Group: 1

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MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Time allowed: 1 hour   Minor 2 Examination   Max. Marks: 25

Name: Shivam Jadhav   Entry Number: 2017CS10378   Signature: Shiv@m

Multiple Selection Questions: **Section 1**   (1 × 5 = 5 marks)

Each of the following questions 1 to 5 has four options out of which one or more options can be correct. Write A, B, C or D which corresponds to the correct option for the first correct answer followed by space and the next correct answer and so on. **1 mark** is awarded if all correct answers are written, **0 mark** for no answer or partial correct answers or any incorrect answer.

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1. Let $X_1 \sim Exp(3)$ and $X_2 \sim Exp(5)$ be two independent random variables. The distribution of $X_{(1)} = \min\{X_1, X_2\}$ is
(A) Exp (8) (B) Gamma (8,3/5) (C) Erlang (8, 3/5) (D) Exp (3/5)
Answer: A

2. Let X and Y be continuous type random variables with joint pdf given by

$$f(x,y) = \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}, \quad -\infty < x < \infty, -\infty < y < \infty.$$

Then, the correlation coefficient between X and Y is (A) 0 (B) 1 (C) 0.5 (D) -1 Answer: A

3. Let (X,Y) be a two-dimensional random variables with joint pdf $f(x,y)$. Then, the pdf of $U = X+Y$ is (A) $g(u) = \int_{-\infty}^{\infty} f(u, u-v)dv$ (B) $g(u) = \int_{-\infty}^{\infty} f(u-v, v)dv$ (C) $g(u) = \int_{-\infty}^{\infty} f(u-v, u-v)dv$ (D) $g(u) = \int_{-\infty}^{\infty} f(v, u-v)dv$ Answer: AB

4. Let N be a positive integer random variable and $X_1, X_2, \ldots$ be a sequence of iid random variables. N is independent of X_i's. Let $S_N = \sum_{i=1}^{N} X_i$. Then, $Var(S_N)$ is
(A) $E(N)Var(X) + Var(N)[E(X)]^2$ (B) $E(N^2)Var(X) + Var(N)E(X^2)$
(C) $E(N)Var(X) + Var(N)E(X)$ (D) $[E(N)]^2 Var(X) + Var(N)[E(X)]^2$ Answer: A

5. Let $(X_1, X_2, \ldots, X_n)$ be a n-dimensional random variables. Let $\sum$ be a $n \times n$ matrix where the entry in row i, column j, $\sum_{ij}$ is given by $Cov(X_i, X_j)$. If y is any $1 \times n$ vector, then
(A) $y\sum y^T < 0$ (B) $y\sum y^T > 0$ (C) $y\sum y^T \leq 0$ (D) $y\sum y^T \geq 0$ Answer: D

Rough Work

$Var(X_1+X_2)$ $\boxed{4 Var(X)}$ $\frac{Cov(X,Y)}{\sqrt{Var(X)}}$ $E(X)E(N) = E(S_N)$ $\Sigma_{ij} = Cov(X_i, X_j)$ $3Var(X_1+X_2)$ $W = Y$ $\Sigma X_i^2 + 2\Sigma\Sigma$ $U - W = Y$ $U - W = X$ $E(N)Var(X) + E(X)^2$ $Var(N)$ $\int_{-\infty}^{\infty} u-w, w \, dw$ $\int_{-\infty}^{\infty} f(w, u-w) dw$ $N=1$ $E(S_N)$ $E(N)E(X^2) - E(N)E(X)^2 + E(N^2)E(X)^2$ $- E(X)^2 E(N)^2$ ≥ 0

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Numeric Type Questions: **Section 2** **(5 × 2 = 10 marks)**

Write the answer upto 4 decimal places (D) or in fraction (F) or the expression (E) for the following questions 6 to 10. **2 marks** are awarded if answer is correct, and **0 mark** for no answer or an incorrect answer.

