Q1

6 Points

A *permutation* of a set S is a bijection from S to S. Given a function $f :$ $S \longrightarrow S$, the associated relation R_f on S is $R_f = \{(x, y) \in S \times S \mid f(x) = y\}$. Note that the transitive closure R_f^+ of the relation R_f is the following relation.

$$R_f^+ = \{(x, y) \in S \times S \mid \exists m \in \mathbb{N} \; \exists z_0, \ldots, z_m \in S \text{ such that}$$

$$x = z_0, \; z_m = y, \text{ and } \forall i \; z_i = f(z_{i-1})\},$$

or equivalently,

$$R_f^+ = \{(x, y) \in S \times S \mid \exists m \in \mathbb{N} \; y = f^m(x)\},$$

where f^m denotes the function f composed m times. Henceforward, we assume that f is a permutation of a finite set S with $|S| = n$. We wish to prove that R_f^+ is an equivalence relation. It is transitive, by definition, already.

Q1.1

3 Points

Prove that R_f^+ is reflexive.

Let us consider any element $x \in S$. We claim that $\exists m \leq n$, where $n = |S|$ such that $f^m(x) = x$. Suppose such an m does not exist, i.e. $f^m(x) \neq x \; \forall m \leq n$.

Note that since f is a bijection, each element of S is mapped to by exactly one element of S under f. Under the above contrapositve assumption we make the following claim: $f^k(x) \neq f^m(x) \; \forall k < m \leq n$. The base case $m = 2$ is trivial to check. Suppose it is true for some m. Then for $m + 1$, suppose $f(f^m(x)) = y$. If $y = f^k(x)$ for some $k < m$, then by bijectivity of f, $f^m(x) = f^{k-1}(x)$ which violates the inductive hypothesis. Hence the claim is true.

Now since $f^k(x) \neq f^m(x) \; \forall k < m \leq n$, after n applications of f on x we must get an element distinct from all the previous elements.

But this means $|S| = n + 1$, which is not possible. Hence our assumption is false and so $\exists m \leq n$, where $n = |S|$ such that $f^m(x) = x$. Thus, R_f^+ is reflexive.

Q1.2
3 Points

Prove that R_f^+ is symmetric.

Suppose $(x, y) \in R_f^+$, i.e. exists $m_1 \in \mathbb{N}$ such that $f^{m_1}(x) = y$. There may exist more than one m_1 but by WOP we choose the smallest such m_1. Now, since f is a bijection, f^{-1} exists and so $(f^{-1})^{m_1}(y) = x$.

By the previous claim, we know that there exists $m_2 \in \mathbb{N}$ such that $f^{m_2}(y) = y$. Apply f^{-1} on both sides of the equation m_1 times to get, $f^{m_2 - m_1}(y) = (f^{-1})^{m_1}(y) = x$, while noting that $m_2 > m_1$ because otherwise the minimality of m_1 is contradicted. So, $\exists m = m_2 - m_1$ such that $f^m(y) = x$, and so R_f^+ is symmetric.

Q2
4 Points

Next, we want to count the number of permutations f of a set of a given size such that R_f^+ has a given number of equivalence classes. Let Q_k^n denote the set of permutations f of the set $\{1, \ldots, n\}$ such that R_f^+ has exactly k equivalence classes. Let $b(n, k) = |Q_k^n|$. We will now derive the following recurrence for the numbers $b(n, k)$.

$$b(n, k) = b(n - 1, k - 1) + (n - 1) \cdot b(n - 1, k).$$

Q2.1
1 Point

Prove that $b(n - 1, k - 1)$, the first term on the right-hand-side, is exactly the number of permutations $f \in Q_k^n$, such that $f(n) = n$.

Suppose we have $f(n) = n$, then the equivalence class of $[n]_R = \{n\}$, since $f^m(n) = n \ \forall \ n \in \mathbb{N}$. We need to create $k - 1$ equivalence classes of the remaining $n - 1$ elements of the set and so $b(n - 1, k - 1)$ is exactly the number of permutations $f \in Q_k^n$ with $f(n) = n$.

Q2.2

3 Points

Prove that $(n - 1) \cdot b(n - 1, k)$, the second term on the right-hand-side, is exactly the number of permutations $f \in Q_k^n$, such that $f(n) \neq n$.

Suppose we have $f(n) \neq n$. Then we should create k equivalence classes from the remaining $n - 1$ elements