**31.** If $f(x, y) = e^x \cdot \sin y$, then $\dfrac{\partial^3 f(0,0)}{\partial x^3}$ = ..........

**32.** Expansion of $e^{xy}$ up to first order tem is ..........

**33.** $f(x, y) = f(1, 2) +$ .........

**C. Indicate True or False for the following statements:**

**1.** If $u, v$ are functions of $r, s$ are themselves functions of $x, y$ then $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{\partial(u,v)}{\partial(r,s)} \times \dfrac{\partial(x,y)}{\partial(r,s)}$

**2.** Geometrically the function $z = f(x, y)$ represents a surface in space.

**3.** If $f(x, y) = ax^2 + 2hxy + by^2$ then $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 2f$.

**4.** If $z$ is a function of two variables then $dz$ is defined as $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy$.

**5.** If $f(x_1, x_2, ..., x_n)$ be a homogeneous function then $x_1\dfrac{\partial f}{\partial x_1} + x_2\dfrac{\partial f}{\partial x_2} + \ldots + x_n\dfrac{\partial f}{\partial x_n}$ $= n(n-1)f$.

**6.** If $u = \dfrac{x^2+y^2}{x^2-y^2} + 4$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 4$.

**7.** When the function $z = f(x, y)$ differentiating (partially) with respect to one variable, other variable is treated (temporarily) as constant.

**8.** To satisfied Euler's theorem the function $f(x, y)$ should not be homogeneous.

**9.** The partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are interpreted geometrically as the slopes of the tangent lines at any point.

**10.** (*i*) The curve $y^2 = 4ax$ is symmetric about $x$-axis.

(*ii*) The curve $x^3 + y^3 = 3axy$ is symmetric about the line $y = -x$.

(*iii*) The curve $x^2 + y^2 = a^2$ is symmetric about both the axis $x$ and $y$.

(*iv*) The curve $x^3 - y^3 = 3axy$ is symmetric about the line $y = x$.

**11.** (*i*) If there is no constant term in the equation then the curve passes through the origin otherwise not.

(*ii*) A point where $\dfrac{d^2y}{dx^2} \neq 0$ is called an inflexion point.

(*iii*) If $r^2$ is negative *i.e.*, imaginary for certain values of $\theta$ then the curve does not exist for those values of $\theta$.

(*iv*) The curve $r = ae^{\theta}$ is symmetric about the line $\theta = \dfrac{\pi}{2}$.

**12.** (*i*) Maclaurin's series expansion is a special case of Taylor's series when the expansion is about the origin (0, 0).

(*ii*) Taylor's theorem is important tool which provide polynomial approximations of real valued functions.

(*iii*) Taylor's theorem fail to expand $f(x, y)$ in an infinite series if any of the functions $f_x(x, y)$, $f_{xx}(x, y)$, $f_{xy}(x, y)$ etc., becomes infinite or does not exist for any value of $x, y$ in the given interval.

(*iv*) $f(x, y) = f(a, b) + \left[(x-a)\frac{\partial}{\partial x} - (x-b)\frac{\partial}{\partial y}\right] f(a, b) + \ldots\ldots$ .

## ANSWERS TO OBJECTIVE TYPE QUESTIONS

**A. Pick the correct answer:**

| | | |
|---|---|---|
| **1.** (*iv*) | **2.** (*iii*) | **3.** (*iii*) |
| **4.** (*iv*) | **5.** (*i*) | **6.** (*i*) |
| **7.** (*iv*) | **8.** (*iv*) | **9.** (*iii*) |
| **10.** (*iv*) | **11.** (*i*) | **12.** (*iii*) |
| **13.** (*iii*) | **14.** (*i*) | **15.** (*iii*) |

**B. Fill in the blanks:**

| | | |
|---|---|---|
| **1.** $2\lfloor n-1$ | **2.** $x \sin\left(x+\frac{n\pi}{2}\right) - n\cos\left(x+\frac{n\pi}{2}\right)$ | |
| **3.** Zero | **4.** Zero | **5.** $\sin hx$ |
| **6.** $e^x(x+n)$ | **7.** Try yourself | **8.** $(n-n^2)\sin\frac{n\pi}{2}$ |
| **9.** Zero | **10.** Zero | **11.** Zero |
| **12.** $\frac{\partial^2 z}{\partial y^2}$ | **13.** $\frac{\partial^2 u}{\partial y \partial x}$ | **14.** 2 $abu$ |
| **15.** $2x + 4y$ | **16.** $n(n-1)f$ | **17.** $3u \log u$ |
| **18.** $u$ | **19.** $\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}$ | **20.** $-\frac{\sin^2\theta}{y}, \frac{\cos^2\theta}{x}$ |
| **21.** $u$ | **22.** 1 | **23.** $t_1$ |
| **24.** initial line | **25.** 0 | **26.** $y = \pm x$ |
| **27.** $-a < x < a$ | **28.** $(0, a)$ and $(0, -$ a) | **29.** $x - \frac{x^2}{2} + xy$ |
| **30.** $-$ 0.823 | **31.** 0 | **32.** $e\{1 + (x-1) + (y-1)\}$ |

**33.** $\left[\left\{(z-1)\frac{\partial f}{\partial x} + (y\cdot 2)\frac{\partial f}{\partial y}\right\} + \frac{1}{\lfloor 2}\left\{(x-1)\frac{\partial f}{\partial x} + (y\cdot 2)\frac{\partial f}{\partial y}\right\}^2 + \ldots\ldots\right]$

**C. True or False:**

| | | | |
|---|---|---|---|
| **1.** *F* | **2.** *T* | **3.** *T* | **4.** *T* |
| **5.** *F* | **6.** *F* | **7.** *T* | **8.** *F* |
| **9.** *T* | | | |
| **10.** (*i*) *T* | (*ii*) *F* | (*iii*) *T* | (*iv*) *F* |
| **11.** (*i*) *T* | (*ii*) *F* | (*iii*) *T* | (*iv*) *F* |
| **12.** (*i*) *T* | (*ii*) *T* | (*iii*) *T* | (*iv*) *F* |

❑❑❑

UNIT II

# Differential Calculus-II

## 2.1 JACOBIAN

The Jacobians* themselves are of great importance in solving for the reverse (inverse functions) derivatives, transformation of variables from one coordinate system to another coordinate system (cartesian to polar etc.). They are also useful in area and volume elements for surface and volume integrals.