6. Let (X, Y) be a two-dimensional continuous type random variables with joint pdf $f(x, y) = \begin{cases} kx(x-y), & -x < y < x, 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$ Find k. Answer(F/D): 1/8

7. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables such that $X_i \sim N(0, 1)$. Define $S_n = \sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$. Then, as $n \to \infty$, $\frac{S_n}{n}$ converges in probability to Answer(F/D): 0

8. Let (X, Y) be a two-dimensional continuous type random variables. Assume that, $E(X), E(Y)$ and $E(XY)$ are exist. Suppose that, $E(X \mid Y = y)$ does not depend on y. Find $E(XY)$. Answer(E): $E(X)E(Y)$

9. It is known that a IIT bus will arrive at random at Nilgiri hostel bus stop sometime between 8:30 A.M. and 8:45 A.M. Rahul decides that he will go at random to this location between these two times and will wait at most 5 minutes for the bus. If he misses it, he will take the cycle rickshaw. What is the probability that he will take the cycle rickshaw? Answer(F/D): ~~2/3~~ 1/3

10. Let (X, Y) be a two-dimensional continuous type random variables with joint pdf

$$f(x, y) = xe^{-x(y+1)} \text{ for } x, y > 0 \text{ and } f(x, y) = 0 \text{ otherwise.}$$

Define $Z = XY$. Then, for $z \geq 0$, $P(Z \leq z)$ is given by: Answer(E): e^{-z}

Rough Work

$P(X=B) = 1/2$ ~~X arrival~~ ~~Y waiting~~ $P(X>5/Y)$

$E(X|Y=y) = \text{const} \cdot C$ ~~$P(X7.5)$~~ 0–15, 7.5, 8:30 – 8:45

5 min

$Z = XY$

$W = X$

$\int_0^2 \int_{-x}^{x} kx(x-y)\,dy\,dx$ $F_X(x) = \int_0^\infty x e^{-x(y+1)} dy$

$9/3$

$\frac{S_n}{n} - \mu$ $\frac{Z}{W} = Y$ $\left[-e^{-x} e^{-xy}\right]_0^\infty$

e^{-2} e^{-wn} e^{-2} $e^{-x} = f_X(x)$

$\left[kx^2 y - \frac{kxy^2}{2}\right]_{-x}^{x}$

$0 \quad 1$ $E(X) = E(C) = C$ $kx^3 - \frac{kx^3}{2} - \left(-kx^3 - \frac{kx^3}{2}\right)$

$\frac{1}{w} - \frac{2}{w^2}$ $w, \frac{z}{w}$

$\left[\frac{kx^4}{2}\right]_0^2$

$YE(X|Y=y) = cY$ $z>0$ $3kx^3$

$\frac{1}{w} \times w\, e^{-w(\frac{z}{w}+1)}$ $z>0$ $w>0$ $8k = 1$

$E(XY) = cE(Y)$ indepe-

e^{-z-w} $w > 0$. $\int_0^\infty e^{-z-w}\,dw$

11. $P(X=x_i, Y=y_i) = \begin{cases} C \cdot x_i \cdot y_i & \text{when, } x_i = 1,2,3 \\ & \quad y_i = 1,2,3 \\ 0 & \text{when otherwise} \end{cases}$

Subjective Type Questions: **Section 3**

Write the answer in the same page provided for the questions 11 and 12. **Full marks** are awarded if all the steps are correct, and **partial marks** for an incorrect answer. (2 × 5 = 10 marks)

~~1~~1. Let (X, Y) be a two-dimensional discrete type random variables with joint pmf $p(x, y) = cxy$ for $x = 1, 2, 3$ and $y = 1, 2, 3$ and equals zero otherwise.

(a) Find c. (b) Find $P(1 \le X \le 2, Y \le 2)$. (c) $P(Y = 3)$. (1 + 2 + 2 marks)

Answer:

$\sum_x \sum_y p(x,y) = 1 \Rightarrow \sum_{x=1}^{3} \sum_{y=1}^{3} cxy = 1$

$\Rightarrow \sum_{x=1}^{3} cx(1+2+3) = 1 \Rightarrow c(36) = 1$

$\Rightarrow c = 1/36.$

$P(Y=3) = \sum_{x=1}^{3} cx(3) = 3 \times \frac{1}{36} \times \sum_{x=1}^{3} x = \frac{3 \times 1 \times 6}{36}$

$(\because P(Y=3) = \sum_{i=1}^{3} P(X=i, Y=3)).$ $= 1/2$

$P(1 \le X \le 2, Y \le 2) = P(X=1, Y=1) + P(X=1, Y=2)$

$+ P(X=2, Y=1) + P(X=2, Y=2)$

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$= c(1)(1) + c(1)(2) + c(2)(1) + c(2)(2)$

