CM3 OPTIMISATION

Exemple : Méthode de Newton

$$f(x) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4 - \frac{1}{2}x_2^2$$

Gradient:

$$\nabla f(x) = \begin{pmatrix} x_1 \\ x_2^3 - x_2 \end{pmatrix}$$

Matrice Hessienne:

$$\nabla^2 f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 3x_2^2 - 1 \end{pmatrix}$$

$$(\nabla^2 f(x))^{-1} = \frac{1}{3x_2^2 - 1}\begin{pmatrix} 3x_2^2 - 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1er cas:

$$X_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to X_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Conclusion : c'est un point selle

2ème cas:

$$X_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Conclusion : on obitent un minimum local.

Exemple : Critère des moindres carrés

$$minf(x) = \frac{1}{2}[(x_i - a)^2 + (y_i - b)^2 - R^2]$$

$$r_i^2 = R^2 - (x_i^2 + a - 2x_i a) - (y_i^2 + b^2 - 2y_i b)$$

$$minf(x) = \frac{1}{2}(r_1^2 + r_2^2 + \ldots + r_m^2)$$

$$r_i = [2x_i \quad 2y_i \quad 1]\begin{bmatrix} a \\ b \\ c \end{bmatrix} - \left[x_i^2 + y_i^2\right] = ax_i + b_i$$

Finalement on obtient :

$$\begin{bmatrix} 2x_1 & 2y_1 & 1 \\ & & \\ 2x_m & 2y_m & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ R^2 - a^2 - b^2 \end{bmatrix} - \begin{bmatrix} x_1^2 + y_1^2 \\ \\ x_m^2 + y_m^2 \end{bmatrix}$$

Exemple : Gauss-Newton

$$f(x,y) = \frac{1}{2}[(x^2 - y)^2 + (1 - x)^2]$$

$$f(x) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4 - \frac{1}{2}x_2^2$$

Gradient:

$$\nabla f(x) = \begin{pmatrix} 2x^3 - 2xy + x - 1 \\ y - x^2 \end{pmatrix}$$

Matrice Hessienne:

$$\nabla^2 f(x) = \begin{pmatrix} 6x^2 - 2y + 1 & -2x \\ -2x & 1 \end{pmatrix}$$

$$\nabla r_1 = \begin{pmatrix} 2x \\ -1 \end{pmatrix};\ \nabla r_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\nabla^2 r_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix};\ \nabla^2 r_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Matrice Jacobienne:

$$J(x) = \begin{pmatrix} 2x & -1 \\ -1 & 0 \end{pmatrix}$$

d'où :

$$\nabla f(x) = \begin{pmatrix} 2x & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x^2 - y \\ 1 - x \end{pmatrix} = \begin{pmatrix} 2x^3 - 2xy + x - 1 \\ y - x^2 \end{pmatrix}$$

On approxime la matrice hessienne :

$$\nabla^2 f(x) = J.J^T = \begin{pmatrix} 4x^2 + 1 & -2x \\ -2x & 1 \end{pmatrix}$$