### 2.1.1 Definition

If $u = u(x, y)$ and $v = v(x, y)$ where $x$ and $y$ are independent, then the determinant

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

is known as the Jacobian of $u$, $v$ with respect to $x$, $y$ and is denoted by

$$\frac{\partial(u, v)}{\partial(x, y)} \text{ or } J(u, v)$$

Similarly, the Jacobian of three functions $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$ is defined as

$$J(u, v, w) = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

* Carl Gustav Jacob Jacobi (1804–1851), German mathematician.

### 2.1.2 Properties of Jacobians

**1.** If $u = u\,(x, y)$ and $v = v\,(x, y)$, then

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1 \text{ or } JJ' = 1 \qquad (U.P.T.U., 2005)$$

where, $$J = \frac{\partial(u,v)}{\partial(x,y)} \text{ and } J' = \frac{\partial(x,y)}{\partial(u,v)}$$

**Proof:** Since
$$u = u\,(x, y) \qquad ...(i)$$
$$v = v\,(x, y) \qquad ...(ii)$$

Differentiating partially equations (*i*) and (*ii*) *w.r.t.* $u$ and $v$, we get

$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\times\frac{\partial y}{\partial u} \qquad ...(iii)$$

$$\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\times\frac{\partial y}{\partial v} \qquad ...(iv)$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\times\frac{\partial y}{\partial u} \qquad ...(v)$$

$$\frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\times\frac{\partial y}{\partial v} \qquad ...(vi)$$

Now, $$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

(By interchanging rows and columns in II determinant)

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix} \quad \text{(multiplying row-wise)}$$

Putting equations (*iii*), (*iv*), (*v*) and (*vi*) in above, we get

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

or $$\boxed{JJ' = 1.}$$ **Hence proved.**

**2. Chain rule:** If $u$, $v$, are function of $r$, $s$ and $r$, $s$ are themselves functions of $x$, $y$ *i.e.*, $u = u\,(r, s)$, $v = v\,(r, s)$ and $r = r\,(x, y)$, $s = s\,(x, y)$

then $$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)}\cdot\frac{\partial(r,s)}{\partial(x,y)}$$

**Proof:** Here
$$u = u\,(r, s),\ v = v\,(r, s)$$
and
$$r = r\,(x, y),\ s = s\,(x, y)$$

Differentiating $u$, $v$ partially w.r.t. $x$ and $y$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} \quad ...(i)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} \quad ...(ii)$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial x} \quad ...(iii)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial y} \quad ...(iv)$$

Now, $$\frac{\partial(u,v)}{\partial(r,s)}\cdot\frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{vmatrix}$$

| By interchanging the rows and columns in second determinant

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial v}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial v}{\partial s}\frac{\partial s}{\partial y} \end{vmatrix}$$ | multiplying row-wise

Using equations (*i*), (*ii*), (*iii*) and (*iv*) in above, we get

$$\frac{\partial(u,v)}{\partial(r,s)}\cdot\frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)}.$$

*Or*

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)}\cdot\frac{\partial(r,s)}{\partial(x,y)}.$$ **Hence proved.**

**Example 1.** Find $\frac{\partial(u,v)}{\partial(x,y)}$, when $u = 3x + 5y$, $v = 4x - 3y$.

**Sol.** We have $u = 3x + 5y$

$v = 4x - 3y$

$\therefore$ $$\frac{\partial u}{\partial x} = 3,\ \frac{\partial u}{\partial y} = 5,\ \frac{\partial v}{\partial x} = 4 \text{ and } \frac{\partial v}{\partial y} = -3$$

Thus, $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 4 & -3 \end{vmatrix} = -9 - 20 = -29.$$

**Example 2.** Calculate the Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ of the following:

$$u = x + 2y + z,\ v = x + 2y + 3z,\ w = 2x + 3y + 5z. \qquad (U.P.T.U., 2007)$$

**Sol.** We have
$$u = x + 2y + z$$
$$v = x + 2y + 3z$$
$$w = 2x + 3y + 5z$$

$\therefore$
$$\frac{\partial u}{\partial x} = 1,\ \frac{\partial u}{\partial y} = 2,\ \frac{\partial u}{\partial z} = 1,\ \frac{\partial v}{\partial x} = 1,\ \frac{\partial v}{\partial y} = 2,\ \frac{\partial v}{\partial z} = 3,$$
$$\frac{\partial w}{\partial x} = 2,\ \frac{\partial w}{\partial y} = 3 \text{ and } \frac{\partial w}{\partial z} = 5.$$

Now,
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix}$$
$$= 1\,(10 - 9) - 2\,(5 - 6) + 1\,(3 - 4) = 2.$$

**Example 3.** Calculate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ if $u = \frac{2yz}{x}$, $v = \frac{3zx}{y}$, $w = \frac{4xy}{z}$.

**Sol.** Given
$$u = \frac{2yz}{x},\ v = \frac{3zx}{y},\ w = \frac{4xy}{z}$$

$\therefore$
$$\frac{\partial u}{\partial x} = \frac{-2yz}{x^2},\ \frac{\partial u}{\partial y} = \frac{2z}{x},\ \frac{\partial u}{\partial z} = \frac{2y}{x},\ \frac{\partial v}{\partial x} = \frac{3z}{y},\ \frac{\partial v}{\partial y} = -\frac{3zx}{y^2},\ \frac{\partial v}{\partial z} = \frac{3x}{y},$$
$$\frac{\partial w}{\partial x} = \frac{4y}{z},\ \frac{\partial w}{\partial y} = \frac{4x}{z} \text{ and } \frac{\partial w}{\partial z} = -\frac{4xy}{z^2}$$

Now,
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{2yz}{x^2} & \frac{2z}{x} & \frac{2y}{x} \\ \frac{3z}{y} & \frac{-3zx}{y^2} & \frac{3x}{y} \\ \frac{4y}{z} & \frac{4x}{z} & \frac{-4xy}{z^2} \end{vmatrix}$$
$$= -\frac{2yz}{x^2}\left[\frac{12x^2yz}{y^2z^2} - \frac{12x^2}{yz}\right] - \frac{2z}{x}\left[\frac{-12xyz}{yz^2} - \frac{12xy}{yz}\right] + \frac{2y}{x}\left[\frac{12xz}{yz} + \frac{12xyz}{zy^2}\right]$$

