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Unit – 3 ⇝ Relation

Introduction

→ In Discrete Mathematics, a relation is a fundamental concept that describes how elements from one set are connected to elements of another (or the same) set.

→ In computer science, a relation is not just a mathematical concept — it's a powerful tool used to model connections, structures, and rules across a wide range of applications.

→ Relations are **mathematical models** for **real-world connections**. Whether you're organizing data, modeling networks, managing users, or designing algorithms, relations help you **structure interactions** logically and efficiently.

→ Relations in **computer science** are used for:

- Organizing and manipulating data (Databases).
- Modelling complex networks (Graphs).
- Managing permissions and access control.
- Defining state transitions and computational logic (Finite State Machines).
- Representing task dependencies (Scheduling).
- Analyzing languages and grammar structure.
- Solving pathfinding and algorithmic problems.

Method – 1 ⇝ Relation

Cartesian Product

→ Cartesian product of sets is set of ordered pair.

→ Cartesian product of sets A and B is denoted by $\mathbf{A \times B}$ which is read as "A cross B" and defined as follow:

$A \times B = \{ (\mathbf{a}, \mathbf{b}) : \mathbf{a} \in \mathbf{A} \text{ and } \mathbf{b} \in \mathbf{B} \}$

→ Cartesian product of sets B and A is denoted by $\mathbf{B \times A}$ which is read as "B cross A" and defined as follow:

$B \times A = \{ (\mathbf{b}, \mathbf{a}) : \mathbf{b} \in \mathbf{B} \text{ and } \mathbf{a} \in \mathbf{A} \}$

→ For example:

Let $A = \{ a, \ b \}$ and $B = \{ 1, \ 2 \}$.

- Cartesian product of set A and B is

$$A \times B = \{ a, \ b \} \times \{ 1, \ 2 \}$$

$$= \{ (a, 1), \ (a, 2), \ (b, 1), \ (b, 2) \}$$

- Similarly, cartesian product of set B and A is

$$B \times A = \{ 1, \ 2 \} \times \{ a, \ b \}$$

$$= \{ (1, a), \ (1, b), \ (2, a), \ (2, b) \}$$

→ Properties of Cartesian product

(1) If $A = \phi$ or $B = \phi$, then $A \times B = \phi$.

(2) If $|A| = m$ and $|B| = n$, then $|A \times B| = m \cdot n$.

(3) Generally, $A \times B \neq B \times A$.

(4) $A \times B = B \times A$ if and only if $A = B$.

Relation or Binary Relation

→ Let A and B be two non-empty sets.

→ **Subset** of Cartesian product $\mathbf{A \times B}$ is known as relation from A to B.

→ It is denoted by **R** and read as "relation R".

i.e.

$\mathbf{R = \{ (a,\ b) \mid a \in A \text{ and } b \in B \} \subseteq A \times B}$

→ Note that, if $|A| = p$ and $|B| = q$,

then total number of relations from A to B or B to A is $\mathbf{2^{pq}}$.

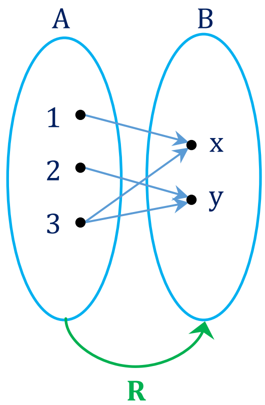
→ In any relation ordered pair of the form $(a,\ a)$ is known as **diagonal pair**.

→ For example:

- Let $A = \{1,\ 2,\ 3\}$ and $B = \{x,\ y\}$.

  $A \times B = \{(1, x),\ (1, y),\ (2, x),\ (2, y),\ (3, x),\ (3, y)\}$.

  Let, $R = \{(1, x),\ (2, y),\ (3, x),\ (3, y)\}$



This diagram is known as **Arrow Diagram** of relation R.

→ For any relation R,

- If $(a, b) \in R$, then it is denoted by $\mathbf{aRb}$ and read as "a is related to b".
- If $(a, b) \notin R$, then it is denoted by $\mathbf{a \not R b}$ and read as "a is **not** related to b".

→ For example:

- $R = \{(1, x),\ (2, y),\ (3, x),\ (3, y)\}$

  $(1,\ x) \in R$ so, it is denoted as $\mathbf{1Rx}$.

  $(1,\ y) \notin R$ so, it is denoted as $\mathbf{1 \not R y}$.

