

MATHER CLASSES AND CONORMAL SPACES OF SCHUBERT VARIETIES IN COMINUSCULE SPACES

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ABSTRACT. Let G/P be a complex cominuscule flag manifold. We prove a type independent formula for the torus equivariant Mather class of a Schubert variety in G/P , and for a Schubert variety pulled back via the natural projection $G/Q \rightarrow G/P$. We apply this to find formulae for the local Euler obstructions of Schubert varieties, and for the torus equivariant localizations of the conormal spaces of these Schubert varieties. We conjecture positivity properties for the local Euler obstructions and for the Schubert expansion of Mather classes. We check the conjectures in many cases, by utilizing results of Boe and Fu about the characteristic cycles of the intersection homology sheaves of Schubert varieties. We also conjecture that certain ‘Mather polynomials’ are unimodal in general Lie type, and log concave in type A.

1. INTRODUCTION

Let X be a complex, projective manifold and let $Y \subset X$ be a closed irreducible subvariety. The Mather class $c_{\text{Ma}}(Y)$ is a non-homogeneous element in the (Chow) homology $A_*(X)$. Its original definition uses the Nash blowup of X along Y , but in this paper we work with the following equivalent definition, going back to Sabbah [Sab85]; see also [Gin86, AMSS17].

Let $T^*(X)$ be the cotangent bundle of X , and let $\iota : X \rightarrow T^*(X)$ be the zero section embedding. The multiplicative group \mathbb{C}^* acts on $T^*(X)$ by fibrewise dilation with character \hbar^{-1} . To the subvariety Y one associates the conormal space $T_Y^* \subset T^*(X)$; this is an irreducible conic Lagrangian cycle in the cotangent bundle. The Mather class $c_{\text{Ma}}(Y)$ is the dehomogenization of the \mathbb{C}^* -equivariant class of the conormal space:

$$c_{\text{Ma}}(Y) := (-1)^{\dim Y} (\iota^*[T_Y^*])_{\hbar=1} \in A_0^{\mathbb{C}^*}(X).$$

For example, it follows from definition that if Y is smooth, then $c_{\text{Ma}}(Y)$ is the push-forward of the homology class $c(T_Y) \cap [Y]$ inside $A_*(X)$. If $Y = X$, then one recovers the well known index formula for the topological Euler characteristic:

$$\chi(X) = (-1)^{\dim X} \int_X \iota^*[T_X^*].$$

An equivariant version of Mather classes was defined by Ohmoto [Ohm06]; we refer to Section 3 below for the precise details.

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Let G be a complex, semisimple Lie group, and fix $T \subset B \subset P \subset G$ a parabolic subgroup P containing a standard Borel subgroup B with a maximal torus T ; let $X = G/P$ be the associated flag manifold. The goal of this paper is to study the T -equivariant Mather class $c_{\text{Ma}}^T(Y) \in H_0^{T \times \mathbb{C}^*}(X)$ for a Schubert variety Y in a cominuscule space G/P , or when Y is a Schubert variety in an arbitrary flag manifold G/Q obtained by pulling back via the natural projection $G/Q \rightarrow G/P$.

The cominuscule spaces are a family of flag manifolds consisting of the ordinary Grassmannian, the maximal orthogonal Grassmannians in Lie types B,D, the Lagrangian Grassmannian in type C, quadrics, and respectively the Cayley plane and the Freudenthal variety in the exceptional Lie types E_6 and E_7 .¹

Let W denote the Weyl group and let W^P be the subset of minimal length representatives. For $w \in W^P$, let $X_w^{P,\circ} = BwP/P$ be the Schubert cell in G/P , and let $X_w^P := \overline{X_w^{P,\circ}}$ be the Schubert variety; let also $X_w^B := \overline{BwB/B}$ be the Schubert variety in G/B .

The Mather class of a Schubert variety is related to Chern-Schwartz-MacPherson (CSM) classes of its Schubert cells via the *local Euler obstruction* coefficients $e_{w,v}$:

$$(1) \quad c_{\text{Ma}}(X_w^P) = \sum_v e_{w,v} c_{\text{SM}}(X_v^{P,\circ}).$$

These coefficients were defined by MacPherson [Mac74] and provide a subtle measure of the singularity of X_w^P at v . For instance, consider the parabolic Kazhdan-Lusztig (KL) polynomial $P_{w,v}(q)$; cf. [Deo87]. Then the equalities

$$(2) \quad e_{w,v} = P_{w,v}(1), \quad \forall v \in W^P,$$

hold if and only if the characteristic cycle of the intersection homology (IH) sheaf of the Schubert variety X_w^P is irreducible. In general, the problem of finding the decomposition of the characteristic cycle of the IH sheaves into irreducible components is open, although some particular cases are known; see e.g. [KL80, KT84, BFL90, BF97, EM99, Bra02, Wil15], and also Section 10 below for more details. We note that since CSM classes of Schubert cells can be explicitly calculated [AM09, AM16, RV18], equation (1) shows that giving an algorithm to calculate Mather classes is equivalent to one for the local Euler obstructions.

To state a precise version of our results, we need to introduce more notation. For $w \in W^P$, let $I(w)$ denote the inversion set of w ; this consists of positive roots α such that $w(\alpha) < 0$, and it may be identified with the *diagram* of w . For a root α we denote by \mathfrak{g}_α the root subspace of $\text{Lie}(G)$ determined by α and by \mathbb{C}_α the one-dimensional B -module of weight α . It follows from [SinBA] (see also [RSW]) that if G/P is cominuscule and $w \in W^P$ then the vector space $T_w := \bigoplus_{\alpha \in I(w)} \mathfrak{g}_{-\alpha}$ has a structure of a B -module. Therefore

$$\mathcal{T}_w := G \times^B T_w$$

is a vector bundle over the complete flag variety G/B . Let $c(\mathcal{T}_w)$ denote its total Chern class.

The following is the main result of our paper; see Theorems 6.1 and 9.2 below.

¹ For the cominuscule property to hold, the maximal orthogonal Grassmannian in type B needs to be regarded as a homogeneous space under the Lie group of type D; see Section 4 below.

Theorem 1.1. *Let G/P be a cominuscule space and projection $\pi : G/B \rightarrow G/P$, and let $w \in W^P$ be a minimal length representative. Then the following hold:*

(a) *The Mather class of X_w^P is given by*

$$c_{Ma}(X_w^P) = \pi_*(c(\mathcal{T}_w) \cap [X_w^B]) = \pi_*\left(\prod_{\alpha \in I(w)} c(G \times^B \mathbb{C}_{-\alpha}) \cap [X_w^B]\right).$$

(b) *Let $Q \subset P$ be any parabolic subgroup, with $\pi_Q : G/Q \rightarrow G/P$ the natural projection. Then the Mather class of the pull-back Schubert variety $\pi^{-1}(X_w^P)$ is*

$$c_{Ma}(\pi_Q^{-1}(X_w^P)) = c(T_{\pi_Q}) \cap \pi_Q^*(c_{Ma}(X_w^P)),$$

where T_{π_Q} is the relative tangent bundle of the projection π_Q .

(c) *The formulae in (a) and (b) hold in the T -equivariant setting.*

We encourage the reader to jump directly to section 6.2 for examples illustrating the formula in part (a) and its equivariant version.

The proof of part (a) exploits the observation that the \mathbb{C}^* -equivariant pull-back $\iota^*[T_{X_w^P}^*(G/P)]$ is essentially given by the Segre class of the conormal space $T_{X_w^P}^*(G/P)$; see Section 2 below. To calculate this Segre class, we utilize a desingularization of the conormal space found by the second named author [SinBA], together with the property that the Segre classes are preserved under birational push forward.

A different proof of the part (a) of Theorem 1.1 may be obtained using the identification by Richmond, Slofstra and Woo [RSW, Thm. 2.1] of the Nash blowup of the Schubert varieties in cominuscule spaces. In this paper we aimed to emphasize the equivalence between Mather classes and the Segre classes of the conormal spaces, a point of view which we believe it will have further benefits for understanding the conormal spaces.

Part (b) follows from the Verdier-Riemann-Roch formula proved by Yokura [Yok99], and from the invariance of Euler obstructions under smooth pull-back. The latter statement is likely well known to experts, but we could not find it in the form we need in the literature. We give it two proofs, one under very general hypotheses in Proposition 3.3, and the second in Section 9 which uses pull-backs of conormal spaces. All constructions are T -equivariant, and part (c) follows.

We give two applications of Theorem 1.1. The first is an explicit localization formula for the conormal spaces $T_{X_w^P}^*(G/P)$ of Schubert varieties in cominuscule spaces; see Theorem 8.3.

The second application is a formula for the local Euler obstructions of Schubert varieties. The proof uses the equation (1) and the identification of the Poincaré duals of CSM classes obtained in [AMSS17]. The resulting formula is given in Theorem 7.1. Based on many calculations in all cominuscule types we conjecture the following positivity properties; cf. Conjectures 10.1 and 10.2 below.

Conjecture 1.2 (Positivity Conjecture). *Let $X = G/P$ be a cominuscule space and let $v, w \in W^P$.*

(a) Consider the Schubert expansion

$$c_{Ma}(X_w^P) = \sum_{v \leq w} a_{w,v}[X_v^P].$$

Then $a_{w,v} \geq 0$. A positivity property also holds for the equivariant Mather classes (cf. Conj. 10.1).

(b) The local Euler obstruction coefficients are non-negative, i.e. $e_{w,v} \geq 0$.

By the equation (1) and positivity of the non-equivariant CSM classes of Schubert cells [Huh16, AMSS17], part (b) implies the non-equivariant positivity from part (a). It is tempting to make this conjecture in *any* flag manifold G/Q , but unfortunately we do not have substantial evidence in this generality. Most of the other cases we can check follow from Proposition 3.3 below, which states that the local Euler obstructions are preserved under smooth pull-backs; therefore one may expand this conjecture to include pull-backs of Schubert varieties from G/P .

By the positivity of KL polynomials and the equation (2), $e_{w,v} > 0$ whenever the characteristic cycle of the IH sheaf of X_w^P is irreducible. This holds for cominuscule spaces in Lie types A and D, by results from [BFL90] and [BF97]. Boe and Fu also prove positivity of Euler obstructions for the odd-dimensional quadrics (in Lie type B). Therefore, Conjecture 1.2 holds in all these cases. See Section 10 below for more details.

We also conjecture a unimodality property for the *Mather polynomial* of $w \in W^P$. The Mather polynomial is obtained from the Schubert expansion of the Mather class by replacing each Schubert class $[X_v^P]$ by $x^{\ell(v)}$. We conjecture that the resulting polynomial is unimodal, in the sense of [Sta89]. For the ordinary Grassmannians, calculations suggest that the polynomial is also log concave; see Section 10.4 for details and examples.

Formulas for the Mather classes and for the local Euler obstructions have been found by B. Jones [Jon10] in the case of Grassmann manifolds, and in [Rai16, Zha18, RP, Tim] for various types of degeneracy loci. Jones' proof is based on the fact that if $\pi' : Z_w \rightarrow X_w^P$ is a small resolution of X_w^P (in the sense of intersection homology) *and* if the characteristic cycle of the IH sheaf of X_w^P is irreducible, then the Mather class satisfies

$$c_{Ma}(X_w^P) = \pi'_*(c(T_{Z_w}) \cap [Z_w]),$$

where $c(T_{Z_w})$ is the total Chern class of the tangent bundle of Z_w . Small resolutions for the Schubert varieties in Grassmannians were constructed by Zelevinsky [Zel83], and Bressler, Finkelberg and Lunts [BFL90] proved that the characteristic cycles of the IH sheaves of Schubert varieties are irreducible. Outside the type A Grassmannian, Schubert varieties may not admit small resolutions; see [SV94] and also [Per07, Example 7.15].

Boe and Fu [BF97] used delicate techniques from geometric analysis to find formulae for the local Euler obstruction $e_{w,v}$ of the Schubert varieties in cominuscule spaces G/P of classical Lie types A–D. Using recursive formulae for the KL polynomials, they were able to show that the identities (2) hold in Lie types A and D, and fail in general for types B, C. We included examples such as Examples 7.4 and 10.7, recovering instances of reducible IH sheaves from [BF97] and [KT84], and

obtained with the formulae from this paper. In future work, we plan to compare our formula from Theorem 7.1 to the formulae in [BF97].

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Throughout the paper we utilize the Chow (co)homology theory [Ful84], and its equivariant version from [EG98]. This is related to the ordinary (possibly equivariant) (co)homology via the cycle map - see [Ful84, Ch. 19] and [EG98, §2.8]; for flag manifolds this map is an isomorphism [Ful84, Ex. 19.1.11]. Finally, we work over the field of complex numbers.

2. SEGRE CLASSES OF CONES

2.1. Segre classes and the pull back via the zero section. The treatment in this section follows largely [Ful84, §4], but we also used [BBM89, §1] and [BBM87]. Let C be a cone in the sense of [Ful84, §4 and Appendix B.5]. For the applications envisioned in this note, we assume in addition that C is a closed subcone of a vector bundle $E \rightarrow X$. We consider the projective completion $\mathbb{P}(E \oplus \mathbb{1})$, and we denote by $\mathcal{O}_E(-1)$ the tautological bundle of lines in $E \oplus \mathbb{1}$. Denote by $\mathcal{O}_C(-1)$ the restriction of $\mathcal{O}_E(-1)$ to the projective completion $\overline{C} = \mathbb{P}(C \oplus \mathbb{1}) \subset \mathbb{P}(E \oplus \mathbb{1})$, and let $q : \mathbb{P}(E \oplus \mathbb{1}) \rightarrow X$ be the natural projection.

The *Segre class* of C is the (non-homogeneous) class in the Chow group $A_*(X)$ defined by:

$$(3) \quad s(C) := q_* \left(\frac{[\overline{C}]}{c(\mathcal{O}_C(-1))} \right) = q_* \left(\sum_{i \geq 0} c_1(\mathcal{O}_C(1))^i \cap [\overline{C}] \right).$$

If $C = E$ is a vector bundle over X then its Segre class is $s(E) = c(E)^{-1} \cap [X]$, see [Ful84, Prop. 4.1].

Suppose now that the cone $C \subset E$ is pure dimensional, with $\dim C = \text{rank}(E)$. (This will be the case for our application, when C is the conormal space of a subvariety.) In this case, the Segre class of C is related to the pull back $\iota^*[C]$ of the class of C via the zero section $\iota : X \rightarrow E$. Observe however that the Segre class is non-homogeneous, while $\iota^*[C]$ is a class in $A_0(X)$. In order to relate the two, one needs to work in the \mathbb{C}^* -equivariant Chow group; this was one of the observations in [AMSS17, §2]. We recall the relevant facts next, referring the reader to [EG98] for details on equivariant Chow groups.

The starting point is the formula from [Ful84, Example 4.1.8], which states that

$$(4) \quad \iota^*[C] = (c(E) \cap s(C))_0,$$

in the Chow group of X , where $(a)_0$ means taking the homogeneous component of degree 0 of the class $a \in A_*(X)$. We will need to ‘homogenize’ this formula.

There is a \mathbb{C}^* -action on E by dilation by a character χ , which extends to an action on $E \oplus \mathbb{1}$ by letting \mathbb{C}^* act trivially on the second component. This induces a \mathbb{C}^* -action on the projective completion $\mathbb{P}(E \oplus \mathbb{1})$. Both C and its closure are \mathbb{C}^* -stable subschemes; the action of \mathbb{C}^* restricted to the base X is trivial. The

character χ determines a class in the equivariant Chow group $A_{\mathbb{C}^*}^1(pt)$ of degree 1, denoted in the same way. Since \mathbb{C}^* acts trivially on X , a class $a \in A_0^{\mathbb{C}^*}(X)$ is equivalent to a *non-homogeneous* class $a_0 + a_1 + \dots \in A_*(X)$ ($a_i \in A_i(X)$) obtained by dehomogenizing a . Conversely, if $a = a_0 + a_1 + \dots \in A_*(X)$ is a non-homogeneous class, its χ -*homogenization* is the class

$$(5) \quad a^\chi := a_0 + a_1\chi + a_2\chi^2 + \dots \in A_0^{\mathbb{C}^*}(X).$$

Observe now that all the classes in the equation (4) are \mathbb{C}^* -equivariant, thus the formula extends to the equivariant context. Further, by [AMSS17, Proposition 2.7],

$$(6) \quad \iota^*[C]_{\mathbb{C}^*} = (c(E) \cap s(C))^\chi \in A_0^{\mathbb{C}^*}(X).$$

(The proposition in *loc.cit.* is stated in terms of the *shadow* of the cone \overline{C} ; by [AMSS17, Lemma 2.1(a) and Lemma 2.2], the shadow equals $c(E) \cap s(C)$.) In Section 3 we will be interested in the pull back $\iota^*[C]_{\mathbb{C}^*}$, and we will use this formula to calculate it.

