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A New Family of Mixed Finite Elements in $\mathbb{R}^3$

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Summary. We introduce two families of mixed finite element on conforming in $H(\mathrm{div})$ and one conforming in $H(\mathrm{curl})$. These finite elements can be used to approximate the Stokes' system.

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1. Introduction

The mixed finite element was introduced first in several papers of Fraeijs De Veubeke. In 1977, Raviart and Thomas [8] used these elements for solving second order equations in two dimension. In 1980 [6], we introduced two families of mixed finite elements in three dimension. The first family generalizes that of P.A. Raviart-J.M. Thomas and is conforming in the space $H(\mathrm{div})$. The second one appears to be completely new and is conforming in the space $H(\mathrm{curl})$.

In 1982, in the reference [7], we used the new finite elements to introduce an approximation of the Stokes equations that generalizes to the $3D$ case, the (ψ, ω) approximation.

In 1984, Brezzi et al. [2] introduced a new mixed finite element conforming in $H(\mathrm{div})$ in two dimensions. This last paper is the starting point of on search for new families of mixed finite elements in three dimension. We introduce here two families of such finite elements. The first one is conforming in the space $H(\mathrm{div})$ and in fact this family is split in three corresponding to the case of tetrahedrons, cubes and prisms. The second family is conforming in the space $H(\mathrm{curl})$ and is also split into three parts. We describe these elements, prove the unisolvence and estimate the interpolation error. When writing this paper, we learned that Brezzi, Douglas, Duran and Fortin obtained similar results for some finite elements in $H(\mathrm{div})$. They obtained results for the case of tetrahedra and cubes. Apparently their degrees of freedom are different from ours.

In the last chapter, we describe the way our elements can be used to approximate the Stokes' system.

Notations

K is a tetrahedron, a cube or a prism, its volume is $\int_K dx$;
∂K is its boundary; n is the normal to this boundary;
f is a face of K, with area $\int_f d\gamma$;
a is an edge of K with length $\int_a ds$;
$L^2(K)$ is the space of square integrable functions defined on $\mathbf{K}$;

$H^m(K)=\{\phi\in L^2(K), \partial^\alpha\phi\in L^2(K); |\alpha|\leq m,$

where $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ is a multi-index; $|\alpha|=\alpha_1+\alpha_2+\alpha_3\}$
$\operatorname{curl} u=\nabla\wedge u$, for each vector $u=(u_1,u_2,u_3)$;
$H(\operatorname{curl})=\{u\in L^2(K))^3;\ \operatorname{curl} u\in(L^2(K))^3\}$;
$\operatorname{div} u=\nabla\cdot u$, for each vector $u=(u_1,u_2,u_3)$;
$H(\operatorname{div})=\{u\in(L^2(K))^3;\ \operatorname{div} u\in(L^2(K))^3\}$;
P_k is the space of polynomials of degree less or equal to k;
$\tilde{P}_k$ is the space of polynomials homogeneous of degree k.

2. Finite Elements in $H(\operatorname{div})$

Let Ω be a bounded domain in $\mathbb{R}^3$. The Hilbert space $H(\operatorname{div})$ is the space of vectors of $\mathbb{R}^3$ defined on Ω that belongs to the space $(L^2(\Omega))^3$ and such that the divergence also belongs to the space $L^2(\Omega)$. We have the

Lemma 1. *Let Ω be the union of two domains K_1 and K_2 with a common face f of normal n. A vector p in the space $(H^1(K_1))^3\cup(H^1(K_2))^3$ is in the space $H(\operatorname{div})$ if and only of the trace $p\cdot n$ is the same on each side of the face f.*

A finite element is defined by the following:

K: a domain which will be a tetrahedron, a prism or a cube.
P: a vector space of polynomials on K of dimension N.
$\mathscr{A}$: a set of N linear functionals acting on P, the elements of $\mathscr{A}$ are called degrees of freedom.

A finite element is said to be *unisolvent* if the set of degrees of freedom determine a unique polynomial in the space P. In that case, we can associate to each *function p* on K an *interpolate πp* in P such that

$$\alpha(p)=\alpha(\pi p);\qquad \forall\alpha\in\mathscr{A}$$

We say that a finite element is conforming in some functional space V if, when Ω is the union of two elements K_1 and K_2 with a common face f, the function defined by $\pi_1 p$ on K_1 and $\pi_2 p$ on K_2, belongs to the functional space $V(\Omega)$.

It follows from Lemma 1 that a finite element is conforming in $H(\operatorname{div})$ if and only if, the nullity of the degrees of freedom depending only on the face f implies the nullity of the normal component of p on this face.

2.1. The Case of Tetrahedra

In the previous work [6], we have introduced some spaces of polynomials and some associated finite elements on tetrahedra.

We introduce the spaces $\mathscr{D}^k$ and $\mathscr{R}^k$

$$\mathscr{D}^k=(P_{k-1})^3\oplus\tilde{P}_{k-1}\cdot r \tag{2.1}$$

where $\tilde{P}_k$ is the space of homogeneous polynomials of degree k and r the vector: $r=(x_1,x_2,x_3)$.

$$\mathscr{R}^k=(P_{k-1})^3\oplus S^k \tag{2.2}$$

$$S^k=\{p\in(\tilde{P}_k)^3;\ (r\cdot p)\equiv 0\} \tag{2.3}$$

The dimension of these spaces are respectively:

$$\dim(\mathscr{D}^k)=\frac{(k+3)(k+1)k}{2} \tag{2.4}$$

$$\dim(\mathscr{R}^k)=\frac{(k+3)(k+2)k}{2} \tag{2.5}$$

$$\dim(\mathscr{S}^k)=k(k+2) \tag{2.6}$$

We introduce now the new finite element.

Definition 1. 1°) K is a tetrahedron

2°) $P=(P_k)^3$

3°) $\mathscr{A}$ is the set of following moments:

$$\int_f (p\cdot n)q\,d\gamma;\qquad \forall q\in P_k(f);\qquad n \text{ normal to the face } f. \tag{2.7}$$

$$\int_K (p\cdot q)\,dx;\qquad \forall q\in\mathscr{R}_{k-1}. \tag{2.8}$$

For this element, the divergence belongs to the space P_{k-1}. We will denote by π^* the orthogonal projection operator in the $L^2(K)$ norm on P_{k-1}.

We have:

Theorem 1. *The finite element given by Definition* 1 *is unisolvent and conforming in $H(\mathrm{div})$. Moreover the interpolation operator π associated is such that*

$$\operatorname{div}\pi p=\pi^*\operatorname{div}p;\qquad \forall p\in H(\mathrm{div}). \tag{2.9}$$

Proof. We first have to check that the total number of degrees of freedom is the dimension of the space P. The dimension of P is

$$\dim(P)=\frac{(k+1)(k+2)(k+3)}{2} \tag{2.10}$$

The number of degrees of freedom of type (2.7) is $4\dim P_k$ that is $2(k+1)(k+2)$. The number of degrees of freedom of type (2.8) is the dimension of $\mathscr{R}_{k-1}$ that is $\frac{(k-1)(k+1)(k+2)}{2}$. To prove the unisolvence now, it suffices to show that p is zero when all the degrees of freedom are zero.

