Name: ____________________ Roll No: ______________

(COL 202) Discrete Mathematics 13 September, 2023

Minor 1

Duration: 120 minutes **(50 points)**

Beware: Be clear in your writing. If you use a statement proved in class or in the problem set, then write down the entire statement before using it. You will not get a new sheet, so make sure you are certain when you write something (maybe use a dark pencil). Make a judicious decision of which tool(s) to use to get a clean and short answer that fits in the space. If you cheat, you will surely get an F in this course.

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1. **($4 \times 3 = 12$ points)** In this question, each sub-question will have zero or more correct answers. You are to circle each correct answer and leave uncircled each incorrect answer. Each problem is worth 3 points and you get points if and only if you circle all of the correct answers and none of the wrong ones. There are no partial points.

   (a) Let w, b and n be propositions where w is "I walk to work", g is "I work in Gurgugram", n is "I work at night". The sentence "When I work nights and I work in Gurgugram, I don't walk to work" could be written using propositions and logical connectives as:

   (1) $(n \wedge g) \implies \neg w$ (2) $(n \vee g) \iff n$ (3) $n \implies \neg(w \wedge g)$ (4) $\neg(w \wedge g) \vee n$

   (b) Identify the *tautologies* among the following:

   (1) $(a \implies b) \iff (\neg a \implies \neg b)$ (2) $(a \implies b) \iff (\neg b \implies \neg a)$
   (3) $(a \implies b) \implies a$ (4) $(a \wedge b \wedge c) \iff (b \wedge c \wedge a)$

   (c) Identify those formulae which are *satisfiable*.

   (1) $(a \vee b) \wedge (a \vee \neg b) \wedge (\neg a \vee b) \wedge (\neg a \vee \neg b)$ (2) $(a \wedge b) \wedge (a \wedge \neg b)$
   (3) $(a \implies b) \implies (\neg b \implies \neg a)$ (4) $(a \wedge b) \implies (a \wedge \neg b)$

   (d) For countably infinite sets A and B, $A \cap B$ can be

   (1) Countably infinite (2) Uncountable (3) Finite (4) Empty

2. **($3 \times 4 = 12$ points)** Answer the following questions with a brief justification.

   (a) Arrange the following functions in a sequence $f_1, f_2, \ldots, f_7$ so that $f_i = O(f_{i-1})$. Additionally, if $f_i = \Theta(f_{i+1})$, indicate that: $n \log n, (\log\log n)^{\log n}, (\log n)^{\log\log n}, n \cdot 2^{\sqrt{\log n}}, (\log n)^{\log\log n}, n^{1+\frac{1}{\log n}}, n^2$. Assume that all the logarithms are to the base 2.

   $n^{1+\frac{1}{\log n}} = O((\log n)^{\log\log n}) = O(n \log n) = O(n \cdot 2^{\sqrt{\log n}}) = O(n^2) = O((\log\log n)^{\log n})$

   (b) How many different ways can you choose 18 muffins from a choice of apple, blueberry, chocolate-chip and date muffins, if there are 9 apple, 8 blueberry, 6 chocolate chip, but an unlimited number of date muffins.
   You can use either inclusion-exclusion or generating functions. For generating functions approach, you can as in Section 16.2.6 in theLLM Book. From inclusion-exclusion, you should get

   $$\binom{18+3}{3} - \left(\binom{8+3}{3} + \binom{9+3}{3} + \binom{11+3}{3}\right) + \left(0 + \binom{1+3}{3} + \binom{2+3}{3}\right) - 0$$

   (c) Count the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 10, x_1 \geq -2, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$. This is the same as the number of non-negative solutions to $y_1 + y_2 + y_3 + y_4 = 12$ which is exactly $\binom{12+3}{3}$

3. **(6 points)** How many 6-character passwords can be made using only the characters from the set $\{A, B, C, D, E, F, 1, 2, 3, 4\}$ if

   (a) The password must contain at least one letter and at least one digit (repeats allowed). $10^6 - 6^6 - 4^6$

(b) The password contains four letters and two digits (in any order and repeats allowed). $\binom{6}{4} \cdot 6^4 \cdot 4^2$

(c) No character is used more than once. $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$

(d) No two letters are adjacent, no two digits are adjacent, and no character is used more than once. Briefly explain your answers for each of the cases. $2 \times (6 \cdot 5 \cdot 4) \times (4 \cdot 3 \cdot 2)$

4. **(5 points)** An 8×8 chessboard is colored in the usual way, but that's boring, so you decide to fix this. You can take any row, column, or 2×2 square, and reverse the colors inside it, switching black to white and white to black. Prove that it's impossible to end up with 63 white squares and 1 black square.

   Any of these operations changes the total number of white squares by an even number. In the case of a row or column, it goes from having k white squares to having $8 - k$ white squares, for a change of $8 - 2k = 2(4 - k)$. In the case of a 2×2 squares, we go from k white squares to $4 - k$ white squares, for a change of $4 - 2k = 2(2 - k)$. Since we start with an even number of white squares, the number of white squares always remains even, so it is impossible to end with 63 white squares.

5. **(5 points)** Recall that for $a, b \in \mathbb{R}$, $[a, b] = \{x \in \mathbb{R} \mid a < x < b\}$ and $(a, b) = \{x \in \mathbb{R} \mid a \leq x \leq b\}$. Find a bijection from $[0, 1]$ to $(0, 1)$.

   Sorry about the mistake in notation - the correct notation for open and closed sets is just the opposite of what I have here, that is $[0, 1] = (0, 1) \cup \{0, 1\}$. This question involves a nice trick: First we identify some nice countable subset S in both $[0, 1], (0, 1)$:

$$\begin{aligned} [0,1] &= \{0, 1, 1/2, 1/3, 1/4, \dots\} \cup S \\ (0,1) &= \{1/2, 1/3, 1/4, \dots\} \cup S \end{aligned}$$

   Now notice that $S = [0, 1] \setminus \{0, 1, 1/2, 1/3, 1/4, \dots\}$ Now the bijection f just maps all elements $xinS$ to x itself, 0 to $1/2$, and $x = 1/n$ to $1/(n + 2)$ (this includes mapping 1 to $1/3$).

6. **(5 points)** If $A = \{a_0, a_1, \dots\}$ and $B = \{b_0, b_1, \dots\}$ are countably infinite sets, Show that their product $A \times B$ is also a countable set by showing how to list the elements of $A \times B$. There at least 3 possible solutions: Proof 1 is as follows: For a fixed $a \in A$, let $B_a = \{(a, b) \in A \times B | b \in B\}$. Since B is countable, each B_a is countable. Note that $\cup_{a \in A} B_a$ is the countable union of countable sets, and hence is countable (we have seen this in class/tutorials). Since $A \times B = \cup_{a \in A} B_a$, we have that $A \times B$ is countable. For Proofs 2 and 3 here.

7. **(5 points)** Find a closed form for $S = \sum_{n=0}^{\infty} \frac{2n}{3^{n+1}}$

   Check this Stackexchange Post. Note that we have deducted marks if you have been careless with manipulation by doing arithmetic with infinity (subtract $S/3$ from S without first proving convergence(where the formal method would have been to define the m-th partial sum and then find its limit). You shouldn't raise a regrade request for this.