

A note on hypergraphs without non-trivial intersecting subgraphs

Xizhi Liu*

July 23, 2020

Abstract

A hypergraph \mathcal{F} is non-trivial intersecting if every two edges in it have a nonempty intersection but no vertex is contained in all edges of \mathcal{F} . Mubayi and Verstraëte showed that for every $k \geq d+1 \geq 3$ and $n \geq (d+1)n/d$ every k -graph \mathcal{H} on n vertices without a non-trivial intersecting subgraph of size $d+1$ contains at most $\binom{n-1}{k-1}$ edges. They conjectured that the same conclusion holds for all $d \geq k \geq 4$ and sufficiently large n . We confirm their conjecture by proving a stronger statement.

They also conjectured that for $m \geq 4$ and sufficiently large n the maximum size of a 3-graph on n vertices without a non-trivial intersecting subgraph of size $3m+1$ is achieved by certain Steiner systems. We give a construction with more edges showing that their conjecture is not true in general.

1 Introduction

We use $[n]$ to denote the set $\{1, \dots, n\}$ and use $\binom{V}{k}$ to denote the collection of all k -subsets of some set V . For a hypergraph \mathcal{H} we use $V(\mathcal{H})$ to denote the vertex set of \mathcal{H} and use $|\mathcal{H}|$ to denote the number of edges in \mathcal{H} .

For $d \geq 2$ a hypergraph \mathcal{F} is d -wise-intersecting if $\bigcap_{i \in [d]} E_i \neq \emptyset$ for all $E_1, \dots, E_d \in \mathcal{F}$, and \mathcal{F} is non-trivial d -wise-intersecting if it is d -wise-intersecting but $\bigcap_{E \in \mathcal{F}} E = \emptyset$. If $d = 2$, then we simply call \mathcal{F} intersecting and non-trivial intersecting, respectively.

A d -simplex is a collection of $d+1$ sets $\{A_1, \dots, A_{d+1}\}$ such that $\bigcap_{i \neq j} A_i \neq \emptyset$ for all $j \in [d+1]$, but $\bigcap_{i \in [d+1]} A_i = \emptyset$. The Chvátal Simplex Conjecture [2] states that for every $k \geq d+1 \geq 3$ and $n \geq (d+1)n/d$ if a hypergraph $\mathcal{H} \subset \binom{[n]}{k}$ does not contain a d -simplex as a subgraph, then $|\mathcal{H}| \leq \binom{n-1}{k-1}$, with equality only if \mathcal{H} is a star, i.e. all sets in \mathcal{H} contain a fixed vertex. The case $k = d+1$ was proved by Chvátal [2]. Mubayi and Verstraëte [17] proved the conjecture for all $k \geq 3$ and $d = 2$. Recently, Currier [4] proved this conjecture for all $k \geq d+1 \geq 3$ and $n \geq 2k$. The Chvátal Simplex Conjecture is still open in general for $n < 2k$ and $3 \leq d \leq k-2$, and we refer the reader to [1, 3, 8, 5, 6, 12, 13, 15] and their references for more results related to this conjecture.

It is easy to see that the family of all d -simplexes is the same as the family of all non-trivial d -wise-intersecting hypergraphs of size $d+1$, and if a hypergraph is d -wise-intersecting, then it is also d' -wise-intersecting for all $2 \leq d' \leq d$.

*Department of Mathematics, Statistics, and Computer Science, University of Illinois, Chicago, IL, 60607 USA. Email: xliu246@uic.edu. Research partially supported by NSF award DMS-1763317.

In the proof for the Chvátal Simplex Conjecture for $d = 2$ Mubayi and Verstraëte actually proved the following stronger result.

Theorem 1.1 (Mubayi and Verstraëte [17]). *Let $k \geq d + 1 \geq 3$ and $n \geq (d + 1)n/d$. Suppose that $\mathcal{H} \subset \binom{[n]}{k}$ contains no non-trivial intersecting subgraph of size $d + 1$. Then $|\mathcal{H}| \leq \binom{n-1}{k-1}$, with equality only if \mathcal{H} is a star.*

Mubayi and Verstraëte also remarked that their proof of Theorem 1.1 actually works for d slightly greater than k as well, and they posed the following conjecture.

Conjecture 1.2 (Mubayi and Verstraëte [17]). *Let $d \geq k \geq 4$ and n be sufficiently large. Suppose that $\mathcal{H} \subset \binom{[n]}{k}$ contains no non-trivial intersecting subgraph of size $d + 1$. Then $|\mathcal{H}| \leq \binom{n-1}{k-1}$, with equality only if \mathcal{H} is a star.*

Let $m \geq 2$. A Steiner $(n, 3, m - 1)$ -system is a 3-graph \mathcal{S} on n vertices such that every pair of vertices in $V(\mathcal{S})$ is contained in exactly $m - 1$ edges of \mathcal{S} . It follows from Keevash's result [11] that if n is a multiple of 3 and sufficiently large, then there exists a Steiner $(n, 3, m - 1)$ -system.

Notice that a Steiner $(n, 3, m - 1)$ -system has size $\frac{m-1}{3} \binom{n}{2}$, which is greater than $\binom{n-1}{2}$ when $m \geq 4$. It was observed by Mubayi and Verstraëte [17] that a Steiner $(n, 3, m - 1)$ -system does not contain non-trivial intersecting subgraph of size $3m + 1$. Therefore, they made the following conjecture for 3-graphs.

Conjecture 1.3 (Mubayi and Verstraëte [17]). *Let $m \geq 4$ and n be sufficiently large. Suppose that $\mathcal{H} \subset \binom{[n]}{3}$ contains no non-trivial intersecting family of size $3m + 1$. Then $|\mathcal{H}| \leq \frac{m-1}{3} \binom{n}{2}$, with equality holds iff \mathcal{H} is a Steiner $(n, 3, m - 1)$ -system.*

In this note, we confirm Conjecture 1.2 by proving a stronger statement (Theorem 1.6), and disprove Conjecture 1.3 by showing a construction with more than $\frac{m-1}{3} \binom{n}{2}$ edges and contains no non-trivial intersecting subgraph of size $3m + 1$.

Let $s \geq 2$. A family $\mathcal{D} = \{D_1, \dots, D_s\}$ is a Δ -system (or a sunflower) if $D_i \cap D_j = C$ for all $\{i, j\} \subset [s]$. The set C is called the center of \mathcal{D} .

Definition 1.4. *Let $k, d \geq p \geq 2$, and $\vec{a} = (a_1, \dots, a_p)$, $\vec{b} = (b_1, \dots, b_p)$ be two sequences of positive integers with $\sum_{i=1}^p a_i = k$.*

- (1) *An \vec{a} -partition of a k -set E is a partition $E = \bigcup_{i \in [p]} A_i$ such that $|A_i| = a_i$ for $i \in [p]$.*
- (2) *A semi- (\vec{a}, \vec{b}) - Δ -system is a collection of sets $\{E_0, E_1^1, \dots, E_1^{b_1}, \dots, E_p^1, \dots, E_p^{b_p}\}$ such that for some \vec{a} -partition of $E_0 = \bigcup_{i \in [p]} A_i$, the family $\{E_0, E_i^1, \dots, E_i^{b_i}\}$ is a Δ -system with center $E_0 \setminus A_i$ for all $i \in [p]$. The set E_0 is called the host of this semi- (\vec{a}, \vec{b}) - Δ -system.*
- (3) *An (\vec{a}, \vec{b}) - Δ -system is a semi- (\vec{a}, \vec{b}) - Δ -system $\{E_0, E_1^1, \dots, E_1^{b_1}, \dots, E_p^1, \dots, E_p^{b_p}\}$ such that sets $E_1^1 \setminus E_0, \dots, E_1^{b_1} \setminus E_0, \dots, E_p^1 \setminus E_0, \dots, E_p^{b_p} \setminus E_0$ are pairwise disjoint.*
- (4) *An (\vec{a}, d) - Δ -system is a (\vec{a}, \vec{b}) - Δ -system for some \vec{b} such that $\sum_{i=1}^p b_i = d$.*

From the definitions one can easily obtain the following observation.

Observation 1.5. *Let $k, d \geq p \geq 3$ and $\vec{a} = (a_1, \dots, a_p)$ be a sequence of integers with $\sum_{i=1}^p a_i = k$. Then an (\vec{a}, d) - Δ -system is a non-trivial $(p-1)$ -wise-intersecting hypergraph with $d+1$ edges.*

An (\vec{a}, d) - Δ -system in which $d = p$, i.e. $b_1 = \dots = b_p = 1$ was studied by Füredi and Özkahya in [10]. In this note we employ a machinery (a complicated version of the delta-system method) developed by them and even earlier by Frankl and Füredi [7], to obtain the following tight bound for the size of a hypergraph without (\vec{a}, d) - Δ -systems for all $d \geq p \geq 2$.

