

# Reconfiguring dominating sets in minor-closed graph classes

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## Abstract

For a graph  $G$ , two dominating sets  $D$  and  $D'$  in  $G$ , and a non-negative integer  $k$ , the set  $D$  is said to  *$k$ -transform* to  $D'$  if there is a sequence  $D_0, \dots, D_\ell$  of dominating sets in  $G$  such that  $D = D_0$ ,  $D' = D_\ell$ ,  $|D_i| \leq k$  for every  $i \in \{0, 1, \dots, \ell\}$ , and  $D_i$  arises from  $D_{i-1}$  by adding or removing one vertex for every  $i \in \{1, \dots, \ell\}$ . We prove that there is some positive constant  $c$  and there are toroidal graphs  $G$  of arbitrarily large order  $n$ , and two minimum dominating sets  $D$  and  $D'$  in  $G$  such that  $D$   $k$ -transforms to  $D'$  only if  $k \geq \max\{|D|, |D'|\} + c\sqrt{n}$ . Conversely, for every hereditary class  $\mathcal{G}$  that has balanced separators of order  $n \mapsto n^\alpha$  for some  $\alpha < 1$ , we prove that there is some positive constant  $C$  such that, if  $G$  is a graph in  $\mathcal{G}$  of order  $n$ , and  $D$  and  $D'$  are two dominating sets in  $G$ , then  $D$   $k$ -transforms to  $D'$  for  $k = \max\{|D|, |D'|\} + \lfloor Cn^\alpha \rfloor$ .

**Keywords:** dominating set; reconfiguration; toroidal graph; minor-closed graph class

**MSC 2020 classification:** 05C69

## 1 Introduction

We consider finite, simple, and undirected graphs, and use standard notation and terminology. Let  $G$  be a graph. A set  $D$  of vertices of  $G$  is a *dominating set* in  $G$  if every vertex of  $G$  belongs to  $D$  or has a neighbor in  $D$ . Let  $D$  and  $D'$  be two dominating sets in  $G$ . The dominating sets  $D$  and  $D'$  are *adjacent* if  $|D \setminus D'| + |D' \setminus D| = 1$ , that is, if  $D'$  arises from  $D$  by adding or removing one vertex. Let  $k$  be a positive integer. We say that  $D$   *$k$ -transforms* to  $D'$ , and write  $D \xleftarrow{k} D'$ , if there is a sequence  $D_0, \dots, D_\ell$  of dominating sets in  $G$  such that  $D = D_0$ ,  $D' = D_\ell$ ,  $|D_i| \leq k$  for every  $i \in \{0, 1, \dots, \ell\}$ , and  $D_{i-1}$  is adjacent to  $D_i$  for every  $i \in \{1, \dots, \ell\}$ . Let  $D_k(G)$  be the graph whose vertices are the dominating sets in  $G$  that are of order at most  $k$ , and whose edges are defined by the above adjacency between dominating sets.

The structure and, in particular, reachability, connectivity, and distance problems in  $D_k(G)$  have been studied in [1, 5, 10] and — with a focus on algorithmic and complexity results — in [6, 11]. In [9] a general survey on reconfiguration problems for several types of sets of vertices in a graph is given. If  $\gamma(G)$  and  $\Gamma(G)$  denote the minimum and maximum order of dominating sets in  $G$  that are minimal with respect to inclusion, respectively, then it is easy to see that  $D_{\Gamma(G)+\gamma(G)}(G)$  is always connected. Answering a question of Haas and Seyffarth [5], Suzuki et al. [10] construct planar graphs  $G$  for which  $D_{\Gamma(G)+1}(G)$  is disconnected. As they point out, it is unknown whether  $D_{\Gamma(G)+2}(G)$  is connected for every graph  $G$ .