Relation on a Set:

→ Relation on a set A is a relation from A to A.

i.e., $R \subseteq A \times A$

→ For example:

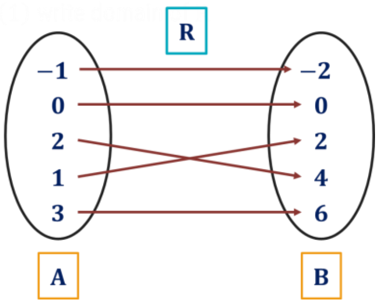
- Let $A = \{ 1, 2, 3 \}$

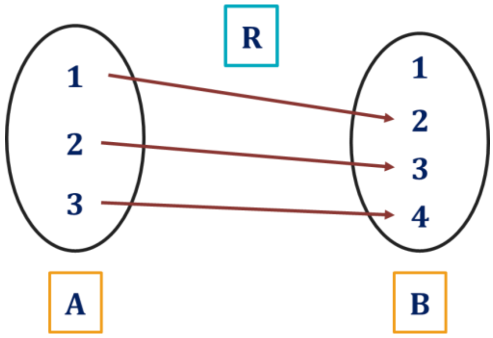
  So, relation R on a set A can be

  $R = \{ (1, 1), \ (1, 3), \ (2, 3), \ (3, 2), \ (3, 3) \}$

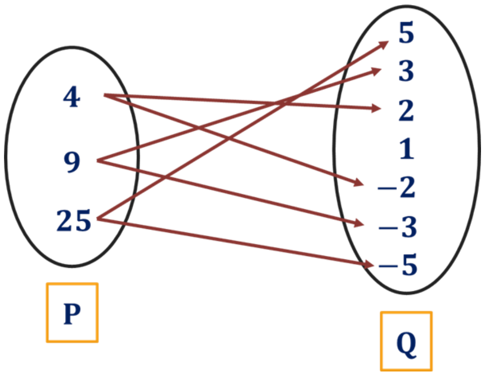
  Here, diagonal pairs are $(1, 1)$ and $(3, 3)$.

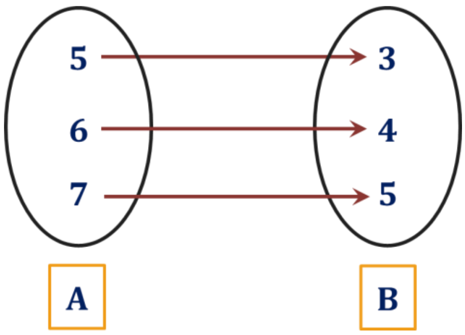
Examples of Method-1: Relation

H	1	Let $A = \{x,\ y,\ z\}$ & $B = \{1,\ 2\}$. Find the number of relations from A to B. **Answer: 64**
C	2	If $A = \{1,\ 2,\ 4\}$, $B = \{3,\ 5\}$, then which of the following are relations from A to B? Give reason in support of your answer. (1) $R_1 = \{(2,3),\ (4,\ 5),\ (1,\ 3)\}$ (2) $R_2 = \{(2,3),\ (3,\ 2),\ (1,\ 5),\ (4,\ 3)\}$ **Answer: R_1 is a relation as $R_1 \subseteq A \times B$, while R_2 is not a relation because $(3,2) \notin A \times B$**
C	3	If a relation $R = \{(0,\ 0),\ (2,\ 4),\ (-1,\ -2),\ (3,\ 6),\ (1,\ 2)\}$, then (1) write domain of R, (2) write range of R (3) write R in set – builder from (4) represent R by an arrow diagram **Answer: (1) Domain of $R = \{0,\ 2,\ -1,\ 3,\ 1\}$** **(2) Range of $R = \{0,\ 4,\ -2,\ 6,\ 2\}$** **(3) Set – builder form of $R = \{(x,\ y) : x \in \mathbb{Z}, -1 \le x \le 3,\ y = 2x\}$** **(4)** 

