Department of Mathematics
MAL 250/MTL 106 (Probability and Stochastic Processes)
Tutorial Sheet No. 1

1. Items coming off a production line are marked defective (D) or non-defective (N). Items are observed and their condition noted. This is continued until two consecutive defectives are produced or four items have been checked, which ever occurs first. Describe the sample space for this experiment.

2. Let $\Omega = \{a, b, c, d\}$. Find three different σ-fields $\{F_n\}$ for $n = 0, 1, 2$ such that $F_0 \subset F_1 \subset F_2$.

3. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. Let $\{A_n\}$ be a nondecreasing sequence of elements in $\mathcal{F}$. Prove that
$$P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

4. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. For any n events in the probability space, show that
$$P(A_1 \cap A_2 \cap \ldots \cap A_n) \geq \sum_{i=1}^{n} P(A_i) - (n-1).$$

5. State **True** or **False** with valid reasons for the following statements.
   (a) The probability that exactly one of the events A or B occurs is equal to $P(A) + P(B) - 2P(A \cap B)$. T
   (b) Let A and B two events with $P(A) = \frac{1}{2}$ and $P(B^c) = \frac{1}{4}$. Then, A and B can be mutually exclusive events. F
   (c) If A and B are two independent events, then A^c and B^c are independent events. T
   (d) Let $\Omega = \{a, b, c\}$. If $F_1 = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$ and $F_2 = \{\emptyset, \{a, b\}, \{c\}, \Omega\}$ are two σ-fields on Ω, then $F_1 \cup F_2$ and $F_1 \cap F_2$ are σ-fields on Ω. F
   (e) Consider a parallel system with identical components each with reliability 0.8. If the reliability of the system is to be at least 0.99, then the minimum number of components in this system is 3. T

6. Let w be a complex cube root of unity with $w \neq 1$. A fair die is thrown three times. If x, y and z are the numbers obtained on the die. Find the probability that $w^x + w^y + w^z = 0$.

7. An urn contains balls numbered from 1 to N. A ball is randomly drawn.
   (a) What is the probability that the number on the ball is divisible by 3 or 4?
   (b) What happens to the probability in the previous question when $N \to \infty$?

8. Consider the flights starting from Delhi to Bombay. In these flights, 90% leave on time and arrive on time, 6% leave on time and arrive late, 1% leave late and arrive on time and 3% leave late and arrive late. What is the probability that, given a flight late, it will arrive on time?

9. Let A and B are two independent events. Prove or disprove that A and B^c, A^c and B^c are independent events.

10. Pick a number x at random out of the integers 1 through 30. Let A be the event that x is even, B that x is divisible by 3 and C that x is divisible by 5. Are the events A, B and C independent?

11. The first generation of particles is the collection of off-springs of a given particle. The next generation is formed by the off-springs of these members. If the probability that a particle has k off springs (splits into k parts) is p_k, where $p_0 = 0.4$, $p_1 = 0.3$, $p_2 = 0.3$, find the probability that there is no particle in second generation. Assume particles act independently and identically irrespective of the generation.

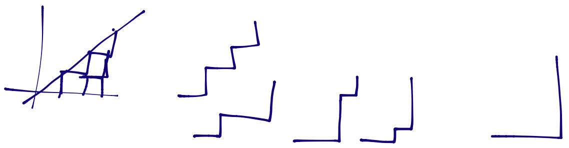
12. A and B throw a pair of unbiased dice alternatively with A starting the game. The game ends when either A or B wins. A wins if he throws 6 before B throws 7. B wins if he throws 7 before A throws 6. What is the probability that A wins the game? Note that "A throws 6" means the sum of values of the two dice is 6. Similarly "B throws 7".

11) $0.4 + 0.3\times0.4 + 0.3\times0.4\times0.4$

12) $\frac{5}{36} + \frac{31}{36}\times\frac{30}{36}\times\frac{5}{36} \cdots\cdots$

$\frac{5}{\cancel{36}} \Big/ \left(36 - \frac{31\times5}{\cancel{36}\times6}\right)$ $\frac{5\times6}{216-155} = \frac{30}{61}$

$\frac{5}{5\times4} = \frac{1}{4}$



RRRLLL RLRRLL
RRLRLL
RRLLRL
RLRLRL

13. In a meeting at the UNO 40 members from under-developed countries and 4 from developed ones sit in a row. What is the probability no two adjacent members are representatives of developed countries.

