

HANKEL DETERMINANTS OF SEQUENCES RELATED TO BERNOULLI AND EULER POLYNOMIALS

KARL DILCHER AND LIN JIU*

ABSTRACT. We evaluate the Hankel determinants of various sequences related to Bernoulli and Euler numbers and special values of the corresponding polynomials. Some of these results arise as special cases of Hankel determinants of certain sums and differences of Bernoulli and Euler polynomials, while others are consequences of a method that uses the derivatives of Bernoulli and Euler polynomials. We also obtain Hankel determinants for sequences of sums and differences of powers and for generalized Bernoulli polynomials belonging to certain Dirichlet characters with small conductors. Finally, we collect and organize Hankel determinant identities for numerous sequences, both new and known, containing Bernoulli and Euler numbers and polynomials.

1. INTRODUCTION

The Bernoulli numbers B_n and polynomials $B_n(x)$ are usually defined by the generating functions

$$(1.1) \quad \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \text{and} \quad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

and the related Euler numbers E_n and polynomials $E_n(x)$ can be defined by

$$(1.2) \quad \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad \text{and} \quad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

These four sequences are among the most important special number and polynomial sequences in mathematics, with numerous applications in number theory, combinatorics, numerical analysis, and other areas. The first few elements of these sequences are listed in Table 1, and their most important properties can be found, e.g., in [17, Ch. 24].

This paper will be concerned with Hankel determinants of sequences related to Bernoulli and Euler numbers and special values of the corresponding polynomials. A *Hankel matrix* or *persymmetric matrix* is a symmetric matrix which has constant

2020 *Mathematics Subject Classification.* Primary 11B68; Secondary 11C20.

Key words and phrases. Bernoulli polynomial, Euler polynomial, Hankel determinant, orthogonal polynomial.

*corresponding author.

Research supported in part by the Natural Sciences and Engineering Research Council of Canada, Grant # 145628481.

entries along its antidiagonals; in other words, it is of the form

$$(1.3) \quad (c_{i+j})_{0 \leq i,j \leq n} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ c_2 & c_3 & c_4 & \cdots & c_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n} \end{pmatrix}.$$

A *Hankel determinant* is then the determinant of a Hankel matrix. Furthermore, given a sequence $\mathbf{c} = (c_0, c_1, \dots)$ of numbers or polynomials, we define the n th Hankel determinant of \mathbf{c} to be

$$(1.4) \quad H_n(\mathbf{c}) = H_n(c_k) = \det_{0 \leq i,j \leq n} (c_{i+j}).$$

If we use the second notation, $H_n(c_k)$, it is always assumed that the sequence begins with $k = 0$; it should be noted that the value of the Hankel determinant depends on this in an essential way. We shall return to this issue in Section 6.

The Hankel determinants of a sequence are closely related to classical orthogonal polynomials; see, e.g., [10, Ch. 2]. In fact, many evaluations of Hankel determinants come from this connection and a related connection with continued fractions. All this has been well-studied; see, e.g., the very extensive treatments in [13, 14, 15], and the numerous references provided there.

Using these connections with orthogonal polynomials and continued fractions, the current authors recently derived some apparently novel evaluations of Hankel determinants of certain subsequences of Bernoulli and Euler polynomials [6]. It is the purpose of this paper to use the results in [6], combined with some other results, to obtain new evaluations of Hankel determinants of various sequences related to Bernoulli and Euler numbers and polynomials.

To put this in perspective and provide an introductory example, we quote the following well-known result for the Euler numbers E_n , due to Al-Salam and Carlitz [1, Eq. (4.2)], namely

$$(1.5) \quad H_n(E_k) = (-1)^{\binom{n+1}{2}} \prod_{\ell=1}^n \ell!^2 \quad (n \geq 0).$$

The closely related sequence (kE_{k-1}) turns out to be quite different. As a corollary of one of the main results in this paper we obtain

$$(1.6) \quad H_{2m+1}(kE_{k-1}) = (-1)^{m+1} 2^{4m(m+1)} \prod_{\ell=1}^m \ell!^8 \quad (m \geq 0),$$

with $H_{2m}(kE_{k-1}) = 0$.

This paper is structured as follows. In Section 2 we quote some identities and known results that will be used later in the paper. Section 3 deals with Hankel determinants of sums and differences of Bernoulli polynomials with the same index, along with various consequences, and in Section 4 we derive analogous results for Euler polynomials. In Section 5 we introduce a method based on the derivatives of Bernoulli and Euler polynomials and obtain several more Hankel determinant evaluations as corollaries. We also recall some necessary facts on orthogonal polynomials in Section 5, and apply this derivative method to a shifted sequence in Section 6. Finally, in Section 7, we collect and organize Hankel determinant identities for numerous sequences containing Bernoulli and Euler numbers and polynomials.

2. SOME KNOWN RESULTS

We begin this section with a few general properties of Bernoulli and Euler polynomials and of Hankel determinants that will be required in later sections. We begin with two identities that connect Bernoulli and Euler numbers with their polynomial analogues:

$$(2.1) \quad B_n(x) = \sum_{j=0}^n \binom{n}{j} B_j x^{n-j}, \quad E_n(x) = \sum_{j=0}^n \binom{n}{j} \frac{E_j}{2^j} \left(x - \frac{1}{2}\right)^{n-j}.$$

These identities follow easily from (1.1), resp. (1.2). The Bernoulli and Euler polynomials are also connected to each other through

$$(2.2) \quad E_{n-1}(x) = \frac{2^n}{n} \left(B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right) \right)$$

(see, e.g., [17, Eq. 24.4.23]), with the related identities

$$(2.3) \quad (2n+1)E_{2n} = 2^{4n+2} B_{2n+1}\left(\frac{3}{4}\right), \quad (n+1)E_n(1) = 2(2^{n+1}-1)B_{n+1} \quad (n \geq 1),$$

which follow easily from [17, Eq. 24.4.31, 24.4.26]. Other important properties are the pair of reflection formulas

$$(2.4) \quad B_n(1-x) = (-1)^n B_n(x), \quad E_n(1-x) = (-1)^n E_n(x);$$

see, e.g., [17, Eq. 24.4.3, 24.4.4], and the zeros

$$(2.5) \quad B_{2k+1}\left(\frac{1}{2}\right) = B_{2k+3}(0) = B_{2k+3}(1) = 0, \quad k = 0, 1, 2, \dots$$

and similarly

$$(2.6) \quad E_{2k+1}\left(\frac{1}{2}\right) = E_{2k+2}(0) = E_{2k+2}(1) = 0, \quad k = 0, 1, 2, \dots$$

Most of these follow from the identities above; see also [17, Sect, 24.4(vi)].

n	B_n	E_n	$B_n(x)$	$E_n(x)$
0	1	1	1	1
1	-1/2	0	$x - \frac{1}{2}$	$x - \frac{1}{2}$
2	1/6	-1	$x^2 - x + \frac{1}{6}$	$x^2 - x$
3	0	0	$x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$	$x^3 - \frac{3}{2}x^2 + \frac{1}{4}$
4	-1/30	5	$x^4 - 2x^3 + x^2 - \frac{1}{30}$	$x^4 - 2x^3 + x$
5	0	0	$x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$	$x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}$
6	1/42	-61	$x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}$	$x^6 - 3x^5 + 5x^3 - 3x$

Table 1: $B_n, E_n, B_n(x)$ and $E_n(x)$ for $0 \leq n \leq 6$.

Next we state some useful properties of Hankel determinants.

Lemma 2.1. *Let (c_0, c_1, \dots) be a sequence and x a variable or a complex number. Then for all $n \geq 0$ we have*

$$(2.7) \quad H_n(x^k c_k) = x^{n(n+1)} H_n(c_k),$$

and if

$$c_k(x) = \sum_{j=0}^k \binom{k}{j} c_j x^{k-j},$$

then

$$(2.8) \quad H_n(c_k(x)) = H_n(c_k).$$

The identity (2.7) is easy to derive by factoring suitable powers of x from the rows and columns of the Hankel determinant. (2.8) can be found, with proof, in [12]; it is also mentioned and used in various other publications, for instance in [13, Lemma 15]. Applying Lemma 2.1 to both identities in (2.1), we obtain

$$(2.9) \quad H_n(B_k(x)) = H_n(B_k),$$

$$(2.10) \quad H_n(E_k(x)) = 2^{-n(n+1)} H_n(E_k).$$

The Hankel determinant on right-hand side of (2.10) was already mentioned in (1.5), while the right-hand side of (2.9) will be recalled in the final section.

The next lemma is about determinants of “checkerboard matrices”, namely matrices in which every other entry vanishes. This result can be found in [5] as Lemmas 5 and 6, and covers more general matrices than just Hankel matrices.

Lemma 2.2. *Let $M = (M_{i,j})_{0 \leq i,j \leq n-1}$ be a matrix. If $M_{i,j} = 0$ whenever $i + j$ is odd, then*

$$(2.11) \quad \det_{0 \leq i,j \leq n-1} (M_{i,j}) = \det_{0 \leq i,j \leq \lfloor (n-1)/2 \rfloor} (M_{2i,2j}) \cdot \det_{0 \leq i,j \leq \lfloor (n-2)/2 \rfloor} (M_{2i+1,2j+1}).$$

If $M_{i,j} = 0$ whenever $i + j$ is even, then for even n we have

$$(2.12) \quad \det_{0 \leq i,j \leq n-1} (M_{i,j}) = (-1)^{n/2} \det_{0 \leq i,j \leq \lfloor (n-1)/2 \rfloor} (M_{2i+1,2j}) \cdot \det_{0 \leq i,j \leq \lfloor (n-2)/2 \rfloor} (M_{2i,2j+1}),$$

while for odd n we have

$$(2.13) \quad \det_{0 \leq i,j \leq n-1} (M_{i,j}) = 0.$$

We note that there is a small typographical error in Lemma 6 of [5]; (2.12) above shows the correct power of (-1) . Lemma 2.2 is best explained through some examples.

