

q -Series congruences involving statistical mechanics partition functions in regime III and IV of Baxter's solution of the hard-hexagon model

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Abstract

For each $s \in \{2, 4\}$, the generating function of $R_s(n)$, the number of partitions of n into odd parts or congruent to $0, \pm s \pmod{10}$, arises naturally in regime III of Rodney Baxter's solution of the hard-hexagon model of statistical mechanics. For each $s \in \{1, 3\}$, the generating function of $R_s^*(n)$, the number of partitions of n into parts not congruent to $0, \pm s \pmod{10}$ and $10 - 2s \pmod{20}$, arises naturally in regime IV of Rodney Baxter's solution of the hard-hexagon model of statistical mechanics. In this paper, we investigate the parity of $R_s(n)$ and $R_s^*(n)$, providing new parity results involving sums of partition numbers $p(n)$ and squares in arithmetic progressions.

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1 Introduction

A partition of a positive integer n is a sequence of positive integers whose sum is n . The order of the summands is unimportant when writing the partitions of n , but for consistency, a partition of n will be written with the summands in a nonincreasing order [2]. As usual, we denote by $p(n)$ the number of the partitions of n . For example, we have $p(5) = 7$ because the partitions of 5 are given as:

$$5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1.$$

The fastest algorithms for enumerating all the partitions of an integer have recently been presented by Merca [8, 9].

The partition function $p(n)$ may be defined by the generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}}.$$

Here and throughout this paper, we use the following customary q -series notation:

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k).$$

Because the infinite product $(a; q)_{\infty}$ diverges when $a \neq 0$ and $|q| \geq 1$, whenever $(a; q)_{\infty}$ appears in a formula, we shall assume $|q| < 1$ and we shall use the compact notation

$$(a_1, a_2, \dots, a_n; q)_{\infty} = (a_1; q)_{\infty}(a_2; q)_{\infty} \cdots (a_n; q)_{\infty}.$$

For $s \in \{2, 4\}$, we denote by $R_s(n)$ the number of partitions of n into odd parts or congruent to 0, $\pm s \pmod{10}$. Elementary techniques in the theory of partitions give the following generating function for $R_s(n)$:

$$\sum_{n=0}^{\infty} R_s(n)q^n = \frac{1}{(q; q^2)_{\infty}(q^s, q^{10-s}; q^{10})_{\infty}}.$$

The generating function for $R_s(n)$ arises naturally in regime III of Rodney Baxter's solution of the hard-hexagon model of statistical mechanics, appearing in the following q -identities of Rogers [1, 3]:

$$\sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(3n+1)/2}}{(q; q)_{2n+1}} = \frac{G(q^2)}{(q; q^2)_{\infty}},$$

and

$$\sum_{n=0}^{\infty} \frac{(-q; q)_n q^{3n(n+1)/2}}{(q; q)_{2n+1}} = \frac{H(q^2)}{(q; q^2)_{\infty}},$$

where

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q, q^4; q^5)_{\infty}}$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{(q^2, q^3; q^5)_{\infty}}$$

are the Rogers-Ramanujan functions.

For $s \in \{1, 3\}$, we denote by $R_s^*(n)$ the number of partitions of n into parts not congruent to 0, $\pm s \pmod{10}$ and $10 - 2s \pmod{20}$. Elementary techniques in the theory of partitions give the following generating function for $R_s^*(n)$:

$$\sum_{n=0}^{\infty} R_s^*(n)q^n = \frac{(q^s, q^{10-s}, q^{10}; q^{10})_{\infty} (q^{10-2s}, q^{10+2s}; q^{20})_{\infty}}{(q; q)_{\infty}}.$$

The generating function for $R_s^*(n)$ arises naturally in regime IV of Rodney Baxter's solution of the hard-hexagon model of statistical mechanics, appearing in the following q -identities of Rogers [1, 3]:

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_{2n}} = \frac{(q, q^9, q^{10}; q^{10})_{\infty} (q^8, q^{12}; q^{20})_{\infty}}{(q; q)_{\infty}}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_{2n+1}} = \frac{(q^3, q^7, q^{10}; q^{10})_{\infty} (q^4, q^{16}; q^{20})_{\infty}}{(q; q)_{\infty}}.$$

In this paper, we shall provide the following q -series congruences.