$\Rightarrow$
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0 + 48 + 48 = 96.$$

But, we have $\frac{\partial(x,y,z)}{\partial(u,v,w)} \times \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$ (Property 1)

$\therefore$
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{96}.$$

**Example 4.** If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$ find $J(x, y, z)$. (*U.P.T.U., 2002*)

**Sol.** Here we calculate $J(u, v, w)$ as follows:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & zx & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= yz(2y - 2z) - zx(2x - 2z) + xy(2x - 2y)$$

$$= 2[y^2z - yz^2 - zx^2 + z^2x + xy(x - y)]$$

$$= 2[-z(x^2 - y^2) + z^2(x - y) + xy(x - y)]$$

$$= 2(x - y)[-zx - zy + z^2 + xy] \qquad \Big| \text{ As } x^2 - y^2 = (x-y)(x+y)$$

$$= 2(x - y)[z(z - x) - y(z - x)]$$

$$= 2(x - y)(z - y)(z - x)$$

$$= -2(x - y)(y - z)(z - x)$$

But $\quad J(x, y, z) \cdot J(u, v, w) = 1$

$\therefore \quad J(x, y, z) =$

**Example 5.** If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and $u = r \sin\theta \cos\phi$,

$v = r \sin\theta \sin\phi$, $w = r\cos\theta$, calculate $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$.

**Sol.** Here $x, y, z$ are functions of $u, v, w$ and $u, v, w$ are functions of $r, \theta, \phi$ so we apply IInd property.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} \qquad ...(i)$$

Consider
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}} \\ \frac{1}{2}\sqrt{\frac{w}{u}} & 0 & \frac{1}{2}\sqrt{\frac{u}{w}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} & 0 \end{vmatrix}$$

$$= \frac{1}{8}\left[\sqrt{\frac{w}{v}\frac{v}{u}\frac{u}{w}} + \sqrt{\frac{v}{w}\frac{w}{u}\frac{u}{v}}\right] = \frac{1}{8}\left[\sqrt{1} + \sqrt{1}\right] = \frac{2}{8} = \frac{1}{4}$$

$$\Rightarrow \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \qquad \text{...}(ii)$$

Next $$\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta \cos\phi\,(r^2 \sin^2\theta \cos\phi) - r\cos\theta\cos\phi\,(-r^2\sin\theta\cos\theta\cos\phi) + r^2\sin\theta\sin\phi\,(\sin^2\theta\sin\phi + \cos^2\theta\sin\phi)$$

$$= r^2\sin\theta\cos^2\phi\,(\sin^2\theta + \cos^2\theta) + r^2\sin\theta\sin^2\phi$$

$$\Rightarrow \quad \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = r^2\sin\theta\cos^2\phi + r^2\sin\theta\sin^2\phi = r^2\sin\theta \qquad \text{...}(iii)$$

Using (*ii*) and (*iii*) in equation (*i*), we get

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{1}{4} \times r^2\sin\theta = \frac{r^2\sin\theta}{4}.$$

**Example 6.** If $u = x(1-r^2)^{-1/2}$, $v = y(1-r^2)^{-1/2}$, $w = z(1-r^2)^{-1/2}$

where $r = \sqrt{x^2+y^2+z^2}$, then find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.

**Sol.** Since $r^2 = x^2 + y^2 + z^2$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

Differentiating partially $u = x(1-r^2)^{-1/2}$ w.r.t. $x$, we get

$$\frac{\partial u}{\partial x} = (1-r^2)^{\frac{-1}{2}} + x\left(\frac{-1}{2}\right)(-2r)\,(1-r^2)^{\frac{-3}{2}} \cdot \frac{\partial r}{\partial x}$$

$$= (1-r^2)^{\frac{-1}{2}} + rx\,(1-r^2)^{\frac{-3}{2}} \cdot \frac{x}{r} = \frac{1}{\sqrt{1-r^2}} + \frac{x^2}{(1-r^2)^{\frac{3}{2}}}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{1-r^2+x^2}{(1-r^2)^{\frac{3}{2}}}$$

Differentiating partially $u$ w.r.t. $y$, we get

$$\frac{\partial u}{\partial y} = x\left(\frac{-1}{2}\right)(1-r^2)^{\frac{-3}{2}} \cdot (-2r)\,\frac{\partial r}{\partial y} = \frac{xr}{(1-r^2)^{\frac{3}{2}}} \cdot \frac{y}{r}$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = \frac{xy}{(1-r^2)^{\frac{3}{2}}}$$

and $$\frac{\partial u}{\partial z} = \frac{xz}{(1-r^2)^{\frac{3}{2}}}$$

Similarly, $$\frac{\partial v}{\partial x} = \frac{yx}{(1-r^2)^{\frac{3}{2}}},\ \frac{\partial v}{\partial y} = \frac{1-r^2+y^2}{(1-r^2)^{\frac{3}{2}}},\ \frac{\partial v}{\partial z} = \frac{yz}{(1-r^2)^{\frac{3}{2}}}$$

$$\frac{\partial w}{\partial x} = \frac{zx}{(1-r^2)^{\frac{3}{2}}},\ \frac{\partial w}{\partial y} = \frac{zy}{(1-r^2)^{\frac{3}{2}}},\ \frac{\partial w}{\partial z} = \frac{1-r^2+z^2}{(1-r^2)^{\frac{3}{2}}}.$$