Example 1. By (2.11) we have

$$\det \begin{pmatrix} a & 0 & b & 0 & c \\ 0 & \mathbf{d} & 0 & \mathbf{e} & 0 \\ f & 0 & g & 0 & h \\ 0 & \mathbf{i} & 0 & \mathbf{j} & 0 \\ k & 0 & l & 0 & m \end{pmatrix} = \det \begin{pmatrix} a & b & c \\ f & g & h \\ k & l & m \end{pmatrix} \cdot \det \begin{pmatrix} \mathbf{d} & \mathbf{e} \\ \mathbf{i} & \mathbf{j} \end{pmatrix},$$

$$\det \begin{pmatrix} a & 0 & b & 0 \\ 0 & \mathbf{d} & 0 & \mathbf{e} \\ f & 0 & g & 0 \\ 0 & \mathbf{i} & 0 & \mathbf{j} \end{pmatrix} = \det \begin{pmatrix} a & b \\ f & g \end{pmatrix} \cdot \det \begin{pmatrix} \mathbf{d} & \mathbf{e} \\ \mathbf{i} & \mathbf{j} \end{pmatrix}.$$

Example 2. By (2.12) and (2.13) we have

$$\det \begin{pmatrix} 0 & \mathbf{d} & 0 & \mathbf{e} \\ f & 0 & g & 0 \\ 0 & \mathbf{i} & 0 & \mathbf{j} \\ k & 0 & l & 0 \end{pmatrix} = \det \begin{pmatrix} f & g \\ k & l \end{pmatrix} \cdot \det \begin{pmatrix} \mathbf{d} & \mathbf{e} \\ \mathbf{i} & \mathbf{j} \end{pmatrix},$$

and the smaller cases

$$\det \begin{pmatrix} 0 & \mathbf{d} \\ a & 0 \end{pmatrix} = -a \cdot \mathbf{d}, \quad \det \begin{pmatrix} 0 & \mathbf{d} & 0 \\ f & 0 & g \\ 0 & \mathbf{i} & 0 \end{pmatrix} = 0.$$

We also recall a well-known property of determinants, which is easy to prove. Let $M = (M_{i,j})$ be an $n \times n$ matrix, and λ be a constant. Then

$$(2.14) \quad \det(\lambda M_{i,j}) = \lambda^n \det(M_{i,j}).$$

This property will be required in the following sections.

We conclude this section by quoting two of the main results from [6].

Theorem 2.3 ([6], Theorem 1.1). *If $b_k = B_{2k+1}(\frac{x+1}{2})$, then for $n \geq 0$ we have*

$$(2.15) \quad H_n(b_k) = (-1)^{\binom{n+1}{2}} \left(\frac{x}{2} \right)^{n+1} \prod_{\ell=1}^n \left(\frac{\ell^4(x^2 - \ell^2)}{4(2\ell+1)(2\ell-1)} \right)^{n+1-\ell}.$$

Theorem 2.4 ([6], Corollary 5.2). *Let $c_k^{(\nu)} = E_{2k+\nu}(\frac{x+1}{2})$ for $\nu = 0, 1, 2$. Then for all $n \geq 0$ we have*

$$(2.16) \quad H_n(c_k^{(\nu)}) = (-1)^{\binom{n+1}{2}} E_\nu(\frac{x+1}{2})^{n+1} \prod_{\ell=1}^n \left(\frac{\ell^2}{4} (x^2 - (2\ell + \nu - 1)^2) \right)^{n+1-\ell},$$

or more explicitly,

$$(2.17) \quad H_n(c_k^{(0)}) = (-1)^{\binom{n+1}{2}} \prod_{\ell=1}^n \left(\frac{\ell^2}{4} (x^2 - (2\ell - 1)^2) \right)^{n+1-\ell},$$

$$(2.18) \quad H_n(c_k^{(1)}) = (-1)^{\binom{n+1}{2}} \left(\frac{x}{2} \right)^{n+1} \prod_{\ell=1}^n \left(\frac{\ell^2}{4} (x^2 - (2\ell)^2) \right)^{n+1-\ell},$$

$$(2.19) \quad H_n(c_k^{(2)}) = (-1)^{\binom{n+1}{2}} \left(\frac{x^2 - 1}{4} \right)^{n+1} \prod_{\ell=1}^n \left(\frac{\ell^2}{4} (x^2 - (2\ell + 1)^2) \right)^{n+1-\ell}.$$

3. SUMS AND DIFFERENCES OF BERNOULLI POLYNOMIALS

While the Hankel determinants of the Euler numbers E_k and Euler polynomials $E_k(x)$ are well-known (see (1.5) and (2.10)), this is not the case for $H_n(b_k)$, where $b_k = k \cdot E_{k-1}(x)$ for $k \geq 1$, and $b_0 = 0$. In order to deal with this case, one could try to use the identity (2.2) in the form

$$(3.1) \quad k \cdot E_{k-1}(x) = -2^k \left(B_k(\frac{x}{2}) - B_k(\frac{x+1}{2}) \right),$$

It is the purpose of this section to show that we can obtain meaningful results for much more general differences, as well as sums, of Bernoulli polynomials than the right-hand side of (3.1).

We fix integers $q \geq 1$ and $0 \leq r < s$, and define

$$(3.2) \quad b_k^\pm(q, r, s; x) := B_k(\frac{x+r}{q}) \pm B_k(\frac{x+s}{q}), \quad k = 0, 1, 2, \dots$$

First we show that, just as in the case of Bernoulli and Euler polynomials, the Hankel determinants of the polynomials in (3.2) do not depend on x . This will greatly simplify further work in this section.

Lemma 3.1. *For any $n \geq 0$, $H_n(b_k^\pm(q, r, s; x))$ is independent of x .*

Proof. Using a well-known identity (see, e.g., [17, Eq. 24.4.12]) for Bernoulli polynomials, we get with (3.2),

$$\begin{aligned} b_k^\pm(q, r, s; x) &= \sum_{j=0}^k \binom{k}{j} B_j\left(\frac{r}{q}\right)\left(\frac{x}{q}\right)^{k-j} \pm \sum_{j=0}^k \binom{k}{j} B_j\left(\frac{s}{q}\right)\left(\frac{x}{q}\right)^{k-j} \\ &= \sum_{j=0}^k \binom{k}{j} \left(B_j\left(\frac{r}{q}\right) \pm B_j\left(\frac{s}{q}\right)\right) \left(\frac{x}{q}\right)^{k-j} \\ &= \sum_{j=0}^k \binom{k}{j} b_k^\pm(q, r, s; 0) \left(\frac{x}{q}\right)^{k-j}. \end{aligned}$$

By Lemma 2.1 this means that for any n the n th Hankel determinant of the sum on the right is independent of x , which completes the proof. \square

The next lemma shows how we can exploit Lemma 3.1 by being able to choose an appropriate value for x .

Lemma 3.2. *For fixed integers $q \geq 1$ and $0 \leq r < s$, we have*

$$(3.3) \quad b_k^-(q, r, s; \frac{q-r-s}{2}) = \begin{cases} 0 & \text{if } k \text{ is even,} \\ 2B_{2\mu+1}(\frac{q+r-s}{2q}) & \text{if } k = 2\mu + 1, \end{cases}$$

and

$$(3.4) \quad b_k^+(q, r, s; \frac{q-r-s}{2}) = \begin{cases} 2B_{2\mu}(\frac{q+r-s}{2q}) & \text{if } k = 2\mu, \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. With (3.2) we get

$$\begin{aligned} b_k^\pm(q, r, s; \frac{q-r-s}{2}) &= B_k\left(\frac{q+r-s}{2q}\right) \pm B_k\left(\frac{q-r+s}{2q}\right) = B_k\left(\frac{q+r-s}{2q}\right) \pm B_k\left(1 - \frac{q+r-s}{2q}\right) \\ &= B_k\left(\frac{q+r-s}{2q}\right) \pm (-1)^k B_k\left(\frac{q+r-s}{2q}\right), \end{aligned}$$

where we have used the reflection formula (2.4). The desired identities (3.3) and (3.4) now follow immediately. \square

The significance of Lemma 3.2 lies in the fact that the Hankel matrices of the sequence in (3.3) are “checkerboard matrices”, and therefore Lemma 2.2 applies. The following is the main result of this section.

Theorem 3.3. *Let $q \geq 1$ and $0 \leq r < s$ be fixed integers and set $b_k := b_k^-(q, r, s; x)$. Then for all integers $m \geq 0$ we have $H_{2m}(b_k) = 0$ and*

$$(3.5) \quad H_{2m+1}(b_k) = (-1)^{m+1} \left(\frac{s-r}{q^{m+1}}\right)^{2m+2} \prod_{\ell=1}^m \left(\frac{\ell^4((s-r)^2 - (q\ell)^2)}{4(2\ell+1)(2\ell-1)}\right)^{2(m+1-\ell)}.$$

Proof. By Lemmas 3.1 and 3.2, we can fix $x = (q - r - s)/2$ for b_k and apply the second part of Lemma 2.2 with $M_{i,j} = b_{i+j}$. Then (2.13) immediately gives the first statement. In order to prove (3.5), we begin by using (2.12) with $n = 2m + 2$, which gives

(3.6)

$$\det_{0 \leq i, j \leq 2m+1} (b_{i+j}) = (-1)^{m+1} \left(\det_{0 \leq i, j \leq m} (b_{2(i+j)+1}) \right)^2 = (-1)^{m+1} H_m(b_{2\mu+1})^2.$$

By (3.3), and using (2.14), we have

$$(3.7) \quad H_m(b_{2\mu+1}) = 2^{m+1} H_m\left(B_{2\mu+1}\left(\frac{q+r-s}{2q}\right)\right).$$

Finally we use Theorem 2.3 with $x = (r-s)/q$, which gives $(x+1)/2 = (q+r-s)/2q$, as required. Combining this with (2.15), (3.7) and (3.6), we readily obtain (3.5). \square

We note that there is no analogue to Theorem 3.3 for $b_k^+(q, r, s; x)$. The main reason for this is the absence of an identity such as (2.15) for even-index Bernoulli polynomials. This fact is briefly discussed in [6, Ch. 4].

We now consider a special case of Theorem 3.3. Returning to (3.1), we get the following results.