$= c + 2c + 2c + 4c = 9c = \frac{9}{36} = \frac{1}{4}.$

Q12.) $\therefore$ Using (1)

$P\left(\sum_{i=1}^{80} Y_i > 60\right)_3 = 1 - \phi\left(\frac{40\sqrt{3}}{\sqrt{560}}\right).$

$\Big(E(X_i^n) = 1^n (P(X_i=1)) + 0^n . (P(X_i=0)) = P(X_i=1) = \frac{7}{9} \Big).$

$\because X_i$ is iid

$\rightarrow Y_i$ is also iid. $\frac{40\sqrt{3}}{4\sqrt{35}}$ $\frac{10\sqrt{3}}{\sqrt{35}}$ $560 = \frac{4 \times 140}{2 \times 7}$ 4×35)

$f_{X_1,X_2}(y,z) =$

12. Let $X_1, X_2, \ldots$ be iid random variables, each having pmf $P(X_i = 1) = \frac{7}{9} = 1 - P(X_i = 0)$. Let $Y_i = X_i + X_i^2$, $i = 1, 2, \ldots$. Use central limit theorem to evaluate $P\left(\sum_{i=1}^{30} Y_i > 60\right)$ approximately. Final answer can be in terms of $\Phi(z)$ where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$.

(5 marks)

Answer:

$P(X_i = 1) = \frac{7}{9} \qquad P(X_i = 0) = \frac{2}{9}$

$E(X_i) = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = \frac{7}{9}$

$E(X_i^2) = 1^2 \cdot P(X_i = 1) + 0^2 \cdot P(X_i = 0) = 7/9$

$Var(X_i) = E(X_i^2) - E(X_i)^2 = \frac{7}{9} - \frac{49}{81} = \frac{63-49}{81} = \frac{14}{81}$

$Y_i = X_i + X_i^2 \Rightarrow E(Y_i) = E(X_i + X_i^2) = E(X_i) + E(X_i^2) = \frac{14}{9}$

$Y_i^2 = X_i^2 + X_i^4 + 2X_i^3$

$E(Y_i^2) = E(X_i^2 + X_i^4 + 2X_i^3) = E(X_i^2) + E(X_i^4) + 2E(X_i^3) = 4 \times \frac{7}{9} = \frac{28}{9}$

$\therefore Var(Y_i) = E(Y_i^2) - E(Y_i)^2 = \frac{28}{9} - \frac{196}{81} = \frac{252 - 196}{81} = \frac{56}{81}$

using CLT $P\left(\frac{\sum_{i=1}^{n} Y_i - E(\sum_{i=1}^{n} Y_i)}{\sqrt{Var(\sum_{i=1}^{n} Y_i)}} \le x \right) = \phi(x)$. ($\because X_i$ is iid $\rightarrow Y_i$ is also iid)

$P\left(\sum_{i=1}^{n} Y_i \le x\sqrt{Var(\sum_{i=1}^{n} Y_i)} + E(\sum_{i=1}^{n} Y_i) \right) = \phi(x)$

$\Rightarrow 1 - P\left((\sum_{i=1}^{n} Y_i) > x\sqrt{Var(\sum_{i=1}^{n} Y_i} + E(\sum_{i=1}^{n} Y_i) \right) = \phi(x)$

$\Rightarrow 1 - \phi(x) = P\left((\sum_{i=1}^{n} Y_i) > x\sqrt{Var(\sum_{i=1}^{n} Y_i)} + E(\sum_{i=1}^{n} Y_i) \right)$ — (1)

$\because X_i$'s are iid

$E(\sum_{i=1}^{n} Y_i) = n E(Y_i) = \frac{14n}{9}$, $Var(\sum_{i=1}^{n} Y_i) = n \times \frac{56}{81} = nVar(Y_i)$

Here $n = 30$, $\Rightarrow 60 = x\sqrt{\frac{56 \times 30}{81}} + \frac{30 \times 14}{9}$

$\left(60 - \frac{140}{3}\right) / \sqrt{\frac{560}{27}} = x \Rightarrow \frac{40\sqrt{3}}{\sqrt{560}} = x$ Continued on pg 3.