Mid-Sem Exam (*Sept. 13, Marks: 20, Duration: 2 hours*) **COL703, Jul-Nov 2023**

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1. **[1+1 = 2 marks]** Prove the following sequents using natural deduction, without using the LEM rule directly or indirectly (i.e., even after deriving it).

(a) $x_1 \to x_2 \lor x_3 \lor x_4,\ x_2 \to \neg x_1 \lor \neg x_4,\ x_3 \lor x_4 \to x_2 \vdash x_1 \to \neg x_4$

(b) $x_1 \to x_2 \lor x_3, x_2 \to \neg x_1 \lor \neg x_4, x_3 \to \neg x_1 \lor \neg x_4, x_4 \to x_1 \land x_5, x_5 \to x_1 \land x_4, x_1 \to x_4 \lor x_5 \vdash \neg x_1$

2. **[2 marks]** Prove, in Hilbert's proof system, that $(\alpha \to \neg\neg\alpha)$.

3. **[2+1 = 3 marks]** Let p and q be atomic propositions, and ϕ_1 and ϕ_2 be propositional logic formulae on p and q.

(a) Consider the following definitions for ϕ_1 and ϕ_2:

- $\phi_1 = (p \to \neg\phi_2)$
- $\phi_2 = (q \to \neg\phi_1)$

Show that there are exactly two pairs of propositional logic formulae (ϕ_1, ϕ_2) which satisfy the above definitions. Justify your answer.

(b) If the definitions of ϕ_1 above is changed to $\phi_1 = (p \to \phi_2)$, and the definition of ϕ_2 is left unchanged, is it possible to find propositional logic formulae on propositions p and q that satisfy the modified definitions? If yes, give the formulae ϕ_1 and ϕ_2. If not, explain why the modified definitions cannot be satisfied.

4. **[3 marks]** Show that for any CNF formula ϕ one can compute in polynomial time an equisatisfiable formula $\psi_1 \land \psi_2$, with ψ_1 a Horn formula and ψ_2 a 2-CNF formula.

5. **[1+2.5+2.5 = 6 marks]** Let us consider formulae in propositional logic with $\to$ as the only propositional connective, and $\perp$ as the only propositional constant. For example, $(x \to (y \to \perp)) \to (\perp \to z)$ is a propositional logic formula that can be constructed with atoms x, y, z, using the allowed connective and constant.

(a) Let ϕ_1 and ϕ_2 be propositional logic formulae using $\to$ as the only connective and $\perp$ as the only constant. Give *semantically equivalent* formula for $\phi_1 \land \phi_2$ and $\neg\phi_1$, such that $\to$ is the only connective and $\perp$ is the only constant in the resulting formulae. Justify your answer.

(b) Your solution to the previous subquestion should convince you that any propositional logic formula can be converted to a semantically equivalent one using only $\to$ and $\perp$. A student now claims that it is possible to prove sequents in this version of propositional logic (with $\to$ as the only connective and $\perp$ as the only constant) using rules $\to_i, \to_e, \perp_e$ of the natural deduction proof system that we studied, in addition to the following special rule, called $(\to \perp)_e$ rule:

$$\frac{(\phi \to \perp) \to \psi \qquad (\phi \to \chi) \qquad (\psi \to \chi)}{\chi}\ (\to \perp)_e$$

Using only the above four proof rules, prove the following sequent:
$(\phi \to \perp) \to \psi, (\phi \to \chi) \vdash (\psi \to \perp) \to \chi$

(c) Are the above four rules, i.e. $\rightarrow_i, \rightarrow_e, \bot_e$, and $(\rightarrow \bot)_e$, complete for the version of propositional logic that uses $\rightarrow$ as the only connective and $\bot$ as the only constant? In other words, given formulas ϕ and ψ, each involving only $\rightarrow$ and $\bot$ apart from propositional atoms, such that $\phi \vDash \psi$, is it always possible to prove the sequent $\phi \vdash \psi$ using only the above four rules? Justify your answer. Assume that you are free to use the *copy* rule even if it is not explicitly given. Recall that the *copy* rule is not really useful for transforming or constructing any formula; it simply allows you to use the premises and the *visible* formulae more than once.

6. **[2+2 = 4 marks]** Recall that α is said to be *consistent* if $\nvdash \neg\alpha$. Suppose that $\vdash \alpha \rightarrow \beta$. For the following statements, answer whether they are true or not, and provide an explanation. Your explanation should not rely on soundness and completeness of propositional logic. Answers with missing or inadequate explanations will not get any marks.

(a) If α is consistent then β is consistent.

(b) If β is consistent then α is consistent.