One of the fundamental properties of the Segre classes is their birational invariance, recalled next. Similar statements can be found in [BBM89, p. 10] (without proof), and, for the *normal cone* of a subvariety, in [Ful84, Proposition 4.2]. For the convenience of the reader we include a proof.

Lemma 2.1. *Let $f : X' \rightarrow X$ be a proper morphism of irreducible non-singular varieties, $E \rightarrow X$ a vector bundle, and C an irreducible subcone of E over a closed subvariety $Y \subset X$. Let $Y' := f^{-1}(Y)$ and assume that C' is a subcone of $f^*(E)$ over Y' such that we have a commutative diagram*

$$\begin{array}{ccc} C' & \xrightarrow{g} & C \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{f} & Y \end{array}$$

where $g : C' \rightarrow C$ is proper and birational. Then

$$(7) \quad f_*(s(C')) = s(C) \in A_*(Y).$$

Proof. The morphism f induces a morphism of vector bundles $f^*(E) \oplus \mathbb{1} \rightarrow E \oplus \mathbb{1}$, which in turn induces a morphism $F : \mathbb{P}(f^*(E) \oplus \mathbb{1}) \rightarrow \mathbb{P}(E \oplus \mathbb{1})$ between projective completions. Let $G : \overline{C}' = \mathbb{P}(C' \oplus \mathbb{1}) \rightarrow \mathbb{P}(C \oplus \mathbb{1})$ be the restriction. There is a commutative diagram

$$\begin{array}{ccc} \mathbb{P}(C' \oplus \mathbb{1}) & \xrightarrow{G} & \mathbb{P}(C \oplus \mathbb{1}) \\ \downarrow q' & & \downarrow q \\ Y' & \xrightarrow{f} & Y \end{array}$$

The birationality of g implies that

$$(8) \quad G_*[\overline{C}'] = [\overline{C}] \in A_*(\mathbb{P}(E \oplus \mathbb{1})).$$

Now, since C' is a subcone of $f^*(E)|_{Y'}$, we have

$$(9) \quad G^*(\mathcal{O}_C(-1)) = \mathcal{O}_{C'}(-1),$$

as both sides are the restriction of $\mathcal{O}_{f^*(E)}(-1) = f^*\mathcal{O}_E(-1)$ to Y' . Following the definition of the Segre class (Eq. (3)), we have,

$$\begin{aligned} f_*(s(C')) &= f_*q'_* \left(\frac{[\bar{C}']}{c(\mathcal{O}_{C'}(-1))} \right) = q_*G_* \left(\frac{[\bar{C}]}{G^*c(\mathcal{O}_C(-1))} \right) \\ &= q_* \left(\frac{[\bar{C}]}{c(\mathcal{O}_C(-1))} \right) = s(C). \end{aligned}$$

Here the third equality uses the projection formula and equations (8) and (9). This finishes the proof. \square

All results extend naturally to the case where X is a variety with an action of a torus T , C is a $T \times \mathbb{C}^*$ -invariant cone, and the map $C \rightarrow X$ is $T \times \mathbb{C}^*$ equivariant (the \mathbb{C}^* acting trivially on X). For instance, in equation (6), the class $\iota^*[C]_{T \times \mathbb{C}^*}$ belongs to $A_0^{T \times \mathbb{C}^*}(X)$, the $T \times \mathbb{C}^*$ equivariant Chow group.

2.2. Conormal spaces. The cones most important in this note are the conormal spaces of subvarieties, whose definition we recall next. Let X be a smooth, irreducible, complex algebraic variety, and let $Y \subset X$ be a closed irreducible subscheme. Let Y^{reg} be any any smooth dense set of Y . The *conormal space* T_Y^*X is the closure of the conormal bundle $T_{Y^{reg}}^*X$ inside the cotangent bundle T^*X . This is a cone in the sense of the previous section, and also a closed subvariety of dimension $\dim X$, contained in the restriction $T^*X|_Y$. In particular, it is stable under the \mathbb{C}^* -dilation on the fibres of T^*X , and also under any group G leaving Y and X invariant. If one regards T^*X as a symplectic manifold, then the conormal space is an irreducible conic Lagrangian cycle. In fact, any irreducible, conic Lagrangian cycle is the conormal cone of some subvariety; see [HTT08, Thm. E.6] (where it is attributed to Kashiwara) and also [Ken90, §1].

3. MATHER CLASSES AND CSM CLASSES

A question with a long and distinguished history is to define analogues of the total Chern class for singular varieties. The Mather classes and the Chern-Schwartz-MacPherson (CSM) classes, considered in this note, are among these classes. We recall their definition next.

3.1. Mather classes. Let X be a smooth complex algebraic variety, and let $Y \subset X$ be a closed irreducible subvariety. The *Mather class* of Y is a non-homogenous homology class $c_{\text{Ma}}(Y) \in A_*(Y)$ with the property that if Y is smooth then $c_{\text{Ma}}(Y) = c(TY) \cap [Y]$. Its original definition involves the Nash blowup of Y , but for the purpose of this note we use a variant of a result of Sabbah [Sab85] (see also [Gin86, PP01]) relating the Mather class to the the class of the conormal space of Y in X . This variant appeared in [AMSS17, Corollary 4.5 and Corollary 3.4]. Further, we work in the equivariant context, using the equivariant Mather class defined by Ohmoto [Ohm06]; the corresponding class is denoted by $c_{\text{Ma}}^T(Y) \in A_*^T(X)$.

Theorem 3.1 (cf. [AMSS17]). *Let $Y \subseteq X$ be a T -stable closed irreducible subvariety of the smooth variety X and assume that \mathbb{C}^* acts by dilation on the cotangent*

bundle T^*X with character \hbar^{-1} . Then the homogenization (cf. (5) above) of the T -equivariant Chern-Mather class satisfies

$$(10) \quad c_{\text{Ma}}^T(Y)^\hbar = (-1)^{\dim Y} \iota^*[T_Y^*X]_{T \times \mathbb{C}^*},$$

as classes in $A_0^{T \times \mathbb{C}^*}(X)$.

It will be convenient to work with a dehomogenized variant of this equation. Recall that by equation (6) above,

$$\iota^*[T_Y^*(X)]_{T \times \mathbb{C}^*} = (c^T(T^*X) \cap s^T(T_Y^*X))^{-\hbar},$$

since the \mathbb{C}^* action is induced by \hbar^{-1} . By equation (10) this implies that

$$(-1)^{\dim Y} c_{\text{Ma}}^T(Y)^{-\hbar} = (c(T^*X) \cap s^T(T_Y^*(X)))^\hbar.$$

After dehomogenizing, i.e. setting $\hbar = 1$, we obtain the expression

$$(11) \quad c_{\text{Ma}}^{T,\vee}(Y) = c^T(T^*X) \cap s^T(T_Y^*(X)),$$

where $c_{\text{Ma}}^{T,\vee}(Y) := ((-1)^{\dim Y} c_{\text{Ma}}^T(Y)^{-\hbar})|_{\hbar=1}$.

In other words, the class $c_{\text{Ma}}^{T,\vee}(Y)$ is obtained from $c_{\text{Ma}}^T(Y)$ by changing signs of each homogeneous component according to its cohomological degree. This is called the *dual Chern-Mather class*; it appears naturally when relating Chern-Mather classes to characteristic cycles on the cotangent bundle; cf. [Sab85].

3.2. Chern-Schwartz-MacPherson classes. Let X be any complex algebraic variety endowed with a Whitney stratification $\{S_i\}$ of smooth constructible subsets. Such a stratification always exists; see [Ver76, Thm. 2.2] for the algebraic context, and [Whi65, Thm. 19.2] for the analytic context. (Later, X will be a Schubert variety with the stratification given by its Schubert cells.)

Denote by $\mathcal{F}(X)$ the group of constructible functions of X . Its elements are finite sums of the form $\sum a_i \mathbf{1}_{W_i}$, where $a_i \in \mathbb{Z}$, the $W_i \subset X$ are constructible subsets, and $\mathbf{1}_{W_i}$ is the indicator function, which equals 1 for points on W_i and 0 otherwise. There are push-forward and pull-back operations defined as follows. If $f : Z \rightarrow X$ is a *proper* morphism, then $f_*(\mathbf{1}_W)(x) = \chi(f^{-1}(x) \cap W)$, where χ denotes the topological Euler characteristic; one extends this further by linearity. For any morphism $f : Z \rightarrow X$, the pull back $f^* : \mathcal{F}(X) \rightarrow \mathcal{F}(Z)$ is defined by $f^*(\varphi)(z) = \varphi(f(z))$, for $\varphi \in \mathcal{F}(X)$.

Proving a conjecture of Grothendieck and Deligne, MacPherson [Mac74] defined a transformation $c_* : \mathcal{F}(X) \rightarrow H_*(X)$ which satisfies $c_*(\mathbf{1}_X) = c(T(X)) \cap [X]$ if X is smooth, and is functorial with respect to proper morphisms $f : Z \rightarrow X$. This means that there is a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(Z) & \xrightarrow{c_*} & H_*(Z) \\ \downarrow f_* & & \downarrow f_* \\ \mathcal{F}(X) & \xrightarrow{c_*} & H_*(X) \end{array}$$

If $W \subset X$ is a constructible subset, the class $c_{\text{SM}}(W) := c_*(\mathbf{1}_W) \in H_*(X)$ is called the *Chern-Schwartz-MacPherson (CSM)* class of W . One may regard the CSM classes as an analogue of the total Chern class of the tangent bundle of X in the case X is singular.

MacPherson's definition of the transformation c_* uses Mather classes, and a constructible function Eu_X on X , called the *local Euler obstruction*. The original definition of the local Euler obstruction in [Mac74] uses transcendental methods (the analytic topology). Later, Gonzalez-Sprinberg and Verdier [GS81], found an algebraic definition, thus extending MacPherson's transformation to one with values in the Chow group $A_*(X)$. More recently, Ohmoto [Ohm06] generalized this to the equivariant context. We recall the following properties of Eu_X - see [Mac74, GS81, BS81]:

- Lemma 3.2.**
- (a) *The local Euler obstruction Eu_X is constant along the strata of any Whitney stratification.*
 - (b) $\text{Eu}_X(x) = 1$ if X is nonsingular at x .
 - (c) If $X = X_1 \times X_2$ as varieties, then $\text{Eu}_{X_1 \times X_2}(x_1, x_2) = \text{Eu}_{X_1}(x_1) \cdot \text{Eu}_{X_2}(x_2)$.

Proof. Property (a) follows from [Mac74, Lemma 2], see also [BS81, Prop. 10.1 and Corollaire 10.2]. The properties (b) and (c) are explicitly stated in [Mac74, §3] and [GS81, §4.2]. \square

We could not find a precise reference for the Proposition below, although we believe it to be known to experts.

Proposition 3.3. *Let $f : Z \rightarrow X$ be a smooth morphism of nonsingular complex varieties, and let $Y \subset X$ be a closed subvariety. Then for any $z \in f^{-1}(Y)$, we have $\text{Eu}_{f^{-1}(Y)}(z) = \text{Eu}_Y(f(z))$, i.e. as constructible functions $f^* \text{Eu}_Y = \text{Eu}_{f^{-1}(Y)}$.*

Proof. Let $z \in f^{-1}(Y)$ and let $d := \dim Z - \dim X$. Since $f : Z \rightarrow X$ is a smooth morphism of relative dimension d , [Sta, Lemma 29.34.20] implies that there exists an open affine neighborhood U of z , an open affine neighborhood V of $f(z)$ such that $f(U) \subset V$, and a commutative diagram

$$\begin{array}{ccccc} Z & \xhookleftarrow{\quad} & U & \xrightarrow{\quad \eta \quad} & \mathbb{A}_V^d \\ \downarrow f & & \downarrow f|_U & & \searrow \\ X & \xhookleftarrow{\quad} & V & & \end{array}$$

where η is étale. From the definition, the local Euler obstruction only depends on the local behavior in the *analytic* topology, and this implies that $\text{Eu}_{f^{-1}(Y)}(z) = \text{Eu}_{f^{-1}(Y \cap V)}(z)$. From the diagram above it follows that η provides a local isomorphism in analytic topology between $f^{-1}(V \cap Y)$ and $(V \cap Y) \times \mathbb{A}^d$. Then the claim follows from the product formula in (c) and again by using the local behavior and part (b). \square

By definition, the Euler obstruction can be written as $\text{Eu}_X = \sum e_i \mathbb{1}_{S_i}$, where $S_i \subset S$ is constructible and $e_i = \text{Eu}_X(x_i)$ for any $x_i \in S_i$. Then the Mather class and the MacPherson transformation are related by

$$(12) \quad c_{\text{Ma}}(X) = c_*(\text{Eu}_X).$$

In terms of CSM classes, this can be expressed as

$$(13) \quad c_{\text{Ma}}(X) = \sum_i e_i c_{\text{SM}}(S_i).$$

For φ a constructible function on X , let

$$s(\varphi) = \frac{c_*(\varphi)}{c(TX)}$$

denote the *Segre-MacPherson (SM)* class. The following Verdier-Riemann-Roch (VRR) type theorem was proved by Yokura [Yok99].

Theorem 3.4. *Assume that $f : Z \rightarrow X$ is a smooth morphism of complex algebraic varieties. Then for any constructible function $\varphi \in \mathcal{F}(X)$, $f^*s(\varphi) = s(f^*(\varphi))$. Equivalently, if T_f denotes the relative tangent bundle of f , then*

$$c_*(f^*(\varphi)) = c(T_f) \cap f^*(c_*(\varphi)),$$

as elements in $A_*(X)$.

Proposition 3.3 implies that if $f : Z \rightarrow X$ is a smooth morphism, then $f^*(\text{Eu}_Y) = \text{Eu}_{f^{-1}(Y)}$. If one takes $\varphi = \text{Eu}_Y$, this implies that in terms of Mather classes

$$(14) \quad c_{\text{Ma}}(f^{-1}(Y)) = c(T_f) \cap f^*(c_{\text{Ma}}(Y)) \in A_*(Z).$$

In Section 9 below we will give another proof of this result, in the case of Mather classes of Schubert varieties, in the case when f the projection between two (generalized) flag manifolds.

As usual, the results from this section can be extended to the case when all varieties have a torus T action, and all morphisms are T -equivariant. The local Euler obstruction is the same, but one uses an equivariant Whitney stratification, and Ohmoto's equivariant version of MacPherson's transformation c_* [Ohm06]; see also [AMSS17].

4. PRELIMINARIES ON FLAG MANIFOLDS AND COMINUSCULE SPACES

4.1. Preliminaries. References for this section are [Kum02] and [Bri05]. Let G be a complex semisimple Lie group and fix a pair of opposite Borel subgroups, B and B^- in G . The opposite Borel subgroups determine a maximal torus $T := B \cap B^-$, and a root system $R \subset \text{Hom}(T, \mathbb{C}^*)$.

Let $R = R^+ \sqcup R^-$ be the decomposition into positive and negative roots, and let $\Delta \subset R^+$ be the set of simple roots. We have a partial order on R^+ , given by $\alpha < \beta$ if $\beta - \alpha$ is a non-negative combination of positive roots.

The Weyl group $W := N_G(T)/T$ associated to (G, T) is a Coxeter group generated by the simple reflections $s_i := s_{\alpha_i}$, for $\alpha_i \in \Delta$. Denote by $\ell : W \rightarrow \mathbb{N}$ the length function and by w_0 the longest element.

Any subset $S \subset \Delta$ determines a standard parabolic subgroup $P \supset B$. We denote by R_P^+ the subset of R^+ consisting of roots whose support is contained in S . The Weyl group W_P of P is generated by the simple reflections s_i , for $\alpha_i \in S$. Denote by w_P the longest element in W_P , and let W^P be the set of minimal length representatives for the cosets in W/W_P . If $w \in W$, the coset wW_P has a unique minimal length representative $w^P \in W^P$ and as usual we set $\ell(wW_P) := \ell(w^P)$.

Let G/P be the generalized flag manifold; this is a projective manifold of dimension $\ell(w_0W_P)$. If $w \in W^P$ is a minimal length representative, the B -orbit $X_w^{P,\circ} = BwP/P$, and the B^- -orbit $(X^P)^{w,\circ} = B^-wP/P$, are opposite Schubert cells for w . With this definition, we have isomorphisms, $X_w^{P,\circ} \simeq \mathbb{C}^{\ell(w)}$ and

$X^{w,P,\circ} \simeq \mathbb{C}^{\dim G/P - \ell(w)}$. The *Schubert varieties* X_w^P and $X^{w,P}$ are the closures of the Schubert cells $X_w^{P,\circ}$ and $X^{w,P,\circ}$ respectively.

