Department of Mathematics
Tutorial Sheet No. 4
MAL 250/MTL 106 (Probability and Stochastic Processes)

1. Let X and Y be independent random variables. The range of X is $\{1, 3, 4\}$ and the range of Y is $\{1, 2\}$. Partial information on the probability mass function is as follows:

$$p_X(3) = 0.50; \quad p_Y(2) = 0.60; \quad p_{X,Y}(4, 2) = 0.18 \ .$$

   (a) Determine p_X, p_Y and $p_{X,Y}$ completely.

   (b) Determine $P(|X - Y| \geq 2)$.

$P_X(4) = 0.3 \Rightarrow P_X(1) = 0.2$

$P_Y(1) = 0.4$

2. Evaluate all possible marginal and conditional distributions if (X, Y) has the following joint probability distribution

   (a) $P(X = j, Y = k) = q^{k-j}p^j, j = 1, 2, \ldots$ and $k = j + 1, j + 2, \ldots \quad q = 1 - p$

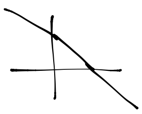
   (b) $P(X = j, Y = k) = \frac{15!}{j!k!(15-j-k)!}(\frac{1}{2})^j(\frac{1}{3})^k(\frac{1}{6})^{15-j-k}$

   for all admissible non negative integral values of j and k.

$P_X = \left(\frac{p}{q}\right)^j \sum_{k=j+1} q^k = \left(\frac{p}{q}\right)^j \frac{q^{j+1}}{1-q}$

3. Show that

$$F(x, y) = \begin{cases} 0, & x < 0, y < 0 \text{ or } x + y < 1 \\ 1, & \text{otherwise} \end{cases}$$



$F(x_1,y_1) + F(x_2,y_2) - F(x_1,y_2) - F(x_2,y_1) \geq 0$

   is not a distribution function.

4. Find k, if the joint probability density of (X_1, X_2) is

$k = 12$

$$f_{X_1X_2}(x_1, x_2) = \begin{cases} ke^{-3x_1-4x_2}, & x_1 > 0, x_2 > 0 \\ 0, & \text{otherwise} \end{cases}$$

   Also find the probability that the value of X_1 falls between 0 and 1 while X_2 falls between 0 and 2.

5. Let X and Y be independent random variables with $X \sim B(3, \frac{1}{3})$ and $Y \sim B(2, \frac{1}{2})$. Find $P(X = Y)$.

$\sum_{r=0}^{2} {}^3C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{3-r} \; {}^2C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{2-r}$

6. Suppose that the two-dimensional random variable (X, Y) has joint pdf

$$f(x, y) = \begin{cases} 1, & 0 < x, y < 1 \\ 0, & \text{otherwise} \end{cases}$$

   Find the pdf of $X + Y$.

$= \text{Pdf } x = 2, \; y = 1 \quad 0<x<1, \; 0<y<1$

$p(Z) = \frac{1}{2} \quad 0<Z<2$

7. Suppose that a two-dimensional random variable (X, Y) has joint probability density function

$$f_{XY}(x, y) = \begin{cases} \frac{e^{-(x+y)}x^8y^4}{8!4!}, & 0 < x < \infty, 0 < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

   (a) Find the pdf of $U = \frac{X}{Y}$.

   (b) Find $E(U)$.

$u = \frac{x}{y} \quad v = y$

$h(u,v) = (uv, v)$

$J = \begin{vmatrix} v & 0 \\ u & 1 \end{vmatrix} = v$

8. Let the execution times X and Y of two independent processes be uniformly distributed in the interval $(0, t_X)$ and $(0, t_Y)$, respectively, with $t_X \leq t_Y$.

   (a) Find the probability that the former process has execution time smaller than the later and the total time of execution of two processes does not exceed $\frac{1}{2}(t_X + t_Y)$.

   (b) Find the distribution of the total time of execution of two processes.

