Name: ____________________ Roll No: ______________

(COL 202) Discrete Mathematics 18 August, 2023

Quiz 1

Duration: 45 minutes **(12 marks)**

- Be clear in your writing.
- If you use a statement proved in class or in the problem set, then write down the entire statement before using it.
- **You will not get a new sheet, so make sure you are certain when you write something.** Make a judicious decision of which tool(s) to use to get a clean and short answer that fits in the space.

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1. **($2 \times 2 = 4$ points)** Translate the following sentences into a predicate formula. The domain of discourse should be the set of students in the class; in addition, the only predicates that you may use are (1) Equality, (2) $E(x, y)$ meaning that "x has sent e-mail to y."

   Note: I have used the predicate Student(x) for clarity, but its not really necessary.

   (a) There is a student who has e-mailed *at most* n other people in the class, besides possibly himself. Let S denote the set of students, the we can say

   $$\exists s \in S, \forall k > n, \left(\bigwedge_{i=1}^{k} (x_i \in S)\right) \implies \neg\left(\bigwedge_{i=1}^{k} E(s, x_i)\right)$$

   (b) There is a student who has emailed *at least* n other people in the class, besides possibly himself. Let S denote the set of students, the we can say

   $$\exists s \in S, \exists k \geq n, \left(\bigwedge_{i=1}^{k} (x_i \in S) \wedge \bigwedge_{p \neq q=1}^{k} \neg(x_p = x_q)\right) \wedge \left(\bigwedge_{i=1}^{k} E(s, x_i)\right)$$

   However many of you have used an extra cardinality predicate to express this, and we have given marks for these as well.

2. **(4 points)** Use the Well Ordering Principle to prove that any integer greater than or equal to 8 can be represented as the sum of nonnegative integer multiples of 3 and 5.

   Proof. For the sake of contradiction assume that the set of counterexamples C is nonempty, i.e.,

   $$C := \{n \geq 8 \mid n \text{ cannot be represented as a linear combination of 3 and 5}\}$$

   By WOP C contains a least element m. Now note that if m can't be represented as a sum of non-negative integer multiples of 3 and 5, then neither can $m - 3$. Therefore $m - 3$ cannot be greater than 10, as if $m \geq 11$ then $m - 3 \geq 8$, and thus $m - 3$ would be in C, which is a contradiction since m is the least element of C. The only remaining cases are $n = 8, 9, 10$, which are clearly not in C. □

3. **(4 points)** Prove by induction that

   $$\sqrt{1\sqrt{2\sqrt{3 \ldots \sqrt{n}}}} < 2$$

Proof. Note that if you try induction naively, it doesn't work: Base case $n = 1 : \sqrt{1} \leq \sqrt{2}$ is fine, but if your induction hypothesis is $\sqrt{1\sqrt{2\sqrt{3\ldots\sqrt{n-1}}}} \leq 2$, it is not clear how to upper bound $\sqrt{1\sqrt{2\sqrt{3\ldots\sqrt{n}}}}$ by 2. We need a stronger induction hypothesis, namely:

$$\forall n \in \mathbb{N}, \forall m \in \mathbb{N} \cup \{0\}, \sqrt{(m+1)\sqrt{(m+2)\sqrt{(m+3)\ldots\sqrt{(m+n)}}}} < (m+2)$$

.

Note that from the stronger hypothesis, when $m = 0$, we recover our simpler hypothesis! Let us proceed to prove the stronger hypothesis by induction:

- Base case: $\forall m \in \mathbb{N} \cup \{0\}$, $\sqrt{m+1} < \sqrt{m+2}$ is clearly true.
- Inductive case: Assume as your induction hypothesis,

$$\forall n \in \mathbb{N}, \forall m \in \mathbb{N} \cup \{0\}, \sqrt{(m+1)\sqrt{(m+2)\sqrt{(m+3)\ldots\sqrt{(m+n-1)}}}} < (m+2)$$

. Now note that

$$\sqrt{(m+2)\sqrt{(m+2)\sqrt{(m+3)\ldots\sqrt{(m+n)}}}} < (m+3)$$

. Therefore we have

$$\begin{aligned} \sqrt{(m+1)\sqrt{(m+2)\sqrt{(m+3)\ldots\sqrt{(m+n)}}}} &< \sqrt{(m+1)(m+3)} \\ &< (m+2) \end{aligned}$$

as required. Note that the last inequality follows by squaring the equation on both sides (this is ok since they are both positive quantities).

□