Hints to problem sheet 10

Ques 1.

a) If the signal $x(t)$ has a Nyquist rate of w_0, then its Fourier Transform $X(jw) = 0$ for $|w|>w_0/2$.

$$Y(t) = x(t) + x(t-1) \overset{FT}{\leftrightarrow} Y(jw) = X(jw) + e^{-jwt}X(jw)$$

Clearly one can guarantee that $Y(jw) = 0$ for $|w| > w_0/2$. Therefore the Nyquist rate for $y(t)$ is also w_0.

b) $y(t)= dx(t)/dt$

$$Y(jw) = jw.X(jw)$$

Clearly one can guarantee that $Y(jw) = 0$ for $|w| > w_0/2$. Therefore the Nyquist rate for $y(t)$ is also w_0.

c) $y(t) = x^2(t)$

$$Y(jw) = (1/2\pi)[X(jw) * X(jw)]$$

Clearly one can guarantee that $Y(jw) = 0$ for $|w| > w_0$. Therefore the Nyquist rate for $y(t)$ is $2w_0$.

d) $y(t) = x(t).\cos w_0 t$

$$Y(jw) = (1/2)X[j(w - w_0)] + (1/2)X[j(w + w_0)]$$

Clearly one can guarantee that $Y(jw) = 0$ for $|w| > w_0 + w_0/2$. Therefore the Nyquist rate for $y(t)$ is $3w_0$.

Ques 2.

$$p(t) \overset{FT}{\leftrightarrow} 2\pi/T \sum_{k=-\infty}^{\infty} \delta(\omega - k\, 2\pi/T)$$

Further,

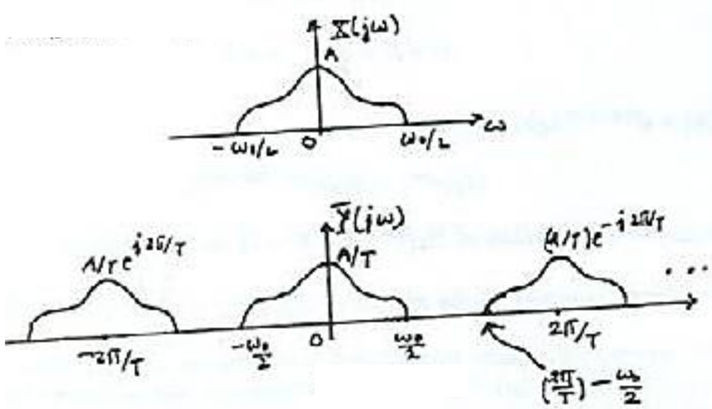
$$p(t-1) \overset{FT}{\leftrightarrow} \frac{2\pi}{T}.e^{-j\omega} \sum_{k=-\infty}^{\infty} \delta(\omega - k\, 2\pi/T) = 2\pi/T \sum_{k=-\infty}^{\infty} \delta(\omega - k\, 2\pi/T)e^{-jk2\pi/T}$$

Since, $y(t) = x(t)p(t-1)$, we have

$$Y(j\omega) = \frac{1}{2\pi}[X(j\omega) * FT\{p(t-1)\}]$$

$$= (1/T)\sum_{k=-\infty}^{\infty} X(j(\omega - k\,2\pi/T))e^{-jk2\pi/T}$$

Therefore, $Y(jw)$ consists of replicas of $X(jw)$ shifted by $k.2\pi/T$ and added to each other. In order to recover $x(t)$ from $y(t)$ we need to be able to get replica of $X(jw)$ from $Y(jw)$.



From the figure, it is clear that this is possible if we multiply $Y(jw)$ with

$$H(jw) = \begin{cases} T, & |w| < w_c \\ 0, & \text{otherwise} \end{cases}$$

where, $(w_0/2) < w < (2\pi/T) - (w_0/2)$

Ques 3.

We know that

$$X_d(e^{j\omega}) \;=\; 1/T\sum_{k=-\infty}^{\infty} Xc(j(\omega - 2\pi k)/T)$$

a) Since $X_d(e^{jw})$ is just formed shifting and summing replicas of $X(jw)$ we may assume that if $X_d(e^{jw})$ is real, then $X(jw)$ must also be real.
b) $X_d(e^{jw})$ consists of replicas of $X(jw)$ which are scaled by $1/T$. Therefore if $X_d(e^{jw})$ has a maximum of 1,then $X(jw)$ will have a maximum of $T = 0.5 \times 10^{-3}$.
c) The region $3\pi/4 \leq |w| \leq \pi$ in the discrete domain corresponds to the region $3\pi/(4T) \leq |w| \leq \pi/T$ in the continuous time domain. Therefore if $X_d(e^{jw}) = 0$ for $3\pi/4 \leq |w| \leq \pi$, then $X(jw) = 0$ for $1500\pi \leq |w| \leq 2000\pi$. But since we already know that $X(jw) = 0$ for $|w| \geq 2000\pi$, we have $X(jw) = 0$ for $|w| \geq 1500\pi$.

d) In this case, since x in discrete time frequency domain corresponds to 2000π in the continuous time frequency domain, this condition translates to $X(jw) = (j(w\text{-}2000\pi))$.

Ques 4.

