Problem 1
Recall that we defined an intersecting family to be a collection of subsets of a given set S such that no two sets in the collection are disjoint. We also proved that when S is a finite set with $|S| = n$, the size of the largets intersecting family of subsets of S is 2^{n-1}. What if we want an intersecting family in which every set has the same given size, say k (a.k.a. a k-uniform intersecting family)? Let us find an answer to this question. Observe that when $k > n/2$, the answer is trivial, so let us assume $k \leq n/2$.

1. [**1 point**] Prove that there exists a k-uniform intersecting family containing $C(n-1, k-1)$ sets ($C(n-1, k-1)$ denotes "$(n-1)$ choose $(k-1)$").
2. [**1 point**] Suppose $A \subseteq S$ and $|A| = k$. Imagine that the elements of S are to be assigned to n distinct places on the circumference of a circle. How many ways are there to do so in such a way that elements of A appear consecutively?
3. [**2 points**] Suppose $\mathcal{F}$ is a family of subsets of S, each of size k, and $|\mathcal{F}| > C(n-1, k-1)$. Prove that the elements of S can be arranged on the circle in such a way that the elements of more than k of the sets in $\mathcal{F}$ appear consecutively. (Hint: Double counting + pigeon-hole.)
4. [**2 points**] Hence argue that if $|\mathcal{F}| > C(n-1, k-1)$, then $\mathcal{F}$ cannot be a k-uniform intersecting family.

Solution: **(1.1)** We will prove by construction. Take out one element (say x) out of the set S. Now S has $n-1$ elements remaining. Let A be a set having $k-1$ elements, each belonging to $S\backslash\{x\}$. There are $\binom{n-1}{k-1}$ ways to form A. Let the collection of all these sets be the set $\mathcal{F}$. Thus $|F| = \binom{n-1}{k-1}$. Now add the element x to all the sets that are elements of $\mathcal{F}$. Thus all the elements of $\mathcal{F}$ have a common element x and each has cardinality k $\blacksquare$.

(1.2) We are given that f is a bijection from the set S to the set $\{0, 1, , 2, ...n-1\}$, such that the elements of the set A ($|A| = k$) have their mappings in the set $\{0, 1, , 2, ...n-1\}$ consecutively .If the elements of A reaches the end i.e, $n-1$ then the next elements mapping should be 0 for consecutiveness according to the definition.

We first partition the elements of S in two parts, A and $S\backslash A$. Since f maps the elements of A in $\{0, 1, , 2, ...n-1\}$ in consecutive order, let us define the *first element* as the $x_0 \in \{0, 1, , 2, ...n-1\}$ such that an element of A is mapped to x_0 and the element of S mapped to the number consecutively to it's left ($n-1$ if the first element is 0) is **not** in A. (For example, if 0 is the first element then the elements of A will be mapped from 0 to $k-1$, if 1 is the first element then elements of A are mapped from 1 to k and so on.) Thus, there are total of n such possible values for first element. Once the first element is chosen there are $k!$ ways to arrange the elements of A and $(n-k)!$ ways to arrange elements of $S\backslash A$.

Therefore, using fundamental principle of multiplication, the total number of ways to do this is $n(n-k)!k!$.

(1.3) Let us suppose that $\mathcal{F}$ is a family of k-sized subsets of S and $|\mathcal{F}| > \binom{n-1}{k-1}$. We arrange the elements of S on a circle having distinct n positions. The number of ways to do this is $n!$. We suppose, to the contrary that a maximum of k of the sets in F appear consecutively over any circle. Let there are m number of sets in $\mathcal{F}$. Also from the previous proof **(1.2)**, we say that for any k-sized set there are a total of $n(n-k)!k!$ number of permutations of the k-sized set over any circle.

Since there are a total of m sets in $\mathcal{F}$ we have $m * n(n-k)!k!$ number of permutations of elements of $\mathcal{F}$ possible such that at least one of the set in $\mathcal{F}$ is consecutive over the circle. Also from our assumption for any particular permutation over the circle there are at most k consecutive appearing sets. Since the total number of permutations possible are $n!$, we get $kn!$ number of permutations in which at least one set in $\mathcal{F}$ is appearing consecutive over the circle. This number should be less than what we previously got (since this is the maximum value). Thus

$$kn! \geq mnk!(n-k)!$$

This implies $m \leq \binom{n-1}{k-1}$ which is contrary to our previous assumption that $|\mathcal{F}| > \binom{n-1}{k-1}$ and hence we arrive at a contradiction. Thus there exists a permutation in which more than k sets in $\mathcal{F}$ are consecutive over the circle.