Every P -representation V determines a G -equivariant vector bundle, $G \times^P V \rightarrow G/P$. The points of $G \times^P V$ are equivalence classes $[g, v]$, for pairs $(g, v) \in G \times V$ such that $(g, v) \simeq (gp^{-1}, pv)$, and the G -action on $G \times^P V$ is given by left multiplication, $g.[g', v] := [gg', v]$. The main examples considered in this note are the following.

If $P = B$ is a Borel subgroup, we will take $V := \mathbb{C}_\lambda$, the one dimensional B -module of character λ . The resulting line bundle is $\mathcal{L}_\lambda := G \times^B \mathbb{C}_\lambda$.

Let \mathfrak{p} and \mathfrak{g} be the Lie algebras of P and G respectively. The group P acts on \mathfrak{p} and \mathfrak{g} via the adjoint action. Setting $V := \mathfrak{g}/\mathfrak{p}$ in the construction above, we obtain the tangent bundle $T(G/P) = G \times^P \mathfrak{g}/\mathfrak{p}$.

Let U_P be the unipotent radical of P . The subspace $\mathfrak{u}_P := \text{Lie}(U_P)$ is stable under the adjoint action of P on \mathfrak{p} . Following [Spr69], we have a P -module isomorphism,

$$\mathfrak{g}/\mathfrak{p} = \mathfrak{u}_P^* = \bigoplus_{\alpha \geq \alpha_P} \mathfrak{g}_{-\alpha},$$

where \mathfrak{g}_α is the one dimensional root subspace of \mathfrak{g} corresponding to the root α .

4.2. Cominuscule spaces. We recall next the basic definitions on cominuscule spaces; see e.g. [BCMP18]. A maximal parabolic subgroup of G is determined upto conjugacy by removing a simple root α_P from Δ . We say that a maximal parabolic subgroup is *cominuscule* if the corresponding simple root α_P appears with coefficient 1 in the highest root from R^+ ; the associated flag manifold G/P is called a *cominuscule space*.

The Dynkin diagrams, along with the possible choices of cominuscule nodes, are listed in Table 1. The cominuscule spaces are classified as follows:

- The Grassmann manifolds $\text{Gr}(k, n)$ if G is of type A .
- The Lagrangian Grassmannian $\text{LG}(n, 2n)$ in type C - this parametrizes vector subspaces of dimension n in \mathbb{C}^{2n} , isotropic with respect to a non-degenerate skew-symmetric quadratic form.
- The maximal orthogonal Grassmannian $\text{OG}(n, 2n)$ in type D - this the connected variety parametrizing vector subspaces of dimension n in \mathbb{C}^{2n} , isotropic with respect to a non-degenerate quadratic form.
- Quadrics in type B and D .
- There are also two cominuscule spaces in the exceptional types E_6 and E_7 called respectively the Cayley plane and the Freudenthal variety.

The n^{th} node of B_n corresponds to the maximal orthogonal Grassmannian $\text{OG}(n, 2n+1)$. This is also cominuscule, but when regarded as a homogeneous space under the type D group. Indeed, this space parametrizes maximal subspaces which are isotropic with respect to a symmetric non-degenerate form in \mathbb{C}^{2n+1} . From this description it follows that $\text{OG}(n, 2n+1)$ is isomorphic to either of the connected components of $\text{OG}(n+1, 2n+2)$, and this isomorphism preserves Schubert classes; see [FP98, p. 68] or e.g. [IMN16, §3.4] for further details. Similarly, the space corresponding to the first node in type C is the projective space \mathbb{P}^{2n-1} . This is a cominuscule space when regarded as a homogeneous space of type A .

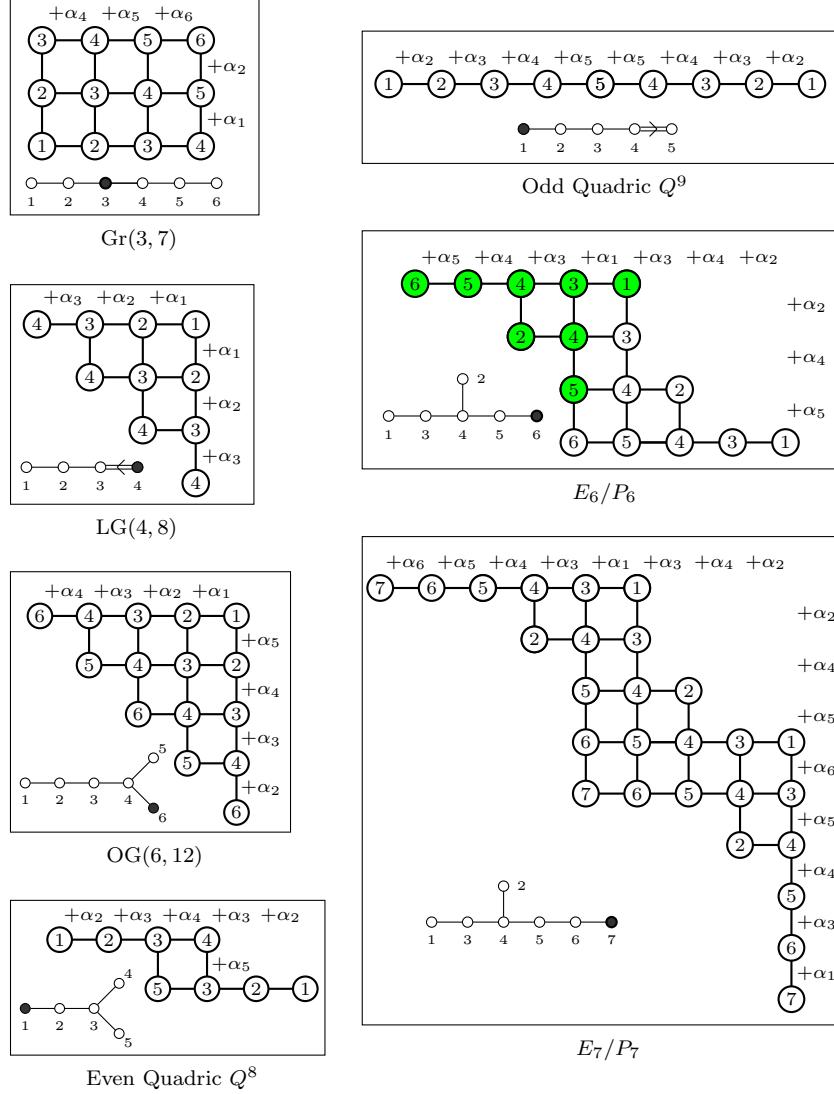


TABLE 1. Hasse diagrams for the lattices $R_{\geq \alpha_P}$. The top left node is the minimal root α_P , and the bottom right node is the highest root. See also Section 11.

5. A RESOLUTION OF SINGULARITIES FOR THE CONORMAL SPACES OF COMINUSCULE SCHUBERT VARIETIES

By a cominuscule Schubert variety, we simply mean a Schubert subvariety of a cominuscule space. We recall here the construction from [SinBA] of a resolution of singularities of the conormal space of a cominuscule Schubert variety. This is the main tool required to calculate (equivariant) Mather classes of these Schubert varieties.

Fix a minimal length representative $w \in W^P$, and let $\underline{w} = (s_{i_1}, \dots, s_{i_k})$ be a reduced word for w . Consider the vector subspace

$$\mathfrak{u}_w = \bigoplus_{\substack{\alpha \geq \alpha_P \\ w(\alpha) > 0}} \mathfrak{g}_\alpha \subset \mathfrak{u}_P.$$

A key fact proved in [SinBA, Lemma 2.1] is that \mathfrak{u}_w is a B -submodule of \mathfrak{u}_P under the adjoint action. To this data we can associate the Bott-Samelson variety $B_{\underline{w}} := P_{i_1} \times^B P_{i_2} \times^B \dots \times^B P_{i_k}/B$, a B -variety under left multiplication, and the vector bundle

$$\mathcal{E}_{\underline{w}} := P_{i_1} \times^B P_{i_2} \times^B \dots \times^B P_{i_k} \times^B \mathfrak{u}_w \rightarrow B_{\underline{w}},$$

see, for example, [BK05, Ch.2]. We have a commutative diagram

$$(15) \quad \begin{array}{ccccccc} \mathcal{E}_{\underline{w}} & \xrightarrow{\theta'_{\underline{w}}} & \overline{BwB} \times^B \mathfrak{u}_w & \xrightarrow{\pi'} & T_{X_w^P}^*(G/P) & \hookrightarrow & T^*(G/P) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B_{\underline{w}} & \xrightarrow{\theta_{\underline{w}}} & X_w^B & \xrightarrow{\pi} & X_w^P & \hookrightarrow & G/P, \end{array}$$

where the maps are defined as follows:

- the morphism $\theta_{\underline{w}}$ is birational, and it is the usual projection from the Bott-Samelson desingularization of the Schubert variety X_w^B ; the middle vertical map is the vector bundle projection;
- the left square is a fibre square, inducing the morphisms $\theta'_{\underline{w}}$ and the left vertical map (also a vector bundle projection);
- the rightmost vertical morphism is the usual projection from the cotangent bundle of G/P ;
- π is the restriction to X_w^B of the usual projection $G/B \rightarrow G/P$;
- π' is obtained by the composition of the morphisms

$$\overline{BwB} \times^B \mathfrak{u}_w \hookrightarrow \overline{BwB} \times^B \mathfrak{u}_P \hookrightarrow G \times^B \mathfrak{u}_P \rightarrow G \times^P \mathfrak{u}_P = T^*(G/P).$$

Each of these morphisms is proper, hence the composition is also proper. It was proved in [SinBA, Thm. A] that π' is birational.

Combining everything proves that the following holds [SinBA]:

Theorem 5.1. *The composition,*

$$\pi' \circ \theta'_{\underline{w}} : \mathcal{E}_w \rightarrow T_{X_w^P}^*(G/P),$$

is proper and birational, hence a resolution of singularities of the conormal space $T_{X_w^P}^(G/P)$.*

6. MATHER CLASSES OF SCHUBERT VARIETIES

In this section we prove the formula calculating the Mather classes of Schubert varieties in cominuscule spaces, and we illustrate the calculation with several examples.

6.1. The formula for Mather classes. Let $\mathcal{U}_w := \overline{BwB} \times^B \mathfrak{u}_w \rightarrow X_w^B$ denote the restriction of the homogeneous bundle $G \times^B \mathfrak{u}_w$ to the Schubert variety X_w^B . There is an exact sequence of homogeneous vector bundles on X_w^B given by

$$(16) \quad 0 \longrightarrow \mathcal{U}_w \longrightarrow \pi^* T^*(G/P)|_{X_w^B} \longrightarrow \mathcal{T}_w^* \longrightarrow 0$$

where $\mathcal{T}_w^* := \overline{BwB} \times^B (\mathfrak{u}_P/\mathfrak{u}_w)$. Observe that \mathcal{T}_w^* restricted to the open Schubert cell in X_w^B is the cotangent bundle of this cell (explaining the choice of notation). Theorem 5.1 and the considerations from diagram (15) imply that the diagram

$$(17) \quad \begin{array}{ccc} \mathcal{U}_w & \xrightarrow{\pi'} & T_{X_w^P}^*(G/P) \\ \downarrow & & \downarrow \\ X_w^B & \xrightarrow{\pi} & X_w^P \end{array}$$

satisfies the hypotheses in Lemma 2.1 with $X' = Y' = X_w^B$ and \mathcal{U}_w a subbundle of $\pi^* T^*(G/P)|_{X_w^P}$. This allows us to calculate the Mather class of the Schubert variety X_w^P . Let \mathcal{T}_w denote the dual of the bundle \mathcal{T}_w^* .

Theorem 6.1. *Let $w \in W^P$ be a minimal length representative. Then the equivariant Mather class of X_w^P is*

$$c_{Ma}^T(X_w^P) = \pi_*(c^T(\mathcal{T}_w) \cap [X_w^B]),$$

where $c^T(\mathcal{T}_w) = \prod_{\alpha \in I(w)} c^T(\mathcal{L}_{-\alpha})$.

Proof. It follows from the T -module decomposition,

$$(\mathfrak{u}_P/\mathfrak{u}_w)^* = \bigoplus_{\substack{\alpha \geq \alpha_P \\ w(\alpha) < 0}} \mathfrak{g}_{-\alpha} = \bigoplus_{\alpha \in I(w)} \mathfrak{g}_{-\alpha},$$

that the total Chern class of the homogeneous vector bundle \mathcal{T}_w has the same localization at T -fixed points $e_v \in G/B$ as the Chern class of the vector bundle

$$\bigoplus_{\alpha \in I(w)} G \times^B \mathbb{C}_{-\alpha} = \bigoplus_{\alpha \in I(w)} \mathcal{L}_{-\alpha}.$$

The result for $c^T(\mathcal{T}_w)$ now follows from Whitney's formula.

To prove the formula for the Mather class, first observe that the birationality property of Segre classes (see Lemma 2.1) applied to the diagram (17) yields

$$(18) \quad s^T(T_{X_w^P}^*(G/P)) = \pi_*(s^T(\mathcal{U}_w) \cap [X_w^B]) = \pi_*(c^T(\mathcal{U}_w)^{-1} \cap [X_w^B]).$$

It follows from the Eq. (11) version of Theorem 3.1 that

$$\begin{aligned} c_{Ma}^{T,\vee}(X_w^P) &= c^T(T^*(G/P)) \cap s^T(T_{X_w^P}^*(G/P)) \\ &= c^T(T^*(G/P) \cap \pi_*(c^T(\mathcal{U}_w)^{-1} \cap [X_w^B])) \\ &= \pi_*(c^T(\mathcal{T}_w^*) \cap [X_w^B]), \end{aligned}$$

where the last equality follows from the projection formula. The proof ends by changing the signs in each homogeneous component; this corresponds to taking the Chern classes of the dual bundle \mathcal{T}_w . \square

Another algorithm to calculate Mather classes, in the case of Grassmannians, was found by B. Jones [Jon10]. He used Zelevinsky's small resolutions for Schubert varieties [Zel83], and equivariant localization, to calculate the *Kazhdan-Lusztig class* (KL) of a Schubert variety. As explained in Section 10 below (see also [AMSS17, §6]) this coincides with the Mather class. Sankaran and Vanchinathan [SV94] found Schubert varieties in the Lagrangian Grassmannian which do not admit small resolutions and Perrin [Per07] characterized the minuscule Schubert varieties with this property.

6.2. Examples. The previous theorem, combined with the (equivariant) Chevalley formula, gives an effective way to calculate the (equivariant) Mather class. We recall the equivariant Chevalley formula, following [BM15, Thm. 8.1]; see also [Kum02, Cor. 11.3.17 and Thm. 11.1.7].

Let λ be a weight, and $\mathcal{L}_\lambda = G \times^B \mathbb{C}_\lambda$ be the associated line bundle. Then

$$c_1^T(\mathcal{L}_\lambda) \cap [X_w^B] = w(\lambda)[X_w^B] + \sum_{\alpha} \langle -\lambda, \alpha^\vee \rangle [X_{ws_\alpha}^B],$$

where the sum is over all positive roots α such that $\ell(ws_\alpha) = \ell(w) - 1$.² Applying repeatedly this formula we can recursively calculate the expression

$$c^T(\mathcal{T}_w) \cap [X_w^B] = \sum_{v \leq w} a_{w,v} [X_v^B],$$

where $a_{w,v} \in A_T^*(pt)$. For instance,

$$a_{w,w} = \prod_{\alpha \in I(w)} (1 - w(\alpha)).$$

Recall that

$$\pi_*[X_v^B] = \begin{cases} [X_v^P] & \text{if } v \in W^P, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the equivariant Mather class of X_w equals $\sum_{v \leq w; v \in W^P} a_{w,v} [X_v^P]$.

Example 6.2. We illustrate this calculation next. Let $G = \mathrm{SL}_4$ and the simple roots $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \alpha_3 = \varepsilon_3 - \varepsilon_4$ (notation as in [Bou02]). The maximal parabolic P associated to α_2 gives the Grassmannian $G/P = \mathrm{Gr}(2, 4)$. One may identify the elements in W^P by Young diagrams, which in turn may be used to read both the inversion set and the reduced word for w ; see e.g. [BCMP18, §3] and Section 11 below. For instance, in the table below, the green portion corresponds to the Schubert divisor $X_{s_1 s_3 s_2}^P \subset \mathrm{Gr}(2, 4)$, with inversion set $I(s_1 s_3 s_2) = \{\alpha_2, \varepsilon_2 - \varepsilon_4, \varepsilon_1 - \varepsilon_3\}$. The Schubert divisor is the smallest example of a singular Schubert variety in $\mathrm{Gr}(2, 4)$: it is a 3 dimensional quadric singular at the point $1.P$.

| | | | |
|---------|---------|---|---|
| α₂ | ε₂ − ε₄ | 2 | 3 |
| ε₁ − ε₃ | ε₁ − ε₄ | 1 | 2 |

²The minus sign is explained by the fact that if ω_i is the i^{th} fundamental weight, then non-equivariantly $c_1(\mathcal{L}_{-\omega_i})$ is the class of the Schubert divisor $X^{s_i, B}$, an effective class.