Thus, $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{1-r^2+x^2}{(1-r^2)^{\frac{3}{2}}} & \frac{xy}{(1-r^2)^{\frac{3}{2}}} & \frac{xz}{(1-r^2)^{\frac{3}{2}}} \\ \frac{yx}{(1-r^2)^{\frac{3}{2}}} & \frac{1-r^2+y^2}{(1-r^2)^{\frac{3}{2}}} & \frac{yz}{(1-r^2)^{\frac{3}{2}}} \\ \frac{zx}{(1-r^2)^{\frac{3}{2}}} & \frac{zy}{(1-r^2)^{\frac{3}{2}}} & \frac{1-r^2+z^2}{(1-r^2)^{\frac{3}{2}}} \end{vmatrix}$$

$$= \frac{1}{(1-r^2)^{\frac{9}{2}}} \begin{vmatrix} 1-r^2+x^2 & xy & xz \\ yx & 1-r^2+y^2 & yz \\ zx & zy & 1-r^2+z^2 \end{vmatrix}$$

$$= (1-r^2)^{\frac{-9}{2}} [(1-r^2+x^2)\{(1-r^2+y^2)(1-r^2+z^2) - y^2z^2\}$$
$$- xy\{xy(1-r^2+z^2) - xyz^2\} + xz\{xy^2z - zx(1-r^2+y^2)\}]$$

$$= (1-r^2)^{\frac{-9}{2}} [(1-r^2+x^2)(1-r^2+y^2)(1-r^2+z^2)$$
$$- (1-r^2)(y^2z^2+x^2y^2+x^2z^2) - x^2y^2z^2]$$

$$= (1-r^2)^{\frac{-9}{2}} [(1-r^2)^3 + (1-r^2)^2(x^2+y^2+z^2)]$$

$$= (1-r^2)^{\frac{-9}{2}} [(1-r^2)^3 + (1-r^2)^2 r^2]$$

$$= (1-r^2)^{\frac{-9}{2}} . (1-r^2)^2 [1-r^2+r^2] = (1-r^2)^{\frac{-5}{2}}.$$

**Example 7.** Verify the chain rule for Jacobians if $x = u$, $y = u \tan v$, $z = w$. (*U.P.T.U., 2008*)

**Sol.** We have
$$x = u \quad \Rightarrow \quad \frac{\partial x}{\partial u} = 1,\ \frac{\partial x}{\partial v} = \frac{\partial x}{\partial w} = 0$$
$$y = u \tan v \quad \Rightarrow \quad \frac{\partial y}{\partial u} = \tan v,\ \frac{\partial y}{\partial v} = u \sec^2 v,\ \frac{\partial y}{\partial w} = 0$$
$$z = w \quad \Rightarrow \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial v} = 0,\ \frac{\partial z}{\partial w} = 1$$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 0 & 0 \\ \tan v & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u \sec^2 v \qquad ...(i)$$

Solving for $u, v, w$ in terms of $x, y, z$, we have

$$u = x$$
$$v = \tan^{-1}\frac{y}{u} = \tan^{-1}\frac{y}{x}$$
$$w = z$$

$\therefore \quad \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0, \frac{\partial v}{\partial x} = -\frac{y}{x^2+y^2}, \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}, \frac{\partial v}{\partial z} = 0, \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0$ and $\frac{\partial w}{\partial z} = 1$

$$J' = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{x}{x^2+y^2} = \frac{1}{x\left(1+\frac{y^2}{x^2}\right)} = \frac{1}{u\sec^2 v} \qquad ...(ii)$$

Hence from (*i*) and (*ii*), we get

$$J.J' = u\sec^2 v.\ \frac{1}{u\sec^2 v} = 1.$$

### 2.1.3 Jacobian of Implicit Functions

If the variables $u, v$ and $x, y$ be connected by the equations

$$f_1(u, v, x, y) = 0 \qquad ...(i)$$
$$f_2(u, v, x, y) = 0 \qquad ...(ii)$$

*i.e.*, $u, v$ are implicit functions of $x, y$.

Differentiating partially (*i*) and (*ii*) w.r.t. $x$ and $y$, we get

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \qquad ...(iii)$$

$$\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial u}{\partial y} = 0 \qquad ...(iv)$$

$$\frac{\partial f_2}{\partial x} + \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \qquad ...(v)$$

$$\frac{\partial f_2}{\partial y} + \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial y} = 0 \qquad ...(vi)$$

Now, $$\frac{\partial(f_1, f_2)}{\partial(u, v)} \times \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial f_1}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v}\frac{\partial v}{\partial y} \\ \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_2}{\partial v}\frac{\partial v}{\partial y} \end{vmatrix}$$

Using (*iii*), (*iv*), (*v*) and (*vi*) in above, we get

$$= \begin{vmatrix} -\dfrac{\partial f_1}{\partial x} & -\dfrac{\partial f_1}{\partial y} \\ -\dfrac{\partial f_2}{\partial x} & -\dfrac{\partial f_2}{\partial y} \end{vmatrix} = (-1)^2 \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix}$$

Thus, $$\frac{\partial(f_1, f_2)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\partial(f_1, f_2)}{\partial(x,y)}$$

$$\Rightarrow \quad \frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\dfrac{\partial(f_1, f_2)}{\partial(x,y)}}{\dfrac{\partial(f_1, f_2)}{\partial(u,v)}}$$

Similarly for three variables $u$, $v$, $w$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u,v,w)}}$$

and so on.

**Example 8.** If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$, show that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{y^2 - x^2}{2uv(u-v)}.$$ (*U.P.T.U., 2006*)

**Sol.** Let $f_1 \equiv u^3 + v^3 - x - y = 0$

$f_2 \equiv u^2 + v^2 - x^3 - y^3 = 0$

Now, $$\frac{\partial(f_1, f_2)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix} = 3(y^2 - x^2)$$

and $$\frac{\partial(f_1, f_2)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix} = 6\,uv\,(u-v)$$

Thus, $$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\dfrac{\partial(f_1, f_2)}{\partial(x,y)}}{\dfrac{\partial(f_1, f_2)}{\partial(u,v)}} = \frac{3(y^2 - x^2)}{6uv(u-v)} = \frac{(y^2 - x^2)}{2uv(u-v)}.$$ Hence Proved.

**Example 9.** If $u$, $v$, $w$ are the roots of the equation in $k$, $\dfrac{x}{a+k} + \dfrac{y}{b+k} + \dfrac{z}{c+k} = 1$, prove that $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = -\dfrac{(u-v)(v-w)(w-u)}{(a-b)(b-c)(c-a)}$.