Corollary 3.4. *For all integers $m \geq 0$ we have $H_{2m}(kE_{k-1}(x)) = H_{2m}(kE_{k-1}) = 0$, and*

$$(3.8) \quad H_{2m+1}(kE_{k-1}(x)) = (-1)^{m+1} \prod_{\ell=1}^m \ell!^8,$$

$$(3.9) \quad H_{2m+1}(kE_{k-1}) = (-1)^{m+1} 2^{4m(m+1)} \prod_{\ell=1}^m \ell!^8.$$

Proof. Comparing (3.1) with (3.2), we see that $kE_{k-1}(x) = -2^k b_k^-(2, 0, 1; x)$. If we use (2.14), (2.7), and (3.5), we get (3.8) after some straightforward manipulations.

To obtain (3.9), we use the well-known identity $E_n = 2^n E_n(\frac{1}{2})$, which is a special case of the second part of (2.1) with $x = 1/2$. Then with Lemma 2.1 we get

$$\begin{aligned} H_{2m+1}(kE_{k-1}) &= H_{2m+1}\left(\frac{1}{2} 2^k kE_{k-1}\left(\frac{1}{2}\right)\right) \\ &= \left(\frac{1}{2}\right)^{2m+2} 2^{(2m+2)(2m+1)} H_{2m+1}\left(kE_{k-1}\left(\frac{1}{2}\right)\right), \end{aligned}$$

and (3.9) follows immediately from (3.8). \square

Theorem 3.3 can also be used to deal with a few special cases of character analogues of Bernoulli numbers and polynomials. Let χ be a primitive character with conductor q . Then the generalized Bernoulli numbers and polynomials belonging to χ are defined by

$$(3.10) \quad \sum_{a=1}^q \frac{\chi(a)te^{at}}{e^{qt}-1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}, \quad B_{n,\chi}(x) = \sum_{k=0}^n \binom{n}{k} B_{k,\chi} x^{n-k},$$

so that $B_{n,\chi}(0) = B_{n,\chi}$ for all $n \geq 0$. These objects contain both the Bernoulli and Euler numbers and polynomials as special cases. Indeed,

$$(3.11) \quad B_n(x) = B_{n,\chi_0}(x-1), \quad E_n(x) = -\frac{2^{1-n}}{n+1} B_{n+1,\chi_4}(2x-1),$$

where χ_0 is the trivial character and χ_4 is the unique (non-trivial) character with conductor 4; see, e.g., [17, Sect. 24.16(ii)]. It was the right-hand identity in (3.11), by the way, that led us to first consider Hankel determinants of $k \cdot E_{k-1}(x)$.

On the other hand, all generalized Bernoulli polynomials can be written in terms of the ordinary Bernoulli polynomials by way of the identity

$$(3.12) \quad B_{n,\chi}(x) = q^{n-1} \sum_{a=1}^q \chi(a) B_n\left(\frac{a+x}{q}\right),$$

which follows easily from comparing the generating functions in (1.1) and (3.10).

There are just three primitive characters that have exactly two nonzero values between $a = 1$ and $a = q$, namely those with conductors $q = 3, 4$, and 6 . In all cases we have $\chi_q(1) = 1$, $\chi_q(q - 1) = -1$, and 0 elsewhere. With (3.12) and (3.2) we then have

$$(3.13) \quad B_{k,\chi_q} = q^{k-1} b_k^-(q, 1, q - 1; 0) \quad (q = 3, 4, 6).$$

With Theorem 3.3 we then get the following result.

Corollary 3.5. *For $q = 3, 4, 6$ and for all integers $m \geq 0$ we have $H_{2m}(B_{k,\chi_q}) = 0$ and*

$$(3.14) \quad H_{2m+1}(B_{k,\chi_q}) = (-1)^{m+1} (q^{m-1}(q-2))^{2m+2} \times \prod_{\ell=1}^m \left(\frac{\ell^4((q-2)^2 - (q\ell)^2)}{4(2\ell+1)(2\ell-1)} \right)^{2(m+1-\ell)}.$$

Proof. We note that by (3.13), and using (2.7) and (2.14), we have

$$\begin{aligned} H_{2m+1}(B_{k,\chi_q}) &= H_{2m+1}(q^{-1}q^k b_k^-(q, 1, q - 1; 0)) \\ &= q^{-(2m+2)} q^{(2m+1)(2m+2)} H_{2m+1}(b_k^-(q, 1, q - 1; 0)). \end{aligned}$$

The desired results now follow directly from Theorem 3.3. \square

Considering the numerator in the right-most fraction in (3.5), we get the following immediate consequence.

Corollary 3.6. *If $q \mid s - r$, then $H_{2m+1}(b_k^-(q, r, s; x)) = 0$ for all $m \geq (s - r)/q$. On the other hand, if $q \nmid s - r$, then $H_{2m+1}(b_k^-(q, r, s; x)) \neq 0$ for all $m \geq 0$.*

We conclude this section with an application of Theorem 3.3 for $q = 1$, which also illustrates Corollary 3.6. Using the well-known identity

$$x^{k-1} + (x+1)^{k-1} + \cdots + (x+s-1)^{k-1} = \frac{1}{k} (B_k(x+s) - B_k(x))$$

for integers $s \geq 1$ (see, e.g., [17, Eq. 24.4.9]) we have by (3.2), with $x = 1$,

$$(3.15) \quad k(1 + 2^{k-1} + \cdots + s^{k-1}) = -b_k^-(1, 0, s; 1).$$

With (2.14) we see that the $-$ sign on the right is irrelevant; Theorem 3.3 now implies the following result.

Corollary 3.7. *Let $S_k(s)$ be the left-hand side of (3.15). Then for all integers $m \geq 0$ we have $H_{2m}(S_k(s)) = 0$ and*

$$H_{2m+1}(S_k(s)) = (-s^2)^{m+1} \prod_{\ell=1}^m \left(\frac{\ell^4(s^2 - \ell^2)}{4(2\ell+1)(2\ell-1)} \right)^{2(m+1-\ell)},$$

and in particular, $H_{2m+1}(S_k(s)) = 0$ for $m \geq s$.

A similar result was earlier obtained by Al-Salam and Carlitz [1, Eq. (7.1)].

4. SUMS AND DIFFERENCES OF EULER POLYNOMIALS

In this section we present “Euler analogues” to some of the results in the previous section. In analogy to (3.2) we fix integers $q \geq 1$ and $0 \leq r < s$ and define

$$(4.1) \quad e_k^\pm(q, r, s; x) := E_k\left(\frac{x+r}{q}\right) \pm E_k\left(\frac{x+s}{q}\right), \quad k = 0, 1, 2, \dots$$

Since the main identity used in the proof of Lemma 3.1 also holds for Euler polynomials (see [17, Eq. 24.4.13]), we have

Lemma 4.1. *For any $n \geq 0$, $H_n(e_k^\pm(q, r, s; x))$ is independent of x .*

Furthermore, since the reflection formulas in (2.4) are identical for both the Bernoulli and Euler polynomials, the following lemma also carries over from the Bernoulli case.

Lemma 4.2. *For fixed integers $q \geq 1$ and $0 \leq r < s$, we have*

$$(4.2) \quad e_k^-(q, r, s; \frac{q-r-s}{2}) = \begin{cases} 0 & \text{if } k \text{ is even,} \\ 2E_{2\mu+1}\left(\frac{q+r-s}{2q}\right) & \text{if } k = 2\mu + 1, \end{cases}$$

and

$$(4.3) \quad e_k^+(q, r, s; \frac{q-r-s}{2}) = \begin{cases} 2E_{2\mu}\left(\frac{q+r-s}{2q}\right) & \text{if } k = 2\mu, \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

While in Theorem 3.3 we only obtained a result in the “–” case, for Euler polynomials we get meaningful results in both cases.

Theorem 4.3. *Let $q \geq 1$ and $0 \leq r < s$ be fixed integers and set $e_k^\pm := e_k^\pm(q, r, s; x)$. Then for all integers $m \geq 0$ we have $H_{2m}(e_k^-) = 0$ and*

$$(4.4) \quad H_{2m+1}(e_k^-) = (-1)^{m+1} \left(\frac{s-r}{q} \right)^{2m+2} \prod_{\ell=1}^m \left(\frac{\ell^2}{4} \left(\left(\frac{s-r}{q} \right)^2 - (2\ell)^2 \right) \right)^{2(m+1-\ell)}.$$

Furthermore, we have

$$(4.5) \quad H_{2m}(e_k^+) = (-1)^m \frac{2^{2m+1}}{m!^2} \prod_{\ell=1}^m \left(\frac{\ell^2}{4} \left(\left(\frac{s-r}{q} \right)^2 - (2\ell-1)^2 \right) \right)^{2(m+1-\ell)}$$

and

$$(4.6) \quad H_{2m+1}(e_k^+) = \prod_{\ell=0}^m \frac{\ell!^4}{16^\ell} \left(\left(\frac{s-r}{q} \right)^2 - (2\ell+1)^2 \right)^{2(m-\ell)+1}.$$

Proof. Fix $x = (q - r - s)/2$. The proof for e_k^- is similar to that of Theorem 3.3. We use again the second part of Lemma 2.2, this time with $M_{i,j} = e_{i+j}^-$; then (2.13) shows that $H_{2m}(e_k^-) = 0$. To prove (4.4), we begin by using (2.12) with $n = 2m+2$, which gives

$$(4.7) \quad \det_{0 \leq i, j \leq 2m+1} (e_{i+j}^-) = (-1)^{m+1} \left(\det_{0 \leq i, j \leq m} (e_{2(i+j)+1}^-) \right)^2 = (-1)^{m+1} H_m(e_{2\mu+1}^-)^2.$$

With (4.2) and (2.14) we have

$$(4.8) \quad H_m(e_{2\mu+1}^-) = 2^{m+1} H_m\left(E_{2\mu+1}\left(\frac{q+r-s}{2q}\right)\right).$$

Finally we use (2.18) with $x = (r - s)/q$. Then with (4.8) and (4.7) we immediately get (4.4).