C	4	If $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$ and $R = \{(x, y) : (x, y) \in A \times B, y = x + 1\}$, then (1) find $A \times B$ (2) write R in roster from (3) write domain and range of R (4) represent R by an arrow diagram **Answer:** (1) $\left\{\begin{matrix}(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),\\(3,1),(3,2),(3,3),(3,4)\end{matrix}\right\}$ (2) $\{(1,2),(2,3),(3,4)\}$ (3) $D_R = \{1,2,3\}$ & $R_R = \{2,3,4\}$ (4) 
H	5	If $R = \{(x,y) : x^2 + y^2 = 100 ; x, y \in \mathbb{W}\}$, then find the domain and the range of R. **Answer:** $D_R = \{0,6,8,10\}$ & $R_R = \{0,6,8,10\}$
H	6	Write the following relations in the roster form: (i) $R_1 = \{(x, x^3) : x$ is a prime number less than 10 $\}$ (ii) $R_2 = \{(x-2, x^2) : x$ is a prime number less than 10 $\}$ **Answer:** $R_1 = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$ $R_2 = \{(0, 4), (1, 9), (3, 25), (5, 49)\}$
H	7	Let R be the relation on $\mathbb{N}$ defined by $R = \{(a,b) : a, b \in \mathbb{N}$ & $a + 3b = 12\}$, then (i) list the elements of R, (ii) find the domain and range of R. **Answer:** (i) $R = \{(9, 1), (6, 2), (3, 3)\}$ (ii) $D(R) = \{3,6,9\}$ & $R(R) = \{1,2,3\}$

H	8	Let $A = \{\, eggs,\ \ milk,\ \ corn \,\}$ and $B = \{\, cows,\ \ goats,\ \ hens \,\}$. Define a relation $R = \{\, (a,\ \ b) : \text{"a is produced by b"}\ ;\ a \in A, b \in B \,\}$. Depict this relationship using roster form. Write down domain and range of relation. **Answer**: $\mathbf{R} = \{\, (\mathbf{eggs},\ \ \mathbf{hens}),\ \ (\mathbf{milk},\ \ \mathbf{cows}),\ \ (\mathbf{milk},\ \ \mathbf{goats}) \,\}$ $\mathbf{Domain} = \mathbf{D_R} = \{\, \mathbf{eggs},\ \ \mathbf{milk} \,\}$ $\mathbf{Range} = \mathbf{R_R} = \{\, \mathbf{cows},\ \ \mathbf{goats},\ \ \mathbf{hens} \,\}$
H	9	Determine the domain and range of the relation R defined by $R = \{\, (x,\ \ x+5) : x \in \{\, 0,\ \ 1,\ \ 2,\ \ 3,\ \ 4,\ \ 5 \,\}\,\}$. Depict this relationship using roster form. Write down domain and range of relation. **Answer**: $\mathbf{R} = \{\, (\mathbf{0},\ \ \mathbf{5}),\ \ (\mathbf{1},\ \ \mathbf{6}),\ \ (\mathbf{2},\ \ \mathbf{7}),\ \ (\mathbf{3},\ \ \mathbf{8}),\ \ (\mathbf{4},\ \ \mathbf{9}),\ \ (\mathbf{5},\ \ \mathbf{10}) \,\}$ $\mathbf{Domain} = \mathbf{D_R} = \{\, \mathbf{0},\ \ \mathbf{1},\ \ \mathbf{2},\ \ \mathbf{3},\ \ \mathbf{4},\ \ \mathbf{5} \,\}$ $\mathbf{Range} = \mathbf{R_R} = \{\, \mathbf{5},\ \ \mathbf{6},\ \ \mathbf{7},\ \ \mathbf{8},\ \ \mathbf{9},\ \ \mathbf{10} \,\}$
T	10	Let $A = \{\, 1,\ \ 2,\ \ 3,\ \ 5 \,\}$ and $B = \{\, 4,\ \ 6,\ \ 9 \,\}$. Define a relation R from A to B by $R = \{\, (x,\ \ y) : \text{the difference of x and y is odd}\ ;\ x \in A,\ \ y \in B \,\}$. Write R in a roster form. **Answer**: $\mathbf{R} = \{\, (\mathbf{1},\ \ \mathbf{4}),\ \ (\mathbf{1},\ \ \mathbf{6}),\ \ (\mathbf{2},\ \ \mathbf{9}),\ \ (\mathbf{3},\ \ \mathbf{4}),\ \ (\mathbf{3},\ \ \mathbf{6}),\ \ (\mathbf{5},\ \ \mathbf{4}),\ \ (\mathbf{5},\ \ \mathbf{6}) \,\}$

<table>
<tr><td>C</td><td>11</td><td>The adjoining diagram shows a relation between the sets P and Q. Write this relation in roster and set builder form. What is domain and range?


