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Department of Mathematics

MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Answer to **Major Examination**

Time: 100 minutes **Max. Marks: 50** **Date: 15/11/2021**

Name: **Entry Number:** **Signature:**

Note: The exam is closed-book, and all the questions are compulsory.

1. **The following questions can have multiple correct answers. The marks will be awarded only if you write all correct answers.** ($5 \times 2 = 10$ marks)

   (a) Let X be an integer-valued random variable having distribution function F, and let Y follows $U(0,1)$. Define, a random variable Z in terms of Y by

   $$Z = n, \quad \text{if and only if } \; F(n-1) < Y \leq F(n),$$

   for any integer n. Then, which of the following statements are TRUE?
   (A) Z is a discrete type random variable (B) Probability distribution of Z and X are same (C) Z follows $U(0,1)$ (D) Z is a continuous type random variable Answer:

   (b) Suppose $X_i, i = 1, 2, \ldots$ are mutually independent random variables such that

   $$P(X_n = 0) = 1 - P(X_n = 1) = 1 - n^{-1} \text{ for } n = 1, 2, \ldots.$$

   Then, which of the following statements are TRUE?
   (A) $EX_n = 0$, for all n (B) $X_n \to 0$ in probability for $n \to \infty$. (C) $X_n \to 0$ a.s. for $n \to \infty$. (D) $X_n \to 0$ in distribution for $n \to \infty$. Answer:

   (c) Consider $\{N(t), t \geq 0\}$ be a Poisson process with parameter λ. Then, which of the following statements are TRUE?
   (A) for $s < t$, $cov(N(s), N(t)) = \lambda s$ (B) for $s < t$, $cov(N(s), N(t) = \lambda t$
   (C) for $s < t$, $cov(N(s), N(t)) = \lambda st$ (D) for $s > t$, $cov(N(s), N(t)) = \lambda t$ Answer:

   (d) Consider a CTMC $\{X(t), t \geq 0\}$ with state space $\{0, 1\}$ and infinitedecimal generator matrix $\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix}$. Then, which of the following statements are TRUE?
   (A) The limiting distribution is $(\frac{3}{5}, \frac{2}{5})$ (B) The limiting distribution is $(\frac{2}{5}, \frac{3}{5})$ (C) Suppose, the initial distribution is $(1, 0)$, then time-dependent distribution is $(\frac{3}{5} + \frac{2}{5}e^{-5t}, \frac{2}{5} - \frac{2}{5}e^{-5t})$ (D) Suppose, the initial distribution is $(0, 1)$, then time-dependent distribution is $(\frac{3}{5} - \frac{3}{5}e^{-5t}, \frac{2}{5} + \frac{3}{5}e^{-5t})$ Answer:

   (e) City Hospital's eye clinic offers free vision tests every Wednesday evening. There are three ophthalmologists on duty. A test takes, on the average, 20 min, and the actual time is found to be approximately exponentially distributed around this average. Clients arrive according to a Poisson process with a mean of 6/hr, and patients are taken on a first-come, first-served basis. Then, which of the following statements are TRUE?
   (A) This system can be modeled as $M/M/3/\infty$ queueing system (B) This system can be modeled as $M/M/1/3$ queueing system (C) This queueing system is a stable system
   (D) This queueing system is a not stable system Answer:

Rough Work

$=80$

$=14.92$

2. **Give the numerical answer to the following questions. The justification of the answers is not required. The step marking is not applicable in these questions.**

(a) Suppose that the weight (in kg) of a person selected at random from some population is normally distributed with mean μ and standard deviation σ. Suppose also that $P(X \leq 80) = \frac{1}{2}$ and $P(X \leq 70) = \frac{1}{4}$. Find μ and σ. What is the probability that the weight of a randomly selected person lie between 80 kg and 90 kg? $= 0.2486$ (0.5 + 0.5 + 1 marks)

Answer:

(b) Suppose X and Y are two independent random variables such that $E(X^4) = 2$, $E(Y^2) = 1$, $E(X^2) = 1$, and $E(Y) = 0$. Compute $\text{Var}(X^2Y)$. $= 2$ (2 marks)

Answer:

(c) The number of customers who visit SBI, IIT Delhi on a Saturday is a random variable with $\mu = 80$ and $\sigma = 5$. Using Chebyshev's inequality, find the lower bound for the probability that there will be more than 55 but fewer than 105 customers in the bank? $= 0.96$ (2 marks)

Answer:

(d) Suppose, you get email according to a Poisson process with $\lambda = 0.4$ messages per hour. You check your email every hour. What is the probability of finding 1 new message? (2 marks)

Answer: $= 0.2682$

(e) Consider a manual car wash station with three bays and no room for waiting (assume that cars that arrive when all the three bays are full leave without getting washed there). Cars arrive according to a Poisson process at an average rate of 3 per minute but not all cars enter. It is known that the long-run fraction of time there were 0, 1, 2, or 3 bays full are 6/15, 5/15, 3/15, and 1/15, respectively. What is the the mean rate in the system for cars that actually "enter" the system (and it does not include cars that were turned away)? (2 marks)

Answer: 2.8

Rough Work

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(a) $P(75 \leq x \leq 85) = 0.258$

(c) 0.96

(d) 0.3032

(e) 3.777

2 (a) $P(90 \leq x \leq 100) = 0.1613$

(c) 0.96

(d) 0.2222

(e) 1.85

2 (a) $P(85 \leq x \leq 95) = 0.212$

(c) 0.96

(d) 0.3292

(e) 0.9375

The following four questions are descriptive type. Detailed answers is required. The step marking is applicable in these following four questions.

3. Let $\{X_t, t \geq 0\}$ be a stochastic process with $X_t = \cos(2\pi At + \alpha)$ where A is a constant and α follows $U(-\pi, \pi)$. Prove that, $\{X_t, t \geq 0\}$ is a wide-sense stationary process. If Y_t is a delayed version of X_t, say $Y_t := X_{t-c}$ where c is a positive constant, determine whether or not $\{Y_t, t \geq 0\}$ is wide-sense stationary process. (3 + 3 marks)

Solution:

$E(X_t) = 0$

$E(X_t^2) = \frac{1}{2}$

$\mathrm{cov}(X_t, X_s) = \frac{1}{2} \cos(2\pi A(t-s))$

$\therefore \{X_t, t \geq 0\}$ is a WSS

$E(Y_t) = 0$

$E(Y_t^2) = \frac{1}{2}$

$\mathrm{cov}(Y_t, Y_s) = \frac{1}{2} \cos(2\pi A(t-s))$

$\therefore \{Y_t, t \geq 0\}$ is also a WSS

(b) $P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $E(\ldots) = 0.4$

(c) $P(\ldots) = 0.6.$

(b) $P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $E(\ldots) = 0.5$

(c) $P(\ldots) = 0.5$

(b) $P =$ " $E(\ldots) = 0.5$

(c) $P(\ldots) = 0.5$

4. Consider a DTMC $\{X_n, n = 0, 1, \ldots\}$ with state space $S = \{0, 1, 2, 3, 4\}$ and one-step transition probability matrix $P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0.6 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

(a) Classify the states of the chain as transient, +ve recurrent or null recurrent.

(b) When $P(X_0 = 1) = 1$, find the expected number of times the Markov chain visit state 3 before being absorbed. $= 0.6$

(c) When $P(X_0 = 2) = 1$, find the probability that the Markov chain absorbs in state 4. $= 0.4$

(3 + 3 + 3 marks)

Solution:

$f_{00} = 1$; $f_{44} = 1$

$f_{11} < 1$; $f_{22} < 1$; $f_{33} < 1$

$\mu_{00} = 1$; $\mu_{44} = 1$

$\therefore \{0, 4\}$ – +ve rec ; $\{1, 2, 3\}$ – transient

5. Consider a taxi station where taxis and customers arrive independently in accordance with Poisson processes with respective rates of 2 and 6 per minute. A taxi will wait no matter how many other taxis are in the system. However, an arriving customer that does not find a taxi waiting leaves. Note that a taxi can accommodate only one customer. Let $\{X(t), t \geq 0\}$ denote the number of taxis waiting for customers at time t.

