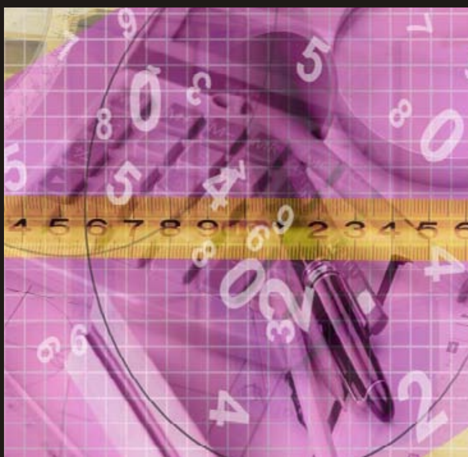

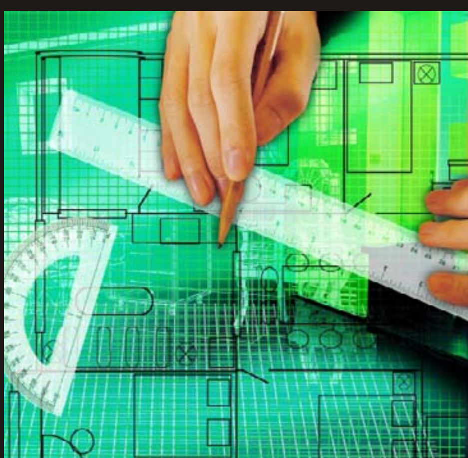

NEW AGE

SECOND EDITION

A Textbook of ENGINEERING MATHEMATICS-I

H.S. Gangwar • Prabhakar Gupta

NEW AGE INTERNATIONAL PUBLISHERS

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(SECOND EDITION)

H.S. Gangwar
M.Sc., Ph.D.
Lecturer
Deptt. of Mathematics
SRMS College of Engineering
and Technology, Bareilly (U.P.)

Prabhakar Gupta
M.Sc. (Math.), M.Tech., Ph.D.
Dean Academics
SRMS College of Engineering
and Technology, Bareilly (U.P.)

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Preface to the Second Revised Edition

This book has been revised exhaustively according to the global demands of the students. Attention has been taken to add minor steps between two unmanageable lines where essential so that the students can understand the subject matter without mental tire.

A number of questions have been added in this edition besides theoretical portion wherever necessary in the book. Latest question papers are fully solved and added in their respective units.

Literal errors have also been rectified which have been accounted and have come to our observation. Ultimately the book is a gift to the students which is now error free and user- friendly.

Constructive suggestions, criticisms from the students and the teachers are always welcome for the improvement of this book.

AUTHORS

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Some Useful Formulae

1. $\sin ix = i \sin hx$

2. $\cos ix = \cos hx$

3. $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

4. $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

5. $\text{Sin } h^2 x = \dfrac{1}{2} \ (\cos h\ 2x - 1)$

6. $\cos h^2 x = \dfrac{1}{2} \ (\cos h\ 2x + 1)$

7. $\int a^x dx = \dfrac{a^x}{\log a} [a \neq 1, a > 0]$

8. $\int \sin hax dx = \dfrac{1}{a} \cos hax$

9. $\int \cos hax dx = \dfrac{1}{a} \sin hax$

10. $\int \tan hax dx = \dfrac{1}{a} \log |\cos hax|$

11. $\int \dfrac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \dfrac{x}{a} = arc \sin \dfrac{x}{a}$

12. $\int \dfrac{1}{\sqrt{x^2 - a^2}} dx = \log \left| x + \sqrt{x^2 - a^2} \right|$

13. $\int \dfrac{1}{x^2 + a^2} dx = \dfrac{1}{a} \tan^{-1} \dfrac{x}{a} = arc \tan \dfrac{x}{a}$

14. $\int \sqrt{a^2 - x^2}\ dx = \dfrac{x}{2} \sqrt{a^2 - x^2} + \dfrac{a^2}{2} \sin^{-1} \dfrac{x}{|a|}$

15. $\int e^{ax} \sin bx\ dx = \dfrac{e^{ax}}{a^2 + b^2} \ (a \sin bx - b \cos bx)$

16. $\int e^{ax} \cos bx\ dx = \dfrac{e^{ax}}{a^2 + b^2} \ (a \cos bx + b \sin bx)$

17. $\int \sec ax\ dx = \dfrac{1}{a} \log \ |\sec ax + \tan ax|$

18. $\int \operatorname{cosec} ax\, dx = \frac{1}{a} \log \left| \operatorname{cosec} ax - \cot ax \right|$

19. $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} \ldots\ldots$

20. $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} \ldots\ldots\ldots$

21. $\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 - \frac{17}{315} x^7 + \ldots\ldots\ldots$

22. $\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$

23. $\log (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\ldots\ldots$

24. $\sin hx = x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \ldots\ldots\ldots$

25. $\frac{d}{dx} a^x = a^x \log_e a$

26. $\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$

27. $\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$

28. $\frac{d}{dx} \operatorname{cosec}^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$

29. $\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e$

30. $\frac{d}{dx} \tan^{-1} \frac{x}{a} = \frac{a}{a^2+x^2}.$

Contents

UNIT I

Differential Calculus-I

1.0 INTRODUCTION

Calculus is one of the most beautiful intellectual achievements of human being. The mathematical study of change motion, growth or decay is calculus. One of the most important idea of differential calculus is derivative which measures the rate of change of a given function. Concept of derivative is very useful in engineering, science, economics, medicine and computer science.

The first order derivative of y denoted by $\frac{dy}{dx}$, second order derivative, denoted by $\frac{d^2y}{dx^2}$ third order derivative by $\frac{d^3y}{dx^3}$ and so on. Thus by differentiating a function $y = f(x)$, n times, successively, we get the nth order derivative of y denoted by $\frac{d^ny}{dx^n}$ or D^ny or $y_n(x)$. Thus, the process of finding the differential co-efficient of a function again and again is called **Successive Differentiation.**

1.1 *n*th DERIVATIVE OF SOME ELEMENTARY FUNCTIONS

1. Power Function $(ax + b)^m$

Let

$$y = (ax + b)^m$$

$$y_1 = ma\,(ax + b)^{m-1}$$

$$y_2 = m\,(m-1)a^2\,(ax + b)^{m-2}$$

....

....

$$y_n = m(m-1)\,(m-2)\,\ldots\,(m - \overline{n - 1})\,a^n\,(ax + b)^{m-n}$$

Case I. When m is positive integer, then

$$y_n = \frac{m\,(m-1)\ldots(m-n+1)(m-n)\ldots3\cdot2\cdot1}{(m-n)\ldots3\cdot2\cdot1}\,a^n(ax + b)^{m-n}$$

$\Rightarrow$

$$\boxed{y_n = \frac{d^n}{dx^n}(ax+b)^m = \frac{\lfloor m}{\lfloor m-n}\,a^n\,(ax+b)^{m-n}}$$

Case II. When $m = n =$ +ve integer

$$y_n = \frac{\lfloor n}{\lfloor 0} a^n (ax+b)^0 = \lfloor n \, a^n \Rightarrow \boxed{\frac{d^n}{dx^n}(ax+b)^n = \lfloor n \, a^n}$$

Case III. When $m = -1$, then

$$y = (ax+b)^{-1} = \frac{1}{(ax+b)}$$

$$\therefore \quad y_n = (-1)(-2)(-3) \ldots (-n)\, a^n (ax+b)^{-1-n}$$

$$\Rightarrow \quad \boxed{\frac{d^n}{dx^n}\left\{\frac{1}{ax+b}\right\} = \frac{(-1)^n \lfloor n \, a^n}{(ax+b)^{n+1}}}$$

Case IV. Logarithm case: When $y = \log(ax+b)$, then

$$y_1 = \frac{a}{ax+b}$$

Differentiating (n–1) times, we get

$$y_n = a^n \frac{d^{n-1}}{dx^{n-1}}(ax+b)^{-1}$$

Using case III, we obtain

$$\Rightarrow \quad \boxed{\frac{d^n}{dx^n}\{\log(ax+b)\} = \frac{(-1)^{n-1} \lfloor (n-1)\, a^n}{(ax+b)^n}}$$

2. Exponential Function

(i) Consider

$$y = a^{mx}$$
$$y_1 = ma^{mx} . \log_e a$$
$$y_2 = m^2 a^{mx} (\log_e a)^2$$

....

....

$$\boxed{y_n = m^n a^{mx} (\log_e a)^n}$$

(ii) Consider $\quad y = e^{mx}$

Putting $\quad a = e$ in above $\quad \boxed{y_n = m^n e^{mx}}$

3. Trigonometric Functions cos ($ax + b$) or sin ($ax + b$)

Let $\quad y = \cos(ax+b)$, then

$$y_1 = -a \sin(ax+b) = a \cos\left(ax+b+\frac{\pi}{2}\right)$$

$$y_2 = -a^2 \cos(ax+b) = a^2 \cos\left(ax+b+\frac{2\pi}{2}\right)$$

$$y_3 = +a^3 \sin(ax+b) = a^3 \cos\left(ax+b+\frac{3\pi}{2}\right)$$

....

....

$$\boxed{y_n = \frac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)}$$

Similarly, $$\boxed{y_n = \frac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)}$$

4. Product Functions e^{ax} sin $(bx + c)$ or e^{ax} cos $(bx + c)$

Consider the function $y = e^{ax} \sin(bx + c)$

$$y_1 = e^{ax} \cdot b \cos(bx + c) + ae^{ax} \sin(bx + c)$$

$$= e^{ax} [b \cos(bx + c) + a \sin(bx + c)]$$

To rewrite this in the form of sin, put

$$a = r \cos \phi, \; b = r \sin \phi, \text{ we get}$$

$$y_1 = e^{ax} [r \sin \phi \cos(bx + c) + r \cos \phi \sin(bx + c)]$$

$$y_1 = re^{ax} \sin(bx + c + \phi)$$

Here, $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}(b/a)$

Differentiating again w.r.t. x, we get

$$y_2 = rae^{ax} \sin(bx + c + \phi) + rbe^{ax} \cos(bx + c + \phi)$$

Substituting for a and b, we get

$$y_2 = re^{ax}.\, r \cos \phi \sin(bx + c + \phi) + re^{ax}\, r \sin \phi \cos(bx + c + \phi)$$

$$y_2 = r^2e^{ax} [\cos \phi \sin(bx + c + \phi) + \sin \phi \cos(bx + c + \phi)]$$

$$= r^2 e^{ax} \sin(bx + c + \phi + \phi)$$

∴ $$y_2 = r^2 e^{ax} \sin(bx + c + 2\phi)$$

Similarly, $$y_3 = r^3e^{ax} \sin(bx + c + 3\phi)$$

....

$$\boxed{y_n = \frac{d^n}{dx^n} e^{ax}\sin(bx+c) = r^n e^{ax} \sin(bx+c+n\phi)}$$

In similar way, we obtain

$$\boxed{y_n = \frac{d^n}{dx^n} e^{ax}\cos(bx+c) = r^n e^{ax} \cos(bx+c+n\phi)}$$

Example 1. Find the nth derivative of $\dfrac{1}{1-5x+6x^2}$

Sol. Let $$y = \frac{1}{1-5x+6x^2} = \frac{1}{(2x-1)(3x-1)}$$

or $$y = \frac{2}{2x-1} - \frac{3}{3x-1} \qquad \text{(By Partial fraction)}$$

∴ $$y_n = 2\frac{d^n}{dx^n}(2x-1)^{-1} - 3\frac{d^n}{dx^n}(3x-1)^{-1}$$

$$= 2\left[\frac{(-1)^n \lfloor n\, 2^n}{(2x-1)^{n+1}}\right] - 3\left[\frac{(-1)^n \lfloor n\, 3^n}{(3x-1)^{n+1}}\right] \quad \left| \text{As } \frac{d^n}{dx^n}(ax+b)^{-1} = \frac{(-1)^n \lfloor n\, a^n}{(ax+b)^{n+1}}\right.$$

or $$y_n = (-1)^n \lfloor n\left[\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{3^{n+1}}{(3x-1)^{n+1}}\right].$$

Example 2. Find the nth derivative of $e^{ax} \cos^2 x \sin x$.

Sol. Let $$y = e^{ax} \cos^2 x \sin x = e^{ax} \frac{(1+\cos 2x)}{2} \sin x$$

$$= \frac{1}{2}e^{ax}\sin x + \frac{1}{2\times 2}e^{ax}(2\cos 2x \sin x)$$

$$= \frac{1}{2}e^{ax}\sin x + \frac{1}{4}e^{ax}\{\sin(3x) - \sin x\}$$

or $$y = \frac{1}{4}e^{ax}\sin x + \frac{1}{4}e^{ax}\sin 3x$$

$\therefore$ $$y_n = \frac{1}{4}\left[r^n e^{ax}\sin\ (x+n\phi)\right] + \frac{1}{4}\left[r_1{}^n e^{ax}\sin\ (3x+n\theta)\right].$$

where $r = \sqrt{a^2+1}$; $\tan \phi = 1/a$

and $r_1 = \sqrt{a^2+9}$; $\tan \theta = 3/a$.

