COL 351

TUTORIAL SHEET 9

1. Let G be a bipartite graph with V_L and V_R denoting the set of vertices on the two sides. Suppose there is a matching M_1 which matches a subset X of vertices in V_L, and there is a matching M_2 which matches a subset Y of vertices in V_R. Show that there is a matching which matches all the vertices in X and Y.

   Solution: Consider the edges in $M_1 \cup M_2$, where we take two copies of an edge if it appears in both the matchings. In this graph H, every connected component will be a path or a cycle. Let X' be the subset of vertices which are unmatched in M_2 and Y' be the subset of Y which are unmatched by M_1. In the graph H, each vertex in $X' \cup Y'$ will have degree 1 and so will be an end-point of a path in H. So now take a path in H: let this path be $v_1, \ldots, v_k$. If $v_1 \in X'$, we take the edges $(v_1, v_2), (v_3, v_4)$... in our matching. Note that if $v_k \in V_R$, then $v_k \in Y'$ and this matching will include v_k also. For every cycle in H, we can take alternate edges forming a matching from this cycle.

2. An edge coloring of a graph with k colors assigns a color from the set $\{1, 2, \ldots, k\}$ to each edge such that no two edges sharing a common vertex receive the same color. Show that a bipartite graph where each vertex has degree exactly k has an edge coloring. Extend this result by showing that if Δ denotes the maximum degree of a vertex in a bipartite graph, then there is an edge coloring with Δ colors.

   Solution: We showed in class that any bipartite graph where each vertex has the same degree k has a perfect matching. We first find such a matching M in G and assign all the edges color 1. Now we remove all these edges. The resulting graph now has the property that all vertices have the same degree, namely $k-1$. So we can again find a perfect matching, color all these edges with color 2, remove them from G and so on.

   For the second part, we will first add some more edges to G such that every vertex has degree exactly Δ and then we will use the first part of this question. So suppose there is a vertex u on the left side whose degree is less than Δ. Then there must be a vertex on the right side whose degree is less than Δ (why?). And so, we can add an edge between these two vertices and repeat the process till every vertex has degree Δ.

3. Suppose we divide the set of 52 playing cards into 13 groups, where each group contains 4 cards. Then show that it is possible to select one card from each group such that the resulting 13 cards have denomination $2, 3, \ldots, 10, J, Q, K, A$.

   Solution: We reduce this problem to bipartite matching problem. Consider a graph with 13 vertices on both sides. On the left side a vertex i is labelled with a suit from $\{2, 3, \ldots, 10, J, Q, K, A\}$ and on the right side, we have one vertex for each group (it is possible that there are parallel edges here, because the same suit can appear several times in a group). We have an edge between a vertex i on left and a vertex j on the right if the suit i appears in the group j. Now each vertex in this graph has degree 4. Therefore it has a perfect matching. Now select the card given by this matching from each group.

4. Let G be a bipartite graph with n vertices on both sides, and let r be the maximum size of any matching in G. Then show that there is a set S of vertices of V_L such that $N(S)$ has size $|S| - r$. Here $N(S)$ denotes the set of vertices in V_R that have at least one edge to a vertex in S.

   Solution: This follows from the proof done in class. As done in class, the set of vertices reachable from s in the directed graph constructed with respect to a maximum matching has this property.

5. Consider the following greedy algorithm for finding a matching in a bipartite graph: repeatedly select edges which do not share a common vertex till we cannot add any more edge. In the class, we saw that this algorithm may not give a maximum matching. However, show that if m is the size of the maximum matching in the graph, then this algorithm gives matching of size at least $m/2$.

   Solution: Let M be a matching produced by this greedy algorithm and M' be an optimal matching. Now if e is an edge in M, it can share a vertex with at most two edges in M'. Therefore if $|M'| > 2|M|$, there is an edge in M' which does not share a vertex with any edge in M. But then, the greedy algorithm should have picked this edge.

6. Let M be a matching in a bipartite graph and suppose the shortest length of any augmenting path with respect to M is at least k. Prove that the maximum matching in the graph has at most $|M| + \frac{n}{k+1}$ edges, where n is the number of vertices in the graph. **Solution:** Let M^* be an optimal matching. Then consider the edges in $M^* \cup M$. If $|M^* \setminus M| = r$, there must be r augmenting paths in this graph. Each such path must have length at least k, and hence at least $k+1$ vertices. Since these paths are disjoint, $r(k+1) \geq n$. Therefore, $r \geq \frac{n}{k+1}$.