

K_{r+1} -saturated graphs with small spectral radius

Jaehoon Kim^{*} Seog-Jin Kim^{†‡} Alexandr V. Kostochka[§] and Suil O[¶]

June 9, 2020

Abstract

For a graph H , a graph G is H -saturated if G does not contain H as a subgraph but for any $e \in E(\overline{G})$, $G + e$ contains H . In this note, we prove a sharp lower bound for the number of paths and walks on length 2 in n -vertex K_{r+1} -saturated graphs. We then use this bound to give a lower bound on the spectral radii of such graphs which is asymptotically tight for each fixed r and $n \rightarrow \infty$.

Keywords: Saturated graphs, complete graphs, spectral radius

AMS subject classification 2010: 05C35, 05C50

1 Introduction

1.1 Notation and preliminaries

In this note we deal with finite undirected graphs with no loops or multiple edges. For a graph H , a graph G is H -saturated if H is not a subgraph of G but after adding to G any edge results in a graph containing H . For a positive integer n and a graph H , the *extremal number* $ex(n, H)$ is the maximum number of edges in an n -vertex graph not containing H . Clearly, an extremal n -vertex graph G not containing H with $|E(G)| = ex(n, H)$ is H -saturated. Thus, one can also say that $ex(n, H)$ is the maximum number of edges in an

^{*}Mathematical Sciences Department, KAIST, jaehoon.kim@kaist.ac.kr

[†]Department of Mathematics Education, Konkuk University, Seoul, 05029, Korea, skim12@konkuk.ac.kr

[‡]This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT)(NRF-2018R1C1B6003786).

[§]Department of Mathematics, University of Illinois, Urbana, IL, 61801, USA and Sobolev Institute of Mathematics, Novosibirsk 630090, Russia, kostochk@math.uiuc.edu. Research of this author is supported in part by NSF grant DMS-1600592 and by grants 18-01-00353 and 19-01-00682 of the Russian Foundation for Basic Research.

[¶]Department of Applied Mathematics and Statistics, The State University of New York, Korea, Incheon, 21985, suil.o@sunykorea.ac.kr. Research supported by NRF-2018K2A9A2A06020345 and by NRF-2020R1F1A1A01048226.

n -vertex H -saturated graph. On the other hand, the *saturation number of H* , $\text{sat}(n, H)$, is the least number of edges in an H -saturated graph with n vertices.

Initiating the study of extremal graph theory, Turán [9] determined the extremal number $\text{ex}(n, K_{r+1})$. He also proved that there is the unique extremal graph, $T_{n,r}$, the n -vertex complete r -partite graph whose partite sets differ in size at most 1. The first result on saturation numbers is due to Erdős, Hajnal and Moon [4]:

Theorem A [4]. *If $2 \leq r < n$, then $\text{sat}(n, K_{r+1}) = (r-1)(n-r+1) + \binom{r-1}{2}$. The only n -vertex K_{r+1} -saturated graph with $\text{sat}(n, K_{r+1})$ edges is the graph $S_{n,r}$ obtained from a copy of K_{r-1} with vertex set S by adding $n-r+1$ vertices, each of which has neighborhood S .*

Graph $S_{n,r}$ has clique number r and no r -connected subgraphs; in particular, $S_{n,2}$ is a star. For an excellent survey on saturation numbers, we refer the reader to Faudree, Faudree, and Schmitt [5].

Recently, there was a series of publications on eigenvalues of H -free graphs. For a graph G , let $A(G)$ be its adjacency matrix, and we index the eigenvalues of $A(G)$ in nonincreasing order, $\lambda_1(G) \geq \dots \geq \lambda_n(G)$. The value $\lambda_1(G)$ is also called the *spectral radius* of G , and denoted by $\rho(G)$.

Studying properties of quasi-random graphs, Chung, Graham, and Wilson [3] proved a theorem implying that, if n is sufficiently large, $0 < c < \frac{1}{2}$ and G is an n -vertex K_r -free graph with $\lceil cn^2 \rceil$ edges, then either $\lambda_n(G) < -c'n$ or $\lambda_2(G) > c'n$, where $c' = c'(r, c)$ is a positive constant. However, the methods in [3] fail to indicate which of the two inequalities actually holds. Bollobás and Nikiforov [1] observed that if G is a dense K_r -free graph, then $\lambda_n(G) < -cn$ for some $c > 0$ independent of n . Nikiforov [7] gave a more precise statement that if G is a K_{r+1} -free graph with n vertices and m edges, then $\lambda_n(G) < -\frac{2^{r+1}m^r}{rn^{2r-1}}$.

