ELL205 Signals and Systems Major Exam *- 80 marks, 120 minutes*

1) (**15 marks**) Let $x[n] = 1+2^2+....+n^2$. *Using properties of the Z-transform*, find a general expression for the sum of square of integers using properties of the Z-transform. You possibly already know the formula for this sum. HINT: How are $x[n+1]$ and $x[n]$ related?

So,

$$x[n] = 1^2 + 2^2 + ... + n^2, \qquad x[n+1] = 1^2 + 2^2 + ... + (n+1)^2. \tag{1}$$

We can easily find out $x[n+1] - x[n]$ as

$$x[n+1] - x[n] = (n+1)^2 \tag{2}$$

Now taking Unilateral Z-Transform on both sides, we get:

$$zX[z] - X[z] = \sum_{n=0}^{\infty}(n+1)^2 z^{-n} = \sum_{n=0}^{\infty} n(n+1)z^{-n} + \sum_{n=0}^{\infty} nz^{-n} + \sum_{n=0}^{\infty} z^{-n} \tag{3}$$

Further, we can write above equation in Z-transform as:

$$zX[z] - X[z] = \sum_{n=-\infty}^{\infty} n(n+1)u[n]z^{-n} + \sum_{n=-\infty}^{\infty} nu[n]z^{-n} + \sum_{n=-\infty}^{\infty} u[n]z^{-n} \tag{4}$$

We know the following Z-transform of sequence $u[n]$ as

$$\sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}. \tag{5}$$

Using differentiation properties of Z-transform, we can write

$$\frac{d}{dz}\sum_{n=0}^{\infty} z^{-n} = \frac{d}{dz}\frac{1}{(1-z^{-1})}. \tag{6}$$

We get

$$\sum_{n=0}^{\infty} nz^{-n-1} = \frac{1}{(z-1)^2}. \tag{7}$$

Further, again performing $\frac{d}{dz}$ on above equation, we get:

$$\sum_{n=0}^{\infty} n(n+1)z^{-n-2} = \frac{2}{(z-1)^3}. \tag{8}$$

Performing $\frac{d}{dz}$ on above equation once again, we get

$$\sum_{n=0}^{\infty} n(n+1)(n+2)z^{-n-3} = \frac{6}{(z-1)^4}. \tag{9}$$

Therefore, using above equations, we get:

$$X[z](z-1) = \frac{2z^2}{(z-1)^3} + \frac{z}{(z-1)^2} + \frac{z}{(z-1)} \tag{10}$$

Further solving, we get:

$$X[z] = \frac{z^3+z^2}{(z-1)^4} = \frac{z^3}{(z-1)^4} + \frac{z^2}{(z-1)^4} \tag{11}$$

Since, $X[z] = \sum_{n=0}^{\infty} x[n]z^{-n}$. The sum $x[n]$ is coefficient of z^{-n} in $X[z]$.

Using (9), we can write:

$$\sum_{n=0}^{\infty} \frac{n(n+1)(n+2)}{6} z^{-n} = \frac{z^3}{(z-1)^4}. \tag{12}$$

$$\sum_{n=0}^{\infty} \frac{n(n+1)(n+2)}{6} z^{-n-1} = \frac{z^2}{(z-1)^4}. \tag{13}$$

Hence X(z) in (11) can be written as:

$$X(z) = \sum_{n=0}^{\infty} \frac{n(n+1)(n+2)}{6} z^{-n} + \sum_{n=0}^{\infty} \frac{n(n+1)(n+2)}{6} z^{-n-1}. \tag{14}$$

The second summation can be re-written as:

$$X(z) = \sum_{n=0}^{\infty} \frac{n(n+1)(n+2)}{6} z^{-n} + \sum_{n=1}^{\infty} \frac{(n-1)(n)(n+1)}{6} z^{-n}. \tag{15}$$

Since, coefficient of z^{-n} will give the required sum $x[n]$. Therefore, coefficient of z^{-n} is:

$$\frac{n(n+1)(n+2)}{6} + \frac{(n-1)(n)(n+1)}{6} = \frac{n(n+1)(2n+1)}{6} \tag{16}$$

2) (**25 marks**) A differentiator needs to be built for a continuous-time signal $x(t)$ whose frequency content does not exceed Ω_c radians/second. However, such differentiators have some practical problems. For this reason, an engineer decides to sample the signal $x(t)$ and use a discrete-time differentiator. She then converts the output of the discrete-time differentiator to a continuous-time signal $y(t)$. The idea is to ensure that $y(t) = \dfrac{dx(t)}{dt}$. Note that this setup has none of the difficulties associated with the continuous-time differentiator.

