Department of Mathematics
MTL 106 (Introduction to Probability and Stochastic Processes)
Tutorial Sheet No. 6
Answer for Selected Problems

1. $Cov(X(s), X(t)) = E[X(s), X(t)] - E[X(s)]E[X(t)]$

   $E[X(t)] = 0, \quad Cov(X(s), X(t)) = 1 + st + s^2t^2$

2. Yes

3. Yes, since $E(Y(t)) = 0$, $E(Y(t)^2) = 0.5E(X(t)^2) < \infty$ and
   $\text{cov}(Y(t), Y(s)) = 0.5\cos(2\pi w(t-s)\text{cov}(X(t), X(s))$

5. State Space={0,1,2,3,4,5}

$$P = \begin{pmatrix} 0.3 & 0.4 & 0.3 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0.3 & 0 & 0 & 0 \\ 0 & 0.3 & 0.4 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 0.4 & 0.3 & 0 \\ 0 & 0 & 0 & 0.3 & 0.4 & 0.3 \\ 0 & 0 & 0 & 0 & 0.3 & 0.7 \end{pmatrix}$$

6. $S = \{0, 1, 2, 3, \ldots\}$

$$\begin{pmatrix} 1-p & p & 0 & 0 & 0 & 0 & \cdots \\ q & 1-(p+q) & p & 0 & 0 & 0 & \cdots \\ 0 & q & 1-(p+q) & p & 0 & 0 & \cdots \\ 0 & 0 & q & 1-(p+q) & p & 0 & \cdots \\ 0 & \cdots & q & 1-(p+q) & p & \cdots & \\ 0 & \cdots & & \cdots & & \cdots & \end{pmatrix}$$

7. $S = \{(0,0), (0,1), (1,0), (1,1)\}$

$$P = \begin{pmatrix} q_{0,0} & 1-q_{0,0} & 0 & 0 \\ 0 & 0 & q_{0,1} & 1-q_{0,1} \\ q_{1,0} & 1-q_{1,0} & 0 & 0 \\ 0 & 0 & q_{1,1} & 1-q_{1,1} \end{pmatrix}$$

   (c) $q_{0,0}^2 + q_{0,1}(1-q_{0,0})$

8. (a) 0.212 $\quad$ (b) 0.0016

9. (a) $C_1 = \{0\}$, $C_2 = \{4\}$, $T = \{1, 2, 3\}$ $\quad$ (b) 1 $\quad$ (c) 0.75

10. $$X_{n+1} = \begin{cases} X_n, & \text{if } X_n = 0 \text{ or } X_n = N \\ X_n + 1, & \text{if } 0 < X_n < N \text{ and the coin turns up heads} \\ X_n - 1, & \text{if } 0 < X_n < N \text{ and the coin turns up tails} \end{cases}.$$

    Since the successive coin tosses are independent, we conclude that $\{X_n, n = 0, 1, \ldots\}$ with state space $\{0, 1, \ldots, N\}$ is a DTMC.

$$P = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1-p & 0 & p & \ldots & 0 & 0 & 0 \\ 0 & 1-p & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & p & 0 \\ 0 & 0 & 0 & \ldots & 1-p & 0 & p \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{bmatrix}.$$

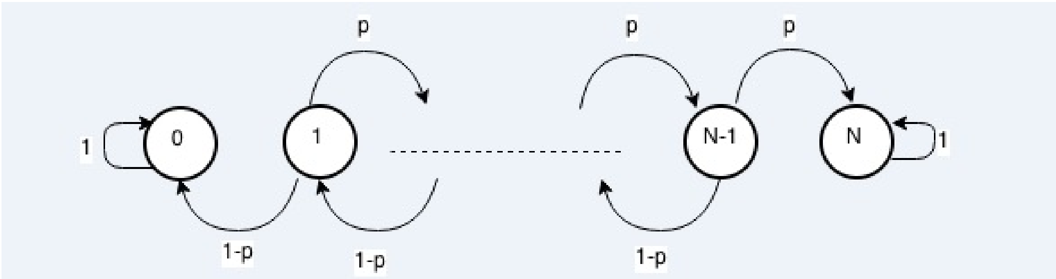

Figure 1: state transition diagram

11. $$\begin{pmatrix} 1-v_0 & 0 & v_0 & 0 & 0 & 0 & \cdots \\ 1-v_1 & 0 & 0 & v_1 & 0 & 0 & \cdots \\ 1-v_2 & 0 & 0 & 0 & v_2 & 0 & \cdots \\ 1-v_3 & 0 & 0 & 0 & 0 & v_3 & \cdots \\ 1-v_4 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1-v_5 & 0 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix}$$

$\{0, 2, 4, \cdots\}-$ Closed Communicating Class

When $\prod_{i=0}^{\infty} v_{2i} = 0, \sum_{i=0}^{\infty} \prod_{n=0}^{i} v_{2n} < \infty$ - positive recurrent states, otherwise null recurrent states

When $\prod_{i=0}^{\infty} v_{2i} > 0$ - transient states
$\{1, 3, 5, \cdots\}-$ transient states

13. $f_{j0}^{(n)} = \begin{cases} {}^{n-1}C_{j-1} p^{n-j} q^j & \text{if } j \leq n \\ 0 & \text{if } j > n \end{cases}$

14. $S = \{0, 1, 2, \ldots\}$
$\{0\}-$ absorbing state
$\{1, 2, 3, \ldots\}$ - transient states.

15. (b) $v_i = p^i(1-p) \quad \forall \quad i = 0, 1, 2, 3, \ldots$

16. (a) $\quad \pi_i = \frac{1}{|S|}, \quad i \in S \quad$ (b) $\quad \pi_i = \frac{1}{K}, \quad i \in \{0, 1, \ldots, K-1\}.$