COL751 - Lecture 24

1 Computing a Tutte-Berge Maximizer

In Lecture 23, we proved that for any set R, $def(G) \geqslant oc(G \setminus R) - |R|$.

We will now show existence of set R that satisfies $def(G) = oc(G \setminus R) - |R|$. Such a set is referred as Tutte–Berge maximizer in G. Existence of such set will prove the following theorem.

Theorem 1 (Tutte-Berge Formula)**.** *For any graph $G = (V, E)$, we have*

$$def(G) \;=\; \max_{R \subseteq V}\; oc(G \setminus R) - |R|$$

Let M be any maximum matching of G, $S = (x_1, \ldots, x_k)$ be set of free vertices under M, and T be an M-*alternating tree* rooted at S. Let $V_{odd}(T)$ and $v_{even}(T)$ be respectively the vertices at odd and even depth in tree T. Further, let $V_{not}(T)$ be those vertices in G that do not lie in T.

$S \leftarrow$ Vertices in V free under M;
$T \leftarrow$ an M-alternating tree rooted at S;
while $\exists (y, z) \in V_{even}(T) \times V_{even}(T)$ **do**
    Contract $\textsc{treepath}\big(LCA(y,z), y\big) \circ (y, z) \circ \textsc{treepath}\big(LCA(y,z), y\big)$ in G, and update tree T;
end
Return the set of vertices in G lying in $V_{odd}(T)$;

Algorithm 1: Tutte-Berge-Maximizer(G, M)

Lemma 1. *Let T_0 be any intermediate tree obtained after zero or more iterations of While loop in Algorithm 1.*

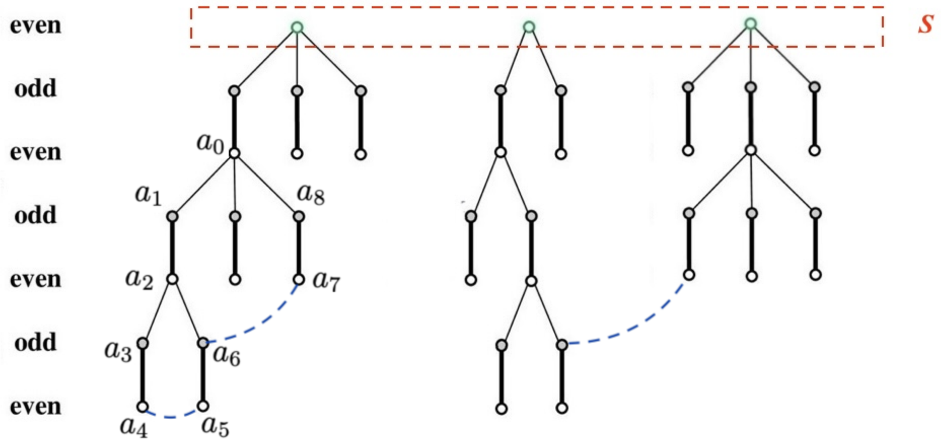
(i) *Each supernode $W \in V_{odd}(T_0)$ has size one.*

(ii) *Each supernode $W \in V_{even}(T_0)$ has odd size.*

(iii) *For each supernode $W \in V_{even}(T_0)$ and each $v \in W$, there is an alternating path starting from s that terminates to v using a matched edge.*

(iv) *For any two supernodes $W_y, W_z \in V_{even}(T_0)$ and each edge $(y, z) \in W_y \times W_z$, $root(y) = root(z)$.*

Proof: Homework. (Use induction on the number of intermediate trees computed). $\square$



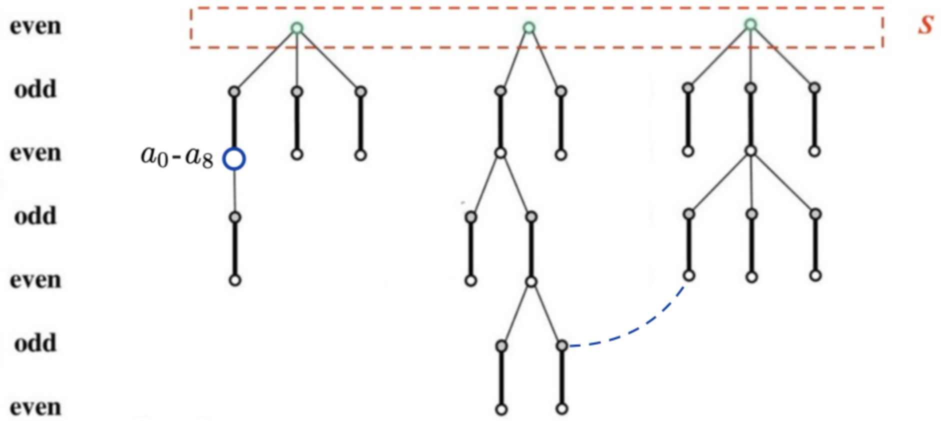


Figure 1: Alternating tree T_{final} obtained after performing all valid cycle contractions. The set R consists of nodes in the odd level.

