PROBLEM SHEET# 7

1.

a) $X_1(j\omega) = 2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)$

$$X_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\omega)e^{j\omega t}d\omega$$

$$= 1 + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) = 1 + cos4\pi t$$

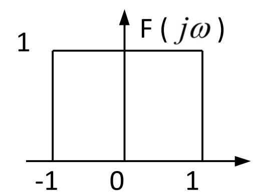
b) $X_2(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(j\omega)e^{j\omega t}d\omega$

$$= \frac{1}{2\pi}\left[\int_{-2}^{0} -2e^{j\omega t}d\omega + \int_{0}^{2} 2e^{j\omega t}d\omega\right.$$

$$\frac{2}{j\pi t}[cos2t - 1]$$

2.

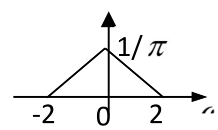
a) Let $f(t) = \frac{sint}{\pi t}$



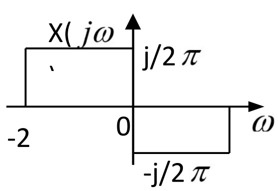
Fourier transform of $f(t)$ is $F(j\omega)$

$$y(t) = f(t) \times f(t) \leftrightarrow \frac{F(\omega) * F(\omega)}{2\pi}$$

Now multiply in time domain



Fourier transform of $t[f(t)]^2$ is $j\frac{d}{d\omega}Y(\omega)$



b) $\int_{-\infty}^{\infty} t^2\left(sin\frac{t}{\pi t}\right)^4 = \frac{1}{2\pi}\int X(j\omega)^2 d\omega = \frac{1}{2\pi} \times \frac{1}{\pi^2} = \frac{1}{2\pi^3}$

3. $Y(\omega) = X(\omega)H(\omega)$

$$Y(\omega) = \frac{1}{9}X\left(\frac{\omega}{3}\right)H\left(\frac{\omega}{3}\right)$$

$$= \frac{1}{9}Y\left(\frac{\omega}{3}\right)$$

$$= \frac{1}{3} \times \frac{1}{3}\ Y\left(\frac{\omega}{3}\right)$$

$$= \frac{1}{3}y(3t)$$

Thus $A = \frac{1}{3}$ $and\ B = 3$

4. $F^{-1}\{(1 + j\omega)X(j\omega)\} = Ae^{-2t}u(t)$

take fourier transform both sides

$$(1 + j\omega)X(j\omega) = \frac{A}{2 + j\omega}$$
$$X(j\omega) = A\left(\frac{1}{1 + j\omega} - \frac{1}{2 + j\omega}\right)$$

Taking inverse Fourier transform

$$x(t) = Ae^{-t}u(t).Ae^{-2t}u(t)$$

Using fact c)

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = 1$$
$$\int_{-\infty}^{\infty} A^2(e^{-2t} + e^{-4t} - 2e^{-3t})u(t)dt = 1$$

Solving and using fact $x(t)$ is non-negative, A=$\sqrt{12}$

5.

1. Fourier transform is purely imaginary. $x(t)$ is real and odd. . Therefore, signals in figures (a) and (d) have this property.

2. $x(t)$ is real and even. Therefore, signals in figures (e) and (f) have this property.

3. For there to exist a real such that $e^{j\propto\omega}X(j\omega)$ is real, we require that $x(t{+}\propto)$ be a real and even signal. Therefore, signals in figures (a), (b), (e), and (f) have this property.

4. For this condition to be true, $x(t = 0) = 0$. Therefore, signals in gures (a), (b), (c), (d), and (f) have this property.

5. For this condition to be true the derivative of $x(t)$ has to be zero at t = 0. Therefore, signals in figures (b), (c), (e), and (f) have this property.

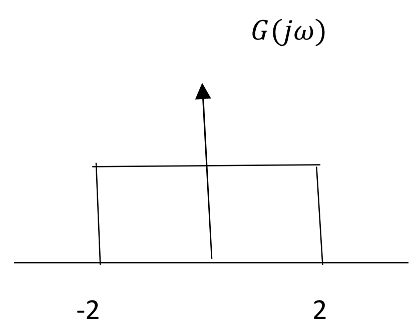
6. The signal in figure (a) and (b) have this property. Figure (a) has this property because it's the integration of shifted square waves. The Fourier transform of the signal that shown in Figure (b) is $X(j\omega) = 2e^{-j\omega}$ which is periodic