Inspired by the cited research and this open problem, we take a slightly different point of view, considering — roughly speaking — how many additional vertices must be allowed in order to transform one given dominating set to another given one. More precisely, if  $D$  and  $D'$  are dominating sets in a

graph  $G$ , then let  $\partial\gamma_G(D, D')$  equal  $k - \max\{|D|, |D'|\}$ , where  $k$  is the smallest positive integer with  $D \xleftarrow{k} D'$ . In particular,

$$D \xleftarrow{\max\{|D|, |D'|\} + \partial\gamma_G(D, D')} D',$$

that is, allowing  $\partial\gamma_G(D, D')$  more vertices than contained in the larger of the two dominating sets, one can transform  $D$  to  $D'$ . It is easy to see that

$$\partial\gamma_G(D, D') \leq \min \left\{ \gamma(G), \frac{n(G)}{2} \right\}$$

for every graph  $G$  of order  $n(G)$ . It seems an interesting problem to determine the best possible upper bound on  $\partial\gamma_G(D, D')$  in terms of the order  $n(G)$ ; for general graphs as well as for graphs from restricted graph classes. Known results [6] imply, for instance, that  $\partial\gamma_G(D, D') \leq 1$  whenever  $G$  is a cograph, a forest, or an interval graph. The results of the present paper were obtained wondering whether  $\partial\gamma_G(D, D')$  can be upper bounded in terms of the maximum degree  $\Delta(G)$  of  $G$ .

Our first result shows that this is not possible.

**Theorem 1.** *There is a positive constant  $c$  such that, for every positive integer  $n$ , there is a 4-regular graph  $G$  of order at least  $n$  that can be embedded on the torus, and there are two dominating sets  $D$  and  $D'$  of  $G$ , both of order  $n(G)/5$ , such that*

$$\partial\gamma_G(D, D') \geq c\sqrt{n(G)}.$$

Our second result shows that the lower bound in Theorem 1 has the right order of magnitude. Rather than considering only graphs embedded on the torus, we consider graphs with sublinear balanced separators. In order to phrase our second result, we need some more definitions: A graph  $G$  has a *balanced separator of order  $k$*  if there is a set  $S$  of at most  $k$  vertices of  $G$  as well as a partition of the vertex set  $V(G)$  of  $G$  into three sets  $S$ ,  $A$ , and  $B$  such that  $|A|, |B| \leq 2n(G)/3$ , and  $G$  contains no edge between  $A$  and  $B$ . A hereditary class  $\mathcal{G}$  of graphs has *balanced separators of order  $n \mapsto n^\alpha$*  if there is some positive constant  $c$  such that every graph  $G$  in  $\mathcal{G}$  has a balanced separator of order  $cn(G)^\alpha$ . It is known that minor-closed graph classes [7] such as planar graphs [8], toroidal graphs, and graphs of bounded genus [3] have balanced separators of order  $n \mapsto \sqrt{n}$ .

Here is our second main result.

**Theorem 2.** *Let  $\mathcal{G}$  be a hereditary class of graphs that has balanced separators of order  $n \mapsto n^\alpha$  for some  $\alpha < 1$ . There is some positive constant  $C$  such that, if  $G$  is a graph in  $\mathcal{G}$ , and  $D$  and  $D'$  are two dominating sets in  $G$ , then*

$$\partial\gamma_G(D, D') \leq Cn(G)^\alpha.$$

The proofs of our two results are given in the following two sections.

## 2 Proof of Theorem 1

The 4-regular graph  $G$  that we construct for this result arises by applying suitable vertex identifications to a sufficiently large subgraph of the infinite grid graph  $\mathbb{Z}^2$  illustrated in Figure 1. In this figure we also illustrate two dominating sets  $D_\square$  and  $D_\circlearrowleft$  of  $\mathbb{Z}^2$ , one indicated by squares  $\square$  and the second indicated by circles  $\circlearrowleft$ . The graph  $G$  will be constructed in such a way that  $D = V(G) \cap D_\square$  and

$D' = V(G) \cap D_{\circlearrowleft}$  are minimum dominating sets of  $G$ . In fact, since  $G$  is 4-regular, every dominating set in  $G$  contains at least  $|D| = |D'| = n(G)/5$  vertices.

