Recap: Structural Induction

Let S be a recursively defined set.

Base cases: $y_1, y_2, \dots \in S$ [there can be infinitely many base cases]

Recursion: $y_1, y_2, \dots, y_k \in S$

$$\Downarrow$$

$\text{Combine}(y_1, y_2, \dots, y_k) \in S$ [Some way to produce a new element using k elements in S]

We want to prove some property P about S.

Let $S \subseteq \mathcal{U}$ (universe), and
$P : \mathcal{U} \to \{T, F\}$. We want to show
$P(x) = T$ for all $x \in S$.

Can be done using structural induction.

1. State that you are using structural ind.
2. Show $P(y_i) = T$ for all $y_i \in$ Base Cases
3. Show that if $P(y_1) \wedge \dots \wedge P(y_k)$, then
   $P(\text{combine}(y_1 \dots y_k)) = T$.

LECTURE 14

We will start with an exercise which hopefully illustrates when to use PMI, when to use structural induction.

Exercise: Let us define a set $S \subseteq \underbrace{\{\otimes, ?, -\}^*}$
as follows:

set of finite length strings with symbols $\otimes$, $?$, $-$

Base case:

$\forall\, y \in \_\!\_^*$ — set of all finite length strings consisting of $\_\!\_$

$y \otimes \_\!\_\ ?\ y \in S$

Recursion:

For all $y, z, w \in \_\!\_^*$,

$$(y \otimes z\ ?\ w \in S) \Rightarrow (y \otimes z\,\_\!\_\ ?\ w\,y \in S)$$

Examples of strings in S:

1. $\_\!\_\ \_\!\_ \otimes \_\!\_\ ?\ \_\!\_\ \_\!\_ \in S$ [Base case]

2. Using (1) and recursion step, we can conclude
$\_\!\_\ \_\!\_ \otimes \_\!\_\ \_\!\_\ ?\ \_\!\_\ \_\!\_\ \_\!\_\ \_\!\_ \in S$

3. Using (2) and recursion step, we can conclude
$\_\_ \otimes \_\_\_ ? \_\_\_\_\_\_ \in S$.

Can we characterize the set S?

__Claim__: Let $T = \left\{ \_^{\alpha} \otimes \_^{\beta} ? \_^{\alpha\beta} : \alpha, \beta \in \mathbb{N} \right\}$

$S = T$.

To prove this claim, it suffices to show
$S \subseteq T$ and $T \subseteq S$.

Claim 1 — we will use str. ind. for this

Claim 2 — we will use regular PMI for this.

__Claim 1__: $S \subseteq T$.

In other words, for every
$\theta \in S$, $\exists\ \alpha, \beta \in \mathbb{N}$ s.t.
$\theta = \_^{\alpha} \otimes \_^{\beta} ? \_^{\alpha\beta}$.

Proof using structural induction.

We will define an appropriate predicate over $\mathcal{U}$.

$P(\theta) = \text{true}$ iff $\exists\ \alpha, \beta \in \mathbb{N}$ s.t.

$$\theta = \_^{\alpha} \otimes \_^{\beta} \ ?\ \_^{\alpha\beta}.$$

If $P(\theta) = \text{true}$, then $\theta \in T$ (by def. of set T)

<u>Base case</u>: Need to show that $P(\theta) = \text{true}$ for every θ in the base case of the def.ⁿ of S.

Base case of S: if $y \in \_^{*}$, then $y \otimes \_ \ ?\ y \in S$.

Since $y = \_^{\alpha}$ for some $\alpha \in \mathbb{N}$,

$$\theta = \_^{\alpha} \otimes \_ \ ?\ \_^{\alpha} \in S.$$

Check that $P(\theta) = \text{true}$ by setting $\beta = 1$.

Induction step: Suppose θ' is derived from θ using the recursion step, and suppose $P(\theta) = \text{true}$.

Since $P(\theta) = \text{true}$, $\exists\ \alpha, \beta \in \mathbb{N}$ s.t.

$$\theta = \underbrace{\_^{\alpha}}_{y} \otimes \underbrace{\_^{\beta}}_{z} \ ?\ \underbrace{\_^{\alpha\beta}}_{w}.$$

The recursion step produces

$$\theta' = \underbrace{\_^{\alpha}}_{y} \otimes \underbrace{\_^{\beta}}_{z}\_ \ ?\ \underbrace{\_^{\alpha\beta}}_{w}\underbrace{\_^{\alpha}}_{y}$$

Check that $P(\theta') = \text{true}$.

Hence using structural induction, $P(\theta) = \text{true}$ for all $\theta \in S$, and therefore $S \subseteq T$.

∎

<u>Claim 2</u>: $T \subseteq S$.

In other words, for all $\alpha, \beta \in \mathbb{N}$,

$$\_\_^{\alpha} \otimes \_\_^{\beta} \;?\; \_\_^{\alpha\beta} \in S.$$

Proof using regular induction.

Predicate $Q(\beta) :=$ for all $\alpha \in \mathbb{N}$,

$$\_\_^{\alpha} \otimes \_\_^{\beta} \;?\; \_\_^{\alpha\beta} \in S.$$

Base case: $\beta = 1$.

Take any $\alpha \in \mathbb{N}$

Using base case of def^{n} of S, it follows that

$$\_\_^{\alpha} \otimes \_\_ \;?\; \_\_^{\alpha} \in S.$$

Induction step: Suppose $Q(\beta) = \text{true}$.

To prove: $Q(\beta+1)$. Take any $\alpha \in \mathbb{N}$.

To show : $\_\_^{\alpha} \otimes \_\_^{\beta+1} \overset{?}{=} \_\_^{\alpha(\beta+1)} \in S.$

Given : $\_\_^{\alpha} \otimes \_^{\beta} \overset{?}{=} \_\_^{\alpha\beta} \in S$ $\quad [Q(\beta) = \text{true}]$

Follows from the induction step.

Hence, using PMI, we conclude $T \subseteq S$.

$\blacksquare$

Concluding Remarks :

- Structural induction can be used for proving that certain property holds for recursively defined sets.

- The same can also be proven using (regular) induction on the number of steps used to 'derive' the element