COL202: Discrete Mathematical Structures **Spring 2023**

Tutorial Sheet 6

Announced on: Feb 16 (Thurs)

1. **[Submission Problem for Group 1]** Based on Sec 12.5 in [LLM17].

   A bipartite graph $G = (V_L, V_R, E)$ with left and right vertex sets V_L and V_R, respectively, is said to be *degree-constrained* if there is an integer $d \geqslant 1$ such that for every vertex $\ell \in V_L$ in the left set and $r \in V_R$ in the right set,

   $$\text{degree}(\ell) \geqslant d \geqslant \text{degree}(r).$$

   Show that any degree-constrained bipartite graph admits a *left-perfect* matching, i.e., a matching that covers every vertex in V_L.

2. **[Submission Problem for Group 2]** Based on Problem 12.27 in [LLM17].

   A simple graph G is said to have width w if and only if its vertices can be arranged in a sequence such that each vertex is adjacent to at most w vertices that precede it in the sequence. For example, if the degree of every vertex is at most d, then the graph has width at most d—just list the vertices in any order.

   a) Prove that every graph with width at most w is $(w + 1)$-colorable.

   b) Prove that the average degree of a graph of width w is at most $2w$.

3. **[Submission Problem for Group 3]** Based on Problem 12.20 in [LLM17].

   Take a regular deck of 52 cards. Each card has a suit and a value. The suit is one of four possibilities: heart, diamond, club, spade. The value is one of 13 possibilities: $A, 2, 3, \ldots, 10, J, Q, K$. There is exactly one card for each of the 4×13 possible combinations of suit and value.

   Ask your friend to lay the cards out into a grid with 4 rows and 13 columns. They can fill the cards in any way they like.

   Show that you can always pick out 13 cards, one from each column of the grid, so that you wind up with cards of all 13 possible values.

   Hint: You may use the result from Problem 1.

4. **[Submission Problem for Group 4]**

   Consider the following stable matching instance for a general (non-bipartite) graph on four vertices a, b, c, d:

$$a : b \succ c \succ d$$
$$b : c \succ a \succ d$$
$$c : a \succ b \succ d$$
$$d : a \succ b \succ c$$

The notation "$a : b \succ c \succ d$" means that vertex a's top choice is b, its next favorite is c, and its least favorite is d. Assume that each vertex prefers being matched over staying unmatched.

Given a matching, a *blocking pair* is a pair of vertices that prefer each other over their assigned partners. A *stable* matching is one that does not have any blocking pair.

How many stable matchings are there in the above instance? Explain your reasoning.

References

[LLM17] Eric Lehman, Tom Leighton, and Albert R Meyer. *Mathematics for Computer Science*. 2017. URL: https://courses.csail.mit.edu/6.042/spring18/mcs.pdf.