Nikiforov [8] also proved that if G ia a K_{r+1} -free graph with n vertices, then $\rho(G) \leq \rho(T_{n,r})$. Since each K_{r+1} -saturated graph is K_{r+1} -free, his theorem implies the following.

Theorem B [7]. *If G is a K_{r+1} -saturated graph with n vertices, then*

$$\rho(G) \leq \rho(T_{n,r}).$$

In this note, we give a new lower bound for the spectral radius of an n -vertex K_{r+1} -saturated graph. This bound is asymptotically tight when r is fixed or grows as $o(n)$. For this, we give a tight lower bound on the sum of the squares of the vertex degrees in an n -vertex K_{r+1} -saturated graph.

1.2 Results

Our main tool will be the following.

Theorem 1.1. *If $n \geq r+1$ and G is a K_{r+1} -saturated graph with n vertices, then*

$$\sum_{v \in V(G)} d^2(v) \geq (n-1)^2(r-1) + (r-1)^2(n-r+1). \quad (1)$$

For $r = 2$, equality in the bound holds only when G is $S_{n,2}$ or a Moore graph with diameter 2. For $r \geq 3$, equality in the bound holds only when G is $S_{n,r}$.

The reason why it is helpful is the following simple observation.

Lemma 1.2. *For every n -vertex graph G with adjacency matrix A ,*

$$\rho^2(A) \geq \frac{1}{n} \sum_{v \in V(G)} d^2(v). \quad (2)$$

Theorem 1.1 together with this observation immediately yield

Theorem 1.3. *If $2 \leq r < n$ and G is a K_{r+1} -saturated graph with n vertices, then*

$$\rho(G) \geq \sqrt{\frac{(n-1)^2(r-1) + (r-1)^2(n-r+1)}{n}}. \quad (3)$$

This bound asymptotically is tight because the spectral radius of $S_{n,r}$ is close to $f(n,r)$, where $f(n,r)$ is the lower bound for $\rho(G)$ in Theorem 1.3. More specifically, note that $\rho(S_{n,2}) = f(2,n)$ and for $r \geq 3$, we have $\rho(S_{n,r}) = f(r,n) + \frac{r-2}{2} + \Theta(\frac{r^{1.5}}{\sqrt{n}})$.

Proposition 1.4. *For integers $2 \leq r < n$,*

$$\rho(S_{n,r}) = \frac{r-2 + \sqrt{(r-2)^2 + 4(r-1)(n-r+1)}}{2}.$$

In the next section we prove Theorem 1.1 (in a somewhat stronger form) and in the last section we present proofs for Lemma 1.2 and Proposition 1.4

For undefined terms, see Brouwer and Haemers [2], Godsil and Royle [6], or West [10].

2 Proof of Theorem 1.1

We will derive Theorem 1.1 from the following slightly stronger statement.

Theorem 2.1. *If $n \geq r+1$ and G is a K_{r+1} -saturated graph with n vertices, then*

$$\sum_{v \in V(G)} (d(v) + 1)(d(v) + 1 - r) \geq (r-1)n(n-r). \quad (4)$$

Proof. Let $m = |E(G)|$ and $\overline{m} = |E(\overline{G})| = \binom{n}{2} - m$. For $v \in V(G)$, let $f(v)$ be the number of pairs of non-adjacent vertices x and y in $N(v)$ such that $G[N(x) \cap N(y) \cap N(v)]$ contains K_{r-2} as a subgraph. Note that if $G[N(v)]$ is a copy of K_{r-1} , then $f(v) = 0$.

Claim 1. $\overline{m} \leq \frac{1}{r-1} \sum_{v \in V(G)} f(v)$.

