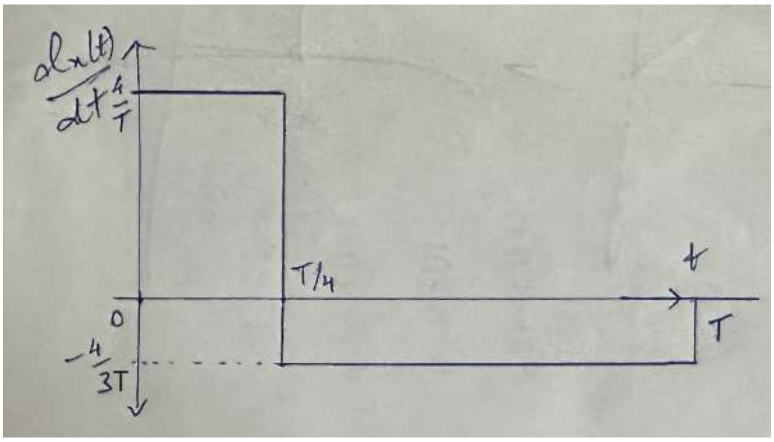
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$$

We know that complex exponential signals are Eigenfunctions of LTI systems. And since linear constant coefficient differential equations represent an LTI system, for an input $e^{jk\omega t}$, the output is $H(jk\omega)e^{jk\omega t}$. Substituting this into the differential equation we get

$$\begin{aligned} &\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t) \\ \implies &H(jk\omega)\frac{d^2e^{jk\omega t}}{dt^2} + 5H(jk\omega)\frac{de^{jk\omega t}}{dt} + 6H(jk\omega)e^{jk\omega t} = e^{jk\omega t} \\ \implies &-k^2\omega^2 H(jk\omega)e^{jk\omega t} + 5jk\omega H(jk\omega)e^{jk\omega t} + 6H(jk\omega)e^{jk\omega t} = e^{jk\omega t} \\ \implies &H(jk\omega) = \frac{1}{5jk\omega + 6 - k^2\omega^2} \end{aligned}$$

Now, we need to find the Fourier coefficients $a_k$ of the signal $x(t)$. If we observe the plots of signals $x(t)$ and $x'(t)$,




Hence, $x'(t)$ is a sum of two rectangular pulse trains(both time shifted). The Fourier coefficients of $x'(t)$ say $c_k$ can be expressed as

$$\begin{aligned}c_k &= \frac{4}{T}\frac{sin(2\pi k/8)}{k\pi}e^{-j2\pi k/8} - \frac{4}{3T}\frac{sin(2\pi \cdot 3k/8)}{k\pi}e^{j2\pi \cdot 3k/8} \\ &= \frac{4}{T}\frac{sin(\pi k/4)}{k\pi}e^{-j\pi k/4} - \frac{4}{3T}\frac{sin(3\pi k/4)}{k\pi}e^{j3\pi k/4}\end{aligned}$$

Hence, the Fourier coefficients of $x(t) = \int_{-\infty}^{t} x'(t)$ can be expressed as

$$a_k = \frac{c_k}{j2\pi k/T} = \frac{2}{j\pi k}\frac{sin(\pi k/4)}{k\pi}e^{-j\pi k/4} - \frac{2}{3j\pi k}\frac{sin(3\pi k/4)}{k\pi}e^{j3\pi k/4}$$

Thus $y(t)$ can be expressed as

$$\begin{aligned}y(t) &= \sum_{k=-\infty}^{\infty} H(j2\pi k/T)\cdot a_k e^{j2\pi kt/T} \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{5jk(2\pi/T)+6-k^2(2\pi/T)^2}\left(\frac{2}{j\pi k}\frac{sin(\pi k/4)}{k\pi}e^{-j\pi k/4} - \frac{2}{3j\pi k}\frac{sin(3\pi k/4)}{k\pi}e^{j3\pi k/4}\right)e^{j2\pi kt/T}\end{aligned}$$