We may express $p(t)$ as $p_1(t) = p_1(t) - p_1(t\text{-}\Delta)$

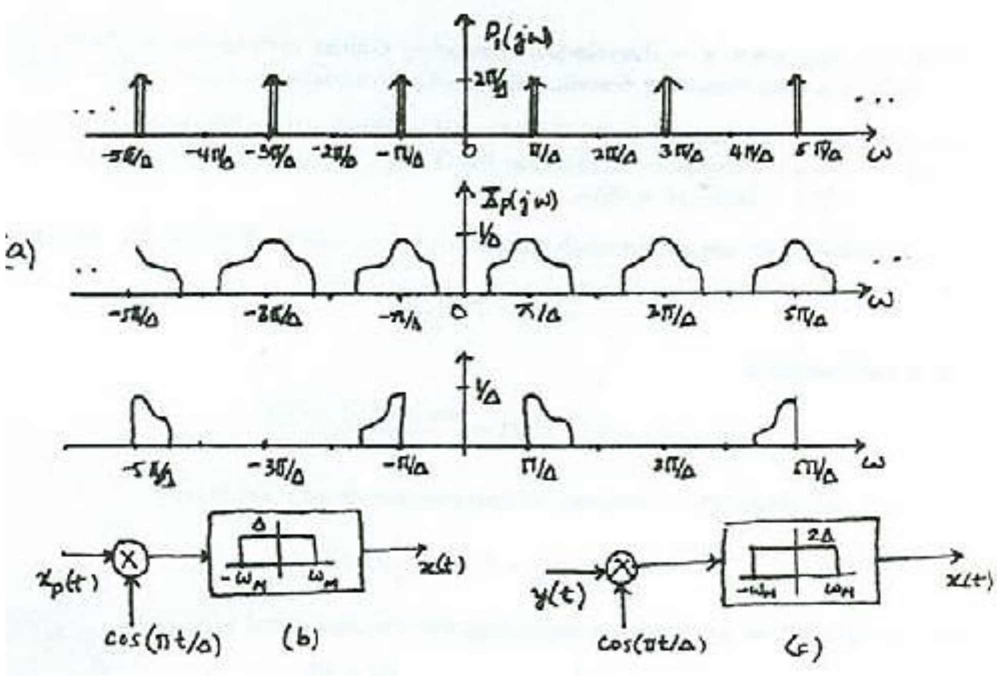
$$\text{where } p_1(t) = \sum_{k=-\infty}^{\infty} \delta(t - k2\Delta)$$

Now,

$$P_1(j\omega) = \pi/\Delta \sum_{k=-\infty}^{\infty} \delta(\omega - \pi/\Delta)$$

Therefore,

$$P(j\omega) = P_1(j\omega) - e^{-j\omega\Delta} P_1(j\omega)$$



Now,

$$Xp(j\omega) = 1/2\pi[X(jw) * P(jw)]$$

Therefore $X_p(jw)$ is as sketched above for $\Delta < \pi/2w_M$. The corresponding $Y(jw)$ is also sketched in the above figure.

b) The system which can be used to recover $x(t)$ from $x_p(t)$ is as shown in the figure.

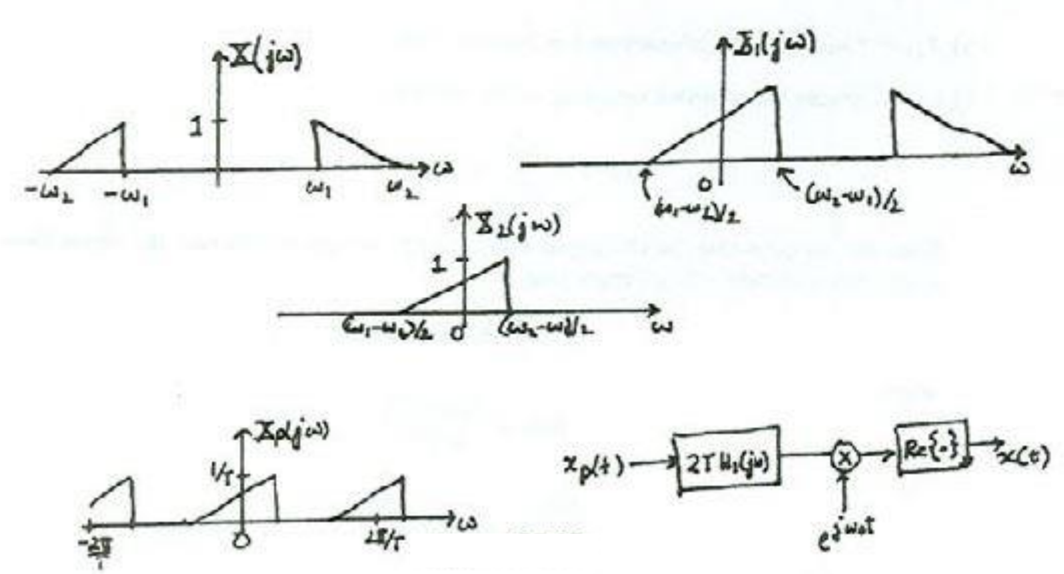
c) The system which can be used to recover $x(t)$ from $x(t)$ is as shown in the figure.

d) We see from the figures sketched in part (a) that aliasing is avoided when $w_M < \pi/\Delta$. Therefore $\Delta_{\max} = \pi/w_M$.

Ques 5.

a) Let $X_1(\mathrm{j}w)$ denote the Fourier Transform of the signal $x_1(t)$ obtained by multiplying $x(t)$ with e^{-jwt} . Let $X_2(\mathrm{j}w)$ be the fourier transform of the signal $x_2(t)$ obtained at the output of the low pass filter. Then $X_1(\mathrm{j}w)$, $X_2(\mathrm{j}w)$ and $X_p(\mathrm{j}w)$ are as shown in the figure below.

b) The Nyquist rate for the signal $x_2(t)$ is $2\times(w_2 - w_1)/2 = w_2 - w_1$. Therefore the sampling period T must be at most $2\pi/(w_2 - w_1)$ in order to avoid aliasing.



c) A system can be used to recover $x(t)$ from $x_p(t)$ is shown in the above figure.