COL751 - Lecture 7

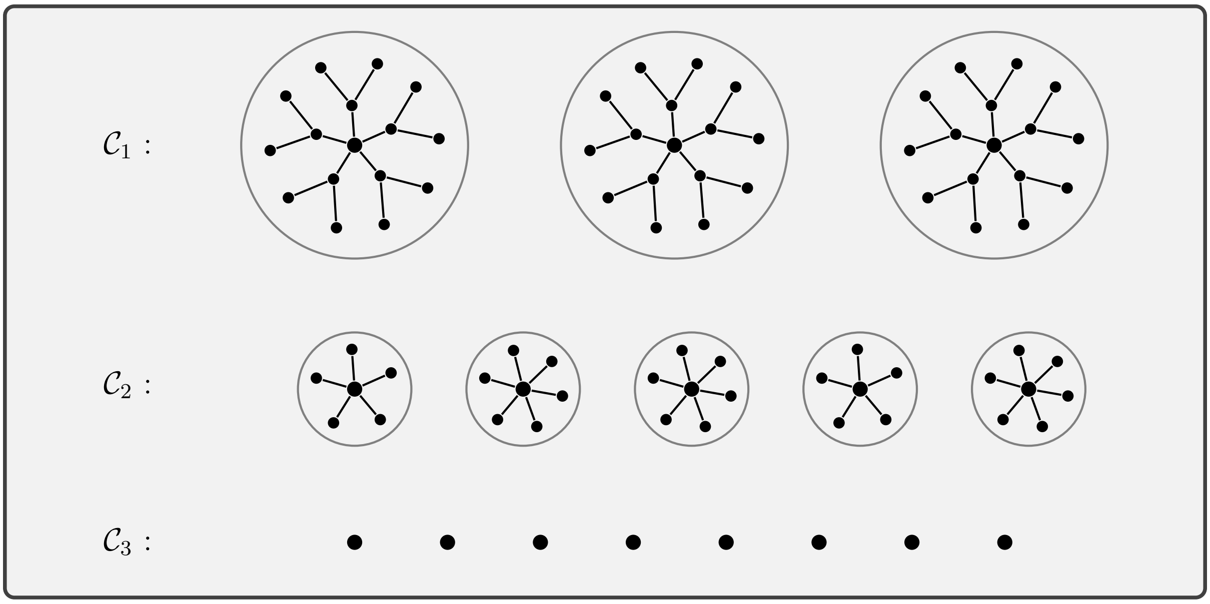
1 A linear time $(2k-1)$-spanner construction

We will see now a linear time construction of $(2k-1)$-spanner of $O(kn^{1+1/k})$ size.
Note: The construction below is slightly different from one presented in class and the modification is we do not enforce that clusters across different layers are vertex disjoint.

Algorithm

1. Initialize H as $(V, \emptyset)$. Let $R_k = V$, and for i going from $k-1$ to 1, let R_i be a random subset of R_{i+1} size $n^{i/k}$. Thus, we have $R_1 \subseteq R_2 \subseteq \cdots \subseteq R_{k-1} \subseteq R_k = V$.

2. Next for $i = 1$ to k, do the following:
   - $\mathcal{C}_i$ = clusters of depth $(k-i)$ centered at vertices in R_i in G.
   - V_i = vertices in clusters in $\mathcal{C}_i$ that do not lie in $\mathcal{C}_1, \ldots, \mathcal{C}_{i-1}$.
   - Add to H edges in clusters in $\mathcal{C}_i$.
   - For each $x \in V_i$ and each cluster $C \in (\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_i)$ satisfying x has a neighbor in C, we add to H exactly one edge from the set $x \times V(C)$.



Lemma 1 *The vertex sets $V_1, \ldots, V_k$ form a partition of V.*

Lemma 2 *The time to compute graph H is $O(k \cdot m)$.*

Lemma 3 *For any $(x, y) \in E(G) \setminus E(H)$, we have $dist(x, y, H) \leqslant 2k - 1$.*

Proof: Let us suppose $x \in V_i$ and $y \in V_j$, for some $j \leqslant i$. Let C be a cluster in level j containing y. Recall that there exists an edge $(x, w) \in E(H)$ such that $w \in C$ as cluster C is adjacent to x.

Therefore, $dist(x, y, H) \leqslant dist(x, w, H) + dist(w, y, H) \leqslant 1 + 2(k - j) \leqslant 2k - 1$. $\square$

Lemma 4 *For $i, j \in [1, k]$ satisfying $i \geqslant j$ the following holds:*
For each $x \in V_i$ the number of clusters $C \in \mathcal{C}_j$ adjacent to x is at most $O(n^{1/k} \log n)$ with probability $1 - 1/n^4$.

Proof: Let $Z_1, \ldots, Z_\alpha$ be clusters in $\mathcal{C}_j$ adjacent to x in G, and let the corresponding roots be $y_1, \ldots, y_\alpha$. Since these clusters have height $k - j$, we have $dist(x, y_1), \ldots, dist(x, y_\alpha) \leqslant k - j + 1$.

Next observe $|R_j| = n^{j/k}$, and R_{j-1} is a random subset of R_j of size $n^{(j-1)/k}$. Thus, if $|\{y_1, \ldots, y_\alpha\}| \geqslant 4n^{1/k} \log n$, then by hitting set argument:

$$Prob\Big(R_{j-1} \cap \{y_1, \ldots, y_\alpha\} \; = \; \emptyset\Big) \; \leqslant \; \Big(1 - \frac{\alpha}{|R_j|}\Big)^{|R_{j-1}|} \; \leqslant \; \frac{1}{n^4}.$$

Further, if a vertex $\tilde{y} \in \{y_1, \ldots, y_\alpha\}$ lies in R_{j-1}, then x must lie in cluster of $\tilde{y}$ in level $j - 1$ as $dist(x, \tilde{y}) \leqslant k - j + 1$, and in such a case (by definition) x cannot lie in V_i. Since $x \in V_i$, we have $R_{j-1} \cap \{y_1, \ldots, y_\alpha\} \; = \; \emptyset$.

This proves that probability (i) $x \in V_i$ and (ii) the number of clusters adjacent to x in level $j (\leqslant i)$ is $\alpha \geqslant 4n^{1/k} \log n$ is at most $1/n^4$. $\square$

Theorem 5 (Baswana, Sen (2003)) *For any n vertex, m edges undirected connected graph G we can compute in $O(m)$ time a $(2k - 1)$-multiplicative spanner. The size of H is $O(kn^{1+1/k} \log n)$ with probability $1 - 1/n^2$.*

Challenge Problem Can you get a single-pass (or a k-pass) streaming algorithm for $(2k - 1)$ spanner construction?