The degrees of type (2.7) imply

$$p\cdot n=0, \quad \text{on each face } f. \tag{2.11}$$

This property is exactly equivalent to the conforming property. Using the Green formula

$$\int_K \operatorname{div} p\,\varphi\,dx = -\int_K (p\cdot\operatorname{grad}\varphi)\,dx + \int_{\partial K}(p\cdot n)\varphi\,d\gamma \tag{2.12}$$

we conclude that

$$\int_K \operatorname{div} p\,\varphi\,dx=0; \quad \forall\varphi\in P_{k-1};$$

which implies

$$\operatorname{div} p\equiv 0. \tag{2.13}$$

Using (2.11) and (2.13) and the formula (2.12), we obtain

$$\int_K (p\cdot\operatorname{grad}\varphi)\,dx=0; \quad \forall\varphi\in H^1(K). \tag{2.14}$$

In the reference [6], we proved the following lemma.

Lemma 2. *The space $(P_{k-1})^3$ is the direct sum of the space $\mathscr{R}_{k-1}$ and the space G_k of gradients of polynomials in $\tilde{P}_k$.*

Using (2.14), the Lemma 2, and the degrees of type (2.8) we obtain

$$\int_K (p\cdot q)\,dx=0; \quad \forall q\in(P_{k-1})^3;$$

The property of unisolvence of the finite element of Definition 5 in reference [6] and (2.11) associated to the last equality implies $p\equiv 0$.

It remains to prove (2.9). Using the Green formula (2.12) for p and πp yields:

$$\int_K \operatorname{div}(p-\pi p)\varphi\,dx = -\int_K (p-\pi p\cdot\operatorname{grad}\varphi)\,dx + \int_{\partial K}(p-\pi p)\cdot n\,\varphi\,d\gamma. \tag{2.15}$$

This relation is true for every φ in the space P_{k-1} and this property is exactly the identity (2.9). □

Proposition 1. *Let π be the interpolation operator associated to the finite element of Definition 1. Let h be the diameter of the tetrahedron K and ρ the diameter of the inscribed sphere, then we have:*

$$\|p-\pi p\|_{(L^2(K))^3}\leqq c h^{k+1}|p|_{(H^{k+1}(K))^3}; \tag{2.16}$$

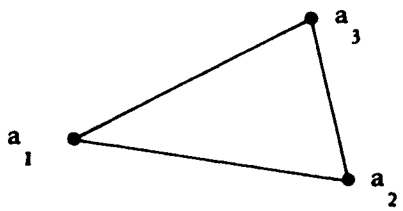
$$\|D(p-\pi p)\|_{(L^2(K))^9}\leqq c\,\frac{h^{k+1}}{\rho}\,|p|_{(H^{k+1}(K))^3} \quad \square \tag{2.17}$$

Proof. We use a reference finite element K and the transformation $x=F(\hat{x}) = B\hat{x}+b$ which is an isomorphism between $\hat{K}$ and K.

We replace the degrees of freedom of type (2.7) by the following

$$\frac{1}{\operatorname{mes}(f)} \int_f (p\cdot v) q\, d\gamma \tag{2.18}$$

with v defined as follow on each face:



$$v=\frac{1}{\det B}\overline{a_1 a_2}\wedge\overline{a_1 a_3}.$$

It can be proved that these degrees are independant of the transformation F if we change the vector p following the rule

$$p(x)=B\hat{p}(\hat{x}). \tag{2.19}$$

This proves that

$$\pi p=\hat{\pi}\hat{p} \tag{2.20}$$

and using this property, the above estimates are classical. □

Let Ω be a bounded polyhedral domain in R^3, and let $\mathscr{C}_h$ be a mesh of tetrahedrons covering Ω. Suppose that each of these tetrahedra is such that

$$\frac{h}{\rho}\leqq c. \tag{2.21}$$

where c is a constant independant of h.

We introduce the following space, which "approximates" the space $H(\operatorname{div})$:

$$V_h=\{u\in H(\operatorname{div}); u|_K\in(P_k)^3; \forall K\in\mathscr{C}_h\}. \tag{2.22}$$

This space is exactly associated to the finite element of Definition 1.

Let N_S, N_a, N_f, N_T be the respective numbers of vertex, edges, faces and tetrahedrons is the mesh. Then the dimension of the space V_h is

$$\dim(V_h)=\frac{(k+1)(k+2)}{2}N_f+\frac{(k-1)(k+1)(k+2)}{2}N_T \tag{2.23}$$

When $k=1$, this dimension is $3N_f$. The corresponding number associated to the finite element of reference [6] when $k=1$ is N_f.

The Proposition 1 shows that we have a better order of approximation, h^2, compared to $o(h)$.

Remark 1. It is possible, as usual, to associate to this new finite element, a corresponding curved finite element.

For simplicity, we describe the curved element when $k=2$.

The classical Lagrange finite element is defined by:

- P_2 is the space of polynomials
- the degrees of freedom are the value on the vertices and on the middles of the edges, this yields 10 degrees of freedom.

Let us be given 10 points in R^3. There exist an application $F(x)$ on a reference finite element such that the image of the vertex and middles of the edge are these 10 points. The image $F(\hat{K})$ is denoted by K and when the 10 points satisfy suitable conditions (see for instance Ciarlet and Raviart [4]) the application F is bijective from $\hat{K}$ onto K.

The curved finite element is now

- K is the geometric element
- $P(x)=DF(\hat{x})(P_2(\hat{x}))^3$, is the space of functions (which are not polynomials because $\hat{x}=F^{-1}(x)$ is not polynomial).
- The degrees of freedom are the image of the degrees of freedom;

For instance, we choose a basis in each face of the reference element, denoted by $\hat{e}_1, \hat{e}_2$. The degrees of type (2.7) are

$$\frac{1}{\operatorname{mes}(\hat{f})}\int_{\hat{f}}(\hat{p}\cdot\hat{e}_1\wedge\hat{e}_2)\hat{q}\,d\hat{\gamma}$$

whose image is

$$\frac{1}{\operatorname{mes}(f)}\int_f (DF(\hat{x})\hat{p}\cdot(DF\hat{e}_1\wedge DF\hat{e}_2))\frac{1}{\det(DF)}(\hat{q}\circ F^{-1})\,d\gamma. \tag{2.24}$$

The degrees of freedom of type (2.8) can be chosen of the form

$$\int_K (DF\hat{p}\cdot DF\hat{q})\,dx. \tag{2.25}$$

2.2 The Case of Cubes

We have introduced in the reference [6] some notations. $Q_{l,m,n}$ is the space of polynomials of three variables (x_1, x_2, x_3) the maximum degree of which are respectively l in x_1, m in x_2, n in x_3. Q_k is the space of polynomials $Q_{k,k,k}$.

The finite element is given by:

Definition 2. 1°) K is the cube C:

$$C=\{x\in\mathbb{R}^3; 0\leqq x_i\leqq 1,\ i=1,2,3\};$$

2°) $P=(Q_k)^3$;

3°) $\mathscr{A}$ is the set of following moments:

$$\int_f (p\cdot n)q\,d\gamma;\quad \forall q\in Q_k(f);\quad n \text{ normal to the face } f; \tag{2.26}$$

$$\int_K (p\cdot q)\,dx;\quad \forall q=(q_1,q_2,q_3);$$

$$q_1\in Q_{k-2,k,k};\quad q_2\in Q_{k,k-2,k};\quad q_3\in Q_{k,k,k-2}. \tag{2.27}$$

Theorem 2. *The finite element given by the Definition 2 is unisolvent and conforming in the space $H(\mathrm{div})$.*

Proof. We have to check first that the dimension of P is the total number of degrees of freedom.