For e_k^+ we use the first part of Lemma 2.2 and distinguish between two cases. First, when $n = 2m + 1$, then by (2.11) we have

$$\det_{0 \leq i, j \leq 2m} (e_{i+j}^+) = \det_{0 \leq i, j \leq m} (e_{2(i+j)}^+) \cdot \det_{0 \leq i, j \leq m-1} (e_{2(i+j)+2}^+),$$

and with (4.3) and (2.14) we get

$$(4.9) \quad H_{2m}(e_k^+) = 2^{2m+1} H_m(E_{2\mu}(\frac{q+r-s}{2q})) \cdot H_{m-1}(E_{2\mu+2}(\frac{q+r-s}{2q})).$$

Then we substitute (2.17) with $n = m$ and (2.19) with $n = m - 1$ into (4.9), both with $x = (r - s)/q$. After some straightforward manipulations we finally obtain (4.5).

Lastly, to prove (4.6) we use (2.11) with $n = 2m + 2$. Then

$$\det_{0 \leq i, j \leq 2m+1} (e_{i+j}^+) = \det_{0 \leq i, j \leq m} (e_{2(i+j)}^+) \cdot \det_{0 \leq i, j \leq m} (e_{2(i+j)+2}^+),$$

and once again using (4.3) and (2.14) we get

$$(4.10) \quad H_{2m+1}(e_k^+) = 4^{m+1} H_m(E_{2\mu}(\frac{q+r-s}{2q})) \cdot H_m(E_{2\mu+2}(\frac{q+r-s}{2q})).$$

We substitute (2.17) and (2.19) into (4.10), both with $n = m$ and $x = (r - s)/q$. After some tedious but straightforward manipulations we get (4.6). \square

As a first consequence of Theorem 4.3 we consider a few more cases of the generalized Bernoulli numbers and polynomials defined in (3.10). In particular, we will deal with certain Dirichlet characters modulo 8 and 12, as given in Table 2.

n	1	3	5	7
$\chi_{8,1}(n)$	1	-1	-1	1
$\chi_{8,2}(n)$	1	1	-1	-1

n	1	5	7	11
$\chi_{12,1}(n)$	1	-1	-1	1
$\chi_{12,2}(n)$	1	1	-1	-1

Table 2: Some characters modulo 8 and 12.

We note that both characters modulo 8 are primitive, and while $\chi_{12,1}$ is also primitive, $\chi_{12,2}$ is induced from the character χ_3 as defined in Section 3 and therefore has conductor 3. However, this does not affect the result that follows.

We begin with the character $\chi_{8,1}$. By (3.12), (2.2), and the definition (4.1) we have

$$\begin{aligned} B_{n,\chi_{8,1}}(x) &= 8^{n-1} (B_n(\frac{x+1}{8}) - B_n(\frac{x+3}{8}) - B_n(\frac{x+5}{8}) + B_n(\frac{x+7}{8})) \\ &= 8^{n-1} \frac{n}{2^n} (-E_{n-1}(\frac{x+1}{4}) + E_{n-1}(\frac{x+3}{4})) \\ &= -\frac{n}{2} \cdot 4^{n-1} \cdot e_{n-1}^-(4, 1, 3; x). \end{aligned}$$

In the same way we can determine expressions for the remaining three cases. Upon setting $n = k + 1$ we summarize the four cases as follows: For $q = 4$ and 6 we have

$$(4.11) \quad b_k^{(1)} := \frac{1}{k+1} B_{k+1,\chi_{2q,1}}(x) = -\frac{1}{2} q^k e_k^-(q, 1, q-1; x),$$

$$(4.12) \quad b_k^{(2)} := \frac{1}{k+1} B_{k+1,\chi_{2q,2}}(x) = -\frac{1}{2} q^k e_k^+(q, 1, q-1; x).$$

Applying the first part of Theorem 4.3 to (4.11) and the second part to (4.12) and using the identities (2.14) and (2.7), we get the following result.

Corollary 4.4. *Let $q = 4$ or 6 , set $\tilde{q} := (q - 2)/q$, and let $b_k^{(1)}, b_k^{(2)}$ be as above. Then for all integers $m \geq 0$ we have $H_{2m}(b_k^{(1)}) = 0$, and*

$$\begin{aligned} H_{2m+1}(b_k^{(1)}) &= (-1)^{m+1} \left(\frac{q-2}{2} q^{2m} \right)^{2m+2} \prod_{\ell=1}^m \left(\frac{\ell^2}{4} (\tilde{q}^2 - (2\ell)^2) \right)^{2(m+1-\ell)}, \\ H_{2m}(b_k^{(2)}) &= (-1)^{m+1} \frac{q^{2m(2m+1)}}{m!^2} \prod_{\ell=1}^m \left(\frac{\ell^2}{4} (\tilde{q}^2 - (2\ell-1)^2) \right)^{2(m+1-\ell)}, \\ H_{2m+1}(b_k^{(2)}) &= \left(\frac{1}{2} q^{2m+1} \right)^{2m+2} \prod_{\ell=0}^m \frac{\ell!^4}{16^\ell} (\tilde{q}^2 - (2\ell+1)^2)^{2(m-\ell)+1}. \end{aligned}$$

As another consequence of Theorem 4.3 we consider the alternating analogue of Corollary 3.7. For integers $s \geq 1$ and $k \geq 0$ we denote

$$(4.13) \quad T_k(s) := 1 - 2^k + 3^k - \cdots + (-1)^{s-1} s^k.$$

There is a well-known connection with Euler polynomials, given by

$$T_k(s) = \frac{1}{2} (E_k(1) - (-1)^s E_k(s+1));$$

see, e.g., [17, Eq. 24.4.10]. With (4.1) this means that

$$T_k(s) = \begin{cases} \frac{1}{2} e_k^-(1, 0, s; 1) & \text{if } s \text{ is even,} \\ \frac{1}{2} e_k^+(1, 0, s; 1) & \text{if } s \text{ is odd.} \end{cases}$$

Using (2.14) and Theorem 4.3, we immediately get the following identities.

Corollary 4.5. *Let $T_k(s)$ be as defined in (4.13).*

(a) *When $s = 2t$ is even, then for all integers $m \geq 0$ we have $H_{2m}(T_k(2t)) = 0$ and*

$$H_{2m+1}(T_k(2t)) = (-t^2)^{m+1} \prod_{\ell=1}^m (\ell^2(t^2 - \ell^2))^{2(m+1-\ell)},$$

and in particular, $H_{2m+1}(T_k(2t)) = 0$ for $m \geq t$.

(b) *When s is odd, then for all $m \geq 0$ we have*

$$\begin{aligned} H_{2m}(T_k(s)) &= \frac{(-1)^m}{m!^2} \prod_{\ell=1}^m \left(\frac{\ell^2}{4} (s^2 - (2\ell-1)^2) \right)^{2(m+1-\ell)}, \\ H_{2m+1}(T_k(s)) &= \frac{1}{4^{m+1}} \prod_{\ell=0}^m \frac{\ell!^4}{16^\ell} (s^2 - (2\ell+1)^2)^{2(m-\ell)+1}, \end{aligned}$$

and these determinants become 0 when $m \geq (s+1)/2$, resp. $m \geq (s-1)/2$.

This result shows again that under certain circumstances all Hankel determinants from a certain index on can vanish. In this connection it would be easy to state an “Euler analogue” to Corollary 3.6. We leave this to the interested reader.

5. DERIVATIVE SEQUENCES

If for a sequence (c_0, c_1, \dots) we know the Hankel determinant $H_n(c_k)$, then by (2.7) and (2.14) we also know $H_n(a \cdot b^k \cdot c_k)$ for any numbers or variables a and b . However, this is generally not the case for $H_n(k \cdot c_k)$ or $H_n((k+1) \cdot c_k)$, or other expressions of this kind. It is the purpose of this section to present a method that allows us to deal with such expressions in some special cases.

We recall that both the Bernoulli and Euler polynomial sequences are *Appell sequences*, that is, they satisfy the derivative property

$$(5.1) \quad B'_n(x) = nB_{n-1}(x), \quad E'_n(x) = nE_{n-1}(x).$$

These identities follow quite easily from the generating functions in (1.1) and (1.2), or from the identities in (2.1). This gives rise to the question whether Hankel determinants of sequences or subsequences of Bernoulli, Euler, or generally Appell polynomials might give rise to Hankel determinants of their derivatives. In general, this would be asking too much; however, under certain circumstances we can indeed pass from a polynomial sequence to its derivative, as the following theorem shows. We will prove it later in this section.

Theorem 5.1. *Let $A_k(x)$, $k \geq 0$, be a sequence of C^1 functions and let $x_0 \in \mathbb{C}$ be such that $A_k(x_0) = 0$ for all $k \geq 0$. Then*

$$(5.2) \quad H_n(A'_k(x_0)) = A'_0(x_0)^{n+1} \lim_{x \rightarrow x_0} \frac{H_n(A_k(x_0))}{A_0(x_0)^{n+1}}.$$

For this result to be useful we need, above all, a sequence of functions whose elements all have a root in common. But this is exactly the case for certain subsequences of Bernoulli and Euler polynomials, as one can see in (2.5) and (2.6). We use this fact in the following corollaries.