Example 3. If $y = \tan^{-1} \dfrac{2x}{1-x^2}$, find y_n. (*U.P.T.U., 2002*)

Sol. We have $$y = \tan^{-1}\frac{2x}{1-x^2}$$

Differentiating y w.r.t. x, we get

$$y_1 = \frac{1}{1+\left(\dfrac{2x}{1-x^2}\right)^2}\cdot\frac{d}{dx}\left(\frac{2x}{1-x^2}\right) = \frac{(1-x^2)^2}{\left(1+x^4-2x^2+4x^2\right)}\cdot\frac{2(1-x^2)+4x^2}{(1-x^2)^2}$$

$$y_1 = \frac{2(1+x^2)}{(1+x^2)^2} = \frac{2}{(1+x^2)} = \frac{2}{(x+i)(x-i)}$$

$$y_1 = \frac{1}{i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right], \text{ (by Partial fractions)}$$

Differentiating both sides $(n–1)$ times w.r. to 'x', we get

$$y_n = \frac{1}{i}\left[\frac{(-1)^{n-1}(\lfloor n-1)}{(x-i)^n} - \frac{(-1)^{n-1}(\lfloor n-1)}{(x+i)^n}\right]$$

$$= \frac{(-1)^{n-1}(\lfloor n-1)}{i}\left[(x-i)^{-n} - (x+i)^{-n}\right]$$

$$= \frac{(-1)^{n-1}(\lfloor n-1)}{i}\left[r^{-n}(\cos\theta - i\sin\theta)^{-n} - r^{-n}(\cos\theta + i\sin\theta)^{-n}\right]$$

(where $x = r\cos\theta$, $1 = r\sin\theta$)

$$= \frac{(-1)^{n-1}(\lfloor n-1) r^{-n}}{i}[\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta]$$

$$y_n = 2(-1)^{n-1} \lfloor n-1 \; r^{-n} \sin n\theta, \text{ where } r = \sqrt{x^2+1}$$

$$\theta = \tan^{-1}\left(\frac{1}{x}\right).$$

Example 4. If $y = x \log \frac{x-1}{x+1}$. Show that

$$y_n = (-1)^{n-2} \lfloor n-2 \left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right] \qquad (\textit{U.P.T.U., 2002})$$

Sol. We have $\quad y = x \log \frac{x-1}{x+1} = x\,[\log(x-1) - \log(x+1)]$

Differentiating w.r. to 'x', we get

$$y_1 = \log(x-1) - \log(x+1) + x\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$$

$$= \log(x-1) - \log(x+1) + \left(1+\frac{1}{x-1}\right) + \left(-1+\frac{1}{x+1}\right)$$

or $$y_1 = \log(x-1) - \log(x+1) + \frac{1}{x-1} + \frac{1}{x+1}$$

Differentiating $(n-1)$ times with respect to $\boldsymbol{x}$, we get

$$y_n = \frac{d^{n-1}}{dx^{n-1}}\log(x-1) - \frac{d^{n-1}}{dx^{n-1}}\log(x+1) + \frac{d^{n-1}}{dx^{n-1}}(x-1)^{-1} + \frac{d^{n-1}}{dx^{n-1}}(x+1)^{-1}$$

$$= \frac{d^{n-2}}{dx^{n-2}}\left\{\frac{d}{dx}\log(x-1)\right\} - \frac{d^{n-2}}{dx^{n-2}}\left\{\frac{d}{dx}\log(x+1)\right\} + \frac{(-1)^{n-1}\lfloor n-1}{(x-1)^n} + \frac{(-1)^{n-1}\lfloor n-1}{(x+1)^n}$$

$$= \frac{d^{n-2}}{dx^{n-2}}\left(\frac{1}{x-1}\right) - \frac{d^{n-2}}{dx^{n-2}}\left(\frac{1}{x+1}\right) + \frac{(-1)^{n-1}\lfloor n-1}{(x-1)^n} + \frac{(-1)^{n-1}\lfloor n-1}{(x+1)^n}$$

$$= \frac{(-1)^{n-2}\lfloor n-2}{(x-1)^{n-1}} - \frac{(-1)^{n-2}\lfloor n-2}{(x+1)^{n-1}} + \frac{(-1)^{n-1}(n-1)\lfloor n-2}{(x-1)^n} + \frac{(-1)^{n-1}(n-1)\lfloor n-2}{(x+1)^n}$$

$$= (-1)^{n-2}\lfloor n-2\left[\frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n} - \frac{(n-1)}{(x-1)^n} - \frac{(n-1)}{(x+1)^n}\right]$$

$$= (-1)^{n-2}\lfloor n-2\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right].$$

Example 5. Find $y_n(0)$ if $y = \frac{x^3}{x^2-1}$.

Sol. We have $\quad y = \frac{x^3}{x^2-1} = \frac{x^3-1+1}{x^2-1} = \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} + \frac{1}{x^2-1}$

or
$$y = \frac{x^2+x+1}{(x+1)} + \frac{1}{(x-1)(x+1)}$$

$$y = \frac{x^2-1+1}{x+1} + 1 + \frac{1}{(x-1)\,(x+1)}$$

$$y = x + \frac{1}{x+1} + \frac{1}{2}\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$$

or
$$y = x + \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x+1}\right]$$

$\therefore$
$$y_n = 0 + \frac{1}{2}\left[\frac{(-1)^n \lfloor\underline{n}}{(x-1)^{n+1}} + \frac{(-1)^n \lfloor\underline{n}}{(x+1)^{n+1}}\right]$$

or
$$y_n = \frac{(-1)^n \lfloor\underline{n}}{2}\left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}}\right]$$

At $x = 0,\ y_n\ (0) = \dfrac{(-1)^n \lfloor\underline{n}}{2}\left[\dfrac{1}{(-1)^{n+1}} + \dfrac{1}{(1)^{n+1}}\right]$

When n is odd, $y_n(0) = \dfrac{(-1)^n \lfloor\underline{n}}{2}[1+1] = -\lfloor\underline{n}$

When n is even, $y_n(0) = \dfrac{(-1)^n \lfloor\underline{n}}{2}[-1+1] = 0.$

EXERCISE 1.1

1. If $y = \dfrac{x^2}{(x-1)^2(x+2)}$, find nth derivative of y. *(U.P.T.U., 2002)*

$$\left[\textbf{Ans. } y_n = \frac{(-1)^n \lfloor\underline{n+1}}{3(x-1)^{n+2}} + \frac{5(-1)^n \lfloor\underline{n}}{9(x-1)^{n+1}} + \frac{4(-1)^n \lfloor\underline{n}}{9(x+2)^{n+1}}\right]$$

2. Find the nth derivative of $\dfrac{x^2}{(x-a)(x-b)}$.

$$\left[\textbf{Ans. } \frac{(-1)^n \lfloor\underline{n}}{(a-b)}\left[\frac{a^2}{(x-a)^{n+1}} - \frac{b^2}{(x-b)^{n+1}}\right]\right]$$

3. Find the nth derivative of $\tan^{-1}\left[\dfrac{1+x}{1-x}\right]$.

[**Ans.** $(-1)^{n-1} \lfloor\underline{n-1} \sin^n\theta \sin n\theta$ where $\theta = \cot^{-1} x$]

4. If $y = \sin^3 x$, find y_n.

$$\left[\textbf{Ans. } \frac{3}{4}\sin\left(x + n\frac{\pi}{2}\right) - \frac{1}{4}.3^n.\sin\left(3x + n\frac{\pi}{2}\right)\right]$$

5. Find nth derivative of $\tan^{-1}\left(\dfrac{x}{a}\right)$.

$$\left[\textbf{Ans. } (-1)^{n-1} \lfloor\underline{n-1}\, a^{-n} \sin^n\theta \sin n\theta\right]$$

6. Find y_n, where $y = e^x.x$. $\left[\textbf{Ans. } e^x(x+n)\right]$

7. Find y_n, when $y = \dfrac{1-x}{1+x}$. $\left[\textbf{Ans. } \dfrac{2(-1)^n \lfloor n}{(x+1)^{n+1}}\right]$

8. Find nth derivative of $\log x^2$. $\left[\textbf{Ans. } (-1)^{n-1} \lfloor n-1 \cdot 2x^{-n}\right]$

9. Find y_n, $y = e^x \sin^2 x$. $\left[\textbf{Ans. } \dfrac{e^x}{2}\left[1 - 5^{n/2}\cos(2x + n\tan^{-1} 2)\right]\right]$

10. If $y = \cos x \cdot \cos 2x \cdot \cos 3x$ find y_n.

[**Hint:** $\cos x \cdot \cos 2x \cdot \cos 3x = \dfrac{1}{4}[(\cos 6x + \cos 4x + \cos 2x + 1)]$

$$\left[\textbf{Ans. } \frac{1}{4}\left[6^n \cos\left(6x + n\frac{\pi}{2}\right) + 4^n \cos\left(4x + n\frac{\pi}{2}\right) + 2^n \cos\left(2x + n\frac{\pi}{2}\right)\right]\right]$$

1.2 LEIBNITZ'S* THEOREM

Statement. If u and v be any two functions of x, then

$$D^n(u.v) = {}^nc_0\, D^n(u).v + {}^nc_1 D^{n-1}(u).\, D(v) + {}^nc_2\, D^{n-2}(u).D^2(v) + \ldots$$
$$+ {}^nc_r\, D^{n-r}(u).D^r(v) + \ldots + {}^nc_n\, u.\, D^n v \quad \ldots(i)$$

(*U.P.T.U., 2007*)

Proof. This theorem will be proved by Mathematical induction.

Now, $$D(u.v) = D(u).v + u.D(v) = {}^1c_0\, D(u).v + {}^1c_1\, u.D(v) \quad \ldots(ii)$$

This shows that the theorem is true for $n = 1$.

Next, let us suppose that the theorem is true for, $n = m$ from (i), we have

$$D^m(u.v) = {}^mc_0\, D^m(u).v + {}^mc_1\, D^{m-1}(u)\, D(v) + {}^mc_2\, D^{m-2}(u)\, D^2(v) + \ldots + {}^mc_r\, D^{m-r}(u)\, D^r(v) + \ldots + {}^mc_m\, u\, D^m(v)$$

Differentiating w.r.t. to x, we have

$$D^{m+1}(uv) = {}^mc_0\{D^{m+1}(u)\cdot v + D^m(u)\cdot D(v)\} + {}^mc_1\, \{D^m(u)D(v) + D^{m-1}(u)D^2(v)\}$$
$$+ {}^mc_2\, \{D^{m-1}(u)D^2(v) + D^{m-2}(u).\, D^3(v)\} + \ldots + {}^mc_r\{D^{m-r+1}(u)D^r v + D^{m-r}(u)D^{r+1}(v)\}$$
$$+ \ldots\ldots + {}^mc_m\{D(u)\cdot D^m(v) + uD^{m+1}(v)\}$$

But from Algebra we know that ${}^mc_r + {}^mc_{r+1} = {}^{m+1}c_{r+1}$ and ${}^mc_0 = {}^{m+1}c_0 = 1$

$$\therefore \quad D^{m+1}(uv) = {}^{m+1}c_0 D^{m+1}(u)\cdot v + ({}^mc_0 + {}^mc_1)D^m(u)\cdot D(v) + ({}^mc_1 + {}^mc_2)D^{m-1}u \cdot D^2 v$$
$$+ \ldots + ({}^mc_r + {}^mc_{r+1})D^{m-r}(u)\cdot D^{r+1}(v) + \ldots + {}^{m+1}c_{m+1} u \cdot D^{m+1}(v)$$

$$\left(\text{As } {}^mc_m = {}^{m+1}c_{m+1} = 1\right)$$

* **Gottfried William Leibnitz (1646–1716)** was born Leipzig (Germany). He was Newton's rival in the invention of calculus. He spent his life in diplomatic service. He exhibited his calculating machine in 1673 to the Royal society. He was linguist and won fame as Sanskrit scholar. The theory of determinants is said to have originated with him in 1683. The generalization of Binomial theorem into multinomial theorem is also due to him. His works mostly appeared in the journal 'Acta eruditorum' of which he was editor-in-chief.

$$\Rightarrow D^{m+1}(uv) = {}^{m+1}c_0 D^{m+1}(u)\cdot v + {}^{m+1}c_1 D^m(u)\cdot D(v) + {}^{m+1}c_2 D^{m-1}(u)\cdot D^2(v) + \ldots$$
$$+ {}^{m+1}c_{r+1} D^{m-r}(u)\cdot D^{r+1}(v) + \ldots + {}^{m+1}c_{m+1} u\cdot D^{m+1}(v) \quad \ldots(iii)$$

Therefore, the equation (*iii*) shows that the theorem is true for $n = m + 1$ also. But from (2) that the theorem is true for $n = 1$, therefore, the theorem is true for $(n = 1 + 1)$ *i.e.*, $n = 2$, and so for $n = 2 + 1 = 3$, and so on. Hence, the theorem is true for all positive integral value of n.