**Sol.** We have $\frac{x}{a+k} + \frac{y}{b+k} + \frac{z}{c+k} = 1$

or $x(b+k)(c+k) + y(a+k)(c+k) + z(a+k)(c+k) = (a+k)(b+k)(c+k)$

or $k^3 + k^2(a+b+c-x-y-z) + k\{bc+ca+ab-(b+c)x-(c+a)$
$y-(a+b)z\} + (abc-bcx-cay-abz) = 0$

Since its roots are given to be $u, v, w$, so we have

$$u+v+w = -(a+b+c-x-y-z)$$
$$uv+vw+wu = bc+ca+ab-(b+c)x-(c+a)y-(a+b)z$$
$$uvw = -(abc-bcx-cay-abz)$$

Let
$$f_1 \equiv u+v+w+a+b+c-x-y-z = 0$$
$$f_2 \equiv uv+vw+wu-bc-ca-ab+(b+c)x+(c+a)y+(a+b)z = 0$$
$$f_3 \equiv uvw+abc-bcx-cay-abz = 0$$

Now,
$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ (b+c) & (c+a) & (a+b) \\ -bc & -ca & -ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ b+c & a-b & a-c \\ bc & c(a-b) & b(a-c) \end{vmatrix} = (a-b)(a-c)(b-c)$$

and
$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ (v+w) & (w+u) & (u+v) \\ vw & wu & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix} \quad (C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1)$$

$$= (u-v)(u-w)(v-w)$$

Thus,
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} = -\frac{(a-b)(a-c)(b-c)}{(u-v)(u-w)(v-w)}$$

$\therefore$
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = -\frac{(u-v)(v-w)(u-w)}{(a-b)(b-c)(a-c)}.$$ **Hence proved.** | As $JJ' = 1$.

**Example 10.** If $u = 2axy$, $v = a(x^2 - y^2)$ where $x = r\cos\theta$, $y = r\sin\theta$, then prove that

$$\frac{\partial(u,v)}{\partial(r,\theta)} = -4a^2r^3.$$

**Sol.** We have $u = 2axy,\ v = a\ (x^2 - y^2)$

Now,
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2ay & 2ax \\ 2ax & -2ay \end{vmatrix} = -4a^2(x^2 + y^2)$$

or
$$\frac{\partial(u,v)}{\partial(x,y)} = -4a^2r^2 \qquad \left| \text{As } x^2 + y^2 = r^2 \right.$$

and
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Hence
$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = (-4a^2r^2).r = -4a^2r^3.$$ **Hence proved.**

**Example 11.** If $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$, $u + v + w = x^2 + y^2 + z^2$, then show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}.$$

**Sol.** Let
$$\begin{aligned} f_1 &\equiv u^3 + v^3 + w^3 - x - y - z = 0 \\ f_2 &\equiv u^2 + v^2 + w^2 - x^3 - y^3 - z^3 = 0 \\ f_3 &\equiv u + v + w - x^2 - y^2 - z^2 = 0 \end{aligned}$$

Now,
$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 \\ -3x^2 & 3(x^2-y^2) & 3(x^2-z^2) \\ -2x & 2(x-y) & 2(x-z) \end{vmatrix} \qquad \left| C_2 \to C_2 - C_1, C_3 \to C_3 - C_1 \right.$$

$$= -6[(x^2 - y^2)(x - z) - (x^2 - z^2)(x - y)]$$
$$= -6(x - y)(x - z)[(x + y) - (x + z)]$$

$$\Rightarrow \quad \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = 6(x - y)(y - z)(z - x)$$

and
$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2v & 2w \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 3u^2 & 3(v^2-u^2) & 3(w^2-u^2) \\ 2u & 2(v-u) & 2(w-u) \\ 1 & 0 & 0 \end{vmatrix} \quad \Big| C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$$

Expand it with respect to third row, we get

$$= 6[(v^2 - u^2)(w - u) - (w^2 - u^2)(v - u)]$$

$$= 6(v - u)(w - u)[(v + u) - (w + u)]$$

$$\Rightarrow \quad \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = -6(u - v)(v - w)(w - u)$$

Hence
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} = + \frac{6(x-y)(y-z)(z-x)}{6(u-v)(v-w)(w-u)}$$

$$= \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}.$$ **Hence proved.**

**Example 12.** $u, v, w$ are the roots of the equation

$$(x - a)^3 + (x - b)^3 + (x - c)^3 = 0, \text{ find } \frac{\partial(u, v, w)}{\partial(a, b, c)}.$$

**Sol.** We have $(x - a)^3 + (x - b)^3 + (x - c)^3 = 0$

$x^3 - a^3 - 3xa(x - a) + x^3 - b^3 - 3xb(x - b) + x^3 - c^3 - 3xc(x - c) = 0$

or $3x^3 - 3x^2(a + b + c) + 3x(a^2 + b^2 + c^2) - (a^3 + b^3 + c^3) = 0$

Since $u, v, w$ are the roots of this equation, we have

$$u + v + w = a + b + c$$

$$uv + vw + wu = a^2 + b^2 + c^2$$

$$uvw = \frac{a^3 + b^3 + c^3}{3}$$

$$\Bigg| \text{As} \quad \alpha + \beta + \gamma = -b/a, \quad \alpha\beta + \beta\gamma + \gamma\alpha = c/a, \quad \alpha\beta\gamma = -d/a$$

Let
$$f_1 \equiv u + v + w - a - b - c = 0$$

$$f_2 \equiv uv + vw + wu - a^2 - b^2 - c^2 = 0$$

$$f_3 = uvw - \frac{a^3 + b^3 + c^3}{3}$$

Now
$$\frac{\partial(f_1, f_2, f_3)}{\partial(a, b, c)} = \begin{vmatrix} -1 & -1 & -1 \\ -2a & -2b & -2c \\ -a^2 & -b^2 & -c^2 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ -2a & 2(a-b) & 2(a-c) \\ -a^2 & (a^2-b^2) & (a^2-c^2) \end{vmatrix}, \begin{pmatrix} c_2 \to c_2 - c_1 \\ c_3 \to c_3 - c_1 \end{pmatrix}$$

$$= -2\{(a - b)(a^2 - c^2) - (a - c)(a^2 - b^2)\} = -2(a - b)(b - c)(c - a)$$

and
$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & v+u \\ vw & wu & uv \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix}, \begin{pmatrix} c_2 \to c_2 - c_1 \\ c_3 \to c_3 - c_1 \end{pmatrix}$$

$$= (u - v)\, v(u - w) - (u - w)\, w(u - v)$$

$$= -(u - v)(v - w)(w - u)$$

Thus $$\frac{\partial(u,v,w)}{\partial(a,b,c)} = (-1)^3 \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(a,b,c)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u,v,w)}} = -\frac{-2(a-b)(b-c)(c-a)}{-(u-v)(v-w)(w-u)}$$

$$= -2\frac{(a-b)(b-c)(c-a)}{(u-v)(v-w)(w-u)}.$$

### 2.1.4 Functional Dependence

Let $u = f_1(x, y)$, $v = f_2(x, y)$ be two functions. Suppose $u$ and $v$ are connected by the relation $f(u, v) = 0$, where $f$ is differentiable. Then $u$ and $v$ are called functionally dependent on one another (*i.e.*, one function say $u$ is a function of the second function $v$) if the $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ are not all zero simultaneously.

