COL751 - Lecture 19

In Lecture 18, we studied a characterization of directed graphs that admit k-edge-disjoint reachability trees rooted at a node s. In particular, we proved that G has k edge-disjoint reachability trees rooted at s if and only if $\textsc{max-flow}(s, v, G) \geqslant k$, for each $v \in V$. We will prove the following related result for undirected graphs.

Theorem 1 (Nash-Williams 1961, Tutte 1961)**.** *Any undirected $2k$-edge-connected graph G on n vertices has k-edge-disjoint spanning trees $T_1, \ldots, T_k$ of size n.*

1 $2k$-edge connected graphs

In order to prove Theorem 1 we will first prove some properties of $2k$-edge-connected graphs.

Consider a graph G obtained as follows. Start with a multigraph consisting of two vertices x and y connected with $2k$ parallel edges. Next repeatedly perform one of the following operations:

1. Add a new edge.
2. Split any set S of k edges. Here **splitting** an edge-set $S = \{(x_1, y_1), \ldots, (x_k, y_k)\}$ refers to adding a new (common) vertex, say w, between endpoints of each edge in S, that is, to replace each edge $(x_i, y_i) \in S$ with the two edges (x_i, w) and (w, y_i).

Lemma 1. *Any graph G obtained by sequence of above two operations is $2k$-edge-connected.*

Proof: Let G_0 be a $2k$-edge-connected graph. Let G_1 be a graph obtained from G_0 by splitting a set S of k edges in G, and let w be the associated vertex added to G_1 in this process. It suffices to show that G_1 is $2k$-edge-connected.

Note for any $x, y \neq w$, $\textsc{max-flow}(x, y, G_1) \geqslant 2k$. Now if G_1 is not $2k$-edge-connected then there must exist a cut (X, X^c) of size at most $2k-1$. Further both X and X^c cannot contain original vertices of G. However, $X, X^c \neq \{w\}$ as degree of w is $2k$, contradicting the existence of cut (X, X^c). $\square$

An interesting question is whether any $2k$-edge-connected graph G can be generated using operations 1 and 2. We will show that this is indeed true. Let us first prove that there is a candidate vertex w in any minimal $2k$-edge-connected graph.

Lemma 2. *Every minimal r-edge-connected graph has a vertex of degree r.*

Proof: Let G be a minimal r-edge-connected graph. Let (X, X^c) be a cut of size r that minimizes $|X|$.

- If $|X| = 1$, then we are done.

- If $|X| > 1$, then there must exist an edge with both endpoints in X, say $e = (a, b)$. Next observe there must exist a cut of size exactly r containing e. Let this cut be (Y, Y^c). Submodularity of cuts states that,

$$\delta(X) + \delta(Y) \geqslant \delta(X \cap Y) + \delta(X \cup Y).$$

  As each cut in G has size at least r, and $\delta(X) = \delta(Y) = r$, we have $X \cap Y$ is also a cut of size r, thereby contradicting the minimality of X. $\square$

Now the next question is whether we can perform **split-off** (i.e. reverse of splitting operation) on vertex w. This is possible due to following result by Lovasz.

Theorem 2 (Lovasz's Splitting Off Theorem)**.** *Let $r \geqslant 2$, G be an undirected graph, and s be a vertex of even degree in G satisfying*

$$\lambda(x, y) \geqslant r, \quad \forall x, y \in V \setminus \{s\}. \tag{1}$$

Then there exists two edges $e_x = (s, x)$ and $e_y = (s, y)$ incident to s such that the graph $G + (x, y) - \{e_x, e_y\}$ also satisfies Eq. 1.

We will prove this theorem in the last section, and proceed assuming the theorem is correct.

Theorem 3. *Every $2k$-edge-connected graph G can be obtained as follows: Start with a multigraph consisting of two vertices x and y connected with $2k$ parallel edges. Next repeatedly perform one of the following operations:*

1. *Add a new edge.*
2. *Split any set S of k edges.*

Proof: Let G be a $2k$-edge-connected graph on n vertices. If G is not a minimal $2k$-edge-connected graph then we can perform operation 1. If G is a minimal $2k$-edge-connected graph then by Lemma 2 we can find a vertex w of degree $2k$, and next perform splitting-off operation on appropriate pairs of edges incident to w to eliminate w. By Theorem 2, the resultant graph will a $2k$-edge-connected graph on $n - 1$ vertices. This process can be repeated until G contains exactly two vertices. $\square$

2 Graph Orientation

Orienting an undirected graph G refers to assigning direction to edges of the graph. We will prove the following result.

Theorem 4 (Nash-Williams, 1960)**.** *An undirected graph G is $2k$-edge-connected iff there exists an orientation of G, say $D(G)$, that is strongly-k-edge-connected.*

Proof: If $D(G)$ is strongly-k-edge-connected, then for any cut (X, X^c) there are k edges in both directions, thereby proving G is $2k$-edge-connected.The reverse claim can be proven inductively using Theorem 3 and is left as an exercise. $\square$

Proof of Theorem 1 By Theorem 4, we have that for any arbitrary vertex s in $D(G)$, $\textsc{max-flow}(s, v, D(G)) \geqslant k$. Using the Edmond's Tree Packing theorem (a.k.a Edmond's Disjoint Reachability Theorem) we get k-edge-disjoint reachability trees $T_1, \ldots, T_k$ rooted at s. Ignoring the edge directions in these k trees gives us the corresponding trees for G.

