Solutions

MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Time allowed: 2 hours   Major Examination   Max. Marks: 52

Name: Entry Number: Signature:

Multiple Selection Questions: Section 1   ($6 \times 2 = 12$ marks)

Each of the following questions 1 to 6 has four options out of which one or more options can be correct. Write A, B, C or D which corresponds to the correct option for the first correct answer followed by space and the next correct answer and so on. **2 marks** is awarded if all correct answers are written, **0 mark** for no answer or partial correct or any incorrect answer.

$\mathcal{F}$ - largest σ-field.

1. Let $(\Omega, \mathcal{F})$ be a sample space and $A \subset \Omega$ fixed. The function $Y : \Omega \to \mathcal{R}$ is a random variable (RV). Which of the following statements are TRUE?

~~(A)~~ $Y(w) = \begin{cases} 0, & w \in A \\ 1, & w \notin A \end{cases}$ (B) $Y(w) = \begin{cases} 1, & w \in A \\ 0, & w \notin A \end{cases}$

~~(C)~~ $Y(w) = \begin{cases} -1, & w \in A \\ 1, & w \notin A \end{cases}$ ~~(D)~~ $Y(w) = \begin{cases} 0.5, & w \in A \\ 0.5, & w \notin A \end{cases}$ Answer: (2 marks)

2. Let X be a non-negative integer valued random variable such that $E(X)$ exists. Which of the following statements are TRUE?

(A) $E(X) = \sum_{k=2}^{\infty} P(X > k)$ (B) $E(X) = \sum_{k=0}^{\infty} P(X \geq k)$

(C) $E(X) = \sum_{k=1}^{\infty} P(X > k)$ ~~(D)~~ $E(X) = \sum_{k=0}^{\infty} P(X > k)$ Answer: (2 marks)

3. If $E[Y/X] = 1$, which of the following statements are NOT TRUE?

(A) $Var[XY] \geq Var[X]$ ~~(B)~~ $Var[XY] \leq Var[X]$

~~(C)~~ $Var[XY] < Var[X]$ ~~(D)~~ $Var[XY] > Var[X]$ Answer: (2 marks)

4. Let $\{X_n, n = 0, 1, 2, \ldots\}$ be a DTMC with finite state space S and $i \in S$ is an absorbing state. Which of the following statements are TRUE?

~~(A)~~ The period of state i, $d_i = 1$. ~~(B)~~ The mean recurrence time of state i, $\mu_i = 1$.

~~(C)~~ State i is a +ve recurrent. (D) State i is a null recurrent. Answer: (2 marks)

5. Let $N(t)$ be the random variable denoting the number of events occurs upto and including time t. Let $\{N(t), t \geq 0\}$ be a Poisson process with parameter λ. Let T_i be the inter-arrival times of the i^{th} event $i = 1, 2, \ldots$. Which of the following statements are TRUE?

~~(A)~~ T_i's are independent. ~~(B)~~ $P(T_2 \leq t) = 1 - e^{-\lambda t}, 0 \leq t < \infty$.

(C) $Cov(T_i, T_j) > 0$, for $i \neq j$ (D) $\sum_{i=1}^{n} T_i$ follows $B(n, \lambda)$. Answer: (2 marks)

6. Consider a $M/M/\cancel{4}^{\,n}/\infty$ queueing model where $X(t)$ denotes the number of customers in the system at any time t. Then, the number of customers undergoing service at time t is

~~(A)~~ $\min\{X(t), \cancel{4}_{\,n}\}$ (B) 4 (C) $\max\{X(t), 4\}$ (D) $\min\{X(t), 1\}$ Answer: (2 marks)

Comprehensive Type Questions: Section 2   ($2 + 2 + 4 + 4 \times 3 = 20$ marks)

Each of the following questions 7 to 13 has some subparts. For each subpart, write the answer upto 4 decimal places (D) or in fraction (F) or the expression (E). **1 mark/2 marks** is awarded if correct answer is written, **0 mark** for no answer or partial correct or any incorrect answer.

7. Let $P[X \leq \overset{K}{0.19}] = 0.75$, where X is a continuous type RV with some CDF defined over (0,1). If $Y = 1 - X$, Find k so that $P[Y \leq k] = 0.25$. Answer(D): $1 - K = 0.81$ (2 marks)

8. Let $0 < q < 1$ and N be a positive integer. Let $X \sim B\left(N, \frac{q}{N}\right)$. Find $\lim_{N\to\infty}\left(1 - \frac{q}{N}\right)^N$, if it exists. Answer(E): e^{-q} For $B(N, \frac{p}{N})$ e^{-p} (2 marks)

9. Let X_1 and X_2 be independent exponential distributed random variables with parameters $\underset{\lambda_1}{3}$ and $\underset{\lambda_2}{4}$ respectively. Define $X_{(1)} = \min\{X_1, X_2\}$ and $X_{(2)} = \max\{X_1, X_2\}$.