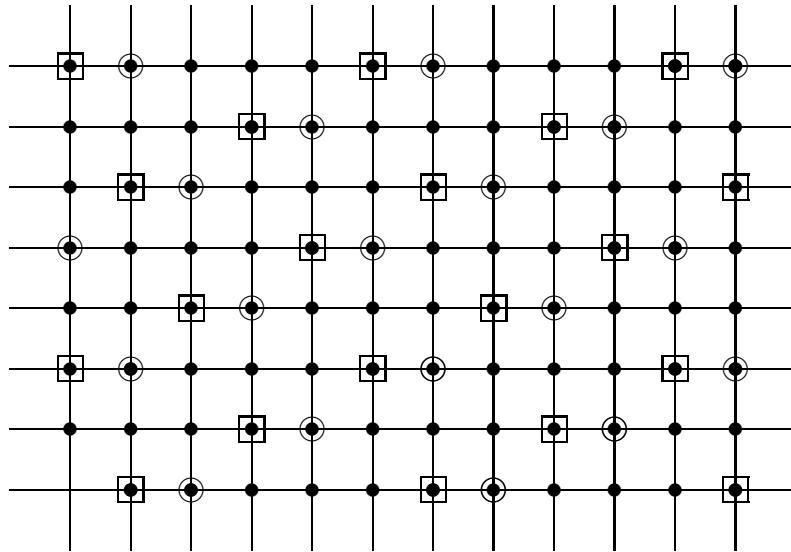


Figure 1: A section of the infinite grid graph  $\mathbb{Z}^2$  and two dominating sets  $D_{\square}$  and  $D_{\circlearrowleft}$ .

The reason why we consider graphs embedded on the torus rather than the more conventional grid graphs  $P_k^2$  is to avoid boundary effects. It is easy to see that  $\gamma(P_k^2) = k^2/5 + \Omega(k)$  [4], that is, the domination number  $\gamma(P_k^2)$  of  $P_k^2$  deviates by a term of the order  $\Omega(k) = \Omega(\sqrt{n(P_k^2)})$  from the lower bound  $n(P_k^2)/(\Delta(P_k^2) + 1)$ , and this disturbing deviation is of the same order of magnitude as our lower bound on  $\partial\gamma_G(D, D')$ .

The elements of  $D_{\square}$  and  $D_{\circlearrowleft}$  come in adjacent pairs  $p = \{u_{\square}, u_{\circlearrowleft}\}$  with  $u_{\square} \in D_{\square}$  and  $u_{\circlearrowleft} \in D_{\circlearrowleft}$ . We consider an infinite auxiliary graph  $H_{\infty}$  whose vertices are these pairs, indicated by diamonds  $\diamond$  in Figure 2, and in which two distinct pairs  $p = \{u_{\square}, u_{\circlearrowleft}\}$  and  $p' = \{u'_{\square}, u'_{\circlearrowleft}\}$  are adjacent if

$$\left( N_{\mathbb{Z}^2}[u_{\square}] \cup N_{\mathbb{Z}^2}[u_{\circlearrowleft}] \right) \cap \left( N_{\mathbb{Z}^2}[u'_{\square}] \cup N_{\mathbb{Z}^2}[u'_{\circlearrowleft}] \right) \neq \emptyset.$$

See Figure 2 for an illustration of  $H_{\infty}$ .