Theorem 1.1. *Let s be a positive integer.*

1. *For $s \in \{2, 4\}$,*

$$\sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(3n+s-1)/2}}{(q; q)_{2n+1}} \equiv \sum_{120n+(3s-5)^2 \text{ square}} q^n \pmod{2}.$$

2. *For $s \in \{1, 3\}$,*

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_{2n+(s-1)/2}} \equiv \sum_{40n+s^2 \text{ square}} q^n \pmod{2}.$$

As a consequence of Theorem 1.1, we immediately deduce the following parity result involving sums of partition numbers $p(n)$ and squares in arithmetic progressions.

Corollary 1.2. *Let n be a nonnegative integer.*

1. *For $s \in \{2, 4\}$,*

$$\sum_{20k+(s-1)^2 \text{ square}} p(n-k) \equiv 1 \pmod{2}$$

if and only if $120n + (3s - 5)^2$ is a square.

2. *For $s \in \{1, 3\}$,*

$$\sum_{15k+(s+1)^2/4 \text{ square}} p(n-k) \equiv 1 \pmod{2}$$

if and only if $40n + s^2$ is a square.

Questions regarding the parity of sums of partition numbers for square values in given arithmetic progressions have been studied recently [6]. As we can see in [6], the cases $s = 1$ and $s = 2$ of Corollary 1.2 are known. The following conjecture is also known: the statement

$$\sum_{ak+1 \text{ square}} p(n-k) \equiv 1 \pmod{2} \quad \text{if and only if } bn+1 \text{ is a square.}$$

is true if and only if

$$(a, b) \in S := \{(6, 8), (8, 12), (12, 24), (15, 40), (16, 48), (20, 120), (21, 168)\}.$$

This paper opens new possibilities for research in this area.

The organization of the paper is as follows. We will first prove Theorem 1.1 in Section 2 considering a truncated theta identity of Gauss [4]. In Section 3, we will provide a proof of Corollary 1.2 considering a decomposition of $R_s(n)$ in terms of Euler partition function $p(n)$. In Section 4, we will introduce new open problems involving the partition function $R_s(n)$, $s \in \{2, 4\}$ and $R_s^*(n)$, $s \in \{1, 3\}$.

2 Proof of Theorem 1.1

Watson's quintuple product identity [5, 10] states that

$$\sum_{n=-\infty}^{\infty} z^{3n} q^{n(3n-1)/2} (1 - zq^n) = (q, z, q/z; q)_{\infty} (qz^2, q/z^2; q^2)_{\infty}. \quad (1)$$

By this identity, with q replaced by $-q^5$ and z replaced by q^2 , we get

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} (-1)^{n(3n-1)/2} \left(q^{n(15n+7)/2} + q^{(3n-2)(5n-1)/2} \right) \\ &= (q^2, -q^3, -q^5; -q^5)_{\infty} \cdot (-q, -q^9; q^{10})_{\infty} \\ &= (q^2, -q^3, -q^5; -q^5)_{\infty} \cdot \frac{(q^2, q^{18}; q^{20})_{\infty}}{(q, q^9; q^{10})_{\infty}} \\ &= (-q^3, -q^5, -q^7; q^{10})_{\infty} \cdot (q^2, q^8, q^{10}; q^{10})_{\infty} \cdot \frac{(q^2, q^{18}; q^{20})_{\infty}}{(q, q^9; q^{10})_{\infty}} \\ &= \frac{(q^6, q^{10}, q^{14}; q^{20})_{\infty}}{(q^3, q^5, q^7; q^{10})_{\infty}} \cdot \frac{(q^2, q^4, q^6, q^8, q^{10}; q^{10})_{\infty}}{(q^4, q^6; q^{10})_{\infty}} \cdot \frac{(q^2, q^{18}; q^{20})_{\infty}}{(q, q^9; q^{10})_{\infty}} \\ &= \frac{(q^2, q^6, q^{10}, q^{14}, q^{18}; q^{20})_{\infty}}{(q, q^3, q^5, q^7, q^9; q^{10})_{\infty}} \cdot \frac{(q^2; q^2)_{\infty}}{(q^4, q^6; q^{10})_{\infty}} \\ &= \frac{(q^2; q^4)_{\infty}}{(q; q^2)_{\infty}} \cdot \frac{(q^2; q^2)_{\infty}}{(q^4, q^6; q^{10})_{\infty}} \\ &= \frac{1}{(q; q^2)_{\infty} (q^4, q^6; q^{10})_{\infty}} \cdot \frac{(q^2; q^2)_{\infty}}{(-q^2; q^2)_{\infty}}. \end{aligned}$$