Example 1. If $y^{1/m} + y^{-1/m} = 2x$, prove that
$(x^2 - 1)\, y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0.$ (*U.P.T.U., 2007*)

Sol. Given $$y^{1/m} + \frac{1}{y^{1/m}} = 2x$$

$$\Rightarrow \quad y^{2/m} - 2xy^{1/m} + 1 = 0$$

or $$(y^{1/m})^2 - 2x(y^{1/m}) + 1 = 0$$

$$\Rightarrow \quad z^2 - 2xz + 1 \qquad (y^{1/m} = z)$$

$$\therefore \quad z = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

$$\Rightarrow \quad y^{1/m} = x \pm \sqrt{x^2 - 1} \Rightarrow y = \left[x \pm \sqrt{x^2 - 1}\right]^m \quad \ldots(i)$$

Differentiating equation (*i*) w.r.t. x, we get

$$y_1 = m\left[x \pm \sqrt{x^2-1}\right]^{m-1}\left[1 \pm \frac{2x}{2\sqrt{x^2-1}}\right] = \frac{m\left[x \pm \sqrt{x^2-1}\right]^m}{\sqrt{x^2-1}}$$

$$\Rightarrow \quad y_1 = \frac{my}{\sqrt{x^2-1}} \Rightarrow y_1\sqrt{x^2-1} = my$$

or $$y_1^2\,(x^2 - 1) = m^2y^2 \quad \ldots(ii)$$

Differentiating both sides equation (*ii*) w.r.t. x, we obtain

$$2y_1y_2(x^2 - 1) + 2xy_1^2 = 2m^2\, yy_1$$

$$\Rightarrow \quad y_2\,(x^2 - 1) + xy_1 - m^2y = 0$$

Differentiating n times by Leibnitz's theorem w.r.t. x, we get

$$D^n\,(y_2)\cdot(x^2 - 1) + {}^nc_1\, D^{n-1}y_2\cdot D^2(x^2-1) + {}^nc_2\, D^{n-2}\, y_2\, D^2\,(x^2 - 1) + D^n\,(y_1)x + {}^nc_1\, D^{n-1}\,(y_1)\, Dx - m^2y_n = 0$$

$$\Rightarrow \quad y_{n+2}\,(x^2 - 1) + ny_{n+1}\cdot 2x + \frac{n(n-1)}{\lfloor 2}\, y_n\cdot 2 + y_{n+1}\cdot x + ny_n - m^2y_n = 0$$

$$\Rightarrow \quad (x^2 - 1)y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - n + n - m^2)y_n = 0$$

$$\Rightarrow \quad (x^2 - 1)\, y_{n+2} + (2n + 1)\, xy_{n+1}\, (n^2 - m^2)\, y_n = 0.$$ **Hence proved.**

Example 2. Find the nth derivative of $e^x \log x$.

Sol. Let $u = e^x$ and $v = \log x$

Then $D^n\,(u) = e^x$ and $D^n\,(v) = \dfrac{(-1)^{n-1}\lfloor n-1}{x^n}$

$$D^n(ax+b)^{-1} = \frac{(-1)^n \lfloor n}{(ax+b)^{n+1}}$$

By Leibnitz's theorem, we have

$$D^n(e^x \log x) = D^n e^x \log x + {}^nc_1 D^{n-1}(e^x) D(\log x) + {}^nc_2 D^{n-2}(e^x) D^2(\log x) + \ldots + e^x D^n(\log x)$$

$$= e^x \cdot \log x + ne^x \cdot \frac{1}{x} + \frac{n(n-1)}{\lfloor 2} e^x\left(-\frac{1}{x^2}\right) + \ldots + e^x \frac{(-1)^{n-1}\lfloor n-1}{x^n}$$

$$\Rightarrow \quad D^n(e^x \log x) = e^x\left[\log x + \frac{n}{x} - \frac{n(n-1)}{2x^2} + \ldots + \frac{(-1)^{n-1}\lfloor n-1}{x^n}\right].$$

Example 3. Find the nth derivative of $x^2 \sin 3x$.

Sol. Let $u = \sin 3x$ and $v = x^2$

$$\therefore \quad D^n(u) = D^n(\sin 3x) = 3^n \sin\left(3x + \frac{n\pi}{2}\right)$$

$$D(u) = 2x,\ D^2(v) = 2,\ D^3(v) = 0$$

By Leibnitz's theorem, we have

$$D^n(x^2 \sin 3x) = D^n(\sin 3x)x^2 + {}^nc_1 D^{n-1}(\sin 3x)\cdot D(x^2) + {}^nc_2 D^{n-2}(\sin 3x)\cdot D^2(x^2)$$

$$= 3^n \sin\left(3x + \frac{n\pi}{2}\right)\cdot x^2 + n3^{n-1}\sin\left(3x + \frac{\overline{n-1}\pi}{2}\right)\cdot 2x + \frac{n(n-1)}{\lfloor 2}\cdot 3^{n-2}\sin\left(3x + \frac{\overline{n-2}\pi}{2}\right)\cdot 2$$

$$= 3^n x^2 \sin\left(3x + \frac{n\pi}{2}\right) + 2nx\cdot 3^{n-1}\sin\left(3x + \frac{\overline{n-1}\pi}{2}\right) + 3^{n-2}n(n-1)\cdot \sin\left(3x + \frac{\overline{n-2}\pi}{2}\right).$$

Example 4. If $y = x \log(1 + x)$, prove that

$$y_n = \frac{(-1)^{n-2}\lfloor n-2\,(x+n)}{(x+1)^n}. \qquad (\textit{U.P.T.U., 2006})$$

Sol. Let $u = \log(1 + x)$, $v = x$

$$D^n(u) = \frac{d^n}{dx^n}\log(1+x) = \frac{d^{n-1}}{dx^{n-1}}\left(\frac{d}{dx}\log(1+x)\right)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\cdot\frac{1}{x+1} = \frac{d^{n-1}}{dx^{n-1}}(x+1)^{-1}$$

$$\Rightarrow \quad D^n(u) = \frac{(-1)^{n-1}\lfloor n-1}{(x+1)^n} \text{ and } D(v) = 1,\ D^2(v) = 0$$

By Leibnitz's theorem, we have

$$y_n = D^n(x \log(1 + x) = D^n(\log(1 + x))\,x + {}^nc_1 D^{n-1}(\log(1 + x))\,Dx$$

$$= x\frac{(-1)^{n-1}\lfloor n-1}{(x+1)^n} + \frac{n(-1)^{n-2}\lfloor n-2}{(x+1)^{n-1}}$$

$$\Rightarrow \qquad y_n = (-1)^{n-2} \lfloor n-2 \left[\frac{-x(n-1)}{(x+1)^n} + \frac{n(x+1)}{(x+1)^n}\right]$$

$$= (-1)^{n-2} \lfloor n-2 \left[\frac{-xn+x+xn+n}{(x+1)^n}\right]$$

$$= (-1)^{n-2} \lfloor n-2 \left[\frac{x+n}{(x+1)^n}\right].$$ **Hence proved.**

Example 5. If $y = a \cos(\log x) + b \sin(\log x)$. Show that

$$x^2y_2 + xy_1 + y = 0$$

and $$x^2y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+1)\, y_n = 0.$$ (*U.P.T.U., 2003*)

Sol. Given $y = a \cos(\log x) + b \sin(\log x)$

$$\therefore \qquad y_1 = -a \sin(\log x)\left(\frac{1}{x}\right) + b \cos(\log x)\left(\frac{1}{x}\right)$$

or $$xy_1 = -a \sin(\log x) + b \cos(\log x)$$

Again differentiating w.r.t. x, we get

$$xy_2 + y_1 = -a \cos(\log x)\left(\frac{1}{x}\right) - b \sin(\log x)\left(\frac{1}{x}\right)$$

$$\Rightarrow \qquad x^2y_2 + xy_1 = -\{a \cos(\log x) + b \sin(\log x)\} = -y$$

$$\Rightarrow \qquad x^2y_2 + xy_1 + y = 0.$$ **Hence proved.** ...(*i*)

Differentiating (*i*) n times, by Leibnitz's theorem, we have

$$y_{n+2} \cdot x^2 + n\, y_{n+1} \cdot 2x + \frac{n(n-1)}{\lfloor 2}\, y_n \cdot 2 + y_{n+1} \cdot x + ny_n + y_n = 0$$

$$\Rightarrow \quad x^2y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 - n + n + 1)\, y_n = 0$$

$$\Rightarrow \quad x^2y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+1)\, y_n = 0.$$ **Hence proved.**

Example 6. If $y = (1-x)^{-\alpha} e^{-\alpha x}$, show that

$$(1-x)y_{n+1} - (n + \alpha x)\, y_n - n\alpha y_{n-1} = 0.$$

Sol. Given $y = (1-x)^{-\alpha}.\, e^{-\alpha x}$

Differentiating w.r.t. x, we get

$$y_1 = \alpha(1-x)^{-\alpha-1} e^{-\alpha x} - (1-x)^{-\alpha} e^{-\alpha x} \cdot \alpha$$

$$y_1 = (1-x)^{-\alpha} e^{-\alpha x} \cdot \alpha \left[\frac{1}{1-x} - 1\right] = y\alpha \left[\frac{x}{1-x}\right]$$

$$= \qquad y_1(1-x) = \alpha xy$$

Differentiating n times w.r.t. x, by Leibnitz's theorem, we get

$$y_{n+1}(1-x) - ny_n = \alpha y_n \cdot x + n\alpha y_{n-1}$$

$$\Rightarrow \quad (1-x)y_{n+1} - (n + \alpha x)\, y_n - n\alpha y_{n-1} = 0.$$ **Hence proved.**

Example 7. If $\cos^{-1}\left(\dfrac{y}{b}\right) = \log\left(\dfrac{x}{m}\right)^m$, prove that $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+m^2)\, y_n = 0.$

Sol. We have $\cos^{-1}\left(\dfrac{y}{b}\right) = \log\left(\dfrac{x}{m}\right)^m = m \log \dfrac{x}{m}$

$$\Rightarrow \qquad y = b\cos\left(m\log\frac{x}{m}\right)$$

On differentiating, we have

$$y_1 = -b\sin\left(m\log\frac{x}{m}\right)\cdot\ \frac{m^2}{x}\cdot\frac{1}{m}$$

$$\Rightarrow \qquad xy_1 = -mb\sin\left(m\log\frac{x}{m}\right)$$

Again differentiating w.r.t. x, we get

$$xy_2 + y_1 = -mb\cos\left(m\log\frac{x}{m}\right)m\cdot\frac{1}{\frac{x}{m}}\cdot\frac{1}{m}$$

$$x(xy_2 + y_1) = -m^2b\cos\left(m\log\frac{x}{m}\right) = -m^2y$$

or $\qquad x^2y_2 + xy_1 + m^2y = 0$

Differentiating n times with respect to x, by Leibnitz's theorem, we get

$$y_{n+2}\cdot x^2 + ny_{n+1}\cdot 2x + \frac{n(n-1)}{\lfloor 2}\cdot 2y_n + xy_{n+1} + ny_n + m^2y_n = 0$$

$\Rightarrow x^2y_{n+2} + (2n+1)\,xy_{n+1} + (n^2 - n + n + m^2)y_n = 0$

$\Rightarrow \quad x^2y_{n+2} + (2n+1)\,xy_{n+1} + (n^2 + m^2)\,y_n = 0.$ **Hence proved.**

Example 8. If $y = (x^2 - 1)^n$, prove that (*U.P.T.U., 2000, 2002*)

$$(x^2 - 1)y_{n+2} + 2xy_{n+1} - n\,(n+1)y_n = 0$$

Hence, if $P_n = \dfrac{d^n}{dx^n}(x^2-1)^n$, show that $\dfrac{d}{dx}\left\{(1-x^2)\dfrac{dP_n}{dx}\right\} + n(n+1)\,P_n = 0.$

Sol. Given $\qquad y = (x^2 - 1)^n$

Differentiating w.r. to x, we get

$$y_1 = n(x^2-1)^{n-1}.\,2x = \frac{2nx\,(x^2-1)^n}{(x^2-1)}$$

$$\Rightarrow \qquad (x^2-1)\,y_1 = 2nxy$$

Again differentiating, w.r.t. x, we obtain

$$(x^2-1)\,y_2 + 2xy_1 = 2nxy_1 + 2ny$$

Now, differentiating n times, w.r.t. x by Leibnitz's theorem

$$(x^2-1)y_{n+2} + 2nxy_{n+1} + \frac{2n(n-1)}{\lfloor 2}y_n + 2xy_{n+1} + 2ny_n = 2nxy_{n+1} + 2n^2y_n + 2ny_n$$

or $\qquad (x^2-1)y_{n+2} + 2xy_{n+1}\,(n+1-n) + (n^2 - n + 2n - 2n^2 - 2n)y_n = 0$

or $\qquad (x^2-1)\,y_{n+2} + 2xy_{n+1} - (n^2+n)y_n = 0$

$\Rightarrow \quad (x^2-1)\,y_{n+2} + 2xy_{n+1} - n(n+1)\,y_n = 0.$ **Hence proved.** ...(*i*)

Second part: Let $y = (x^2 - 1)^n$

$\therefore \quad P_n = \frac{d^n}{dx^n} y = y_n$

Now

$$\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}y_n\right\} = \frac{d}{dx}\{(1-x^2)y_{n+1}\}$$

$$= (1-x^2)y_{n+2} - 2xy_{n+1} = -\left[(x^2-1)y_{n+2} + 2xy_{n+1}\right]$$

$$\Rightarrow \quad \frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_n\right\} = -\left[n(n+1)y_n\right] \qquad \text{[Using equation } (i)]$$

or $\quad \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1)y_n = 0.$ **Hence proved.**

Example 9. Find the nth derivative of $y = x^{n-1} \log x$ at $x = \frac{1}{2}$.

Sol. Differentiating

$$y_1 = (n-1)\, x^{n-1-1} \log x + x^{n-1}\, \frac{1}{x}$$

or $\quad y_1 = \frac{(n-1)x^{n-1} \cdot \log x}{x} + \frac{x^{n-1}}{x} \Rightarrow xy_1 = (n-1)y + x^{n-1}$

Differentiating (n–1) times by Leibnitz's theorem, we get

$$xy_n + {}^{n-1}c_1\, y_{n-1} = (n-1)y_{n-1} + \lfloor n-1 \qquad \left|\ \frac{d^{n-1}}{dx^{n-1}} x^{n-1} = (n-1)(n-2)\ldots 2.1 = \lfloor n-1\right.$$

$$\Rightarrow \quad xy_n + (n-1)y_{n-1} = (n-1)y_{n-1} + \lfloor n-1$$

$$\Rightarrow \quad xy_n = \lfloor n-1 \quad \textit{i.e.}\ y_n = \frac{\lfloor n-1}{x}$$

At $\quad x = \frac{1}{2}$

$$y_n\left(\frac{1}{2}\right) = 2\,\lfloor n-1.$$

Example 10. If $y = (1 - x^2)^{-1/2} \sin^{-1}x$, when $-1 < x < 1$ and $-\frac{\pi}{2} < \sin^{-1}x < \frac{\pi}{2}$, then show that $(1 - x^2)y_{n+1} - (2n + 1)\, xy_n - n^2\, y_{n-1} = 0$.

