No. 190367N Date.

MA 2033 - Quiz 13.

$A \in \mathbb{C}^{n\times n}$ and $A_{n\times n} = U_{n\times n} \Sigma_{n\times n} V^H_{n\times n}$ where U, V are orthonomal & Σ is diagonal. consider

$$A^HA = (U\Sigma V^H)^H (U\Sigma V^H)$$
$$= V\Sigma^H U^H U \Sigma V^H \quad ; (U^HU = I, \text{orthonormal})$$
$$(A^HA)V = V\Sigma^H \Sigma V^H V$$
$$(A^HA)V = V(\Sigma^H\Sigma) \quad (V^HV = I, \text{ orthonormal})$$

This is an eigan value problem for $B = A^HA$. Since B is harmitian, $(B^H = B)$, eigan values are real valued and positive. Hence, their square roots, which are the singular values of A, are also real valued and chosen positive.

suppose $\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_n \end{pmatrix}_{n\times n}$ then $\Sigma^H\Sigma = \begin{pmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & & \sigma_n^2 \end{pmatrix}_{n\times n}$

where $\sigma_i \geq 0$ are the singular values of A

$\Rightarrow \rho(A^HA) = \max\{\sigma_i^2\}$

$\Rightarrow \sqrt{\rho(A^HA)} = \sqrt{\max\{\sigma_i^2\}} = \max\{\sigma_i\}$

$\Rightarrow \|A\|_2 = \max\{\sigma_i\}$ —— (1)

Given $\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$

let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_{n\times n} = (w_1 \; w_2 \; \cdots \; w_n)$ where each $w_i \in \mathbb{C}^{n\times 1}$

$$A^HA = \begin{pmatrix} w_1^H \\ w_2^H \\ \vdots \\ w_n^H \end{pmatrix} (w_1 \; w_2 \; \cdots \; w_n)$$

$$A^HA = \begin{pmatrix} w_1^Hw_1 & w_1^Hw_2 & \cdots & w_1^Hw_n \\ w_2^Hw_1 & w_2^Hw_2 & \cdots & w_2^Hw_n \\ \vdots & \vdots & & \vdots \\ w_n^Hw_1 & w_n^Hw_2 & \cdots & w_n^Hw_n \end{pmatrix}$$

$$\Rightarrow \text{trace}(A^HA) = w_1^Hw_1 + w_2^Hw_2 + \cdots + w_n^Hw_n$$
$$= \sum_{i=1}^n \bar{a}_{i1} a_{i2} + \sum_{i=1}^n \bar{a}_{i2} a_{i2} + \cdots + \sum_{i=1}^n \bar{a}_{in} a_{in}$$
$$= \sum_{i,j=1}^n \bar{a}_{ij} a_{ij} = \sum_{i,j=1}^n |a_{ij}|^2$$

$$\therefore \|A\|_F = \cancel{\sqrt{\sum_{i,j} |a_{ij}|^2}} = \sqrt{\sum_{i,j} |a_{ij}|^2} = \sqrt{\text{trace}(A^HA)}$$

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by theorem, $trace(A) = \sum_i \lambda_i$

$$\Rightarrow trace(A^H A) = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2$$

$$\Rightarrow trace(A^H A) = \sum_{i=1}^{n} \sigma_i^2$$

$$\Rightarrow \sqrt{trace(A^H A)} = \sqrt{\sum_i a_i^2}$$

$$\Rightarrow \|A\|_E = \sqrt{\sum_i \sigma_i^2} \text{ ——— } (2)$$

By $\rho(A) \leq \|A\|$ for any $\|\cdot\|$ theorem,

$$\rho(A^H A) \leq \|A^H A\|_E \leq \|A^H\|_E \|A\|_E \quad (\because \|AB\| \leq \|A\|\|B\|)$$

$$\Rightarrow \rho(A^H A) \leq \sqrt{\sum_{i,j} |\bar{a}_{ij}|^2} \sqrt{\sum_{i,j} |a_{ij}|^2}$$

Since $\forall z \in \mathbb{C}$,

$|z| = |\bar{z}|$, $\Rightarrow \rho(A^H A) \leq \sum_{i,j} |a_{ij}|^2$

$$\Rightarrow \sqrt{\rho(A^H A)} \leq \sqrt{\sum_{i,j} |a_{ij}|^2}$$

$$\Rightarrow \|A\|_2 \leq \|A\|_E \text{ ——— } (3)$$

by (1), (2), (3) we have,

$$\|A_2\| = \max\{\sigma_i\} \leq \|A\|_E = \sqrt{\sum_i \sigma_i^2}$$

where $\sigma_i \geq 0$ are the singular values of A //.