(1.4) Let $\mathcal{F}$ be the k-uniform intersecting family with $|\mathcal{F}| > \binom{n-1}{k-1}$.

Claim: Any bijection, say f from S to $P = \{0, 1, 2, ...n-1\}$ can have at most k sets of $\mathcal{F}$ appearing consecutively.

Proof: Let us suppose any set $X = \{a_1, a_2, ...a_k\}$ have its mappings in f consecutive in the set P. Note that $i, i+1, ...n-1, 0, 1, ...$ is also considered as consecutive according to the definition. We define the **leftmost mapping** of any set T that has its elements mapped consecutive over P as the leftmost number in P to which it's elements are mapped. Suppose if the elements in T are mapped to 3,4,....3+k mod n . Then 3 is the leftmost mapping and so on. Now we define **first set** as the set such that its leftmost mapping in P is such that there is not set in F that appears consecutively on P and mapping of any element of that set is immediate left of the leftmost mapping of the first set.

<u>Proof of existence of such a first set</u> :

Let us say that any set $X \in \mathcal{F}$ is the first set. So $X = \{a_1, a_2, ...a_k\}$. W.l.o.g we assume that a_1 is the leftmost mapping of X in P. (Note that left of 0 is n-1)
Now say that $a \in P = f(a_1)$. Let Y be a set, Y $\in$ F such that Y has an element that maps to $a - 1$ mod n (element to the immediate left of a) and elements of Y maps consecutively in P .Two cases arises here

Case 1: If $a - 1$ is the rightmost mapping of elements of Y then Y will never have any mapping common to X since X and Y both have k elements and n $\geq$ 2*k. This is not possible since $\mathcal{F}$ is a k-uniform intersecting family.

Case 2: If $a - 1$ is not the rightmost mapping of Y then X is not the first set. Hence out assumption was wrong and now we assume Y is the first set. Since total sets in $\mathcal{F}$ is finite, repeating this procedure we eventually reach a first set. This proves the existence of such a first set.

Now since a is the leftmost mapping and every other mapping should have something common with $\{a, a+1 \text{ mod } n, ...a+k \text{ mod } n\}$, therefore we can have $k-1$ sets each having leftmost mapping starting from $a, a+1$ mod $n, ...a+k$ mod n. Thus we have at most $k-1$ such sets, and one set is X itself. Thus at most k sets in $\mathcal{F}$ can have consecutive mapping over P for any bijection f.

However in part 1.3 it has been proved that if $\mathcal{F}$ is a set of k-size sets and $|\mathcal{F}| > \binom{n-1}{k-1}$ then there exist a bijection f such that more that k of the sets in $\mathcal{F}$ appear consecutively. Thus we arrive at a contradiction here and thus there can be no k-uniform intersecting family $\mathcal{F}$ with $|F| > \binom{n-1}{k-1}$.

Problem 2
Consider the poset $(2^S, \subseteq)$, where $S = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$. A non-empty chain $\{A_1, A_2, \ldots, A_k\}$ of this poset, where $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k$, is said to be a *symmetric chain* if $|A_1| + |A_k| = n$ and $|A_{i+1}| = |A_i| + 1$ for each $i = 1, \ldots, k-1$.

1. **[2 points]** Prove that the set 2^S can be partitioned into symmetric chains. (Hint: Induction on n.)

2. **[2 points]** Using the above result, find the size of the largest antichain in 2^S as a function of n, and prove your answer.

Solution: **(2.1)** We wish to prove that 2^S can be partitioned into symmetric chains. We will prove this claim by induction on n.

Base Case: $n = 1$. The set $S = \{1\}$ and so $2^S = \{\phi, \{1\}\}$. We have, 2^S itself is a symmetric chain, since $|A_1| + |A_2| = 0 + 1 = 1$ and $|A_2| = |A_1| + 1 = 1$ as required.

Induction Hypothesis: Suppose the claim is true for some n.

Induction Step: Now, when $S = \{1, 2, ... \ n + 1\}$. We know that $2^{\{1,2,...n\}}$ can be partitioned into symmetric chains. We will construct a partition of 2^S into symmetric chains from this partition $\mathcal{P}$ of $2^{\{1,2,...n\}}$.