We construct an auxiliary bipartite graph H with parts A and B as follows. Let $A = E(\overline{G})$ and $B = V(G)$. The graph H has an edge between $xy \in A$ and $v \in B$ iff $x, y \in N(v)$ and $G[N(x) \cap N(y) \cap N(v)]$ contains K_{r-2} as a subgraph. Then for each $v \in B$, we have $|N_H(v)| = f(v)$. Also, since G is K_{r+1} -saturated, for each $xy \in A$, $G + xy$ contains K_{r+1} as a subgraph. Thus there exist at least $r - 1$ vertices v such that $x, y \in N(v)$ and $G[N(x) \cap N(y) \cap N(v)]$ contains K_{r-2} as a subgraph, which implies

$$|N_H(xy)| \geq r - 1. \quad (5)$$

By (5),

$$(r - 1)\overline{m} \leq \sum_{xy \in A} d_H(xy) = |E(H)| = \sum_{v \in V(G)} f(v). \quad (6)$$

This proves Claim 1.

Claim 2. For each $v \in V(G)$, we have $f(v) \leq \binom{d(v) - r + 2}{2}$.

Let $H_v = G[N(v)]$, and let $d(v) = p$. Since G contains no K_{r+1} , the graph H_v has no K_r . Partition the pairs of vertices in $N(v)$ into the sets E_1, E_2 and E_3 as follows:

- (i) $E_1 = E(H_v)$,
- (ii) E_2 is the set of the edges $xy \in E(\overline{H_v})$ such that $H_v + xy$ does not contain K_r ,
- (iii) E_3 is the set of the edges $xy \in E(\overline{H_v})$ such that $H_v + xy$ contains K_r .

Let $m_i = |E_i|$ for $1 \leq i \leq 3$. By definition, $m_3 = f(v)$ and $m_1 + m_2 + m_3 = \binom{p}{2}$. As any K_r -free graph is a subgraph of K_r -saturated graph on the same vertex set, there exists a K_r -saturated graph H' with vertex set $N(v)$ containing H_v . Then $E(H') \supseteq E_1$. Furthermore, since H' is K_r -free and contains E_1 , $E(H') \cap E_3 = \emptyset$. By Theorem A, $|E(H')| \geq (r - 2)(p - r + 2) + \binom{r-2}{2}$. Hence

$$m_3 \leq \binom{p}{2} - |E(H')| \leq \binom{p}{2} - (r - 2)(p - r + 2) - \binom{r-2}{2} = \binom{p - r + 2}{2}.$$

This proves Claim 2.

Now we are ready to prove the theorem. By Claims 1 and 2,

$$\binom{n}{2} = m + \overline{m} \leq m + \frac{1}{r-1} \sum_{v \in V(G)} f(v) \leq \sum_{v \in V(G)} \left[\frac{d(v)}{2} + \frac{1}{r-1} \frac{(d(v) - r + 2)(d(v) - r + 1)}{2} \right].$$

Multiplying both sides by $2(r - 1)$, we get

$$(r - 1)n(n - 1) \leq \sum_{v \in V(G)} [(r - 1)d(v) + (d(v) + 1)(d(v) - r + 1) - (r - 1)(d(v) - r + 1)].$$

This yields

$$\sum_{v \in V(G)} (d(v) + 1)(d(v) + 1 - r) \geq (r - 1)n(n - 1) - (r - 1)^2n = (r - 1)n(n - r),$$

and Theorem 2.1 is proved. \square

To obtain Theorem 1.1, observe that (4) implies

$$\sum_{v \in V(G)} d^2(v) \geq (r-1)n(n-r) + (r-1)n + (r-2)2m.$$

So, by Theorem A,

$$\begin{aligned} \sum_{v \in V(G)} d^2(v) &\geq (r-1)n(n-r) + (r-1)n + 2(r-2) \left[\binom{n}{2} - \binom{n-r+1}{2} \right] \\ &= (n-1)^2(r-1) + (r-1)^2(n-r+1). \end{aligned} \quad (7)$$

This proves the first part of Theorem 1.1. Furthermore, for $r \geq 3$, equality in the bound requires equality in Theorem A. Thus equality holds only for $S_{n,r}$.

Suppose now $r = 2$ and G is an n -vertex K_3 -saturated graph for which (1) holds with equality. As G is K_3 -saturated, G has diameter 2. Equality in the bound requires equality in (6), and hence equality in (5) for every $xy \in E(\overline{G})$. This means G has no C_4 , which implies that G has girth at least 5. If G has no cycles, then G is a copy of $S_{n,2}$. Otherwise, G is a Moore graph with diameter 2.

Recall that there are at most four Moore graphs with diameter 2: C_5 , the Petersen graph, the Hoffman-Singleton graph, and possibly one 57-regular graph of girth 5 with 3250 vertices.

3 Spectral radius

We will use the following standard tool.