Corollary 5.2. *For all $n \geq 0$ we have*

$$(5.3) \quad H_n((2k+1)E_{2k}) = 2^{2n(n+1)} \prod_{\ell=1}^n \ell!^4.$$

Proof. We set $A_k(x) := E_{2k+1}(\frac{1+x}{2})$. Then $A_k(0) = E_{2k+1}(\frac{1}{2}) = 0$ for all $k \geq 0$ and by (5.1),

$$A'_k(0) = \frac{2k+1}{2} E_{2k}(\frac{1}{2}) = \frac{2k+1}{2^{2k+1}} E_{2k},$$

where we have also used the right-hand identity in (2.1). Now, by (2.14) and (2.7) we have

$$H_n((2k+1)E_{2k}) = H_n(2^{2k+1} A'_k(0)) = 2^{n+1} 4^{n(n+1)} H_n(A'_k(0)).$$

Next, since $A_0(x) = E_1(\frac{1+x}{2}) = \frac{x}{2}$ and $A'_0(0) = \frac{1}{2}$, we get with (2.18) and (5.2),

$$\begin{aligned} H_n((2k+1)E_{2k}) &= 2^{n+1} 4^{n(n+1)} \left(\frac{1}{2}\right)^{n+1} (-1)^{\binom{n+1}{2}} \prod_{\ell=1}^n \left(\frac{\ell^2}{4} (0 - (2\ell)^2)\right)^{n+1-\ell} \\ &= 4^{n(n+1)} \prod_{\ell=1}^n (\ell^4)^{n+1-\ell}, \end{aligned}$$

and this is easily seen to be equivalent to (5.3). \square

The identity (5.3) can also be obtained by two alternative means: First, we can use (2.3) and (2.15) with $x = 1/4$, again applying (2.14) and (2.7). And second, Corollary 3.4 shows that $H_n(kE_{k-1})$ is of “checkerboard type”; this means that we can use Lemma 2.2 combined with (3.9), and easily obtain (5.3) again.

As a second application of Theorem 5.1 we follow along the same lines as in the proof of Corollary 5.2.

Corollary 5.3. *For all $n \geq 0$ we have*

$$(5.4) \quad H_n((2^{2k+2} - 1)B_{2k+2}) = \frac{(n+1)!}{2^{n+1}} \prod_{\ell=1}^n \ell!^4.$$

Proof. Here we set $A_k(x) := E_{2k+2}(\frac{1+x}{2})$ and $x_0 = 1$. Then for all $k \geq 0$ we have $A_k(1) = E_{2k+2}(1) = 0$ and also

$$A'_k(1) = (k+1)E_{2k+1}(1) = (2^{2k+2} - 1)B_{2k+2},$$

where we have used the second identity in (2.3). We also have

$$A'_0(1) = 3B_2 = \frac{1}{2}, \quad A_0(x) = E_2(\frac{1+x}{2}) = \frac{x^2 - 1}{4}.$$

Substituting everything, including (2.19), into (5.2), we get

$$\begin{aligned} H_n((2^{2k+2} - 1)B_{2k+2}) &= (\frac{1}{2})^{n+1} (-1)^{\binom{n+1}{2}} \prod_{\ell=1}^n \left(\frac{\ell^2}{4} (1 - (2\ell+1)^2) \right)^{n+1-\ell} \\ &= \frac{1}{2^{n+1}} \prod_{\ell=1}^n (\ell^3(\ell+1))^{n+1-\ell}. \end{aligned}$$

Finally, a straightforward manipulation shows that this is equivalent to (5.4). \square

Corollary 5.4. *For all $n \geq 0$ we have*

$$(5.5) \quad H_n((2k+1)B_{2k}(\frac{1}{2})) = \prod_{\ell=1}^n \frac{\ell!^8}{(2\ell)!(2\ell+1)!}.$$

Proof. We take $A_k(x) := B_{2k+1}(\frac{1+x}{2})$ and $x_0 = 0$. Then $A_k(0) = B_{2k+1}(\frac{1}{2}) = 0$ for all $k \geq 0$. Furthermore,

$$A'_k(0) = \frac{2k+1}{2} B_{2k}(\frac{1}{2}), \quad A'_0(0) = \frac{1}{2} B_0(\frac{1}{2}) = \frac{1}{2}, \quad A_0(x) = B_1(\frac{1+x}{2}) = \frac{x}{2}.$$

With (2.14) we now get

$$H_n((2k+1)B_{2k}(\frac{1}{2})) = H_n(2A'_k(0)) = 2^{n+1} H_n(A'_k(0)),$$

and then with (5.2) and (2.15),

$$\begin{aligned} H_n((2k+1)B_{2k}(\frac{1}{2})) &= 2^{n+1} (\frac{1}{2})^{n+1} (-1)^{\binom{n+1}{2}} \prod_{\ell=1}^n \left(\frac{\ell^4(0 - \ell^2)}{4(2\ell+1)(2\ell-1)} \right)^{n+1-\ell} \\ &= \prod_{\ell=1}^n \left(\frac{\ell^6}{4(2\ell+1)(2\ell-1)} \right)^{n+1-\ell}. \end{aligned}$$

Once again, a straightforward manipulation shows that this is equivalent to (5.5). \square

To prove Theorem 5.1 and to derive some further consequences, we need some basics from the classical theory of orthogonal polynomials. Suppose we are given a sequence $\mathbf{c} = (c_0, c_1, \dots)$; then under certain conditions there exists a positive Borel measure μ on \mathbb{R} with infinite support such that

$$(5.6) \quad c_k = \int_{\mathbb{R}} y^k d\mu(y), \quad k = 0, 1, 2, \dots$$

We summarize several well-known facts and state them as a lemma, with a few consequences; see, e.g., [10, Ch. 2], or [6, Sect. 3] for a somewhat extended summary.

Lemma 5.5. *If μ is the measure in (5.6), there exists a unique sequence of monic polynomials $P_n(y)$ of degree n , $n = 0, 1, \dots$, and a sequence of positive numbers $(\zeta_n)_{n \geq 0}$, with $\zeta_0 = 1$, such that*

$$(5.7) \quad \int_{\mathbb{R}} P_m(y) P_n(y) d\mu(y) = \zeta_n \delta_{m,n},$$

where $\delta_{m,n}$ is the Kronecker delta function. Furthermore, for all $n \geq 1$ we have $\zeta_n = H_n(\mathbf{c})/H_{n-1}(\mathbf{c})$, and for $n \geq 0$,

$$(5.8) \quad P_n(y) = \frac{1}{H_{n-1}(\mathbf{c})} \det \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & y & \cdots & y^n \end{pmatrix},$$

where the polynomials $P_n(y)$ satisfy the 3-term recurrence relation $P_0(y) = 1$, $P_1(y) = y + s_0$, and

$$(5.9) \quad P_{n+1}(y) = (y + s_n)P_n(y) - t_n P_{n-1}(y) \quad (n \geq 1),$$

for some sequences $(s_n)_{n \geq 0}$ and $(t_n)_{n \geq 1}$.

We continue with a couple of important consequences, summarized as a second lemma.

Lemma 5.6. *With the sequence (c_k) and the polynomials $P_n(y)$ as in Lemma 5.5, we have for $0 \leq r \leq n-1$*

$$(5.10) \quad y^r P_n(y) \Big|_{y^k=c_k} = 0.$$

Furthermore, with the sequence (t_n) as in (5.9), we have

$$(5.11) \quad H_n(\mathbf{c}) = c_0^{n+1} t_1^n t_2^{n-1} \cdots t_{n-1}^2 t_n \quad (n \geq 0).$$

There is also an interesting and important connection with certain continued fractions (J -fractions in this case). However, this will not be needed here; it can be found in various relevant publications, for instance, in [13, p. 20], or [6, Sect. 3].

We are now ready to prove Theorem 5.1. Given a sequence of C^1 functions $A_k(x)$, by Lemma 5.6 there is a sequence $P_n(y; x)$ of monic orthogonal polynomials satisfying

$$(5.12) \quad y^r P_n(y; x) \Big|_{y^k=A_k(x)} = 0 \quad (0 \leq r \leq n-1).$$

This polynomial sequence $P_n(y; x)$ is sometimes called the *monic orthogonal polynomials with respect to $A_k(x)$* . We begin by proving the following key property.

Lemma 5.7. *Let $A_k(x)$ be a sequence of C^1 functions, and let $P_n(y; x)$ be the corresponding monic orthogonal polynomials. If $A_k(x_0) = 0$ for some $x_0 \in \mathbb{C}$ and for all $k \geq 0$, then $P_n(y; x_0)$ are the monic orthogonal polynomials with respect to the sequence $A'_k(x_0)$.*

Proof. If we write

$$P_n(y; x) = \sum_{j=0}^n \alpha_{n,j}(x)y^j,$$

then by (5.12) we have for $0 \leq r \leq n - 1$,

$$(5.13) \quad \sum_{j=0}^n \alpha_{n,j}(x)A_{j+r}(x) = 0.$$

Since $A_k(x_0) = 0$ for all $k \geq 0$, we have

$$\sum_{j=0}^n \alpha_{n,j}(x_0)A_{j+r}(x_0) = 0.$$

Subtracting this from (5.13) and dividing by $x - x_0$, we get

$$\sum_{j=0}^n \alpha_{n,j}(x) \frac{A_{j+r}(x) - A_{j+r}(x_0)}{x - x_0} = 0.$$

Finally, taking the limit as $x \rightarrow x_0$, we get

$$\sum_{j=0}^n \alpha_{n,j}(x_0)A'_{j+r}(x_0) = 0;$$

this, with (5.12), proves the lemma. \square

Proof of Theorem 5.1. By Lemma 5.7, the sequences $A_k(x_0)$ and $A'_k(x_0)$ share the same monic orthogonal polynomial. This means, in particular, that the terms t_1, t_2, \dots in (5.9) are the same, and therefore, by (5.11) we have

$$\lim_{x \rightarrow x_0} \frac{H_n(A_k(x_0))}{A_0(x_0)^{n+1}} = \frac{H_n(A'_k(x_0))}{A'_0(x_0)^{n+1}}.$$

This immediately leads to (5.2), and the proof is complete. \square

6. SHIFTED SEQUENCES

In the previous section we used the facts that $E_{2k+1}(\frac{1}{2}) = E_{2k+2}(1) = B_{2k+1}(\frac{1}{2}) = 0$ for all $k \geq 0$ to obtain the identities (5.3)–(5.5), respectively. We did that by applying Theorem 5.1 and using Theorems 2.3 and 2.4. Apart from equivalent forms, there is one more sequence with a common root we have not yet exploited, namely $B_{2k+1}(0) = 0$. The problem here is that this holds only for $k \geq 1$ since $B_1(x) = x + \frac{1}{2}$. Therefore we cannot simply combine Theorem 2.3 with Theorem 5.1, as we did in the proof of Corollary 5.4.