Theorem 1.6. *Let $k > p \geq 2$, $d \geq p$, and $\vec{a} = (a_1, \dots, a_p)$ be a sequence of positive integers with $\sum_{i=1}^p a_i = k$. Suppose that $n \geq n_0(k, d)$ is sufficiently large and $\mathcal{H} \subset \binom{[n]}{k}$ does not contain a (\vec{a}, d) - Δ -system as a subgraph. Then $|\mathcal{H}| \leq \binom{n-1}{k-1}$, with equality only if \mathcal{H} is a star.*

Remark: Our proof of Theorem 1.6 uses the delta-system method and Theorem 3.3 due to Füredi, so our lower bound for $n_0(k, d)$ is at least exponential in k and d . It would be interesting to find the minimum value of $n_0(k, d)$ such that the statement in Theorem 1.6 holds for all $n \geq n_0(k, d)$.

The following result is an immediate consequence of Theorem 1.6 and Observation 1.5.

Theorem 1.7. *Let $k > p \geq 3$, $d \geq p$. Suppose that $n \geq n_0(k, d)$ is sufficiently large and $\mathcal{H} \subset \binom{[n]}{k}$ does not contain a non-trivial $(p-1)$ -wise-intersecting subgraph of size $d+1$. Then $|\mathcal{H}| \leq \binom{n-1}{k-1}$, with equality only if \mathcal{H} is a star.*

Note that Conjecture 1.2 is a special case of Theorem 1.7, i.e. $p = 3$.

We are also able to prove the following stability version of Theorem 1.6.

Theorem 1.8. *Let $k > p \geq 2$, $d \geq p$, and $\vec{a} = (a_1, \dots, a_p)$ be a sequence of positive integers with $\sum_{i=1}^p a_i = k$. For every $\delta > 0$ there exists $\epsilon > 0$ and $n_0(k, d, \delta)$ such that the following holds for all $n \geq n_0(k, d, \delta)$. Suppose that $\mathcal{H} \subset \binom{[n]}{k}$ does not contain a (\vec{a}, d) - Δ -system as a subgraph, and $|\mathcal{H}| \geq (1 - \epsilon) \binom{n-1}{k-1}$. Then there exists a vertex $v \in [n]$ such that v is contained in all but at most δn^{k-1} edges in \mathcal{H} .*

For 3-graphs the following result shows that Conjecture 1.3 is not true in general.

Theorem 1.9. *Let $m \geq 4$, n be a multiple of 3 and sufficiently large. Then there exists a 3-graph \hat{S} on n vertices with $\frac{m-1}{3} \binom{n}{2} + \frac{n}{3}$ edges and contains no non-trivial intersecting subgraph of size $3m+1$.*

In Section 2 we present a construction that proves Theorem 1.9. In Section 3 we present some preliminary lemmas for the proof of Theorems 1.6 and 1.8, and in Section 4 we prove Theorems 1.6 and 1.8.

2 Constructions

In this section we give a construction to show that Conjecture 1.3 is not true in general. We need the following structural theorem of intersecting 3-graphs due to Kostochka and Mubayi [14]. Define

- $EKR(n) = \left\{ A \in \binom{[n]}{3} : 1 \in A \right\}.$
- $H_0(n) = \left\{ A \in \binom{[n]}{3} : |A \cap [3]| \geq 2 \right\}.$
- $H_1(n) = \left\{ A \in \binom{[n]}{3} : 1 \in A \text{ and } |A \cap \{2, 3, 4\}| \geq 1 \right\} \cup \{234\}.$
- $H_2(n) = \left\{ A \in \binom{[n]}{3} : 1 \in A \text{ and } |A \cap \{2, 3\}| \geq 1 \right\} \cup \{234, 235, 145\}.$
- $H_3(n) = \left\{ A \in \binom{[n]}{3} : \{1, 2\} \in A \right\} \cup \{134, 135, 145, 234, 235, 245\}.$
- $H_4(n) = \left\{ A \in \binom{[n]}{3} : \{1, 2\} \in A \right\} \cup \{134, 156, 235, 236, 245, 246\}.$
- $H_5(n) = \left\{ A \in \binom{[n]}{3} : \{1, 2\} \in A \right\} \cup \{134, 156, 136, 235, 236, 246\}.$

Theorem 2.1 (Kostochka and Mubayi [14]). *Every intersecting 3-graph with at least 11 edges is contained in one of $EKR(n), H_0(n), H_1(n), \dots, H_5(n)$.*

For a 3-graph \mathcal{H} and $\{u, v\} \subset V(\mathcal{H})$ let $\deg_{\mathcal{H}}(uv)$ denote the number of edges in \mathcal{H} that contain both u and v . Let $\Delta_2(\mathcal{H}) = \max\{\deg_{\mathcal{H}}(uv) : \{u, v\} \subset V(\mathcal{H})\}.$

Observation 2.2. *Let \mathcal{H} be a 3-graph with e edges. If $\mathcal{H} \subset H_0(n)$, then $\Delta_2(\mathcal{H}) \geq \lceil \frac{e}{3} \rceil$. If $\mathcal{H} \subset H_2(n)$, then $\Delta_2(\mathcal{H}) \geq \lceil \frac{e-3}{2} \rceil$. If \mathcal{H} is contained in $H_3(n)$, $H_4(n)$, or $H_5(n)$, then $\Delta_2(\mathcal{H}) \geq e - 6$.*

Now we define the construction. Let n be a multiple of 3 and sufficiently large. Let $\mathcal{S} \subset \binom{[n]}{3}$ be a Steiner $(n, 3, m-1)$ -system. Then the complement of \mathcal{S} , which is $\widehat{\mathcal{S}} := \binom{[n]}{3} \setminus \mathcal{S}$, satisfies that $d_{\widehat{\mathcal{S}}}(uv) = n - m + 1$ for all $\{u, v\} \subset V(\mathcal{S})$. Therefore, by the Rödl-Ruciński-Szemerédi Theorem [18], $\widehat{\mathcal{S}}$ contains a matching \mathcal{M} with $n/3$ edges. Let $\widehat{\widehat{\mathcal{S}}} = \mathcal{S} \cup \mathcal{M}$. Then it is easy to see that

$$|\widehat{\widehat{\mathcal{S}}}| = |\mathcal{S}| + |\mathcal{M}| = \frac{m-1}{3} \binom{n}{2} + \frac{n}{3}.$$

The following proposition proves Theorem 1.9.

Proposition 2.3. *Let $m \geq 4$. Then $\widehat{\widehat{\mathcal{S}}}$ does not contain a non-trivial intersecting subgraph of size $3m + 1$.*

Proof. Suppose not. Let $\mathcal{F} \subset \widehat{\widehat{\mathcal{S}}}$ be a non-trivial intersecting subgraph with $3m + 1 \geq 11$ edges. By Theorem 2.1, \mathcal{F} is contained in one of $H_0(n), H_1(n), \dots, H_5(n)$. Notice that $\Delta_2(\mathcal{F}) \leq \Delta_2(\widehat{\widehat{\mathcal{S}}}) = m$. If \mathcal{F} is contained in one of $H_0(n), H_2(n), \dots, H_5(n)$, then by Observation 2.2, $\Delta_2(\mathcal{F}) \geq \min\{\lceil \frac{3m+1}{3} \rceil, \lceil \frac{3}{2}m - 1 \rceil, 3m - 5\} > m$, a contradiction. Therefore, $\mathcal{F} \subset H_1(n)$. Then \mathcal{F} contains four vertices v_0, v_1, v_2, v_3 such that $\deg_{\widehat{\widehat{\mathcal{S}}}}(v_0v_1) + \deg_{\widehat{\widehat{\mathcal{S}}}}(v_0v_2) + \deg_{\widehat{\widehat{\mathcal{S}}}}(v_0v_3) \geq 3m$, which implies $\deg_{\widehat{\widehat{\mathcal{S}}}}(v_0v_1) = \deg_{\widehat{\widehat{\mathcal{S}}}}(v_0v_2) = \deg_{\widehat{\widehat{\mathcal{S}}}}(v_0v_3) = m$. However, this is impossible because the set $\{\{u, v\} \subset V(\mathcal{S}) : \deg_{\widehat{\mathcal{S}}}(uv) = m\}$ consists of $n/3$ copies of pairwise vertex-disjoint triangles. ■

3 Lemmas

In this section we present some preliminary lemmas for the proofs of Theorems 1.6 and 1.8. Our first lemma shows that a sufficiently large semi- (\vec{a}, \vec{c}) - Δ -system contains an (\vec{a}, \vec{b}) - Δ -system.

Lemma 3.1. *Let $k, d \geq p \geq 2$ and $\vec{a} = (a_1, \dots, a_p)$, $\vec{b} = (b_1, \dots, b_p)$, $\vec{c} = (c_1, \dots, c_p)$ be sequences of positive integers with $\sum_{i=1}^p a_i = k$. Suppose that $c_i \geq b_i + \sum_{j=1}^{i-1} a_j b_j$ for $i \in [p]$. Then every semi- (\vec{a}, \vec{c}) - Δ -system contains an (\vec{a}, \vec{b}) - Δ -system. In particular, if $c_1 \geq 1$ and $c_i \geq kd$ for $2 \leq i \leq p$, then every semi- (\vec{a}, \vec{c}) - Δ -system contains an (\vec{a}, d) - Δ -system.*

Proof. Let $\mathcal{F} = \{E_0, E_1^1, \dots, E_1^{c_1}, \dots, E_p^1, \dots, E_p^{c_p}\}$ be a semi- (\vec{a}, \vec{c}) - Δ -system. Our goal is to choose $\{F_i^1, \dots, F_i^{b_i}\} \subset \{E_i^1, \dots, E_i^{c_i}\}$ for all $i \in [p]$ so that sets $E_0, F_1^1, \dots, F_1^{b_1}, \dots, F_p^1, \dots, F_p^{b_p}$ form a (\vec{a}, \vec{b}) - Δ -system.

