

**Department of Mathematics**  
**MTL 106 (Introduction to Probability and Stochastic Processes)**  
**Tutorial Sheet No. 2**

1. Consider a probability space  $(\Omega, \mathcal{F}, P)$  with  $\Omega = \{0, 1, 2\}$ ,  $\mathcal{F} = \{\emptyset, \{0\}, \{1, 2\}, \Omega\}$ ,  $P(\{0\}) = 0.5 = P(\{1, 2\})$ . Give an example of a real-valued function on  $\Omega$  that is NOT a random variable. Justify your answer.

2. Do the following functions define distribution functions.

$$(a) F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq \frac{1}{2} \\ 1, & x > \frac{1}{2} \end{cases} \quad (b) F(x) = \left(\frac{1}{\pi}\right)\tan^{-1}x, \quad -\infty < x < \infty \quad (c) F(x) = \begin{cases} 0, & -\infty < x < 0 \\ 1 - e^{-x}, & 0 \leq x < \infty \end{cases}$$

3. Consider the random variable  $X$  that represents the number of people who are hospitalized or die in a single head-on collision on the road in front of a particular spot in a year. The distribution of such random variables are typically obtained from historical data. Without getting into the statistical aspects involved, let us suppose that the cumulative distribution function of  $X$  is as follows:

$x$	0	1	2	3	4	5	6	7	8	9	10
$F(x)$	0.250	0.546	0.898	0.932	0.955	0.972	0.981	0.989	0.995	0.998	1.000

Find (a)  $P(X = 10)$  (b)  $P(X \leq 5/X > 2)$ .

4. Let  $X$  be a Poisson random variable with parameter  $\lambda$ . Show that  $P(X = i)$  increases monotonically and then decreases monotonically as  $i$  increases, reaching its maximum when  $i$  is the largest integer not exceeding  $\lambda$ .
5. For what values of  $\alpha, p$  does the following function represent a probability mass function  $p_X(x) = \alpha p^x, x = 0, 1, 2, \dots$ . Prove that the random variable having such a probability mass function satisfies the following memoryless property  $P(X > a + s/X > a) = P(X \geq s)$ .
6. Let  $X$  be a random variable such that  $P(X = 2) = \frac{1}{4}$  and its distribution function is given by

$$F_X(x) = \begin{cases} 0, & x < -3 \\ \alpha(x+3), & -3 \leq x < 2 \\ \frac{3}{4}, & 2 \leq x < 4 \\ \beta x^2, & 4 \leq x < 8/\sqrt{3} \\ 1, & x \geq 8/\sqrt{3} \end{cases}.$$

(a) Find  $\alpha, \beta$  if 2 is the only jump discontinuity of  $F$ .

(b) Compute  $P(X < 3/X \geq 2)$ .

7. Consider a random experiment of choosing a point in the annular disc of inner radius  $r_1$  and outer radius  $r_2$  ( $r_1 < r_2$ ). Let  $X$  be the distance of chosen point from the center of annular disc. Find the pdf of  $X$ .
8. Let  $X$  be an absolutely continuous random variable with density function  $f$ . Prove that the random variables  $X$  and  $-X$  have the same distribution function if and only if  $f(x) = f(-x)$  for all  $x \in \mathbb{R}$ .
9. A student arrives to the bus stop at 6:00 AM sharp, knowing that the bus will arrive in any moment, uniformly distributed between 6:00 AM and 6:20 AM.
- (a) What is the probability that the student must wait more than five minutes?
- (b) If at 6:10 AM the bus has not arrived yet, what is the probability that the student has to wait at least five more minutes?
10. Accidents in Delhi roads involving BlueLine buses obey Poisson process with 9 per month of 30 days. In a randomly chosen month of 30 days.
- (a) What is the probability that there are exactly 4 accidents in the first 15 days?
- (b) Given that exactly 4 accidents occurred in the first 15 days, what is the probability that all the four occurred in the last 7 days out of these 15 days?

11. In a torture test a light switch is turned on and off until it fails. If the probability that the switch will fail any time it is turned 'on' or 'off' is 0.001, what is probability that the switch will fail after it has been turned on or off 1200 times?.
12. The time to failure of certain units is exponentially distributed with parameter  $\lambda$ . At time  $t = 0$ ,  $n$  identical units are put in operation. The units operate, so that failure of any unit is not affected by the behavior of the other units. For any  $t > 0$ , let  $N_t$  be the random variable whose value is the number of units still in operation time  $t$ . Find the distribution of the random variable  $N_t$ .
13. The life time (in hours) of a certain piece of equipment is a continuous random variable  $X$ , having pdf

$$f_X(x) = \begin{cases} \frac{xe^{-x/100}}{10^4}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}.$$

If four pieces of this equipment are selected independently of each other from a lot, what is the probability that atleast two of them have life length more than 20 hours?.

14. Suppose that  $f$  and  $g$  are density function and that  $0 < \lambda < 1$  is a constant. (a) Is  $\lambda f + (1 - \lambda)g$  a probability density function? (b) Is  $fg$  a probability density function? Explain.
15. The probability of hitting an aircraft is 0.001 for each shot. Assume that the number of hits when  $n$  shots are fired is a random variable having a binomial distribution. How many shots should be fired so that the probability of hitting with two or more shots is above 0.95?
16. An airline knows that 5 percent of the people making reservation on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. Assume that passengers come to airport are independent with each other. What is the probability that there will be a seat available for every passenger who shows up?
17. Let  $X$  be a random variable with cumulative distribution function given by:

$$F_X(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{25}, & 1 \leq x < 2 \\ \frac{x^2}{10}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}.$$

Determine the cumulative discrete distribution functions  $F_d$  and one continuous  $F_c$  such that:  $F_X(x) = \alpha F_d(x) + \beta F_c(x)$ .

18. Suppose the duration (measured in minutes) of a telephone conversation between two persons is a random variable  $X$  with cumulative distribution function

$$P(X \leq t) = \begin{cases} 0, & -\infty < t < 0 \\ 1 - e^{-0.04t}, & 0 \leq t < \infty \end{cases}$$

Given that the conversation has been going on for 20 minutes, compute the probability that it continues for at least another 10 minutes.

19. Suppose that the life length of two electronic devices say  $D_1$  and  $D_2$  have normal distributions  $N(40, 36)$  and  $N(45, 9)$  respectively. (a) If a device is to be used for 45 hours, which device would be preferred? (b) If it is to be used for 42 hours which one should be preferred?