COL751 - Lecture 14

For any undirected graph $G = (V, E, c)$ and any two vertices $s, t \in V$, we denote the capacity of an (s,t)-min-cut in G by $\lambda_{s,t,G}$. For any tree T and any two vertices x, y in T, we denote the set of edges in an (x,y) path in T by $P_{x,y,T}$. When the tree T is clear from the context we omit the term T from the subscript.

Below we define the notion of Gomory Hu trees that are sparse structures representing all-pairs min-cut values in an undirected graph.

Definition 1 (Gomory Hu Tree) *A tree $T = (V, E^*, c^*)$ is said to be a Gomory-Hu tree for a graph $G = (V, E, c)$ if it satisfies that for any distinct $x, y \in V$:*

1. $\lambda_{x,y,G} \ = \ \lambda_{x,y,T} \ = \ \min_{e \in P_{x,y}} c^*(e)$.
2. *If $e \ = \ \arg\min_{e \in P_{x,y}} c^*(e)$, then $T - e$ corresponds to an (x,y)-min-cut in G.*

We will prove in Lecture 14 and 15 that for any graph G we can construct a Gomory-Hu tree by invoking just n computations of max-flow in either G or a graph derived from G.

1 Some Fundamental Properties of Cuts

Property 1 *For any sequence of distinct vertices $(x = x_1, x_2, \ldots, x_k = y)$ of size $k \geqslant 2$, we have*

$$\lambda_{x,y,G} \ \geqslant \ \min_{i<k} \lambda_{x_i,x_{i+1},G}.$$

Proof: Let (A, A^c) be a (x,y)-min-cut in G. Let i be largest index such that $x_i \in A$. Then (A, A^c) is also an (x_i, x_{i+1})-cut. Thus, $\lambda_{x,y,G} \geqslant \lambda_{x_i,x_{i+1},G}$, which proves our claim. $\square$

Lemma 2 (Submodularity) *For any two cuts (A, A^c) and (B, B^c) in an undirected graph $G = (V, E, c)$, we have $c(A) + c(B) \ \geqslant \ c(A \cap B) + c(A \cup B)$.*

Proof: Partition edges of G into six sets, namely, $E_1, \ldots, E_6$ as shown in Figure 1. For any $\mathcal{E} \subseteq E$, define $c(\mathcal{E}) = \sum_{e \in \mathcal{E}} c(e)$. Observe,

$$c(A) = c(E_1) + c(E_2) + c(E_5) + c(E_6),$$

$$c(B) = c(E_1) + c(E_2) + c(E_3) + c(E_4).$$

Further,

$$c(A \cap B) = c(E_2) + c(E_3) + c(E_5),$$
$$c(A \cup B) = c(E_2) + c(E_4) + c(E_6).$$

By a simple counting argument we obtain that $c(A) + c(B) = 2c(E_1) + c(A \cap B) + c(A \cup B)$, which directly proves our claim. $\square$

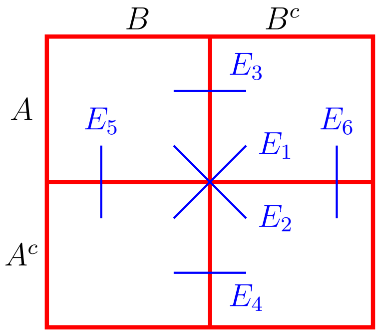

Figure 1: Partition of edges into sets $E_1, \ldots, E_6$.

Property 2 *Let $s, t \in V$ be distinct vertices and (A, A^c) an (s,t)-min-cut in G. Then, for any two distinct vertices $x, y \in A$ there is a (x,y)-min-cut (B, B^c) in G such that either $B \subseteq A$ or $B^c \subseteq A$.*

(In other words, the (x,y)-min-cut is unaffected on considering A^c as a supernode.)

Proof: Let (A, A^c) be an (s,t)-min-cut in G. Further, let (B, B^c) be a minimum-cut separating x, y in G, i.e. either $(x,y) \in B \times B^c$ or $(x,y) \in B^c \times B$. Without loss of generality assume that $t \in A^c \cap B^c$.

Then, $(A \cup B, A^c \cap B^c)$ is an (s,t)-cut in G, implying $c(A \cup B) \geqslant c(A)$. By Submodulaity of Cuts, we get $c(A \cap B) \leqslant c(B)$. Thus, $(A \cap B, A^c \cup B^c)$ is an (x,y) or (y,x) min-cut such that $A \cap B$ lies completely inside set A. $\square$

2 Algorithm

Below is pseudo-code to compute a Gomory Hu Tree.