The space G/B is the complete flag manifold $\mathrm{Fl}(4)$. The Chevalley formula in $A^*(\mathrm{Fl}(4))$ gives that

$$(19) \quad \begin{aligned} c(\mathcal{T}_w) \cap [X_{s_1 s_3 s_2}^B] &= [X_{s_1 s_3 s_2}^B] + 3[X_{s_3 s_2}^B] + 4[X_{s_3 s_1}^B] + 3[X_{s_3}^B] + 3[X_{s_1 s_2}^B] \\ &\quad + 8[X_{s_2}^B] + 3[X_{s_1}^B] + 6[X_{id}^B]. \end{aligned}$$

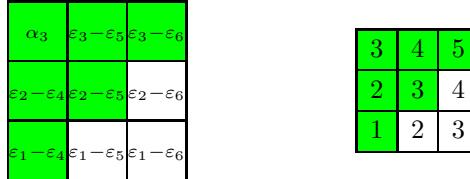
Pushing forward to $\mathrm{Gr}(2, 4)$, we obtain the Mather class:

$$(20) \quad \begin{aligned} c_{\mathrm{Ma}}(X_{s_1 s_3 s_2}^P) &= [X_{s_1 s_3 s_2}^P] + 3[X_{s_3 s_2}^P] + 3[X_{s_1 s_2}^P] + 8[X_{s_2}^P] + 6[X_{id}^P] \\ &= \square + 3 \square + 3 \square + 8 \square + 6 \emptyset. \end{aligned}$$

(We simply used λ for the Schubert class indexed by λ .) The equivariant calculation is more involved, and we present only the final answer.

$$(21) \quad \begin{aligned} c_{\mathrm{Ma}}^T(X_{s_1 s_3 s_2}^P) &= (1 + \alpha_1)(1 + \alpha_3)(1 + \alpha_1 + \alpha_2 + \alpha_3) \square \\ &\quad + (1 + \alpha_3)(3 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3) \square \\ &\quad + (1 + \alpha_1)(3 + 2\alpha_1 + 2\alpha_2 + \alpha_3 + 3) \square \\ &\quad + (8 + 2\alpha_1 + 4\alpha_2 + 2\alpha_3) \square + 6 \emptyset. \end{aligned}$$

Example 6.3. In our next example we consider $G = \mathrm{SL}_6$ and the Grassmannian $G/P = \mathrm{Gr}(3, 6)$. We consider the Schubert variety indexed by the green portion in the diagrams below.

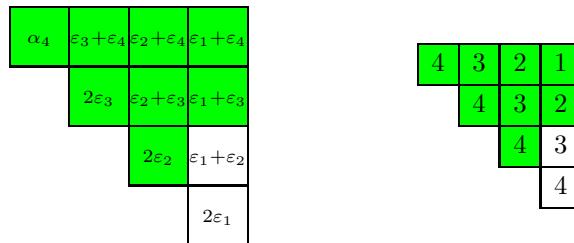


Then $w = s_1 s_3 s_2 s_5 s_4 s_3$ and the inversion set consists of the roots in the green boxes. The Mather class equals

$$\begin{aligned} c_{\mathrm{Ma}}(\square) &= \square + 4\square + 4\square + 4\square + 15\square + 15\square + 15\square \\ &\quad + 17\square + 52\square + 17\square + 54\square + 54\square + 60\square + 24 \emptyset. \end{aligned}$$

This is in accordance [Jon10, Table 2]. In fact, we were able to recover all calculations from *loc. cit.* - see Section 11 below.

Example 6.4. We now consider the symplectic group Sp_8 and the Lagrangian Grassmannian $\mathrm{LG}(4, 8)$. The elements in W^P are indexed by strict partitions included in the $(4, 3, 2, 1)$ staircase. Take $\lambda = (4, 3, 1)$ corresponding to the green boxes in the diagram below. As usual, the notation for the roots follows [Bou02]; we refer the reader to [BCMP18, §3] and Section 11 below for further combinatorial details.



In this case the reduced word is $w = s_4s_2s_3s_4s_1s_2s_3s_4$, and the inversion set consists of the entries in the green boxes of the first diagram. The Mather class is given by

$$\begin{aligned} c_{\text{Ma}}(\text{Diagram}) &= \text{Diagram} + 4\text{Diagram} + 7\text{Diagram} + 27\text{Diagram} + 25\text{Diagram} + 60\text{Diagram} + 92\text{Diagram} \\ &\quad + 45\text{Diagram} + 241\text{Diagram} + 183\text{Diagram} + 269\text{Diagram} + 246\text{Diagram} + 132\text{Diagram} + 24\emptyset. \end{aligned}$$

Remark 6.5. We observe that the bundle \mathcal{T}_w is *not* globally generated, even when restricted to X_w^B ; if it were, then multiplying its total non-equivariant Chern class by any Schubert class would have to be effective. To see that this is not the case, consider the situation above when $G/P = \text{Gr}(2, 4)$, $w = s_1s_3s_2 \in W^P$, and take $u = s_3$. Then $X_u^B \subset X_w^B$ and $c(\mathcal{T}_w) \cap [X_u^B] = [X_{s_3}^B] - [X_{id}^B]$. Despite this, examples suggest that the Mather classes are effective; see Section 10 below for more about this.

More examples are included in Section 11 below.

7. A COHOMOLOGICAL FORMULA FOR THE LOCAL EULER OBSTRUCTION

The goal of this section is to prove Theorem 7.1, which gives a formula for the local Euler obstruction function for cominuscule Schubert varieties. Different formulae in the classical Lie types were also obtained by Boe and Fu [BF97].

Because the CSM class of the Schubert cell $c_{\text{SM}}(X_u^{P,\circ})$ is a deformation of the fundamental class $[X_u^P]$ of the Schubert variety, the CSM classes $c_{\text{SM}}(X_u^{P,\circ})$ ($u \in W^P$) form a basis of the Chow group $A_*(G/P)$. It was proved in [AMSS17] that the (Poincaré) dual basis of the CSM basis of Schubert cells with respect to the intersection pairing is the family of the *Segre-Schwartz-MacPherson* (SSM) classes $s_{\text{SM}}((X^P)^{v,\circ})$ ($v \in W^P$) of the opposite Schubert cells.

We recall next the relevant definitions. The SSM class is defined by

$$s_{\text{SM}}((X^P)^{v,\circ}) = \frac{c_{\text{SM}}((X^P)^{v,\circ})}{c(T(G/P))}.$$

The Chern class in the quotient is invertible because $c(T(G/P)) = 1 + \kappa$, where κ is a nilpotent element. Then by [AMSS17, Thm.9.4],

$$(22) \quad \langle c_{\text{SM}}(X_u^{P,\circ}), s_{\text{SM}}((X^P)^{v,\circ}) \rangle = \int_{G/P} c_{\text{SM}}(X_u^{P,\circ}) \cdot s_{\text{SM}}((X^P)^{v,\circ}) = \delta_{u,v}.$$

Combined with Theorem 6.1, this duality implies a cohomological formula for the Euler obstruction coefficients $e_{w,v}$ from equation (32); we record this next.

Theorem 7.1. *Let $v, w \in W^P$ and assume that $v \leq w$. Then the local Euler obstruction coefficient $e_{w,v}$ is given by*

$$e_{w,v} = \sum \int_{G/B} \frac{c(\mathcal{T}_w) \cdot [X_w^B]}{c(T(G/B))} \cdot c_{\text{SM}}((X^B)^{u,\circ}) = \sum \int_{G/B} c(\mathcal{T}_w) \cdot [X_w^B] \cdot c_{\text{SM}}^V((X^B)^{u,\circ}),$$

where the sum is over $u \in W; v \leq u \leq w$ such that $uW_P = vW_P$.

Proof. By the duality equation (22), and the projection formula, the Euler obstruction coefficients are given by

$$e_{w,v} = \int_{G/P} \pi_*(c(\mathcal{T}_w) \cdot [X_w^B]) \cdot s_{\text{SM}}((X^P)^{v,\circ}) = \int_{G/B} c(\mathcal{T}_w) \cdot [X_w^B] \cdot \pi^* s_{\text{SM}}((X^P)^{v,\circ}).$$

The Verdier-Riemann-Roch formula implies that

$$\pi^* s_{\text{SM}}((X^P)^{v,\circ}) = s_{\text{SM}}(\pi^{-1}((X^P)^{v,\circ})) = \sum s_{\text{SM}}((X^B)^{u,\circ}),$$

where the sum is over all $u \geq v$ such that $uW_P = vW_P$. Further, it is proved in [AMSS17, Cor. 7.4] that for any $u \in W$,

$$s_{\text{SM}}((X^B)^{u,\circ}) = c_{\text{SM}}^\vee((X^B)^{u,\circ}).$$

Since the class $c_{\text{SM}}^\vee(X^B)^{u,\circ}$ is supported on the Schubert variety $X^{B,u}$, it follows that the product $[X_w^B] \cdot [X^{B,u}] = 0$ unless $u \leq w$. The claim follows by combining the three equations above. \square

We note that an explicit calculation for the CSM classes was obtained in [AM16]. Therefore the integrals from Theorem 7.1 can be explicitly computed in small examples, using software such as the *Equivariant Schubert Calculator* by A. Buch.

Example 7.2. We continue with the example $G/P = \text{Gr}(2, 4)$ and $X_w^P = X_{s_1 s_3 s_2}^P$ the Schubert divisor. Recall from equation (19):

$$c(\mathcal{T}_w) \cap [X_{s_1 s_3 s_2}^B] = [X_{s_1 s_3 s_2}^B] + 3[X_{s_3 s_2}^B] + 4[X_{s_3 s_1}^B] + 3[X_{s_3}^B] + 3[X_{s_1 s_2}^B] + 8[X_{s_2}^B] + 3[X_{s_1}^B] + 6[X_{id}^B].$$

If we take $v = s_3 s_2$, then $u = v$ and we obtain that $c_{\text{SM}}^\vee(X^{B,s_3 s_2,\circ})$ equals

$$\begin{aligned} & [X^{B,s_3 s_2}] - [X^{B,s_3 s_2 s_1}] - 2[X^{B,s_3 s_1 s_2}] + [X^{B,s_3 s_1 s_2 s_1}] + 4[X^{B,s_2 s_3 s_1 s_2}] \\ & - 2[X^{B,s_2 s_3 s_1 s_2 s_1}] - [X^{B,s_2 s_3 s_2}] + [X^{B,s_2 s_3 s_2 s_1}] + 3[X^{B,s_1 s_2 s_3 s_2}] - 2[X^{B,s_1 s_2 s_3 s_2 s_1}] \\ & + [X^{B,s_1 s_2 s_3 s_1 s_2 s_1}] - 3[X^{B,s_1 s_2 s_3 s_1 s_2}]. \end{aligned}$$

Since $\int_{G/B} [X_{v_1}^B] \cdot [X^{B,v_2}] = \delta_{v_1, v_2}$ we obtain that

$$e_{s_1 s_3 s_2, s_3 s_2} = -2 + 3 = 1.$$

Of course, this was expected, as the Schubert divisor is only singular at the base point $1.P$.

Example 7.3. Consider the Lagrangian Grassmannian $\text{LG}(2, 4)$. This is isomorphic to a 3 dimensional quadric in \mathbb{P}^4 . The set W^P indexing the Schubert varieties is in bijection with the strict partitions in the 2×2 rectangle: (0) , (1) , (2) and $(2, 1)$. The only singular Schubert variety is the divisor X_{\square}^P , with singularity at the point $X_{\emptyset}^P = 1.P$. One calculates that

$$e_{(2),(2)} = e_{(2),(1)} = 1; \quad e_{(2),(0)} = 0.$$

This verifies examples from [BF97, p. 456]. Using the isomorphism of $\text{LG}(2, 4)$ to the 3-dimensional quadric, it also verifies one instance of [BF97, Eq.(6.3.3)].

Example 7.4. We now consider the Lagrangian Grassmannian $\text{LG}(3, 6)$. In this case the Schubert varieties are indexed by strict partitions in the 3×3 rectangle. An interesting example is obtained by considering the divisor $X_{(3,2)}^P$. In this case, the Euler obstructions are:

$$\begin{aligned} e_{(3,2),(3,2)} &= e_{(3,2),(3,1)} = e_{(3,2),(2,1)} = e_{(3,2),(0)} = 1 \\ e_{(3,2),(3)} &= e_{(3,2),(2)} = e_{(3,2),(1)} = 0. \end{aligned}$$

Observe that the Euler obstruction at $X_{id}^P = 1.P$ is 1, even though the variety is singular at that point.

8. LOCALIZATION OF CONORMAL SPACES

The goal of this section is to use Theorem 6.1 to obtain formulae for the localization of conormal spaces.

Denote by $\iota : G/P \hookrightarrow T^*(G/P)$ the zero section. By equation (10),

$$(23) \quad \iota^*[T_{X_w^P}^*(G/P)]_{T \times \mathbb{C}^*} = (-1)^{\ell(w)} c_{\text{Ma}}^T(X_w^P)^\hbar = (-1)^{\ell(w)} \pi_*((c^T(\mathcal{T}_w) \cap [X_w^B])^\hbar).$$

This is a class in $A_0^{T \times \mathbb{C}^*}(G/P)$ and it is obtained by homogenizing the equivariant (homology) class $c_{\text{Ma}}^T(X_w^P)$. We can use (23) to calculate the localization at the fixed points of the conormal space. Fix $u \in W^P$ such that $u \leq w$. If we write

$$c_{\text{Ma}}^T(X_w^T) = \sum a_v [X_v^P],$$

where $a_v \in A_T^*(pt)$, then localizing at $u \in W$ gives

$$c_{\text{Ma}}^T(X_w^T)|_u = \sum a_v [X_v^P]|_u.$$

The coefficients a_v are not homogeneous, thus homogenizing each term $(a_v [X_v^P])|_u$ amounts to multiplying each homogeneous component by the appropriate power of $\hbar \in A_{T \times \mathbb{C}^*}^1(pt)$ such that the total cohomological degree equals $\dim G/P$ (i.e. homological degree 0.)

A more explicit formula is obtained by analyzing the homogenization $(c^T(\mathcal{T}_w) \cap [X_w^B])^\hbar$. Let $a_1, \dots, a_{\ell(w)}$ be the T -equivariant Chern roots of \mathcal{T}_w . Then

$$c(\mathcal{T}_w) \cap [X_w^B] = \left(\prod_{i=1}^{\ell(w)} (1 + a_i) \right) \cap [X_w^B] = \sum_{i \geq 0} e_i(a_1, \dots, a_{\ell(w)}) \cap [X_w^B],$$

where e_i denote the elementary symmetric functions. The term $e_i(a_1, \dots, a_{\ell(w)}) \cap [X_w^B]$ belongs to the equivariant Chow group $A_{\ell(w)-i}^T(G/B)$, therefore its homogenization by \hbar is

$$(24) \quad (c(\mathcal{T}_w) \cap [X_w^B])^\hbar = \sum_{i \geq 0} \hbar^{\ell(w)-i} e_i(a_1, \dots, a_{\ell(w)}) \cap [X_w^B] = \left(\prod_{i=1}^{\ell(w)} (\hbar + a_i) \right) \cap [X_w^B].$$

One key observation is that the quantity $(-1)^{\ell(w)} \prod_{i=1}^{\ell(w)} (\hbar + a_i)$ has a geometric interpretation. If one considers the \mathbb{C}^* -action on the cotangent bundle $T^*(G/B)$ with character \hbar^{-1} , then the elements $-\hbar - a_i$ are the $T \times \mathbb{C}^*$ -equivariant Chern roots of \mathcal{T}_w^* . Thus

$$(25) \quad (-1)^{\ell(w)} \prod_{i=1}^{\ell(w)} (\hbar + a_i) = c_{\ell(w)}^{T \times \mathbb{C}^*}(\mathcal{T}_w^*).$$

The fibre of \mathcal{T}_w^* over the fixed point e_v is

$$v \cdot \bigoplus_{\alpha \in I(w)} \mathfrak{g}_\alpha \otimes \mathbb{C}_{-\hbar} = \bigoplus_{\alpha \in I(w)} \mathfrak{g}_{v(\alpha)} \otimes \mathbb{C}_{-\hbar},$$

and from this we deduce the formula for the localization:

$$(c_{\ell(w)}^{T \times \mathbb{C}^*}(\mathcal{T}_w^*))|_v = \prod_{\alpha \in I(w)} (-\hbar + v(\alpha)).$$

Combining with the equation (24), this proves the following Lemma:

Lemma 8.1. *Let $v \leq w$. Then the following holds in $A_{T \times \mathbb{C}^*}^{\dim G/B}(pt)$:*

$$(-1)^{\ell(w)}((c(\mathcal{T}_w) \cap [X_w^B])^\hbar)|_v = \prod_{\alpha \in I(w)} (-\hbar + v(\alpha)) \cdot [X_w^B]|_v.$$

We note that since \mathbb{C}^* acts trivially on G/B , the $T \times \mathbb{C}^*$ -localization $[X_w^B]|_v$ is the same as the T -equivariant localization. A formula for the latter can be found in [Kum02, Thm. 11.1.7]; see also [AJS94, App. D] or [Bil99].