**Necessary and sufficient condition for functional dependence (Jacobian for functional dependence functions):**

Let $u$ and $v$ are functionally dependent then

$$f(u, v) = 0 \qquad \text{...}(i)$$

Differentiate partially equation (*i*) w.r.t. $x$ and $y$, we get

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = 0 \qquad \text{...}(ii)$$

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = 0 \qquad \text{...}(iii)$$

There must be a non-trivial solution for $\dfrac{\partial f}{\partial u} \neq 0$, $\dfrac{\partial f}{\partial v} \neq 0$ to this system exists.

Thus, $$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0$$ | For non-trivial solution $|A| = 0$

or $$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0$$ | Changing all rows in columns

or $$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

Hence, two functions $u$ and $v$ are **"functionally dependent"** if their Jacobian is equal to zero.

**Note:** The functions $u$ and $v$ are said to be "functionally independent" if their Jacobian is not equal to zero *i.e.*, $J(u, v) \neq 0$

Similarly for three functionally dependent functions say $u$, $v$ and $w$.

$$J(u, v, w) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$$

**Example 13.** Show that the functions $u = x + y - z$, $v = x - y + z$, $w = x^2 + y^2 + z^2 - 2yz$ are not independent of one another. Also find the relation between them.

**Sol.** Here $u = x + y - z$, $v = x - y + z$ and $w = x^2 + y^2 + z^2 - 2yz$

Now,
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2y-2z & 2z-2y \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x & 2y-2z & 0 \end{vmatrix} \quad (C_3 \to C_3 + C_2)$$

$= 0$. Hence $u$, $v$, $w$ are not independent.

Again
$$u + v = x + y - z + x - y + z = 2x$$
$$u - v = x + y - z - x + y - z = 2(y - z)$$
$$\therefore \quad (u + v)^2 + (u - v)^2 = 4x^2 + 4(y - z)^2$$
$$= 4(x^2 + y^2 + z^2 - 2yz) = 4w$$
$$\Rightarrow \quad (u + v)^2 + (u - v)^2 = 4w$$
or
$$2(u^2 + v^2) = 4w \text{ or } u^2 + v^2 = 2w.$$

**Example 14.** Find Jacobian of $u = \sin^{-1} x + \sin^{-1} y$ and $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$. Also find relation between $u$ and $v$.

**Sol.** We have $u = \sin^{-1} x + \sin^{-1} y$, $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

Now,
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix}$$

$$= -\frac{xy}{\sqrt{1-x^2}\,\sqrt{1-y^2}} + 1 - 1 + \frac{xy}{\sqrt{1-x^2}\,\sqrt{1-y^2}} = 0.$$ Hence $u$ and $v$ are dependent.

Next,
$$u = \sin^{-1} x + \sin^{-1} y \Rightarrow u = \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}$$

$$\left|\text{As } \sin^{-1}A + \sin^{-1}B = \sin^{-1}\left\{A\sqrt{1-B^2} + B\sqrt{1-A^2}\right\}\right.$$

$$\Rightarrow \quad \sin u = x\sqrt{1-y^2} + y\sqrt{1-x^2} = v$$
or
$$v = \sin u.$$

**Example 15.** Show that $ax^2 + 2hxy + by^2$ and $Ax^2 + 2Hxy + By^2$ are independent unless $\frac{a}{A} = \frac{h}{H} = \frac{b}{B}$.

**Sol.** Let
$$u = ax^2 + 2hxy + by^2$$
$$v = Ax^2 + 2Hxy + By^2$$

If $u$ and $v$ are not independent, then $\frac{\partial(u,v)}{\partial(x,y)} = 0$

or
$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2ax+2hy & 2hx+2by \\ 2Ax+2Hy & 2Hx+2By \end{vmatrix} = 0$$

$\Rightarrow$ $(ax + hy)(Hx + By) - (hx + by)(Ax + Hy) = 0$

$\Rightarrow$ $(aH - hA)x^2 + (aB - bA)xy + (hB - bH)y^2 = 0$

But variable $x$ and $y$ are independent so the coefficients of $x^2$ and $y^2$ must separately vanish and therefore, we have

$$aH - hA = 0 \text{ and } hB - bH = 0 \text{ i.e., } \frac{a}{A} = \frac{h}{H} \text{ and } \frac{h}{H} = \frac{b}{B}$$

*i.e,*
$$\frac{a}{A} = \frac{h}{H} = \frac{b}{B}.$$ **Hence proved.**

**Example 16.** If $u = x^2 e^{-y} \cos hz$, $v = x^2 e^{-y} \sin hz$ and $w = 3x^4 e^{-2y}$ then prove that $u, v, w$ are functionally dependent. Hence establish the relation between them.

**Sol.** We have $u = x^2 e^{-y} \cos hz$, $v = x^2 e^{-y} \sin hz$, $w = 3x^4 e^{-2y}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix} = \begin{vmatrix} 2xe^{-y}\cos hz & -x^2e^{-y}\cos hz & x^2e^{-y}\sin hz \\ 2xe^{-y}\sin hz & -x^2e^{-y}\sin hz & x^2e^{-y}\cos hz \\ 12x^3e^{-2y} & -6x^4e^{-2y} & 0 \end{vmatrix}$$

$$= 2x\, e^{-y} \cos hz\{0 + 6x^6 e^{-3y} \cos hz\} + x^2 e^{-y} \cos hz\{0 - 12x^5 e^{-3y} \cos hz\}$$
$$+ x^2 e^{-y} \sin hz\{-12x^5 e^{-3y} \sin hz + 12x^5 e^{-3y} \sin hz\}$$
$$= 12x^7 e^{-4y} \cos h^2 z - 12x^7 e^{-4y} \cos h^2 z = 0$$

Thus $u, v$ and $w$ are functionally dependent.