5) (**5 marks**) Sketch the block diagram of an efficient (smallest number of integrators and summers) practical implementation (without using differentiators) of a system described by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) \quad = \quad \frac{dx(t)}{dt} + x(t)$$

Let $v(t) = x(t) + \frac{dx(t)}{dt}$. Then we have

$$\begin{aligned}\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) &= v(t)\\ \Rightarrow \frac{dy(t)}{dt} + 5y(t) &= \int (v(t) - 6y(t))dt\\ \Rightarrow y(t) &= \int \left(\int (v(t) - 6y(t))dt - 5y(t)\right) dt\end{aligned}$$

Now, we can recognize that both these systems (that generates $v(t)$ and the one that generates $y(t)$ are both LTI. So we can interchange the two.
So, on interchanging, we get (for the equations below, $u(t)$ is the intermediate signal)

$$y(t) = u(t) + \frac{du(t)}{dt}$$

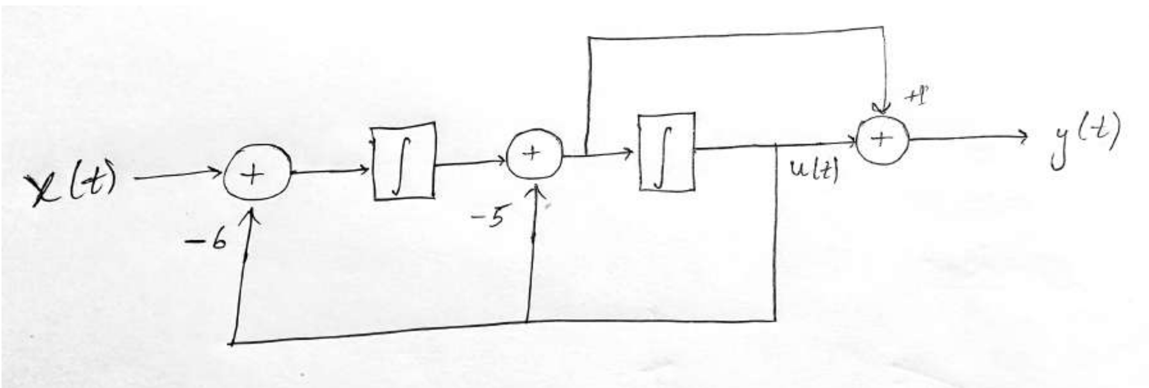
Also,

$$u(t) = \int \left(\int (x(t) - 6u(t))dt - 5u(t)\right) dt$$

Hence, we remove the differential term from the expression of y(t) by using the above expression

$$\begin{aligned}y(t) &= u(t) + \frac{d}{dt}\int \left(\int (x(t) - 6u(t))dt - 5u(t)\right) dt\\ &= u(t) + \left(\int (x(t) - 6u(t))dt - 5u(t)\right)\end{aligned}$$

It can be seen that the second term in the above equation is same as the input to the integrator that outputs $u(t)$. Hence, we can draw the system using integrators and summers as given below.




6) (**3 marks**) An LTI system has an impulse response $h(t) = \frac{u(t)}{t^2+1}$, where $u(t)$ is the unit-step function. Is this system stable?
Stability requires $\| h(t) \|_1 = \int_{-\infty}^{\infty} |h(t)|dt < \infty$. So, we get