In a similar way, letting $q \rightarrow q^5$ and setting $z = -q$ in (1), we obtain

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} (-1)^{n(n-1)/2} \left(q^{n(15n+1)/2} + q^{(3n-1)(5n-2)/2} \right) \\ &= \frac{1}{(q; q^2)_{\infty} (q^2, q^8; q^{10})_{\infty}} \cdot \frac{(q^2; q^2)_{\infty}}{(-q^2; q^2)_{\infty}}. \end{aligned}$$

Thus, for $s \in \{2, 4\}$, we deduce that

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} (-1)^{n((s-1)n-1)/2} \left(q^{n(15n+3s-5)/2} + q^{(3n-s/2)(5n-3+s/2)/2} \right) \\ &= \frac{1}{(q; q^2)_{\infty} (q^s, q^{10-s}; q^{10})_{\infty}} \cdot \frac{(q^2; q^2)_{\infty}}{(-q^2; q^2)_{\infty}}. \end{aligned} \quad (2)$$

On the other hand, the Jacobi triple product identity says that (cf. [7, Eq. (1.6.1)])

$$\sum_{n=-\infty}^{\infty} (-z)^n q^{n(n-1)/2} = (z, q/z, q; q)_{\infty}. \quad (3)$$

By this identity, with q replaced by $-q^5$ and z replaced by $-q$, we derive

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} (-1)^{n(n-1)/2} q^{n(5n-3)/2} \\ &= (-q, q^4, -q^5; -q^5)_{\infty} \\ &= (q^4, q^6, q^{10}; q^{10})_{\infty} (-q, -q^5, -q^9; q^{10})_{\infty} \\ &= (q^4, q^6, q^{10}; q^{10})_{\infty} \frac{(q^2, q^{10}, q^{18}; q^{20})_{\infty}}{(q, q^5, q^9; q^{10})_{\infty}} \\ &= \frac{(q^2, q^4, q^6, q^8, q^{10}; q^{10})_{\infty}}{(q, q^2, q^5, q^8, q^9; q^{10})_{\infty}} \cdot \frac{(q^2, q^6, q^{10}, q^{14}, q^{18}; q^{20})_{\infty}}{(q^6, q^{14}; q^{20})_{\infty}} \\ &= \frac{(q^2; q^2)_{\infty}}{(q, q^2, q^5, q^8, q^9; q^{10})_{\infty}} \cdot \frac{(q^2; q^4)_{\infty}}{(q^6, q^{14}; q^{20})_{\infty}} \\ &= \frac{(q^3, q^4, q^6, q^7, q^{10}; q^{10})_{\infty}}{(q; q)_{\infty}} \cdot \frac{(q^2, q^2, q^4; q^4)_{\infty}}{(q^6, q^{14}; q^{20})_{\infty}} \\ &= \frac{(q^3, q^4, q^7, q^{10}, q^{13}, q^{16}, q^{17}, q^{20}; q^{20})_{\infty}}{(q; q)_{\infty}} \cdot (q^2, q^2, q^4; q^4)_{\infty} \\ &= \frac{(q^3, q^7, q^{10}; q^{10})_{\infty} (q^4, q^{16}; q^{20})_{\infty}}{(q; q)_{\infty}} \cdot \frac{(q^2; q^2)_{\infty}}{(-q^2; q^2)_{\infty}}. \end{aligned}$$

In a similar way, letting $q \rightarrow -q^5$ and setting $z = -q^3$ in (3), we obtain

$$\sum_{n=-\infty}^{\infty} (-1)^{n(n-1)/2} q^{n(5n+1)/2} = \frac{(q, q^9, q^{10}; q^{10})_{\infty} (q^8, q^{12}; q^{20})_{\infty}}{(q; q)_{\infty}} \cdot \frac{(q^2; q^2)_{\infty}}{(-q^2; q^2)_{\infty}}.$$