One possibility would be to consider $B_{2k+3}(x)$, which does indeed vanish for $x = 0$ and for all $k \geq 0$. But we still have the problem that there is no analogue of Theorem 2.3 for the shifted sequence $B_{2k+3}(\frac{1+x}{2})$; however, this can be resolved as follows.

Given a sequence $\mathbf{c} = (c_0, c_1, \dots)$, let $P_n(y)$, $n = 0, 1, \dots$, be the monic polynomials orthogonal with respect to \mathbf{c} , as in Lemmas 5.5 and 5.6. With the coefficients s_n and t_n as in (5.9), for all $n \geq 0$ we consider the determinant

$$(6.1) \quad d_n := \det \begin{pmatrix} -s_0 & 1 & 0 & \cdots & 0 \\ t_1 & -s_1 & 1 & \cdots & 0 \\ 0 & t_2 & -s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_n & -s_n \end{pmatrix};$$

thus, in particular, $d_0 = -s_0$. These determinants play an important role in connecting the Hankel determinant of a given sequence with that of a shifted sequence.

Lemma 6.1 ([16, Prop. 1.2]). *With notations as above, we have, for a given sequence (c_0, c_1, \dots) ,*

$$(6.2) \quad H_n(c_{k+1}) = d_n \cdot H_n(c_k).$$

We now consider the special case $c_k = B_{2k+1}(\frac{1+x}{2})$. Then the coefficients s_n , t_n will be functions of x and therefore $d_n = d_n(x)$ will also be a function of x . In fact, in [6, Theorem 4.1] we showed that

$$(6.3) \quad s_n = \binom{n+1}{2} - \frac{x^2 - 1}{4}, \quad t_n = \frac{n^4(n^2 - x^2)}{4(2n+1)(2n-1)}.$$

We can now state and prove the following result.

Lemma 6.2. *Consider the sequence $c_k = B_{2k+1}(\frac{1+x}{2})$ and let $d_n(x)$ be defined as in (6.1), which depends on x due to (6.3). Then for all $n \geq 2$ we have*

$$(6.4) \quad \lim_{x \rightarrow -1} \frac{d_n(x)}{x^2 - 1} = \frac{(-1)^n 2^{n-2}}{3^n} \prod_{\ell=2}^n \left(\frac{(\ell+1)^2(2\ell-1)}{\ell(\ell-1)(2\ell+1)} \right)^{n+1-\ell}.$$

Proof. Using elementary determinant operations, we get from (6.1) the recurrence relation

$$(6.5) \quad d_{n+1}(x) = -s_{n+1}d_n(x) - t_{n+1}d_{n-1}(x).$$

We prove (6.4) by induction on n . By direct computation, using (6.3) and (6.1), we obtain

$$\begin{aligned} \lim_{x \rightarrow -1} \frac{d_2(x)}{x^2 - 1} &= \frac{3}{10} = \frac{(-1)^2 2^{2-2}}{3^2} \cdot \frac{(2+1)^2(4-1)}{2(2-1)(4+1)}, \\ \lim_{x \rightarrow -1} \frac{d_3(x)}{x^2 - 1} &= -\frac{36}{35} = \frac{(-1)^3 2^{3-2}}{3^3} \left(\frac{(2+1)^2(4-1)}{2(2-1)(4+1)} \right)^2 \frac{(3+1)^2(6-1)}{3(3-1)(6+1)}, \end{aligned}$$

which is the induction beginning. Suppose now that (6.4) is true up to some n . Then we divide both sides of (6.5) by $x^2 - 1$ and use the induction hypothesis (6.4) along with (6.3). After some straightforward but tedious manipulations we obtain an expression for $\lim_{x \rightarrow -1} d_{n+1}(x)/(x^2 - 1)$, which is the same as the right-hand side of (6.4), but with n replaced by $n + 1$. This completes the proof by induction. \square

We are now ready to prove the desired fourth consequence of Theorem 5.1.

Corollary 6.3. *For all $n \geq 0$ we have*

$$(6.6) \quad H_n((2k+3)B_{2k+2}) = \frac{1}{2^{n+1}} \prod_{\ell=1}^n \left(\frac{\ell^3(\ell+1)^3}{4(2\ell+1)^2} \right)^{n+1-\ell}.$$

Proof. For $n = 0$ and 1 , the identity (6.6) is easy to verify by direct calculation; we may therefore assume that $n \geq 2$. We set $A_k(x) := B_{2k+3}(\frac{1+x}{2})$ and $x_0 = -1$. Then we have $A_k(-1) = B_{2k+3}(0) = 0$ for all $k \geq 0$, and also

$$A'_k(x) = \frac{2k+3}{2} B_{2k+2}(\frac{1+x}{2}), \quad A'_k(-1) = \frac{2k+3}{2} B_{2k+2},$$

as well as (see Table 1)

$$A'_0(-1) = \frac{3}{2} B_2 = \frac{1}{4}, \quad A_0(x) = B_3(\frac{1+x}{2}) = \frac{x(x^2-1)}{8}.$$

With (2.14) and (5.2) we therefore get

$$(6.7) \quad \begin{aligned} H_n((2k+3)B_{2k+2}) &= H_n(2A'_k(-1)) = 2^{n+1} H_n(A'_k(-1)) \\ &= 2^{n+1} \left(\frac{1}{4} \right)^{n+1} \lim_{x \rightarrow -1} \frac{H_n(B_{2k+3}(\frac{1+x}{2}))}{\left(\frac{x(x^2-1)}{8} \right)^{n+1}}. \end{aligned}$$

Now by Lemma 6.1 we have

$$(6.8) \quad \frac{H_n(B_{2k+3}(\frac{1+x}{2}))}{\left(\frac{x(x^2-1)}{8} \right)^{n+1}} = 4 \cdot \frac{H_n(B_{2k+1}(\frac{1+x}{2}))}{\left(\frac{x}{2} \right)^{n+1} \left(\frac{x^2-1}{4} \right)^n} \cdot \frac{d_n(x)}{x^2-1}.$$

Next, from (2.15) we get

$$\lim_{x \rightarrow -1} \frac{H_n(B_{2k+1}(\frac{1+x}{2}))}{\left(\frac{x}{2} \right)^{n+1} \left(\frac{x^2-1}{4} \right)^n} = (-1)^{\binom{n+1}{2}} \frac{1}{3^n} \prod_{\ell=2}^n \left(\frac{\ell^4(1-\ell^2)}{4(2\ell+1)(2\ell-1)} \right)^{n+1-\ell}.$$

Finally, substituting this and (6.4) into (6.8), and then (6.8) into (6.7), we obtain the desired identity (6.4) after some easy manipulations. \square

7. A COLLECTION OF HANKEL DETERMINANT FORMULAS

As indicated in the Introduction, Hankel determinants of Bernoulli and Euler numbers have been studied for many years, and numerous identities were derived by different authors, often with differing notations and in different but equivalent forms. In this section we attempt to collect all the identities we are aware of and present them, as far as possible, in a unified format.

First we recall that in writing $H_n(b_k)$ it is assumed that we use the definition (1.4) and (1.3) and that the sequence (b_k) begins with $k = 0$. Next, there is the issue of the close connection between Bernoulli and Euler polynomials, given by identities such as (2.2). Thus we have, for instance,

$$(7.1) \quad B_{2k}(\frac{1}{2}) = (2^{1-2k} - 1) B_{2k},$$

$$(7.2) \quad (2^{2k+2} - 1) B_{2k+2} = (k+1) E_{2k+1}(1),$$

$$(7.3) \quad E_k(1) = \frac{2}{k+1} (2^{k+1} - 1) B_{k+1} \quad (k \geq 1),$$

$$(7.4) \quad (2k+1) E_{2k} = 2^{4k+2} B_{2k+1}(\frac{3}{4}),$$

where (7.1) can be found in [17, Eq. 24.4.27], and (7.2)–(7.4) come from (2.3). The left-hand sides of (7.1)–(7.4) are included in the tables below since they could be considered somewhat simpler than the right-hand sides. In the case of (7.2) it is not clear which side could be considered “simpler”, and in fact, we have also included the right-hand side (multiplied by 2).

Finally, we need to be aware of the fact that the products that occur in all identities for Hankel determinants can usually be written in at least two different forms. One could argue, for instance, that the identities in the statements of Corollaries 5.2–5.4 are simpler and therefore preferable to the ones at the end of the corresponding proofs. However, due to the close connection between Hankel determinants and orthogonal polynomials, especially as given by (5.11), it makes sense to use the latter forms as standard format. The following identities may serve to easily pass from one form to the other:

$$(7.5) \quad \prod_{\ell=1}^n \ell! = \prod_{\ell=1}^n \ell^{n+1-\ell};$$

$$(7.6) \quad \prod_{\ell=1}^n (2\ell + \nu)! = \nu!^n \prod_{\ell=1}^n ((2\ell - 1 + \nu)(2\ell + \nu))^{n+1-\ell}, \quad \nu = 0, 1, 2;$$

$$(7.7) \quad \prod_{\ell=1}^n (4\ell + \nu)! = \nu!^n \prod_{\ell=1}^n ((4\ell - 3 + \nu) \cdots (4\ell + \nu))^{n+1-\ell}, \quad \nu = 0, 1, 2, 3.$$

These identities, which actually hold in greater generality, can be verified without much difficulty.

We are now ready to list the identities for Hankel determinants, mostly given in a standard format and organized in a couple of tables. The references provided are not necessarily the first occurrences in the literature.