The number of degrees of freedom of type (2.26) is six times the dimension of $Q_k(f)$ that is: $6(k+1)^2$.

The number of degrees of freedom of type (2.27) is: $3(k-1)(k+1)^2$.

The dimension of P is:

$$\dim(Q_k)^3 = 3(k+1)^3$$

and the result is a simple computation.

We only have to prove now that when all the degrees of freedom are zero, the vector p is zero. The degrees of type (2.26) imply

$$p \cdot n \equiv 0, \qquad \text{on each face } f. \tag{2.28}$$

This implies that for instance,

$$p_1 = x_1(1-x_1)q_1; \qquad q_1 \in Q_{k-2,k,k};$$

and using the degrees of freedom of type (2.27), this implies $p \equiv 0$.

To prove (2.28), we use the Green' formula (2.15) with a function φ in the space $(Q_k)^3$. □

As usual, we can use an affine transformation to associate to this reference finite element, a new one defined on a parallelotope.

Using a transformation whose coordinates are in Q_k, we associate a curved finite element. In order to do that, we take, as degrees of freedom, the images of the usual ones, using formulas similar to (2.24) and (2.25).

Remark 2. The property of Theorem 1 that $\pi^* \mathrm{div} = \mathrm{div}\, \pi$ is not true in the case of the finite element of Theorem 2. The divergence of a vector in $(Q_k)^3$ is contained in Q_k. The Green formula (2.15) implies that for any φ in the space Q_{k-1}:

$$\int_K \mathrm{div}(p - \pi p)\varphi\, dx = 0; \qquad \forall \varphi \in Q_{k-1}.$$

which is weaker that $\pi^* \mathrm{div} = \mathrm{div}\, \pi$.

2.3. *The Case of Prisms*

We introduce some notations:

$P_{l,m}$ denotes the space of polynomials of degrees l in the two variables x_1 and x_2, and of degree m in the variable x_3. Its dimension is

$$\dim(P_{l,m}) = (m+1)\frac{(l+1)(l+2)}{2}.$$

$\mathcal{R}_{l,m}$ (resp. $\mathcal{D}_{l,m}$) will denote the spaces of pairs of polynomials which are in $\mathcal{R}_l$ (resp. $\mathcal{D}_l$) for x_3 fixed, and are of degree m in x_3.

We can defined the new finite element:

Definition 3. 1°) K is a prism whose base is a triangle in the (x_1, x_2) plane, with three vertical edges parallel to the x_3 axis.

2°) The space P of polynomials is

$$P = (P_{k,k})^3$$

3°) The degrees of freedom are of four types:

$$\int_f (p \cdot n) q \, d\gamma, \quad \forall q \in P_k; \quad \text{for the two horizontal faces;} \tag{2.29}$$

$$\int_f (p \cdot n) q \, d\gamma; \quad \forall q \in Q_k; \quad \text{for the three vertical faces;} \tag{2.30}$$

$$\int_K p_3 q_3 \, dx; \quad \forall q_3 \in P_{k,k-2}; \tag{2.31}$$

$$\int_K (p_1 q_1 + p_2 q_2) \, dx; \quad \forall q = (q_1, q_2) \in \mathcal{R}_{k-1,k}. \tag{2.32}$$

Theorem 3. *The finite element given by Definition 3 is unisolvent and conforming in the space $H(div)$.*

Proof. We first prove the equality of the dimension of P and of the total number of degrees of freedom. We have:

$$\dim P = \tfrac{3}{2}(k+1)^2(k+2).$$

The number of degrees of type (2.29) is twice the dimension of P_k:

$$2 \times \frac{(k+1)(k+2)}{2}.$$

The number of degrees of freedom of type (2.30) is three time the dimension of Q_k: $3(k+1)^2$. The number of degrees of freedom of type (2.31) is: $\frac{(k+2)(k+1)(k-1)}{2}$. The dimension of $\mathcal{R}_{k-1,k}$ is the dimension of $\mathcal{R}_{k-1}$ in $2D$ multiply by the dimension of P_k in $1D$ that is: $(k+1)^2(k-1)$.

The result on equality is just an easy verification.

To prove anisolvence, it is sufficient to prove that, when all the degrees of freedom are zero, the vector p is also zero.

We use first the degrees of type (2.29) which implies:

$$p \cdot n = 0 \quad \text{on each horizontal face.}$$

This lead to

$$p_3 = x_3(1 - x_3) q_3, \quad q_3 \in P_{k,k-2}. \tag{2.33}$$

Now using the degrees of type (2.31), we obtain

$$P_3 \equiv 0, \tag{2.34}$$

and using the degrees of type (2.20), we have:

$$p \cdot n = 0 \quad \text{on each face of the prism.} \tag{2.35}$$

The formula of Stokes (2.11) give us

$$\int_K \operatorname{div} p \varphi \, dx = -\int_K (p \cdot \operatorname{grad} \varphi) \, dx + \int_{\partial K} (p \cdot n) \varphi \, d\gamma \tag{2.36}$$

which is zero for φ in the space $P_{k-1,k}$ because then the first two components of $\operatorname{grad}\varphi$ are in $\mathscr{R}_{k-1,k}$.

Thus, as $\operatorname{div} p$ is also in this space, we have proved

$$\operatorname{div} p = 0. \tag{2.37}$$

Using again (2.36), we have:

$$\int_K (p \cdot \operatorname{grad} \varphi) \, dx = 0; \quad \forall \varphi \in H^1(K). \tag{2.38}$$

We know that in $2D$, $(P_{k-1})^2$ is the direct sum of $\mathscr{R}_{k-1}$ and the $\operatorname{grad}(P_k)$. As a consequence, we obtain that $(P_{k-1,k})^2$ is the direct sum of $\mathscr{R}_{k-1,k}$ and the space $\operatorname{grad}(P_{k,k})$, for the first two components.

Using this property, (2.38) and the degrees of type (2.32) we obtain

$$\int_K (p \cdot q) \, dx = 0; \quad \forall q = (q_1, q_2); \quad q_1, q_2 \in P_{k-1,k}. \tag{2.39}$$

Now (2.35) and (2.39) constitute the degrees of freedom of the $H(\operatorname{div})$ finite element in $2D$ from reference [6] for the degree $k+1$ so that they imply $p \equiv 0$. $\square$

We can prove, for this finite element, some error estimates similar to those contained in Proposition 1. We can also associate a curved finite element. This is particularly usefull in the case $k=1$ using distorded prisms.

Remark 3. Using the Green' formula (2.36), we obtain:

$$\int_K \operatorname{div}(p - \pi p) \cdot \varphi \, dx = 0; \quad \forall \varphi \in P_{k-1,k}.$$

This identity is not sufficient to conclude that the divergence of πp is the projection of the divergence of p, because the divergence of πp belong to a bigger space than $P_{k-1,k}$.

In the reference [6], we omitted the description of the finite element in $H(\operatorname{div})$ corresponding to prisms. We repair here this omission.

