Department of Mathematics
Tutorial Sheet No. 4
MTL 106 (Introduction to Probability and Stochastic Processes)

1. Let X and Y be independent random variables. The range of X is $\{1,3,4\}$ and the range of Y is $\{1,2\}$. Partial information on the probability mass function is as follows:

$$p_X(3) = 0.50; \quad p_Y(2) = 0.60; \quad p_{X,Y}(4,2) = 0.18 \ .$$

   (a) Determine p_X, p_Y and $p_{X,Y}$ completely.
   (b) Determine $P(|X-Y| \geq 2)$.

2. Evaluate all possible marginal and conditional distributions if (X,Y) has the following joint probability distribution
(a) $P(X=j, Y=k) = q^{k-j}p^j, j = 1,2,\ldots$ and $k = j+1, j+2, \ldots \quad q = 1-p$
(b) $P(X=j, Y=k) = \frac{15!}{j!k!(15-j-k)!}(\frac{1}{2})^j(\frac{1}{3})^k(\frac{1}{6})^{15-j-k}$
for all admissible non negative integral values of j and k.

3. Show that

$$F(x,y) = \begin{cases} 0, & x<0, y<0 \text{ or } x+y<1 \\ 1, & \text{otherwise} \end{cases}$$

is not a distribution function.

4. Find k, if the joint probability density of (X_1, X_2) is

$$f_{X_1X_2}(x_1,x_2) = \begin{cases} ke^{-3x_1-4x_2}, & x_1>0, x_2>0 \\ 0, & \text{otherwise} \end{cases}$$

Also find the probability that the value of X_1 falls between 0 and 1 while X_2 falls between 0 and 2.

5. Consider a transmitter sends out either a 0 with probability p, or a 1 with probability $(1-p)$, independently of earlier transmissions. Assume that the number of transmissions within a given time interval is Poisson distributed with parameter λ. Find the distribution of number of 1's transmitted in that same time interval?

6. Suppose that the two-dimensional random variable (X,Y) has joint pdf

$$f(x,y) = \begin{cases} 1, & 0<x,y<1 \\ 0, & \text{otherwise} \end{cases}$$

Find the pdf of $X+Y$.

7. Romeo and Juliet have a date at a given time, and each, independently, will be late by an amount of time, denoted by X and Y respectively, that is exponentially distributed with parameter λ. Find the pdf of, $X-Y$, the difference between their times of arrival?

8. Let X, Y and Z be independent and identically distributed random variables each having a uniform distribution over the interval $[0,1]$. Find the joint density function of (V,W) where $V = XY$ and $W = Z^2$.

9. The random variable X represents the amplitude of cosine wave; Y represents the amplitude of a sine wave. Both are independent and uniformly distributed over the interval (0,1). Let R represent the amplitude of their resultant, i.e., $R^2 = X^2 + Y^2$ and θ represent the phase angle of the resultant, i.e., $\theta = \tan^{-1}(\mathrm{Y/X})$. Find the joint and marginal pdfs of θ and R.

10. Let X and Y be continuous random variables having joint distribution which is uniform over the square which has corners at $(2,2), (-2,2), (-2,-2)$ and $(2,-2)$. Determine $P(|Y| > |X|+1)$.

11. Consider the metro train arrives at the station near your home every quarter hour starting at 5:00 AM. You walk into the station every morning between 7:10 and 7:30 AM, with the time in this interval being a uniform random variable, that is $\mathcal{U}([7:10, 7:30])$.

    (a) Find the distribution of time you have to wait for the first train to arrive?

    (b) Also, find its mean waiting time?

12. Suppose that the two-dimensional random variable (X, Y) has joint pdf

$$f_{XY}(x,y) = \begin{cases} kx(x-y), & 0 < x < 2, -x < y < x \\ 0, & \text{otherwise} \end{cases}$$

    Find k. Evaluate $P(X < 1/Y = \frac{1}{2})$ and $P(Y < \frac{3}{2}/X = 1)$.

13. Let A, B and C be independent random variables each with uniform distributed on interval $(0, 1)$. What is the probability that $Ax^2 + Bx + C = 0$ has real roots?

14. Aditya and Aayush work independently on a problem in Tutorial Sheet 4 of Probability and Stochastic Processes course. The time for Aditya to complete the problem is exponential distributed with mean 5 minutes. The time for Aayush to complete the problem is exponential distributed with mean 3 minutes.

    (a) What is the probability that Aditya finishes the problem before Aayush?

    (b) Given that Aditya requires more than 1 minutes, what is the probability that he finishes the problem before Aayush?

    (c) What is the probability that one of them finishes the problem a minute or more before the other one?

15. Let X_1 and X_2 be two iid random variables each $N(0, 1)$ distributed.

    (a) Are $X_1 + X_2$ and $X_1 - X_2$ independent random variables? Justify your answers.

    (b) Obtain $E[X_1^2 + X_2^2 \mid X_1 + X_2 = t]$.

    (c) Calculate $E[(X_1 + X_2)^4 / (X_1 - X_2)]$.

16. Let X and Y be two identically distributed random variables with $\text{Var}(X)$ and $\text{Var}(Y)$ exist. Prove or disprove that $\text{Var}\left(\frac{X+Y}{2}\right) \leq \text{Var}(X)$

17. Let X and Y be two random variables such that $\rho(X, Y) = \frac{1}{2}$, $Var(X) = 1$ and $Var(Y) = 4$. Compute $Var(X - 3Y)$.

18. Let $X_1, X_2, \ldots, X_5$ be i.i.d random variables each having uniform distributions in the interval $(0, 1)$.

    (a) Find the probability that $\min(X_1, X_2, \ldots, X_5)$ lies between $(1/4, 3/4)$.

    (b) Find the probability that X_1 is the minimum and X_5 is the maximum among these random variables.

19. Let $X_1, X_2, \ldots, X_n$ be iid random variables with $E(X_1) = \mu$ and $Var(X_1) = \sigma^2$. Define

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$$

    Find (a) $Var(\overline{X})$ (b) $E[S^2]$.

20. Pick the point (X, Y) uniformly in the triangle $\{(x, y) \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x\}$. Calculate $E[(X-Y)^2/X]$.