   (a) Find $Var(X_{(1)})$? Answer(F): $\frac{1}{(\lambda_1+\lambda_2)^2}$ (1 mark)

   (b) Find the distribution of $X_{(1)}$? Answer(E): $Exp(\lambda_1+\lambda_2)$ (1 mark)

   (c) Find $E(X_{(2)})$? Answer(F): $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2}$ (2 marks)

10. Let X_0 be an integer-valued random variable, $P(X_0 = 0) = 1$, that is independent of the i.i.d. sequence $Z_1, Z_2, \ldots$, where Z_n can take values in the set $\{-1, 1\}$ such that $P(Z_n = -1) = \underset{q}{\frac{1}{4}}, P(Z_n = 1) = \underset{p}{\frac{3}{4}}$. Let $X_n = X_{n-1} + Z_n$, $n = 1, 2, \ldots$.

    (a) Find $P(X_3 = 1)$? Answer(F): $3p^2q$ (1 mark)

    (b) Find the value of $P(X_5 = -1 \mid X_2 = 0)$? Answer(F): $3pq^2$ (2 marks)

11. Let us assume that cars arrive according to a Poisson process at rate $\overset{\lambda}{7}$ per hour. Assume each car will pick up a hitchhiker (usually strangers, for a ride in their car) with probability $\frac{1}{7}$. You are $\underset{x}{\text{third}}$ in line.

    $\frac{1}{\lambda p}$

    (a) What is the probability that no cars arrive in first 15 minutes (t_1 minutes)? Answer(E): $P(N(\frac{t_1}{60}) = 0) = e^{-\frac{\lambda t_1}{60}}$ (1 mark)

    (b) What is the probability that you will have to wait for more than 3 hours (t_2 hours)?

    For $x = 3$

    Answer(E): $P(N(t_2) \leq x-1) = P(N(t_2)=0) + P(N(t_2)=1) + \cdots + P(N(t_2)=x-1)$ (2 marks)

    $P(N(t_2)=0) + P(N(t_2)=1) + P(N(t_2)=2)$

12. Assume the life times of $N = 100$ soldiers are iid following an exponential distribution with parameter μ, then the process of the number of surviving soldiers by time t, $\{X(t), t \geq 0\}$, is a pure death process with death rates $\mu_i = i\mu$, $i = 1, 2, \ldots, N$. Assume that, $X(0) = N$.

    (a) Find $P(X(t) = N - 1)$? Answer(E): $N \times (e^{-\mu t})^{N-1}(1 - e^{-\mu t})$ (2 marks)

    (b) Let S_N be the time of the death of the last member of the population, i.e., S_N is the time to extinction. Find $E(S_N)$? Answer(E): $S_N = X_1 + X_2 + \cdots X_N$ (1 mark) $\sum_{i=1}^{N} \frac{1}{i\mu}$

    $X_i \sim exp(i\mu)$ $E(S_N) = \sum_{i=1} E(X_i) = \sum_{i=1} \frac{1}{i\mu}$

13. Patients visit a doctor in accordance with a Poisson process at the rate of $\overset{\lambda}{5}$ per hour, and the time doctor takes to examine any patient is independent exponential distributed with mean 4 minutes (n minutes). All arriving patients attended by the doctor.

    (a) Write the Kendall notaion for the underlying queueing model.
    Answer(E): M|M|1 $\lambda = \lambda$, $\mu = \frac{60}{n}$ per hour $= 15$ (1 mark)

    (b) Find the expected waiting time (in minutes) in queue of any patient.
    Answer(F): $\frac{\rho^2}{\lambda(1-\rho)}$ hour $= \frac{\lambda}{\mu(\mu-\lambda)}$ hour $= \frac{5}{15(15-5)} = \frac{1}{30}$ hour (2 marks)

    $= \frac{1}{30} \times 60 = 2$ minutes

Q11 (b) $\lambda' = \text{Rate} = \lambda \times \frac{1}{p} = 1$

Req. probability $= e^{-\lambda' t_2} + e^{-\lambda' t_2} \cdot \lambda' t_2 + \frac{e^{-\lambda' t_2}(\lambda' t_2)^2}{2!}$

Subjective Type Questions: **Section 3** **($4 \times 5 = 20$ marks)**

Write the answer in the same page provided for the questions 14 to 17. **Full marks** are awarded if all the steps are correct, and **partial marks** for an incorrect answer with wrong steps.