Lemma 2. *The output of Algorithm 1 is a Tutte-Berge Maximizer.*

Proof: Let T_{final} be the tree obtained after all iterations of While loop in Algorithm 1. Then following holds:

- By Lemma 1 (i - ii), we have that each node $W \in T_{final}$ corresponds to a subset of V of odd size such that $G[W]$ is connected.
- By Lemma 1 (iv), we have that induced subgraph $G[V_{even}(T_{final})]$ has zero edges.
- Root of T_{final} has size $|S|$ as by Lemma 1 (iv) no contraction occurs across trees.

On setting $R = \bigcup_{W \in V_{odd}(T_{final})} W$, we get

$$oc(G \setminus R) = \{G[W] \mid W \in V_{even}(T_{final})\}.$$

This is because induced graph $G[V_{not}(T_{final})]$ has no odd-sized components and is disconnected from the remaining vertices in $G \setminus R$. Therefore, $oc(G \setminus R) = |R| + |S|$. This proves that $def(G) = |S| = oc(G \setminus R) - |R|$. $\square$

Remark If G has a perfect matching, then the above described approach cannot be used as there are no free vertices. However, we can simply take set $R = \emptyset$ as a Tutte-Berge Maximizer in G.

We thus get the following theorem.

Theorem 2. *Given a maximum matching M of an n vertex, m edges graph $G = (V, E)$ we can compute a Tutte-Berge maximizer in $O(mn)$ time.*

2 Applications of Tutte–Berge Formulae

2.1 Maximum matching certificate

If a graph $G = (V, E)$ has a maximum matching M then a Tutte-Berge maximizer R would satisfy that $n - 2|M| = oc(G \setminus R) - |R|$. Thus a Tutte-Berge maximizer can be used as a certificate to verify that a given matching is maximum.

2.2 Perfect matching

The following characterization for graphs having perfect matching follows as a corollary of Tutte-Berge Formula stated in Theorem 1.

Theorem 3 (Tutte's theorem)**.** *An undirected graph $G = (V, E)$ has a perfect matching if and only if for every set $R \subseteq V$ we have $|R| \geqslant oc(G \setminus R)$.*

2.3 Hall's theorem

We will provide here an alternate proof of Hall's theorem that follows from Tutte's theorem.

Theorem 4 (Hall's theorem)**.** *Let $G = (X, Y, E)$ be a bipartite graph satisfying $|X| = |Y|$. Then G has a perfect matching if and only if for all $S \subseteq X$, $|NS)| \geqslant |S|$.*

Proof: It is easy to verify that if G has a perfect matching then it satisfies the Hall's condition. To prove the converse let us assume G satisfies the Hall's condition. Let $H = (X \cup Y, E_H)$ be a graph obtained from G by connecting every pair of vertices in Y. Observe that G has a perfect matching if and only if H has a perfect matching. So in to prove that G has a perfect matching it suffices to prove that H satisfies Tutte's condition.

Consider a set $R \subseteq V$. Let S be a subset of those vertices $v \in X \setminus R$ for which $N(v) \subseteq R$. As Hall's condition is satisfied for G, we have:

$$|S| \leqslant |N(S)| \leqslant |R|. \tag{1}$$

The above equation can be further strengthened as:

$$1_{parity(S) \neq parity(R)} + |S| \leqslant |N(S)| \leqslant |R|, \tag{2}$$

where, two sets have same parity iff either both have odd size or both have even size.

Also, observe that

$$\begin{aligned} oc(G-R) &= |S| + 1_{\text{Graph } G[(X\cup Y)-(S\cup R)] \text{ has odd number of vertices}} \quad (3) \\ &= |S| + 1_{parity(S)\neq parity(R)} \quad (4) \end{aligned}$$

On combining Eq. 2 and Eq. 4 we get that H satisfies Tutte's condition. Thus graph H has a perfect matching, thereby implying that G has a perfect matching. □