Since $c_1 \geq b_1$, we can simply let $F_1^j = E_1^j$ for $j \in [b_1]$. Now suppose that we have defined sets $\{F_1^1, \dots, F_1^{b_1}, \dots, F_i^1, \dots, F_i^{b_i}\}$ for some $i \in [p-1]$. We are going to define sets $F_{i+1}^1, \dots, F_{i+1}^{b_{i+1}}$. Note that for every $1 \leq j \leq i$ and $1 \leq \ell \leq b_j$ the set $F_j^\ell \setminus E_0$ can have nonempty intersection with at most a_j sets in $\{E_{i+1}^1, \dots, E_{i+1}^{c_{i+1}}\}$. Since $c_{i+1} \geq b_{i+1} + \sum_{j=1}^i a_j b_j$, there exist at least b_{i+1} sets in $\{E_{i+1}^1, \dots, E_{i+1}^{c_{i+1}}\}$ that have empty intersection with all sets in $\{F_1^1 \setminus E_0, \dots, F_1^{b_1} \setminus E_0, \dots, F_i^1 \setminus E_0, \dots, F_i^{b_i} \setminus E_0\}$, and choose any b_{i+1} sets from them to form $\{F_{i+1}^1, \dots, F_{i+1}^{b_{i+1}}\}$. The process terminates when $i = p$, and clearly, sets $E_0, F_1^1, \dots, F_1^{b_1}, \dots, F_p^1, \dots, F_p^{b_p}$ form an (\vec{a}, \vec{b}) - Δ -system.

Now suppose that $c_1 \geq 1$ and $c_i \geq kd$ for $2 \leq i \leq p$. Let $b_1 = 1$ and $b_i \geq 1$ for $2 \leq i \leq p$ such that $\sum_{i=2}^p b_i = d - 1$. Since $c_i \geq kd \geq b_i + \sum_{j=1}^{i-1} a_j b_j$, by the previous argument, \mathcal{F} contains an (\vec{a}, \vec{b}) - Δ -system, which is an (\vec{a}, d) - Δ -system. \blacksquare

For a hypergraph \mathcal{H} and $E \in \mathcal{H}$. The intersection structure of E with respect to \mathcal{H} is

$$\mathcal{I}(E, \mathcal{H}) := \{E \cap E' : E' \in \mathcal{H} \setminus \{E\}\}.$$

A hypergraph $\mathcal{H} \subset \binom{[n]}{k}$ is k -partite if there exists a partition $[n] = V_1 \cup \dots \cup V_k$ such that $|E \cap V_i| = 1$ for all $i \in [k]$. Suppose that \mathcal{H} is k -partite with k parts V_1, \dots, V_k . Then for every $S \subset [n]$, its projection is $\Pi(S) := \{i : S \cap V_i \neq \emptyset\}$. For every family $\mathcal{F} \subset 2^{[n]}$, its projection is $\Pi(\mathcal{F}) := \{\Pi(F) : F \in \mathcal{F}\}$.

Definition 3.2. *Let $s \geq 2$. A hypergraph $\mathcal{H} \subset \binom{[n]}{k}$ is s -homogeneous if it satisfies the following conditions.*

- (1) \mathcal{H} is k -partite.
- (2) There exists a family $\mathcal{J} \subset 2^{[k]} \setminus \{[k]\}$ such that $\Pi(\mathcal{I}(E, \mathcal{H})) = \mathcal{J}$ for all $E \in \mathcal{H}$, where \mathcal{J} is called the intersection pattern of \mathcal{H} .
- (3) \mathcal{J} is closed under intersection, i.e. if $A, B \in \mathcal{J}$, then $A \cap B \in \mathcal{J}$.
- (4) For every $E \in \mathcal{H}$ every set in $\mathcal{I}(E, \mathcal{H})$ is the center of a Δ -system \mathcal{D} of size s formed by edges of \mathcal{H} and containing E , i.e. $E \in \mathcal{D} \subset \mathcal{H}$.

A hypergraph $\mathcal{H} \subset \binom{[n]}{k}$ is homogeneous if it is s -homogeneous for some $s \geq 2$.

Füredi [9] showed that for every $s \geq 2$, every hypergraph contains a large s -homogeneous subgraph.

Theorem 3.3 (Füredi [9]). *For every $s, k \geq 2$, there exists a constant $c(k, s) > 0$ such that every hypergraph $\mathcal{H} \subset \binom{[n]}{k}$ contains a s -homogeneous subgraph \mathcal{H}^* with $|\mathcal{H}^*| \geq c(k, s)|\mathcal{H}|$.*

For a family $\mathcal{J} \subset 2^{[k]} \setminus \{[k]\}$ the rank of \mathcal{J} is

$$r(\mathcal{J}) := \min\{|A| : A \subset [k], A \notin \mathcal{J} \text{ and } \exists B \in \mathcal{J} \text{ such that } A \subset B\}.$$

It is easy to see from the definition that $r(\mathcal{J}) = k$ iff $\mathcal{J} = 2^{[k]} \setminus \{[k]\}$.

For a hypergraph $\mathcal{H} \subset \binom{[n]}{k}$ and $1 \leq i \leq k-1$ the i -th shadow of \mathcal{H} is

$$\partial_i \mathcal{H} := \left\{ A \in \binom{[n]}{k-i} : \exists E \in \mathcal{H} \text{ such that } A \subset E \right\}.$$

For convention, let $\partial_0 \mathcal{H} = \mathcal{H}$.

The following lemma gives an upper bound for the size of a homogeneous hypergraph \mathcal{H} in terms of the rank of its intersection pattern and its shadow.

Lemma 3.4. *Let $\mathcal{H} \subset \binom{[n]}{k}$ be a homogeneous hypergraph with intersection pattern $\mathcal{J} \subset 2^{[k]} \setminus \{[k]\}$. Then $|\mathcal{H}| \leq |\partial_{k-r(\mathcal{J})} \mathcal{H}|$.*

Proof. Let $r = r(\mathcal{J})$. By the definition of rank, there exists an r -set $S \subset [k]$ that is not contained in \mathcal{J} , and moreover, every $T \subset [k]$ that contains S is also not contained in \mathcal{J} . Since $\Pi(\mathcal{I}(E, \mathcal{H})) = \mathcal{J}$ for all $E \in \mathcal{H}$, there exists an r -set in every $E \in \mathcal{H}$ that is not contained in any other edges in \mathcal{H} . Therefore, $|\mathcal{H}| \leq |\partial_{k-r} \mathcal{H}|$. ■

The following lemma shows that if a hypergraph is s -homogeneous for sufficiently large s and does not contain an (\vec{a}, d) - Δ -system as a subgraph, then the rank of its intersection pattern is at most $k-1$.

Lemma 3.5. *Let $d \geq p \geq 2$, $k > p$, $s \geq kd+1$, and $\vec{a} = (a_1, \dots, a_p)$ be a sequence of positive integers with $\sum_{i=1}^p a_i = k$. Let $\mathcal{H} \subset \binom{[n]}{k}$ be a s -homogeneous hypergraph with intersection pattern $\mathcal{J} \subset 2^{[k]} \setminus \{[k]\}$. If $r(\mathcal{J}) = k$, then \mathcal{H} contains an (\vec{a}, d) - Δ -system.*

Proof. Since $r(\mathcal{J}) = k$, $\mathcal{J} = 2^{[k]} \setminus \{[k]\}$. Let $E \in \mathcal{H}$ and let $\bigcup_{i=1}^p A_i = E$ be an \vec{a} -partition of E . Since $\Pi(\mathcal{I}(E, \mathcal{H})) = \mathcal{J}$, we have $E \setminus A_i \in \mathcal{I}(E, \mathcal{H})$ for all $i \in [p]$. Since \mathcal{H} is s -homogeneous, there exists a Δ -system \mathcal{D}_i of size s with center $E \setminus A_i$ for $i \in [p]$. By assumption, $s \geq kd+1$, therefore, by Lemma 3.1, \mathcal{H} contains an (\vec{a}, d) - Δ -system. ■

Lemmas 3.5 and 3.4, and Theorem 3.3 implies that following proposition.