14. Let $\Omega = \{a, b, c, d\}$.

(a) Find three different σ-algebras $\{\mathcal{F}_n\}$ for $n = 1, 2, 3$ such that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$. (3 marks)

(b) Further, create a set function $P : \mathcal{F}_3 \to \mathcal{R}$ such that, $(\Omega, \mathcal{F}_3, P)$ is a probability space. (2 marks)

(a) $F_1 = \{\phi, \Omega\}$, $F_2 = \{\phi, \{a\}, \{b, c, d\}, \Omega\}$

F_3 = Power set of Ω or $F_3 = \underbrace{\{\phi, \{a\}, \{b\}, \{a,b\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}, \Omega\}}_{A}$

There can be many possible F_1, F_2, and F_3. But they should satisfy the three properties of sigma field and they should be such that $F_1 \subseteq F_2 \subset F_3$.

Properties 1.) $\phi \in F$ 2.) If $A \in F$ then $A^c \in F$

3.) If $A_1, A_2, \ldots A_n \in F$ then $A_1 \cup A_2 \cup \ldots A_n \in F$.

Similarly for $\Omega = \{1, 2, 3, 4\}$ taking $1 \leftrightarrow a$, $2 \leftrightarrow b$, $3 \leftrightarrow c$, $4 \leftrightarrow d$.

(b) If F_3 is power set then defining on the single ton elements will be enough and that sum should be 1

For e.g. $P(\{1\}) = 0.2$ $P(\{2\}) = 0.2$ $P(\{3\}) = 0.3$ $P(\{4\}) = 0.4$

Then $P(\Omega) = 0.2 + 0.2 + 0.3 + 0.4 = 1$

$P(\{1, 2, 3\}) = P\{1\} + P\{2\} + P\{3\} = 0.2 + 0.2 + 0.3 = 0.7$

Similar for other elements

If F_3 is like A. Then define on the single tons in F_3 and one set ~~negation~~ complement of their union and from their probability for all other elements can be obtained.

For e.g. for $F_3 = A$

Define P as $P\{a\} = 0.2$ $P\{b\} = 0.2$ $P\{c,d\} = 0.6$

Then $P\{a,b\} = 0.2 + 0.2$ $P\{a,c,d\} = P\{a\} + P\{c,d\}$

$= 0.2 + 0.6 = 0.8$

15. Consider a DTMC $\{X_n, n = 0, 1, \ldots\}$ with $S = \{1, 2, 3, 4\}$ and its one-step transition probability matrix

$$P = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1/4 & 1/2 & 1/4 & 0 \\ 0 & 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix} \end{array}$$

(a) Classify the states as transient, + recurrent or null recurrent. (2 marks)

(b) Find the stationary distribution, $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$, if it exists. (3 marks)

(a) Since all the states communicate with each other, therefore the markov chain is irreducible.
Now, in a finite state markov chain atleast one state is positive recurrent. and here since ~~all~~ chain is irreducible, therefore, all states are positive recurrent.

(b) The chain is positive recurrent, irreducible and also aperiodic as period of state 1 is 1 and hence of all states. This implies the chain is ergodic.

Since $\sum_{j=1}^{4} P_{ij} = \sum_{i=1}^{4} P_{ij} = 1$, therefore, the chain is doubly stochastic.

An ergodic markov chain has a unique stationary distribution, and since the chain is doubly stochastic that distribution will be the uniform one i.e

$$\pi_i = \frac{1}{4}, \quad i = 1, 2, 3, 4$$

OR

Solve $\pi = \pi P$ and $\sum_i \pi_i = 1$

You will get $\pi_1 = \pi_2 = \pi_3 = \pi_4$ and since $\sum \pi_i = 1$

$\Rightarrow \pi_i = \frac{1}{4} \quad i = 1, 2, 3, 4.$

16. Let $\{X(t), t \geq 0\}$ be a Poisson process with parameter λ and $X(0) = j$ where j is a positive integer. Consider the random variable $T_j = \inf\{t : X(t) = j+1\}$, i.e., T_j is the time of occurrence of the first jump after the jth jump, $j = 1, 2, \ldots$.

(a) Find the distribution of T_1. (2 marks)

(b) Find the joint distribution of $(T_{2019}, T_{2018}, T_{2017})$. (3 marks)

(a) The interarrival time in a poisson process follows exponential with parameter same as the parameter of the poisson process.

Here T_1 is the interarival time between 1st and 2nd jump, therefore, $T_1 \sim exp(\lambda)$.

$$f_{T_1}(t) = \lambda e^{-\lambda t} \quad , t \geq 0$$

(b) The interarrival times in a poisson process are independent and exponential.

