Quiz4B (COL 351)

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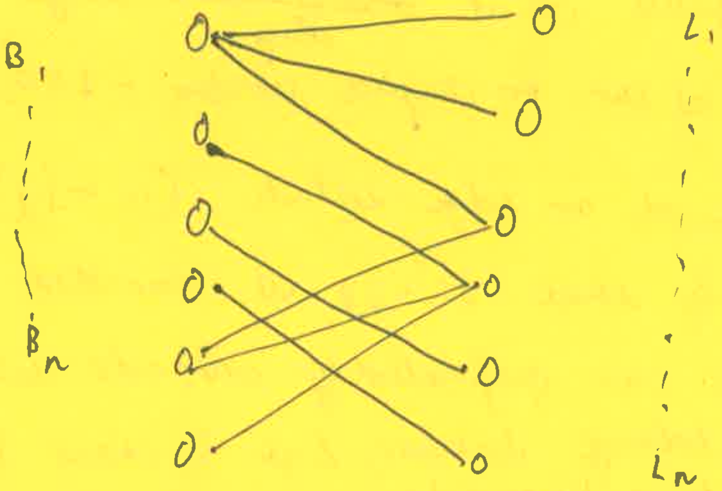
You are given a set of n borrowers $B_1, \ldots, B_n$ and n lenders $L_1, \ldots, L_n$. For each pair of borrower B_i and lender A_j, you are given the amount of money borrowed by B_i from the lender A_j (this amount could be 0 also). You are also given that the following condition holds: for each borrower $B_i, i = 1, \ldots, n$, the total amount of money borrowed by B_i (from all the lenders) is Rs 100. Similarly, for each lender $L_j, j = 1, \ldots, n$, the total amount of money lent by L_j (to all the borrowers) is Rs 100. We want to find a permutation $\sigma_1, \sigma_2, \ldots, \sigma_n$ of $1, 2, ..., n$ such that borrower B_i has borrowed non-zero amount of money from lender L_{σ_i} for each $i = 1, \ldots, n$.

Example: Suppose there are 3 lenders and 3 borrowers. B_1 has borrowed Rs 100, Rs 0, Rs 0 from L_1, L_2, L_3 respectively. B_2 has borrowed Rs 0, Rs 60, Rs 40 from L_1, L_2, L_3 respectively. B_3 has borrowed Rs 0, Rs 40, Rs 60 from L_1, L_2, L_3 respectively. Then one valid permutation is $(1, 3, 2)$ because B_1 has borrowed non-zero amount from L_1, B_2 has borrowed non-zero amount from L_3, and B_3 has borrowed non-zero amount from L_2.

Show that this problem can be formulated as a bipartite matching problem. Then prove that such a permutation always exists.

We will formulate the bipartite graph as follows:-



Draw edge for every B_i & lender L_j such that B_i borrowed non zero amount from L_j

$\exists\, e \in G$, ~~set~~ $B_i - L_j$ s.t B_i borrows non zero from L_j.

Note that this graph is bipartite as each edge exists from ~~one~~ vertex from $B_1 \,--\, B_n$ & one from $L_1 \,---\, L_n$.

Now we claim selecting / maximum matching of this bipartite graph will lead a permutation of vertices $1, 2 \,--\, n$.

For the permutation :- each L_j matched to B_i, ~~on left~~ in our maximum matching $\Longleftrightarrow$ $\sigma_i = j$. (as each edge connects no 2 same vertex)

There will exist atmost n edges in the matching, we prove that there will be exactly n edges in the matching

~~Each edge is connected to a~~

Each vertex on left & right is connected to atmost one edge.
So we get a permutation

We will show ~~that~~ by contradiction
that it is not possible for any vertex on the
left side to be unmatched.

Consider B_i s.t no edge exists from B_i to L_j in
the maximum matching, there will exist some L_v which maps to no vertex.

Consider set of edges from B_i to $L_{j_1}, L_{j_2} \dots L_{j_k}$ in
the bipartite graph.

Now, as B_i is not matched, $\Rightarrow j_1, j_2 \dots j_k$ are
matched to some other vertices on the left side. (otherwise we could select it).

Sum of edges $(i-j_1) \dots (i-j_k)$ is 100.

Since $L_{j_1}, L_{j_2} \dots L_{j_k}$ are matched to other vertices,
it is not possible that ~~all of~~ any the edge weights
is 100. (as sum on right vertex = 100)

$\therefore$ there will exist an edge which $(i-j_l)$ which
has the property that $i'-j_l$ is connected in the
matching, we can repeatedly cascade until
we get a matching between L_v & some B_t.
& hence get a permutation

Consider a greedy algorithm (similar to the one discussed in class)
which repeatedly picks edges with the minimum
weight possible.

It will