Next, $3u^2 - 3v^2 = 3(x^4 e^{-2y} \cos h^2 z - x^4 e^{-2y} \sin h^2 z) = 3x^4 e^{-2y} (\cos h^2 z - \sin h^2 z)$
$$= 3x^4 e^{-2y}$$

$\Rightarrow$ $3u^2 - 3v^2 = w.$

## EXERCISE 2.1

1. If $x = r \cos \theta$, $y = r \sin \theta$ find $\frac{\partial(x,y)}{\partial(r,\theta)}$. $\left[\textbf{Ans. } \frac{1}{r}\right]$

2. If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$ show that the Jacobian of $y_1, y_2, y_3$ with respect to $x_1$, $x_2$, $x_3$ is 4. (*U.P.T.U., 2004(CO), 2002*)

**3.** If $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$, show that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 \sin\theta.$$ (*U.P.T.U., 2000*)

**4.** If $u = x + y + z$, $uv = y + z$, $uvw = z$, evaluate $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$. (*U.P.T.U., 2003*) $\left[\textbf{Ans. } u^2v\right]$

**5.** If $u = \dfrac{y^2}{2x}$, $v = \dfrac{\left(x^2+y^2\right)}{2x}$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$. $\left[\textbf{Ans. } -\dfrac{y}{2x}\right]$

**6.** If $x = a \cos h\,\xi\ \cos\eta$, $y = a \sin h\,\xi \sin\eta$, show that

$$\frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{1}{2}a^2(\cos h\,2\xi - \cos 2\eta).$$

**7.** If $u^3 + v + w = x + y^2 + z^2$, $u + v^3 + w = x^2 + y + z^2$, $u + v + w^3 = x^2 + y^2 + z$, then evaluate $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$. $\left[\textbf{Ans. } \dfrac{(1-4xy-4yz-4zx+6xyz)}{27u^2v^2w^2+2-3\left(u^2+v^2+w^2\right)}\right]$

**8.** If $u, v, w$ are the roots of the equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ in $\lambda$, find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.

(*U.P.T.U., 2001*) $\left[\textbf{Ans. } \dfrac{-2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}\right]$

**9.** If $u = x_1 + x_2 + x_3 + x_4$, $uv = x_2 + x_3 + x_4$, $uvw = x_3 + x_4$ and $uvwt = x_4$, show that

$$\frac{\partial(x_1,x_2,x_3,x_4)}{\partial(u,v,w,t)} = u^3v^2w.$$

**10.** Calculate $J = \dfrac{\partial(u,v)}{\partial(x,y)}$ and $J' = \dfrac{\partial(x,y)}{\partial(u,v)}$. Verify that $JJ' = 1$ given

(*i*) $u = x + \dfrac{y^2}{x}$, $v = \dfrac{y^2}{x}$. $\left[\textbf{Ans. } J = \dfrac{2y}{x}, J' = \dfrac{x}{2y}\right]$

(*ii*) $x = e^u \cos v$, $y = e^u \sin v$. $\left[\textbf{Ans. } J = e^{2u}, J' = e^{-2u}.\right]$

**11.** Show that $\dfrac{\partial(u,v)}{\partial(r,\theta)} = 6r^3 \sin 2\theta$ given $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$.

**12.** If $X = u^2v$, $Y = uv^2$ and $u = x^2 - y^2$, $v = xy$, find $\dfrac{\partial(X,Y)}{\partial(x,y)}$. $\left[\textbf{Ans. } 6x^2y^2\left(x^2+y^2\right)\left(x^2-y^2\right)^2\right]$

**13.** Find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$, if $u = x^2$, $v = \sin y$, $w = e^{-3z}$. $\left[\textbf{Ans. } -6e^{-3z}x\cos y\right]$

**14.** Find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$, if $u = 3x + 2y - z$, $v = x - y + z$, $w = x + 2y - z$. $\left[\textbf{Ans. } -2\right]$

**15.** Find $J(u, v, w)$ if $u = xyz$, $v = xy + yz + zx$, $w = x + y + z$. $\left[\textbf{Ans. } (x-y)(y-z)(z-x)\right]$

**16.** Prove that $u, v, w$ are dependent and find relation between them if $u = xe^y \sin z$, $v = xe^y \cos z$, $w = x^2e^{2y}$. $\left[\textbf{Ans. } \text{dependent}, u^2+v^2=w\right]$

**17.** $u = \dfrac{3x^2}{2(y+z)}$, $v = \dfrac{2(y+z)}{3(x-y)^2}$, $w = \dfrac{x-y}{x}$. $\left[\textbf{Ans. } \text{dependent}, uvw^2=1\right]$

**18.** If $X = x + y + z + u$, $Y = x + y - z - u$, $Z = xy - zu$ and $U = x^2 + y^2 - z^2 - u^2$, then show that $J = \dfrac{\partial(X, Y, Z, U)}{\partial(x, y, z, u)} = 0$ and hence find a relation between $X, Y, Z$ and $U$.

$\left[\textbf{Ans. } XY = U + 2Z\right]$

**19.** If $u = \dfrac{x}{y-z}$, $v = \dfrac{y}{z-x}$, $w = \dfrac{z}{x-y}$, then prove that $u, v, w$ are not independent and also find the relation between them. $\left[\textbf{Ans. } uv+vw+wu+1=0\right]$

**20.** If $u = x + 2y + z$, $v = x - 2y + 3z$, $w = 2xy - xz + 4yz - 2z^2$, show that they are not independent. Find the relation between $u, v$ and $w$. $\left[\textbf{Ans. } 4w = u^2 - v^2\right]$

**21.** If $u = \dfrac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$. Are $u$ and $v$ functionally related? If yes find the relationship. [**Ans.** yes, $u = \tan v$]

**22.** If $u = x + y + z$, $uv = y + z$, $uvw = z$, show that $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$.

**23.** If $x^2 + y^2 + u^2 - v^2 = 0$ and $uv + xy = 0$ prove that $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{x^2-y^2}{u^2+v^2}$.

