1. **[1+1 = 2 marks]** Prove the following sequents using natural deduction, without using the LEM rule directly or indirectly (i.e., even after deriving it).

(a) $x_1 \to x_2 \vee x_3 \vee x_4,\ x_2 \to \neg x_1 \vee \neg x_4,\ x_3 \vee x_4 \to x_2 \vdash x_1 \to \neg x_4$

Ans:

1. $x_1 \to x_2 \vee x_3 \vee x_4$	premise
2. $x_2 \to \neg x_1 \vee \neg x_4$	premise
3. $x_3 \vee x_4 \to x_2$	premise
4. x_1	assume
5. $x_2 \vee x_3 \vee x_4$	MP 4, 1
6. x_2	assume
7. x_2	copy 6
8. $x_3 \vee x_4$	assume
9. x_2	MP 8, 3
10. x_2	$\vee_e$ 5, 6-7, 8-9
11. $\neg x_1 \vee \neg x_4$	MP 10,2
12. $\neg x_1$	assume
13. $\bot$	$\neg_e$ 4, 12
14. $\neg x_4$	$\bot_e$
15. $\neg x_4$	assume
16. $\neg x_4$	copy 15
17. $\neg x_4$	$\vee_e$ 11, 12-14, 15-16
18. $x_1 \to \neg x_4$	$\to_i$ 4-17

(b) $x_1 \to x_2 \vee x_3, x_2 \to \neg x_1 \vee \neg x_4, x_3 \to \neg x_1 \vee \neg x_4, x_4 \to x_1 \wedge x_5, x_5 \to x_1 \wedge x_4, x_1 \to x_4 \vee x_5 \vdash \neg x_1$

Ans:

1. $x_1 \to x_2 \vee x_3$	premise
2. $x_2 \to \neg x_1 \vee \neg x_4$	premise
3. $x_3 \to \neg x_1 \vee \neg x_4$	premise
4. $x_4 \to x_1 \wedge x_5$	premise
5. $x_5 \to x_1 \wedge x_4$	premise
6. $x_1 \to x_4 \vee x_5$	premise
7. $\neg\neg x_1$	assume
8. x_1	$\neg\neg_e$ 6
9. $x_2 \vee x_3$	MP 8, 1
10. x_2	assume
11. $\neg x_1 \vee \neg x_4$	MP 10, 2
12. x_3	assume
13. $\neg x_1 \vee \neg x_4$	MP 12, 3
14. $\neg x_1 \vee \neg x_4$	$\vee_e$ 9, 10-11, 12-13
15. $\neg x_1$	assume
16. $\bot$	$\neg_e$ 8, 15
17. $\neg x_4$	$\bot_e$
18. $\neg x_4$	assume
19. $\neg x_4$	copy 18
20. $\neg x_4$	$\vee_e$ 14, 15-17, 18-19
21. $x_4 \vee x_5$	MP 8, 6

22. x_4 assume
23. x_4 copy 22
24. x_5 assume
25. $x_1 \wedge x_4$ MP 24, 5
26. x_4 $\wedge_{e,2}$ 25
27. $\perp$ $\neg_e$ 26, 20
28. $\neg x_1$ PBC

2. **[2 marks]** Prove, in Hilbert's proof system, that $(\alpha \rightarrow \neg\neg\alpha)$.

   Ans. Recall the following facts that we had derived in the class (see slide no. 10/20, slides-week-4.pdf):

   (a) $\neg\neg\alpha \rightarrow \alpha$

   (b) $(\neg\beta \rightarrow \neg\alpha) \rightarrow (\alpha \rightarrow \beta)$

   From (a), substituting $\neg\alpha$ in place of α, we get $\neg\neg\neg\alpha \rightarrow \neg\alpha$. From (b), $\neg\neg\alpha$ in place of β, we get $(\neg\neg\neg\alpha \rightarrow \neg\alpha) \rightarrow (\alpha \rightarrow \neg\neg\alpha)$.

   Applying *modus ponens* once on these two formulae gives us the desired result.