$\underbrace{T_{2019}}_{X_1}, \underbrace{T_{2018}}_{X_2}, \underbrace{T_{2017}}_{X_2}$ are interarrival times and since are independent and each one is $exp(\lambda)$.

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = f_{X_1}(x_1)\, f_{X_2}(x_2)\, f_{X_3}(x_3)$$

$$= \lambda e^{-\lambda x_1} \times \lambda e^{-\lambda x_2} \times \lambda e^{-\lambda x_3} \qquad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

$$= \lambda^3 e^{-\lambda(x_1 + x_2 + x_3)} \quad , x_1, x_2, x_3 \geq 0.$$

17. Consider a $M/M/1/2$ queuing system. Let $X(t)$ be a random variable denoting the number of customers in the system at any time t.

(a) Draw the state transition diagram for the process $\{X(t), t \geq 0\}$. (1 mark)

(b) Let $p_n(t)$ be the probability that there are n customers in the system at time t given that there was no customers at time 0. Find the time-dependent probabilities $p_n(t)$ for $n = 0, 1, 2$. (4 marks)

(a)

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "\mu"] \end{tikzcd}$$

(b) $Q = \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -(\mu+\lambda) & \lambda \\ 0 & \mu & -\mu \end{bmatrix}$

$p_n'(t) = p_n(t)\, Q \Rightarrow$

& $p_0(t) + p_1(t) + p_2(t) = 1$

$p_0'(t) = -\lambda p_0(t) + \mu p_1(t)$

$p_1'(t) = \mu p_0(t) - (\mu+\lambda) p_1(t) + \mu p_2(t)$

$p_2'(t) = \lambda p_1(t) - \mu p_2(t)$

$p_0(t) + p_1(t) + p_2(t) = 1$ Also $p_0(0) = 1$, $p_1(0) = 0$, $p_2(0) = 0$

Taking Laplace transform

$s p_0^*(s) - 1 = -\lambda p_0^*(s) + \mu p_1^*(s)$

$s p_1^*(s) = \mu p_0^*(s) - (\mu+\lambda) p_1^*(s) + \mu p_2^*(s)$

$s p_2^*(s) = \lambda p_1^*(s) - \mu p_2^*(s)$

$p_0^*(s) + p_1^*(s) + p_2^*(s) = \frac{1}{s}$

Solve above eqn^{n}s to obtain

$$p_0^*(s) = \frac{\lambda^2}{s(s^2 + 2s\mu + 2\lambda s + \mu^2 + \lambda\mu + \lambda^2)}, \quad p_1^*(s) = \frac{\lambda(s+\mu)}{s(s^2 + 2s\mu + 2\lambda s + \mu^2 + \lambda^2 + \lambda\mu)}$$

$$p_2^*(s) = \frac{s^2 + 2s\mu + s\lambda + \mu^2}{s(s^2 + 2s\mu + 2\lambda s + \mu^2 + \lambda\mu + \lambda^2)}$$

Applying the partial fractional techniques and then taking the inverse Laplace transform for $\lambda \neq \mu$ we get

$$p_0(t) = \frac{\lambda^2}{\lambda^2 + \lambda\mu + \mu^2} + \frac{\lambda^2}{2\lambda\mu - 2(\lambda+\mu)\sqrt{\lambda\mu}} e^{-(\lambda+\mu-\sqrt{\lambda\mu})t} + \frac{\lambda^2}{2\lambda\mu + 2(\lambda+\mu)\sqrt{\lambda\mu}} e^{-(\lambda+\mu+\sqrt{\lambda\mu})t}$$

$$p_1(t) = \frac{\lambda\mu}{\lambda^2 + \lambda\mu + \mu^2} + \frac{\lambda(-\lambda+\sqrt{\lambda\mu})}{2\lambda\mu - 2(\lambda+\mu)\sqrt{\lambda\mu}} e^{-(\lambda+\mu-\sqrt{\lambda\mu})t} + \frac{\lambda(-\lambda-\sqrt{\lambda\mu})}{2\lambda\mu - 2(\lambda+\mu)\sqrt{\lambda\mu}} e^{-(\lambda+\mu+\sqrt{\lambda\mu})t}$$

$$p_2(t) = \frac{\mu^2}{\lambda^2 + \lambda\mu + \mu^2} + \frac{\lambda\sqrt{\lambda\mu}}{2\lambda\mu - 2(\lambda+\mu)\sqrt{\lambda\mu}} e^{-(\lambda+\mu-\sqrt{\lambda\mu})t} + \frac{\lambda\sqrt{\lambda\mu}}{2\lambda\mu - 2(\lambda+\mu)\sqrt{\lambda\mu}} e^{-(\lambda+\mu+\sqrt{\lambda\mu})t}.$$