Definition 4. We define a finite element by the following

1°) K is the prism of Definition 3

2°) The space P of polynomials is

$$P=\{p=(p_1,p_2,p_3);(p_1,p_2)\in\mathscr{D}_{k,k-1},p_3\in P_{k-1,k}\}$$

3°) The degrees of freedom are of four types:

$$\int_f (p\cdot n)q\,d\gamma;\quad \forall q\in P_{k-1},\quad \text{for the two horizontal faces;} \tag{2.40}$$

$$\int_f (p\cdot n)q\,d\gamma;\quad \forall q\in Q_{k-1},\quad \text{for the three vertical faces;} \tag{2.41}$$

$$\int_K (p_1q_1+p_2q_2)\,dx;\quad (q_1,q_2)\in(P_{k-2,k-1})^2; \tag{2.42}$$

$$\int_K p_3q_3\,dx;\quad q_3\in P_{k-1,k-2}. \tag{2.43}$$

Theorem 4. *The finite element given by Definition 4 is unisolvent and conforming in the space $H(\mathrm{div})$. Moreover, if π is the operator of interpolation associated and π^* the L^2 projector on $P_{k-1,k-1}$, we have:*

$$\pi^*\operatorname{div}v=\operatorname{div}\pi v;\quad \forall v\in(H^1(K))^3 \tag{2.44}$$

Proof. We first check that the number of degrees of freedom is exactly the dimension of the space P.

The dimension of P is: $k^2(k+2)+\frac{k(k+1)^2}{2}$.

The number of degrees of type (2.40) is: $k(k+1)$;
The number of degrees of type (2.41) is: $3k^2$;
The number of degrees of type (2.42) is: $(k-1)k^2$;
The number of degrees of type (2.43) is: $\frac{1}{2}(k-1)k(k+1)$.

The degrees of freedom on the faces are such that we know that when they are zero, we have:

$$p\cdot n=0. \tag{2.45}$$

This property proves the conformity in $H(\mathrm{div})$.

It follows from (2.45), that we have:

$$p_3=x_3(1-x_3)q_3,\quad q_3\in P_{k-1,k-2}.$$

Using the degrees of type (2.43) this implies

$$P_3\equiv 0. \tag{2.46}$$

We then use the formula (2.36) and the degrees of type (2.42) to find:

$$\int_K \operatorname{div}p\,\varphi\,dx=0;\quad \forall\varphi\in P_{k-1,k-1}.$$

But the space P is such that $\operatorname{div} p$ is in $P_{k-1,k-1}$, so that

$$\operatorname{div} p = 0. \tag{2.47}$$

Now, for each x_3 fixed, we have:

$$\left.\begin{aligned} p_1 &= \frac{\partial \varphi}{\partial x_2} \\ p_2 &= \frac{\partial \varphi}{\partial x_1} \end{aligned}\right\}; \quad \varphi \in P_{k,k-1} \tag{2.48}$$

and from (2.45) on the vertical faces, we have $\varphi = 0$; on the three vertical faces; so that

$$\varphi = \lambda_1 \lambda_2 \lambda_3 \psi, \qquad \psi \in P_{k-3,k-1}. \tag{2.49}$$

Using the degrees of type (2.42), we find:

$$\int_K \lambda_1 \lambda_2 \lambda_3 \psi \left(\frac{\partial q_2}{\partial x_1} - \frac{\partial q_1}{\partial x_1} \right) dx = 0 \tag{2.50}$$

which completes the proof.

It remain to prove the formula (2.44).

From the formula (2.36), we have:

$$\int_K \operatorname{div}(p - \pi p) \varphi \, dx = -\int_K ((p - \pi p) \cdot \operatorname{grad} \varphi) \, dx + \int_{\partial K} ((p - \pi p) \cdot n) \varphi \, d\gamma. \tag{2.51}$$

The degrees of freedom are such that the right handside is zero when φ is in the space $P_{k-1,k-1}$. This property is exactly the expression of (2.44).

3. Finite Element in H(curl)

The Hilbert space H(curl) is the space of vectors of $\mathbb{R}^3$ defined on the bounded domain Ω, which are in the space $(L^2(\Omega))^3$ and such the vorticity is also in $(L^2(\Omega))^3$. We have the

Lemma 3. *Let Ω be the union of two domains K_1 and K_2 with a common face f of normal n. A vector p in the space $(H^1(K_1))^3 \cup (H^1(K_2))^3$ is in H(curl) if and only if the trace $p \wedge n$ is the same on each side of the face f.*

3.1. The Case of Tetrahedra

We introduced in reference [6] a family of finite element in the space H(curl). We are going to describe here a new family that in some sense is complementary to the previous one.

Definition 5. We define a finite element by the following

1°) K is a tetrahedron

2°) $P=(P_k)^3$

3°) $\mathscr{A}$ is the following moments:

$$\int_a (p\cdot\tau)\varphi\,ds; \qquad \forall\varphi\in P_k(a), \tag{3.1}$$

where a is any of the six edges of the tetrahedron and τ is the unit tangent vector along the edge a;

$$\int_f (p\cdot q)\,d\gamma; \qquad \forall q\in\mathscr{D}_{k-1}(f) \tag{3.2}$$

and tangent to the face f for each of the four faces of the tetrahedron; $\mathscr{D}_{k-1}$ is defined by (2.1);

$$\int_K (p\cdot q)\,dx; \qquad \forall q\in\mathscr{D}_{k-2}. \tag{3.3}$$

Theorem 5. *The finite element given in Definition 5 is unisolvent and conforming in the space $H(\mathrm{curl})$.*

Proof. We verify first the equality between the number of degrees of freedom and the dimension of P.

$$\dim(P_k)^3=\frac{(k+1)(k+2)(k+3)}{2} \tag{3.4}$$

The number of degrees of type (3.1) is six times the dimension of P_k that is:

$$6(k+1).$$

The number of degrees of type (3.2) is four times the dimension of $\mathscr{D}_{k-1}(f)$ that is:

$$4(k+1)(k-1).$$

The number of degrees of freedom of type (3.3) is the dimension of $\mathscr{D}_{k-2}$ that is:

$$\frac{(k+1)(k-1)(k-2)}{2}.$$

We can now just check that the total number of degrees is $\dim P$.

To prove the conformity in the space $H(\mathrm{curl})$, we use the degrees of type (3.1) and (3.2) relative to one face.

The degrees of type (3.1) imply

$$(p\cdot\tau)=0 \qquad \text{on each edge.} \tag{3.5}$$

We have the Stokes' formula on this face (n is the normal to the face):

$$\int_f (p\wedge n\cdot\operatorname{grad}\varphi)\,dy+\int_f \varphi\,\mathrm{curl}_f\,p\,dy=\int_{\partial f}(p\cdot\tau)\varphi\,ds. \tag{3.6}$$

Using (3.5) and the degrees of type (3.2) we obtain:

$$\int_f \varphi \operatorname{curl}_f p \, d\gamma = 0; \quad \forall \varphi \in P_{k-1}(f). \tag{3.7}$$

This implies now:

$$\operatorname{curl}_f p = 0,$$

or

$$p = \operatorname{grad}_f \psi, \quad \text{for a function } \psi \text{ in the space } P_{k+1}(f). \tag{3.8}$$

The relation (3.5) show that ψ is constant along ∂f and so can be chosen to be zero of ∂f. We can then write ψ as a product of the barycentric coordinates on ∂f by a polynomial of degree $k-2$:

$$\psi = \lambda_1 \lambda_2 \lambda_3 \varphi; \quad \varphi \in P_{k-2}(f). \tag{3.9}$$

We next use Green' formula on the face to obtain:

$$\int_f \lambda_1 \lambda_2 \lambda_3 \varphi \operatorname{div} q \, d\gamma = -\int_f (p \cdot q) \, d\gamma; \quad \forall q \in \mathscr{D}_{k-1}(f). \tag{3.10}$$

Finally we know that the image of $\mathscr{D}_{k-1}(f)$ by the operator divergence is exactly $P_{k-2}(f)$ so that (3.10) implies $\varphi = 0$ and

$$p \wedge n = 0; \quad \text{on } f. \tag{3.11}$$

This proves the conforming property.