Sol. Given $y = (1 - x^2)^{-1/2} \sin^{-1}x$

Differentiating

$$y_1 = -\frac{1}{2}(1-x^2)^{-3/2}(-2x)\sin^{-1}x + (1-x^2)^{-1/2}\cdot\frac{1}{\sqrt{1-x^2}}$$

$$y_1 = \frac{x(1-x^2)^{-\frac{1}{2}}\sin^{-1}x}{(1-x^2)} + \frac{1}{(1-x^2)} = \frac{xy+1}{(1-x^2)}$$

$$\Rightarrow \quad y_1(1-x^2) = xy + 1$$

Differentiating n times w.r.t. x, by Leibnitz's theorem, we get

$$y_{n+1}(1-x^2) + ny_n(-2x) + \frac{n(n-1)}{\lfloor 2}\, y_{n-1}\cdot(-2) = xy_n + ny_{n-1}$$

$$(1-x^2)y_{n+1} - (2n+1)xy_n - (n^2-n+n)y_{n-1} = 0$$

$\Rightarrow$ $(1-x^2)y_{n+1} - (2n+1)xy_n - n^2y_{n-1} = 0.$ **Hence proved.**

Example 11. If $y = x^n \log x$, then prove that

(*i*) $y_{n+1} = \dfrac{\lfloor n}{x}$ (*ii*) $y_n = ny_{n-1} + \lfloor (n-1)$.

Sol. (*i*) We have $y = x^n \log x$

Differentiating w.r. to x, we get

$$y_1 = nx^{n-1}\cdot \log x + \frac{x^n}{x}$$

$\Rightarrow$ $$xy_1 = nx^n . \log x + x^n$$

$$xy_1 = ny + x^n \qquad ...(i)$$

Differentiating equation (*i*) n times, we get

$$xy_{n+1} + ny_n = ny_n + \lfloor n$$

$\Rightarrow$ $$y_{n+1} = \frac{\lfloor n}{x} \quad \textbf{Proved.}$$

(*ii*) $$y_n = \frac{d^n}{dx^n}\left(x^n.\log x\right) = \frac{d^{n-1}}{dx^{n-1}}\left(\frac{d}{dx}x^n.\log x\right)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left(\frac{x^n}{x} + nx^{n-1}.\log x\right)$$

$$= n\frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1}.\log x\right) + \frac{d^{n-1}}{dx^{n-1}}\,.\,x^{n-1}$$

$$= ny_{n-1} + \lfloor (n-1). \quad \textbf{Proved.}$$

As $y_n = \dfrac{d^n}{dx^n}\left(x^n \log x\right)$

$\therefore$ $y_{n-1} = \dfrac{d^{n-1}}{dx^{n-1}}\left(x^{n-1}\log x\right)$

EXERCISE 1.2

Find the nth derivative of the following:

1. $e^x \log x$. $\left[\textbf{Ans. } e^x\left[\log x + {}^nc_1\cdot\frac{1}{x} - {}^nc_2\cdot\frac{1}{x^2} + \lfloor 2\,{}^nc_3\cdot\frac{1}{x^3} + \ldots + (-1)^{n-1}\lfloor n-1\,{}^nc_n x^{-n}\right]\right]$

2. $x^2 e^x$. $\left[\textbf{Ans. } e^x\left[x^2 + 2nx + n(n-1)\right]\right]$

3. $x^3 \log x$. $\left[\textbf{Ans. } \dfrac{6(-1)^n \lfloor n-4}{x^{n-3}}\right]$

4. $\dfrac{1-x}{1+x}$. $\left[\textbf{Ans. } \dfrac{2(-1)^n \lfloor n}{(1+x)^{n+1}}\right]$

5. $x^2 \sin 3x$.

$\left[\textbf{Ans. } 3^n x^2 \sin\left(3x+\dfrac{n\pi}{2}\right)+2nx\cdot 3^{n-1}\left[\sin\{3x+\dfrac{1}{2}(n-1)\pi\}\right]+3^{n-2}n(n-1)\sin\left\{3x+\dfrac{\pi}{2}(n-2)\right\}\right]$

6. $e^x (2x+3)^3$. $\left[\textbf{Ans. } e^x\left\{(2x+3)^2+6n(2x+3)^2+12(n-1)(2x+3)+8n(n-1)(n-2)\right\}\right]$

7. If $x = \tan y$, prove that *(U.P.T.U., 2006)*

$(1+x^2)\, y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$.

8. If $y = e^x \sin x$, prove that $y''-2y'+2y = 0$.

9. If $y = \sin(m \sin^{-1}x)$, prove that

$(1-x^2)y_{n+2} - (2n+1)x\cdot y_{n+1} + (m^2-n^2)y_n = 0$. *(U.P.T.U., 2004, 2002)*

10. If $x = \cos h\left[\left(\dfrac{1}{m}\right)\log y\right]$, prove that $(x^2-1)y_2 + xy_1 - m^2y = 0$ and $(x^2-1)y_{n+2} + (2n+1)xy_{n+1}$ $+ (n^2-m^2)y_n = 0$.

11. If $\cos^{-1}\left(\dfrac{y}{b}\right) = \log\left(\dfrac{x}{n}\right)^n$, prove that $x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$.

12. If $y = e^{\tan^{-1}x}$, prove that $(1+x^2)y_{n+2} + \{2(n+1)x-1\}y_{n+1} + n(n+1)y_n = 0$.

13. If $\sin^{-1}y = 2\log(x+1)$, show that

$(x+1)^2y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$.

14. If $y = C_1\left(x+\sqrt{x^2-1}\right)^n + C_2\left(x-\sqrt{x^2-1}\right)^n$, prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} = 0$.

15. If $x = \cos[\log(y^{1/a})]$, then show that $(1-x^2)y_{n+2} - (2n+1)\, xy_{n+1} - (n^2+a^2)\, y_n = 0$.

1.2.1 To Find $(y_n)_0$ *i.e.*, *n*th Differential Coefficient of *y*, When $x = 0$

Sometimes we may not be able to find out the *n*th derivative of a given function in a compact form for general value of *x* but we can find the *n*th derivative for some special value of *x* generally $x = 0$. The method of procedure will be clear from the following examples:

Example 1. Determine $y_n(0)$ where $y = e^{m.\cos^{-1}x}$.

Sol. We have $y = e^{m.\cos^{-1}x}$ ...(*i*)

Differentiating w.r.t. *x*, we get

$$y_1 = e^{m\cos^{-1}x}\, m\left(\frac{-1}{\sqrt{1-x^2}}\right) \Rightarrow \sqrt{1-x^2}\,.\,y_1 = -\,me^{m\cos^{-1}x}$$

or

$$\sqrt{1-x^2}\; y_1 = -\,my \Rightarrow (1-x^2)y_1^2 = m^2y^2$$

Differentiating again

$$(1 - x^2)\, 2y_1\, y_2 - 2xy_1^2 = 2m^2 yy_1$$

$$\Rightarrow \qquad (1 - x^2)y_2 - xy_1 = m^2 y \qquad ...(ii)$$

Using Leibnitz's rule differentiating n times w.r.t. x

$$(1 - x^2)y_{n+2} - 2nxy_{n+1} - \frac{2n(n-1)}{\lfloor 2}\, y_n - xy_{n+1} - ny_n = m^2 y_n$$

or $$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + m^2)y_n = 0$$

Putting $x = 0$

$$y_{n+2}(0) - (n^2 + m^2)y_n(0) = 0$$

$$\Rightarrow \qquad y_{n+2}(0) = (n^2 + m^2)y_n(0) \qquad ...(iii)$$

replace n by $(n - 2)$

$$y_n(0) = \{(n - 2)^2 + m^2\}\, y_{n-2}(0)$$

replace n by $(n - 4)$ in equation (iii), we get

$$y_{n-2}(0) = \{(n - 4)^2 + m^2\}\, y_{n-4}(0)$$

$$\therefore \qquad y_n(0) = \{(n - 2)^2 + m^2\}\, \{(n - 4)^2 + m^2\}\, y_{n-4}(0)$$

Case I. When n is odd:

$$y_n(0) = \{(n - 2)^2 + m^2\}\, \{(n - 4)^2 + m^2\} \ldots (1^2 + m^2)y_1(0) \qquad ...(iv)$$

[The last term obtain putting $n = 1$ in eqn. (iii)]

Now we have $$y_1 = -e^{m\cos^{-1}x}\, m \cdot \frac{1}{\sqrt{1-x^2}}$$

At $x = 0$, $y_1(0) = -me^{\frac{m\pi}{2}}$ ...(v) $\quad \left|\ \text{As } \cos^{-1} 0 = \frac{\pi}{2}\right.$

Using (v) in (iv), we get

$$y_n(0) = -\left\{(n-2)^2 + m^2\right\}\left\{(n-4)^2 + m^2\right\} \ldots \left(1^2 + m^2\right) me^{\frac{m\pi}{2}}.$$

Case II. When n is even:

$$y_n(0) = \{(n - 2)^2 + m^2\}\, \{(n - 4)^2 + m^2\} \ldots (2^2 + m^2)y_2(0) \qquad ...(vi)$$

[The last term obtain by putting $n = 2$ in (iii)]

From (ii), $$y_2(0) = m^2(y)_0$$

$$\therefore \qquad y_2(0) = m^2 e^{m\pi/2} \qquad ...(vii)$$

$\left|\ \text{As } y = e^{m\cos^{-1}x}\right.$

$\left|\ \therefore\ y(0) = e^{m\cos^{-1}0} = e^{m\pi/2}\right.$

From eqns. (vi) and (vii), we get

$$y_n(0) = \{(n - 2)^2 + m^2\}\, \{(n - 4)^2 + m^2\} \ldots (2^2 + m^2)\, m^2 e^{m\pi/2}.$$

Example 2. If $y = (\sin^{-1}x)^2$. Prove that $y_n(0) = 0$ for n odd and $y_n(0) = 2.2^2.4^2....(n - 2)^2$, $n \neq 2$ for n even. *(U.P.T.U., 2005, 2008)*

Sol. We have $$y = (\sin^{-1} x)^2 \qquad ...(i)$$

On differentiating $$y_1 = 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}} \Rightarrow y_1\sqrt{1-x^2} = 2\sqrt{y}, \left(\text{As } \sqrt{y} = \sin^{-1} x\right)$$

Squaring on both sides,

$$y_1^2 (1 - x^2) = 4y$$

Again differentiating

$$2(1-x^2)\,y_1 y_2 - 2xy_1^2 = 4y_1$$

or $$(1-x^2)\,y_2 - xy_1 = 2 \qquad ...(ii)$$

Differentiating n times by Leibnitz's theorem

$$(1-x^2)\,y_{n+2} - 2nxy_{n+1} - \frac{2n(n-1)}{\lfloor 2}\,y_n - xy_{n+1} - ny_n = 0$$

or $(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - n^2y_n = 0$

Putting $x = 0$ in above equation

$$y_{n+2}(0) - n^2\,y_n(0) = 0$$

$\Rightarrow$ $$y_{n+2}(0) = n^2\,y_n(0) \qquad ...(iii)$$

replace n by $(n-2)$

$$y_n(0) = (n-2)^2\,y_{n-2}(0)$$

Again replace n by $(n-4)$ in (iii) and putting the value of $y_{n-2}(0)$ in above equation

$$y_n(0) = (n-2)^2\,(n-4)^2\,y_{n-4}(0)$$

Case I. If n is odd, then

$$y_n(0) = (n-2)^2\,(n-4)^2\,(n-6)^2 \ldots 1^2\cdot y_1(0)$$

But $$y_1(0) = 2\sin^{-1}0\cdot\frac{1}{\sqrt{1-0}} = 0$$

$\therefore$ $y_n(0) = 0.$ **Hence proved.**

Case II. If n is even, then

$$y_n(0) = (n-2)^2\,(n-4)^2 \ldots 2^2\cdot y_2(0) \qquad ...(iv)$$

From (ii) $$y_2(0) = 2$$

Using this value in eqn. (iv), we get

$$y_n(0) = (n-2)^2\,(n-4)^2 \ldots 2^2\cdot 2$$

or $y_n(0) = 2.2^2.4^2 \ldots (n-2)^2,\ n \neq 2$ otherwise 0. **Proved.**

Example 3. If $y = \left[x+\sqrt{1+x^2}\right]^m$, find y_n 0).

