

Problem 1

An IPL tournament is played between n cricket teams, where each team plays exactly one match with every other team. How many matches are played? (This is easy.) Assume that no match ends in a tie. We say that a subset S of teams is *consistent* if it is possible to order teams in S as $T_1, \dots, T_{|S|}$ (think of this as the strongest to weakest ordering) such that for every i, j with $1 \leq i < j \leq |S|$, T_i beats T_j . Prove that irrespective of the outcomes of the matches, there always exists a consistent subset S with $|S| \geq \log_2 n$.

Solution: The number of matches played is given by $\binom{n}{2} = \frac{n(n-1)}{2}$. For the second claim, we proceed using the Strong Induction. Consider the claim:

$p(k)$: In a tournament of k teams, always exists a consistent subset S such that $|S| \geq \log_2(n+1)$

Base Case: For $n = 2$, there are only two teams T_1 and T_2 . Say T_1 wins against T_2 . Then $\{T_1, T_2\}$ form a consistent subset with $T_1 T_2$ being the required ordering. $|S| = 2 \geq \log_2 3$. So $P(2)$ is true. For $n = 3$, consider the outcome of any one match. Say T_1 wins against T_2 . Consider the set $S = \{T_1, T_2\}$, which is consistent under the ordering $T_1 T_2$. Also, $|S| = 2 \geq \log_2(n+1) = \log_2 4$.

Induction Hypothesis: Let $P(k)$ be true $\forall k < n, k \geq 2$.

Induction Step: Consider a tournament of n teams. We will construct a consistent subset for this tournament. Note that there will be $\frac{n(n-1)}{2}$ wins distributed amongst n teams, and so by Pigeon-Hole principle, exists a team T such that T wins at least $\lceil \frac{n(n-1)}{2n} \rceil = \lceil \frac{n-1}{2} \rceil$ matches.

Suppose the set of teams T wins against is X . Since $\lceil \frac{n-1}{2} \rceil < |X| < n$, by the induction hypothesis, there exists a consistent subset S of the teams in X such that,

$$|S| \geq \log_2(|X| + 1)$$

Thus, \exists ordering $T_1 T_2 \dots T_{|S|}$ such that T_i beats $T_j \forall i < j$. Note that T wins against $T_k \forall T_k \in S$ and so the set $S' = \{T\} \cup X$ is also consistent, with the required ordering $T T_1 T_2 \dots T_{|S|}$. Now we have, $|S'| = |S| + 1 \geq \log_2(|X| + 1) + 1$. Since $\lceil \frac{n-1}{2} \rceil < |X|$ we have,

$$|S'| \geq \log_2 \left(\left\lceil \frac{n-1}{2} \right\rceil + 1 \right) + 1 \geq \log_2 \left(\frac{n-1}{2} + 1 \right) + 1 \geq \log_2(n+1)$$

Thus, we have $S' = T \cup S$ is the required consistent subset. ■

Problem 2

Call a non-empty subset S of integers *nice* if, for every $x, y \in S$ and every two integers a, b , we have $ax + by \in S$. Observe that the set of multiples of any integer is a nice set. Prove that, in fact, these are the only nice sets. In other words, prove that for every nice set S there exists an integer x such that $S = \{ax \mid a \in \mathbb{Z}\}$.

Solution: We will prove by construction, i.e. finding an x such that $S = \{ax \mid a \in \mathbb{Z}\}$. Recall the result from problem 8 of the tutorial:

$$\text{For every } x, y \in \mathbb{Z} \exists a, b \in \mathbb{Z} \text{ such that } \gcd(x, y) = ax + by$$

By construct of S , $ax + by \in S \forall x, y \in \mathbb{S}$ and $a, b \in \mathbb{Z}$. Thus we have, $\gcd(x, y) \in S \forall x, y \in S$. Note that 0 is always in S . Further, consider the set $T = \{|s| \mid s \neq 0, s \in S\}$. Note that $T \subseteq S$. Also since $\gcd(x, y) \geq 0 \forall x, y$, we must have $\gcd(x, y) \in T \forall x, y \in S$.

Now, $T \subset \mathbb{N}$, thus by Well-Ordering Principle, exists a minimum element (say t_0) in T . Consider $\gcd(t_0, s)$ for any $s \in S$. We know that

$$\gcd(t_0, s) \leq t_0$$

But since $\gcd(t_0, s) \in T$ by minimality of t_0 we must have $\gcd(t_0, s) = t_0 \forall s \in S$. Thus, for all $s \in S$ we may write $s = at_0$ for some $a \in \mathbb{Z}$, or in other words, $S = \{at_0 \mid a \in \mathbb{Z}\}$ ■

Problem 3

Prove “Claim 2” from the proof of Schröder-Bernstein Theorem discussed in Lecture 5. Here is the statement of the claim. Let A and B be infinite sets, f be an injection from A to B , and g be an injection from B to A . Let $B' = \{b \in B \mid \exists b^* \in B \setminus \text{Im}(f) \exists k \in \mathbb{N} \cup \{0\} : b = (f \circ g)^k(b^*)\}$, and $A' = \{g(b) \mid b \in B'\}$. Then for every $b \in B$, the following statements are equivalent.