$X \sim U(0, t_x)$

$Y \sim U(0, t_y)$

$f_{XY}(x,y) = \frac{1}{t_x} \frac{1}{t_y}$

a) $P\left(X \leq Y, X + Y \leq \frac{1}{2}(t_x + t_y)\right)$

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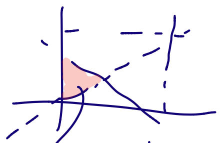


$= \frac{1}{2} \times \left(\frac{t_x + t_y}{2}\right)\left(\frac{t_x + t_y}{4}\right) \times \frac{1}{t_x t_y}$

$T = X + Y, \quad V = Y \qquad X = U - V, \; Y = V$

$f_{UV}(u,v) = f_{XY}(u-v, v)\,|J| \quad ; \quad 0 < u - v < t_x, \; 0 < v < t_y$

$= \frac{1}{t_x t_y} \times |J| = \frac{1}{t_x t_y}$

$f_U(u) = \int f_{UV}(u,v)\,dv$

$= \begin{cases} \frac{1}{t_x t_y} \\ \ldots \end{cases}$

9. The random variable X represents the amplitude of cosine wave; Y represents the amplitude of a sine wave. Both are independent and uniformly distributed over the interval (0,1). Let R represent the amplitude of their resultant, i.e., $R^2 = X^2 + Y^2$ and θ represent the phase angle of the resultant, i.e., $\theta = \tan^{-1}(\mathrm{Y/X})$. Find the joint and marginal pdfs of θ and R.

10. Let X and Y be two independent continuous random variables. Show that

$$P[X \leq Y] = \int_{-\infty}^{\infty} F_X(y) f_Y(y) dy$$

where f_Y is the probability density function of Y and F_X is the cumulative distribution function of X. Also, find the value of $P[X \leq Y]$ when X and Y are i.i.d. random variables with common density function

$$f_X(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

11. Let X and Y be continuous random variables having joint distribution which is uniform over the square which has corners at $(2, 2), (-2, 2), (-2, -2)$ and $(2, -2)$. Determine $P(|Y| > |X| + 1)$.

12. Suppose that the two-dimensional random variable (X, Y) has joint pdf

$$f_{XY}(x, y) = \begin{cases} kx(x-y), & 0 < x < 2, -x < y < x \\ 0, & \text{otherwise} \end{cases}$$

Find k. Evaluate $P(X < 1/Y = \frac{1}{2})$ and $P(Y < \frac{3}{2}/X = 1)$.

13. Let A, B and C be independent random variables each with uniform distributed on interval $(0, 1)$. What is the probability that $Ax^2 + Bx + C = 0$ has real roots?

14. Let $\{X(t), t \geq 0\}$ be a time homogeneous Poisson process with $X(0) = j$. Consider the random variable

$$T_j = \inf\{t : X(t) = j + 1\}, \quad j = 0, 1, \ldots$$

T_j is the time of occurrence of the first jump after the jth. Find the distribution of T_j for $j \geq 0$. Also, find the joint distribution of $(T_{2012}, T_{2013}, T_{2014})$.

15. Let X_1 and X_2 be two iid random variables each $N(0, 1)$ distributed.
    (a) Are $X_1 + X_2$ and $X_1 - X_2$ independent random variables? Justify your answers.
    (b) Obtain $E[X_1^2 + X_2^2 \mid X_1 + X_2 = t]$

16. Prove that the correlation coefficient between any two random variables X and Y lies in the interval $[-1, 1]$.

17. Given

$$E(X_1) = 3\ ,\ E(X_2) = 2\ ,\ E(X_3) = 1$$

$$Var(X_1) = \frac{3}{2}\ ,\ Var(X_2) = \frac{4}{3}\ ,\ Var(X_3) = \frac{5}{6}$$

$$cov(X_1, X_2) = -1\ ,\ cov(X_1, X_3) = \frac{1}{3} \text{ and } cov(X_2, X_3) = \frac{-1}{3}.$$

Determine the following
(a) $E(X_1^2 + X_2^2 + X_3^2)$ (b) $Var(X_1 - X_2 + X_3)$ (c) $cov(X_1 + X_2, X_3)$.