3. **[2+1 = 3 marks]** Let p and q be atomic propositions, and ϕ_1 and ϕ_2 be propositional logic formulae on p and q.

   (a) Consider the following definitions for ϕ_1 and ϕ_2:

   - $\phi_1 = (p \rightarrow \neg\phi_2)$
   - $\phi_2 = (q \rightarrow \neg\phi_1)$

   Show that there are exactly two pairs of propositional logic formulae (ϕ_1, ϕ_2) which satisfy the above definitions. Justify your answer.

   Ans: From the given definitions, we know that ϕ_1 and ϕ_2 must evaluate to *true* when, respectively, p and q take the value *false*. Further, when p is *true* and q is false, because ϕ_2 evaluates to *true*, $\neg\phi_2$ must be *false*. Thus, ϕ_1 must evaluate to *false*. Similarly, ϕ_2 must evaluate to *false* when p is *false* and q is *true*.

   When both p and q are *true*, the only constraint that we have is that ϕ_1 and ϕ_2 are negations of one another. This gives us the two pairs of formulas satisfying the above definitions.

   (b) If the definitions of ϕ_1 above is changed to $\phi_1 = (p \rightarrow \phi_2)$, and the definition of ϕ_2 is left unchanged, is it possible to find propositional logic formulae on propositions p and q that satisfy the modified definitions? If yes, give the formulae ϕ_1 and ϕ_2. If not, explain why the modified definitions cannot be satisfied.

   Ans: With the modified definitions, when both p and q are *true*, the first definition forces ϕ_1 and ϕ_2 to take the same value, while the second definition forces them to take complementary values. This cannot be possible.

4. **[3 marks]** Show that for any CNF formula ϕ one can compute in polynomial time an equisatisfiable formula $\psi_1 \wedge \psi_2$, with ψ_1 a Horn formula and ψ_2 a 2-CNF formula.

   Ans. We know that we can convert any CNF formula into an equisatisfiable 3-CNF formula in polynomial time (see slide nos. 13/33 to 18/33, slides-week-5.pdf). So, let us assume that our formula only has clauses with three literals.

If all the clauses are Horn (i.e., they have at most one positive literal) then the result follows trivially. In case the formula has a clause with two or three positive literals, here is what we may do: $(\neg p \vee q \vee r)$ can be written as $(\neg p \vee q \vee \neg s) \wedge (s \vee r)$, and $(p \vee q \vee r)$ can be written as $(p \vee t) \wedge (\neg t \vee q \vee r)$ (and the latter can again be rewritten as shown just above). Thus, by introducing clauses with two literals, we can always ensure that the clauses of size three are always Horn.

Clearly, this can be done in polynomial time and leads to an equisatisfiable formula.

5. **[1+2.5+2.5 = 6 marks]** Let us consider formulae in propositional logic with $\rightarrow$ as the only propositional connective, and $\perp$ as the only propositional constant. For example, $(x \rightarrow (y \rightarrow \perp)) \rightarrow (\perp \rightarrow z)$ is a propositional logic formula that can be constructed with atoms x, y, z, using the allowed connective and constant.

   (a) Let ϕ_1 and ϕ_2 be propositional logic formulae using $\rightarrow$ as the only connective and $\perp$ as the only constant. Give *semantically equivalent* formula for $\phi_1 \wedge \phi_2$ and $\neg\phi_1$, such that $\rightarrow$ is the only connective and $\perp$ is the only constant in the resulting formulae. Justify your answer.

   Ans: $\neg\phi_1$ is semantically equivalent to $(\phi_1 \rightarrow \perp)$. This formula evaluates to true iff ϕ_1 evaluates to false, iff $\neg\phi_1$ evaluates to true.

   $\phi_1 \wedge \phi_2$ is semantically equivalent to $\neg(\neg\phi_1 \vee \neg\phi_2)$ which is semantically equivalent to $\neg(\phi_1 \rightarrow \neg\phi_2)$. Re-writing $\neg\phi$ as $(\phi \rightarrow \perp)$, we get $((\phi_1 \rightarrow (\phi_2 \rightarrow \perp)) \rightarrow \perp)$.

