Name: ______________________ Roll No: ______________

(COL 202) Discrete Mathematics November 21, 2023

Major (   )

Duration: 2 hours **(35 points)**

- Be clear in your writing. If you use a statement proved in class or in the problem set, then write down the entire statement before using it.
- **You will not get a new sheet, so make sure you are certain when you write something.** Make a judicious decision of which tool(s) to use to get a clean and short answer that fits in the space.
- Every question in this paper is worth 7 points.
- This major paper contains two question papers in one: one **regualar paper and one easy paper** as follows. Every question has two parts: **A** and **B**. You can **either answer Part A for all questions** or **answer Part B for all questions**, i.e., you can either **answer 1A, 2A, 3A, 4A, 5A** or you can **answer 1B, 2B, 3B, 4B, 5B**. You **CANNOT mix**: for e.g., **if you have answered 1A, 2B, 3A, 4B, 5A, then your questions 2 and 4 will not be graded**, i.e, if you mix, I will assume you chose to answer part A or B based on whichever part the majority of your answers come from. **Part B of every question will be considerably easy**. So if you want to choose the easy paper, you should attempt just the questions from Part B for all the questions. **However if you choose to answer Part B, the maximum final grade you will be eligible for is a D, even if your pre-major + major score is eligible for a better grade.**
- Before you turn in your paper, **indicate which part (A or B) you have attempted** in this paper in the top of this page in the space provided, i.e., **Major (   )**
- If you cheat, you will surely get an F in this course.

Notation and some helpful information.

- Recall from tutorial 9 that a *tournament* is a directed graph $G = (V, E)$ where, for every pair of vertices $u, v \in V$, exactly one of the following holds: (a) $u = v$; (b) $u \to v$; (c) $v \to u$. We can think of the vertices of a tournament as players in a round-robin tournament without ties or rematches. Each player plays against every other player exactly once; $u \to v$ indicates that player u beat player v.
- A **Hamilton cycle** in a digraph is a (directed) cycle of length n, i. e., a cycle that passes through all vertices. G is **Hamiltonian** if it has a Hamilton cycle. A **Hamilton path** in a digraph is a (directed) path of length $n-1$, i. e., a path that passes through all vertices.

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1. **(A)** Show that if the minimum degree of any vertex of a graph G is greater than or equal to $\frac{n-1}{2}$, then G must be connected.

   By means of contradiction, assume that the minimum vertex degree is $(n-1)/2$ and that G is not connected. Then G has (at least) two connected components. In each of the components, the minimum vertex degree is still $(n-1)/2$ and that means that each connected component must have at least $(n-1)/2 + 1$ vertices. Since there are at least two components, that means that the graph has at least $2((n-1)/2 + 1) = n + 1$ vertices, which is a contradiction Note: Many of you have tried to prove it using induction which roughly goes as follows: You have assumed as your induction hypothesis "if the minimum degree of any vertex of a graph G is greater than or equal to $\frac{n-1}{2}$, then G must be connected." and then for a graph on $n+1$ vertices, you have tried to show that it is connected. There were several variations on this theme that I cannot list here. Notice the premise guarantees your minimum degree to be $n/2$. Pick a vertex v and look at $G' = G \setminus \{v\}$. Once you remove a vertex your degree drops by 1, so the new minimum degree is $n = n - 2/2$ which is lower than what is guaranteed by the induction hypothesis for n-vertex graphs. Similarly you cannot argue that I construct my new graph on $n+1$ vertices by adding so and so edges to my old graph - it seems like you are giving an algorithm to get a connected graph on $n+1$ vertices starting from a connected graph on n vertices. This is not what the question is asking you to prove. If you raise an arbitrary regrade request for your wrong induction proof, you will directly get a -7 on the paper. So do make sure you are confident of convincing me before you raise a regrade request.

2. **(A)** For a graph G, let $L(G)$ denote the so-called line graph of G, given by

$$L(G) = (E, \{\{e, e'\} : e, e' \in E(G), e \cap e' \neq \emptyset\})$$

. Show that G is connected if and only if $L(G)$ is connected.