18. Let $X_1, X_2, \ldots, X_5$ be i.i.d random variables each having uniform distributions in the interval $(0, 1)$.
    (a) Find the probability that $\min(X_1, X_2, \ldots, X_5)$ lies between $(1/4, 3/4)$.
    (b) Find the probability that X_1 is the minimum and X_5 is the maximum among these random variables?

19. Let X, Y, Z be i.i.d random variables each having uniform distribution in the interval (1,2). Find $Var\left(\frac{4X}{3Y} + \frac{3Y}{2Z}\right)$.

1 a) $P_X = \begin{cases} 0.2 & x=1 \\ 0.5 & x=3 \\ 0.3 & x=4 \end{cases}$

$P_Y = \begin{cases} 0.4 & y=1 \\ 0.6 & y=2 \end{cases}$

$P_{X,Y} = P_X P_Y$

$\iint xy \, dx\,dy$

b) $P(|X-Y| \geq 2)$

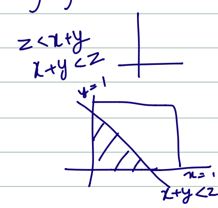
1,3,4 , 1,2

3, 1 = 0.2

4, 1 = 0.12

4, 2 = 0.18

$\overline{\quad 0.50 \quad}$



$z < 1.$

7)

$f_{u,v}(u,v) = \dfrac{v e^{-(uv+v)} u^8 v^{12}}{8!4!}$

$u = x+y$

$v = y$

$h \quad u-v, v$

$J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$

$\dfrac{u^8 \int e^{-(u+1)v} v^{13}}{8!\cancel{4}!}$

$\int_0^\infty e^{-(u+1)v} v^{13} dv = \dfrac{1}{(u+1)^{14}} \int_0^\infty e^{-t} t^{13} dt = \dfrac{\Gamma(14)}{(u+1)^{14}} = \dfrac{\lfloor 13}{(u+1)^{14}}$

$\Rightarrow f(u) = \dfrac{\lfloor 13 \, u^8}{(u+1)^{14} 8!4!} \qquad u > 0$

$\quad\quad 0 \qquad\qquad \text{o}\omega$

c) $E(u) = \dfrac{\lfloor 13}{\lfloor 8 \lfloor 4} \int_0^\infty \dfrac{u^9}{(u+1)^{14}} du$

$u = \tan^2\theta$

$\dfrac{\lfloor 13}{\lfloor 8 \lfloor 4} \int_0^{\pi/2} \dfrac{2 \tan^{18}\theta \; \tan\theta \sec^2\theta \, d\theta}{\sec^{28}\theta}$

$\dfrac{\lfloor 13}{\lfloor 8 \lfloor 4} \times 2 \times \int_0^{\pi/2} \sin^{19}\theta \; \cos^7\theta \, d\theta$

$\dfrac{\cancel{\lfloor 13}}{\lfloor 8 \lfloor 4} \qquad \dfrac{\Gamma_{10} \Gamma_4}{\cancel{\Gamma_{14}}}$

$= \dfrac{\lfloor 9 \; \lfloor 3}{\lfloor 8 \; \lfloor 4} = \dfrac{9}{4}$

9) $X = U(0,1)$

$Y = U(0,1)$

$R = \sqrt{X^2 + Y^2}$

$\theta = \tan^{-1}(Y/X)$

$X^2 + Y^2 = R^2$

$Y = X\tan\theta$

$X^2(\sec^2\theta) = R^2 \quad \Rightarrow X = R\cos\theta$

$Y = R\sin\theta$

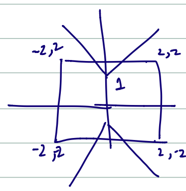
$J = \begin{vmatrix} \cos\theta & \sin\theta \\ -R\sin\theta & R\cos\theta \end{vmatrix} = R$