Consider any chain $C_i \in \mathcal{P}$. Suppose that $C_i = \{A_1, A_2, ...A_k\}$. Then we will construct two symmetric chains C_{1i} and C_{2i} from C as follows:

- $C_{1i} = \{A_1, A_2, ..., A_{k-1}, A_k, A_k \cup \{n + 1\}\}$
- $C_{2i} = \{A_1 \cup \{n + 1\}, A_2 \cup \{n + 1\}, ..., A_{k-1} \cup \{n + 1\}\}$, $k \geq 2$.

We have, C_{1i} is symmetric, since $|A_1| + |A_{k+1}| = |A_1| + |A_k| + 1 = n + 1$ by induction hypothesis, and $|A_{i+1}| = |A_i| + 1 \ \forall \ i \in [1, k - 1]$.

Similarly, C_{2i} is symmetric since $|A_1| + |A_{k-1}| = |A_1| + 1 + |A_{k-1}| + 1 = |A_1| + 2 + |A_k| - 1 = n + 1$, and again $|A_{i+1}| = |A_i| + 1 \ \forall \ i \in [1, k - 1]$.

We now want to prove $\mathcal{P}' = \{C_{1i}, C_{2i} \mid 1 \leq i \leq n\}$ is a partition. Note that,

1. Since $\mathcal{P}$ is a partition of $\{1, 2, ...n\}$, $C_i \neq \phi$ for any i. By our construction, $C_{1i} \neq \phi$, and $C_{2i} \neq \phi$ since we only construct it for $k \geq 2$. So $\phi \notin \mathcal{P}'$.

2. We have, $\bigcup_{1 \leq i \leq |\mathcal{P}|} C_i = 2^{\{1,2,...n\}}$. By our construction, for any $A \in C_i$, $A \in C_{1i} \cup C_{2i}$ and $A \cup \{n+1\} \in C_{1i} \cup C_{2i}$, and therefore,
$$\bigcup_{1 \leq i \leq |P|} C_{1i} \cup C_{2i} = 2^S$$

3. We clearly have $C_{1i} \cap C_{2i} = \phi$ by our construction. Also, since $\mathcal{P}$ is a partition of $2^{\{1,2,...n\}}$,$i \neq j \implies C_i \cap C_j = \phi$ and so $C_{ai} \cap C_{bj} = \phi \ \ \forall \ a, b = 1, 2$ and $1 \leq i < j \leq n$, since for any set $A \in C_i$, neither A nor $A \cup \{n + 1\}$ lies in any other C_j, $i \neq j$.

Thus, $\mathcal{P}'$ is the required partition of 2^S into symmetric chains. ■

(2.2) According to Dilworth's Theorem, for a finite poset $(\mathcal{S}, R)$, the size of a largest antichain of $\mathcal{S}$ equals the minimum number of chains into which $\mathcal{S}$ can be partitioned. Let the size of the largest antichain of 2^S be N. First, we will try to find the number of symmetric chains which 2^S can be partitioned into.

Consider any chain $\{A_1, A_2, ...A_k\}$ belonging to the partition $\mathcal{P}$ of 2^S into symmetric chains. We know that $|A_1| + |A_k| = n$ and $|A_{i+1}| = |A_i| + 1 \ \forall i$. From here, we can get that $|A_1| = \frac{n-k+1}{2}$ and $|A_k| = \frac{n+k-1}{2}$. Since $k \geq 1$, $\frac{n-k+1}{2} \leq \lfloor \frac{n}{2} \rfloor \leq \frac{n+k-1}{2}$. Therefore, $\exists \ m \in [1, k]$ such that $|A_m| = \lfloor \frac{n}{2} \rfloor$. Therefore each chain contains a distinct subset of 2^S of size $\lfloor \frac{n}{2} \rfloor$, and so there must be exactly $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ chains in the partition $\mathcal{P}$.

Since N is the *minimum* number of chains into which 2^S can be partitioned, we must have $N \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. If we prove we can create an antichain of 2^S of length $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ then we are done. For this purpose, we can simply consider the set B of all subsets of S of size $\lfloor \frac{n}{2} \rfloor$. Then for any distinct $X, Y \in B$, since $|X| = |Y|$, $X \subseteq Y \implies X = Y$, which is a contradiction. So B is an antichain, and $|B| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Thus, we have that the size of the largest antichain of 2^S is given by,

$$N = \binom{n}{\lfloor \frac{n}{2} \rfloor}$$