To prove the unisolvence we suppose that all the degrees are zero. We already know that we have (3.11) on the faces. We use first Stokes' formula in the tetrahedron (using (3.11)):

$$\int_K (\operatorname{curl} p \cdot q) \, dx = \int_K (p \cdot \operatorname{curl} q) \, dx = 0; \quad \forall q \in (P_{k-2})^3. \tag{3.12}$$

This proves that:

$$p = \operatorname{grad} \psi; \quad \psi \in P_{k+1}.$$

Using (3.11), we see that

$$\psi = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \varphi; \quad \varphi \in P_{k-3}.$$

Using Green's formula we obtain

$$\int_K \psi \operatorname{div} q \, dx = -\int_K (\operatorname{grad} \varphi \cdot q) \, dx = 0; \quad \forall q \in \mathscr{D}_{k-2}.$$

Using next the property that $\operatorname{div}(\mathscr{D}_{k-2})$ is P_{k-3}, we obtain that φ is zero an the proof is complete. □

From Theorem 4, we know that there is an interpolation operator denoted by π, associated to the finite element of Definition 4.

We have some links between this finite element and the finite element of Definition 1 for the degree $k-1$. Let π^* be the interpolation operator associate to the latter element. We have

Proposition 2. *For every p in $(H^2(K))^3$, we have*

$$\pi^* \operatorname{curl} p = \operatorname{curl} \pi p. \tag{3.13}$$

Proof. First we remark that $\operatorname{curl} \pi p$ belong to $(P_{k-1})^3$. Thus it is suffisant to prove that the degrees of freedom of type (2.7) and (2.8) are the same (for $k-1$). Using the identity (3.6), we obtain, for every face of K

$$\int_f \operatorname{curl}_f(p-\pi p)\varphi \, d\gamma = \int_{\partial f} ((p-\pi p)\cdot\tau)\varphi \, ds - \int_f ((p-\pi p)\wedge n \cdot \operatorname{grad}\varphi) \, d\gamma \tag{3.14}$$

This identity yields

$$\int_f (\operatorname{curl}(p-\pi p)\cdot n)\varphi \, d\gamma = 0; \qquad \forall \varphi \in P_{k-1}(f).$$

Now, using the identity (3.12) we obtain,

$$\int_K (\operatorname{curl}(p-\pi p)\cdot q) \, dx = \int_K (p-\pi p \cdot \operatorname{curl} q) \, dx - \int_{\partial K} ((p-\pi p)\wedge n \cdot q) \, d\gamma \tag{3.15}$$

From this identity and the definition of the operator π, we have

$$\int_K (\operatorname{curl}(p-\pi p)\cdot q) \, d\gamma = 0; \qquad \forall \mathrm{q} \in (P_{k-2})^3.$$

Remark 4. We have proved a result apparently more precise, for we have obtained all the degrees of freedom of the finite element given by the Definition 5 of reference [6]. But in fact the vector $\operatorname{curl} p$ is divergence-free and it is easy to check that for such a vector these two finite elements are identical.

Proposition 3. *For every p in $(H^{k+1}(K))^3$;*

$$\|p-\pi p\|_{(L^2(K))^3} \leqq c h^{k+1} |p|_{(H^{k+1}(K))^3}; \tag{3.16}$$

$$\|D(p-\pi p)\|_{(L^2(K))^9} \leqq c \frac{h^{k+1}}{\rho} |p|_{(H^{k+1}(K))^3}. \quad \square \tag{3.17}$$

The proof is similar to the proof of Proposition 1 except the fact that it is necessary to use the transformation

$$p(x) = B^{*-1} p(x) \tag{3.18}$$

in order to obtain invariant degrees of freedom and the fundamental property: $\pi p = \hat{\pi}\hat{p}$.

Example. Let us examine the finite element in the case $k=1$. There exist in that case only degrees of freedom of type (3.1)

$$\int_a (p\cdot\tau)\varphi \, ds; \qquad \varphi \in P_1(a). \tag{3.19}$$

They can be replaced by two values of $(p\cdot\tau)$ taken at two points of the edge a.

If we compare this element with the corresponding one of reference [6] it can be seen that the number of unknowns for a given mesh will be multiplied by two, when going from the old finite element to the new one. If we consider the error estimate, we have an error $o(h)$ compare to an error $o(h^2)$ for the vector. However the error in the curl is the same for the two elements, and we have seen above that the curl is in fact the same for the two elements.

3.2. The Case of Cubes

We define the corresponding finite element by

Definition 6

1°) K is the cube C

2°) $P=(Q_k)^3$

3°) The set $\mathscr{A}$ of degrees of freedom are the following moments

$$\int_a (p\cdot\tau)\varphi\,ds;\qquad \forall\varphi\in P_k(a), \tag{3.20}$$

where a is each of the twelve edges.

$$\int_f (p\cdot q)\,d\gamma; \tag{3.21}$$

for every vector $q\in Q_{k,k-2}\times Q_{k-2,k}$ tangent to the face f for each of the six faces.

$$\int_K (p\cdot q)\,dx;\qquad \forall q=(q_1,q_2,q_3); \tag{3.22}$$

$$q_1\in Q_{k,k-2,k-2},\qquad q_2\in Q_{k-2,k,k-2},\qquad q_3\in Q_{k-2,k-2,k}\quad\square$$

We have the

Theorem 6. *The finite element of Definition 6 is unisolvent and conforming in* H(curl).

Proof. The total number of degrees of freedom is:

type (3.20): $12(k+1)$,
type (3.21): $12(k+1)(k-1)$,
type (3.22): $3(k+1)(k-1)^2$,

the sum is: $3(k+1)^3$ which is the dimension of P.

Suppose now that the degrees of type (3.20) and (3.21) relative to one face are zero. The degrees of type (3.20) imply:

$$p\cdot\tau=0, \tag{3.23}$$

and using this property, we obtain (the face is supposed to be $x_3=0$)

$$\begin{aligned} p_1&=x_2(1-x_2)\varphi_1;\\ p_2&=x_1(1-x_1)\varphi_2. \end{aligned} \tag{3.24}$$

Then the degrees of type (3.21) implies now that φ_1 and φ_2 are zero. This proves the conformity of the element. We now have

$$p \wedge n = 0 \qquad \text{for each face.} \tag{3.25}$$

From these identifies, it result that

$$\begin{aligned} p_1 &= x_2(1-x_2)x_3(1-x_3)\varphi_1 \\ p_2 &= x_1(1-x_1)x_3(1-x_3)\varphi_2 \\ p_3 &= x_1(1-x_1)x_2(1-x_2)\varphi_3 \end{aligned} \tag{3.26}$$

Using finally the degrees of type (3.22), we conclude the proof. □

3.3. *The Case of Prisms*

We define the corresponding finite element by

Definition 7

1°) K is the prism of definition 3.

