COL202: Discrete Math	Tutorial 9
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The solutions for the ($\star$) marked problems must be submitted on Gradescope by 11:59 am on 8th November.
The $\blacklozenge$ marked problems will be discussed in the tutorial.

In this tutorial, we will discuss graphs. Throughout this tutorial, n represents the number of vertices, and m the number of edges. Unless specified otherwise, the graphs have no self-loops or multi-edges.

1 Tutorial Submission Problem

1.1. ($\star$) Let $G = (V, E)$ be an undirected graph, and let $\mathbf{A}$ denote the corresponding adjacency matrix. Prove that for all integers $k > 0$, and for any $u, v \in V$, $\mathbf{A}^k[u, v]$ is equal to the number of walks from u to v of length exactly k.

2 The Probabilistic Method (on Graphs)

The probabilistic method is extensively used for proving graph-theoretic properties. In fact, one of the first applications of the probabilistic method was for proving a lower bound on the Ramsey number, which you saw in Tutorial 7 (exercise 2.6). Here, we will see a few more applications of the probabilistic method (especially in the context of graphs).

2.1. ($\blacklozenge$: [AS92], Theorem 1.2.2) Let $G = (V, E)$ be an undirected graph where every vertex has degree at least δ. A dominating set in a graph is a subset of vertices $U \subseteq V$ such that every vertex in $V \setminus U$ has at least one neighbor in U. Clearly, if we take the entire vertex set V, then this is a dominating set. How small can the dominating set be? Show that there exists a dominating set of size at most $n(1 + \ln(\delta + 1))/(\delta + 1)$.

Solution: Pick a random subset $X \subseteq V$ by sampling each vertex, independently with probability p (the parameter p will be fixed later). The sampled set may not be a dominating set. Let Y be the set of all vertices in $V \setminus X$ which have no neighbor in X. Let $U = X \cup Y$. The expected size of X is np, and the expected size of Y is $n(1-p)^{\delta+1}$. As a result, $\mathbb{E}[|U|] = np + n(1-p)^{\delta+1}$. Hence, there exists a dominating set of size at most $np + n(1-p)^{\delta+1}$. Since this holds for all p, we find the p that minimizes this expression. Note that $np + n(1-p)^{\delta+1} \le np + ne^{-p(\delta+1)}$, and this expression is minimum at $p = \frac{\ln(\delta+1)}{\delta+1}$.

2.2. ([AS92], Ex 4) Let $G = (V, E)$ be a graph where every vertex has degree $\delta > 10$. Show that V can be partitioned into two disjoint subsets A, B such that $|A| \le O(n \ln \delta/\delta)$ and every vertex in B has at least one neighbor in A and at least one neighbor in B.

Solution: The approach used for previous problem should also work here. You can see the solution here (we have not proof-read this solution).

2.3. Let $G = (V, E)$ be an undirected graph where every vertex has degree exactly d. An independent set $S \subseteq V$ is a subset of vertices such that for every $a, b \in S$, $\{a, b\} \notin E$. Show that there exists an independent set of size at least $n/2d$.

Solution: Consider the following process: delete each vertex (together with all the edges incident to it), independently, with probability p. In the residual graph, there will be a few edges remaining. For each edge, delete one of the endpoints.

Let X denote the number of vertices left after the first round. Let Y denote the number of vertices deleted in the second round. The size of the independent set is $X - Y$. As a result, using the probabilistic method, there exists an independent set of size $\mathbb{E}[X] - \mathbb{E}[Y]$.

First, note that $\mathbb{E}[X] = n(1 - p)$. Next, note that the number of edges in the original graph is at least $m = nd/2$.

We will be deleting one vertex for every edge left in the graph after the first step. An edge remains in the graph after the first step if both its endpoints were not deleted in the first step. Therefore, $\mathbb{E}[Y] = nd(1 - p)^2/2$.

For $p = 1 - 1/d$, $\mathbb{E}[X] - \mathbb{E}[Y] = n/d - n/(2d) = n/(2d)$.

3 General Properties of Graphs

3.1. (♦) Matrix multiplication can be performed in time $o(n^3)$. Use this fact to give an $o(n^3)$ time algorithm for the following problem: given an undirected graph G, check if there exist three vertices $a, b, c \in V$ such that $\{a, b\}, \{b, c\}$ and $\{c, a\}$ are all edges in E. The naive algorithm (checking all triplets) takes time $O(n^3)$.

Solution: As discussed in class, exponentiating the adjacency matrix A of a graph allows us to determine walks of a specific length. Specifically, each entry $(A^k)_{i,j}$ in the matrix A^k represents the number of distinct walks of length k from vertex i to vertex j.

Consider a graph with edges $\{a, b\}$, $\{b, c\}$, and $\{c, a\}$, which together form a walk of length 3. Let $B = A^3$. Then, $B[i, i] \neq 0$ if and only if there exists a walk of length 3 that starts and ends at vertex i. Thus, computing A^3 helps us determine if there exists a walk of length 3 from any vertex back to itself.

Since matrix multiplication can be computed in $O(n^3)$ time, A^3 can be obtained within this time complexity, which completes the proof.