Proposition 3.6. *Let $d \geq p \geq 2$, $k > p$, and $\vec{a} = (a_1, \dots, a_p)$ be a sequence of positive integers with $\sum_{i=1}^p a_i = k$. Let $\mathcal{H} \subset \binom{[n]}{k}$ be a hypergraph that contains no (\vec{a}, d) - Δ -systems. Then there exists a constant $c(k, d) > 0$ such that $|\partial \mathcal{H}| \geq c(k, d)|\mathcal{H}|$.*

Proof. Let $s = kd + 1$ and \mathcal{H}^* be a maximum s -homogeneous subgraph of \mathcal{H} with intersection pattern \mathcal{J} . Then by Theorem 3.3, $|\mathcal{H}^*| \geq c(k, d)|\mathcal{H}|$ for some constant $c(k, d) > 0$. Since \mathcal{H}^* contains no (\vec{a}, d) - Δ -systems, by Lemma 3.5, $r(\mathcal{J}) \leq k - 1$. So by Lemma 3.4, $|\mathcal{H}^*| \leq |\partial\mathcal{H}^*|$. Therefore, $|\partial\mathcal{H}| \geq |\partial\mathcal{H}^*| \geq |\mathcal{H}^*| \geq c(k, d)|\mathcal{H}|$. \blacksquare

The next lemma gives another condition that guarantees a hypergraph to contain an (\vec{a}, d) - Δ -system as a subgraph.

Lemma 3.7. *Let $d \geq p \geq 2$, $k > p$, $s \geq kd + 1$, and $\vec{a} = (a_1, \dots, a_p)$ be a sequence of positive integers with $\sum_{i=1}^p a_i = k$. Let $\mathcal{H} \subset \binom{[n]}{k}$ and \mathcal{H}^* be a s -homogeneous subgraph of \mathcal{H} . Let $E_0 \in \mathcal{H}^*$ and $\bigcup_{i \in [p]} A_i = E_0$ be an \vec{a} -partition of E_0 . Suppose that there exists $i_0 \in [p]$ such that $E \setminus A_{i_0} \in \mathcal{I}(E, \mathcal{H}^*)$ for all $i \in [p] \setminus \{i_0\}$, and there exists $E \in \mathcal{H}$ such that $E \cap E_0 = E_0 \setminus A_{i_0}$. Then \mathcal{H} contains an (\vec{a}, d) - Δ -system.*

Proof. Without loss of generality, we may assume that $i_0 = 1$. By assumption, $E \setminus A_i$ is the center of Δ -system of size $s \geq kd + 1$ in $\mathcal{H}^* \subset \mathcal{H}$ for $2 \leq i \leq p$, and $E_0 \setminus A_1$ is the center of a Δ -system of size 2 in \mathcal{H} , i.e. $\{E_0, E\}$. Therefore, by Lemma 3.1, \mathcal{H} contains an (\vec{a}, d) - Δ -system. \blacksquare

Lemma 3.8. *Let $d \geq p \geq 2$, $k > p$, $s \geq kd + 1$, and $\vec{a} = (a_1, \dots, a_p)$ be a sequence of positive integers with $\sum_{i=1}^p a_i = k$. Let $\mathcal{H} \subset \binom{[n]}{k}$ be a hypergraph that does not contain (\vec{a}, d) - Δ -systems. Let \mathcal{H}^* be a s -homogeneous subgraph of \mathcal{H} with intersection pattern \mathcal{J} . Suppose that $r(\mathcal{J}) = k - 1$ and \mathcal{J} contains exactly $k - 1$ $(k - 1)$ -sets. Let $v \in E \in \mathcal{H}^*$ be the vertex that is contained in all $(k - 1)$ -sets in $\mathcal{I}(E, \mathcal{H}^*)$. Then $v \in F$ for all $F \in \mathcal{H}$ that satisfies $|F \cap E| \geq k - a_1$.*

Proof. Let $E = \{v_1, \dots, v_k\} \in \mathcal{H}^*$ and suppose that v_1 is contained in all $(k - 1)$ -sets in $\mathcal{I}(E, \mathcal{H}^*)$. Let $F \in \mathcal{H}$ and suppose that $|E \cap F| = k - t$ for some $1 \leq t \leq a_1$, but $v_1 \notin F$. If $t = a_1$, then let $\bigcup_{i \in [p]} A_i = E$ be an \vec{a} -partition such that $A_1 = E \setminus F$. For $2 \leq i \leq p$ since $v_1 \in E \setminus A_i$, $E \setminus A_i \in \mathcal{I}(E, \mathcal{H}^*)$. Therefore, $E \setminus A_i$ is the center of a Δ -system of size s in \mathcal{H}^* for $2 \leq i \leq p$. Since $E \setminus A_1$ is the center of a Δ -system of size 2, i.e. $\{E, F\}$, by Lemma 3.1, \mathcal{H} contains an (\vec{a}, d) - Δ -system, a contradiction. So, $t < a_1$.

Let $M \subset E$ such that $E \setminus F \subset M$ and $|M| = k - a_1 + t$. Since $|M| \leq k - 1$ and $v_1 \in M$, $M \in \mathcal{I}(E, \mathcal{H}^*)$. Therefore, M is the center of a Δ -system of size s in \mathcal{H}^* , which means that there exists $E_1 \in \mathcal{H}^*$ such that $E_1 \cap E = M$ and $(E_1 \setminus E) \cap F = \emptyset$. This implies that $E_1 \cap F = M \setminus (E \setminus F)$ and $|E_1 \cap F| = k - a_1$. Since $\Pi(\mathcal{I}(E_1, \mathcal{H}^*)) = \Pi(\mathcal{I}(E, \mathcal{H}^*))$, applying the same argument as above to E_1 and F we obtain that \mathcal{H} contains an (\vec{a}, d) - Δ -system, a contradiction. \blacksquare

For a hypergraph \mathcal{H} and $E \in \mathcal{H}$ the weight of E is

$$\omega_{\mathcal{H}}(E) := \sum_{E' \subset E, |E'|=k-1} \frac{1}{\deg_{\mathcal{H}}(E')},$$

where $\deg_{\mathcal{H}}(E')$ is the number of edges in \mathcal{H} containing E' . We have the following identity:

$$\sum_{E \in \mathcal{H}} \omega_{\mathcal{H}}(E) = \sum_{E \in \mathcal{H}} \sum_{E' \subset E, |E'|=k-1} \frac{1}{\deg_{\mathcal{H}}(E')} = \sum_{E' \in \partial\mathcal{H}} \sum_{E \in \mathcal{H}, E' \subset E} \frac{1}{\deg_{\mathcal{H}}(E')} = |\partial\mathcal{H}|. \quad (1)$$

The following lemma gives a lower bound for $\omega_{\mathcal{H}}(E)$ regarding the intersection structure of E in a homogeneous subgraph of \mathcal{H} .

Lemma 3.9. *Let $d \geq p \geq 2$, $k > p$, $s \geq kd + 1$, and $\vec{a} = (a_1, \dots, a_p)$ be a sequence of positive integers with $\sum_{i=1}^p a_i = k$. Let $\mathcal{H} \subset \binom{[n]}{k}$ be a hypergraph that does not contain (\vec{a}, d) - Δ -systems. Let \mathcal{H}^* be a s -homogeneous subgraph of \mathcal{H} with intersection pattern \mathcal{J} . Suppose that $r(\mathcal{J}) = k - 1$. Then the followings hold.*

- (1) *If \mathcal{J} contains exactly $k - 1$ $(k - 1)$ -sets, then every $E \in \mathcal{H}^*$ contains a $(k - 1)$ -subset that is not contained in any other edges in \mathcal{H} . In particular, $\omega_{\mathcal{H}}(E) \geq 1$ for all $E \in \mathcal{H}^*$.*
- (2) *If \mathcal{J} contains at most $k - 2$ $(k - 1)$ -sets, then $\omega_{\mathcal{H}}(E) \geq \frac{k}{k-1}$ for all $E \in \mathcal{H}^*$.*

Proof. We prove (1) first. We may assume that $a_1 \geq \dots \geq a_k$, and note that $a_1 \geq 2$ since $\sum_{i=1}^p a_i = k > p$. Let $E = \{v_1, \dots, v_k\} \in \mathcal{H}^*$. Since $\Pi(\mathcal{I}(E, \mathcal{H}^*)) = \mathcal{J}$, by assumption, there are exactly $k - 1$ $(k - 1)$ -sets in $\mathcal{I}(E, \mathcal{H}^*)$. Without loss of generality we may assume that $E \setminus \{v_i\} \in \mathcal{I}(E, \mathcal{H}^*)$ for $2 \leq i \leq k$. We claim that $\{v_2, \dots, v_k\}$ is not contained in any set in $\mathcal{H} \setminus \{E\}$. Indeed, if there exists $E_1 \in \mathcal{H}$ such that $\{v_2, \dots, v_k\} \subset E_1$, then $|E_1 \cap E| \geq k - 1$. So, by Lemma 3.8, $v_1 \in E_1$, which implies that $E_1 = E$.

Now we prove (2). Suppose that \mathcal{J} has exactly $k - t$ $(k - 1)$ -sets for some $2 \leq t \leq k$. Let $E = \{v_1, \dots, v_k\} \in \mathcal{H}^*$. Without loss of generality, we may assume that $E \setminus \{v_i\} \in \mathcal{I}(E, \mathcal{H}^*)$ for $t + 1 \leq i \leq k$.