Sol. Given $$y = \left[x+\sqrt{1+x^2}\right]^m \qquad ...(i)$$

$\therefore$ $$y_1 = m\left[x+\sqrt{1+x^2}\right]^{m-1}\left[1+\frac{x}{\sqrt{1+x^2}}\right]$$

$$= \frac{m\left[x+\sqrt{1+x^2}\right]^m}{\sqrt{1+x^2}} = \frac{my}{\sqrt{1+x^2}}$$

or $$y_1\sqrt{1+x^2} = my$$

Squaring $$y_1^2(1+x^2) = m^2y^2 \qquad ...(ii)$$

Again differentiating, $y_1^2\,(2x) + (1+x^2)2y_1y_2 = m^2\cdot 2yy_1$

or $$y_2\,(1+x^2) + xy_1 - m^2y = 0 \qquad ...(iii)$$

Differentiating n times by Leibnitz's theorem

$$(1+x^2)y_{n+2} + 2nxy_{n+1} + \frac{2n(n-1)}{\lfloor 2}\,y_n + xy_{n+1} + ny_n - m^2y_n = 0$$

or $$(1 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0$$

Putting $x = 0$, we get

$$y_{n+2}(0) + (n^2 - m^2)\, y_n(0) = 0$$

$$\Rightarrow \quad y_{n+2}(0) = -(n^2 - m^2)\, y_n(0) \qquad ...(iv)$$

replace n by $n - 2$

$$y_n(0) = -\{(n-2)^2 - m^2\}\, y_{n-2}(0)$$

Again replace n by $(n - 4)$ in (iv) and putting $y_{n-2}(0)$ in above equation

$$y_n(0) = (-1)^2 \{(n-2)^2 - m^2\} \{(n-4\}^2 - m^2\}\, y_{n-4}(0)$$

Case I. If n is odd

$$y_n(0) = -\{(n-2)^2 - m^2\} \{(n-4)^2 - m^2\} \ldots \{1^2 - m^2\}\, y_1(0)$$

But $$y_1(0) = my(0)$$

or $$y_1(0) = m \qquad (\text{As } y(0) = 1)$$

$$\Rightarrow \quad y_n(0) = \{m^2 - (n-2)^2\}\{m^2 - (n-4)^2\} \ldots (m^2 - 1^2)\cdot m.$$

Case II. If n is even

$$y_n(0) = \{m^2 - (n-2)^2\} \{m^2 - (n-4)^2\} \ldots (m^2 - 2^2)\, y_2(0)$$

$$\Rightarrow \quad y_n(0) = \{m^2 - (n-2)^2\} \{m^2 - (n-4)^2\} \ldots (m^2 - 2^2)\cdot m^2.$$

$$(\text{As } y_2(0) = m^2).$$

Example 4. Find the nth differential coefficient of the function on $\cos(2\cos^{-1} x)$ at the point $x = 0$.

Sol. Let $$y = \cos(2\cos^{-1} x) \qquad ...(i)$$

On diffentiating, $$y_1 = -\sin(2\cos^{-1} x)\left[\frac{-2}{\sqrt{1-x^2}}\right]$$

or $$y_1\sqrt{1-x^2} = 2\sin(2\cos^{-1} x)$$

Squaring on both sides, we get

$$y_1^2(1 - x^2) = 4\sin^2\left(2\cos^{-1} x\right)$$

$$= 4\left\{1 - \cos^2\left(2\cos^{-1} x\right)\right\}$$

or $$y_1^2(1 - x^2) = 4(1 - y^2)$$

Again differentiating w.r.t. x, we get

$$2y_1y_2(1 - x^2) - 2xy_1^2 = -8yy_1$$

or $$y_2(1 - x^2) - xy_1 + 4y = 0 \qquad ...(ii)$$

Differentiating n times by Liebnitz's theorem

$$(1 - x^2)\, y_{n+2} - 2nxy_{n+1} - \frac{2n(n-1)}{\lfloor 2}\, y_n - xy_{n+1} - ny_n + 4y_n = 0$$

$$(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - (n^2 - 4)\, y_n = 0$$

Putting $x = 0$ in above equation, we get

$$y_{n+2}(0) - (n^2 - 4)\, y_n(0) = 0$$

or $$y_{n+2}(0) = (n^2 - 4)\, y_n(0) \qquad ...(iii)$$

Replace n by $n - 2$, we get

$$y_n(0) = \{(n-2)^2 - 4\}\, y_{n-2}(0)$$

Again replace n by $(n - 4)$ in (*iii*) and putting $y_{n-2}(0)$ in above then, we get

$$y_n(0) = \{(n-2)^2 - 4\}\{(n-4)^2 - 4\}\, y_{n-4}(0)$$

Case I. If n is odd

$$y_n(0) = \{(n-2)^2 - 4\}\{(n-4)^2 - 4\} \ldots\ldots\ldots\ (1^2 - 4)\, y_1(0)$$

But $\quad y_1(0) = 2\sin(2\cos^{-1} 0) = 2\sin(\pi) = 0$

$\therefore \quad y_n(0) = 0.$

Case II. If n is even

$$y_n(0) = \{(n-2)^2 - 4\}\{(n-4)^2 - 4\} \ldots\ldots\ldots\ \{2^2 - 4\}\, y_2(0)$$

$$y_n(0) = 0$$

Hence for all values of n, even or odd,

$$y_n(0) = 0.$$

Example 5. Find the nth derivative of $y = x^2 \sin x$ at $x = 0$. (*U.P.T.U., 2008*)

Sol. We have $\quad y = x^2 \sin x = \sin x . x^2 \quad$ ...(*i*)

Differentiate n times by Leibnitz's theorem, we get

$$y_n = n_{C_0} D^n (\sin x). x^2 + n_{C_1} D^{n-1} (\sin x)\, D(x^2) + n_{C_2} D^{n-2} (\sin x)\, D^2 (x^2) + 0$$

$$= x^2 . \sin\left(x + \frac{n\pi}{2}\right) + 2nx . \sin\left(x + \frac{n-1}{2}\pi\right) + n(n-1)\sin\left(x + \frac{n-2}{2}\pi\right)$$

$$= x^2 \sin\left(x + \frac{n\pi}{2}\right) + 2nx \sin\left(x + \frac{n\pi}{2} - \frac{\pi}{2}\right) + n(n-1)\sin\left(x + \frac{n\pi}{2} - \pi\right)$$

$$= x^2 \sin\left(x + \frac{n\pi}{2}\right) - 2nx \cos\left(x + \frac{n\pi}{2}\right) - n(n-1)\sin\left(x + \frac{n\pi}{2}\right)$$

$$y_n = (x^2 - n^2 + n)\sin\left(x + \frac{n\pi}{2}\right) - 2nx\cos\left(x + \frac{n\pi}{2}\right)$$

Putting $x = 0$, we obtain

$$y_n(0) = (n - n^2)\sin\frac{n\pi}{2}.$$

Example 6. If $y = \sin(a \sin^{-1} x)$, prove that

$$(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - a^2)y_n = 0$$

Also find nth derivative of y at $x = 0$. [*U.P.T.U. (C.O.), 2007*]

Sol. We have $\quad y = \sin(a \sin^{-1} x)$

Differentiating w.r.t. x, we get

$$y_1 = \cos(a \sin^{-1} x). \frac{a}{\sqrt{1-x^2}} \quad \text{...}(i)$$

or $\quad y_1\sqrt{1-x^2} = a\cos(a \sin^{-1} x)$

Squaring on both sides, we obtain

$$y_1^2 (1 - x^2) = a^2 \cos^2 (a \sin^{-1} x) = a^2 [1 - \sin^2 (a \sin^{-1} x)]$$

or $\quad y_1^2 (1 - x^2) = a^2 (1 - y^2) \quad$ ...(*ii*)

Differentiating again, we get

$$2y_1\, y_2\,(1-x^2) - 2xy_1^{\,2} = -2\,a^2yy_1$$

or $$y_2\,(1-x^2) - xy_1 = -a^2y \qquad ...(iii)$$

Differentiating n times by Leibnitz's theorem

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{2n(n-1)}{\underline{|2}}\; y_n - xy_{n+1} - ny_n = -a^2y_n$$

or $$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-a^2)y_n = 0 \qquad ...(iv)$$

Putting $x = 0$ in relation (iv), we get

$$y_{n+2}\,(0) - (n^2-a^2)y_n\,(0) = 0$$

or $$y_{n+2}\,(0) = (n^2-a^2)y_n\,(0) \qquad ...(v)$$

Replace n by $(n-2)$ in relation (v), we get

$$y_n\,(0) = \{(n-2)^2 - a^2\}y_{n-2}\,(0)$$

Again replace n by $(n-4)$ in equation (v) and putting $y_{n-2}\,(0)$ in above relation, we get

$$y_n\,(0) = \{(n-2)^2 - a^2\}\,\{(n-4)^2 - a^2\}y_{n-4}\,(0)$$

Case I. When n is odd:

$$y_n\,(0) = \{(n-2)^2 - a^2\}\,\{(n-4)^2 - a^2\}\,.....\,\{1^2 - a^2\}y_1\,(0) \qquad ...(vi)$$

[The last term in (vi) obtain by putting $n = 1$ in equation (v)]

Putting $x = 0$, in equation (i), we get

$$y_1\,(0) = \cos(a\,\sin^{-1} 0).\, a = \cos 0\,.\,a \Rightarrow y_1\,(0) = a$$

Hence, $$y_n\,(0) = \{(n-2)^2 - a^2\}\,\{(n-4)^2 - a^2\}\,.....\,\{1^2 - a^2\}.\,a$$

Case II. When n is even:

$$y_n\,(0) = \{(n-2)^2 - a^2\}\,\{(n-4)^2 - a^2\}\,.....\,\{2^2 - a^2\}y_2\,(0).$$

Putting $x = 0$ in (iii), we get

[The last term obtain by putting $n = 2$ in equation (v)]

$$y_2\,(0) = -\,a^2\,y(0) = -\,a^2\,x_0 = 0 \qquad (\text{As } y(0) = 0)$$

Hence, $$y_n\,(0) = 0.$$

EXERCISE 1.3

1. If $y = \tan^{-1} x$, find the value of $y_7\,(0)$ and $y_8\,(0)$. **[Ans.** $\underline{|6}$ and 0.]

2. If $y = e^{a\,\sin^{-1} x}$, prove that $(1-x^2)\,y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)\,y_n = 0$ and hence find the value of y_n when $x = 0$.

   $\Big[$**Ans.** n is odd, $y_n\,(0) = \left\{(n-2)^2 + a^2\right\}\,\left\{(n-4)^2 + a^2\right\}\,...\,(3^2 + a^2)\,(1^2 + a^2)a$ n is even,

   $y_n\,(0) = \left\{(n-2)^2 + a^2\right\}\left\{(n-4)^2 + a^2\right\}......\,(4^2 + a^2)(2^2 + a^2)\,a^2.\Big]$

3. If $\log y = \tan^{-1}x$, show that $(1+x^2)\,y_{n+2} + \{2(n+1)x - 1\}\,y_{n+1} + n\,(n+1)\,y_n = 0$ and hence find y_3, y_4 and y_5 at $x = 0$. [*U.P.T.U. (C.O.), 2003*]

   $\Big[$**Ans.** $y_3\,(0) = -1,\ y_4\,(0) = -1,\ y_5\,(0) = 5\Big]$

4. If $f(x) = \tan x$, then prove that

$$f^n(0) - {}^nc_2\, f^{n-2}(0) + {}^nc_4\, f^{n-4}(0) - \ldots\ldots = \text{Sin}\left(\frac{n\pi}{2}\right).$$

5. If $y = \sin^{-1}x$, find $y_n(0)$.

$$\left[\textbf{Ans. } n \text{ is odd, } y_n(0) = (n-2)^2 (n-4)^2 \ldots 5^2 \,.\, 3^2 \,.\, 1 \;\; n \text{ is even, } y_n(0) = 0.\right]$$

6. Find $y_n(0)$ when $y = \sin(m \sin h^{-1} x)$.

$$\left[\textbf{Ans. } n \text{ is odd, } y_n(0) = (-1)^{\left(\frac{n-1}{2}\right)} \left\{(n-2)^2 + m^2\right\}\left\{(n-4)^2 + m^2\right\} \ldots (1^2 + m^2)\, m \cdot n \text{ is even, } y_n = (0) = 0.\right]$$

7. If $y = \left[\log\left\{x + \sqrt{1+x^2}\right\}\right]^2$, show that

$$y_{n+2}(0) = -\,n^2\, y_n(0) \text{ hence find } y_n(0).$$

$$\left[\textbf{Ans. } n \text{ is odd, } y_n(0), = 0 \;\; n \text{ is even } y_n(0) = (-1)^{\frac{n-2}{2}} (n-2)^2 (n-4)^2 \ldots 4^2\, 2^2.2.\right]$$

PARTIAL DIFFERENTIATION

Introduction

Real world can be described in mathematical terms using parametric equations and functions such as trigonometric functions which describe cyclic, repetitive activity; exponential, logrithmic and logistic functions which describe growth and decay and polynomial functions which approximate these and most other functions.

The problems in computer science, statistics, fluid dynamics, economics etc., deal with functions of two or more independent variables.

1.3 FUNCTION OF TWO VARIABLES

If $f(x, y)$ is a unique value for every x and y, then f is said to be a function of the two independent variables x and y and is denoted by

$$z = f(x, y)$$

Geometrically the function $z = f(x, y)$ represents a surface.

The graphical representation of function of two variables is shown in Figure 1.1.

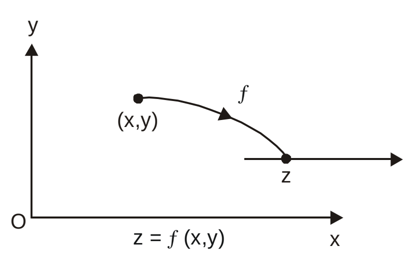

Fig. 1.1

1.4 PARTIAL DIFFERENTIAL COEFFICIENTS

The partial derivative of a function of several variables is the ordinary derivative with respect to any one of the variables whenever, all the remaining variables are held constant. The difference between partial and ordinary differentiation is that while differentiating (partially) with respect to one variable, all other variables are treated (temporarily) as constants and in ordinary differentiation no variable taken as constant,

Definition: Let $z = f(x, y)$

Keeping y constant and varying only x, the partial derivative of z w.r.t. 'x' is denoted by $\frac{\partial z}{\partial x}$ and is defined as the limit

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

Partial derivative of z, w.r.t. y is denoted by $\frac{\partial z}{\partial y}$ and is defined as

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}.$$

Notation: The partial derivative $\frac{\partial z}{\partial x}$ is also denoted by $\frac{\partial f}{\partial x}$ or f_x similarly $\frac{\partial z}{\partial y}$ is denoted by $\frac{\partial f}{\partial y}$ or f_y. The partial derivatives for higher order are calculated by successive differentiation.