Note: The same method cannot be directly applied to determine whether there exists a walk of length 4 from a vertex a back to itself. (Think why and when it fails ?)

3.2. Consider an undirected graph $G = (V, E)$ where every vertex has degree at least d. Prove that G has a path of length at least d.

Solution: Claim: The problem statement can be strengthened to at least $d + 1$.

Let the longest path in the graph have length p. Consider the last vertex in this path. Since this vertex has degree at least d, all its neighbors must be included in the path. Otherwise, we could extend the path by adding any missing neighbor, contradicting the assumption that the path is the longest.

Therefore, the longest path must contain at least $d + 1$ vertices.

3.3. (♦ - [MN09], pg 130, Exercise 16) Consider any connected, undirected graph $G = (V, E)$ where, for any pair of distinct vertices u, v, either u and v have no common neighbors, or have exactly 2 common neighbors. Prove that all vertices of G have the same degree.

(Harder version) : Suppose you are given that for any pair of distinct vertices u, v, either u and v have no common neighbors, or have exactly 5 common neighbors. Prove that all vertices have the same degree.

Solution: Solution to the easier version:

The graph is connected, which is an important condition for the problem, as we want to use the constraint that each pair of vertices has either 0 or 2 common neighbors. Let us take two adjacent vertices u and v.

Consider every neighbor x_i of v (other than u, denoted by $N(v) \setminus \{u\}$). Since each x_i and u must have a common neighbor other than v (because they have exactly 2 common neighbors), let each x_i have a unique neighbor y_i. For instance, y_1 and v are common neighbors of x_1.

We now show that no two y_i are the same. For example, $y_5 \neq y_7$, meaning y_5 is not a common neighbor of both x_5 and x_7. If it were, then y_5 would have x_5, x_7, and u as neighbors, violating the condition on common neighbors. Thus, each x_i has a unique y_i.

This implies that $|N(u)| \geq |N(v)|$ and, by symmetry, $|N(v)| \geq |N(u)|$. Together, these imply $\deg(u) = \deg(v)$ for all adjacent vertices. Since the graph is connected, this implies the graph is regular.

Solution for the harder part:

Consider any edge $e = (x, y)$ in G. We claim that $deg(x) = deg(y)$. Consider the sets A, B and C defined as follows :

- $A = N(x) - (N(y) \cup \{y\})$
- $B = N(x) \cap N(y)$
- $C = N(y) - (N(x) \cup \{x\})$

Clearly A, B and C are pairwise disjoint sets. For a node u and a set S, let $X(u,S) = \{v \mid (u,v) \in E(G), v \in S\}$. and for 2 sets S and T let $X(S,T) = \{(u,v) \in E(G) \mid u \in S, v \in T\}$.

Claim 1. *$\forall v \in A$, $|X(v, B \cup C)| = 4$ and $\forall v \in C$, $|X(v, B \cup A)| = 4$.*

Proof. Consider $v \in A$. Since $x \in N(y) \cup N(v)$, y and v have exactly 5 neighbors. Since $B \cup C = N(y) - \{x\}$, $|X(v, B \cup C)| = 4$. Similarly $|X(v, B \cup A)| = 4 \, \forall \, v \in C$. □

Claim 2. *$|X(A,B)| + |X(A,C)| = 4|A|$ and $|X(C,B)| + |X(C,A)| = 4|C|$.*

Proof. $|X(A,B)| + |X(A,C)| = \sum_{v \in A}(|X(v,B)| + |X(v,C)|) = 4 \cdot |A|$ by 1. Similarly $|X(C,B)| + |X(C,A)| = \sum_{v \in C}(|X(v,B)| + |X(v,A)|) = 4 \cdot |C|$. □

Claim 3. *$\forall v \in B, |X(v,A)| = |X(v,C)|$.*

Proof. Since $v \in N(x) \cap N(y)$, v and x have exactly 5 common neighbors $\Rightarrow$ $|X(v, A \cup B)| = 4$. Similarly $|X(v, C \cup B)| = 4$. Thus $|X(v,A)| + |X(v,B)| = |X(v, A \cup B)| = |X(v, C \cup B)| = |X(v,C)| + |X(v,B)| \Rightarrow |X(v,A)| = |X(v,C)|$. □

Claim 4. *$|X(B,A)| = |X(B,C)|$.*

Proof. $|X(B,A)| = \sum_{v \in B} |X(v,A)|$ and $|X(B,C)| = \sum_{v \in B} |X(v,C)|$. Thus $|X(B,A)| = |X(B,C)|$ by 3. □

Thus using 2 and 4, we get $|A| = |C|$. Hence $|A \cup B| = |C \cup B| \Rightarrow |N(x)| = |N(y)|$. Since G is connected, applying this argument to all edges in G, gives us G is a $k-$regular graph for some k.

Note: The constants 2 and 5 are arbitrary and can be replaced with any fixed values as the argument remains valid.

References

[AS92] Noga Alon and Joel Spencer. *The Probabilistic Method.* John Wiley, 1992.

[MN09] Jirí Matousek and Jaroslav Nesetril. *Invitation to Discrete Mathematics (2. ed.).* Oxford University Press, 2009.