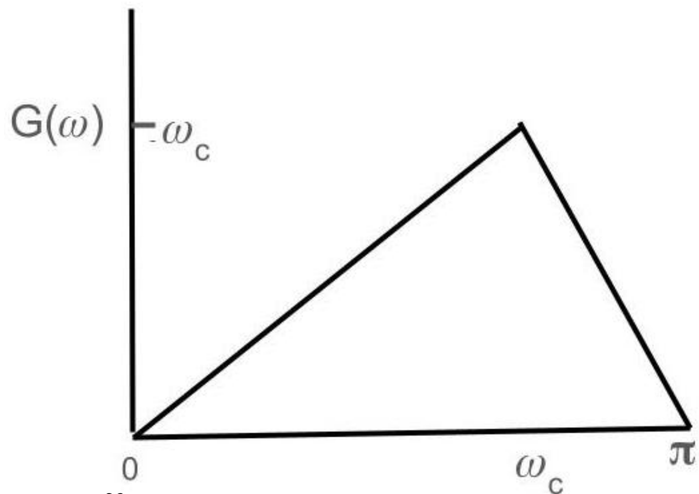
a) (**1 mark**) Using concepts learnt in the course explain why the continuous-time differentiator is prone to noise (thermal noise for example).

b) (**1 mark**) What should be the minimum sampling rate used to ensure no aliasing?

c) (**7 marks**) What should the frequency response $H(e^{j\omega})$ of the discrete-time differentiator be? What is the impulse response $h[n]$ when the minimum sampling rate is used?

d) (**3 marks**) Suppose the impulse response obtained above is truncated for $|n| > N$. What is the energy in the error in representation of the frequency response $H(e^{j\omega})$ over a 2π interval?

e) (**9 marks**) To produce an impulse response that is easier to implement, the student decided to use a filter with frequency response $H(e^{j\omega}) = jG(\omega)$, $0 \leq \omega \leq \pi$ as depicted below (note that the frequency response is depicted only for $0 \leq \omega \leq \pi$). What is the impulse response of the filter? What is the sampling rate required to ensure that the overall system still works as a continuous-time differentiator? What is the impulse response for the specific case of $\omega_c = \pi/2$? Can you see an implementation advantage?



f) (**4 marks**) Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ using concepts from this course.

ANS: a) Any thermal noise present in the system with a high frequency component gets amplified by the frequency response of the differentiator $j\omega$, which causes large noise amplification.

b) Clearly, $\dfrac{2\pi}{T_s} > 2\Omega_c$ should be the sampling rate.

c) Clearly: $H(e^{j\omega}) = j\omega - \pi < \omega < \pi$. Using Fourier series concepts and the duality of the DTFT and the CTFS, we have:

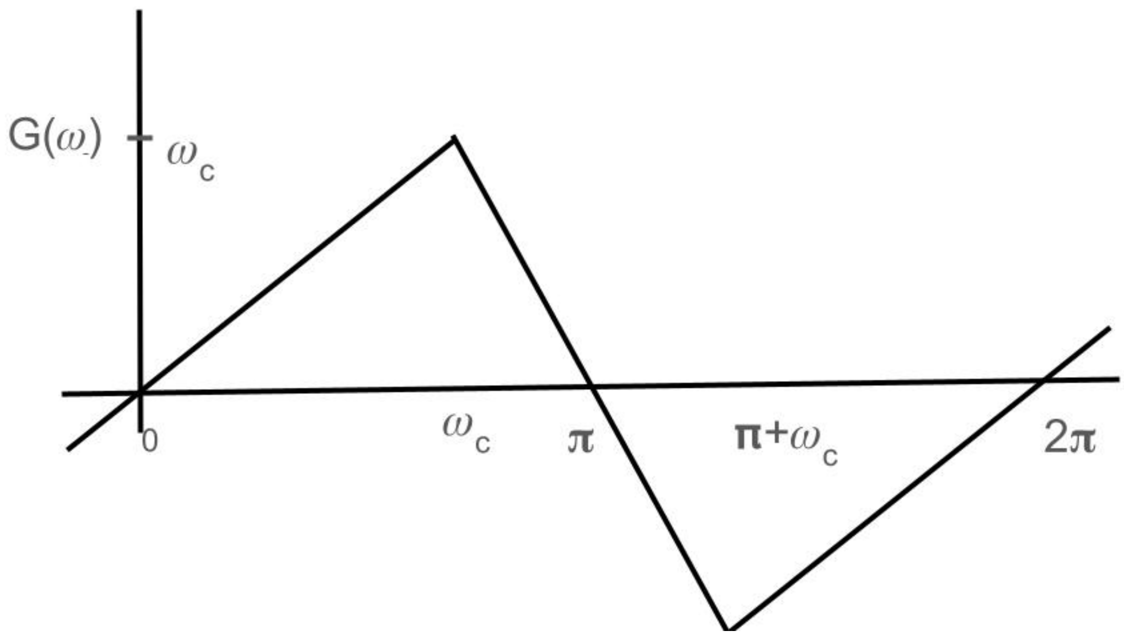
$$h[n] = \frac{(-1)^n}{n}$$

You need to use properties of the DTFT or CTFS to derive the above, these details are omitted (direct evaluation is not acceptable).

d) Denote the approximate impulse response by $\hat{h}[n]$ and its DTFT by $\hat{H}(e^{j\omega})$. Then, the following follows from the Parsevals theorem:

$$\int_{-\pi}^{\pi} \left| H(e^{j\omega}) - \hat{H}(e^{j\omega}) \right|^2 d\omega \;=\; 2\pi \sum_n |h[n] - \hat{h}[n]|^2 = 2\pi \sum_{n \notin [-N,N]} |h[n]|^2$$

e) We need to note that $h[n]$ is real, which implies that $H(e^{j\omega}) = H^*(e^{-j\omega})$. This implies that $G(\omega)$ is odd (its value from $-\pi$ to π is known. $H(e^{j\omega})$ is now specified from $-\pi$ to π. To find $h[n]$, we use several approaches (FS concepts or properties of the DTFT for example). All you need to



do now is to invert $H(e^{j\omega}) = jG(\omega)$...This can be done in several ways... I suggest something like double differentiation and use of differentiation property... When $\omega_c = \pi/2$ you an see that it has a triangular form...The impulse response will be squared $\sin(x)/x$, and therefore decays fast. It is therefore easy to implement in practice. You can check to see that these type of impulse response is not implementable by a pole-zero system (constant coefficient linear difference equation). This makes the ability to truncate the response very important. The flip side? The sampling rate has to be larger! Earlier it was $1/T_s > \dfrac{\Omega_c}{\pi}$. Now it becomes $1/T_s > \dfrac{\Omega_c}{\omega_c}$, and $\Omega_c < \pi$ implies that the sampling rate has increased.

f) We can use the Parseval's relation and use the fact that $\omega_c = \pi$ here to get:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \omega^2 d\omega \tag{17}$$

3) (**10 marks**) You are given that $x(t) = 0$ for $t < 0$. Solve for $x(t)$ if you are also given that

$$x(t) \;=\; e^{-t} + \int_0^t \cosh(t-\tau) x(\tau) d\tau, \quad t \geq 0$$

Do you need to impose any constraints on $x(t)$?

The integral can be written as:

$$x(t) = e^{-t} + \cosh tu(t) * x(t), \qquad t \geq 0. \tag{18}$$

Taking Unilateral Laplace transform on both sides, we get:

$$X(s) = \int_{t=0}^{\infty} e^{-t} e^{-st} dt + \int_{t=0}^{\infty} (\cosh tu(t) * x(t)) u(t) e^{-st} dt. \tag{19}$$

Further, above equation can be written as:

$$X(s) = \int_{t=-\infty}^{\infty} e^{-t} u(t) e^{-st} dt + \sum_{n=-\infty}^{\infty} (\cosh tu(t) * x(t)) u(t) e^{-st} dt. \tag{20}$$

Now, using property of Laplace transform and using $\mathcal{L}\{\cosh at\} = \frac{s}{(s^2-a^2)}$, we get:

$$X(s) = \frac{1}{(s+1)} + X(s) \frac{s}{(s^2-1)}. \tag{21}$$

On solving, we get

$$X(s) = \frac{s-1}{(s^2-s-1)}. \tag{22}$$

We can rewrite this as:

$$X(s) = \frac{(s-1/2)}{((s-1/2)^2 - 5/4)} - \frac{1/2}{((s-1/2)^2 - 5/4)}. \tag{23}$$

Using, $\mathcal{L}\{\cosh at\} = \frac{s}{(s^2-a^2)}$ and $\mathcal{L}\{\sinh at\} = \frac{a}{(s^2-a^2)}$ and properties of Laplace transform, we get:

$$x(t) = \cosh \frac{\sqrt{5}t}{2} e^{-t/2} u(t) - \frac{\sqrt{5}}{4} \sinh \frac{\sqrt{5}t}{2} e^{-t/2} u(t) \tag{24}$$