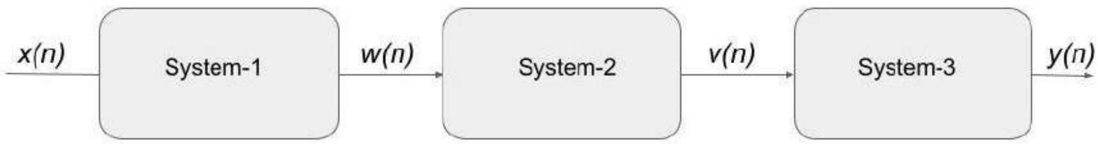
$$\begin{aligned}\int_{-\infty}^{\infty} |h(t)|dt &= \int_{-\infty}^{\infty} |\frac{u(t)}{t^2+1}|dt\\ &= \int_0^{\infty} \frac{dt}{t^2+1}\\ &= tan^{-1}(\infty) - tan^{-1}(0)\\ &= \frac{\pi}{2} - 0\\ &= \frac{\pi}{2} < \infty\end{aligned}$$

Hence, the given LTI system is stable.

7) (**10 marks**) Consider the cascade of three systems below. System-1 is described by an input-output relation:

$$w[n] = \begin{cases} x[n/3] & n \text{ is a multiple of } 3 \\ 0 & \text{Otherwise} \end{cases}$$

System-2 is an LTI system with impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$, where $u[n]$ denotes the discrete-time unit-step sequence. System-3 has an input-output relation: $y[n] = v[3n]$.




a) (**8 marks**) Express $y[n]$ in terms of $x[n]$.

The trick is to see that $w[n] = \sum_{l=-\infty}^{\infty} x[l]\delta[n-3l]$. Now, we try and find a relation between $v[n]$ and $w[n]$, We know that $v[n] = \sum_{k=-\infty}^{\infty} h[k]w[n-k]$ as we are already given that h[n] is the impulse response of System-2. Hence,

$$\begin{aligned} v[n] &= \sum_{k=-\infty}^{\infty} h[k]w[n-k] \\ &= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u[k] \sum_{l=-\infty}^{\infty} x[l]\delta[n-k-3l] \\ &= \sum_{l=-\infty}^{\infty} x[l] \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \delta[n-k-3l] \end{aligned}$$

We see that the inner sum contains the unit impulse function which evaluates to 1 only when the input to the function is zero. Hence, out of all the values of $k$ from 0 to $\infty$, only value of k for which this will hold true is $k = n - 3l$. **NOTE:** As our k ranges from 0 to $\infty$, $n - 3l \geq 0$ necessarily for the inner term to not be zero and hence $l <= \lfloor \frac{n}{3} \rfloor$. So, we get

$$v[n] = \sum_{l=-\infty}^{\lfloor \frac{n}{3} \rfloor} \left(\frac{1}{2}\right)^{n-3l} x[l]$$

Now, $y[n] = v[3n]$, so we finally get,

$$\begin{aligned} y[n] &= v[3n] \\ &= \sum_{l=-\infty}^{\lfloor \frac{3*n}{n} \rfloor} \left(\frac{1}{2}\right)^{3n-3l} x[l] \\ &= \sum_{l=-\infty}^{n} \left(\frac{1}{8}\right)^{n-l} x[l] \end{aligned}$$

b) (**1 marks**) Is the overall system (with input $x[n]$ and output $y[n]$) causal?

As we can see from the above expression, that the value of y[n] only depends on the values upto x[n] and doesn't depend on any future values, hence the overall system is causal.

c) (**1 marks**) Is the overall system (with input $x[n]$ and output $y[n]$) time-invariant?

If we time shift y[n] by $n_o$, we get

$$y[n-n_o] = \sum_{l=-\infty}^{n-n_o} \left(\frac{1}{8}\right)^{n-n_o-l} x[l]$$

If we time shift x[n] by the same amount $n_o$, we get the expression of $y[n-n_o]$ as

$$\begin{aligned} y[n-n_o] &= \sum_{l=-\infty}^{n} \left(\frac{1}{8}\right)^{n-l} x[l-n_o] \\ &= \sum_{l=-\infty}^{n-n_o} \left(\frac{1}{8}\right)^{n-n_o-l} x[l] \end{aligned}$$

As both the expression for $y[n-n_o]$ match, the above system is time invariant.