Thus, for $s \in \{1, 3\}$, we deduce that

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} (-1)^{n(n+s)/2} q^{n(5n-s)/2} \\ &= \frac{(q^2, q^{10-s}, q^{10}; q^{10})_{\infty} (q^{10-2s}, q^{10+2s}; q^{20})_{\infty}}{(q; q)_{\infty}} \cdot \frac{(q^2; q^2)_{\infty}}{(-q^2; q^2)_{\infty}}. \end{aligned} \quad (4)$$

The following theta identity is often attributed to Gauss [2, p. 23, eqs. (2.2.12)]:

$$1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} = \frac{(q; q)_{\infty}}{(-q; q)_{\infty}}.$$

In [4], the authors considered this theta identity and proved the following truncated form:

$$\begin{aligned} & \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \left(1 + 2 \sum_{j=1}^k (-1)^j q^{j^2} \right) \\ &= 1 + 2(-1)^k \frac{(-q; q)_k}{(q; q)_k} \sum_{j=0}^{\infty} \frac{q^{(k+1)(k+j+1)} (-q^{k+j+2}; q)_{\infty}}{(1 - q^{k+j+1})(q^{k+j+2}; q)_{\infty}}. \end{aligned}$$

By this identity, with q replaced by q^2 , we obtain

$$\begin{aligned} & (-1)^k \left(\frac{(-q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \left(1 + 2 \sum_{j=1}^k (-1)^j q^{2j^2} \right) - 1 \right) \\ &= 2 \frac{(-q^2; q^2)_k}{(q^2; q^2)_k} \sum_{j=0}^{\infty} \frac{q^{2(k+1)(k+j+1)} (-q^{2(k+j+2)}; q^2)_{\infty}}{(1 - q^{2(k+j+1)})(q^{2(k+j+2)}; q^2)_{\infty}}. \end{aligned} \quad (5)$$

Multiplying both sides of this identity by (2), we obtain

$$\begin{aligned} & \left(1 + 2 \sum_{n=1}^k (-1)^n q^{2n^2} \right) \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(3n+s-1)/2}}{(q; q)_{2n+1}} \\ & - \sum_{n=-\infty}^{\infty} (-1)^{n((s-1)n-1)/2} \left(q^{n(15n+3s-5)/2} + q^{(3n-s/2)(5n-3+s/2)/2} \right) \\ &= 2(-1)^k \frac{(-q^2; q^2)_k}{(q^2; q^2)_k} \sum_{n=k+1}^{\infty} \frac{q^{2n(k+1)} (-q^{2(n+1)}; q^2)_{\infty}}{(1 - q^{2n})(q^{2(n+1)}; q^2)_{\infty}} \times \\ & \times \sum_{n=-\infty}^{\infty} (-1)^{n((s-1)n-1)/2} \left(q^{n(15n+3s-5)/2} + q^{(3n-s/2)(5n-3+s/2)/2} \right). \end{aligned} \quad (6)$$

For each $s \in \{2, 4\}$, it is elementary to see that squares that are congruent to $(3s-5)^2$ modulo 120 are of the form $n(15n+3s-5)/2$ or $(3n-s/2)(5n-3+s/2)/2$ with $k \in \mathbb{Z}$. The first identity is proved.

In a similar way, multiplying both sides of (5) by (4), we obtain

$$\begin{aligned} & \left(1 + 2 \sum_{n=1}^k (-1)^n q^{2n^2}\right) \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_{2n+(s-1)/2}} - \sum_{n=-\infty}^{\infty} (-1)^{n(n+s)/2} q^{n(5n-s)/2} \\ &= 2(-1)^k \frac{(-q^2; q^2)_k}{(q^2; q^2)_k} \sum_{n=k+1}^{\infty} \frac{q^{2n(k+1)} (-q^{2(n+1)}; q^2)_{\infty}}{(1-q^{2n})(q^{2(n+1)}; q^2)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^{n(n+s)/2} q^{n(5n-s)/2}. \end{aligned} \quad (7)$$

For each $s \in \{1, 3\}$, it is elementary to see that squares that are congruent to s^2 modulo 40 are of the form $n(5n-s)/2$ with $n \in \mathbb{Z}$. Thus we deduce the second identity.