**24.** Find Jacobian of $u, v, w$ w.r.t. $x, y, z$ when $u = \dfrac{yz}{x}$, $v = \dfrac{zx}{y}$, $w = \dfrac{xy}{z}$. [**Ans.** 4]

## 2.2 APPROXIMATION OF ERRORS

Let $u = f(x, y)$ then the total differential of $u$, denoted by $du$, is given by

$$du = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \qquad ...(i)$$

If $\delta x$ and $\delta y$ are increments in $x$ and $y$ respectively then the total increment $\delta u$ in $u$ is given by

$$\delta u = f(x + \delta x, y + \delta y) - f(x, y)$$

or

$$f(x + \delta x, y + \delta y) = f(x, y) + \delta u \qquad ...(ii)$$

But

$$\delta u \approx du,\ \delta x \approx dx \text{ and } \delta y \approx dy$$

$\therefore$ From (*i*)

$$\delta u \approx \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial y}\,\delta y \qquad ...(iii)$$

Using (*iii*) in (*ii*), we get the approximate formula

$$\boxed{f(x+\delta x,\, y+\delta y) \approx f(x, y) + \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial y}\,\delta y} \qquad ...(iv)$$

Thus, the approximate value of the function can be obtained by equation (*iv*). Hence

$$\delta x \text{ or } dx = \text{Absolute error}$$

$$\frac{\delta x}{x} \text{ or } \frac{dx}{x} = \text{Proportional or Relative error}$$

and $$100 \times \frac{dx}{x} \text{ or } 100 \times \frac{dx}{x} = \text{Percentage error in } x.$$

**Example 1.** If $f(x, y) = x^2 y^{\frac{1}{10}}$, compute the value of $f$ when $x = 1.99$ and $y = 3.01$.

(*U.P.T.U., 2007*)

**Sol.** We have $f(x, y) = x^2 y^{\frac{1}{10}}$

$$\therefore \quad \frac{\partial f}{\partial x} = 2xy^{\frac{1}{10}}, \quad \frac{\partial f}{\partial y} = \frac{1}{10}x^2 y^{-\frac{9}{10}}$$

Let $x = 2,\ \delta x = -0.01$

$y = 1,\ \delta y = 2.01$

As $x+\delta x = 2+(-0.01) = 1.99$
$y+\delta y = 1+(2.01) = 3.01$

Now, $f(x+\delta x,\, y+\delta y) = f(x, y) + \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial y}\,\delta y$

$$\Rightarrow f\{2+(-0.01),\, 1+2.01\} = f(2, 1) + 2\times 2(1)^{\frac{1}{10}} \times (-0.01) + \frac{1}{10}(2)^2.\ (1)^{-\frac{9}{10}} \times (2.01)$$

$$\Rightarrow \quad f(1.99, 3.01) \approx 2^2 \times 1^{\frac{1}{10}} + (-0.04) + 0.804$$
$$\approx 4 - 0.04 + 0.804 = 4.764.$$

**Example 2.** The diameter and height of a right circular cylinder are measured to be 5 and 8 cm. respectively. If each of these dimensions may be in error by ± 0.1 cm, find the relative percentage error in volume of the cylinder.

**Sol.** Let diameter of cylinder = $x$ cm.

height of cylinder = $y$ cm.

then $$V = \frac{\pi x^2 y}{4} \quad \left(\text{radius} = \frac{x}{2}\right)$$

$$\therefore \quad \frac{\partial V}{\partial x} = \frac{\pi x y}{2}, \quad \frac{\partial V}{\partial y} = \frac{\pi x^2}{4}$$

$$\Rightarrow \quad dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy$$

$$\Rightarrow \qquad dV = \frac{1}{2}\pi\, xy.\, dx + \frac{1}{4}\,\pi x^2.\, dy$$

or
$$\frac{dV}{V} = \frac{\frac{1}{2}\pi xy.dx}{\frac{\pi x^2 y}{4}} + \frac{\frac{1}{4}\pi x^2 dy}{\frac{\pi x^2 y}{4}} = 2.\ \frac{dx}{x} + \frac{dy}{y}$$

or
$$100 \times \frac{dV}{V} = 2\left(100\times\frac{dx}{x}\right) + 100 \times \frac{dy}{y}$$

Given $x = 5$ cm., $y = 8$ cm. and error $dx = dy = \pm\, 0.1$. So $100 \times \dfrac{dV}{V} = \pm\, 100\left(2\times\dfrac{0.1}{5}+\dfrac{0.1}{8}\right)$ $= \pm\, 5.25$.

Thus, the percentage error in volume = ± 5.25.

**Example 3.** A balloon is in the form of right circular cylinder of radius 1.5 m and length 4 m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and the length by 0.05 m, find the percentage change in the volume of the balloon. [*U.P.T.U., 2005 (Comp.), 2002*]

**Sol.** Let radius $= r = 1.5$ m, $\delta r = 0.01$ m

height $= h = 4$ m, $\delta h = 0.05$ m

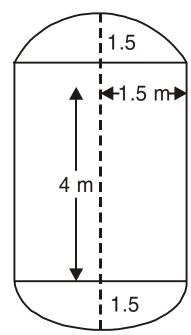


**Fig. 2.1**

$$\text{volume } (V) = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3$$

$$\therefore \qquad \frac{\partial V}{\partial r} = 2\pi rh + 4\pi r^2, \quad \frac{\partial V}{\partial h} = \pi r^2$$

$$dV = \frac{\partial V}{\partial r}\, dr + \frac{\partial V}{\partial h}\, dh = (2\pi rh + 4\pi r^2)\, dr + \pi r^2 dh$$

or
$$\frac{dV}{V} = \frac{2\pi r(h+2r)}{\pi r^2\left(h+\frac{4r}{3}\right)}\, dr + \frac{\pi r^2}{\pi r^2\left(h+\frac{4r}{3}\right)}\, dh$$

$$= \frac{3\times 2(h+2r)}{r(3h+4r)}\, dr + \frac{3}{(3h+4r)}\, dh = \frac{3}{r(3h+4r)}\,[2(h + 2r)\, dr + rdh]$$

$$= \frac{3}{1.5(12+6)}\,[2(4 + 3)(0.01) + 1.5\,(0.05)] \qquad \left|\begin{array}{l}\delta r = dr\\ \delta h = dh\end{array}\right.$$

$$= \frac{1}{9}\,[0.14 + 0.075] = \frac{0.215}{9}$$

$$\Rightarrow \qquad 100 \times \frac{dV}{V} = 100 \times \frac{0.215}{9} = 2.389\%$$

Thus, change in the volume = 2.389%.