Claim 3.10. *There does not exist a $(t - 1)$ -set $I \subset [t]$ and $t - 1$ distinct vertices $\{u_i : i \in I\}$, such that $(E \setminus \{v_i\}) \cup \{u_i\} \in \mathcal{H}$ for all $i \in I$.*

Proof of Claim 3.10. Suppose not, and without loss of generality we may assume that $F_i := (E \setminus \{v_i\}) \cup \{u_i\} \in \mathcal{H}$ for all $2 \leq i \leq t$, where u_2, \dots, u_t are distinct vertices.

By assumption $\mathcal{I}(E, \mathcal{H}^*)$ contains all $(k - 1)$ -sets that contain $\{v_1, \dots, v_t\}$. Since $\mathcal{I}(E, \mathcal{H}^*)$ is closed under intersection, $\mathcal{I}(E, \mathcal{H}^*)$ contains all proper subsets of E that contain $\{v_1, \dots, v_t\}$, i.e. if $A \subset \{v_{t+1}, \dots, v_k\}$, then $E \setminus A \in \mathcal{I}(E, \mathcal{H}^*)$.

On the other hand, since $r(\mathcal{I}(E, \mathcal{H}^*)) = k - 1 \geq k - 2$ and $E \setminus \{v_i\}, E \setminus \{v_j\} \notin \mathcal{I}(E, \mathcal{H}^*)$ for $i, j \in [t]$, we have $E \setminus \{v_i, v_j\} \in \mathcal{I}(E, \mathcal{H}^*)$ for all $\{i, j\} \subset [t]$. This together with the previous argument and the property that $\mathcal{I}(E, \mathcal{H}^*)$ is closed under intersection imply that if $|A \cap \{v_1, \dots, v_t\}| \geq 2$, then $E \setminus A \in \mathcal{I}(E, \mathcal{H}^*)$.

Let $i_0 \in [p]$ such that $\sum_{i=1}^{i_0-1} a_i < t \leq \sum_{i=1}^{i_0} a_i$, and let $\ell = t - \sum_{i=1}^{i_0-1} a_i$. Recall that $a_1 \geq \dots \geq a_p \geq 1$ and $a_1 \geq 2$. Suppose that $\ell \geq 2$. Then there exists an \vec{a} -partition $E = \bigcup_{i \in [p]} A_i$ such that $A_1, \dots, A_{i_0-1} \subset \{v_1, \dots, v_t\}$, $|A_{i_0} \cap \{v_1, \dots, v_t\}| \geq \ell \geq 2$, and $A_{i_0+1}, \dots, A_p \subset \{v_{t+1}, \dots, v_k\}$. Since $a_1 \geq \dots \geq a_{i_0-1} \geq a_{i_0} \geq 2$, by the argument above, $E \setminus A_i \in \mathcal{I}(E, \mathcal{H}^*)$ for all $i \in [p]$. Therefore, $E \setminus A_i$ is the center of a Δ -system of size $s \geq kd + 1$ in \mathcal{H}^* for $i \in [p]$, so by Lemma 3.1, \mathcal{H} contains an (\vec{a}, d) - Δ -system, a contradiction. Therefore, $\ell = 1$.

Suppose that $a_{i_0} = 1$. Then let $E = \bigcup_{i \in [p]} A_i$ be an \vec{a} -partition such that $\bigcup_{i \in [i_0]} A_i = \{v_1, \dots, v_t\}$ and $v_1 \in A_1$. Since $A_1 \subset \{v_1, \dots, v_t\}$ and $|A_1| \geq 2$, $E \setminus A_1 \in \mathcal{I}(E, \mathcal{H}^*)$. So $E \setminus A_1$ is the center of a Δ -system of size s . Without loss of generality we may assume that $a_2 = \dots = a_{i_0} = 1$ since other cases can be proved similarly. For $i_0 + 1 \leq i \leq p$ since $A_i \subset \{v_{t+1}, \dots, v_k\}$, $E \setminus A_i \in \mathcal{I}(E, \mathcal{H}^*)$. So $E \setminus A_i$ is the center of a Δ -system of size s for $i_0 + 1 \leq i \leq p$. Notice that by assumption for every $2 \leq i \leq i_0$ there exists $F_{j_i} \in \mathcal{H}$ such that $F_{j_i} \cap E = E \setminus \{v_{j_i}\}$ for $2 \leq j_i \leq t$, and moreover, $\{F_{j_i} \setminus E : 2 \leq i \leq i_0\}$ are distinct. Therefore, by a similar argument as in the proof of Lemma 3.1, \mathcal{H} contains an (\vec{a}, d) - Δ -system, a contradiction. Therefore, $a_{i_0} \geq 2$.

Suppose that $a_1 \geq 3$. Then let $E = \bigcup_{i \in [p]} A_i$ be an \vec{a} -partition such that $A_2 \cup \dots \cup A_{i_0-1} \subset \{v_1, \dots, v_t\}$, $v_{t+1} \in A_1$ and $A_1 \setminus \{v_{t+1}\} \subset \{v_1, \dots, v_t\}$, and $\{v_1, \dots, v_t\} \setminus \left(\bigcup_{i \in [i_0-1]} A_i\right) \subset A_{i_0}$. Then $|A_i \cap \{v_1, \dots, v_t\}| \geq 2$ for all $i \in [i_0]$ and $A_j \subset \{v_{t+1}, \dots, v_k\}$ for all $i_0+1 \leq j \leq p$. Therefore, $E \setminus A_i$ is the center of a Δ -system of size $s \geq kd+1$ in \mathcal{H}^* for $i \in [p]$, so by Lemma 3.1, \mathcal{H} contains an (\vec{a}, d) - Δ -system, a contradiction. Therefore, $a_1 = 2$.

Suppose that $a_p = 1$. Then let $E = \bigcup_{i \in [p]} A_i$ be an \vec{a} -partition such that $A_1 \cup \dots \cup A_{i_0-1} = \{v_1, \dots, v_{t-1}\}$ and $A_p = \{v_t\}$. Then $E \setminus A_i$ is the center of a Δ -system of size $s \geq kd+1$ in \mathcal{H}^* for $i \in [p-1]$ and $E \setminus A_p$ is the center of a Δ -system of size 2 in \mathcal{H} , i.e. $\{E, F_t\}$. Therefore, by Lemma 3.1, \mathcal{H} contains an (\vec{a}, d) - Δ -system, a contradiction. Therefore, $a_p = 2$.

Now we have $a_1 = \dots = a_p = 2$ and $\ell = 1$. Then t is odd, $t \geq 3$, and k is even, $k > t$. Since $E \setminus \{v_k\} \in \mathcal{I}(E, \mathcal{H}^*)$, $E \setminus \{v_k\}$ is the center of a Δ -system of size s in \mathcal{H}^* . So there exists $F_k := (E \setminus \{v_k\}) \cup \{u_k\} \in \mathcal{H}^*$ such that $u_k \notin \{u_2, \dots, u_t\}$. Let $F_k = \bigcup_{i \in [p]} A_i$ be an \vec{a} -partition such that $A_1 = \{v_2, u_k\}$, $A_2 = \{v_1, v_3\}$, and $A_i = \{v_{2i-2}, v_{2i-1}\}$ for $3 \leq i \leq p$. Then for every $i \in [p] \setminus \{1\}$, either $A_i \subset \{v_1, \dots, v_t\}$ or $A_i \subset \{v_{t+1}, \dots, v_{k-1}\}$. Since $\Pi(\mathcal{I}(F_k, \mathcal{H}^*)) = \Pi(\mathcal{I}(E, \mathcal{H}^*))$, $F_k \setminus A_i$ is the center of a Δ -system of size $s \geq kd+1$ in \mathcal{H}^* for $i \in [p] \setminus \{1\}$. Since $V \setminus A_1$ is the center of a Δ -system of size 2 in \mathcal{H} , i.e. $\{F_k, F_2\}$. Therefore, by Lemma 3.1, \mathcal{H} contains an (\vec{a}, d) - Δ -system, a contradiction. This completes the proof of Claim 3.10. \blacksquare

Define a bipartite graph G with two parts $L = \{v_1, \dots, v_t\}$ and $R = [n] \setminus E$, and for every $v_i \in L$ and $u \in R$, $v_i u$ is an edge in G iff $(E \setminus v_i) \cup \{u\} \in \mathcal{H}$. Claim 3.10 implies that there are at most $t-2$ pairwise disjoint edges in G . Therefore, by the König-Hall theorem, G contains a vertex cover S with $|S| \leq t-2$. Let $\ell = |L \setminus S| \geq 2$. Then $|S \cap R| \leq \ell-2$. For every $v \in L \setminus S$ since $N_G(v) \subset S \cap R$, we obtain

$$\deg_{\mathcal{H}}(E \setminus \{v\}) = \deg_G(v) + 1 \leq \ell - 1,$$

which implies that

$$\omega_{\mathcal{H}}(E) = \sum_{E' \subset E, |E'|=k-1} \frac{1}{\deg_{\mathcal{H}}(E')} > \sum_{v \in L \setminus S} \frac{1}{\deg_{\mathcal{H}}(E \setminus \{v\})} \geq \frac{\ell}{\ell-1} \geq \frac{k}{k-1}.$$

This completes the proof of Lemma 3.9. \blacksquare

4 Proofs

In this section we prove Theorems 1.6 and 1.8. First, let us prove Theorem 1.8.