7.1. Identities with nonzero terms for all n . Most identities have nonzero Hankel determinants for all positive integers n ; we present them in the format

$$(7.8) \quad H_n(b_k) = (-1)^{\varepsilon(n)} \cdot a^{n+1} \cdot \prod_{\ell=1}^n b(\ell)^{n+1-\ell}.$$

Here, the column for $\varepsilon(n)$ could be eliminated by incorporating a $-$ sign in a or $b(\ell)$, as appropriate. However, we decided to make the sign pattern more explicit.

b_k	$\varepsilon(n)$	a	$b(\ell)$	Reference
B_k	$\binom{n+1}{2}$	1	$\frac{\ell^4}{4(2\ell+1)(2\ell-1)}$	[13, (3.56)]
B_{k+1}	$\binom{n+2}{2}$	$\frac{1}{2}$	$\frac{\ell^2(\ell+1)^2}{4(2\ell+1)^2}$	[13, (3.57)]
B_{k+2}	$\binom{n+1}{2}$	$\frac{1}{6}$	$\frac{\ell(\ell+1)^2(\ell+2)}{4(2\ell+1)(2\ell+3)}$	[13, (2.38)]
B_{2k+2}	0	$\frac{1}{6}$	$\frac{\ell^3(\ell+1)(2\ell-1)(2\ell+1)^3}{(4\ell-1)(4\ell+1)^2(4\ell+3)}$	[13, (3.59)]

B_{2k+4}	$n + 1$	$\frac{1}{30}$	$\frac{\ell(\ell+1)^3(2\ell+1)^3(2\ell+3)}{(4\ell+1)(4\ell+3)^2(4\ell+5)}$	[13, (3.60)]
$B_{2k}(\frac{1}{2})$	0	1	$\frac{\ell^4(2\ell-1)^4}{(4\ell-3)(4\ell-1)^2(4\ell+1)}$	[4, (41)]
$(2^{2k+2} - 1)B_{2k+2}$	0	$\frac{1}{2}$	$\ell^3(\ell+1)$	Cor. 5.3
$(2k+1)B_{2k}(\frac{1}{2})$	0	1	$\frac{\ell^6}{4(2\ell+1)(2\ell-1)}$	Cor. 5.4
$(2k+3)B_{2k+2}$	0	$\frac{1}{2}$	$\frac{\ell^3(\ell+1)^3}{4(2\ell+1)^2}$	Cor. 6.3
$B_{2k+1}(\frac{x+1}{2})$	$\binom{n+1}{2}$	$\frac{x}{2}$	$\frac{\ell^4(x^2 - \ell^2)}{4(2\ell+1)(2\ell-1)}$	[6, Thm. 1.1]
E_k	$\binom{n+1}{2}$	1	ℓ^2	[1, (4.2)]
$E_k(x)$	$\binom{n+1}{2}$	1	$\frac{\ell^2}{4}$	[1, (5.2)]
$E_{k+1}(1)$	$\binom{n+1}{2}$	$\frac{1}{2}$	$\frac{\ell(\ell+1)}{4}$	[9, (H4)]
E_{2k}	0	1	$(2\ell-1)^2(2\ell)^2$	[13, (3.52)]
$E_{2k+1}(1)$	0	$\frac{1}{2}$	$\frac{\ell^2(2\ell-1)(2\ell+1)}{4}$	[15, (4.56)]
E_{2k+2}	$n + 1$	1	$(2\ell)^2(2\ell+1)^2$	[13, (3.53)]
$E_{2k+3}(1)$	$n + 1$	$\frac{1}{4}$	$\frac{\ell(\ell+1)(2\ell+1)^2}{4}$	[15, (4.57)]
$(2k+1)E_{2k}$	0	1	$(2\ell)^4$	Cor. 5.2
$(2k+2)E_{2k+1}(1)$	0	1	$\ell^3(\ell+1)$	Cor. 5.3
$\frac{E_{k+1}(1)}{(k+1)!}$	$\binom{n+1}{2}$	$\frac{1}{2}$	$\frac{1}{4(2\ell-1)(2\ell+1)}$	[9, (H12)]
$\frac{E_{2k+1}(1)}{(2k+1)!}$	0	$\frac{1}{2}$	$\frac{1}{16(4\ell-3)(4\ell-1)^2(4\ell+1)}$	[9, (H13)]
$\frac{E_{2k+3}(1)}{(2k+3)!}$	$n + 1$	$\frac{1}{24}$	$\frac{1}{16(4\ell-1)(4\ell+1)^2(4\ell+3)}$	[9, (H22)]

$E_{2k}(\frac{x+1}{2})$	$\binom{n+1}{2}$	1	$\frac{\ell^2}{4}(x^2 - (2\ell - 1)^2)$	[6, Cor. 5.2]
$E_{2k+1}(\frac{x+1}{2})$	$\binom{n+1}{2}$	$\frac{x}{2}$	$\frac{\ell^2}{4}(x^2 - (2\ell)^2)$	[6, Cor. 5.2]
$E_{2k+2}(\frac{x+1}{2})$	$\binom{n+1}{2}$	$\frac{x^2-1}{4}$	$\frac{\ell^2}{4}(x^2 - (2\ell + 1)^2)$	[6, Cor. 5.2]

7.2. Identities with zero terms for all even n . A second class of identities have zero Hankel determinants for all positive even integers, that is,

$$(7.9) \quad H_{2m}(b_k) = 0;$$

we then present the nonzero terms in the format

$$(7.10) \quad H_{2m+1}(b_k) = (-1)^{m+1} \cdot a^{2(m+1)} \cdot \prod_{\ell=1}^m b(\ell)^{2(m+1-\ell)}.$$

b_k	a	$b(\ell)$	Reference
E_{k+1}	1	$(2\ell)^2(2\ell + 1)^2$	[9, (H8)]
$E_{k+2}(1)$	$\frac{1}{4}$	$\frac{\ell(\ell + 1)(2\ell + 1)^2}{4}$	[9, (H11)] ¹
$(0, E_1(1), E_2(1), \dots)$	$\frac{1}{2}$	$\frac{\ell^2(2\ell - 1)(2\ell + 1)}{4}$	[9, (H9)]
$kE_{k-1}(x)$	1	ℓ^4	(3.8)
kE_{k-1}	1	$(2\ell)^4$	(3.9)
$\left(0, \frac{E_1(1)}{1!}, \frac{E_2(1)}{2!}, \dots\right)$	$\frac{1}{2}$	$\frac{1}{16(4\ell - 3)(4\ell - 1)^2(4\ell + 1)}$	[9, (H15)]
$\frac{E_{k+2}(1)}{(k+2)!}$	$\frac{1}{24}$	$\frac{1}{16(4\ell - 1)(4\ell + 1)^2(4\ell + 3)}$	[9, (H14)]
$B_k(\frac{x+r}{q}) - B_k(\frac{x+s}{q})$	$\frac{s-r}{q^{m+1}}$	$\frac{\ell^4((s-r)^2 - (q\ell)^2)}{4(2\ell - 1)(2\ell + 1)}$	(3.5)
$E_k(\frac{x+r}{q}) - E_k(\frac{x+s}{q})$	$\frac{s-r}{q}$	$\frac{\ell^2((s-r)^2 - (2q\ell)^2)}{4q^2}$	(4.4)

¹it appears that the author intended the determinant to be H_{2n+1} .

7.3. Miscellaneous identities. We now collect a number of identities that do not fit into Subsections 7.1 or 7.2. We begin with a few that are, however, closely related to some identities in the two tables above. The first of these identities was adapted from Andrews and Wimp [2, p. 441]:

$$(7.11) \quad H_n \left(\frac{B_k}{k!} \right) = (-1)^{\binom{n+1}{2}} (n+1) \prod_{\ell=1}^n \left(\frac{1}{4(2\ell-1)(2\ell+1)} \right)^{n+1-\ell}.$$

The next three identities are due to Krattenthaler and were published in [7, p. 346].

$$(7.12) \quad H_n \left(\frac{B_{2k+2}}{(2k+2)!} \right) = \left(\frac{1}{4} \right)^{(n+1)^2} \prod_{\ell=1}^{2n+1} \left(\frac{1}{2\ell+1} \right)^{2n+2-\ell},$$

$$(7.13) \quad H_n \left(\frac{B_{2k+4}}{(2k+4)!} \right) = \left(\frac{-1}{36} \right)^{n+1} \left(\frac{1}{4} \right)^{(n+1)^2} \prod_{\ell=1}^{2n+1} \left(\frac{1}{2\ell+3} \right)^{2n+2-\ell},$$

$$(7.14) \quad H_n \left(\frac{B_{2k+6}}{(2k+6)!} \right) = \frac{(n+2)(2n+5)}{3 \cdot 60^{2n+2}} \left(\frac{1}{4} \right)^{(n+1)^2} \prod_{\ell=1}^{2n+1} \left(\frac{1}{2\ell+5} \right)^{2n+2-\ell}.$$

The identity (7.14) was slightly changed from its original form. The following three identities were adapted from (H21), (H23), and (H24), respectively, in [9].

$$(7.15) \quad H_n \left(\frac{E_{k+3}(1)}{(k+3)!} \right) = (-1)^{\binom{n+2}{2}} \left(\frac{1}{24} \right)^{n+1} \prod_{\ell=1}^n \left(\frac{1}{4(2\ell+1)(2\ell+3)} \right)^{n+1-\ell} \\ \times \begin{cases} \binom{n+3}{2}, & (n \text{ odd}), \\ \binom{n+2}{2}, & (n \text{ even}); \end{cases}$$

$$(7.16) \quad H_n \left(\frac{E_{2k+5}(1)}{(2k+5)!} \right) = \left(\frac{1}{2 \cdot 6!} \right)^{n+1} \binom{2n+4}{2} \\ \times \prod_{\ell=1}^n \left(\frac{1}{16(4\ell+1)(4\ell+3)^2(4\ell+5)} \right)^{n+1-\ell};$$

$$(7.17) \quad H_n \left(\frac{E_{2k+7}(1)}{(2k+7)!} \right) = \left(\frac{-1}{5 \cdot 8!} \right)^{n+1} \frac{4n^2 + 18n + 17}{3} \binom{2n+6}{4} \\ \times \prod_{\ell=1}^n \left(\frac{1}{16(4\ell+3)(4\ell+5)^2(4\ell+7)} \right)^{n+1-\ell}.$$

Another identity of a similar nature is

$$(7.18) \quad H_n(B_{k+2}(-1)) = (-1)^{\binom{n+1}{2}} \left(\frac{1}{6} \right)^{n+1} ((n+1)(n+2)^2(n+3)+1) \\ \times \prod_{\ell=1}^n \left(\frac{\ell(\ell+1)^2(\ell+2)}{4(2\ell+1)(2\ell+3)} \right)^{n+1-\ell},$$

which can be found in [8, Eq. (7.2)], in a slightly different form. In this connection Fulmek and Krattenthaler also showed that

$$H_n(B_k - 2B_{k+1} + B_{k+2}) = H_n(B_{k+2}(-1)) \quad (n \geq 0).$$

Furthermore, they derived identities for

$$H_{2m}(B_{k+2}(-\frac{1}{2})) \quad \text{and} \quad H_{2m+1}(B_{k+2}(-\frac{1}{2})),$$

involving ${}_4F_3$ hypergeometric functions [8, Eq. (7.3), (7.4)] and identities for

$$H_{2m}\left(B_{k+2}\left(\frac{1}{2}\right)\right) \quad \text{and} \quad H_{2m+1}\left(B_{k+2}\left(\frac{1}{2}\right)\right),$$

which involve certain finite sums [8, Eq. (7.5), (7.6)]. At this point we also mention the identities for

$$H_{2m}\left(E_k\left(\frac{x+r}{q}\right) + E_k\left(\frac{x+s}{q}\right)\right) \quad \text{and} \quad H_{2m+1}\left(E_k\left(\frac{x+r}{q}\right) + E_k\left(\frac{x+s}{q}\right)\right),$$

which were obtained in (4.5) and (4.6) above.