14. A particle starts at the origin and moves to and fro on a straight line. At any move it jumps either 1 unit to the right or 1 unit to the left each with probability $\frac{1}{2}$. All successive moves are independent. Given that the particle is at the origin at the completion of 6th move, find the probability that it never occupied a position to the left of origin during previous moves?

15. The coefficients a, b and c of the quadratic equation $ax^2 + bx + c = 0$ are determined by rolling a fair die three times in a row. What is the probability that both the roots of the equation are real? What is the probability that both roots of the equation are complex?

16. An electronic assembly consists of two subsystems, say A and B. From previous testing procedures, the following probabilities assumed to be known: $P(A \text{ fails}) = 0.20$, $P(A \text{ and } B \text{ both fail}) = 0.15$, $P(B \text{ fails alone}) = 0.15$. Evaluate the following probabilities (a) $P(A \text{ fails}/B \text{ has failed})$ (b) $P(A \text{ fails alone } /A \text{ or } B \text{ fail})$.

1/2 1/7

17. An aircraft has four engines in which two engines in each wing. The aircraft can land using atleast two engines. Assume that the reliability of each engine is $R = 0.93$ to complete a mission, and that engine failures are independent.
a) Obtain the mission reliability of the aircraft.
b) If at least one functioning engine must be on each wing, what is the mission reliability?

18. Suppose that in answering a question on a multiple choice test an examinee either knows the answer or he guesses. Let p be the probability that he will know the answer, and let $1 - p$ be the probability that he will guess. Assume that the probability of answering a question correctly is unity for an examinee who knows the answer and $\frac{1}{m}$ for an examinee who guesses where m is the number of multiple choice alternatives. Find the conditional probability that an examinee knew answer to a question, given that he has correctly answered it.

19. Along a line segment ab two points l and m are randomly marked. Find the probability that l is closer to a than m, $(al < am)$.

20. Four lamps are located in circular. Each lamp can fail with probability q, independently of all the others. The system is operational if no two adjacent lamps fail. Obtain an expression for system reliability?.

21. An urn contains b black balls and r red balls. One of the ball is drawn at random, but when it is put back in the urn c additional balls of the same colour are put in with it. Now suppose that we draw another ball. Find the probability that the first ball drawn was black given that the second ball drawn was red?

22. The base and altitude of a right triangle are obtained by picking points randomly from $[0, a]$ and $[0, b]$ respectively. Find the probability that the area of the triangle so formed will be less than $ab/4$?

23. Find the probability that the sum of two randomly chosen positive real numbers (both ≤ 1) will not exceed 1 and that their product will be $\leq 2/9$.

24. A batch of N transistors is dispatched from a factory. To control the quality of the batch the following checking procedure is used; a transistor is chosen at random from the batch, tested and placed on one side. This procedure is repeated until either a pre-set number $n(n < N)$ of transistors have passed the test (in which case the batch is accepted) or one transistor fails (in this case the batch is rejected). Suppose that the batch actually contains exactly D faulty transistors. Find the probability that the batch will be accepted.

Department of Mathematics
MAL 250/MTL 106 (Probability and Stochastic Processes)
Tutorial Sheet No. 1
Answer for selected Problems

1. $\mid \Omega \mid = 12$
   $\Omega = DD, NDD, NDND, NNDD, NNDN, NNNN, NNND, NDNN,$
   $DNNN, DNDN, DNND, DNDD$

5. a)T b)F c)T d)F e)T

6. $\dfrac{6 \times 4 \times 2}{6 \times 6 \times 6} = \dfrac{2}{9}$

7. a)$\dfrac{[\frac{N}{3}] + [\frac{N}{4}] - [\frac{N}{12}]}{N}$ where [ ] =greatest integer function b)$\dfrac{1}{2}$

8. $\frac{1}{4}$

10. A, B, C are independent

11. $p_0 + p_1 \;\; p_0 + p_2 \;\; (p_0)^2$

12. $\frac{30}{61}$

13. $\frac{40! \times \; ^{41}P_4}{44!}$

14. $\frac{3}{8}$

15. $\frac{43}{216}, \frac{173}{216}$

16. (a)$\frac{1}{2}$ (b)$\frac{1}{7}$

17. $a) R^4 + \; ^4C_3 R^3 (1-R) + \; ^4C_2 R^2 (1-R)^2$
    $b) R^4 + \; ^2C_1 R(1-R) \times \; ^2C_1 R(1-R)$