Thus, $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}, \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} \text{ and so on.}$$

Geometrical interpretation of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

Let $z = f(x, y)$ represents the equation of a surface in xyz-coordinate system. Suppose APB is the curve which a plane through any point P on the surface $\parallel$ to the xz-plane, cuts. As point P moves along this curve APB, its coordinates z and x vary while y remains constant. The slope of the tangent line at P to APB represents the rate at which z-changes w.r.t. x.

Hence, $$\frac{\partial z}{\partial x} = \tan\theta \text{ (slope of the curve } APB \text{ at the point } P)$$

and $$\frac{\partial z}{\partial y} = \tan\phi \text{ (slope of the curve } CPD \text{ at point } P)$$

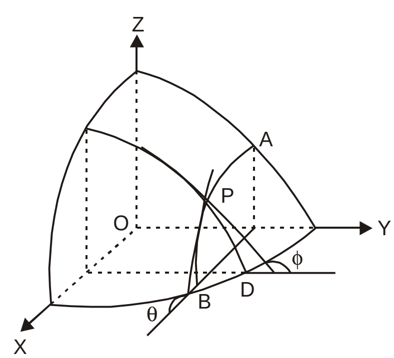

Fig. 1.2

Example 1. Verify that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ where $u(x, y) = \log_e \left(\dfrac{x^2+y^2}{xy}\right)$ (*U.P.T.U., 2007*)

Sol. We have
$$u(x, y) = \log_e \left(\frac{x^2+y^2}{xy}\right)$$

$$\Rightarrow \quad u(x, y) = \log (x^2 + y^2) - \log x - \log y \qquad \text{...}(i)$$

Differentiating partially w.r.t. x, we get
$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} - \frac{1}{x}$$

Now differentiating partially w.r.t. y.
$$\frac{\partial^2 u}{\partial y \partial x} = -\frac{4xy}{\left(x^2+y^2\right)^2} \qquad \text{...}(A)$$

Again differentiate (i) partially w.r.t. y, we obtain
$$\frac{\partial u}{\partial y} = \frac{2y}{\left(x^2+y^2\right)} - \frac{1}{y}$$

Next, we differentiate above equation w.r.t. x.
$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{4xy}{\left(x^2+y^2\right)^2} \qquad \text{...}(B)$$

Thus, from (A) and (B), we find
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$ **Hence proved.**

Example 2. If $f = \tan^{-1}\left(\dfrac{y}{x}\right)$, verify that $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$.

Sol. We have
$$f = \tan^{-1}\left(\frac{y}{x}\right) \qquad \text{...}(i)$$

Differentiating (i) partially with respect to x, we get
$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\dfrac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \left(\frac{-y}{x^2+y^2}\right) \qquad \text{...}(ii)$$

Differentiating (i) partially with respect to y, we get
$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\dfrac{y}{x}\right)^2}\frac{1}{x} = \frac{x}{x^2+y^2} \qquad \text{...}(iii)$$

Differentiating (ii) partially with respect to y, we get
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{\left(x^2+y^2\right)(-1)-(-y)(2y)}{\left(x^2+y^2\right)^2}$$

$$= \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2} \qquad ...(iv)$$

Differentiating (*iii*) partially with respect to x, we get

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) = \frac{\left(x^2 + y^2\right)(1) - x(2x)}{\left(x^3 + y^2\right)^2}$$

$$= \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2} \qquad ...(v)$$

$\therefore$ From eqns. (*iv*) and (*v*), we get $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$. **Hence proved.**

Example 3. If $u(x + y) = x^2 + y^2$, prove that $\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right)$.

Sol. Given $$u = \frac{x^2 + y^2}{x + y}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{(x+y)(2x) - \left(x^2 + y^2\right)(1)}{(x+y)^2} = \frac{x^2 + 2xy - y^2}{(x+y)^2}$$

and $$\frac{\partial u}{\partial y} = \frac{(x+y)\,(2y) - \left(x^2 + y^2\right)(1)}{(x+y)^2} = \frac{y^2 + 2xy - x^2}{(x+y)^2}$$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{4xy}{(x+y)^2}$$

or $$1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 1 - \frac{4xy}{(x+y)^2} = \frac{(x-y)^2}{(x+y)^2} \qquad ...(i)$$

and $$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = \frac{\left(x^2 + 2xy - y^2\right) - \left(y^2 + 2xy - x^2\right)}{(x+y)^2}$$

$$= \frac{2\left(x^2 - y^2\right)}{(x+y)^2} = \frac{2\,(x-y)}{(x+y)} \qquad ...(ii)$$

$\therefore$ From (*ii*), we get

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = \frac{4\,(x-y)^2}{(x+y)^2} = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right), \text{ from } (i).$$ **Hence proved.**

Example 4. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Sol. Given $$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right) \quad ...(i)$$

$\therefore$ $$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{\left\{1-\left(\frac{x}{y}\right)^2\right\}}} \cdot \frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right)$$

or $$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{(y^2-x^2)}} - \frac{yx}{x^2+y^2} \quad ...(ii)$$

and from (*i*), $$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{\left\{1-\left(\frac{x}{y}\right)^2\right\}}}\left(-\frac{x}{y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

or $$y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{(y^2-x^2)}} + \frac{xy}{x^2+y^2} \quad ...(iii)$$

Adding (*ii*) and (*iii*), we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$. **Hence proved.**

Example 5. If $f(x, y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ then prove that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Sol. $$\frac{\partial f}{\partial x} = 2x \cdot \tan^{-1}\left(\frac{y}{x}\right) + x^2 \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(-\frac{y}{x^2}\right) - y^2 \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(\frac{1}{y}\right)$$

or $$\frac{\partial f}{\partial x} = 2x \cdot \tan^{-1}\left(\frac{y}{x}\right) - \frac{yx^2}{x^2+y^2} - \frac{y^3}{x^2+y^2} = 2x\tan^{-1}\left(\frac{y}{x}\right) - y$$

Differentiating both sides with respect to y, we get
$$\frac{\partial^2 f}{\partial y \partial x} = 2x \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{1}{x}\right) - 1 = \frac{2x^2}{x^2+y^2} - 1 = \frac{x^2-y^2}{x^2+y^2} \quad ...(i)$$

Again $$\frac{\partial f}{\partial y} = x^2 \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} - 2y\tan^{-1}\left(\frac{x}{y}\right) - y^2 \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(-\frac{x}{y^2}\right)$$

or $$\frac{\partial f}{\partial y} = \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right) + \frac{xy^2}{x^2+y^2}$$

$$= \frac{x\left(x^2+y^2\right)}{x^2+y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right) = x - 2y \tan^{-1}\left(\frac{x}{y}\right).$$

Differentiating both sides with respect to x, we get

$$\frac{\partial^2 f}{\partial x\, \partial y} = 1 - 2y \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right) = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} \qquad ...(ii)$$

Thus, from (*i*) and (*ii*), we get

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$ **Hence proved.**

Example 6. If $V = (x^2 + y^2 + z^2)^{-1/2}$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Sol. Given $\quad V = (x^2 + y^2 + z^2)^{-1/2}$. ...(*i*)

$$\therefore \quad \frac{\partial V}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\, 2x = -x\,(x^2+y^2+z^2)^{-3/2}$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} = -\left[x\left\{-\frac{3}{2}\left(x^2+y^2+z^2\right)^{-5/2}\cdot 2x\right\}+\left(x^2+y^2+z^2\right)^{-3/2.}\cdot 1\right]$$

$$= 3x^2\,(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$= (x^2+y^2+z^2)^{-5/2}\,[3x^2 - (x^2+y^2+z^2)]$$

or $$\frac{\partial^2 V}{\partial x^2} = (x^2+y^2+z^2)^{-5/2}\,(2x^2 - y^2 - z^2) \qquad ...(ii)$$

Similarly from (*i*), we can find

$$\frac{\partial^2 V}{\partial y^2} = (x^2+y^2+z^2)^{-5/2}\,(2y^2 - x^2 - z^2) \qquad ...(iii)$$

and $$\frac{\partial^2 V}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}\,(2z^2 - x^2 - y^2) \qquad ...(iv)$$

Adding (*ii*), (*iii*) and (*iv*), we get

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}\,[(2x^2-y^2-z^2) + (2y^2-x^2-z^2) + (2z^2-x^2-y^2)]$$

$$= (x^2+y^2+z^2)^{-5/2}\,[0] = 0.$$ **Hence proved.**

Example 7. If $u = f(r)$, where $r^2 = x^2 + y^2$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r).$$ (*U.P.T.U., 2001, 2005*)

Sol. Given $r^2 = x^2 + y^2$ ...(*i*)

Differentiating both sides partially with respect to x, we have

$$2r\frac{\partial r}{\partial x} = 2x \text{ or } \frac{\partial r}{\partial x} = \frac{x}{r} \quad ...(ii)$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r} \quad ...(iii)$$

Now, $u = f(r)$

$$\therefore \quad \frac{\partial u}{\partial x} = f'(r).\frac{\partial r}{\partial x} = f'(r)\cdot\frac{x}{r}, \text{ from } (ii)$$

Again differentiating partially w.r.to x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{f'(r)x}{r}\right] = \frac{r\left[f'(r).1 + xf''(r)(\partial r/\partial x)\right] - xf'(r)(\partial r/\partial x)}{r^2}$$

or $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r^2}\left[rf'(r) + x^2 f''(r) - \frac{x^2}{r}f'(r)\right], \text{ from } (ii).$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[rf'(r) + y^2 f''(r) - \frac{y^2}{r}f'(r)\right]$$

Adding, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[2rf'(r) + (x^2+y^2)f''(r) - \frac{(x^2+y^2)}{r}f'(r)\right]$$

$$= \frac{1}{r^2}[2rf'(r) + r^2 f''(r) - rf'(r)], \text{ from } (i)$$

$$= \frac{1}{r}f'(r) + f''(r).$$ **Hence proved.**

Example 8. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$; show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$ (*U.P.T.U., 2003*)

Sol. Given $u = \log(x^3 + y^3 + z^3 - 3xyz)$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} \quad ...(i)$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} \quad ...(ii)$$

and $$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}. \quad ...(iii)$$

Adding eqns. (*i*), (*ii*) and (*iii*), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right)}$$

$$\left| \text{As } a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \right.$$

or $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}. \quad ...(iv)$$

Now, $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right), \text{ from } (iv)$$

$$= 3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$= 3\left[-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}\right] = \frac{-9}{(x+y+z)^2}.$$

Hence proved.

Example 9. If $u = \tan^{-1}\dfrac{xy}{\sqrt{(1+x^2+y^2)}}$, show that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{\left(1+x^2+y^2\right)^{3/2}}.$$

Sol. Given $$u = \tan^{-1}\frac{xy}{\sqrt{(1+x^2+y^2)}}.$$

$\therefore$ $$\frac{\partial u}{\partial y} = \frac{1}{\left[1+\left\{x^2y^2/\left(1+x^2+y^2\right)\right\}\right]} \times x\left[\frac{\sqrt{\left(1+x^2+y^2\right)}.1 - y\frac{1}{2}\left(1+x^2+y^2\right)^{-1/2}2y}{\left(1+x^2+y^2\right)}\right]$$

$$= \frac{x}{1+x^2+y^2+x^2y^2}\cdot\frac{\left(1+x^2+y^2\right)-y^2}{\sqrt{\left(1+x^2+y^2\right)}}$$

or $$\frac{\partial u}{\partial y} = \frac{x}{(1+x^2)(1+y^2)} \cdot \frac{1+x^2}{\sqrt{(1+x^2+y^2)}} = \frac{x}{(1+y^2)\sqrt{(1+x^2+y^2)}}$$

Again differentiating partially w.r.to x

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{x}{(1+y^2)\sqrt{(1+x^2+y^2)}}\right] = \frac{1}{(1+y^2)}\frac{\partial}{\partial x}\left[\frac{x}{\sqrt{(1+x^2+y^2)}}\right]$$

$$= \frac{1}{(1+y^2)}\left[\frac{\sqrt{(1+x^2+y^2)}1 - x\frac{1}{2}(1+x^2+y^2)^{-1/2}2x}{(1+x^2+y^2)}\right]$$

$$= \frac{1}{(1+y^2)} \cdot \frac{(1+x^2+y^2)-x^2}{(1+x^2+y^2)^{3/2}} = \frac{1}{(1+x^2+y^2)^{3/2}}.$$ **Hence proved.**

Example 10. If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$ (*U.P.T.U., 2002*)

Sol. We have

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \qquad ...(i)$$

where u is a function of x, y and z

Differentiating (*i*) partially with respect to x, we get

$$= \frac{2x}{a^2+u} - \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x} = 0$$

or $$\frac{\partial u}{\partial x} = \frac{2x/(a^2+u)}{\left[x^2/(a^2+u)^2 + y^2/(b^2+u)^2 + z^2/(c^2+u)^2\right]} = \frac{2x/(a^2+u)}{\sum\left[x^2/(a^2+u)^2\right]}$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{2y/(b^2+u)}{\sum\left[x^2/(a^2+u)^2\right]}; \quad \frac{\partial u}{\partial z} = \frac{2z/(c^2+u)}{\sum\left[x^2/(a^2+u)^2\right]}$$

Adding with square

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{4\left[x^2/(a^2+u)^2 + y^2/(b^2+u)^2 + z^2/(c^2+u)^2\right]}{\left[\sum\left\{x^2/(a^2+u)^2\right\}\right]^2}$$

or $$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = \frac{4}{\sum\left[x^2/\left(a^2+u\right)^2\right]} \qquad ...(ii)$$

Also, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{1}{\sum\left[x^2/\left(a^2+u\right)^2\right]}\left[\frac{2x^2}{\left(a^2+u\right)}+\frac{2y^2}{\left(b^2+u\right)}+\frac{2z^2}{\left(c^2+u\right)}\right]$$

$$= \frac{2}{\sum\left[x^2/\left(a^2+u\right)^2\right]} \text{ [1], from } (i) \qquad ...(iii)$$

From (*ii*) and (*iii*), we have

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$ **Hence proved.**

Example 11. If $x^x\, y^y\, z^z = c$, show that at $x = y = z$,

$$\frac{\partial^2 z}{\partial x\,\partial y} = -(x \log ex)^{-1}.$$ Where z is a function of x and y.