Proof of Theorem 1.8. Let $k > p \geq 2$, $d \geq p$, and $\vec{a} = (a_1, \dots, a_p)$ be a sequence of integers such that $a_1 \geq \dots \geq a_p \geq 1$ and $\sum_{i \in [p]} a_i = k$. Let $\epsilon > 0$ and n be sufficiently large. Let $\mathcal{H} \subset \binom{[n]}{k}$ be a hypergraph that contains no (\vec{a}, d) - Δ -systems and $|\mathcal{H}| \geq (1 - \epsilon) \binom{n-1}{k-1}$.

Let $s = kd+1$ and let \mathcal{H}_1 be a maximum s -homogeneous subgraph of \mathcal{H} . Suppose now we have defined $\mathcal{H}_1, \dots, \mathcal{H}_i$ for some $i \geq 1$. Let \mathcal{H}_{i+1} be the maximum s -homogeneous subgraph of $\mathcal{H} \setminus \left(\bigcup_{j=1}^i \mathcal{H}_j\right)$. This process terminates if $\mathcal{H} \setminus \left(\bigcup_{j=1}^m \mathcal{H}_j\right) = \emptyset$ or the intersection pattern of \mathcal{H}_{m+1} has rank at most $k-2$ for some $m \geq 1$. Let \mathcal{J}_i denote the intersection

pattern of \mathcal{H}_i for $i \in [m]$, and note that by definition and Lemma 3.5, $r(\mathcal{J}_i) = k - 1$ for $i \in [m]$. Let

$$\begin{aligned}\widehat{\mathcal{H}}_1 &= \bigcup_i \{\mathcal{H}_i : i \in [m] \text{ and } \mathcal{J}_i \text{ contains exactly } k - 1 \text{ } (k - 1)\text{-sets}\}, \\ \widehat{\mathcal{H}}_2 &= \bigcup_i \{\mathcal{H}_i : i \in [m] \text{ and } \mathcal{J}_i \text{ contains at most } k - 2 \text{ } (k - 1)\text{-sets}\}, \\ \widehat{\mathcal{H}}_3 &= \mathcal{H} \setminus (\widehat{\mathcal{H}}_1 \cup \widehat{\mathcal{H}}_2) = \mathcal{H} \setminus \left(\bigcup_{i \in [m]} \mathcal{H}_i \right).\end{aligned}$$

Our first step is to show that the sizes of $\widehat{\mathcal{H}}_2$ and $\widehat{\mathcal{H}}_3$ are small.

Claim 4.1. $|\widehat{\mathcal{H}}_2| + |\widehat{\mathcal{H}}_3| < 3\epsilon k \binom{n-1}{k-1}$.

Proof of Claim 4.1. First we show that $\widehat{\mathcal{H}}_3 = O(n^{k-2})$. We may assume that $\widehat{\mathcal{H}}_3 \neq \emptyset$. Recall that \mathcal{H}_{m+1} is a maximum s -homogeneous subgraph of $\widehat{\mathcal{H}}_3$ with intersection pattern \mathcal{J}_{m+1} . By Theorem 3.3, there exists a constant $c(k, s) > 0$ such that $|\mathcal{H}_{m+1}| \geq c(k, s)|\widehat{\mathcal{H}}_3|$. By definition, $r(\mathcal{J}_{m+1}) \leq k - 2$, so by Lemma 3.4, $|\mathcal{H}_3| \leq |\partial_2 \mathcal{H}_3| \leq \binom{n}{k-2}$. Therefore, $|\widehat{\mathcal{H}}_3| \leq \frac{1}{c(k, s)} \binom{n}{k-2}$.

Next we show that $\widehat{\mathcal{H}}_2 = O(n^{k-2})$. By Lemma 3.7 and Equation (1),

$$|\partial \mathcal{H}| = \sum_{E \in \mathcal{H}} \omega_{\mathcal{H}}(E) = \sum_{E \in \widehat{\mathcal{H}}_1} \omega_{\mathcal{H}}(E) + \sum_{E \in \widehat{\mathcal{H}}_2} \omega_{\mathcal{H}}(E) \geq |\widehat{\mathcal{H}}_1| + \frac{k}{k-1} |\widehat{\mathcal{H}}_2|.$$

Therefore, $|\widehat{\mathcal{H}}_1| + \frac{k}{k-1} |\widehat{\mathcal{H}}_2| \leq \binom{n}{k-1}$, which implies that

$$\begin{aligned}|\widehat{\mathcal{H}}_2| &= (k-1) \left(|\widehat{\mathcal{H}}_1| + \frac{k}{k-1} |\widehat{\mathcal{H}}_2| + |\widehat{\mathcal{H}}_3| - |\mathcal{H}| \right) \\ &\leq (k-1) \left(\binom{n}{k-1} + |\widehat{\mathcal{H}}_3| - (1-\epsilon) \binom{n-1}{k-1} \right) < 2\epsilon k \binom{n-1}{k-1}.\end{aligned}$$

This completes the proof of Claim 4.1. ■

Note that the proof of Claim 4.1 also shows that

$$|\mathcal{H}| \leq |\widehat{\mathcal{H}}_1| + \frac{k}{k-1} |\widehat{\mathcal{H}}_2| + |\widehat{\mathcal{H}}_3| \leq \binom{n}{k-1} + O(n^{k-2}). \quad (2)$$

Claim 4.1 implies that

$$|\widehat{\mathcal{H}}_1| = |\mathcal{H}| - (|\widehat{\mathcal{H}}_2| + |\widehat{\mathcal{H}}_3|) > (1 - 4\epsilon k) \binom{n-1}{k-1}. \quad (3)$$

By definition, for every $E \in \widehat{\mathcal{H}}_1$ there exists a unique s -homogeneous hypergraph \mathcal{H}_i for some i such that $E \in \mathcal{H}_i$, moreover, $r(\mathcal{J}_i) = k - 1$ and \mathcal{J}_i contains exactly $k - 1$ $(k - 1)$ -sets. Therefore, $\mathcal{I}(E, \mathcal{H}_i)$ contains a unique vertex $c \in E$ such that every $(k - 1)$ -subset of E that contains c is contained in $\mathcal{I}(E, \mathcal{H}_i)$. Let $c(E)$ denote this unique vertex c for every $E \in \widehat{\mathcal{H}}_1$. Define $\mathcal{G}_i = \{E \in \widehat{\mathcal{H}}_1 : c(E) = i\}$ for $i \in [n]$, and notice that $\bigcup_{i \in [n]} \mathcal{G}_i = \widehat{\mathcal{H}}_1$ is a partition. Let $\mathcal{G}_i(i) = \{E \setminus \{i\} : E \in \mathcal{G}_i\}$ for $i \in [n]$. From the proof of Lemma 3.9 (1), for every $i \in [n]$ and every $E \in \mathcal{G}_i$ the set $E \setminus \{i\}$ is not contained in any set in $\mathcal{H} \setminus \{E\}$. Therefore, $\mathcal{G}_i(i) \cap \mathcal{G}_j(j) = \emptyset$ for all $\{i, j\} \subset [n]$.

Claim 4.2. $\partial\mathcal{G}_i(i) \cap \partial\mathcal{G}_j(j) = \emptyset$ for all $\{i, j\} \subset [n]$.

Proof of Claim 4.2. Suppose not. Without loss of generality we may assume that there exists $A \in \partial\mathcal{G}_1(1) \cap \partial\mathcal{G}_2(2)$. Then there exists $E_1 \in \mathcal{G}_1$ and $E_2 \in \mathcal{G}_2$ such that $E_1 = \{1, u\} \cup A$ and $E_2 = \{2, v\} \cup A$ for some $u, v \in [n]$. Since \mathcal{G}_1 is s -homogeneous and $|E_2 \cap E_1| \geq k-2 \geq k-a_1$, by Lemma 3.8, $1 \in E_2$. Similarly, we obtain $2 \in E_1$. Therefore, $E_1 = E_2 = \{1, 2\} \cup A$, which implies that $\{1, 2\} \cup A \in \mathcal{G}_1 \cap \mathcal{G}_2$, a contradiction. \blacksquare