We conclude this list of identities with a very general formula, which is also due to Fulmek and Krattenthaler [8, Eq. (5.3)]. Here a well-known symbolic notation is used, where after expansion each power \mathcal{B}^j is replaced by the Bernoulli number B_j . Also, the shifted factorial $(a)_j$ is defined by $(a)_j := a(a+1) \cdots (a+j-1)$ for $j \geq 1$, and $(a)_0 = 1$. Thus, for example, we have

$$\mathcal{B}^3(\mathcal{B}+1)_2 = \mathcal{B}^3(\mathcal{B}+1)(\mathcal{B}+2) = \mathcal{B}^5 + 3\mathcal{B}^4 + 2\mathcal{B}^3 \equiv B_5 + 3B_4 + 2B_3.$$

We can now state the identity in question, which is, in fact, again of the form (7.8):

For integers $a, b \geq 1$ and $c, d \geq 0$, we have

$$(7.19) \quad \begin{aligned} & H_n(\mathcal{B}^{k+2}(\mathcal{B}+1)_{a-1}(\mathcal{B}+1)_{b-1}(-\mathcal{B}+1)_{c-1}(-\mathcal{B}+1)_{d-1}) \\ &= (-1)^{\binom{n+1}{2}} \left(\frac{(a+c-1)!(b+c-1)!(a+d-1)!(b+d-1)!}{(a+b+c+d-1)!} \right)^{n+1} \\ & \times \prod_{\ell=1}^n \left(\frac{\ell(a+c+\ell-1)(b+c+\ell-1)(a+d+\ell-1)}{(a+b+c+d+2\ell-3)(a+b+c+d+2\ell-2)^2} \right. \\ & \quad \left. \times \frac{(b+d+\ell-1)(a+b+c+d+\ell-2)}{(a+b+c+d+2\ell-1)} \right)^{n+1-\ell}, \end{aligned}$$

where in the case $c = 0$ or $d = 0$ we interpret $(-\mathcal{B}+1)_{-1}$ as $1/(-\mathcal{B})$.

As mentioned in [8, p. 626], the cases $a = b = 1$, $c = d = 0$ and $a = b = c = d = 1$ give the first and third entries, respectively, in the table in Subsection 7.1. Similarly, $a = b = c = 1$, $d = 0$, would give the second entry in this table.

7.4. Other related sequences. We made the conscious decision to restrict ourselves to Bernoulli and Euler numbers and polynomials in Subsections 7.1–7.3. We conclude this section with a few remarks on related sequences.

1. While number theorists and researchers in special functions tend to favor the definition (1.2) for Euler numbers, combinatorists typically prefer the alternative sequence \mathbf{E}_n defined by

$$(7.20) \quad \tan t + \sec t = \sum_{n=0}^{\infty} \mathbf{E}_n \frac{t^n}{n!},$$

where we use a different font to avoid possible confusion. The first few terms, starting with \mathbf{E}_0 , are 1, 1, 1, 2, 5, 16, 61, 272, 1385; they are all positive integers. By comparing the generating function (7.20) with (1.2), it is not difficult to see that, for all $k \geq 0$,

$$(7.21) \quad \mathbf{E}_{2k} = (-1)^k E_{2k},$$

$$(7.22) \quad \mathbf{E}_{2k+1} = (-1)^k 2^{2k+1} E_{2k+1}(1).$$

Using these identities and (2.14), any Hankel determinant identity for the sequences on the right immediately give identities for the ones on the left, and vice versa; in

fact, this is how we imported the numerous identities from Han's recent paper [9]. There are eight more identities in [9] for the "mixed" sequence (\mathbf{E}_k) , namely for

$$H_n(\mathbf{E}_{k+\mu}), \mu = 0, 1, 2, \quad \text{and} \quad H_n(\mathbf{E}_{k+\nu}/(k+\nu)!), \nu = 0, 1, 2, 3, 4.$$

2. By (7.20) it is clear that the numbers \mathbf{E}_{2k-1} , $k \geq 1$, are the same as the *tangent numbers* (or *tangent coefficients*) T_k , which are also known to have the form

$$T_k = (-1)^{k-1} 2^k \frac{2^{2k} - 1}{2k} B_{2k}$$

(see, e.g., [17, Eq. 4.19.3]). By (7.22) we have $T_k = (-1)^{k-1} 2^{2k-1} E_{2k-1}(1)$; therefore all identities for $E_k(1)$ can also be seen as identities for tangent numbers.

3. Euler numbers of an integer order $p \geq 1$ are defined by a generating function that is the p th power of the left-hand identity in (1.2). Already Al-Salam and Carlitz [1] found the Hankel determinants of the sequence of these higher-order Euler numbers. More recently, Han [9] dealt with other related sequences of higher-order Euler numbers, and the second author and Shi [11] determined the orthogonal polynomials of higher-order Euler *polynomials*, which also led to relevant Hankel determinants. Higher-order *Bernoulli* numbers, however, are more challenging; see the remarks in [11, p. 401].

4. Among other generalizations of Bernoulli and Euler numbers for which Hankel determinants have been computed are the q -Bernoulli-Carlitz numbers [3], the median Bernoulli numbers [4], and some character analogues (Corollaries 3.5 and 4.4 above). We refer the interested reader to three very extensive studies by Krattenthaler [13, 14] and Milne [15] for numerous other Hankel determinant evaluations. Extensive references to the vast literature are provided in [13, pp. 47–48], [14, p. 122], and [15, pp. 54–57].

REFERENCES

- [1] W. A. Al-Salam and L. Carlitz, Some determinants of Bernoulli, Euler and related numbers, *Portug. Math.* **18** (1959), 91–99.
- [2] G. Andrews and J. Wimp, Some q -orthogonal polynomials and related Hankel determinants, *Rocky Mountain J. Math.* **32** (2002), 429–442.
- [3] F. Chapoton and J. Zeng, Nombres de q -Bernoulli-Carlitz et fractions continues, *J. Théor. Nombres Bordeaux* **29** (2017), 347–368.
- [4] K.-W. Chen, A summation on Bernoulli numbers, *J. Number Theory* **111** (2005), 372–391.
- [5] J. Cigler and C. Krattenthaler, Some determinants of path generating functions, *Adv. in Appl. Math.* **46** (2011), 144–174.
- [6] K. Dilcher and L. Jiu, Orthogonal polynomials and Hankel determinants for certain Bernoulli and Euler Polynomials. *Preprint*, 2020. Available at <http://arxiv.org/abs/2006.15236>.
- [7] S. Fukuhara and Y. Yang, Period polynomials and explicit formulas for Hecke operators on $\Gamma_0(2)$, *Math. Proc. Cambridge Philos. Soc.* **146** (2009), 321–350.
- [8] M. Fulmek and C. Krattenthaler, The number of rhombus tilings of a symmetric hexagon which contain a fixed rhombus on the symmetry axis, II, *European J. Combin.* **21** (2000), 601–640.
- [9] G. N. Han, Hankel continued fractions and Hankel determinants of the Euler numbers, *Trans. Amer. Math. Soc.* **373** (2020), 4255–4283.
- [10] M. E. H. Ismail, *Classical and quantum orthogonal polynomials in one variable. With two chapters by Walter Van Assche*. Encyclopedia of Mathematics and its Applications, 98. Cambridge University Press, Cambridge, 2005.
- [11] L. Jiu and D. Y. Shi, Orthogonal polynomials and connection to generalized Motzkin numbers for higher-order Euler polynomials, *J. Number Theory* **199** (2019), 389–402.
- [12] A. Junod, Hankel determinants and orthogonal polynomials, *Expo. Math.* **21** (2003), 63–74.

- [13] C. Krattenthaler, Advanced determinant calculus, *Séminaire Lotharingien Combin.* **42** (“The Andrews Festschrift”) (1999), Article B42q.
- [14] C. Krattenthaler, Advanced determinant calculus: a complement, *Linear Algebra Appl.* **411** (2005), 68–166.
- [15] S. C. Milne, Infinite families of exact sums of squares formulas, Jacobi elliptic functions, continued fractions, and Schur functions, *Ramanujan J.* **6** (2002), 7–149.
- [16] L. Mu, Y. Wang, and Y. Yeh, Hankel determinants of linear combinations of consecutive Catalan-like numbers, *Discrete Math.* **340** (2017), 3097–3103.
- [17] F. W. J. Olver et al. (eds.), *NIST Handbook of Mathematical Functions*, Cambridge Univ. Press, New York, 2010. Online version: <http://dlmf.nist.gov>.

DEPARTMENT OF MATHEMATICS AND STATISTICS, DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, B3H 4R2, CANADA

E-mail address: dilcher@mathstat.dal.ca

DEPARTMENT OF MATHEMATICS AND STATISTICS, DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, B3H 4R2, CANADA

E-mail address: lin.jiu@dal.ca