

Tutorial 10

1. [Submission Problems for Group 1] Problem 12.24 and 12.65 in LLM Book
2. [Submission Problems for Group 2] Problem 12.28 and 12.59 in LLM Book
3. [Submission Problems for Group 3] Problem 12.11 and 12.34 in LLM Book
4. [Submission Problems for Group 4] Problem 12.17 and 12.38 in LLM Book
5. [Bonus] Some of the problems could be hard and may not fit into the tutorial slot - use piazza for discussing those.

- (a) Problem 12. 19, 12.24, 12.30, and 12.67 in LLM Book
- (b) A graph is self-complementary if it is isomorphic to its complement. (a) Construct a self-complementary graph with 4 vertices. (b) Construct a self-complementary graph with 5 vertices. (c) Prove: if a graph with n vertices is self-complementary then $n \equiv 0$ or $1 \pmod{4}$. Prove: if $(\forall v \in V)(\text{outdegree}(v) = \text{indegree}(v))$ and G is weakly connected then G is strongly connected.
- (c) A rooted tree is a tree with a special node called the *root*. Derive a generating function for the number of rooted trees on n nodes. Given two rooted trees (T, r_1) and (T, r_2) show that you can check if they are isomorphic in $O(n)$ time.
- (d) Prove: If G is a triangle-free graph (i.e. it does not contain K_3 as a subgraph) then the number of edges in G is at most $\lfloor n^2/4 \rfloor$. Show that this bound is tight for every n .

Proceed by induction in increments of 2.

Hint: State a lemma about the sum of the degrees of a pair of adjacent vertices. Then

- (e) Let d_1, \dots, d_n be positive integers such that $\sum_{i=1}^n d_i = 2n - 2$. Consider those spanning trees of K_n which have degree d_i at vertex i . Count these spanning trees; show that their number is

$$\frac{(n-2)!}{\prod_{i=1}^n (d_i - 1)!}$$

Use this to show that the number of spanning trees of K_n is n^{n-2} .

- (f) The diameter of a graph is the maximum distance between all pairs of vertices. So if a graph has diameter d then $(\forall x, y \in V)(\text{dist}(x, y) \leq d)$ and $(\exists x, y \in V)(\text{dist}(x, y) = d)$. (a) Disprove the following statement: “the diameter of a graph is the length of its longest path.” (b) For every n , find the maximum ratio between the length of the longest path and the diameter. (c) Prove that the statement is true for trees: the diameter is the length of the longest path.
- (g) The girth of a graph is the length of its shortest cycle. If a graph has no cycles then its girth is said to be infinite. (a) For every $g \geq 3$ find a trivalent graph (a regular graph of degree 3) of girth $\geq g$. (b) For every $g \geq 3$ and $d \geq 3$ find a d -regular graph of girth $\geq g$.