Theorem 3.1 (Rayleigh Quotient Theorem). *For a real matrix A*

$$\rho(A) = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x^T Ax}{x^T x}. \quad (8)$$

First, we present a proof of Lemma 1.2. By (8),

$$\rho^2(A) = \rho(A^2) = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x^T A^2 x}{x^T x} = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{(x^T A^T)(Ax)}{x^T x} \geq \frac{(\mathbf{1}^T A^T)(A\mathbf{1})}{\mathbf{1}^T \mathbf{1}} = \frac{1}{n} \sum_{v \in V(G)} d^2(v).$$

Thus, (2) holds. Together with Theorem 1.1, this implies Theorem 1.3.

To show that Theorem 1.3 is asymptotically tight, we will determine the spectral radius of $S_{n,r}$, i.e. prove Proposition 1.4. We will need a new notion. Consider a partition $V(G) = V_1 \cup \dots \cup V_s$ of the vertex set of a graph G into s non-empty subsets. For $1 \leq i, j \leq s$, let $q_{i,j}$ denote the average number of neighbors in V_j of the vertices in V_i . The quotient matrix

Q of this partition is the $s \times s$ matrix whose (i, j) -th entry equals $q_{i,j}$. The eigenvalues of the quotient matrix interlace the eigenvalues of G . This partition is *equitable* if for each $1 \leq i, j \leq s$, each vertex $v \in V_i$ has exactly $q_{i,j}$ neighbors in V_j . In this case, the eigenvalues of the quotient matrix are eigenvalues of G and the spectral radius of the quotient matrix equals the spectral radius of G (see [2], [6] for more details).

[Proof of Proposition 1.4]

Partition $V(S_{n,r})$ into sets A and B such that $S_{n,r}[A]$ is a copy of K_{r-1} and $S_{n,r}[B]$ is an independent set with $n - r + 1$ vertices. Each vertex in A is adjacent to all vertices in B . The quotient matrix of the partitions A and B is

$$\begin{pmatrix} r-2 & n-r+1 \\ r-1 & 0 \end{pmatrix}.$$

The characteristic polynomial of the matrix is $x^2 - (r-2)x - (r-1)(n-r+1) = 0$. Since the partition $V(S_{n,r}) = A \cup B$ is equitable,

$$\rho(S_{n,r}) = \frac{r-2 + \sqrt{(r-2)^2 + 4(r-1)(n-r+1)}}{2}.$$

This completes the proof of Proposition 1.4. \square

Note that $\rho(S_{n,2}) = \sqrt{n-1}$. Thus for $r = 2$, equality in Theorem 1.3 holds if and only if G is $S_{n,2}$ or a Moore graph. For $r \geq 3$, the bound in Theorem 1.3 may be improved, and we guess that the spectral radius of $S_{n,r}$ is the minimum of $\rho(H)$ among all n -vertex K_{r+1} -saturated graphs H .

Acknowledgement. We thank Xuding Zhu for helpful comments.

References

- [1] B. Bollobás, V. Nikiforov, Graphs and Hermitian matrices: eigenvalue interlacing, *Discrete Math.* **289** (2004) 119–127.
- [2] A. Brouwer and W. Haemers, *Spectra of Graphs*, Springer, New York, (2011).
- [3] F.R.K. Chung, R.L. Graham, R.M. Wilson, Quasi-random graphs, *Combinatorica* **9** (1989) 345–362.
- [4] P. Erdős, A. Hajnal, and J. W. Moon. A problem in graph theory, *Amer. Math. Monthly* **71** (1964) 1107–1110.
- [5] J.R. Faudree, R.J. Faudree, J.R. Schmitt, A survey of minimum saturated graphs, *Electron. J. Combin.* **18** (2011), Dynamic Survey 19, 36 pages.

- [6] C. Godsil and G. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics, 207. Springer-Verlag, New York, 2001.
- [7] V. Nikiforov, The smallest eigenvalue of K_r -free graphs, *Discrete Math.* **306** (2006), no. 6, 612–616.
- [8] V. Nikiforov, Bounds on graph eigenvalues. II. *Linear Algebra Appl.* **427** (2007), no. 2–3, 183–189.
- [9] P. Turán, Eine Extremalaufgabe aus der Graphentheorie, *Mat. Fiz. Lapok* **48** (1941) 436–452.
- [10] D.B. West, *Introduction to Graph Theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 2001.