The last step to calculate the localization of the class of the conormal space $T_{X_w^P}^*(G/P)$ is to relate the localization of the class from Lemma 8.1 to the localization of its push forward. For that, we need a generalization of the localization formula. For $v \in W$, let $[e_v]$ denote the T -equivariant class of the fixed point e_v .

Lemma 8.2. *Let $u \in W^P$ and let $\kappa \in H_T^*(G/B)$ be any equivariant cohomology class. Then*

$$\pi_*(\kappa)|_{uW_P} = \sum \frac{[e_{uW_P}]|_{uW_P}}{[e_v]|_v} \kappa|_v$$

in an appropriate localization of $A_T^*(G/B)$, where the sum is over $v \in W$ such that $uW_P = vW_P$.

Proof. By injectivity of localization map, the classes $[e_v]$ form a basis for the T -equivariant cohomology of G/B , localized at the prime ideal generated by the equivariant parameters $A_T^*(pt)$. Thus we can expand $\kappa = \sum c_v [e_v]$ where the sum is over $v \in W$. Localizing both sides at $v \in W$, we obtain that $\kappa|_v = c_v [e_v]|_v$. Pushing forward and localizing at uW_P one obtains

$$\pi_*(\kappa)|_{uW_P} = \sum_{v \in W} c_v \pi_*([e_v])|_{uW_P} = \sum_{v \in W} c_v [e_{vW_P}]|_{uW_P} = \sum_{v \in W} \frac{[e_{vW_P}]|_{uW_P}}{[e_v]|_v} \kappa|_v.$$

Since $[e_{vW_P}]|_{uW_P}$ is nonzero only when $uW_P = vW_P$, the last sum is as in the statement of the lemma, and this finishes the proof. \square

Theorem 8.3. *Let $u, w \in W^P$ such that $u \leq w$, and let \mathbb{C}^* act on $T^*(G/P)$ by character \hbar^{-1} . Then the $T \times \mathbb{C}^*$ -localization of the conormal space $T_{X_w^P}^*(G/P)$ is given by:*

$$[T_{X_w^P}^*(G/P)]|_{uW_P} = \sum_{\substack{v \leq w \\ vW_P = uW_P}} \frac{\prod_{\alpha \in I(w)} (-\hbar + v(\alpha)) \cdot \prod_{\alpha \in R^+ \setminus R_P^+} u(-\alpha)}{\prod_{\alpha \in R^+} v(-\alpha)} [X_w^B]|_v.$$

Proof. Let $\kappa = (-1)^{\ell(w)}(c^T(\mathcal{T}_w) \cap [X_w^B])^\hbar$, regarded as a cohomology class in $A_T^*(G/B)[\hbar]$. By equation (23), the left hand side equals to $\pi_*(\kappa)|_{uW_P}$. Since \mathbb{C}^* acts trivially on G/B , we have $A_{T \times \mathbb{C}^*}^*(G/B) = A_T^*(G/B)[\hbar]$, and further, the projection π_* is $A_{T \times \mathbb{C}^*}^*(pt)$ -linear. To finish the proof, apply Lemmas 8.1 and 8.2, using the fact that the $T \times \mathbb{C}^*$ equivariant Euler classes $[e_v]|_v$ and $[e_{uW_P}]|_{uW_P}$ coincide with the T -equivariant ones; further, the latter equal $[e_v]|_v = \prod_{\alpha \in R^+} v(-\alpha)$ and $[e_{uW_P}]|_{uW_P} = \prod_{\alpha \in R^+ \setminus R_P^+} u(-\alpha)$. \square

Example 8.4. Let $u = w$. The only v satisfying the requirements is $v = w$. Then

$$\iota_w^* [T_{X_w^P}^*(G/P)]_{T \times \mathbb{C}^*} = \frac{\prod_{\alpha \in I(w)} (-\hbar + v(\alpha)) \cdot [X_w^B]|_w}{\prod_{\alpha \in R_P^+} v(-\alpha)} = \prod_{\alpha \in I(w)} (-\hbar + w(\alpha)) \cdot [X_w^P]|_w,$$

where the last equality follows from standard manipulations of (equivariant) Euler classes, for example by using Lemma 8.2.

In [LS17], Lakshmibai and Singh identified certain conormal spaces as open subsets of affine Schubert varieties. It would be interesting to obtain localization formulae for the conormal spaces using localization for affine Schubert varieties.

9. MATHER CLASSES OF PULL BACKS OF SCHUBERT VARIETIES

In this section we let P, Q be two arbitrary parabolic subgroups such that $B \subset Q \subset P$; we remove the cominuscule hypothesis. Our goal is to give an alternative proof of the formula (14) for the Mather classes of pull back Schubert varieties via the projection $\pi : G/Q \rightarrow G/P$. Instead of analyzing the Euler obstruction, this proof focuses on the conormal spaces of Schubert varieties, and their relation to Mather classes.

Fix $w \in W^P$ and set $C := T_{X_w^P}^*(G/P) \subset T^*(G/P)$, the conormal space of the Schubert variety X_w^P . Consider the commutative diagram

$$(26) \quad \begin{array}{ccccc} \rho_\pi \omega_\pi^{-1}(C) & \xlongequal{\qquad\qquad} & \omega_\pi^{-1}(C) & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ T^*(G/Q) & \xleftarrow{\rho_\pi} & G/Q \times_{G/P} T^*(G/P) & \xrightarrow{\omega_\pi} & T^*(G/P) \\ \uparrow \iota^Q & \searrow \iota^P & \downarrow \iota^P & & \uparrow \iota^P \\ G/Q & \xrightarrow{\pi} & G/P & & \end{array}$$

Here the downward vertical maps on the bottom right square, and the diagonal maps, are projections, the upward maps ι, ι^Q, ι^P are the zero sections, and the right squares are fibre squares. The morphism ρ_π is defined by $\rho_\pi(x, \xi) = (x, \xi \circ d\pi(x))$, where $d\pi(x) : T_x(G/P) \rightarrow T_{\pi(x)}(G/P)$ is the differential of π at x . Since π is a smooth morphism, ω_π is smooth by base change, and ρ_π is a closed embedding; see e.g. [HTT08, p.65].

The following Lemma is well known; see e.g. [KT84, Lemma 3] for a special case, or [Dim04, Prop. 4.3.3] for more general cases, referring to [KS94, pag. 231-232]. For completeness we include a proof.

Lemma 9.1. *Let $w \in W^P$. Then*

$$\rho_\pi \omega_\pi^{-1}(T_{X_w^P}^*(G/P)) = T_{\pi^{-1}(X_w^P)}^*(G/Q)$$

is the conormal space of the pull-back Schubert variety $\pi^{-1}(X_w^P)$.

Proof. The morphism π is a locally trivial fibration with smooth connected fibre $F \simeq P/Q$. Therefore ω_π is again a locally trivial fibration with fibre F , by base change. Combining this with the fact that the conormal space $C := T_{X_w^P}^*(G/P) \subset T^*(G/P)$ is an irreducible (conic) Lagrangian cycle, we obtain that $\omega_\pi^{-1}(C)$ is irreducible of dimension

$$\dim \omega_\pi^{-1}(C) = \dim \pi^*(T^*(G/P)) - \dim G/P = \dim G/Q.$$

It is easy to check that $\omega_\pi^{-1}(C)$ contains the conormal space of the pull back $\pi^{-1}(X_w^{P,\circ})$ of the Schubert cell $X_w^{P,\circ}$. The latter conormal space is an open set

both in $\omega_\pi^{-1}(C)$ and in $T_{\pi^{-1}(X_w^P)}^*(G/Q)$. The claim follows since $\omega_\pi^{-1}(C)$ is irreducible. \square

Denote by T_π the relative tangent bundle associated to the smooth morphism π . This is a bundle on G/Q of rank $\dim P/Q$, determined by the following exact sequence of bundles on G/Q :

$$0 \longrightarrow T_\pi \longrightarrow T(G/Q) \longrightarrow \pi^*(T(G/P)) \longrightarrow 0$$

Next we give another proof of the equation (14) for Schubert varieties, using conormal spaces.

Theorem 9.2. *Let $w \in W^P$. Then*

$$c_{Ma}^T(\pi^{-1}(X_w^P)) = c^T(T_\pi) \cap \pi^*(c_{Ma}^T(X_w^P)).$$

Proof. Denote by $n := \dim Q/P$. We will use the notation from diagram (26). Since ρ_π is a closed embedding, and ω_π is smooth, we may define the push-forward $(\rho_\pi)_*$ and the pull back ω_π^* in the appropriate Chow groups. To prove the claim we utilize the formula (10), and we calculate $(\iota^Q)^*[T_{\pi^{-1}(X_w^P)}^*(G/Q)]_{T \times \mathbb{C}^*}$:

$$(\iota^Q)^*[T_{\pi^{-1}(X_w^P)}^*(G/Q)]_{T \times \mathbb{C}^*} = \iota^* \rho_\pi^*[T_{\pi^{-1}(X_w^P)}^*(G/Q)]_{T \times \mathbb{C}^*} = \iota^* \rho_\pi^* (\rho_\pi)_* \omega_\pi^*[C]_{T \times \mathbb{C}^*},$$

where the last equality follows from Lemma 9.1. By the self-intersection formula [Ful84, Cor. 6.3]

$$\rho_\pi^* (\rho_\pi)_* \omega_\pi^*[C]_{T \times \mathbb{C}^*} = c_n^{T \times \mathbb{C}^*}(N) \cap \omega_\pi^*[C]_{T \times \mathbb{C}^*},$$

where N is the normal bundle of $G/Q \times_{G/P} \pi^*(T^*(G/P)) \subset T^*(G/Q)$. As a $T \times \mathbb{C}^*$ -equivariant bundle, the normal bundle is just the pull-back $N = p^* T_\pi^*$, where $p : G/Q \times_{G/P} T^*(G/P) \rightarrow G/Q$ is the projection. Combining the last two equations and using that ι^* is a ring homomorphism, it follows that

$$\begin{aligned} (\iota^Q)^*[T_{\pi^{-1}(X_w^P)}^*(G/Q)]_{T \times \mathbb{C}^*} &= \iota^* (c_n^{T \times \mathbb{C}^*}(p^* T_\pi^*)) \cap \iota^* (\omega_\pi^*[C]_{T \times \mathbb{C}^*}) \\ &= c_n^{T \times \mathbb{C}^*}(T_\pi^*) \cap \iota^* (\omega_\pi^*[C]_{T \times \mathbb{C}^*}); \end{aligned}$$

the last equality follows because $p \circ \iota = id_{G/Q}$. Finally, we use that $\omega_\pi \circ \iota = \iota^P \circ \pi$ to obtain:

$$\begin{aligned} (\iota^Q)^*[T_{\pi^{-1}(X_w^P)}^*(G/Q)]_{T \times \mathbb{C}^*} &= c_n^{T \times \mathbb{C}^*}(T_\pi^*) \cap \iota^* (\omega_\pi^*[C]_{T \times \mathbb{C}^*}) \\ &= c_n^{T \times \mathbb{C}^*}(T_\pi^*) \cap \pi^*(\iota^P)^*[C] \\ &= (-1)^{\dim X_w^P} c_n^{T \times \mathbb{C}^*}(T_\pi^*) \cap \pi^*(c_{Ma}^T(X_w^P)^\hbar). \end{aligned}$$

Since $\dim \pi^{-1}(X_w^P) = \dim X_w^P + n$, this can be rewritten as

$$(27) \quad (-1)^n c_{Ma}^T(\pi^{-1}(X_w^P))^\hbar = c_n^{T \times \mathbb{C}^*}(T_\pi^*) \cap \pi^*(c_{Ma}^T(X_w^P)^\hbar).$$

Let a_1, \dots, a_n denote the Chern roots of the T -equivariant bundle T_π . Now, since \mathbb{C}^* acts on the fibres of various vector bundles with character \hbar^{-1} , we deduce that the Chern roots of the $T \times \mathbb{C}^*$ -equivariant bundle T_π^* are $-\hbar - a_1, \dots, -\hbar - a_n$. This implies that

$$c_n^{T \times \mathbb{C}^*}(T_\pi^*) = (-\hbar - a_1) \cdot \dots \cdot (-\hbar - a_n) = (-1)^n c^T(T_\pi)^\hbar,$$

where $c^T(T_\pi)^\hbar \in A_T^n(G/Q)$ denotes the homogenization of the total Chern class of T_π by the character \hbar . The theorem now follows from combining this with equation (27) and setting $\hbar = 1$. \square

Remark 9.3. The same proof works when π is replaced by a smooth morphism $f : Z \rightarrow X$ of complex manifolds, and $Y \subset X$ is a irreducible closed subvariety with conormal space $C := T_Y^*(X)$, such that $\omega_f^{-1}(C)$ is irreducible in $Z \times_X f^*(T^*(X))$.

Example 9.4. Consider the divisor $X_{s_1 s_3 s_2}^P \subset \mathrm{Gr}(2, 4)$. Consider $\pi : \mathrm{Fl}(4) \rightarrow \mathrm{Gr}(2, 4)$. Then $\pi^{-1}X_{s_1 s_3 s_2}^P = X_{s_1 s_2 s_3 s_2 s_1}^B$, and using the equation (20) and Theorem 9.2 we obtain:

$$\begin{aligned} c_{\mathrm{Ma}}(X_{1,2,3,2,1}^B) &= [X_{1,2,3,2,1}] + 3[X_{2,3,2,1}] + 3[X_{1,2,3,1}] + 10[X_{2,3,1}] + 28[X_{3,1}] \\ &\quad + 2[X_{1,2,3,2}] + 8[X_{2,3,2}] + 4[X_{1,2,3}] + 16[X_{2,3}] + 28[X_3] \\ &\quad + 2[X_{3,1,2,1}] + 4[X_{3,2,1}] + 8[X_{1,2,1}] + 16[X_{2,1}] + 28[X_1] \\ &\quad + 4[X_{3,1,2}] + 12[X_{3,2}] + 12[X_{1,2}] + 32[X_2] + 24[X_{id}]. \end{aligned}$$

(To ease notation, we omitted the B superscript and the s 's from the reduced words.)

A calculation involving the Mather class of the pull-back divisor from the Lagrangian Grassmannian $\mathrm{LG}(2, 4)$ and its relation to Kazhdan-Lusztig classes, is given in example 10.7 below.

10. POSITIVITY AND UNIMODALITY OF MATHER CLASSES

In this section we discuss positivity conjectures for the Euler obstruction and for the Mather class of a Schubert variety, and we prove them in some cases; we also record a positivity result for Segre-Mather classes. Finally, we make a unimodality and log concavity conjecture for Mather classes, similar to the one for CSM classes [AMSS]. The proofs are based on positivity properties proved in [Huh16, AMSS17] for CSM classes, and the results from [BF97] and [BFL90] for the local Euler obstruction.

10.1. Positivity conjectures. In [Jon10, Rmk. 5.7], B. Jones conjectured that all Mather classes for Grassmannians are nonnegative. Based on substantial experimentation for all cominuscule spaces we make the following conjecture:

Conjecture 10.1 (Strong Positivity conjecture of Mather classes). *Let X_w^P be a Schubert variety in a cominuscule space G/P . Consider the Schubert expansion of the Mather class*

$$(28) \quad c_{\mathrm{Ma}}(X_w^P) = \sum_v a_v [X_v^P].$$

Then $a_v > 0$.