4) (**18 marks**) Evaluate the following integrals using concepts learnt in this course:

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} \frac{2\sin(Wt)}{\pi t \left(1 + \left(\frac{t}{2}\right)^2\right)} dt \\ I_2 &= \int_{-\pi}^{\pi} \frac{\sin\left(2\omega\left(N_1 + \frac{1}{2}\right)\right) \sin\left(\omega\left(N_2 + \frac{1}{2}\right)\right)}{\sin(\omega)\sin(\omega/2)} d\omega, \quad N_1 > N_2 \end{aligned}$$

ANS: Using the time-scaling property, it is easy to see that:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1 + \left(\frac{\omega}{2}\right)} e^{j\omega t} d\omega = 2e^{-2|t|}$$

Using duality, by interchanging $-\omega$ and t we can see that:

$$4\pi e^{-2|\omega|} = \int_{-\infty}^{\infty} \frac{2}{1+\left(\frac{t}{2}\right)} e^{-j\omega t} dt$$

From the generalized Parseval's theorem, we have:

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)Y^*(j\omega)d\omega$$

Using $x(t) = \dfrac{\sin(Wt)}{\pi t}$ and $y(t) = \dfrac{2}{1+\left(\frac{t}{2}\right)^2}$, it can be seen that integral I_1 is $\int_{-W}^{W} \frac{1}{2}e^{-2|\omega|}d\omega = \frac{1}{4}\int_{-2W}^{2W} e^{-|\omega|}d\omega = \frac{1}{2}\int_{0}^{2W} e^{-|\omega|}d\omega$. This evaluates to $\frac{1}{2}\left(1-e^{-2W}\right)$

To evaluate I_2 we use:

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

Noting that

$$y[n] = \begin{cases} 1 & -N_2 \le n \le N_2 \\ 0 & \text{Otherwise} \end{cases} \iff \frac{\sin\left(\omega\left(N_2+\frac{1}{2}\right)\right)}{\sin(\omega/2)} \tag{25}$$

and that

$$x[n] = \begin{cases} x_1[n/2] & \text{n even} \\ 0 & \text{n odd} \end{cases} \iff \frac{\sin\left(2\omega\left(N_1+\frac{1}{2}\right)\right)}{\sin(\omega)} \tag{26}$$

It can be inferred using $N_1 > N_2$ that

$$I_2 = 2\pi\sum_{n=-N_2}^{N_2} x[n]y[n] = 2N_2 - 1 \tag{27}$$

5) (**7 marks**) Find a sequence $x[n]$ whose Z-transform is $X(z) = e^{-2z}$ (also specify the ROC that results in the sequence).

ANS: use the Taylor $\exp(-2z) = \sum_{n=0}^{\infty}\frac{(-2z)^n}{n!}$. Using $-n$ for n, we have $X(z) = \sum_{n=-\infty}^{0}\frac{(-2)^{-n}}{(-n)!}z^{-n} = \sum_{n=-\infty}^{\infty}\frac{(-2)^{-n}}{(-n)!}u[-n]z^{-n}$ from which we can identify the discrete sequence as $x[n] = \frac{(-2)^{-n}}{(-n)!}u[-n]$. In the above, the negative factorial has to be interpreted correctly.

6) (**5 marks**) Consider $X(t) = A\cos(\omega_c t + \Phi)$ where A and ω_c are constants, and Φ is a uniformly distributed random variable that takes values between $-\pi$ and π. Find the mean and autocorrelation of $X(t)$, and comment on the nature of $X(t)$.

ANS: Mean $\mu(t) = E\{X(t)\} = A\cos(\omega_c t + \Phi) = A2\pi \displaystyle\int_{-\pi}^{\pi} \cos(\omega_c t + \phi)d\phi = 0$ (constant, and not time-dependent).

Similarly, $E\{X(t)X(t+\tau)\} = A^2 E\{\cos(\omega_c t+\Phi)\cos(\omega_c(t+\tau)+\Phi)\} = A^2 \int_{\pi}^{\pi} \cos(\omega_c t+\phi)\cos(\omega_c(t+\tau) + \phi)d\phi$ Using Trigonometric formulae, it can be shown that the above reduces to $\dfrac{A^2}{2}\cos(\omega_c \tau)$. Clearly, $X(t)$ is wide-sense stationary.