Sol. Given $x^x.y^y.z^z = c$, where z is a function of x and y.

Taking logarithms, $x \log x + y \log y + z \log z = \log c$. ...(*i*)

Differentiating (*i*) partially with respect to x, we get

$$= \left[x\left(\frac{1}{x}\right)+(\log x)1\right]+\left[z\left(\frac{1}{z}\right)+(\log z)1\right]\frac{\partial z}{\partial x} = 0$$

or $$\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)} \qquad ...(ii)$$

Similarly, from (*i*), we have $$\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{(1+\log z)} \qquad ...(iii)$$

$$\therefore \quad \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[-\left(\frac{1+\log y}{1+\log z}\right)\right], \text{ from } (iii)$$

or $$\frac{\partial^2 z}{\partial x\partial y} = -(1+\log y)\cdot\frac{\partial}{\partial x}\left[(1+\log z)^{-1}\right]$$

$$= -(1+\log y)\cdot\left[-(1+\log z)^{-2}\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial x}\right]$$

or $$\frac{\partial^2 z}{\partial x\partial y} = \frac{(1+\log y)}{z(1+\log z)^2}\cdot\left\{-\left(\frac{1+\log x}{1+\log z}\right)\right\}, \text{ from } (iii)$$

$\therefore$ At $x = y = z$, we have $$\frac{\partial^2 z}{\partial x\partial y} = -\frac{(1+\log x)^2}{x\,(1+\log x)^3}$$

$$\Rightarrow \qquad \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(1+\log x)} = -\frac{1}{x(\log_e e + \log x)} = \text{(As } \log_e e = 1)$$

$$= -\frac{1}{x\log(ex)} = -\{x \log (ex)\}^{-1}. \quad \textbf{Hence proved.}$$

Example 12. If $u = \log (x^3 + y^3 - x^2y - xy^2)$ then show that

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}.$$

Sol. We have $\qquad u = \log (x^3 + y^3 - x^2y - xy^2)$

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 2xy - y^2}{(x^3 + y^3 - x^2y - xy^2)} \qquad ...(i)$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - x^2 - 2xy}{(x^3 + y^3 - x^2y - xy^2)} \qquad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{(3x^2 - 2xy - y^2) + (3y^2 - x^2 - 2xy)}{(x^3 + y^3 - x^2y - xy^2)}$$

$$= \frac{2\,(x-y)^2}{(x+y)\,(x^2 + y^2 - 2xy)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{2\,(x-y)^2}{(x+y)(x-y)^2} = \frac{2}{(x+y)} \qquad ...\ (iii)$$

Now, $\quad \dfrac{\partial^2 u}{\partial x^2} + 2\,\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = \left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}\right)^2 u$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) . \frac{2}{x+y} \quad \text{(from } iii)$$

$$= 2\,\frac{\partial}{\partial x}\left(\frac{1}{x+y}\right) + 2\,\frac{\partial}{\partial y}\left(\frac{1}{x+y}\right)$$

$$= -\frac{2}{(x+y)^2} - \frac{2}{(x+y)^2} = -\frac{4}{(x+y)^2}. \quad \textbf{Hence proved.}$$

Example 13. If $u = e^{xyz}$, show that

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}.$$

Sol. We have $u = e^{xyz}$ $\therefore$ $\dfrac{\partial u}{\partial z} = e^{xyz}.xy$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}(e^{xyz}\cdot xy) = e^{xyz}\, x^2yz + e^{xyz}.x$$

or $$\frac{\partial^2 u}{\partial y \partial z} = (x^2yz + x)\, e^{xyz}$$

Hence $$\frac{\partial^3 u}{\partial x \partial y \partial z} = (2xyz + 1)\, e^{xyz} + (x^2yz + x)\, e^{xyz}\cdot yz$$

$$= (1 + 3xyz + x^2y^2z^2)\, e^{xyz}.$$ **Hence proved.**

Example 14. If $u = \log r$, where $r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r^2}.$$

Sol. Given $r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$, ...(*i*)

Differentiating partially with respect to x, we get

$$2r\frac{\partial r}{\partial x} = 2\,(x - a) \text{ or } \frac{\partial r}{\partial x} = \left(\frac{x-a}{r}\right). \qquad ...(ii)$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{(y-b)}{r} \text{ and } \frac{\partial r}{\partial y} = \frac{(z-c)}{r}$$

Now, $u = \log r.$

$\therefore$ $$\frac{\partial u}{\partial x} = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{1}{r}\left(\frac{x-a}{r}\right), \text{ from } (ii)$$

or $$\frac{\partial u}{\partial x} = \frac{x-a}{r^2}$$

$\therefore$ $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x-a}{r^2}\right) = \frac{r^2(1) - (x-a)\,2r(\partial r/\partial x)}{r^4}$$

or $$\frac{\partial^2 u}{\partial x^2} = \frac{r^2 - 2\,(x-a)^2}{r^4}, \text{ from } (ii)$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{r^2 - 2\,(y-b)^2}{r^4};\ \frac{\partial^2 u}{\partial z^2} = \frac{r^2 - 2\,(z-c)^2}{r^4}.$$

$\therefore$ $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{3r^2 - 2\left\{(x-a)^2 + (y-b)^2 + (z-c)^2\right\}}{r^4}$$

$$= \frac{3r^2 - 2r^2}{r^4}, \text{ from } (i) = \frac{1}{r^2}.$$ **Hence proved.**

Example 15. If $u = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$, prove that

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}.$$

Sol. Given $u = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$.

Differentiating partially with respect to x, we get

$$\frac{\partial u}{\partial x} = x^2 \frac{1}{1+(y/x)^2} \cdot \left(-\frac{y}{x^2}\right) + 2x \tan^{-1}\left(\frac{y}{x}\right) - y^2 \cdot \frac{1}{1+(x/y)^2} \cdot \frac{1}{y}$$

$$= -\frac{x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2} + 2x \tan^{-1} \frac{y}{x}$$

$$= -\frac{y\left(x^2+y^2\right)}{\left(x^2+y^2\right)} + 2x \tan^{-1}\left(\frac{y}{x}\right) = -y + 2x \tan^{-1}\left(\frac{y}{x}\right)$$

Again differentiating partially with respect to y, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left\{-y + 2x \tan^{-1}\left(\frac{y}{x}\right)\right\} = -1 + 2x \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

$$= -1 + \frac{2x^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}.$$ **Hence proved.**

Example 16. If $z = f(x - by) + \phi(x + by)$, prove that

$$b^2 \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}.$$

Sol. Given $z = f(x - by) + \phi(x + by)$ ...(*i*)

$\therefore$ $$\frac{\partial z}{\partial x} = f'(x - by) + \phi'(x + by)$$

and $$\frac{\partial^2 z}{\partial x^2} = f''(x - by) + \phi''(x + by).$$ ...(*ii*)

Again from (*i*), $$\frac{\partial z}{\partial y} = -bf'(x - by) + b\phi'(x + by)$$

and $$\frac{\partial^2 z}{\partial y^2} = b^2 f''(x - by) + b^2 \phi''(x + by) = b^2 \frac{\partial^2 z}{\partial x^2},$$ from (*ii*). **Hence proved.**

Example 17. If $u(x, y, z) = \log(\tan x + \tan y + \tan z)$. Prove that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2.$$ (*U.P.T.U., 2006*)

Sol. $$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y + \tan z}$$

$$\frac{\partial u}{\partial z} = \frac{\sec^2 z}{\tan x + \tan y + \tan z}$$

$$\therefore \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$= \frac{\sin 2x \sec^2 x + \sin 2y \sec^2 y + \sin 2z \sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)}$$

$= 2.$ **Hence proved.**

EXERCISE 1.4

1. Find $\dfrac{\partial^3 u}{\partial x \partial y \partial z}$ if $u = e^{x^2+y^2+z^2}$. **[Ans.** $8xyzu$**]**

2. Find the first order derivatives of

(*i*) $u = x^{xy}$. $\left[\textbf{Ans. } \dfrac{\partial u}{\partial x} = x^{xy}(y \log x + y);\ \dfrac{\partial u}{\partial y} = x^{xy+1} \log x\right]$

(*ii*) $u = \log\left(x + \sqrt{x^2 - y^2}\right)$ $\left[\textbf{Ans. } \dfrac{\partial u}{\partial x} = \dfrac{1}{\sqrt{x^2 - y^2}}\ \dfrac{\partial u}{\partial y} = -y\left(x^2 - y^2\right)^{\frac{-1}{2}}\left(x + \sqrt{x^2 - y^2}\right)^{-1}\right]$

3. If $u = \sin^{-1}\left(\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)$, show that $\dfrac{\partial u}{\partial x} = -\dfrac{y}{x}\dfrac{\partial u}{\partial y}$.

4. If $u = e^x (x \cos y - y \sin y)$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

5. If $u = a \log (x^2 + y^2) + b \tan^{-1}\left(\dfrac{y}{x}\right)$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

6. If $z = \tan (y - ax) + (y + ax)^{3/2}$, prove that $\dfrac{\partial^2 z}{\partial x^2} - a^2 \dfrac{\partial^2 z}{\partial y^2} = 0$.

7. If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

8. If $u = 2(ax + by)^2 - (x^2 + y^2)$ and $a^2 + b^2 = 1$, find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$. [**Ans.** 0]

9. If $u = \log (x^2 + y^2 + z^2)$, find the value of $\dfrac{\partial^2 u}{\partial y \partial z}$. $\left[\textbf{Ans. } \dfrac{-4yz}{(x^2+y^2+z^2)^2}\right]$

10. If $u = \left(x^2+y^2+z^2\right)^{\frac{1}{2}}$, then prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{2}{u}$.

11. If $z = f(x + ay) + \phi(x - ay)$, prove that $\dfrac{\partial^2 z}{\partial y^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$.

12. If $u = \cos^{-1} \left[(x-y)/(x+y)\right]$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

13. If $u = \log (x^2 + y^2)$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

14. If $u = x^2y + y^2z + z^2x$, show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = (x + y + z)^2$.

15. Prove that $f(x, t) = a \sin bx . \cos bt$ satisfies $\dfrac{\partial^2 f}{\partial x^2} = b^2 \dfrac{\partial^2 f}{\partial t^2}$.

16. If $u = r^m$, where $r = \sqrt{x^2+y^2+z^2}$ find $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$. [**Ans.** $m(m + 1)r^{m-2}$]

17. If $u = (x^2 + y^2 + z^2)^{-1}$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 2(x^2 + y^2 + z^2)^{-2}$.

18. If $\theta = t^n e^{-\frac{r^2}{4t}}$, find the value of n, when $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$. $\left[\textbf{Ans. } n = -\dfrac{3}{2}\right]$

19. For $n = 2$ or -3 show that $u = r^n (3\cos^2\theta - 1)$ satisfies the differential equation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) = 0.$$

20. If $u = e^{a\theta} \cos (a \log r)$, show that $\left(\dfrac{\partial^2 u}{\partial r^2}\right) + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0$.

21. If $e^{-\frac{z}{(x^2-y^2)}} = (x - y)$, show that $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = x^2 - y^2$.

[**Hint:** Solve for $z = (y^2 - x^2) \log (x - y)$].

22. If $u = \dfrac{1}{r}$ and $r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$.

23. If $u\,(x, y, z) = \cos 3x \cos 4y \sin h\, 5z$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$.

24. If $x^2 = au + bv$; $y^2 = au - bv$, prove that $\left(\dfrac{\partial u}{\partial x}\right)_y \left(\dfrac{\partial x}{\partial u}\right)_v = \dfrac{1}{2}\left(\dfrac{\partial v}{\partial y}\right)_x \left(\dfrac{\partial y}{\partial v}\right)_u$.

25. If $x = r\cos\theta$, $y = r\sin\theta$, find $\left(\dfrac{\partial x}{\partial r}\right)_\theta$, $\left(\dfrac{\partial x}{\partial \theta}\right)_{r,}$ $\left(\dfrac{\partial \theta}{\partial x}\right)_{y,}$ $\left(\dfrac{\partial \theta}{\partial y}\right)_{x,}$ $\left(\dfrac{\partial y}{\partial x}\right)_r$.

[**Ans.** $\cos\theta, -r\sin\theta, -r^{-1}\sin\theta, r^{-1}\cos\theta, -\cot\theta$.]

1.5 HOMOGENEOUS FUNCTION

A polynomial in x and y *i.e.,* a function $f\,(x, y)$ is said to be homogeneous if all its terms are of the same degree. Consider a homogeneous polynomial in x and y

$$f\,(x, y) = a_0 x^n + a_1\, x^{n-1} y + a_2\, x^{n-2} y^2 + \ldots.. + a_n\, y^n$$

$$= x^n \left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \ldots + a_n\left(\frac{y}{x}\right)^n\right]$$

or

$$\boxed{f\,(x, y) = x^n\, F\left(\frac{y}{x}\right)}$$

Hence every homogeneous function of x and y of degree n can be written in above form.

NOTE: Degree of Homogeneous function = degree of numerator – degree of denominator.