More generally, consider the Schubert expansion of the equivariant Mather class

$$c_{\mathrm{Ma}}^T(X_w^P) = \sum_v a_v(t) [X_v^P]_T.$$

Then $a_v(t) = a_v(\alpha_1, \dots, \alpha_r) \in A_T^(pt)$ is a polynomial in positive simple roots $\alpha_1, \dots, \alpha_r$ with non-negative coefficients.*

We will refer to the situation when $a_v \geq 0$ simply as the ‘Positivity conjecture’, and emphasize ‘Strong’ whenever we can claim it. A similar positivity conjecture was given in [AM09, AM16] for the CSM classes and it was proved in the non-equivariant case in [Huh16] for Grassmannians and [AMSS17] in general. Computer evidence suggests a more refined conjecture, for the local Euler obstructions.

Conjecture 10.2. *Let $v, w \in W^P$ such that $v \leq w$. Then the local Euler obstruction $\text{Eu}_{X_w^P}(e_v) \geq 0$.*

As we explain below, Boe and Fu [BF97] calculated local Euler obstructions for cominuscule spaces of classical Lie types, and in the process proved Conjecture 10.2 for the cominuscule spaces of types A, B and D.

Note that for general spaces the Euler obstruction may be negative. For instance, if C is a cone over a nonsingular plane curve of degree d with vertex O , then $\text{Eu}_O(C) = 2d - d^2$, cf. [Mac74]. Also, the Euler obstruction may be 0 even for cominuscule spaces - see Examples 7.3 and 7.4 above.

It is tempting to conjecture the positivity statements above for Schubert varieties in *all* homogeneous spaces G/P . Unfortunately, outside the cominuscule cases, there are very few instances where we have algorithms to calculate (non-trivial) Euler obstructions and Mather classes. By allowing calculations of Mather classes of pull-back Schubert varieties, Proposition 3.3 and Theorem 9.2 provide some evidence on this matter.

10.2. Kazhdan-Lusztig classes, Mather classes, and positivity. Unless otherwise specified, in this section P is an arbitrary parabolic subgroup. We refer to [AMSS17, §6] for more details about the material below.

Let $IH(X_w^P)$ denote the characteristic cycle of the intersection homology of the Schubert variety X_w^P . The characteristic cycle $IH(X_w^P)$ is an effective, conic, Lagrangian cycle in the cotangent bundle $T^*(G/P)$. Its irreducible components are conormal spaces of Schubert varieties; see e.g. [HTT08, Thm. E.3.6]. Therefore there is a expansion

$$(29) \quad [IH(X_w^P)]_{T \times \mathbb{C}^*} = \sum_v m_{w,v} [T_{X_v^P}^*(G/P)]_{T \times \mathbb{C}^*} \in A_{\dim G/P}^{T \times \mathbb{C}^*}(T^*(G/P))$$

Define the *Kazhdan-Lusztig (KL) class* $KL_w^P \in A_*^T(G/P)$ to be the $\hbar = 1$ dehomogenization of

$$(-1)^{\ell(w)} (\iota^P)^* [IH(X_w^P)] \in A_0^{T \times \mathbb{C}^*}(G/P).$$

By equation (10) this is the same as

$$(30) \quad KL_w^P = \sum_v (-1)^{\ell(w)-\ell(v)} m_{w,v} c_{\text{Ma}}^T(X_v^P) \in A_*^T(G/P).$$

At the same time, from the proof of the Kazhdan-Lusztig conjectures [BK81, BB81] and the calculations of ι^* from [AMSS17] (see especially equation (25)³), it follows that

$$(31) \quad KL_w^P = \sum_v P_{w,v}(1) c_{\text{SM}}^T(X_v^{P,\circ}),$$

³ The results from [AMSS17, §6] are stated for G/B , but everything extends using parabolic versions of the objects considered. The formulae for the pull back via the zero section of various characteristic cycles hold for any smooth projective variety; cf. [AMSS17, Thm.4.3].

where $P_{w,v}$ is the parabolic Kazhdan-Lusztig polynomial; see e.g. [Deo87, Prop. 3.4]. (Observe that we could have taken (31) to be the definition of the KL class.)

Consider the expansion of the Mather class into (equivariant) CSM classes of Schubert cells:

$$(32) \quad c_{\text{Ma}}^T(X_w^P) = \sum_v e_{w,v} c_{\text{SM}}^T(X_v^{P,\circ}).$$

Recall that the coefficient $e_{w,v} = \text{Eu}_{X_w^P}(e_v)$ is the local Euler obstruction evaluated at the fixed point e_v . Combining equations (30), (31) and (32), it follows that the characteristic cycle $IH(X_w^P)$ is irreducible if and only if the local Euler obstruction satisfies

$$(33) \quad e_{w,v} = P_{w,v}(1)$$

for all $v \in W^P$. These considerations lead to the following conditional statements.

Proposition 10.3. *Let $X = G/P$ be a homogeneous space, and let $v, w \in W^P$ such that $v \leq w$.*

- (a) *If Conjecture 10.2 holds for X , then the non-equivariant positivity Conjecture 10.1 holds, i.e. in Eq. (28), the coefficients $a_v \geq 0$.*
- (b) *If $e_{w,v} > 0$ for all $v \in W^P$ such that $v \leq w$, then the non-equivariant strong positivity Conjecture 10.1 holds for X .*
- (c) *If the intersection homology characteristic cycle $IH(X_w^P)$ is irreducible, then conjecture 10.2 holds.*

Proof. Parts (a) and (b) follow from the equation (32) (for non-equivariant classes), using that the non-equivariant CSM classes of Schubert cells are nonnegative [Huh16, AMSS17], and that the initial term of $c_{\text{SM}}(X_v^{P,\circ})$ is $[X_v^P]$. Part (c) follows from equation (33), using that the Kazhdan-Lusztig polynomials $P_{w,v}$ ($v \leq w$) have non-negative integer coefficients, and have constant term equal to 1. \square

The following instances of Conjecture 10.2 follow from results of Bressler, Finkelberg and Lunts [BFL90] in type A, and by Boe and Fu [BF97] in classical Lie types.

Theorem 10.4 ([BFL90, BF97]). *Let $X = G/P$ be a cominuscule space of classical Lie type A-D except for the Lagrangian Grassmannian $\text{LG}(n, 2n)$ for $n \geq 3$. Then the Euler obstruction $e_{w,v} > 0$ if G is of Lie type A or D, and $e_{w,v} \geq 0$ in general.*

Proof. The strict positivity part follows from (33) because the Schubert varieties in cominuscule spaces of type A and D have irreducible characteristic cycles. This is proved by Bressler, Finkelberg and Lunts [BFL90] in type A, and by Boe and Fu [BF97] in type D (they also reprove the statement for type A). For the odd quadrics (in type B), Boe and Fu calculated the Euler obstructions explicitly - see [BF97, §6.3], especially equations (6.3.3) and (6.3.5) - and found them to be non-negative. Finally, $\text{LG}(2, 4)$ is a 3-dimensional quadric, isomorphic to the type B_2 cominuscule space. This finishes the proof. \square

Corollary 10.5. *Let $X = G/P$ be a cominuscule space of Lie type A – D, except the Lagrangian Grassmannian $\text{LG}(n, 2n)$ for $n \geq 3$. Let $\pi : G/B \rightarrow G/P$ be the natural projection.*

- (a) *The strong positivity conjecture 10.1 holds for all Schubert varieties in X in Lie types A and D, and the weak positivity conjecture holds for the odd quadric in type B.*
- (b) *Let $w \in W^P$. Then the Mather class $c_{Ma}(\pi^{-1}(X_w^P)) \in A_*(G/B)$ has the same (strong/weak) positivity property as $c_{Ma}(X_w^P)$ from part (a).*

Proof. Part (a) follows from Proposition 10.3 and Theorem 10.4. Part (b) follows because the Euler obstructions for the pull backs $\pi^{-1}(X_w^P)$ coincide with those for X_w^P by Proposition 3.3. This proves (b) and it finishes the proof. \square

Remark 10.6. The problem of finding the multiplicities of the characteristic cycle seems to be very difficult. Kazhdan and Lusztig [KL80] conjectured the irreducibility of characteristic cycles of the IH sheaf in type A. However, Kashiwara and Tanisaki [KT84], then Kashiwara and Saito [KS97] found counterexamples for the full flag manifolds of Lie type B and type A respectively. See also [Bra02, Wil15] for more about this. Boe and Fu [BF97] also found that the characteristic cycles of the Schubert varieties in cominuscule spaces of Lie types B, C are in general reducible.

In the next example, we use the methods of this paper to recover an example of Kashiwara and Tanisaki of a reducible IH characteristic cycle.

Example 10.7. Consider the Lagrangian Grassmannian $LG := LG(2, 4)$ and the Schubert divisor $X_{1,2}^P \subset LG(2, 4)$ (s_2 corresponds to the long simple root). Let $SF := SF(1, 2; 4)$ be the complete flag manifold of type C_2 ; it parametrizes flags $F_1 \subset F_2 \subset \mathbb{C}^4$ where F_i is isotropic with respect to a symplectic form. Let $\pi : SF \rightarrow LG$ be the projection. This is a \mathbb{P}^1 -bundle, and the preimage $\pi^{-1}(X_{1,2}^P)$ is the Schubert divisor indexed by $X_{1,2,1}^B \subset SF$. A calculation of the Kazhdan-Lusztig polynomials using e.g. SAGE shows that $P_{121,v} = 1$ for any $v \leq s_1 s_2 s_1$. Thus the non-equivariant KL class of $X_{1,2,1}^B$ is:

$$KL_{1,2,1}^B = c_{SM}(X_{1,2,1}^{B,\circ}) + c_{SM}(X_{1,2}^{B,\circ}) + c_{SM}(X_{2,1}^{B,\circ}) + c_{SM}(X_2^{B,\circ}) + c_{SM}(X_1^{B,\circ}) + c_{SM}(X_{id}^B).$$

Using now the calculation for the local Euler obstructions from Example 7.3 we obtain that

$$c_{Ma}(X_{1,2}^P) = c_{SM}(X_{1,2}^{P,\circ}) + c_{SM}(X_2^{P,\circ}).$$

Then from Theorem 9.2 and the Verdier-Riemann-Roch Theorem 3.4 (or Proposition 3.3) it follows that

$$\begin{aligned} c_{Ma}(X_{1,2,1}^B) &= c_{SM}(\pi^{-1}X_{1,2}^{P,\circ}) + c_{SM}(\pi^{-1}X_2^{P,\circ}) \\ &= c_{SM}(X_{1,2,1}^{B,\circ}) + c_{SM}(X_{1,2}^{B,\circ}) + c_{SM}(X_{2,1}^{B,\circ}) + c_{SM}(X_2^{B,\circ}). \end{aligned}$$

Using that $c_{Ma}(X_1^B) = c_{SM}(X_1^{B,\circ}) + c_{SM}(X_{id}^B)$ (as $X_1^B \simeq \mathbb{P}^1$), we deduce that

$$KL_{1,2,1}^B = c_{Ma}(X_{1,2,1}^B) + c_{Ma}(X_1^B).$$

By Theorem 3.1 and the definition of the KL class, this shows that the IH characteristic cycle $IH(X_{1,2,1}^B) \subset T^*(G/B)$ satisfies

$$IH(X_{1,2,1}^B) = [T_{X_{1,2,1}^B}^*(SF)] + [T_{X_1^B}^*(SF)],$$

in accordance to [KT84, p. 194].⁴

We end by observing that by equation (33), the characteristic cycle $IH(\pi^{-1}(X_w^P))$ must be reducible every time the Euler obstruction $\text{Eu}_{X_w^P}(e_v) \leq 0$ for some v . For more such examples, see Example 7.4. It would be interesting to find criteria when this happens and compare with the reducibility criteria from [BF97].

10.3. Segre-Mather classes are alternating. We also record a positivity result of Segre-Mather classes of Schubert varieties in cominuscule spaces. This result can be proved in full G/P generality using [AMSS19, Thm. 1.1]. Here we restrict to the cominuscule case, but we provide a self-contained proof based on Theorem 6.1.

Recall the definition of the *Segre-Mather class*:

$$s_{\text{Ma}}(X_w^P) := \frac{c_{\text{Ma}}(X_w^P)}{c(T(G/P))} \quad (w \in W^P).$$

Our next result shows that these classes are alternating.

Proposition 10.8. *Let G/P be a cominuscule space, and let $w \in W^P$. Consider the expansion of the non-equivariant Segre-Mather class in its homogeneous components:*

$$s_{\text{Ma}}(X_w^P) = c_0 + c_1 + \dots,$$

where $c_i \in A_i(G/P)$. Then $s_{\text{Ma}}(X_w^P)$ is alternating, i.e. for each component $(-1)^{\ell(w)-i} c_i$ is effective.

Proof. By Theorem 6.1,

$$\frac{c_{\text{Ma}}(X_w^P)}{c(T(G/P))} = \pi_*(c(\mathcal{U}_w^*)^{-1} \cap [X_w^B]).$$

But \mathcal{U}_w^* is globally generated, because it is a quotient of the bundle $\pi^*T(G/P)$. Then the Segre class $c(\mathcal{U}_w^*)^{-1} \cap [X_w^B]$ is alternating. \square

10.4. Unimodality and log concavity of Mather polynomials. For $w \in W^P$, consider the Schubert expansion

$$c_{\text{Ma}}(X_w^P) = \sum a_{w,v}[X_v^P].$$

The *Mather polynomial* associated to w is

$$M_w(x) = \sum a_{w,v}x^{\ell(v)}.$$

For instance, the Mather polynomial of the Schubert variety $X_{(4,3,1)} \subset \text{LG}(4,8)$ from Example 6.4 is

$$M_{(4,3,1)}(x) = x^8 + 11x^7 + 52x^6 + 152x^5 + 286x^4 + 452x^3 + 246x^2 + 132x + 24.$$

Following [Sta89], we say that a polynomial $a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ is *unimodal* if $a_0 \leq a_1 \leq \dots \leq a_k \geq a_{k+1} \geq \dots \geq a_n$ for some index k . It is *log concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all i (by convention $a_{-i} = a_{n+i} = 0$ for all $i \geq 1$). If one assumes that the polynomial has strictly positive coefficients, then any log concave polynomial is also unimodal.

Substantial amount of calculations for all Lie types supports the following:

⁴The example in [KT84] is in type B_2 , but the corresponding complete flag variety is isomorphic to the variety SF.

Conjecture 10.9. *Let $X = G/P$ be a cominuscule space and $w \in W^P$.*

- (a) *The Mather polynomial M_w has strictly positive coefficients and it is unimodal.*
- (b) *Assume in addition that $G/P = \mathrm{Gr}(k, n)$. Then M_w is log concave.*

We checked this conjecture for the Grassmannians $\mathrm{Gr}(k, n)$ where $n \leq 8$, the cominuscule spaces of Lie types C_n and D_n where $n \leq 5$, and all but 5 Mather classes in the Cayley plane E_6/P_6 . The log concavity fails outside type A. For instance, the Mather polynomial of the 5 dimensional quadric $\mathrm{OG}(1, 7)$ is

$$x^5 + 5x^4 + 11x^3 + 26x^2 + 18x + 6.$$

(This Mather class is the same as the total Chern class). Similarly, the Mather classes of the Lagrangian Grassmannian $\mathrm{LG}(5, 10)$ and of the Orthogonal Grassmannian $\mathrm{OG}(4, 8)$ are not log concave.

The unimodality and log concavity properties of characteristic classes of singular varieties seem to be new and unexplored phenomena. For instance, in analogy to the Mather polynomial one may define two flavors of a *CSM polynomial*: one obtained from the CSM class of a Schubert cell, and the other from the CSM class of a Schubert variety. This is conjectured to satisfy an analog of Conjecture 10.9; more details will be discussed in the upcoming note [AMSS]. Log concavity has also been conjectured for certain coefficients of motivic Chern classes of Schubert cells [FRW, §6.2]. It would be interesting to know whether these phenomena fit into the (Hodge-Riemann and Hard Lefschetz) framework from [Huh18] or [HMMSD].

11. TABLES

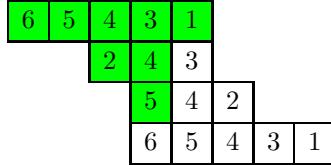
In this section, we aggregate our computations of the Chern-Mather classes and local Euler obstructions. The computations of the Euler obstructions in $\mathrm{LG}(4, 8)$ rely on the recurrence relations obtained by Boe and Fu [BF97] and have been checked using the results of this note.

11.1. Schubert Varieties in Cominuscule spaces. We recall some facts about diagrams indexing the Schubert varieties in cominuscule spaces. Our main reference is [BCMP18].