2°) The space P of polynomials is

$$P = (P_{k,k})^3$$

3°) The degrees of freedom are of five types:

$$\int_a (p \cdot \tau) q \, ds; \qquad \forall q \in P_k(a); \tag{3.27}$$

τ tangent to the edge a for each of the nine edges.

$$\int_f (p \cdot q) \, d\gamma; \tag{3.28}$$

$\forall q$ tangent to the face f and:

1°) $q \in \mathscr{D}_{k-1}(f)$ for the two horizontal faces

2°) $q \in Q_{k,k-2} \times Q_{k-2,k}$ for the three vertical faces.

$$\int_K p_3 q_3 \, dx; \qquad \forall q_3 \in P_{k-3,k} \tag{3.29}$$

$$\int_K (p_1 q_1 + p_2 q_2) \, dx; \qquad q = (q_1, q_2), \quad q \in \mathscr{D}_{k-1,k-2} \tag{3.20}$$

which are the polynomials of degrees $k-2$ in x_3 which restriction for x_3 fixed are in $\mathscr{D}_{k-1}$.

We have the corresponding theorem

Theorem 7. *The finite element of Definition 7 is conforming in* $H(\text{curl})$ *and unisolvent.*

Proof. Each face of this finite element is either a triangle or a rectangle. The degrees on the triangular faces are exactly those of the corresponding tetrahedron element and the degrees on the rectangular faces are exactly those of the corresponding cubic element. It result from Theorem 4 and 5 that the nullity of these degrees implies

$$p \wedge n = 0, \qquad \text{on these faces.} \tag{3.31}$$

This argument show that this finite element is conforming in $H(\text{curl})$.

To prove the unisolvence, we first check that the dimension of P which is $\frac{3}{2}(k+1)^2(k+2)$ is the total number of degrees of freedom.

type (3.27): $9(k+1)$;

type (3.28) 1°): $2(k+1)(k-1)$;

type (3.28) 2°): $6(k+1)(k-1)$;

type (3.29): $(k+1)\dfrac{(k-2)(k-1)}{2}$;

type (3.30): $(k+1)(k-1)^2$.

It is easy to check that the sum is $\dim(P)$.

To prove that p is zero when all the degrees are zero we already know that we have (3.31) for all the faces. This implies

$$\begin{aligned} p_1 &= x_3(1-x_3)\varphi_1;\\ p_2 &= x_3(1-x_1)\varphi_2; \end{aligned} \tag{3.32}$$

$$p_3 = \lambda_1\lambda_2\lambda_3\varphi_3. \tag{3.33}$$

Using the relation (3.33) and the degrees of type (3.29), we obtain:

$$p_3 = 0. \tag{3.34}$$

Using Stokes' formula, we have, for every horizontal triangle T_{x_3}

$$\int_{T_{x_3}} \left(\frac{\partial p_2}{\partial x_1} - \frac{\partial p_1}{\partial x_2}\right) q_3\, d\gamma = \int_{T_{x_3}} \left(p_1 \frac{\partial q_3}{\partial x_2} - p_2 \frac{\partial q_3}{\partial x_1}\right) d\gamma \tag{3.35}$$

We choose q_3 in the space $P_{k-1,k-2}$ and using the degrees of type (3.30) we obtain (taking the integral in x_3)

$$\int_K \left(\frac{\partial p_2}{\partial x_1} - \frac{\partial p_1}{\partial x_2}\right) q_3\, dx = 0; \qquad \forall q_3 \in P_{k-1,k-2}. \tag{3.36}$$

Using now (3.32) and (3.36), we have

$$\frac{\partial p_2}{\partial x_1} - \frac{\partial p_1}{\partial x_2} = 0. \tag{3.37}$$

This equality implies

$$\begin{aligned} p_1 &= \frac{\partial \varphi}{\partial x_1} \\ p_2 &= \frac{\partial \varphi}{\partial x_2} \end{aligned}; \qquad \varphi \in P_{k+1,k}. \tag{3.38}$$

Using (3.31), we can choose φ so that

$$\varphi = \psi(x_3); \qquad \text{on the three vertical faces;} \tag{3.39}$$

But φ is defined up an additive function of the variable x_3 so that we can choose

$$\varphi|_{\partial K} = 0. \tag{3.40}$$

This implies

$$\varphi = \lambda_1 \lambda_2 \lambda_3 x_3 (1 - x_3) \psi; \qquad \psi \in P_{k-2,k-2}. \tag{3.41}$$

Now using Green' formula, we have

$$\int_K \varphi \operatorname{div} q \, dx = -\int_K (p_1 q_1 + p_2 q_2) dx; \qquad \forall q \in \mathscr{D}_{k-1,k-2} \tag{3.42}$$

Using the $2D$ result that $\operatorname{div}(\mathscr{D}_{k-1})$ is P_{k-2}, we can conclude that $\psi = 0$ and that completes the proof. □

One might think that a property similar to that of Proposition 2 is true for this finite element and that of Definition 3. This is not the case and similarly it is also not true for the finite elements of Definition 2 and Definition 6.

The finite element of Definition 7 is in the spirit of those of Definition 5 and 6. We introduce here a different one that belongs to the family of the reference [6].

Definition 8

1°) K is the prism of Definition 3

2°) The space P of polynomials is

$$P = \{p_1, p_2, p_3; (p_1, p_2) \in \mathscr{R}_{k,k}, p_3 \in P_{k,k-1}\}$$

$\mathscr{R}_{k,l}$ are the polynomials of degree l in x_3 which are in $\mathscr{R}_k$ for x_3 fixed.

3°) The degrees of freedom are of five types:

$$\int_a (p \cdot \tau) q \, ds; \qquad \forall q \in P_{k-1}(a); \tag{3.43}$$

τ tangent to the edge a for each of the nine edges.

$$\int_f (p \cdot q) d\gamma; \tag{3.44}$$

$\forall q$ tangent to the face f and

1°) $q \in (P_{k-2})^2$, for the two horizontal faces.

2°) $q \in Q_{k-1,k-2} \times Q_{k-2,k-1}$, for the three vertical faces.

$$\int_K p_3 q_3 \, dx; \quad \forall q_3 \in P_{k-3,k-1}. \tag{3.45}$$

$$\int_K (p_1 q_1 + p_2 q_2) \, dx; \quad \forall (q_1, q_2) \in (P_{k-2,k-2})^2 \tag{3.46}$$

We have the

Theorem 8. *The finite element of Definition 8 is conforming in* $H(\text{curl})$ *and unisolvent.*

Proof. We first remark that the degrees of freedom on one horizontal face are exactly those of the finite element of Definition 4 reference [6]. The degrees on the vertical faces are those of the finite element of Definition 6 reference [6].

It result from these facts that when these degress are zero, the tangential components of p on these faces are zero and that yields the conforming property.

To prove the unisolvence, we first check that the dimension of the space P which is $\frac{3}{2}k(k+1)(k+2)$ is the total number of degrees of freedom.

type (3.43): $9k$

type (3.44) 1°): $2(k-1)k$

type (3.44) 2°): $6(k-1)k$

type (3.45): $\frac{1}{2}(k-2)(k-1)k$

type (3.46): $(k-1)^2 k$

It is easy to check that the sum is $\dim(P)$.