6.

a) $G(j\omega) = \frac{(X(j(\omega-1)) + X(j(\omega+1))}{2}$

$g(t) = \int_{-\infty}^{\infty} G(j\omega)e^{j\omega t}d\omega = \frac{sin2t}{\pi t} = x(t)cost$

$$x(t) = \frac{2sint}{\pi t}$$



b) $G(j\omega) = \frac{\left(X_1(j\left(\omega - \frac{2}{3}\right)) + X_1(j(\omega + \frac{2}{3})\right)}{2}$

$$x_1(t) = \frac{g(t)}{cos\frac{2}{3}t} = \frac{sin2t}{\pi t cos\frac{2}{3}t}$$

7.

a) $H(j\omega) = \frac{1}{2\pi}(H(j\omega) * (\frac{1}{j\omega} + \pi\delta(\omega))$

$H(j\omega) = \frac{1}{2}H(j\omega) + \frac{1}{2\pi}\left(H(j\omega) * \frac{1}{\omega}\right)$

$$H(j\omega) = \frac{1}{\pi j}\left(H(j\omega) * \frac{1}{\omega}\right)$$

$$H(j\omega) = \frac{1}{\pi j}\int_{-\infty}^{\infty}\frac{H(j\eta)}{\omega - \eta}d\eta$$

$$H_R(jw) + jH_I(jw) = \frac{1}{\pi j}\int_{-\infty}^{\infty}\frac{H_R(j\eta) + jH_I(j\eta)}{w-\eta}d\eta = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H_I(j\eta) - jH_R(j\eta)}{w-\eta}$$

$$H_R(jw) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H_I(j\eta)}{w-\eta}d\eta \quad \text{and} \quad H_I(jw) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H_R(j\eta)}{w-\eta}d\eta$$

b) $y(t) = x(t) * \frac{1}{\pi t}$

$$u(t) \stackrel{FT}{\longleftrightarrow} \frac{1}{jw} + \pi\delta(w)$$

$$2u(t) - 1 \stackrel{FT}{\longleftrightarrow} 2\frac{1}{jw}.$$

Using the duality property, we have

$$\frac{2}{jt} \stackrel{FT}{\longleftrightarrow} 2\pi[2u(-w) - 1]$$

$$\frac{1}{\pi t} \stackrel{FT}{\longleftrightarrow} j[2u(-w) - 1]$$

$$Y(jw) = X(jw)H(jw)$$

$$H(jw) = j[2u(-w) - 1] = \begin{cases} -j, w > 0 \\ j, w < 0 \end{cases}$$

c) Let $y(t)$ be Hilbert transform of $x(t)$

$$Y(jw) = X(jw)H(jw) = \pi[\delta(w-3) + \delta(w+3)]H(jw) = -j\pi\delta(w-3) + j\pi\delta(w+3).$$

Therefore $y(t) = sin3t$

8.

a) Since $H(j\omega)$ is real and even $h(t)$ is also real and even.

$$|h(t)| = \left|\frac{1}{2\pi}\int_{-\infty}^{\infty}H(jw)e^{jwt}dw\right| \le \frac{1}{2\pi}\int_{-\infty}^{\infty}|H(jw)|\,e^{jwt}dw$$

b) B.W is 2ω

c) $B_w H(j0) =$ Area under $H(j\omega)$

$$B_w = \frac{1}{H(j0)} \int_{-\infty}^{\infty} H(jw)dw.$$

d) $t_r = \frac{s(\infty)}{h(0)} = \frac{\int_{-\infty}^{\infty} h(t)dt}{\frac{1}{2\pi}\int_{-\infty}^{\infty} H(jw)dw} = \frac{H(j0)}{\frac{1}{2\pi}\int_{-\infty}^{\infty} H(jw)dw} = \frac{2\pi}{B_w}$

9.

a)$X(\omega) = \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t}\, dt$

making perfect square

$$\int_{-\infty}^{\infty} (e^{-a(t+\frac{j\omega}{2a})^2}\, dt) e^{-\frac{\omega^2}{4a}}$$

$$\sqrt{a}\left(t + \frac{jw}{2a}\right) = t'; \ \sqrt{a}dt = dt'$$

$$\frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-(t')^2} dt' e^{\frac{-w^2}{4a}}$$

$$\sqrt{\frac{\pi}{a}} e^{\frac{-w^2}{4a}}$$