Let $x_i \in \mathbb{R}$ such that $|\mathcal{G}_i| = |\mathcal{G}_i(i)| = \binom{x_i}{k-1}$ for $i \in [n]$. Without loss of generality we may assume that $x_1 \geq \dots \geq x_n \geq 0$. By the Kruskal-Katona theorem (e.g. see [16]),

$$|\mathcal{G}_i(i)| \leq \frac{\binom{x_i}{k-1}}{\binom{x_i}{k-2}} |\partial\mathcal{G}_i(i)| = \frac{x_i - k + 2}{k-1} |\partial\mathcal{G}_i(i)|,$$

for $i \in [n]$. Therefore by (3) and Claim 4.2,

$$\begin{aligned} (1 - 4\epsilon k) \binom{n-1}{k-1} &< |\widehat{\mathcal{H}}_1| = \sum_{i \in [n]} |\mathcal{G}_i| = \sum_{i \in [n]} |\mathcal{G}_i(i)| \leq \sum_{i \in \mathcal{H}} \frac{x_i - k + 2}{k-1} |\partial\mathcal{G}_i(i)| \\ &\leq \frac{x_1 - k + 2}{k-1} \sum_{i \in \mathcal{H}} |\partial\mathcal{G}_i(i)| \leq \frac{x_1 - k + 2}{k-1} \binom{n}{k-2}, \end{aligned}$$

which implies that

$$x_1 \geq (k-1) \frac{(1 - 4\epsilon k) \binom{n-1}{k-1}}{\binom{n}{k-2}} + k - 2 > (1 - 5\epsilon k)n.$$

Therefore,

$$|\mathcal{G}_1| = \binom{x_1}{k-1} > \binom{(1 - 5\epsilon k)n}{k-1} > (1 - 5\epsilon k^2) \binom{n-1}{k-1},$$

which together with (2) implies that all but at most $5\epsilon k^2 n^{k-1}$ edges in \mathcal{H} contain the vertex 1. \blacksquare

Now we prove Theorem 1.6.

Proof of Theorem 1.6. Let $d \geq p \geq 2$, $k > p$, $s = kd + 1$, and $\vec{a} = (a_1, \dots, a_p)$ be a sequence of positive integers such that $a_1 \geq \dots \geq a_p$ and $\sum_{i \in [p]} a_i = k$. Let $n \geq n_0(k, d)$ be sufficiently large. Let $\mathcal{H} \subset \binom{[n]}{k}$ be a hypergraph that contains no (\vec{a}, d) - Δ -systems and $|\mathcal{H}| = \binom{n-1}{k-1}$. It suffices to show that all edges in \mathcal{H} contain a fixed vertex.

From the proof of Theorem 1.8 we know that \mathcal{H} contains a subgraph \mathcal{G}_1 such that all edges in \mathcal{G}_1 contains a fixed vertex (we may assume that this vertex is 1), moreover, \mathcal{G}_1 consists of pairwise edge-disjoint s -homogeneous hypergraphs whose intersection patterns have rank $k-1$ and contain all $(k-1)$ -subsets of $[k]$ that contain 1.

Define

$$\mathcal{B}_0 = \{E \in \mathcal{H} : 1 \notin E\},$$

$$\mathcal{B}_1 = \{E \in \mathcal{H} : 1 \in E \text{ and } |E \cap B| \geq k - a_1 \text{ for some } B \in \mathcal{B}_0\},$$

$$\mathcal{G} = \{E \in \mathcal{H} \setminus \mathcal{B}_1 : 1 \in E, \forall S \subset E \text{ with } 1 \in S \text{ is the center of a } \Delta\text{-system in } \mathcal{H} \text{ of size } s\},$$

$$\mathcal{B}_2 = \{E \in \mathcal{H} : 1 \in E\} \setminus (\mathcal{B}_1 \cup \mathcal{G}).$$

Note that $\mathcal{G}_1 \subset \mathcal{G}$. Let

$$\mathcal{B}_1(1) = \{E \setminus 1 : E \in \mathcal{B}_1\}, \quad \mathcal{G}(1) = \{E \setminus 1 : E \in \mathcal{G}\}, \quad \text{and} \quad \mathcal{B}_2(1) = \{E \setminus 1 : E \in \mathcal{B}_2\}.$$

Let $\mathcal{B}_1^*(1), \mathcal{B}_2^*(1)$ be maximum s -homogeneous subgraphs of $\mathcal{B}_1(1), \mathcal{B}_2(1)$, respectively. Then by Theorem 3.3, $|\mathcal{B}_i^*(1)| \geq c(k, s)|\mathcal{B}_i(1)|$ for some constant $c(k, s) > 0$ and $i = 1, 2$. Recall that for every $E \in \partial\mathcal{G}(1)$, $\deg_{\mathcal{G}(1)}(E)$ is the number of edges in $\mathcal{G}(1)$ that contain E . Since $\sum_{E \in \partial\mathcal{G}(1)} \deg_{\mathcal{G}(1)}(E) = (k-1)|\mathcal{G}(1)|$ and $\deg_{\mathcal{G}(1)}(E) \leq n-k+1$, we have

$$|\partial\mathcal{G}(1)| \geq \frac{k-1}{n-k+1}|\mathcal{G}(1)|. \quad (4)$$

Claim 4.3. $|\mathcal{G}| + 4|\mathcal{B}_0| \leq \binom{n-1}{k-1}$.

Proof of Claim 4.3. Notice that by definition $|E \cap B| \leq k - a_1 - 1 \leq k - 3$ for all $E \in \mathcal{G}(1)$ and $B \in \mathcal{B}_0$. Therefore, $\partial\mathcal{G}(1) \cap \partial_2\mathcal{B}_0 = \emptyset$, and hence $|\partial\mathcal{G}(1)| + |\partial_2\mathcal{B}_0| \leq \binom{n-1}{k-2}$. Let $x \in \mathbb{R}$ such that $|\partial\mathcal{B}_0| = \binom{x}{k-1}$, then by the Kruskal-Katona theorem and Proposition 3.6,

$$|\partial_2\mathcal{B}_0| \geq \frac{k-1}{x-k+1}|\partial\mathcal{B}_0| \geq \frac{k-1}{x-k+1}c(k, s)|\mathcal{B}_0|.$$

Therefore, together with (4) we obtain

$$\frac{k-1}{n-k+1}|\mathcal{G}(1)| + \frac{k-1}{x-k+1}c(k, s)|\mathcal{B}_0| \leq \binom{n-1}{k-2},$$

which implies $|\mathcal{G}| + c(k, s)\frac{n-k+1}{x-k+1}|\mathcal{B}_0| \leq \binom{n-1}{k-1}$. By Theorem 1.8, $\binom{x}{k-1} = |\partial\mathcal{B}_0| \leq k|\mathcal{B}_0| \leq \delta n^{k-1}$ for all sufficiently small $\delta > 0$ (as long as n is sufficiently large), so $x < \delta' n$ for some sufficiently small $\delta' > 0$ (depending on δ). Choosing $\delta' \ll c(k, s)$ we obtain $c(k, s)\frac{n-k+1}{\delta' n - k + 1} > 4$, this completes the proof of Claim 4.3. \blacksquare

Claim 4.4. Every $E \in \mathcal{B}_1^*(1)$ has a $(k-2)$ -subset that is not contained in any other set in $\mathcal{B}_1^*(1) \cup \mathcal{G}'$.

Proof of Claim 4.4. Suppose not. Let $E = \{v_1, \dots, v_{k-1}\} \in \mathcal{B}_1^*(1)$ such that $E \setminus \{v_i\}$ is contained in some set in $\mathcal{B}_1^*(1) \cup \mathcal{G}(1)$ for $1 \leq i \leq k-1$. Without loss of generality we may assume that $E \setminus \{v_i\} \in \mathcal{I}(E, \mathcal{G}(1))$ for $1 \leq i \leq \ell$, and $E \setminus \{v_i\} \in \mathcal{I}(E, \mathcal{B}_1^*(1))$ for $\ell+1 \leq i \leq k-1$.

Let $\mathcal{J}_{\mathcal{B}_1^*(1)}$ be the intersection pattern of $\mathcal{B}_1^*(1)$. Let $\mathcal{B}_1^* = \{E \cup \{1\} : E \in \mathcal{B}_1^*(1)\}$, and note that \mathcal{B}_1^* is also s -homogeneous with intersection pattern $\mathcal{J}_{\mathcal{B}_1^*} := \{A \cup \{1\} : A \in \mathcal{J}_{\mathcal{B}_1^*(1)}\}$. Let $\widehat{E} = E \cup \{1\} \in \mathcal{B}_1^*$.

If $\ell = 0$, then $\mathcal{J}_{\mathcal{B}_1^*(1)} = 2^{[k-1]} \setminus \{[k-1]\}$, and hence $r(\Pi(\mathcal{I}(\widehat{E}, \mathcal{B}_1^*))) = k-1$ and $\Pi(\mathcal{I}(\widehat{E}, \mathcal{B}_1^*))$ contains all $(k-1)$ -subsets of \widehat{E} that contain 1. By definition there exists $B \in \mathcal{B}_0$ such that $|B \cap \widehat{E}| \geq k - a_1$. However, by Lemma 3.8, $1 \in B$, a contradiction. Therefore, $\ell \geq 1$.