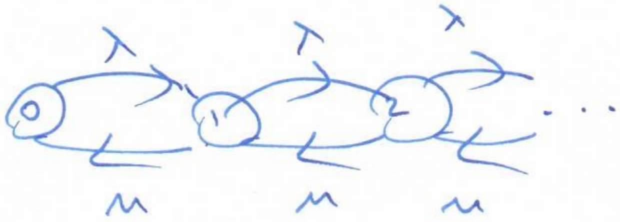
   (a) Draw the state transition diagram for this process.

   (b) Find the average time a taxi waits in the system.

   (c) Find the proportion of arriving customers that get taxis. (2 + 3 + 3 marks)

Solution:

(a)



$\lambda = 2;\ \mu = 6$

$\lambda = 2;\ \mu = 4$

$\lambda = 3;\ \mu = 6$

$\lambda = 1;\ \mu = 3$

(b)
$$E(Q) = \frac{L_q}{\lambda} = \frac{\rho^2/(1-\rho)}{\lambda} = \begin{cases} \frac{1}{12}; & \lambda = 2;\ \mu = 6 \\ \frac{1}{4}; & \lambda = 2;\ \mu = 4 \\ \frac{1}{6}; & \lambda = 3;\ \mu = 6 \\ \frac{1}{6}; & \lambda = 1;\ \mu = 3 \end{cases}$$

(c)
$$= 1 - (1 - \rho) = \rho = \frac{\lambda}{\mu} = \begin{cases} \frac{1}{3}; & \lambda = 2;\ \mu = 6 \\ \frac{1}{2}; & \lambda = 2;\ \mu = 4 \\ \frac{1}{2}; & \lambda = 3;\ \mu = 6 \\ \frac{1}{3}; & \lambda = 1;\ \mu = 3 \end{cases}$$

(c) $\frac{L_q}{\lambda} + \frac{1}{\mu} = \begin{cases} 0.033351 ; \lambda = 20 ; \mu = 30 \\ 0.0335 ; \lambda = 30 ; \mu = 30 \\ 0.05034 ; \lambda = 20 ; \mu = 20 \\ \cancel{0.05704} ; \lambda = 30 ; \mu = 20 \\ 0.05149 \end{cases}$

6. New Delhi International Airport with landing of flights is under consideration. Suppose that, it has four runway. Airplanes have been found to arrivals at the rate of 20 per hour. It is estimated that each landing takes 2 minutes. Assume that a Poisson process for arrivals and an exponential distribution for landing times. Without loss of generality, assume that the system is modeled as a birth and death process.

   (a) What is the steady state probability that the no waiting time to land?

   (b) What is the expected number of airplanes waiting to land?

   (c) Find the expected total time to land?

(3 + 2 + 2 marks)

Solution:

(a) $\pi_0 + \pi_1 + \pi_2 + \pi_3 = \begin{cases} \frac{785}{789} ; \lambda = 20 ; \mu = 30 \\ \frac{48}{49} ; \lambda = 30, \mu = 30 \\ \frac{48}{49} ; \lambda = 20 ; \mu = 20 \\ \frac{335}{362} ; \lambda = 30 ; \mu = 20 \end{cases}$

$\pi_n = \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \pi_0 & ; 1 \le n \le 4 \\ \frac{1}{4^{n-4}} \frac{1}{4!}\left(\frac{\lambda}{\mu}\right)^n \pi_0 & n > 4 \end{cases}$

$\pi_0 = \left\{ \sum_{n=0}^{3} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{4!}\left(\frac{\lambda}{\mu}\right)^4 \left(\frac{4\mu}{4\mu - \lambda}\right) \right\}^{-1}$

(b) $L_q = \sum_{n=5}^{\infty} (n-4)\pi_n = \begin{cases} 0.00101 ; \lambda = 20 ; \mu = 30 \\ \frac{1}{147} ; \lambda = 30 ; \mu = 30 \\ \frac{1}{147} ; \lambda = 20 ; \mu = 20 \\ \frac{81}{1810} ; \lambda = 30 ; \mu = 20 \end{cases}$

$= \frac{\rho}{4!(1-\rho)^2}\left(\frac{\lambda}{\mu}\right)^4 \pi_0$

$\rho = \frac{\lambda}{4\mu}$;

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