1. $b \in B'$.
2. If $f^{-1}(b)$ exists, then it is in A' .
3. $g(b) \in A'$.

Solution: (1 \implies 2) Consider any $b \in B'$. By definition, suppose $a = f^{-1}(b)$, or $f(a) = b$. Now by the definition of B' , we have that $b = (f \circ g)^k(b^*)$ for some $k \in \mathbb{N} \cup \{0\}$ and $b^* \in B \setminus \text{Im}(f)$. Thus we may write,

$$f(a) = (f \circ g)^k(b^*)$$

By definition of B' , $b \neq b^*$ since $b \in \text{Im}(f)$ and so $k \geq 1$. Also, by injectivity of f we must have,

$$a = g\left((f \circ g)^{k-1}(b^*)\right) = g(\bar{b}), \bar{b} \in B'$$

Hence, $a \in A'$.

(2 \implies 3) If $f^{-1}(b)$ exists and in A' , then we have $a = f^{-1}(b) = g(\bar{b})$ for some $\bar{b} \in B'$. Write $\bar{b} = (f \circ g)^k(b^*)$ for some $k \in \mathbb{N} \cup \{0\}$ and $b^* \in B \setminus \text{Im}(f)$, and so we have,

$$b = f(a) = f \circ g\left((f \circ g)^k(b^*)\right) = (f \circ g)^{k+1}(b^*)$$

Hence, $b \in B'$ and $g(b) \in A'$.

(3 \implies 1) Since $g(b) \in A'$, by definition of A' it follows that $b \in B'$ ■

Problem 4

Given a set A , the set of finite length strings over A is denoted by A^* . Prove that if A is a finite set, then A^* is necessarily countable. What can you say about the cardinality of A^* if A is countably infinite instead?

Solution: (a) Let s_P denoted a string over a set P , let s_i represent the i -th character of the string, and l_s denote the length of a string s . Define the set A_i as follows:

$$A_i = \{s_A \mid l_s = i\}$$

Note that,

$$A^* = \bigcup_{i=1}^{\infty} A_i$$

Given that A is a finite set, let us suppose it's cardinality is n . Then, cardinality of each A_i is, $|A_i| = n^i$. Thus, each A_i is finite.

We recall that if we have a countably infinite collection of sets, each of which is countable, then $\bigcup_{i=1}^{\infty} A_i$ is countable. Therefore, A^* is countable. ■

(b) By virtue of the countability of A we impose an ordering a_1, a_2, \dots on its elements. Define the set,

$$B_i = \{s_A \mid s_j \in \{a_1 a_2 \dots a_i\} \forall j < l_s\}$$

B_i is the set of finite length strings over the finite set $\{a_1 a_2 \dots a_i\}$. By part (a), B_i is countable. And we have,

$$A^* = \bigcup_{i=1}^{\infty} B_i$$

Once again, we have a countably infinite collection of sets, each of which is countable, and thus $\bigcup_{i=1}^{\infty} B_i$ is countable. Therefore, A^* is countable. ■

Problem 5

Prove by mathematical induction that every graph has at least two vertices having equal degree.

Solution: We proceed by induction on the number of vertices n of the graph. Consider the claim:

$p(k)$: A graph with k vertices at least two vertices with the same degree.

Base Case: $n = 2$. If the two vertices of the graph are connected by an edge, then both have degree 1, else both have degree 0. So $P(2)$ is true.

Induction Hypothesis: Let $P(n - 1)$ be true for some $n \geq 3$, $n \in \mathbb{N}$.

Induction Step: Consider a graph with n vertices. First consider the subgraph G' including any $n - 1$ vertices, say $V = \{v_1, v_2, \dots, v_{n-1}\}$. By the induction hypothesis, \exists vertices $v_i, v_j \in V$ of equal degree. Now when we consider the connection of the n -th vertex v_n there are three cases possible:

Case 1: v_n is connected to neither v_i nor v_j . Then their degrees remains unchanged and hence equal.

Case 2: v_n is connected to both v_i and v_j . Then their degrees are both increased by 1 and hence are still equal.

Case 3: If v_n is connected to v_i but not v_j (say). Let the degree of v_k be represented by d_k . Note that $d_k \in \{0, 1, 2, \dots, n - 1\} \forall k \leq n$. However, if there exists $v : d_v = 0$ then cannot exist a vertex of degree $n - 1$. Similarly if there exists $v : d_v = n - 1$ then cannot exist a vertex of degree 0. So, $d_k \in \{0, 1, 2, \dots, n - 2\}$ or $d_k \in \{1, 2, \dots, n - 1\}$. So we have here $n - 1$ pigeons (possible degrees) and n pigeonholes (vertices). Hence, by PHP, there must exist two vertices which have the same degree. ■