Suppose G is connected. Then for any two vertices $u, v \in G$, there is a path $P = u \to x_1 \to x_2 \to \cdots \to x_n \to v$ where the x_i's are some other vertices of G. By the existence of such a path P, we can say that u is adjacent to x_1, x_i is adjacent to x_{i+1} (for $1 \leq i < n$) and x_n is adjacent to v By the definition of the line graph $L(G)$, this means $(u, x_1), (x_i, x_{i+1})$ and (x_n, v) are vertices of $L(G)$ for $1 \leq i < n$. Now, since (u, x_1) and (x_1, x_2) share a common endpoint x_1, they must be adjacent in $L(G)$. Similarly, (x_i, x_{i+1}) is adjacent to (x_{i+1}, x_{i+2}) for $1 \leq i < n-1$ and (x_{n-1}, x_n) is adjacent to $L(G)$. So, we have a path L_P in $L(G)$ as $L_P = (u, x_1) \to (x_1, x_2) \to \cdots \to (x_{n-1}, x_n) \to (x_n, v)$ Hence, for every path P in G, there is a path L_P in $L(G)$. Suppose (a, b) and (A, B) are two vertices of $L(G)$. We show that there is a path between them: By definition of $L(G)$, all four of a, b, A, B are vertices of G with a adjacent to b, ie, $a \to b$ and A adjacent to B, ie, $A \to B$; since G is connected, there must be a path P_{bA} from b to A which translates to a path $L_{P_{bA}}$ in $L(G)$ which gives us a path $a \to b \to P_{bA} \to A \to B$ in G and a path $(a, b) \to L_{P_{bA}} \to (A, B)$ in $L(G)$. The converse is false! (as many of your figured out during the exam :-)). Consider the graph $G = (\{1, 2, 3, 4\}, \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$. Here 4 is an isolated vertex, yet $L(G)$ will be a connected graph on 3 vertices.

3. **(A)** Let $G = (V, E)$ be a simple, undirected graph with $2n$ vertices and $2n$ edges, for $n \geq 3$. The graph consists of two disjoint cycles with n edges each. A pair of vertices u and v from G is selected uniformly at random from pairs of distinct vertices with no edge between them. Let $G' = (V, E \cup \{(u, v)\})$. What is the probability that G' is connected? What if k pairs of vertices from G are selected uniformly at random from the pairs of distinct vertices with no edge between them (Repetitions allowed, i.e., it is possible, for example, that the same pair appears multiple times in the set of k pairs). Let G'' be the same as G, except that there are k new edges: the edges that correspond to the k selected pairs. What is the probability that G'' is not connected?

G' is connected if and only if u and v come from different cycles. There are n^2 pairs of vertices consisting of vertices in different cycles. In all, there are $\binom{2n}{2} - 2n$ pairs of vertices with no edge between them, since there are $\binom{2n}{2}$ pairs of vertices and $2n$ of these pairs have an edge between them. The desired probability p can be computed as follows:

$$p = \frac{n^2}{\binom{2n}{2}} = \frac{n^2}{2n^2 - n - 2n} = \frac{n}{2n-3}$$

Note that the probability of not connecting the graph in one sampling of two nonadjacent vertices is $p = 1 - \frac{n}{2n-3}$. Now since we are able to choose the same pair many times, we are simply taking independent samples of two nonadjacent vertices. Furthermore, in k samples, the graph is not connected if and only if none of the pairs chosen have connected the graph. The probability of this happening is $p^k = (1 - \frac{n}{2n-3})^k$

4. **(A)** Prove that every tournament has a Hamilton path (see Page 1 for definitions). See here.

5. **(A)** Define the Double Fibonacci numbers $D_0, D_1 \ldots$ are defined recursively by the rules $D_n = 2D_{n-1} + D_{n-2}$ for $n \geq 2$ and initial conditions $D_0 = D_1 = 1$. Find the generating function of the sequences D_n (expressed as a ratio of two polynomials).

   Let $D(x) = \sum_{n \geq 0} D_n x^n$ be the generating function for the sequence. Consider $2xD(x)$ and $x^2 D(x)$: Notice that they satisfy $D(x)(1 - 2x - x^2) = 1 - x$. Therefore $D(x) = \frac{1-x}{1-2x-x^2}$.