Let G/P be a cominuscule space corresponding to the simple root α_P . Recall from Section 4 that the set of positive roots R^+ is equipped with a partial order $<$. Let $R_{\geq \alpha_P}$ be the subset of those roots $\alpha \in R^+$ such that $\alpha \geq \alpha_P$. This is a lattice under the partial order $<$. A *lower-order ideal* in $R_{\geq \alpha_P}$ is a subset satisfying the following condition: for any pair of elements $i, j \in R_{\geq \alpha_P}$ with $i \in I$ and $j \leq i$, we have $j \in I$. Following [Pro84], the Weyl group elements $w \in W^P$ are in bijection with the lower-order ideals in $R_{\geq \alpha_P}$.

Fix a Hasse diagram for $R_{\geq \alpha_P}$, (cf. e.g. Table 1). Then the lower-order ideal I_w of $w \in W^P$ gives the *diagram of w* . This generalizes the usual Young diagram associated to a Schubert variety in the classical Lie types; for an equivalent model using quivers, see [Per07]. In particular, the nodes of I_w , which are also the boxes of the diagram of w , are given by positive roots in $R_{\geq \alpha_P}$. These are precisely the positive roots in the inversion set of w ; thus the length of w equals the number of boxes in the diagram of w . As explained in [BCMP18, §3], each box in the diagram

of w may also be labelled by a simple reflection, and these labels can be used to obtain a minimal word for w . For instance, the Hasse diagram for the Cayley plane E_6/P_6 , and the associated diagram, are given in the Table 1. The shaded parts give the diagram $(5, 2, 1)$, corresponding to the element $w \in W^P$ with a reduced expression $w = s_5 s_4 s_2 s_1 s_3 s_4 s_5 s_6$; see also Section 6.2.



11.2. Type A: Grassmannians. The Schubert subvarieties of $\mathrm{Gr}(k, n)$ are indexed by Young diagrams (or partitions) $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ such that $\lambda_1 \leq n - k$ and $\lambda_k \geq 0$. The Schubert variety X_λ has dimension $|\lambda| = \lambda_1 + \dots + \lambda_k$.

Table 4 lists the Mather classes of Schubert varieties in $\mathrm{Gr}(3, 7)$. The expansion of the Chern-Mather class of a Schubert variety in terms of ordinary Schubert classes is given in the column indexed by the corresponding partition. Following Section 11.5, we see that this table also contains the Mather classes of all $\mathrm{Gr}(k, n)$ with $k \leq 3$ and $n - k \leq 4$.

11.3. Type C: Lagrangian Grassmannians. Let $G/P = \mathrm{LG}(n, 2n)$, the variety parametrizing the Lagrangian subspaces of a $2n$ dimensional symplectic vector space. The Schubert subvarieties of G/P are indexed by *strict partitions* $\lambda = (\lambda_1 > \lambda_2 > \dots > \lambda_k)$, where $\lambda_1 \leq n$, $\lambda_k > 0$, and $0 \leq k \leq n$. As before, we have $\dim X_\lambda = |\lambda|$.

Table 2 lists the Mather classes of Schubert varieties in $\mathrm{LG}(4, 8)$; The expansion of the Chern-Mather class of a Schubert variety in terms of ordinary Schubert classes is given in the column indexed by the corresponding partition. Table 3 list the Euler obstructions of the Schubert varieties in $\mathrm{LG}(4, 8)$.

11.4. Type E_6 : the Cayley plane. Table 9 lists the Mather classes of some Schubert subvarieties of the cominuscule space E_6/P_6 ; The expansion of the Chern-Mather class of a Schubert variety in terms of ordinary Schubert classes is given in the column indexed by the corresponding partition.

11.5. Stability. Our ‘homological’ indexing conventions for Schubert varieties gives a stability property for the Mather classes in the ordinary Grassmannians, and for the maximal isotropic Grassmannians in type C and D. We explain this for the ordinary Grassmannian, following [AM09, §2.1].

Fix a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ included in the $k \times (n - k)$ rectangle. Fix the standard flag $F_\bullet : F_1 \subset \dots \subset F_n = \mathbb{C}^n$ where $F_i = \langle e_1, \dots, e_i \rangle$. The Schubert variety $X_\lambda \subset \mathrm{Gr}(k, n)$ is defined by

$$X_\lambda = \{V : \dim V \cap F_{\lambda_{k+1-i}+i} \geq i\}.$$

If the $k \times (n - k)$ diagram is included in the $k' \times (n' - k')$ diagram then one can define an embedding $i : \mathrm{Gr}(k, n) \hookrightarrow \mathrm{Gr}(k', n')$ by $i(V) = \langle e_1, \dots, e_{k'-k} \rangle \oplus \tilde{V}$, where

\tilde{V} is obtained from V by shifting the indices of basis elements according to the rule $e_j \mapsto e_{j+k'-k}$. With this definition, $i(X_\lambda) = X_\lambda$, and it follows that

$$i_* c_{\text{Ma}}(X_\lambda) = c_{\text{Ma}}(X_\lambda) \in A_*(\text{Gr}(k', n')).$$

For instance, the Schubert variety $X_\square \subset \text{Gr}(1, 3)$ is $\{\langle ae_1 + be_2 \rangle \subset \mathbb{C}^3 \mid [a : b] \in \mathbb{P}^1\}$. Under the inclusion $\text{Gr}(1, 3) \hookrightarrow \text{Gr}(3, 7)$ the image of X_\square is the Schubert variety parametrizing the dimension 3 subspaces $\langle e_1, e_2, ae_3 + be_4 \rangle \in \text{Gr}(3, 7)$.

One may define similar embeddings, $i : \text{LG}(n, 2n) \hookrightarrow \text{LG}(n', 2n')$, and $i : \text{OG}(n, 2n) \hookrightarrow \text{OG}(n', 2n')$ with $n \leq n'$. In all these cases, we have $i(X_\lambda) = X_\lambda$, (cf. [LR08, §6.2, §7.2]), and hence $i_* c_{\text{Ma}}(X_\lambda) = c_{\text{Ma}}(X_\lambda)$. We leave it to the reader to check the details in other types.

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| | () | 1 | 2 | 21 | 3 | 4 | 31 | 41 | 32 | 42 | 321 | 43 | 421 | 431 | 432 | 4321 |
|------|----|---|---|----|---|----|----|----|----|----|-----|-----|-----|-----|-----|------|
| () | 1 | 2 | 2 | 4 | 4 | 4 | 8 | 8 | 4 | 8 | 8 | 12 | 16 | 24 | 8 | 16 |
| 1 | 0 | 1 | 4 | 8 | 9 | 16 | 20 | 34 | 18 | 40 | 36 | 64 | 80 | 132 | 64 | 128 |
| 2 | 0 | 0 | 1 | 3 | 5 | 14 | 14 | 37 | 23 | 64 | 46 | 114 | 128 | 246 | 172 | 344 |
| 21 | 0 | 0 | 0 | 1 | 0 | 0 | 5 | 14 | 18 | 58 | 37 | 114 | 120 | 269 | 268 | 536 |
| 3 | 0 | 0 | 0 | 0 | 1 | 6 | 3 | 17 | 7 | 36 | 15 | 80 | 76 | 183 | 176 | 352 |
| 4 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 3 | 0 | 7 | 0 | 19 | 15 | 45 | 52 | 105 |
| 31 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 6 | 6 | 34 | 15 | 90 | 82 | 241 | 336 | 674 |
| 41 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 6 | 0 | 21 | 15 | 60 | 102 | 210 |
| 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 6 | 4 | 25 | 23 | 92 | 190 | 386 |
| 42 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 7 | 4 | 27 | 68 | 147 |
| 321 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 6 | 25 | 88 | 184 |
| 43 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 4 | 14 |
| 421 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 7 | 32 | 76 |
| 431 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 8 | 24 |
| 432 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 5 |
| 4321 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

TABLE 2. Chern-Mather classes for $\text{LG}(4,8)$.

| | () | 1 | 2 | 21 | 3 | 31 | 32 | 321 | 4 | 41 | 42 | 421 | 43 | 431 | 432 | 4321 |
|------|----|---|---|----|---|----|----|-----|---|----|----|-----|----|-----|-----|------|
| () | 1 | 1 | 0 | 1 | 1 | 2 | 1 | 1 | 0 | 1 | 2 | 2 | 2 | 3 | 0 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 2 | 0 | 1 | 1 | 1 | 0 | 2 | 1 | 3 | 1 | 1 |
| 2 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 2 | 1 | 2 | 1 | 1 |
| 21 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 2 | 2 | 1 | 2 | 0 | 1 |
| 3 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 2 | 1 | 1 |
| 31 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 2 | 0 | 1 |
| 32 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 |
| 321 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 2 | 1 |
| 41 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 0 | 1 |
| 42 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| 421 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 43 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 431 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 432 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 4321 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

TABLE 3. Local Euler obstructions for $\text{LG}(4,8)$. The Euler obstruction of the Schubert variety X_v at the point u is given in row u and column v .

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| | (0) | 1 | 2 | 11 | 3 | 21 | 111 | 31 | 22 | 211 | 32 | 311 | 221 | 33 | 321 | 222 | 331 | 322 | 332 | 333 |
|-----|-----|---|---|----|---|----|-----|----|----|-----|----|-----|-----|----|-----|-----|-----|-----|-----|-----|
| (0) | 1 | 2 | 3 | 3 | 4 | 6 | 4 | 8 | 6 | 8 | 12 | 12 | 12 | 10 | 24 | 10 | 20 | 20 | 30 | 20 |
| 1 | 0 | 1 | 3 | 3 | 6 | 8 | 6 | 15 | 12 | 15 | 27 | 27 | 27 | 30 | 60 | 30 | 66 | 66 | 108 | 90 |
| 2 | 0 | 0 | 1 | 0 | 4 | 3 | 0 | 11 | 7 | 6 | 23 | 21 | 17 | 35 | 54 | 25 | 82 | 74 | 144 | 150 |
| 11 | 0 | 0 | 0 | 1 | 0 | 3 | 4 | 6 | 7 | 11 | 17 | 21 | 23 | 25 | 54 | 35 | 74 | 82 | 144 | 150 |
| 3 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 3 | 0 | 0 | 7 | 6 | 0 | 15 | 17 | 0 | 37 | 25 | 69 | 90 |
| 21 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 4 | 4 | 4 | 15 | 15 | 15 | 30 | 52 | 30 | 98 | 98 | 210 | 270 |
| 111 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 3 | 0 | 6 | 7 | 0 | 17 | 15 | 25 | 37 | 69 | 90 |
| 31 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 4 | 4 | 0 | 11 | 15 | 0 | 40 | 30 | 93 | 146 |
| 22 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 4 | 0 | 4 | 12 | 15 | 12 | 42 | 42 | 108 | 174 |
| 211 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 4 | 4 | 0 | 15 | 11 | 30 | 40 | 93 | 146 |
| 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 5 | 4 | 0 | 19 | 12 | 54 | 108 |
| 311 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 4 | 0 | 11 | 11 | 36 | 66 |
| 221 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 4 | 5 | 12 | 19 | 54 | 108 |
| 33 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 4 | 0 | 12 | 32 |
| 321 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 5 | 5 | 24 | 58 |
| 222 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 4 | 12 | 32 |
| 331 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 5 | 17 |
| 322 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 5 | 17 |
| 332 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 6 |
| 333 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

TABLE 4. Chern-Mather classes for $\text{Gr}(3, 6)$.

| | 1111 | 2111 | 3111 | 2211 | 3211 | 2221 | 2222 | 3311 | 3221 | 3222 | 3321 | 3322 | 3331 | 3332 | 3333 |
|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| () | 5 | 10 | 15 | 15 | 30 | 20 | 15 | 30 | 40 | 30 | 60 | 45 | 40 | 60 | 35 |
| 1 | 10 | 24 | 42 | 42 | 92 | 64 | 60 | 108 | 140 | 130 | 228 | 210 | 190 | 300 | 210 |
| 2 | 0 | 10 | 34 | 29 | 89 | 56 | 65 | 141 | 163 | 180 | 315 | 341 | 330 | 544 | 455 |
| 11 | 10 | 26 | 48 | 51 | 117 | 88 | 105 | 153 | 203 | 240 | 351 | 411 | 360 | 624 | 525 |
| 3 | 0 | 0 | 10 | 0 | 29 | 0 | 0 | 66 | 56 | 65 | 155 | 185 | 205 | 351 | 350 |
| 21 | 0 | 10 | 36 | 36 | 119 | 82 | 120 | 216 | 258 | 360 | 541 | 738 | 690 | 1266 | 1260 |
| 111 | 5 | 14 | 27 | 30 | 71 | 58 | 90 | 99 | 139 | 215 | 251 | 387 | 305 | 621 | 630 |
| 31 | 0 | 0 | 10 | 0 | 36 | 0 | 0 | 94 | 82 | 120 | 252 | 375 | 397 | 768 | 896 |
| 22 | 0 | 0 | 0 | 10 | 36 | 36 | 67 | 97 | 121 | 214 | 300 | 502 | 483 | 968 | 1141 |
| 211 | 0 | 5 | 19 | 19 | 67 | 49 | 91 | 128 | 166 | 297 | 369 | 652 | 547 | 1193 | 1407 |
| 32 | 0 | 0 | 0 | 0 | 10 | 0 | 0 | 46 | 36 | 67 | 157 | 281 | 318 | 678 | 938 |
| 311 | 0 | 0 | 5 | 0 | 19 | 0 | 0 | 51 | 49 | 91 | 157 | 298 | 278 | 645 | 868 |
| 221 | 0 | 0 | 0 | 5 | 19 | 24 | 58 | 54 | 86 | 199 | 229 | 505 | 434 | 1056 | 1470 |
| 33 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 10 | 0 | 0 | 36 | 67 | 100 | 228 | 376 |
| 321 | 0 | 0 | 0 | 0 | 5 | 0 | 0 | 24 | 24 | 58 | 110 | 257 | 257 | 672 | 1076 |
| 222 | 0 | 0 | 0 | 0 | 0 | 5 | 18 | 0 | 19 | 65 | 54 | 174 | 136 | 408 | 680 |
| 331 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 0 | 0 | 24 | 58 | 80 | 224 | 427 |
| 322 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 18 | 24 | 83 | 78 | 257 | 497 |
| 332 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 18 | 29 | 101 | 238 |
| 333 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 18 |
| 1111 | 1 | 3 | 6 | 7 | 17 | 15 | 31 | 25 | 37 | 77 | 69 | 145 | 90 | 245 | 301 |
| 2111 | 0 | 1 | 4 | 4 | 15 | 11 | 26 | 30 | 40 | 93 | 93 | 218 | 146 | 423 | 588 |
| 3111 | 0 | 0 | 1 | 0 | 4 | 0 | 0 | 11 | 11 | 26 | 36 | 88 | 66 | 198 | 302 |
| 2211 | 0 | 0 | 0 | 1 | 4 | 5 | 16 | 12 | 19 | 59 | 54 | 163 | 108 | 368 | 604 |
| 3211 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 5 | 5 | 16 | 24 | 75 | 58 | 207 | 378 |
| 2221 | 0 | 0 | 0 | 0 | 0 | 1 | 6 | 0 | 4 | 23 | 12 | 66 | 32 | 168 | 336 |
| 2222 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 4 | 0 | 12 | 0 | 32 | 80 |
| 3311 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 5 | 16 | 17 | 65 | 141 |
| 3221 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 6 | 5 | 29 | 17 | 95 | 215 |
| 3222 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 5 | 0 | 17 | 49 |
| 3321 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 6 | 6 | 35 | 98 |
| 3322 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 6 | 23 |
| 3331 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 6 | 24 |
| 3332 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 7 |
| 3333 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

TABLE 5. Chern-Mather classes for $\text{Gr}(3, 7)$.