It remain to prove that p is zero when the degrees are zero. We already know that $p \wedge n$ is zero on each face. This implies

$$\begin{aligned} p_1 &= x_3(1-x_3)\varphi_1; \\ p_2 &= x_3(1-x_3)\varphi_2; \end{aligned} \quad (\varphi_1, \varphi_2) \in \mathscr{R}_{k,k-2}; \tag{3.47}$$

$$p_3 = \lambda_1 \lambda_2 \lambda_3 \varphi_3, \quad \varphi_3 \in P_{k-3,k-1}. \tag{3.48}$$

Using the degrees (3.45), we obtain $p_3 = 0$.

From the identity (3.35), we obtain, using the degrees of type (3.46):

$$\int_K \left(\frac{\partial p_2}{\partial x_1} - \frac{\partial p_1}{\partial x_2} \right) q_3 \, dx = 0; \quad \forall q_3 \in P_{k-1,k-2}. \tag{3.49}$$

Using (3.47), we then have

$$\frac{\partial p_2}{\partial x_1} - \frac{\partial p_1}{\partial x_2} = 0. \tag{3.50}$$

The Poincaré theorem and the properties of $\mathscr{R}_k$ lead to:

$$\begin{aligned} p_1 &= \frac{\partial \varphi}{\partial x_1}; \\ p_2 &= \frac{\partial \varphi}{\partial x_2} \end{aligned} \quad \varphi \in P_{k,k}. \tag{3.51}$$

We can choose φ such that $\varphi|_{\partial K}=0$, so that

$$\varphi=\lambda_1\lambda_2\lambda_3 x_3(1-x_3)\psi; \qquad \psi\in P_{k-3,k-2} \tag{3.52}$$

Using the formula (3.42), we conclude that $\psi=0$. □

Proposition 4. *Let π be the operator of interpolation associated to the finite element of Definition 8 and let π^* be the operator of interpolation associated to the finite element of Definition 4; we have:*

$$\operatorname{curl}\pi p=\pi^*\operatorname{curl}p; \qquad \forall p\in(H^2(K))^3. \tag{3.53}$$

Proof. For every face of the prism K, we use the identity (3.14). This identity yields for the two horizontal faces

$$\int_f (\operatorname{curl}(p-\pi p)\cdot n)\varphi\,d\gamma=0; \qquad \forall\varphi\in P_{k-1}(f); \tag{3.54}$$

and for the three vertical faces

$$\int_f (\operatorname{curl}(p-\pi p)\cdot n)\varphi\,d\gamma=0; \qquad \forall\varphi\in Q_{k-1}(f). \tag{3.55}$$

Furthermore, we can use the identity (3.15)

$$\int_K (\operatorname{curl}(p-\pi p)\cdot q)\,dx=\int_K((p-\pi p)\cdot\operatorname{curl}q)\,dx-\int_{\partial K}((p-\pi p)\wedge n\cdot q)\,d\gamma;$$

to obtain

$$\int_K (\operatorname{curl}(p-\pi p)\cdot q)\,dx=0; \tag{3.56}$$

for every $q=(q_1,q_2,q_3)$, $(q_1,q_2)\in(P_{k-2,k-1})^2$, $q_3\in P_{k-1,k-2}$.
This last identity finishes the proof of the proposition.

4. Application to the Equation of Stokes

In the reference [7], we have introduce a variational formulation of the Stokes' equation using a stream vector potential.

We start from the Stokes equation in a bounded domain Ω of $\mathbb{R}^3$.

$$\begin{aligned} -\nu\Delta u+\operatorname{grad}p&=f, && \text{in } \Omega,\\ \operatorname{div}u&=0, && \text{in } \Omega,\\ u|_\Gamma&=0; && \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

We introduce

$$w=\operatorname{curl}u \tag{4.1}$$

and the potential ψ, solution of

$$\begin{aligned} -\Delta\phi&=\operatorname{curl}u, \quad \text{in } \Omega,\\ \operatorname{div}\phi|_{\partial\Omega}&=0,\\ \phi\wedge n|_{\partial\Omega}&=0. \end{aligned} \tag{4.2}$$

Then the system (4.2) has a anique solution when the domain Ω is connected and simply connected, and we have

$$\begin{aligned} u &= \operatorname{curl} \phi, && \text{in } \Omega; \\ \operatorname{div} \phi &= 0, && \text{in } \Omega; \\ \phi \wedge n &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{4.3}$$

We introduce the following Hilbert spaces:

$$H(\operatorname{div}^0) = \{v \in (L^2(\Omega))^3; \operatorname{div} v = 0 \text{ in } \Omega; v \cdot n|_{\partial\Omega} = 0\};$$

$$K = \{\psi \in H(\operatorname{curl}); \operatorname{div} \psi = 0 \text{ in } \Omega; \psi \wedge n|_{\partial\Omega} = 0\}.$$

Then a variational formulation of the equation linking ϕ and ω, given by (4.1) and (4.3), is:

$$\begin{aligned} &\nu \int_\Omega (\operatorname{curl} \omega \cdot \operatorname{curl} \psi)\, dx = \int_\Omega (f \cdot \operatorname{curl} \psi)\, dx; \quad \forall \psi \in \mathscr{K}; \\ &\int_\Omega (\omega \cdot \pi)\, dx - \int_\Omega (\operatorname{curl} \phi \cdot \operatorname{curl} \pi)\, dx = 0; \quad \forall \pi \in H(\operatorname{curl}). \end{aligned} \tag{4.4}$$

We will also use the equivalent variational formulation:

Find $u \in H(div^0)$, $\omega \in H(\operatorname{curl})$ such that

$$\begin{aligned} &\nu \int_\Omega (\operatorname{curl} \omega \cdot v)\, dx = \int_\Omega (f \cdot v)\, dx; \quad \forall v \in H(\operatorname{div}^0); \\ &\int_\Omega (\omega \cdot \pi)\, dx - \int_\Omega (u \cdot \operatorname{curl} \pi)\, dx = 0; \quad \forall \pi \in H(\operatorname{curl}). \end{aligned} \tag{4.5}$$

In order to approximate the solution of the Equation (4.4) using the finite element introduced in this work, we suppose that we have a mesh $\mathscr{C}_h$ covering the domain Ω. We will suppose here that we are using tetrahedrons.

We introduce some finite element spaces:

$$W_h = \{w_h \in H(\operatorname{curl}); w_h|_K \in (P_k)^3, \forall K \in \mathscr{C}_h\};$$

$$W_h^0 = \{w_h \in W_h; w_h \wedge n|_{\partial\Omega} = 0\};$$

$$V_h = \{v_h \in H(\operatorname{div}); v_h|_K \in (P_{k-1})^3; \forall K \in \mathscr{C}_h\};$$

$$U_h = V_h \cap H(div^0);$$

$$U_h^* = \{v_h \in V_h; \operatorname{div} v_h = 0 \text{ in } \Omega\}.$$

Now the approximation of the equation (4.5) is

$$\begin{aligned} &\nu \int_\Omega (\operatorname{curl} \omega_h \cdot v_h)\, dx = \int_\Omega (f \cdot v_h)\, dx; \quad \forall v_h \in U_h; \\ &\int_\Omega (\omega_h \cdot \pi_h)\, dx - \int_\Omega (u_h \cdot \operatorname{curl} \pi_h)\, dx = 0; \quad \forall \pi_h \in W_h. \end{aligned} \tag{4.6}$$

In order to go from this approximation to an equivalent one associated to the variational formulation (4.4), we solve in the discrete spaces an equation similar to (4.3).