3 Proof of Corollary 1.2

For each $s \in \{2, 4\}$, the generating function for $R_s(n)$ can be written as follows:

$$\sum_{n=0}^{\infty} R_s(n) q^n = \frac{(q^{6-s}, q^{4+s}, q^{10}; q^{10})_{\infty}}{(q; q)_{\infty}}.$$

By the Jacobi triple product identity (3), with q replaced by q^{10} and z replaced by q^{6-s} , we get

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+1-s)} = (q^{6-s}, q^{4+s}, q^{10}; q^{10})_{\infty}.$$

Thus we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} R_s(n) q^n &= \left(\sum_{n=0}^{\infty} p(n) q^n \right) \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+1-s)} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=-\infty}^{\infty} (-1)^k p(n-k(5k+1-s)) \right) q^n. \end{aligned}$$

Equating the coefficient of q^n in this equation, we obtain the following decomposition of $R_s(n)$ in terms of the partition function $p(n)$:

$$R_s(n) = \sum_{k=-\infty}^{\infty} (-1)^k p(n-k(5k+1-s)).$$

It is an easy exercise to show that the squares congruent to $(s-1)^2$ modulo 20 are of the form $k(5k+1-s)$ with $k \in \mathbb{Z}$.

For each $s \in \{1, 3\}$, considering Watson's quintuple product identity (1), with q replaced by q^{10} and z replaced by q^s , we get

$$\sum_{n=0}^{\infty} R_s^*(n) q^n = \frac{1}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} q^{15n^2-(5-3s)n} (1-q^{10n+s})$$

$$= \left(\sum_{n=0}^{\infty} p(n) q^n \right) \left(\sum_{n=-\infty}^{\infty} (q^{15n^2 - (5-3s)n} - q^{15n^2 + (5+3s)n+s}) \right).$$

Equating the coefficient of q^n in this equation, we obtain

$$R_s^*(n) = \sum_{k=-\infty}^{\infty} \left(p(n - (15k^2 - (5-3s)k)) - p(n - (15k^2 + (5+3s)k + s)) \right).$$

It is an easy exercise to show that the squares congruent to $(s+1)^2/4$ modulo 15 are of the form $15k^2 - (5-3s)k$ or $15k^2 + (5+3s)k + s$ with $k \in \mathbb{Z}$.

4 Open problems

Related to relations (6) and (7), we remark that there is a substantial amount of numerical evidence to state the following conjecture.

Conjecture 1. *Let k be a positive integer.*

1. *For $s \in \{2, 4\}$,*

$$\begin{aligned} & \left(1 + 2 \sum_{n=1}^k (-1)^n q^{2n^2} \right) \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(3n+s-1)/2}}{(q; q)_{2n+1}} \\ & - \sum_{n=-\infty}^{\infty} (-1)^{n((s-1)n-1)/2} \left(q^{n(15n+3s-5)/2} + q^{(3n-s/2)(5n-3+s/2)/2} \right) \end{aligned}$$

has nonnegative coefficients if k is even and nonpositive coefficients if k is odd.

2. *For $s \in \{1, 3\}$,*

$$\left(1 + 2 \sum_{n=1}^k (-1)^n q^{2n^2} \right) \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_{2n+(s-1)/2}} - \sum_{n=-\infty}^{\infty} (-1)^{n(n+s)/2} q^{n(5n-s)/2}$$

has nonnegative coefficients if k is even and nonpositive coefficients if k is odd.

Assuming this conjecture, we derive the following families of linear inequalities involving $R_s(n)$ and $R_s^*(n)$.

Conjecture 2. *Let k be a positive integer.*

1. *For $s \in \{2, 4\}$,*

$$(-1)^k \left(R_s(n) + 2 \sum_{j=1}^k (-1)^j R_s(n - 2j^2) - \rho_s(n) \right) \geq 0,$$

where

$$\rho_s(n) = \begin{cases} (-1)^{m((s-1)m-1)/2}, & \text{if } n = m(15m+3s-5)/2 \text{ or} \\ & n = (3m-s/2)(5m-3+s/2)/2, m \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

2. For $s \in \{1, 3\}$,

$$(-1)^k \left(R_s^*(n) + 2 \sum_{j=1}^k R_s^*(n - 2j^2) - \rho_s^*(n) \right) \geq 0,$$

where

$$\rho_s^*(n) = \begin{cases} (-1)^{m(m+s)/2}, & \text{if } n = m(5m-s)/2, m \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

It would be very appealing to obtain other q -series congruences that involve other statistical mechanics partition functions appearing in Baxter's solution of the hard-hexagon model.

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