$P(R,\theta) = \begin{cases} R \\ 0 \end{cases}$

10) $P(X \leq Y)$

$$\int_{-\infty}^{\infty} F(X)\, \cancel{ff}(y) \quad Y = y$$

11)



$|Y| > |X| + 1$

$2 \times \frac{1}{2} \times \sqrt{2} \times \sqrt{2} \times \frac{1}{16} = \frac{1}{8}$

12) $\begin{cases} kx(x-y) & 0<x<2, \quad -x<y<x \\ 0 & \text{ow} \end{cases}$

$$k\int_0^2 \int_{-x}^{x} x(x-y)\, dy\, dx = 1$$

$$k\int_0^2 2x^3\, dx = k\frac{2}{4} \times 2^4 = 1$$

$\Rightarrow k = 1/8$

$$P(X<1 / Y=\tfrac{1}{2}) = \frac{\int_{1/2}^{1} x(x-\frac{1}{2})\, dx}{\int_{1/2}^{2} x(x-\cancel{\tfrac{1}{2}})\, dx} = \frac{\frac{1-\frac{1}{8}}{3} - \frac{1-\frac{1}{4}}{4}}{\frac{8-\frac{1}{8}}{3} - \frac{4-\frac{1}{4}}{4}} = \frac{\frac{7}{24} - \frac{3}{16}}{\frac{\cancel{63}\ 21}{8\times 8} - \frac{15}{16}} = \frac{5}{81}$$

13) $Ax^2 + Bx + C = 0$

$B^2 - 4AC \geq 0$

$B^2 \geq 4AC$ $\quad 1$

$$\int_0^1 \int_0^1 \int_{\sqrt{4AC}}^1 1$$

fix limits.

14)

15) $X_1 \sim N(0,1)$
$X_2 \sim N(0,2)$

$$\frac{e^{-\frac{x^2+y^2}{2}}}{2}$$

$u = x+y \qquad v = x-y$

$\frac{u+v}{2}, \ \frac{u-v}{2} \qquad J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

$$\frac{1}{4} e^{-\frac{2}{2\times 4}(u^2+v^2)} = \frac{e^{-\frac{1}{4}(u^2+v^2)}}{4} \qquad N(0,2)$$

Yes.

17) $E(X_1^2) = \frac{21}{2} \qquad E(X_2^2) = \frac{16}{3} \qquad E(X_3^2) = \frac{11}{6}$

a) $53/3$

$E([X_1 - X_2 + X_3 - (2)]^2)$

$E(\{X_1-3\} - [X_2-2] + [X_3-1])^2$

$(X_1-3)^2 + (X_2-2)^2 + (X_3-1)^2 \quad -2(X_1-3)(X_2-2) \quad -2(X_2-2)(X_3-1) + 2(X_1-3)(X_3-1)$

$\frac{3}{2} + \frac{4}{3} + \frac{5}{6} \qquad + 2 \qquad + \frac{2}{3} + \frac{2}{3}$

$\frac{3}{2} + \frac{8}{3} + \frac{5}{6} + 2 = \frac{9+16+5}{6} \quad + 2 \quad = 7$

$E((X_1+X_2-5)(X_3-1))$

$\frac{1}{3} - \frac{1}{3} = 0$

18) $\int_{1/4}^{3/4} \int_x^1 \int_x^1 \int_x^1 \int_x^1 1$

a) $\frac{\frac{3}{4}^5 - \frac{1}{4}^5}{5} \times 5$

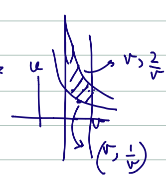
b) $1/25$.

19) $X, Y, Z \sim U(1,2)$

$\frac{4X}{3Y}$

$u = X/Y \quad v = y$
$X = uv \quad y = v$
$J = \begin{vmatrix} v & 0 \\ u & 1 \end{vmatrix} = v$

$$\begin{cases} v & \frac{1}{v} < u < \frac{2}{v} \quad 1 < v < 2 \\ 0 & \text{ow} \end{cases}$$



$f(u) = \int_1 v$