We introduce some spaces and some notations:

$$\Theta_e = \{\theta_e \in H^1(\Omega); \theta_e|_K \in P_{k+1}; \forall K \in \mathscr{C}_e\}; \qquad \Theta_e^0 = \Theta_e \cap H_0^1(\Omega).$$

N_s is the number of vertex of the triangulation $\mathscr{C}_h$; N_a is the number of edges of the triangulation $\mathscr{C}_h$; N_f is the number of faces of the triangulation $\mathscr{C}_h$; N_T is the number of tetrahedrons of the triangulation $\mathscr{C}_h$.

A triangulation is said to be regular if we have:

$$\frac{h}{\rho} \leqq C \quad \text{for every } K \quad \text{in } \mathscr{C}_h,$$

and if the diameters h are uniformly equivalent.

We have the

Theorem 7. *When the triangulation is regular, for every element v_h in the space U_h^* (respectively U_h), there exist a unique element w_h in the space W_h (respectively W_h^0) such that:*

$$\operatorname{curl} w_h = v_h; \tag{4.7}$$

$$\int_\Omega (w_h \cdot \operatorname{grad} \theta_h) dx = 0; \qquad \forall \theta_h \in \Theta_h (\textit{respectively } \Theta_h^0). \tag{4.8}$$

Moreover, we have

$$\|w_h\|_{H(\mathrm{curl})} \leqq C \|v_h\|_{(L^2(\Omega))^3}. \tag{4.9}$$

Proof. We first prove the uniqueness. We just have to show that the system of equations

$$\begin{aligned} &\operatorname{curl} w_h = 0; \qquad w_h \in W_h (\text{resp. } W_h^0); \\ &\int_\Omega (w_h \cdot \operatorname{grad} \theta_h) dx = 0; \qquad \forall \theta_h \in \Theta_h (\text{resp. } \Theta_h^0); \end{aligned} \tag{4.10}$$

has zero for unique solution.

The Stokes' theorem yields

$$w_h = \operatorname{grad} \alpha_h \tag{4.11}$$

and locally on each tetrahedron, we have:

$$\alpha_h|_K \in P_{k+1}. \tag{4.12}$$

This property implies that α_h is in the space Θ_h(resp. Θ_h^0). So that we can choose θ_h as α_h in (4.10) which yields $w_h = 0$.

We have proven that the operator curl is injective from the subspace of vectors in W_h(resp. W_h^0) satisfying (4.8) into the space U_h^*(resp. U_h).

In order to prove the surjectivity, it is sufficient to prove that these two spaces are of same dimension. The dimension of W_h is

$$\dim W_h = (k+1)N_a + (k+1)(k-1)N_f + \frac{(k+1)(k-1)(k-2)}{2} N_T. \tag{4.13}$$

The dimension of Θ_h is:

$$\dim \Theta_h = N_s + kN_a + \frac{k(k-1)}{2} N_f + \frac{k(k-1)(k-2)}{6} N_T. \tag{4.14}$$

The dimension of the subspace of W_h is

$$\begin{aligned} M_k &= \dim W_h - \dim \Theta_h + 1 \\ &= -N_s + N_a + \frac{(k-1)(k+2)}{2} N_f + \frac{(k-1)(k-2)(2k+2)}{6} N_T + 1 \end{aligned}$$

The dimension of U_e^* is the dimension of V_e minus the number of conditions imposed by the zero divergence. This dimension has been already computed in the reference [7], and we have:

$$\dim U_h^* = \frac{k(k+1)}{2} N_f + \frac{k(k+1)(2k-5)}{6} N_T. \tag{4.15}$$

We conclude this part of the proof using the Euler-Poincaré identity

$$N_s - N_a + N_f - N_T = 1 \tag{4.16}$$

In the case of the spaces W_h^0 and U_h, we must modify the above numbers. n_s is the number of vertex of $\mathscr{C}_n$ on $\partial\Omega$; n_a is the number of edges of $\mathscr{C}_h$ on $\partial\Omega$; n_f is the number of faces of $\mathscr{C}_h$ on $\partial\Omega$.

We have

$$\dim W_h^0 = \dim W_h - (k+1)n_a - (k+1)(k-1)n_f; \tag{4.17}$$

$$\dim \Theta_h^0 = \dim \Theta_h - n_s - kn_a - \frac{k(k-1)}{2} n_f. \tag{4.18}$$

Now the dimension of the subspace is M_k^0

$$M_k^0 = \dim W_h^0 - \dim \Theta_h^0 = M_k + n_s - n_a - \frac{(k-1)(k+2)}{2} n_f - 1 \tag{4.19}$$

The zero divergence conditions are linked by

$$\int_{\partial\Omega} (u_h \cdot n)\, d\gamma = 0, \tag{4.20}$$

and we have

$$\dim U_h = \dim U_h^* = \frac{k(k+1)}{2} n_f + 1. \tag{4.21}$$

So that we obtain

$$M_k^0 - \dim U_h = M_k - \dim U_h^* + n_s - n_a + n_f - 2. \tag{4.22}$$

We conclude this part of the proof using the Euler-Poincaré identity on the closed surface $\partial\Omega$:

$$n_s - n_a + n_f = 2. \tag{4.23}$$

The proof of inequality (4.9) is very similar to the same proof in reference [7] and will not be given here. □

From the Theorem 7, we can derive some error estimates for the approximation of the Stokes' equation (4.5).

Proposition 3. *Let* u, w *be the solution of* (4.5) *and* (u_h, w_h) *the solution of* (4.6), *we have*

$$\|u-u_h\|_{(L^2(\Omega))^3}+\|w-w_h\|_{(L^2(\Omega))^3}\leqq ch^{k-1}. \tag{4.24}$$

Example. In the case $k=1$, the dimension of the space W_h is $2N_a$. The dimension of the space U_h^*(resp. U_h) is in fact the same. We have changed the space W_e(resp. W_e^0) and this multiplies by two the corresponding number of unknowns. The error estimate appears to be unchanged. However the estimate (4.24) is not optimal in the variable w_e and we may think that a better estimate is valid for this variable.

The property of Theorem 7 doesn't seen to be true for the corresponding spaces associated to the finite elements of Definition 2 and Definition 6. The operator curl does not operate from Q_k into Q_{k-1}. In fact, this property seems to be related to the property of Proposition 2 which provide an alternative proof of Theorem 7. Let us sketch this proof.

Let v_h be given in U_h^*. We solve

$$\begin{aligned} \operatorname{curl} w &= v_h;\\ \operatorname{div} w &= 0;\\ w\wedge n|_{\Gamma} &= 0. \end{aligned} \tag{4.25}$$

The domain Ω being connex, connected and simply connected, there exist a unique solution of this equation in the space $(H^2(\Omega))^3$. Now, the interpolate $\pi_h w$ is such that (by the Proposition 2)

$$\operatorname{curl}(\pi_h w)=v_h. \tag{4.26}$$

We built the w_h of Theorem 7 by adding a gradient:

$$w=\pi_h w+\operatorname{grad}\alpha_h; \qquad \alpha_h\in\Theta_h$$

This gradient can be chosen such that we have (4.8).

This proof yields an indirect (and very complicated) proof of the Euler-Poincaré identity. This direct construction of w_h allows also a simpler proof of (4.9).

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