Department of Mathematics
MTL 106 (Introduction to Probability and Stochastic Processes)
Tutorial Sheet No. 1

1. Let $\Omega = \{a, b, c, d\}$. Find three different σ-fields $\{F_n\}$ for $n = 0, 1, 2$ such that $F_0 \subset F_1 \subset F_2$.

2. Let $\Omega = \{0, 1, 2, \ldots\}$. Let $\mathcal{F}$ be the collection of subsets of Ω that are either finite or whose complement is finite. Is $\mathcal{F}$ a σ-field? Justify your answer.

3. Let $\Omega = \{s_1, s_2, s_3, s_4\}$ and $P\{s_1\} = \frac{1}{6}$, $P\{s_2\} = \frac{1}{5}$, $P\{s_3\} = \frac{1}{3}$, $P\{s_4\} = \frac{3}{10}$.
   Define,
   $$A_n = \begin{cases} \{s_1, s_3\} & \text{if } n \text{ is odd} \\ \\ \{s_2, s_4\} & \text{if } n \text{ is even} \end{cases}$$
   Find $P(\liminf A_n), P(\limsup A_n)$.

4. Consider $\Omega = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$. Let $\mathcal{F}$ be the largest σ-field over Ω. Define $P(R) =$ area of $R = (b-a)(d-c)$ where R is the rectangular region that is a subset of Ω of the form $R = \{(u, v) : a \le u < b, c \le v < d\}$. Let T be the triangular region $T = \{(x, y) : x \ge 0, y \ge 0, x + y < 1\}$. Show that T is an event, and find $P(T)$, using the axiomatic definition of probability.

5. State **True** or **False** with valid reasons for the following statements.
   (a) The probability that exactly one of the events A or B occurs is equal to $P(A) + P(B) - 2P(A \cap B)$.
   (b) Let A and B two events with $P(A) = \frac{1}{2}$ and $P(B^c) = \frac{1}{4}$. Then, A and B can be mutually exclusive events.
   (c) If A and B are two independent events, then A^c and B^c are independent events.
   (d) Let $\Omega = \{a, b, c\}$. If $F_1 = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$ and $F_2 = \{\emptyset, \{a, b\}, \{c\}, \Omega\}$ are two σ-fields on Ω, then $F_1 \cup F_2$ and $F_1 \cap F_2$ are σ-fields on Ω.
   (e) A box contains a double-headed coin, a double-tailed coin and an unbiased coin. A coin is picked at random and flipped. It shows a head. The conditional probability that it is the double-headed coin is 0.5 .
   (f) Consider a parallel system with identical components each with reliability 0.8. If the reliability of the system is to be at least 0.99, then the minimum number of components in this system is 3.

6. Let w be a complex cube root of unity with $w \neq 1$. A fair die is thrown three times. If x, y and z are the numbers obtained on the die. Find the probability that $w^x + w^y + w^z = 0$.

7. An urn contains balls numbered from 1 to N. A ball is randomly drawn.
   (a) What is the probability that the number on the ball is divisible by 3 or 4?
   (b) What happens to the probability in the previous question when $N \to \infty$?

8. Consider the flights starting from Delhi to Bombay. In these flights, 90% leave on time and arrive on time, 6% leave on time and arrive late, 1% leave late and arrive on time and 3% leave late and arrive late. What is the probability that, given a flight late, it will arrive on time?

9. Let A and B are two independent events. Prove or disprove that A and B^c, A^c and B^c are independent events.

10. Pick a number x at random out of the integers 1 through 30. Let A be the event that x is even, B that x is divisible by 3 and C that x is divisible by 5. Are the events A, B and C pairwise independent? Further, are the events A, B and C mutually independent?

11. The first generation of particles is the collection of off-springs of a given particle. The next generation is formed by the off-springs of these members. If the probability that a particle has k off springs (splits into k parts) is p_k, where $p_0 = 0.4$, $p_1 = 0.3$, $p_2 = 0.3$, find the probability that there is no particle in second generation. Assume particles act independently and identically irrespective of the generation.

12. A and B throw a pair of unbiased dice alternatively with A starting the game. The game ends when either A or B wins. A wins if he throws 6 before B throws 7. B wins if he throws 7 before A throws 6. What is the probability that A wins the game? Note that "A throws 6" means the sum of values of the two dice is 6. Similarly "B throws 7".

13. In a meeting at the UNO 40 members from under-developed countries and 4 from developed ones sit in a row. What is the probability no two adjacent members are representatives of developed countries.

14. A random walker starts at 0 on the x-axis and at each time unit moves 1 step to the right or 1 step to the left with probability 0.5. Find the probability that, after 4 moves, the walker is more than 2 steps from the starting position.

15. The coefficients a, b and c of the quadratic equation $ax^2 + bx + c = 0$ are determined by rolling a fair die three times in a row. What is the probability that both the roots of the equation are real? What is the probability that both roots of the equation are complex?

16. An electronic assembly consists of two subsystems, say A and B. From previous testing procedures, the following probabilities assumed to be known: $P(A \text{ fails}) = 0.20$, $P(A \text{ and } B \text{ both fail}) = 0.15$, $P(B \text{ fails alone}) = 0.15$. Evaluate the following probabilities (a) $P(A \text{ fails}/B \text{ has failed})$ (b) $P(A \text{ fails alone }/A \text{ or } B \text{ fail})$.

17. An aircraft has four engines in which two engines in each wing. The aircraft can land using atleast two engines. Assume that the reliability of each engine is $R = 0.93$ to complete a mission, and that engine failures are independent.
a) Obtain the mission reliability of the aircraft.
b) If at least one functioning engine must be on each wing, what is the mission reliability?

18. Along a line segment ab two points l and m are randomly marked. Find the probability that l is closer to a than m, $(al < am)$.

19. Four lamps are located in circular. Each lamp can fail with probability q, independently of all the others. The system is operational if no two adjacent lamps fail. Obtain an expression for system reliability?.

20. An urn contains b black balls and r red balls. One of the ball is drawn at random, but when it is put back in the urn c additional balls of the same colour are put in with it. Now suppose that we draw another ball. Find the probability that the first ball drawn was black given that the second ball drawn was red?

21. The base and altitude of a right triangle are obtained by picking points randomly from $[0, a]$ and $[0, b]$ respectively. Find the probability that the area of the triangle so formed will be less than $ab/4$?

22. Find the probability that the sum of two randomly chosen positive real numbers (both ≤ 1) will not exceed 1 and that their product will be $\leq 2/9$.

23. A batch of N transistors is dispatched from a factory. To control the quality of the batch the following checking procedure is used; a transistor is chosen at random from the batch, tested and placed on one side. This procedure is repeated until either a pre-set number $n (n < N)$ of transistors have passed the test (in which case the batch is accepted) or one transistor fails (in this case the batch is rejected). Suppose that the batch actually contains exactly D faulty transistors. Find the probability that the batch will be accepted.