   (b) Your solution to the previous subquestion should convince you that any propositional logic formula can be converted to a semantically equivalent one using only $\rightarrow$ and $\perp$. A student now claims that it is possible to prove sequents in this version of propositional logic (with $\rightarrow$ as the only connective and $\perp$ as the only constant) using rules $\rightarrow_i, \rightarrow_e, \perp_e$ of the natural deduction proof system that we studied, in addition to the following special rule, called $(\rightarrow \perp)_e$ rule:

$$\frac{(\phi \rightarrow \perp) \rightarrow \psi \qquad (\phi \rightarrow \chi) \qquad (\psi \rightarrow \chi)}{\chi} \; (\rightarrow \perp)_e$$

   Using only the above four proof rules, prove the following sequent:
   $(\phi \rightarrow \perp) \rightarrow \psi, (\phi \rightarrow \chi) \vdash (\psi \rightarrow \perp) \rightarrow \chi$

   Ans: Here is the proof:

1. $(\phi \rightarrow \perp) \rightarrow \psi$	premise
2. $\phi \rightarrow \chi$	premise
3. $\psi \rightarrow \perp$	assume
4. $\phi \rightarrow \perp$	assume
5. ψ	$\rightarrow_e$ 4, 1
6. $\perp$	$\rightarrow_e$ 5, 3
7. $(\phi \rightarrow \perp) \rightarrow \perp$	$\rightarrow_i$ 4–6
8. $\perp$	assume
9. χ	$\perp_e$
10. $(\perp \rightarrow \chi)$	$\rightarrow_i$ 8–9
11. χ	$(\rightarrow \perp)_e$ 7, 2, 10
12. $(\psi \rightarrow \perp) \rightarrow \chi$	$\rightarrow_i$ 3–11

   (c) Are the above four rules, i.e. $\rightarrow_i, \rightarrow_e, \perp_e$, and $(\rightarrow \perp)_e$, complete for the version of propositional logic that uses $\rightarrow$ as the only connective and $\perp$ as the only constant? In other words, given formulas ϕ and ψ, each involving only $\rightarrow$ and $\perp$ apart from propositional atoms, such that $\phi \vDash \psi$, is it always possible to prove the sequent $\phi \vdash \psi$ using only the above four rules? Justify your answer. Assume that you are free to use the *copy* rule even if it is not explicitly given. Recall that the *copy* rule is not really useful for transforming or constructing any formula; it simply allows you to use the premises and the *visible* formulae more than once.

Ans: We argue that these rules are complete, by showing that all the derivations made by the basic proof rules of natural deduction can be made here. We have the rules $\to_i, \to_e, \perp_e$, and $(\to \perp)_e$ in our system already. The $(\to \perp)_e$ rule in nothing but the $\vee_e$ rule of natural deduction. Consider the $\neg_e$ (or, $\perp_i$) rule. It allows us to derive $\perp$ from ϕ and $\neg\phi$. In our case, we should have a way to derive $\perp$ from ϕ and $(\phi \to \perp)$ (which is the same as $\neg\phi$). This can be done with $\to_e$.

For $\neg\neg_e$, we should be able to derive ϕ from $\neg\neg\phi$, i.e. $(\phi \to \perp) \to \perp$. But that can be done with $(\to \perp)_e$ if we have $\phi \to \phi$ and $\perp \to \phi$. Both of these can be obtained trivially from the $\to_e$ and $\perp_e$ rules.

We can also derive what the or-introduction rules, $\vee_{i,1}$ and $\vee_{i,2}$, let us derive. Here is how we can get $\phi_1 \vee \phi_2$ (or, in our case, $(\phi_1 \to \perp) \to \phi_2$) from ϕ_1:

1. ϕ_1	premise
2. $\phi_1 \to \perp$	assume
3. $\perp$	$\to_e$ 1, 2
4. ϕ_2	$\perp_e$
5. $(\phi_1 \to \perp) \to \phi_2$	$\to_i$ 2–4

Deriving $(\phi_1 \to \perp) \to \phi_2)$ from ϕ_2 is trivial (can be done with *copy* and $\to_i$).