Answer:
In Roster Form: $\{\,(\mathbf{4},\ \mathbf{2}),\ (\mathbf{4},\ -\mathbf{2}),\ (\mathbf{9},\ \mathbf{3}),\ (\mathbf{9},\ -\mathbf{3}),\ (\mathbf{25},\ \mathbf{5}),\ (\mathbf{25},\ -\mathbf{5})\,\}$
In Set – Builder Form: $\mathbf{R} = \{\,(\mathbf{x},\ \mathbf{y}) : \mathbf{x} = \mathbf{y}^2,\ \mathbf{x} \in \mathbf{P},\ \mathbf{y} \in \mathbf{Q}\,\}$
$\mathbf{D_R} = \{\,\mathbf{4},\ \ \mathbf{9},\ \ \mathbf{25}\,\}$ & $\mathbf{R_R} = \{\,-\mathbf{2},\ \mathbf{2},\ -\mathbf{3},\ \mathbf{3},\ -\mathbf{5},\ \mathbf{5}\,\}$</td></tr>
<tr><td>H</td><td>12</td><td>The adjoining diagram shows a relationship between the sets P and Q. Write this relation in roster and set – builder form.


Answer: In Roster Form $\rightsquigarrow \{\,(\mathbf{5},\mathbf{3}),(\mathbf{6},\mathbf{4}),(\mathbf{7},\mathbf{5})\,\}$
In Set – Builder Form $\rightsquigarrow \{\,(\mathbf{x},\mathbf{y}) : \mathbf{y} = \mathbf{x} - \mathbf{2}, \mathbf{x} \in \mathbb{N}, \mathbf{5} \le \mathbf{x} \le \mathbf{7}\,\}$</td></tr>
<tr><td>T</td><td>13</td><td>Let $R = \{\,(x,y) : x,y \in \mathbb{Z}, y = 2x - 4\,\}$. If $(a,-2)$ and $(4,b^2)$ belong to R, find the values of a and b.
Answer: $\mathbf{a} = \mathbf{1}, \mathbf{b} = \pm\mathbf{2}$</td></tr>
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Method – 2 ⇝ Properties of Relation

Properties of Relation

→ There are several properties that are used to classify relations on a set.

(1) Reflexive Relation

(2) Irreflexive Relation

(3) Symmetric Relation

(4) Asymmetric Relation

(5) Anti-symmetric Relation

(6) Transitive Relation

Reflexive Relation

→ A relation R on a set A is **reflexive** if $(a, a) \in R$, $\forall a \in A$. ["∀" means "for every"]

→ For example:

- Let R_1 and R_2 be relations on a set $A = \{1, 2, 3, 4\}$.
  - $R_1 = \{(1, 1), (1, 2), (1, 4), (2, 2), (2, 3), (3, 3), (3, 4), (4, 4)\}$

    Here, $(1, 1), (2, 2), (3, 3), (4, 4) \in R_1$

    Hence, R_1 is reflexive.
  - $R_2 = \{(1, 1), (1, 4), (2, 2), (2, 4), (3, 3), (3, 1)\}$

    Here, $(4, 4) \notin R_2$

    Hence, R_2 is **not** reflexive.

Irreflexive Relation

→ A relation R on a set A is **irreflexive** if $(a, a) \notin R$, $\forall a \in A$.

→ For example:

- Let R_1 and R_2 be relations on a set $A = \{1, 2, 3, 4\}$.
  - $R_1 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$

    Here, $(1, 1), (2, 2), (3, 3), (4, 4) \notin R_1$.

    Hence, R_1 is irreflexive.

- $R_2 = \{ (2, 3), (2, 4), (4, 4) \}$

  Here, $(4, 4) \in R_1$.

  Hence, R_2 is **not** irreflexive.

Symmetric Relation

→ A relation R on a set A is **symmetric** if whenever $(\mathbf{a}, \mathbf{b}) \in \mathbf{R}$, then $(\mathbf{b}, \mathbf{a}) \in \mathbf{R}$, $\forall$ a, b $\in$ R.

→ For example:

- Let R_1 and R_2 be relations on a set $A = \{ 1, 2, 3, 4 \}$.
  - $R_1 = \{ (1, 2), (2, 1), (2, 3), (3, 2), (3, 3)\}$

    Here, $(1, 2), (2, 1) \in R_1$, $(2, 3), (3, 2) \in R_1$ and $(3, 3) \in R_1$.

    Hence, R_1 is symmetric.
  - $R_2 = \{ (2, 1), (2, 3) \}$

    Here, $(2, 1) \in R_2$ but $(1, 2) \notin R_2$.

    Hence, R_2 is **not** symmetric.