18. $\dfrac{p}{\dfrac{(1-p)}{m} + p}$

19. $\frac{1}{2}(1 + \ln 2)$

20. $p^4 + 4p^3 q + 3p^2 q^2$

22. $\dfrac{\frac{(b-a)^2}{2}}{(b-a)^2} = \dfrac{1}{2}$

23. $\frac{2}{9} \ln 2 + \frac{1}{3}$

24. $\frac{^{(N-D)}C_n}{^{N}C_n}$

1) DD _ _ DNNN DNDN
NDD _ NDNN DNND
NNDD NNDN NDND
DNDD NNND
NNNN

2)

3)

4) $P(A_1 \cap A_2 \ldots A_n) \geq \sum P(A_i) - (n-1)$

$P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$

$P(A_1 \ldots A_{n-1}) \geq \sum_{i=1}^{n-1} P(A_i) - n + 2$

$P(A_1 \ldots A_{n-1} \cap A_n) = P(A_1 \ldots A_{n-1}) + P(A_n) - P(A_1 \cup \ldots \cup A_n)$

$\geq \sum_{i=1}^{n} P(A_i) - n + 2 - P(A_1 \cup \ldots A_n)$

$\geq \sum_{i=1}^{n} P(A_i) - (n-1)$

6) x, y, z

1,4 2,5 3,6

$\frac{3 \times 2 \times 2 \times 2 \times 2}{6 \times 6 \times 6} = \frac{2}{9}$

7) $\frac{\lfloor \frac{N}{3} \rfloor + \lfloor \frac{N}{4} \rfloor - \lfloor \frac{N}{12} \rfloor}{N} = \frac{1}{2}$

10) Independent.

13) $\frac{40! \times {}^{41}C_4 \times 4!}{44!}$

14) $\frac{5}{{}^6C_3}$

15) $b^2 \geq 4ac$

1,2,1 2,4,2 2,5,3 × 2 3,6,3
2,3,1 1,4,3 ×2 1,5,6 ×2 1,6,6 ×2
1,3,2 1,4,4 ×2 2,5,2 1,6,5 ×2
1,3,1 1,4,1 1,5,2 ×2 1,6,4 ×2
1,4,2 ×2 1,5,3 ×2 1,6,3 ×2
1,5,4 ×2 1,6,2 ×2
1,5,5 ×2 1,6,1
1,5,1 2,6,4 ×2 3,6,3
2,6,3 ×2
2,6,2

$\frac{30 + 13}{6^3}$

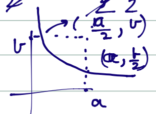
17) $0.93^4 + 4\times 0.93^3 \times 0.07 + 6\times 0.93^2 \times 0.07^2$

18) $$\frac{p}{p+(1-p)\frac{1}{m}}$$

20) $m^4 + 4m^3(1-m) + 2m^2(1-m)$

21) $$\frac{\frac{b}{b+r}}{\frac{b}{b+r}\times\frac{r}{b+r+c} + \frac{r}{b+r}\times\frac{r+c}{b+r+c}}$$

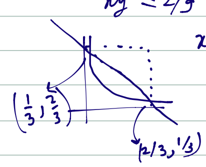
22) $\frac{1}{\cancel{2}} xy < \frac{ab}{\cancel{2}\,2}$



$$\frac{a}{2}\times b + \int_{a/2}^{a} \frac{ab}{2x}\,dx$$

$$= \frac{\frac{ab}{2} + \frac{ab}{2}\ln 2}{ab} = \frac{1+\ln 2}{2}$$

23) $x, y \le 1$
$x+y \le 1$
$xy \le 2/9$



$x - x^2 = \frac{2}{9}$

$x^2 - x + \frac{2}{9} = 0 \quad (x-\frac{2}{3})(x-\frac{1}{3}) = 0$

$$\int_{1/3}^{2/3} \frac{2}{9x}\,dx \qquad \frac{\frac{2}{9}\ln 2 + \frac{1}{2}\times\frac{1}{3}^2 + \frac{1}{2}\times\frac{5}{3}\times\frac{1}{3}}{}$$

$$= \frac{2}{9}\ln 2 + \frac{1}{18} + \frac{5}{18} = \frac{1}{3} + \frac{2}{9}\ln 2$$

3) $\lim_{n\to\infty} A_n = \bigcup_{i=1}^{\infty} A_i$

$$P(\lim_{n\to\infty} A_n) = P\left(\bigcup^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right)$$

$B_1 = A_1$
$B_2 = A_2 - A_1$
$B_3 = A_3 - A_2$

$$= \sum_{i=1}^{\infty} P(B_i)$$

$\Rightarrow$

$$\lim_{n\to\infty} \sum_{i=1}^{n} P(B_i) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} P(A_n)$$