Remark 1: If $f\,(x, y) = a_0 x^n + a_1 x^{n+1} \cdot y^{-1} + a_2 x^{n+2} \,.\, y^{-2} + \ldots\ldots + a_n x^{n+n}\, y^{-n}$

$$= x^n \left\{a_0 + a_1 \frac{x}{y} + a_2\left(\frac{x}{y}\right)^2 + \ldots. + a_n\left(\frac{x}{y}\right)^n\right\}$$

$\Rightarrow$ $f\,(x, y) = x^n\, F\left(\dfrac{x}{y}\right)$; degree $= n$

Remark 2: If $f\,(x, y) = a_0 y^{-n} + a_1 y^{-n-1} \,.\, x + \ldots\ldots + a_n y^{-n-n} \,.\, x^n$

$$= y^n \left\{a_0 + a_1\left(\frac{x}{y}\right) + \ldots. + a_n\left(\frac{x}{y}\right)^n\right\}$$

$\Rightarrow$ $f\,(x, y) = y^{-n}\, F\left(\dfrac{x}{y}\right)$; degree $= -n$

Another forms are also possible *i.e.,*

$$f\,(x, y) = y^n\, F\left(\frac{x}{y}\right);\; f\,(x, y) = y^n\, F(y/x)$$

1.6 EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

Statement: If f is a homogeneous function of x, y of degree n then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf. \qquad \textit{(U.P.T.U., 2006)}$$

Proof. Since f is a homogeneous function

$$\therefore \qquad f(x, y) = x^n F\left(\frac{y}{x}\right) \qquad ...(i)$$

Differentiating partially w.r.t. x and y, we get

$$\frac{\partial f}{\partial x} = nx^{n-1} F\left(\frac{y}{x}\right) + x^n F'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \qquad ...(ii)$$

$$\frac{\partial f}{\partial y} = x^n F'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \qquad ...(iii)$$

Multiplying (*ii*) by x and (*iii*) by y and adding, we have

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nx^n F\left(\frac{y}{x}\right) - x^{n-1} y F'\left(\frac{y}{x}\right) + x^{n-1} y F'\left(\frac{y}{x}\right)$$

$$= nx^n F\left(\frac{y}{x}\right)$$

$$\Rightarrow \qquad \boxed{x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f} \quad \text{(from } (i)).$$

In general if $f(x_1, x_2, ..., x_n)$ be a homogeneous function in $x_1, x_2, ..., x_n$ then $x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2}$

$$+ ... + x_n \frac{\partial f}{\partial x_n} = nf.$$

Corollary 1. If f is a homogeneous function of degree n, then

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n-1)f.$$

Proof. We have

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f \qquad ...(i)$$

Differentiating (*i*) w.r.t. x and y respectively, we get

$$\frac{\partial f}{\partial x} + x \frac{\partial^2 f}{\partial x^2} + y \frac{\partial^2 f}{\partial x \partial y} = n \frac{\partial f}{\partial x} \qquad ...(ii)$$

and

$$x \frac{\partial^2 f}{\partial y \partial x} + \frac{\partial f}{\partial y} + y \frac{\partial^2 f}{\partial y^2} = n \frac{\partial f}{\partial y} \qquad ...(iii)$$

Multiplying (*ii*) by x and (*iii*) by y and adding, we have

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{xy^2} + x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n\left(x \frac{\partial f}{\partial x} + y \frac{\partial t}{\partial y}\right)$$

$$\Rightarrow \qquad x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n^2 f - nf = n(n-1) f$$

Example 1. Verify Euler's theorem for the function

$$f(x, y) = ax^2 + 2hxy + by^2.$$

Sol. Here the given function $f(x, y)$ is homogeneous of degree $n = 2$. Hence the Euler's theorem is

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2f \qquad \text{...}(i)$$

Now, we are to prove equation (*i*) as follows:

$$\frac{\partial f}{\partial x} = 2\,ax + 2\,hy, \quad \frac{\partial f}{\partial y} = 2hx + 2by$$

$$\therefore \qquad x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2ax^2 + 2hxy + 2hxy + 2by^2$$

$$= 2(ax^2 + 2\,hxy + by^2) = 2f$$

$$\Rightarrow \qquad x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2f, \text{ which proves equation } (i).$$

Example 2. Verify Euler's theorem for the function $u = x^n \sin\left(\frac{y}{x}\right)$.

Sol. Since u is homogeneous function in x and y of degree n, hence we are to prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu \qquad \text{...}(i)$$

We have

$$u = x^n \sin\left(\frac{y}{x}\right)$$

$$\therefore \qquad \frac{\partial u}{\partial x} = nx^{n-1} \sin\left(\frac{y}{x}\right) + x^n \cos\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

or

$$x \frac{\partial u}{\partial x} = nx^n \sin\left(\frac{y}{x}\right) - x^{n-1}\, y \cos\left(\frac{y}{x}\right) \qquad \text{...}(ii)$$

Similarly,

$$y \frac{\partial u}{\partial y} = yx^{n-1} \cos\left(\frac{y}{x}\right) \qquad \text{...}(iii)$$

Adding (*ii*) and (*iii*), we get

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nx^n \sin\left(\frac{y}{x}\right)$$

$$\Rightarrow \qquad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

which verifies Euler's theorem.

Example 3. If $u = \sin^{-1}\left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, show by Euler's theorem that

$$\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}.$$

Sol. We have $$u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right) \Rightarrow \sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

Let $$f = \sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

Here, f is a homogeneous function in x and y

where, $$\text{degree } n = \frac{1}{2} - \frac{1}{2} = 0$$

$\therefore$ By Euler's theorem, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0.\, f = 0$$

or $$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = 0 \quad \Big| \text{ As } f = \sin u$$

or $$x\cos u \cdot \frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 0$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}.$$ **Hence proved.**

Example 4. If $u = \log[(x^4 + y^4)/(x + y)]$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3.$$ (*U.P.T.U., 2000*)

Sol. We have $$u = \log_e \frac{x^4+y^4}{x+y} \Rightarrow e^u = \frac{x^4+y^4}{x+y}$$

Let $$f = e^u = \frac{x^4+y^4}{x+y}$$

Here the function f is a homogeneous function in x and y of degree, $n = 4 - 1 = 3$

$\therefore$ By Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf = 3f$$

$$\Rightarrow \quad x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = 3f$$

$$\Rightarrow \qquad \left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)e^u = 3e^u$$

$$\Rightarrow \qquad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 3.$$ **Hence proved.**

Example 5. Verify Euler's theorem for

$$u = \sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right).$$ (*U.P.T.U., 2006*)

Sol. Here u is a homogeneous function of degree,

$n = 1 - 1 = 0$; hence by Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 0$$

Now,
$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\frac{1}{y}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$$

or
$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}}-\frac{xy}{x^2+y^2} \qquad ...(i)$$

and
$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(-\frac{x}{y^2}\right)+\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{1}{x}\right)$$

or
$$y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}}+\frac{xy}{\left(x^2+y^2\right)} \qquad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 0,$$ hence Euler's theorem is verified.

Example 6. If $u = x\sin^{-1}\left(\frac{x}{y}\right)+y\sin^{-1}\left(\frac{y}{x}\right)$, find the value of

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}.$$ (*U.P.T.U., 2007*)

Sol. We have

$$u = x\sin^{-1}\left(\frac{x}{y}\right)+y\sin^{-1}\left(\frac{y}{x}\right)$$

$$\therefore \qquad \frac{\partial u}{\partial x} = \sin^{-1}\left(\frac{x}{y}\right)+\frac{x}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{1}{y}\right)+y\cdot\frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\left(-\frac{y}{x^2}\right)$$

$$\Rightarrow \qquad x \frac{\partial u}{\partial x} = x \sin^{-1}\left(\frac{x}{y}\right) + \frac{x^2}{\sqrt{y^2 - x^2}} - \frac{y^2}{\sqrt{x^2 - y^2}} \qquad \ldots(i)$$

and
$$\frac{\partial u}{\partial y} = x \cdot \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right) + \sin^{-1}\left(\frac{y}{x}\right) + y \frac{1}{\sqrt{1 - \frac{y^2}{x^2}}}\left(\frac{1}{x}\right)$$

$$\Rightarrow \qquad y \frac{\partial u}{\partial y} = -\frac{x^2}{\sqrt{y^2 - x^2}} + y \sin^{-1}\left(\frac{y}{x}\right) + \frac{y^2}{\sqrt{x^2 - y^2}} \qquad \ldots(ii)$$

Adding (*i*) and (*ii*), we get

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x \sin^{-1}\left(\frac{x}{y}\right) + y \sin^{-1}\left(\frac{y}{x}\right) = u \qquad \ldots(iii)$$

Differentiating (*iii*) partially w.r.t. x and y respectively.

$$\frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x} \Rightarrow x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = 0 \qquad \ldots(iv)$$

and
$$x \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial y} \Rightarrow y \frac{\partial^2 u}{\partial y^2} + x \frac{\partial^2 u}{\partial x \partial y} = 0 \qquad \ldots(v)$$

Multiplying equation (*iv*) by x and (*v*) by y and adding, we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

Example 7. If $u = \tan^{-1} \frac{x^3 + y^3}{x - y}$, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$ and evaluate

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}.$$

Sol. We have

$$u = \tan^{-1} \frac{x^3 + y^3}{x - y} \Rightarrow \tan u = \frac{x^3 + y^3}{x - y}$$

$$\therefore \qquad \text{Let } f = \tan u = \frac{x^3 + y^3}{x - y}$$

Since $f(x, y)$ is a homogeneous function of degree

$$n = 3 - 1 = 2$$

By Euler's theorem, we have

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf \Rightarrow x \frac{\partial}{\partial x} (\tan u) + y \frac{\partial}{\partial y} (\tan u) = 2 \tan u$$

$$\Rightarrow \quad x \frac{\partial u}{\partial x} \cdot \sec^2 u + y \frac{\partial u}{\partial y} \cdot \sec^2 u = 2 \tan u$$

or $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \frac{\tan u}{\sec^2 u} = \sin 2u. \quad \textbf{Proved.} \qquad ...(i)$$

Differentiating (i) partially w.r.t. x, we get

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = 2 \cos 2u \cdot \frac{\partial u}{\partial x}$$

or $$x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = (2 \cos 2u - 1) \frac{\partial u}{\partial x}$$

Multiplying by x, we obtain

$$x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} = x (2 \cos 2u - 1) \frac{\partial u}{\partial x} \qquad ...(ii)$$

Again differentiating equation (i) partially w.r.t. y, we get

$$x \frac{\partial^2 u}{\partial y \partial x} + y \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 2 \cos 2u \frac{\partial u}{\partial y}$$

or $$y \frac{\partial^2 u}{\partial y^2} + x \frac{\partial^2 u}{\partial y \partial x} = (2 \cos 2u - 1) \frac{\partial u}{\partial y}$$

or $$y^2 \frac{\partial^2 u}{\partial y^2} + xy \frac{\partial^2 u}{\partial x \partial y} = y (2 \cos 2u - 1) \frac{\partial u}{\partial y} \text{(multiply by } y) \qquad ...(iii)$$

Adding (ii) and (iii), we get

$$\begin{aligned} x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} &= (2\cos 2u - 1) \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right) \\ &= (2\cos 2u - 1) \sin 2u, \text{ (from } (i)) \\ &= (2\sin 2u \cos 2u - \sin 2u) \\ &= \sin 4u - \sin 2u \\ &= 2 \cos \left(\frac{4u+2u}{2} \right) \cdot \cos \left(\frac{4u-2u}{2} \right). \end{aligned}$$

Hence, $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 2 \cos 3u \cdot \cos u.$

Example 8. If $z = x^m f\left(\dfrac{y}{x}\right) + x^n g\left(\dfrac{x}{y}\right)$, prove that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + mnz = (m + n - 1) \left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} \right).$$

Sol. Let $$u = x^m f\left(\frac{y}{x}\right), \; v = x^n g\left(\frac{x}{y}\right)$$

then $$z = u + v \qquad ...(i)$$

Now, u is homogeneous function of degree m. Therefore with the help of (Corollary 1, on page 36), we have

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = m(m-1)u \qquad ...(ii)$$

Similarly for $v = x^n g\left(\frac{x}{y}\right)$, we have

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = n(n-1)v \qquad ...(iii)$$

Adding (*ii*) and (*iii*), we get

$$x^2 \frac{\partial^2}{\partial x^2}(u+v) + 2xy \frac{\partial^2}{\partial x \partial y}(u+v) + y^2 \frac{\partial}{\partial y^2}(u+v) = m(m-1)u + n(n-1)v$$

$$\Rightarrow \quad x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = m(m-1)u + n(n-1)v \text{ (As } z = u+v). \qquad ...(iv)$$

Again from Euler's theorem, we get

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = mu \text{ and } x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = nv$$

Adding $x \frac{\partial}{\partial x}(u+v) + y \frac{\partial}{\partial y}(u+v) = mu + nv$

$$\Rightarrow \quad x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = mu + nv \qquad ...(v)$$

Now,
$$\begin{aligned} m(m-1)u + n(n-1)v &= (m^2 u + n^2 v) - (mu + nv) \\ &= m(m+n)u + n(m+n)v - mn(u+v) - (mu+nv) \\ &= (mu+nv)(m+n) - (mu+nv) - mnz \\ &= (mu+nv)(m+n-1) - mnz \\ &= (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) - mnz, \text{ from } (v) \end{aligned}$$

Putting this value in equation (*iv*), we get

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) - mnz$$

$$\Rightarrow \quad x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + mnz = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right).$$ **Hence proved.**

Example 9. If $u = \sin^{-1}\left[\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$, show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} + 3 \tan u = 0.$$ (*U.P.T.U., 2003*)