Asymmetric Relation

→ A relation R on a set A is **asymmetric** if whenever $(\mathbf{a}, \mathbf{b}) \in \mathbf{R}$, then $(\mathbf{b}, \mathbf{a}) \notin \mathbf{R}$, $\forall$ a, b $\in$ R.

→ Asymmetric relation does not contain diagonal pairs.

→ For example:

- Let R_1 and R_2 be relations on a set $A = \{ 1, 2, 3, 4 \}$.
  - $R_1 = \{ (1, 2), (2, 3), (3, 2) \}$

    Here, $(2, 3), (3, 2) \in R_1$.

    Hence, R_1 is **not** asymmetric.
  - $R_2 = \{ (2, 1), (2, 3) \}$

    Here, $(2, 1) \in R_2$ but $(1, 2) \notin R_2$ and $(2, 3) \in R_2$ but $(3, 2) \notin R_2$.

    Hence, R_2 is asymmetric.

Antisymmetric Relation

→ A relation R on a set A is **antisymmetric** if whenever $(a, b) \in R$ and $(b, a) \in R$, then $a = b$, $\forall\, a, b \in R$.

→ If $(a, b) \in R$ and $(b, a) \notin R$, then there is no need to discuss $a = b$ or $a \neq b$.

OR

→ A relation R on a set A is antisymmetric if whenever $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$, $\forall\, a, b \in R$.

→ Antisymmetric relation may contain diagonal pairs.

→ For example:

- Let R_1 and R_2 be relations on a set $A = \{1, 2, 3, 4\}$.
  - $R_1 = \{(1, 2), (2, 3), (2, 2)\}$

    Here, $(1, 2) \in R_1$ but $(2, 1) \notin R_1$, $(2, 3) \in R_1$ but $(3, 2) \notin R_1$.

    So, no need to discuss $a = b$.

    Hence, R_1 is antisymmetric.
  - $R_2 = \{(2, 1), (1, 2)\}$

    Here, $(2, 1), (1, 2) \in R_2$ but $1 \neq 2$.

    Hence, R_2 is **not** antisymmetric.

Transitive Relation

→ A relation R on a set A is **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, $\forall\, a, b, c \in R$.

→ If $(a, b) \in R$ and $(b, c) \notin R$, then there is no need to discuss $(a, c) \in R$ or $(a, c) \notin R$.

→ For example:

- Let R_1, R_2 and R_3 be relations on a set $A = \{1, 2, 3, 4\}$.
  - $R_1 = \{(1, 2), (2, 3), (2, 2)\}$

    Here, $(1, 2), (2, 3) \in R_1$ but $(1, 3) \notin R_1$.

    Hence, R_1 is **not** transitive.
  - $R_2 = \{(2, 1), (3, 2), (3, 1)\}$

    Here, for $(3, 2), (2, 1) \in R_2$, $(3, 1) \in R_2$.

    Hence, R_2 is transitive.

- $R_3 = \{ (2, 3), (2, 1), (4, 1)\}$

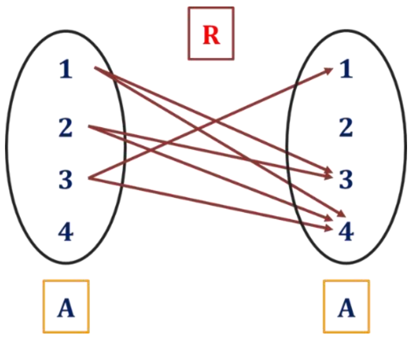
  Here, for $(a, b) \in R_2$, $(b, c) \notin R_2$.

  So, there is no need to discuss about (a, c).

  Hence, R_3 is transitive.

Examples of Method-2: Properties of Relation

C	1	For each of these relations on the set $A = \{ 1, 2, 3, 4 \}$, determine whether it is reflexive, irreflexive, symmetric, asymmetric, antisymmetric or transitive. **(1)** $R_1 = \{ (1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4) \}$ **(2)** $R_2 = \{ (1, 1), (2, 2), (3, 3), (4, 4) \}$ **Answer: R_1 is reflexive, symmetric and transitive** **R_2 is reflexive, symmetric, antisymmetric and transitive**
H	2	For each of these relations on the set $\{ 1, 2, 3, 4 \}$, determine whether it is reflexive, irreflexive, symmetric, asymmetric, antisymmetric or transitive. **(1)** $R_1 = \{ (1, 1), (2, 2), (3, 3) \}$ **(2)** $R_2 = \{ (1, 1), (1, 3), (1, 2), (3, 1), (3, 2), (3, 3), (4, 4) \}$ **(3)** $R_3 = \{ (1, 2), (1, 3), (1, 4), (2, 3), (3, 1), (4, 1), (3, 4) \}$ **(4)** $R_4 = A \times A$ **Answer: $R_1 \rightsquigarrow$ Reflexive, Symmetric, Transitive** **$R_2 \rightsquigarrow$ Transitive** **$R_3 \rightsquigarrow$ Irreflexive** **$R_4 \rightsquigarrow$ Reflexive, Irreflexive, Symmetric, Asymmetric, Antisymmetric, Transitive**