Let $E_i \in \mathcal{G}$ such that $E_i \cap \widehat{E} = \widehat{E} \setminus \{v_i\}$ for $1 \leq i \leq \ell$. Let $B \in \mathcal{B}_0$ such that $|B \cap \widehat{E}| \geq k - a_1$ and suppose that $|B \cap \widehat{E}| = k - t$ for some $1 \leq t \leq a_1$. Then for $1 \leq i \leq \ell$ we have $|B \cap E_i| \geq k - t - 1$. However, by the definition of \mathcal{G} , $|B \cap E_i| \leq k - a_1 - 1$ for $1 \leq i \leq \ell$. Therefore, $|B \cap \widehat{E}| = k - a_1$ and $v_i \in B$ for all $1 \leq i \leq \ell$. Let $\bigcup_{i \in [p]} A_i = \widehat{E}$ be an \vec{a} -partition such that $A_1 = \widehat{E} \setminus B$. Note that for $2 \leq i \leq p$, either $1 \in \widehat{E} \setminus A_i \subset E_{j_i}$ for some $1 \leq j_i \leq \ell$, which by the definition of \mathcal{G} , is the center of some Δ -system of size s in \mathcal{H} , or $\{1, v_1, \dots, v_\ell\} \subset \widehat{E} \setminus A_i$, which implies that $\widehat{E} \setminus A_i \in \mathcal{I}(\widehat{E}, \mathcal{B}_1^*)$ and hence is the center of some Δ -system of size s in \mathcal{B}_1^* . Note that $E \setminus A_1$ is the center of a Δ -system of size 2, i.e. $\{\widehat{E}, B\}$. Therefore, by Lemma 3.1, \mathcal{H} contains an (\vec{a}, d) - Δ -system, a contradiction. \blacksquare

By Claim 4.4, we obtain $|\partial\mathcal{G}(1)| + |\mathcal{B}_1^*(1)| \leq \binom{n-1}{k-2}$, which implies $|\mathcal{G}| + c(k, s) \frac{n-k+1}{k-1} |\mathcal{B}_1| \leq \binom{n-1}{k-1}$. Note that $c(k, s) \frac{n-k+1}{k-1} \gg 1$, so

$$|\mathcal{G}| + 4|\mathcal{B}_1| \leq \binom{n-1}{k-1}.$$

Claim 4.5. *Every $E \in \mathcal{B}_2^*(1)$ has a $(k-2)$ -subset that is not contained in any other set in $\mathcal{B}_2^*(1) \cup \mathcal{G}'$.*

Proof of Claim 4.5. Suppose not. Let $E = \{v_1, \dots, v_{k-1}\} \in \mathcal{B}_2^*(1)$ such that $E \setminus \{v_i\}$ is contained in some set in $\mathcal{B}_2^*(1) \cup \mathcal{G}(1)$ for $1 \leq i \leq k-1$. Without loss of generality we may assume that $E \setminus \{v_i\} \in \mathcal{I}(E, \mathcal{G}(1))$ for $1 \leq i \leq \ell$, and $E \setminus \{v_i\} \in \mathcal{I}(E, \mathcal{B}_2^*(1))$ for $\ell+1 \leq i \leq k-1$.

Let $\mathcal{J}_{\mathcal{B}_2^*(1)}$ be the intersection pattern of $\mathcal{B}_2^*(1)$. Let $\mathcal{B}_2^* = \{E \cup \{1\} : E \in \mathcal{B}_2^*(1)\}$, and note that \mathcal{B}_2^* is also s -homogeneous with intersection pattern $\mathcal{J}_{\mathcal{B}_2^*} := \{A \cup \{1\} : A \in \mathcal{J}_{\mathcal{B}_2^*(1)}\}$. Let $\widehat{E} = E \cup \{1\} \in \mathcal{B}_2^*$.

If $\ell = 0$, then $\mathcal{J}_{\mathcal{B}_2^*(1)} = 2^{[k-1]} \setminus \{[k-1]\}$, and hence $r(\Pi(\mathcal{I}(\widehat{E}, \mathcal{B}_2^*))) = k-1$ and $\Pi(\mathcal{I}(\widehat{E}, \mathcal{B}_2^*))$ contains all $(k-1)$ -subsets of \widehat{E} that contain 1. Since $\mathcal{I}(\widehat{E}, \mathcal{B}_2^*)$ is closed under intersection, all proper subsets of \widehat{E} that contain 1 is contained in $\mathcal{I}(\widehat{E}, \mathcal{B}_2^*)$, which by definition, implies that $\widehat{E} \in \mathcal{G}$, a contradiction. Therefore, $\ell \geq 1$.

Let $E_i \in \mathcal{G}$ such that $E_i \cap \widehat{E} = \widehat{E} \setminus \{v_i\}$ for $1 \leq i \leq \ell$. For every proper subset $S \subset \widehat{E}$ with $1 \in S$, if $v_i \notin S$ for some $1 \leq i \leq \ell$, then $S \subset E_i$, which, by the definition of \mathcal{G} , means that S is the center of some Δ -system of size s in \mathcal{H} . If $\{v_1, \dots, v_\ell\} \subset S$, then $S \in \mathcal{I}(\widehat{E}, \mathcal{B}_2^*)$ and hence S is the center of some Δ -system of size s in \mathcal{B}_2^* . Therefore, every proper subset $S \subset \widehat{E}$ with $1 \in S$ is the center of some Δ -system of size s in \mathcal{H} , which by definition, implies that $\widehat{E} \in \mathcal{G}$, a contradiction. \blacksquare

Similarly, we obtain

$$|\mathcal{G}| + 4|\mathcal{B}_2| \leq \binom{n-1}{k-1}.$$

Therefore, by the assumption that $|\mathcal{H}| = \binom{n-1}{k-1}$ we obtain

$$\begin{aligned} 3 \binom{n-1}{k-1} &\leq 3|\mathcal{H}| + |\mathcal{B}_0| + |\mathcal{B}_1| + |\mathcal{B}_2| \\ &= |\mathcal{G}| + 4|\mathcal{B}_0| + |\mathcal{G}| + 4|\mathcal{B}_1| + |\mathcal{G}| + 4|\mathcal{B}_2| \leq 3 \binom{n-1}{k-1}, \end{aligned}$$

which implies that $|\mathcal{G}| = \binom{n-1}{k-1}$ and $\mathcal{B}_0 = \mathcal{B}_1 = \mathcal{B}_2 = \emptyset$. This completes the proof of Theorem 1.6. \blacksquare

References

- [1] J.-C. Bermond and P. Frankl. On a conjecture of Chvátal on m -intersecting hypergraphs. *Bull. London Math. Soc.*, 9(3):310–312, 1977.

- [2] V. Chvátal. An extremal set-intersection theorem. *J. London Math. Soc. (2)*, 9:355–359, 1974/75.
- [3] R. Csákány and J. Kahn. A homological approach to two problems on finite sets. *J. Algebraic Combin.*, 9(2):141–149, 1999.
- [4] G. Currier. New results on simplex-clusters in set systems. *arXiv preprint arXiv:2001.01812*, 2020.
- [5] P. Frankl. On Sperner families satisfying an additional condition. *J. Combinatorial Theory Ser. A*, 20(1):1–11, 1976.
- [6] P. Frankl. On a problem of Chvátal and Erdős on hypergraphs containing no generalized simplex. *J. Combin. Theory Ser. A*, 30(2):169–182, 1981.
- [7] P. Frankl and Z. Füredi. Forbidding just one intersection. *J. Combin. Theory Ser. A*, 39(2):160–176, 1985.
- [8] P. Frankl and Z. Füredi. Exact solution of some Turán-type problems. *J. Combin. Theory Ser. A*, 45(2):226–262, 1987.
- [9] Z. Füredi. On finite set-systems whose every intersection is a kernel of a star. *Discrete Math.*, 47(1):129–132, 1983.
- [10] Z. Füredi and L. Özkahya. Unavoidable subhypergraphs: \mathbf{a} -clusters. *J. Combin. Theory Ser. A*, 118(8):2246–2256, 2011.
- [11] P. Keevash. The existence of designs II. *arXiv preprint arXiv:1802.05900*, 2018.
- [12] P. Keevash and D. Mubayi. Set systems without a simplex or a cluster. *Combinatorica*, 30(2):175–200, 2010.
- [13] N. Keller and N. Lifshitz. The junta method for hypergraphs and the Erdős-Chvátal simplex conjecture. *arXiv preprint arXiv:1707.02643*, 2017.
- [14] A. Kostochka and D. Mubayi. The structure of large intersecting families. *Proc. Amer. Math. Soc.*, 145(6):2311–2321, 2017.
- [15] N. Lifshitz. On set systems without a simplex-cluster and the junta method. *J. Combin. Theory Ser. A*, 170:105139, 20, 2020.
- [16] L. Lovász. *Combinatorial problems and exercises*. North-Holland Publishing Co., Amsterdam, second edition, 1993.
- [17] D. Mubayi and J. Verstraëte. Proof of a conjecture of Erdős on triangles in set-systems. *Combinatorica*, 25(5):599–614, 2005.
- [18] V. Rödl, A. Ruciński, and E. Szemerédi. Perfect matchings in large uniform hypergraphs with large minimum collective degree. *J. Combin. Theory Ser. A*, 116(3):613–636, 2009.