| | 4111 | 4211 | 4311 | 4221 | 4222 | 4411 | 4321 | 4322 |
|------|------|------|------|------|------|------|------|------|
| () | 20 | 40 | 60 | 60 | 45 | 45 | 120 | 90 |
| 1 | 64 | 140 | 228 | 228 | 210 | 210 | 480 | 440 |
| 2 | 76 | 188 | 345 | 333 | 355 | 405 | 762 | 808 |
| 11 | 76 | 188 | 333 | 345 | 405 | 355 | 762 | 888 |
| 3 | 44 | 118 | 243 | 219 | 245 | 375 | 556 | 636 |
| 21 | 84 | 264 | 555 | 555 | 750 | 750 | 1370 | 1830 |
| 111 | 44 | 118 | 219 | 243 | 375 | 245 | 556 | 856 |
| 31 | 46 | 155 | 374 | 340 | 480 | 653 | 969 | 1383 |
| 22 | 0 | 84 | 272 | 272 | 466 | 466 | 814 | 1324 |
| 211 | 46 | 155 | 340 | 374 | 653 | 480 | 969 | 1685 |
| 32 | 0 | 46 | 201 | 157 | 281 | 453 | 644 | 1102 |
| 311 | 24 | 86 | 215 | 215 | 388 | 388 | 644 | 1182 |
| 221 | 0 | 46 | 157 | 201 | 453 | 281 | 644 | 1386 |
| 33 | 0 | 0 | 46 | 0 | 0 | 153 | 157 | 281 |
| 321 | 0 | 24 | 110 | 110 | 257 | 257 | 478 | 1074 |
| 222 | 0 | 0 | 0 | 46 | 153 | 0 | 157 | 493 |
| 331 | 0 | 0 | 24 | 0 | 0 | 83 | 110 | 257 |
| 322 | 0 | 0 | 0 | 24 | 83 | 0 | 110 | 363 |
| 332 | 0 | 0 | 0 | 0 | 0 | 0 | 24 | 83 |
| 4 | 10 | 29 | 66 | 56 | 65 | 138 | 155 | 185 |
| 41 | 10 | 36 | 94 | 82 | 120 | 216 | 252 | 375 |
| 42 | 0 | 10 | 46 | 36 | 67 | 140 | 157 | 281 |
| 411 | 5 | 19 | 51 | 49 | 91 | 119 | 157 | 298 |
| 43 | 0 | 0 | 10 | 0 | 0 | 56 | 36 | 67 |
| 421 | 0 | 5 | 24 | 24 | 58 | 75 | 110 | 257 |
| 44 | 0 | 0 | 0 | 0 | 0 | 10 | 0 | 0 |
| 431 | 0 | 0 | 5 | 0 | 0 | 29 | 24 | 58 |
| 422 | 0 | 0 | 0 | 5 | 18 | 0 | 24 | 83 |
| 441 | 0 | 0 | 0 | 0 | 0 | 5 | 0 | 0 |
| 432 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 18 |
| 1111 | 10 | 29 | 56 | 66 | 138 | 65 | 155 | 327 |
| 2111 | 10 | 36 | 82 | 94 | 216 | 120 | 252 | 587 |
| 3111 | 5 | 19 | 49 | 51 | 119 | 91 | 157 | 378 |
| 2211 | 0 | 10 | 36 | 46 | 140 | 67 | 157 | 465 |
| 3211 | 0 | 5 | 24 | 24 | 75 | 58 | 110 | 335 |
| 2221 | 0 | 0 | 0 | 10 | 56 | 0 | 36 | 193 |
| 2222 | 0 | 0 | 0 | 0 | 10 | 0 | 0 | 36 |
| 3311 | 0 | 0 | 5 | 0 | 0 | 18 | 24 | 75 |
| 3221 | 0 | 0 | 0 | 5 | 29 | 0 | 24 | 134 |
| 3222 | 0 | 0 | 0 | 0 | 5 | 0 | 0 | 24 |
| 3321 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 29 |
| 3322 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 |
| 4111 | 1 | 4 | 11 | 11 | 26 | 26 | 36 | 88 |
| 4211 | 0 | 1 | 5 | 5 | 16 | 16 | 24 | 75 |
| 4311 | 0 | 0 | 1 | 0 | 0 | 6 | 5 | 16 |
| 4221 | 0 | 0 | 0 | 1 | 6 | 0 | 5 | 29 |
| 4222 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 5 |
| 4411 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 4321 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 6 |
| 4322 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

TABLE 6. Chern-Mather classes for $\text{Gr}(4,8)$.

| | 4421 | 4331 | 4422 | 4332 | 4431 | 4333 | 4432 | 4441 | 4433 | 4442 | 4443 | 4444 |
|-----|------|------|------|------|------|------|------|------|-------|-------|-------|-------|
| () | 90 | 80 | 90 | 120 | 120 | 70 | 180 | 70 | 105 | 105 | 140 | 70 |
| 1 | 440 | 400 | 480 | 630 | 630 | 440 | 990 | 440 | 690 | 690 | 960 | 560 |
| 2 | 888 | 786 | 1032 | 1286 | 1356 | 1060 | 2208 | 1140 | 1805 | 1855 | 2680 | 1820 |
| 11 | 808 | 786 | 1032 | 1356 | 1286 | 1140 | 2208 | 1060 | 1855 | 1805 | 2680 | 1820 |
| 3 | 856 | 684 | 1044 | 1152 | 1386 | 1090 | 2322 | 1410 | 2170 | 2370 | 3544 | 2800 |
| 21 | 1830 | 1734 | 2520 | 3144 | 3144 | 3090 | 5652 | 3090 | 5520 | 5520 | 8568 | 6720 |
| 111 | 636 | 684 | 1044 | 1386 | 1152 | 1410 | 2322 | 1090 | 2370 | 2170 | 3544 | 2800 |
| 31 | 1685 | 1439 | 2457 | 2715 | 3097 | 3022 | 5787 | 3624 | 6383 | 6766 | 10942 | 9863 |
| 22 | 1324 | 1310 | 2066 | 2578 | 2578 | 3002 | 4996 | 3002 | 5763 | 5763 | 9532 | 8582 |
| 211 | 1383 | 1439 | 2457 | 3097 | 2715 | 3624 | 5787 | 3022 | 6766 | 6383 | 10942 | 9863 |
| 32 | 1386 | 1234 | 2292 | 2547 | 2943 | 3366 | 5964 | 4070 | 7744 | 8220 | 14264 | 14672 |
| 311 | 1182 | 1094 | 2208 | 2466 | 2466 | 3198 | 5502 | 3198 | 7130 | 7130 | 12760 | 12992 |
| 221 | 1102 | 1234 | 2292 | 2943 | 2547 | 4070 | 5964 | 3366 | 8220 | 7744 | 14264 | 14672 |
| 33 | 493 | 410 | 850 | 903 | 1194 | 1422 | 2551 | 1990 | 3825 | 4263 | 7868 | 9246 |
| 321 | 1074 | 1074 | 2344 | 2706 | 2706 | 4176 | 6672 | 4176 | 10200 | 10200 | 19768 | 22900 |
| 222 | 281 | 410 | 850 | 1194 | 903 | 1990 | 2551 | 1422 | 4263 | 3825 | 7868 | 9246 |
| 331 | 363 | 349 | 820 | 943 | 1090 | 1730 | 2841 | 2038 | 5003 | 5281 | 10942 | 14408 |
| 322 | 257 | 349 | 820 | 1090 | 943 | 2038 | 2841 | 1730 | 5281 | 5003 | 10942 | 14408 |
| 332 | 83 | 134 | 275 | 446 | 446 | 1014 | 1412 | 1014 | 3084 | 3084 | 7304 | 10946 |
| 333 | 0 | 24 | 0 | 83 | 83 | 248 | 275 | 248 | 784 | 784 | 2112 | 3656 |
| 4 | 327 | 205 | 415 | 351 | 559 | 350 | 959 | 692 | 985 | 1198 | 1850 | 1701 |
| 41 | 587 | 397 | 900 | 768 | 1146 | 896 | 2217 | 1592 | 2658 | 3102 | 5216 | 5376 |
| 42 | 465 | 318 | 814 | 678 | 1069 | 938 | 2257 | 1754 | 3163 | 3721 | 6764 | 7926 |
| 411 | 378 | 278 | 737 | 645 | 825 | 868 | 1918 | 1232 | 2659 | 2895 | 5388 | 6126 |
| 43 | 193 | 100 | 348 | 228 | 510 | 376 | 1131 | 1028 | 1798 | 2310 | 4482 | 6016 |
| 421 | 335 | 257 | 768 | 672 | 897 | 1076 | 2322 | 1600 | 3782 | 4152 | 8442 | 10928 |
| 44 | 36 | 0 | 67 | 0 | 100 | 0 | 228 | 256 | 376 | 604 | 1236 | 1909 |
| 431 | 134 | 80 | 315 | 224 | 429 | 427 | 1167 | 951 | 2157 | 2608 | 5690 | 8421 |
| 422 | 75 | 78 | 254 | 257 | 291 | 497 | 939 | 613 | 1855 | 1924 | 4466 | 6551 |
| 441 | 24 | 0 | 58 | 0 | 80 | 0 | 224 | 228 | 427 | 656 | 1508 | 2552 |
| 432 | 29 | 29 | 101 | 101 | 163 | 238 | 547 | 444 | 1252 | 1450 | 3656 | 6160 |
| 442 | 5 | 0 | 18 | 0 | 29 | 0 | 101 | 109 | 238 | 373 | 993 | 1924 |
| 433 | 0 | 5 | 0 | 18 | 29 | 56 | 101 | 112 | 304 | 376 | 1088 | 2144 |
| 443 | 0 | 0 | 0 | 0 | 5 | 0 | 18 | 34 | 56 | 119 | 360 | 832 |
| 444 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 0 | 18 | 56 | 160 |

TABLE 7. (cont.) Chern-Mather classes for $\text{Gr}(4,8)$.

| | 4421 | 4331 | 4422 | 4332 | 4431 | 4333 | 4432 | 4441 | 4433 | 4442 | 4443 | 4444 |
|------|------|------|------|------|------|------|------|------|------|------|------|-------|
| 1111 | 185 | 205 | 415 | 559 | 351 | 692 | 959 | 350 | 1198 | 985 | 1850 | 1701 |
| 2111 | 375 | 397 | 900 | 1146 | 768 | 1592 | 2217 | 896 | 3102 | 2658 | 5216 | 5376 |
| 3111 | 298 | 278 | 737 | 825 | 645 | 1232 | 1918 | 868 | 2895 | 2659 | 5388 | 6126 |
| 2211 | 281 | 318 | 814 | 1069 | 678 | 1754 | 2257 | 938 | 3721 | 3163 | 6764 | 7926 |
| 3211 | 257 | 257 | 768 | 897 | 672 | 1600 | 2322 | 1076 | 4152 | 3782 | 8442 | 10928 |
| 2221 | 67 | 100 | 348 | 510 | 228 | 1028 | 1131 | 376 | 2310 | 1798 | 4482 | 6016 |
| 2222 | 0 | 0 | 67 | 100 | 0 | 256 | 228 | 0 | 604 | 376 | 1236 | 1909 |
| 3311 | 83 | 78 | 254 | 291 | 257 | 613 | 939 | 497 | 1924 | 1855 | 4466 | 6551 |
| 3221 | 58 | 80 | 315 | 429 | 224 | 951 | 1167 | 427 | 2608 | 2157 | 5690 | 8421 |
| 3222 | 0 | 0 | 58 | 80 | 0 | 228 | 224 | 0 | 656 | 427 | 1508 | 2552 |
| 3321 | 18 | 29 | 101 | 163 | 101 | 444 | 547 | 238 | 1450 | 1252 | 3656 | 6160 |
| 3322 | 0 | 0 | 18 | 29 | 0 | 109 | 101 | 0 | 373 | 238 | 993 | 1924 |
| 3331 | 0 | 5 | 0 | 29 | 18 | 112 | 101 | 56 | 376 | 304 | 1088 | 2144 |
| 3332 | 0 | 0 | 0 | 5 | 0 | 34 | 18 | 0 | 119 | 56 | 360 | 832 |
| 3333 | 0 | 0 | 0 | 0 | 0 | 5 | 0 | 0 | 18 | 0 | 56 | 160 |
| 4111 | 88 | 66 | 222 | 198 | 198 | 302 | 600 | 302 | 946 | 946 | 1962 | 2416 |
| 4211 | 75 | 58 | 231 | 207 | 207 | 378 | 736 | 378 | 1374 | 1374 | 3168 | 4458 |
| 4311 | 29 | 17 | 91 | 65 | 95 | 141 | 356 | 215 | 754 | 830 | 2074 | 3336 |
| 4221 | 16 | 17 | 91 | 95 | 65 | 215 | 356 | 141 | 830 | 754 | 2074 | 3336 |
| 4222 | 0 | 0 | 16 | 17 | 0 | 49 | 65 | 0 | 197 | 141 | 517 | 944 |
| 4411 | 5 | 0 | 16 | 0 | 17 | 0 | 65 | 49 | 141 | 197 | 517 | 944 |
| 4321 | 6 | 6 | 35 | 35 | 35 | 98 | 198 | 98 | 542 | 542 | 1656 | 3070 |
| 4322 | 0 | 0 | 6 | 6 | 0 | 23 | 35 | 0 | 132 | 98 | 425 | 904 |
| 4421 | 1 | 0 | 6 | 0 | 6 | 0 | 35 | 23 | 98 | 132 | 425 | 904 |
| 4331 | 0 | 1 | 0 | 6 | 6 | 24 | 35 | 24 | 136 | 136 | 512 | 1128 |
| 4422 | 0 | 0 | 1 | 0 | 0 | 0 | 6 | 0 | 23 | 23 | 106 | 262 |
| 4332 | 0 | 0 | 0 | 1 | 0 | 7 | 6 | 0 | 41 | 24 | 160 | 416 |
| 4431 | 0 | 0 | 0 | 0 | 1 | 0 | 6 | 7 | 24 | 41 | 160 | 416 |
| 4333 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 6 | 0 | 24 | 80 |
| 4432 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 7 | 7 | 48 | 152 |
| 4441 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 6 | 24 | 80 |
| 4433 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 7 | 31 |
| 4442 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 7 | 31 |
| 4443 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 8 |
| 4444 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

TABLE 8. (cont.) Chern-Mather classes for $\text{Gr}(4,8)$.

| | () | 1 | 2 | 3 | 31 | 4 | 5 | 41 | 51 | 42 | 52 | 421 | 53 |
|-----|----|---|---|---|----|----|----|----|----|----|-----|-----|-----|
| () | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 10 | 12 | 10 | 18 | 10 | 14 |
| 1 | 0 | 1 | 3 | 6 | 10 | 10 | 15 | 22 | 33 | 26 | 54 | 32 | 52 |
| 2 | 0 | 0 | 1 | 4 | 10 | 10 | 20 | 28 | 55 | 44 | 106 | 68 | 130 |
| 3 | 0 | 0 | 0 | 1 | 5 | 5 | 15 | 22 | 60 | 48 | 144 | 92 | 225 |
| 31 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 5 | 15 | 16 | 55 | 40 | 108 |
| 4 | 0 | 0 | 0 | 0 | 0 | 1 | 6 | 5 | 27 | 16 | 79 | 40 | 159 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 5 | 0 | 16 | 0 | 42 |
| 41 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 6 | 6 | 33 | 22 | 86 |
| 51 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 6 | 0 | 22 |
| 42 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 6 | 7 | 23 |
| 52 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 7 |
| 421 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 53 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

| | 521 | 4211 | 531 | 5211 | 532 | 5311 | 533 | 5321 | 5322 |
|------|-----|------|-----|------|-----|------|------|------|------|
| () | 19 | 10 | 28 | 20 | 23 | 30 | 18 | 40 | 27 |
| 1 | 68 | 40 | 108 | 85 | 100 | 135 | 96 | 190 | 150 |
| 2 | 164 | 100 | 288 | 240 | 310 | 421 | 360 | 644 | 600 |
| 3 | 274 | 160 | 552 | 470 | 705 | 948 | 960 | 1611 | 1770 |
| 31 | 138 | 86 | 318 | 292 | 488 | 672 | 768 | 1276 | 1659 |
| 4 | 184 | 86 | 450 | 372 | 691 | 912 | 1080 | 1760 | 2271 |
| 5 | 40 | 0 | 132 | 86 | 242 | 298 | 432 | 680 | 1020 |
| 41 | 112 | 62 | 322 | 296 | 604 | 840 | 1092 | 1840 | 2814 |
| 51 | 22 | 0 | 94 | 62 | 216 | 272 | 456 | 734 | 1308 |
| 42 | 39 | 29 | 139 | 151 | 326 | 507 | 690 | 1285 | 2334 |
| 52 | 7 | 0 | 46 | 29 | 138 | 180 | 354 | 601 | 1284 |
| 421 | 6 | 8 | 23 | 45 | 70 | 162 | 180 | 465 | 1014 |
| 53 | 0 | 0 | 7 | 0 | 30 | 29 | 98 | 147 | 384 |
| 521 | 1 | 0 | 7 | 8 | 30 | 53 | 100 | 215 | 564 |
| 4211 | 0 | 1 | 0 | 6 | 0 | 23 | 0 | 70 | 186 |
| 531 | 0 | 0 | 1 | 0 | 8 | 8 | 38 | 61 | 206 |
| 5211 | 0 | 0 | 0 | 1 | 0 | 7 | 0 | 30 | 100 |
| 532 | 0 | 0 | 0 | 0 | 1 | 0 | 9 | 8 | 39 |
| 5311 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 8 | 38 |
| 533 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 5321 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 9 |
| 5322 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

TABLE 9. Chern-Mather classes for all but 5 Schubert varieties in the Cayley plane, the cominuscule space E_6/P_6 .