C	3	The given diagram shows a relation on the set A. Write this relation in tabular form and determine whether it is reflexive, irreflexive, symmetric, asymmetric, antisymmetric or transitive.  **Answer: R is only irreflexive relation.**
C	4	Let $A = \{1, 2, 3, 4, 5, 6\}$ and define a relation R on A as $R = \{(x, y) \mid y \text{ is divisible by } x\}$ Check whether R is reflexive, symmetric or transitive. **Answer: R is reflexive and transitive but not symmetric.**
H	5	Let $A = \{1, 2, 3, 4, 5, 6\}$ and define a relation R on A as $R = \{(a, b) : b - a = 2\}$ Check whether R is reflexive, irreflexive, symmetric, asymmetric, antisymmetric or transitive. **Answer: R is irreflexive, asymmetric and antisymmetric**
H	6	Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$. Check whether R is reflexive, symmetric, transitive or not. **Answer: R is reflexive, symmetric and transitive**
T	7	Check whether R is reflexive, symmetric, transitive or not. Relation $R = \{(x, y) : x + 4y = 10 ;\ x, y \in \mathbb{N}\}$ **Answer: R is transitive but neither reflexive nor symmetric.**

C	8	Determine whether the relation R on the set A of all people is reflexive, symmetric, asymmetric, antisymmetric or transitive, where xRy if and only if x is a father of y. **Answer: R is only asymmetric and antisymmetric**
H	9	Determine whether the relation R on the set of all set of all people is reflexive, symmetric, antisymmetric or transitive, where aRb if and only if **(1)** a is taller than b. **(2)** a is 3 inches shorter than b. **(3)** a and b were born on the same day. **(4)** a has the same first name as b. **(5)** a is grandparent of b. **(6)** a is brother of b. **Answer: (1) R is transitive and antisymmetric** **(2) R is antisymmetric** **(3) R is reflexive, symmetric and transitive** **(4) R is reflexive, symmetric and transitive** **(5) R is antisymmetric** **(6) R is transitive**
T	10	Determine whether the relation R on the set of all Web pages are reflexive, symmetric, asymmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if a) everyone who has visited Web page a has also visited Web page b. b) there are no common links found on both Web page a and Web page b. c) there is at least one common link on Web page a and Web page b. d) there is a Web page that includes links to both Web page a and Web page b. **Answer: a) reflexive and transitive b), c), d) symmetric only**

T	11	Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric or transitive, where $(x, y) \in R$ if and only if **(1)** $x \neq y$ **(2)** $xy \geq 1$ **(3)** $x = y^2$ **(4)** $xy = 0$ **Answer: (1) R is symmetric (2) R is symmetric and transitive** **(3) R is antisymmetric (4) R is symmetric**
T	12	Let relation R defined on $\mathbb{R}$ as $R = \{ (a, b) : a \leq b^3;\ a, b \in \mathbb{R} \}$. Check whether R is reflexive, symmetric, transitive or not. **Answer: R is neither reflexive nor symmetric nor transitive**
T	13	Let R be a relation defined as $R = \{ (a, b) \in \mathbb{R}^2 : a - b \leq 3 \}$. Determine whether R is reflexive, symmetric, antisymmetric and transitive. **Answer: R is reflexive only**
C	14	Prove that if a relation R on a set A is transitive and irreflexive, then it is asymmetric.
H	15	Prove that if a relation R on a set A is transitive and asymmetric, then it is irreflexive.
T	16	Prove that if a relation R on a set A is both symmetric and antisymmetric, then R is a subset of the identity relation on A.
T	17	Give an example of a relation which is **(a)** reflexive and transitive but not symmetric **(b)** symmetric and transitive but not reflexivec **(c)** reflexive and symmetric but not transitive **(d)** reflexive and transitive but neither symmetric nor antisymmetric **(e)** neither symmetric nor antisymmetric **Hint: Refer Theory**