Finally, the $\wedge$ rules. We need to show that ϕ_1 and ϕ_2 let us derive $(\phi_1 \to (\phi_2 \to \perp)) \to \perp$ ($\wedge_i$).

1. ϕ_1	premise
2. ϕ_2	premise
3. $(\phi_1 \to (\phi_2 \to \perp))$	assume
4. $(\phi_2 \to \perp)$	$\to_e$ 1, 3
5. $\perp$	$\to_e$ 2, 4
6. $(\phi_1 \to (\phi_2 \to \perp)) \to \perp$	$\to_i$ 3–5

And that $(\phi_1 \to (\phi_2 \to \perp)) \to \perp$ lets us derive both ϕ_1 ($\wedge_{e,1}$) and ϕ_2 ($\wedge_{e,2}$).

1. $(\phi_1 \to (\phi_2 \to \perp)) \to \perp$	premise
2. $\phi_1 \to \perp$	assume
3. ϕ_1	assume
4. $\perp$	$\to_e$ 3, 2
5. $\phi_2 \to \perp$	$\perp_e$
6. $\phi_1 \to (\phi_2 \to \perp)$	$\to_i$ 3–5
7. $\perp$	$\to_e$ 6, 1
8. $((\phi_1 \to \perp) \to \perp)$	$\to_i$ 2–7
9. ...	apply $\neg\neg_e$ as explained above

1. $(\phi_1 \to (\phi_2 \to \perp)) \to \perp$	premise
2. $\phi_2 \to \perp$	assume
3. ϕ_1	assume
4. $\phi_2 \to \perp$	copy 2
5. $(\phi_1 \to (\phi_2 \to \perp))$	$\to_i$ 3–4
6. $\perp$	$\to_e$ 5, 1
7. $((\phi_2 \to \perp) \to \perp)$	$\to_i$ 2–6
8. ...	apply $\neg\neg_e$ as explained above

6. **[2+2 = 4 marks]** Recall that α is said to be *consistent* if $\nvdash \neg\alpha$. Suppose that $\vdash \alpha \to \beta$. For the following statements, answer whether they are true or not, and provide an explanation. Your explanation should not rely on soundness and completeness of propositional logic. Answers with missing or inadequate explanations will not get any marks.

   (a) If α is consistent then β is consistent.

   Ans: True. We need to prove that if $\nvdash \neg\alpha$ then $\nvdash \neg\beta$. We will prove the contrapositive: if $\neg\beta$ is derivable, then $\neg\alpha$ is also derivable.

   From $\neg\beta$, we can get $\neg\neg\alpha \to \neg\beta$ (from A1 and modus ponens). Further, we know (proved in class) that: $\neg\neg\alpha \to \alpha$. From this, and $\alpha \to \beta$ (given), we know: $\neg\neg\alpha \to \beta$.

   In A3, substitute $\neg\alpha$ for β and β for α, and then use *modus ponens* to derive $\neg\alpha$.

   (b) If β is consistent then α is consistent.

   Ans: False. Consider α to be $\neg(\neg\beta \to \neg\beta)$. Claim: $\vdash \alpha \to \beta$. Reason: we know that $\neg\beta \to (\neg\beta \to \neg\beta)$ from A1, and then we have shown in the class that $(\gamma \to \delta) \to (\neg\delta \to \neg\gamma)$, and also that $\neg\neg\gamma \to \gamma$.

   Now, irrespective of β, we know that α is not consistent, because because $\neg\alpha$ which is $(\neg\beta \to \neg\beta)$ is actually derivable (we have seen a derivation in class).

   Thus, the given statement is false (unless it is *vacuously* true because of an unsatisfiable premise, i.e. when we cannot find any β that is consistent).

   Note: We can argue that we can find a consistent formula β, but that may require soundness of propositional logic.