$\mathcal{T}_1 = (\{V\}, \emptyset)$.
for $i = 2$ *to* n **do**
  Let $X \in V(\mathcal{T}_{i-1})$ be a set of size at least two.
  Take any two vertices s, t in X.
  Let $C_1, \ldots, C_k$ be connected-components in $\mathcal{T}_{i-1} - X$.
  Let H be a graph obtained from G by contracting $C_1, \ldots, C_k$ into k super-nodes.
  Compute an (s,t)-min-cut, say (S_H, T_H), in H and let (S, T) be an (s,t)-cut in G obtained from (S_H, T_H) on uncontracting $C_1, \ldots, C_k$.
  Split node X into two nodes $X_S = S \cap X$ and $X_T = T \cap X$, and for $j \in [1, k]$, connect C_j to X_S if $V(C_j) \subseteq S$ and X_T otherwise, to obtain tree $\mathcal{T}_i$.
  Set $c^*(X_S, X_T) = \lambda_{s,t,G}$.
end
Return $\mathcal{T}_n$.

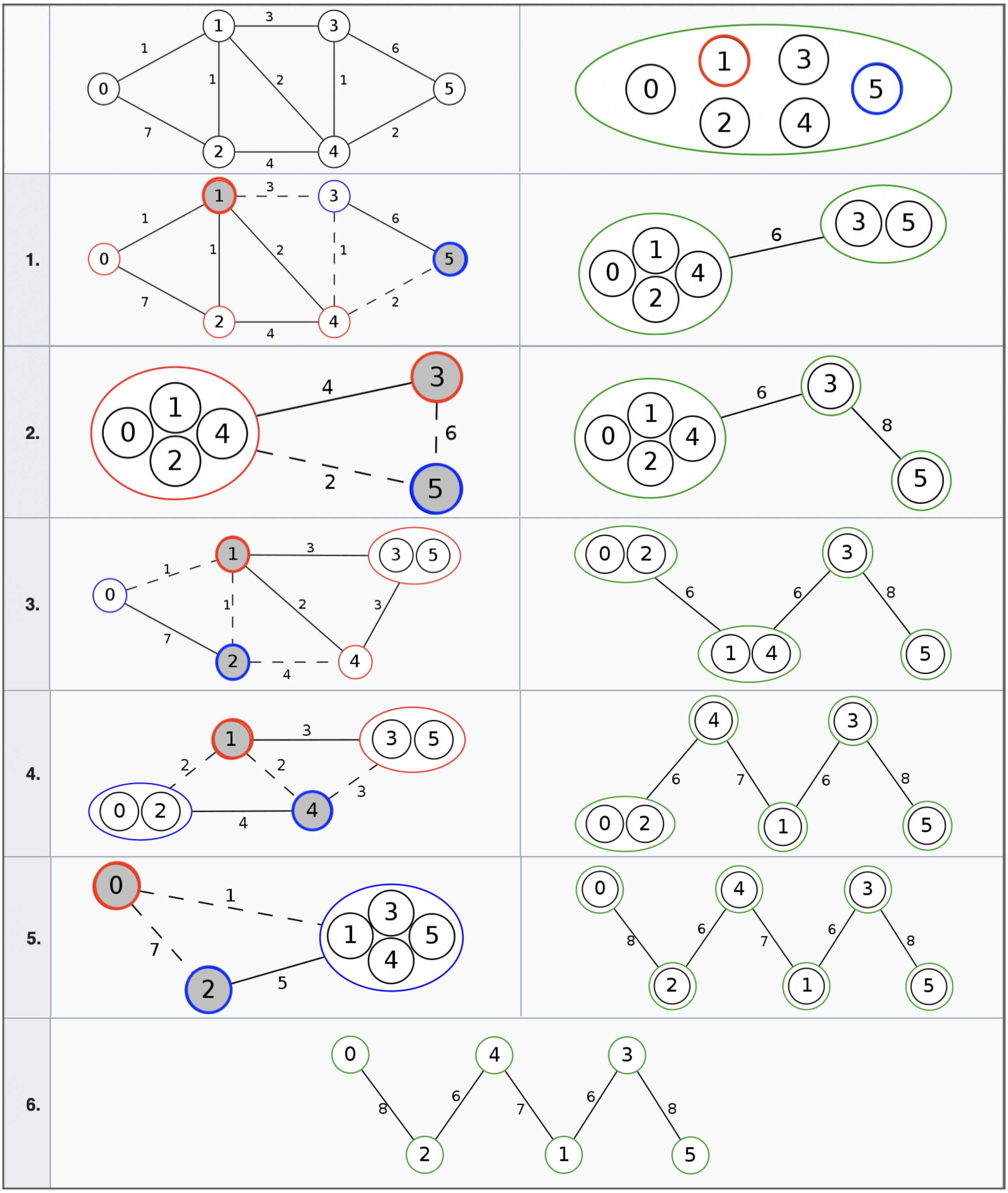

Figure 2: Construction of Gomory-Hu Tree (*Source: Wikipedia*)