3 Lovasz's Splitting Off Theorem

Reminder of Theorem 2. *Let $k \geqslant 2$, G be an undirected graph, and s be a vertex of even degree in G satisfying*

$$\lambda(a, b) \geqslant k, \quad \forall a, b \in V \setminus \{s\}. \tag{2}$$

Then there exists two edges $e_x = (s, x)$ and $e_y = (s, y)$ incident to s such that the graph $G + (x, y) - \{e_x, e_y\}$ also satisfies Eq. 2.

Proof: Let us fix an edge (s, x) incident to s. We will prove that there exists a neighbor $y(\neq x)$ of s for which $G + (x, y) - \{(s, x), (s, y)\}$ satisfies Eq. 2. Let us suppose this is not true. Then for each $y \in Y$ there must exist a cut (Y, Y^c) of size $k+1$ satisfying $x, y \in Y$, $s \in Y^c$, and $|Y^c| \geqslant 2$. Let '$\mathcal{C}$' be a minimal collection of such Y's whose union covers neighbors of s.

Observe that for $Y \in \mathcal{C}$, $\delta(Y) \leqslant k + 1$ and $\delta(Y \cup \{s\}) \geqslant k$. This implies

$$deg(s, Y) \leqslant deg(s)/2.$$

This together with the fact that $x \in Y$ implies $|\mathcal{C}| \geqslant 3$. Let Y_1, Y_2, Y_3 be three elements in $\mathcal{C}$. We have

$$\begin{aligned} x &\in Y_1 \cap Y_2 \cap Y_3, \\ Y_1 &\not\subseteq (Y_2 \cup Y_3), \\ Y_2 &\not\subseteq (Y_1 \cup Y_3), \\ Y_3 &\not\subseteq (Y_1 \cup Y_2). \end{aligned}$$

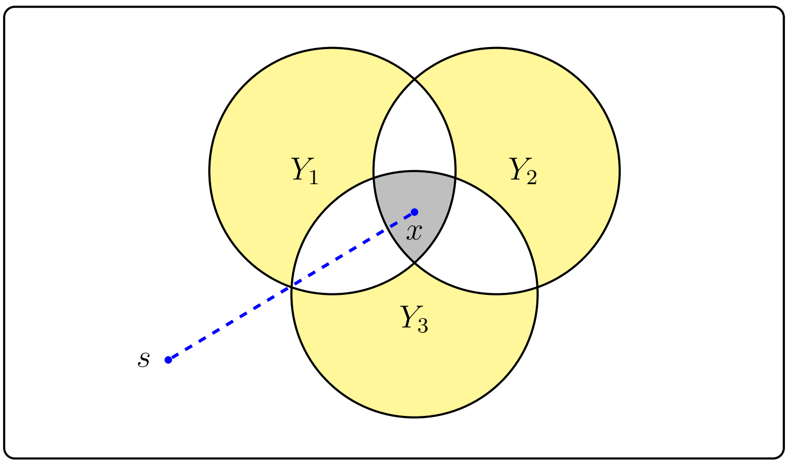

Figure 1: Depiction of sets Y_1, Y_2, Y_3 and edge (s, x) separated by cuts (Y_i, Y_i^c), $i \in [1, 3]$.

By three-way submodularity we have,

$$\begin{aligned}\delta(Y_1)+\delta(Y_2)+\delta(Y_3) \;\geqslant\; & \delta(Y_1\cap Y_2\cap Y_3)\\ & +\delta(Y_1\setminus(Y_2\cup Y_3))\\ & +\delta(Y_2\setminus(Y_1\cup Y_3))\\ & +\delta(Y_3\setminus(Y_1\cup Y_2)).\end{aligned}$$

This along with the fact that edge (s,x) is covered thrice in left-hand side, and exactly once in right-hand side implies the following stronger relation.

$$\begin{aligned}\delta(Y_1)+\delta(Y_2)+\delta(Y_3) \;\geqslant\; & \delta(Y_1\cap Y_2\cap Y_3)\\ & +\delta(Y_1\setminus(Y_2\cup Y_3))\\ & +\delta(Y_2\setminus(Y_1\cup Y_3))\\ & +\delta(Y_3\setminus(Y_1\cup Y_2))\\ & +2.\end{aligned}$$

In above inequality, every term on the left-hand side is at most $k+1$ (by the definition of Y's) and every term on the right-hand side is at least k (by our assumption of max-flow being at least k between pairs in $V\setminus\{v\}$). So, we have

$$3k+3 \;\geqslant\; 4k+2,$$

implying $k\leqslant 1$. This contradicts our assumption. Hence, there must exist a neighbor $y(\neq x)$ of s for which $G+(x,y)-\{(s,x),(s,y)\}$ satisfies Eq. 2. $\square$