$H_{\infty}$  is a tilted grid graph sitting within  $\mathbb{Z}^2$  at an angle of  $\arcsin\left(\frac{1}{\sqrt{5}}\right) \approx 26,56^\circ$ . The finite subgraph of  $H_{\infty}$  corresponding to the pairs  $p = \{u_{\square}, u_{\circlearrowleft}\}$  where  $u_{\square}$  and  $u_{\circlearrowleft}$  both belong to  $G$  will be denoted by  $H$ . At some point we want to apply to  $H$  an isoperimetric inequality for the discrete torus  $\mathbb{Z}_k^2$  with even  $k$  that was shown by Bollobás and Leader [2]. Therefore, our construction of  $G$ , and hence of  $H$ , ensures that  $H$  is isomorphic to  $\mathbb{Z}_k^2$  for some sufficiently large integer  $k$  that is a multiple of 4. More precisely, in order to construct  $G$  and  $H$ , we select, for some sufficiently large integer  $k \equiv 0 \pmod{4}$ , a  $(k+1) \times (k+1)$  grid subgraph of  $H_{\infty}$ , as illustrated in Figure 3 for  $k=8$ , and identify

- the left border with the right border both from top to bottom, and
- the top border with the bottom border both from left to right.

This yields a graph  $H$  isomorphic to  $\mathbb{Z}_k^2$  as well as a graph  $G$  of order  $5 \times k^2$ , both embeddable on the torus, for which  $D = V(G) \cap D_{\square}$  and  $D' = V(G) \cap D_{\circlearrowleft}$  are two minimum dominating sets of order  $k^2$ .

Now, let

$$D_0, \dots, D_{\ell}$$

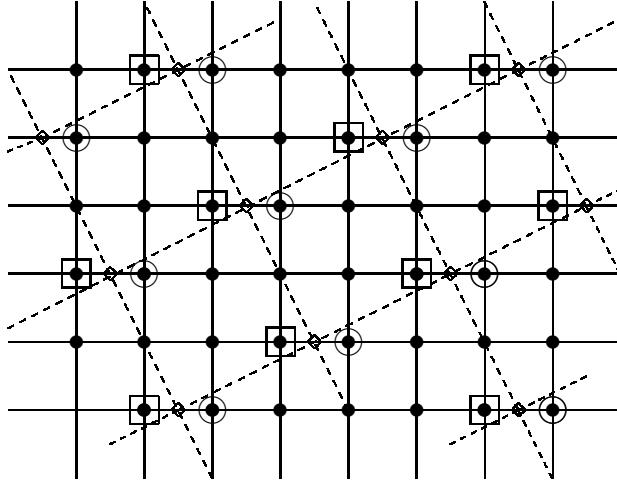


Figure 2: A section of the infinite auxiliary graph  $H_\infty$  whose vertices are indicated by diamonds  $\diamond$  and whose edges are indicated by dashed lines.

be a sequence of dominating sets in  $G$  such that  $D = D_0$ ,  $D' = D_\ell$ , and  $D_{i-1}$  is adjacent to  $D_i$  for every  $i \in [\ell]$ , where  $[\ell] = \{1, \dots, \ell\}$ . For every pair  $p = \{u_\square, u_\circlearrowright\}$  that is a vertex of  $H$ , we say that  $p$  is

- of *type left in  $D_i$*  if  $\{u_\square, u_\circlearrowright\} \cap D_i = \{u_\square\}$ ,
- of *type right in  $D_i$*  if  $\{u_\square, u_\circlearrowright\} \cap D_i = \{u_\circlearrowright\}$ ,
- of *type 0 in  $D_i$*  if  $\{u_\square, u_\circlearrowright\} \cap D_i = \emptyset$ , and
- of *type 2 in  $D_i$*  if  $\{u_\square, u_\circlearrowright\} \cap D_i = \{u_\square, u_\circlearrowright\}$ , respectively.

See Figure 4 for an illustration of these four possibilities.

Let  $[\ell]_0 = \{0, 1, \dots, \ell\}$ . For an integer  $i \in [\ell]_0$  and a type  $t \in \{\text{left}, \text{right}, 0, 2\}$ , let  $P(i, t)$  be the set of all vertices  $p = \{u_\square, u_\circlearrowright\}$  of  $H$  that are of type  $t$  in  $D_i$ , and let  $n(i, t) = |P(i, t)|$ . Trivially,

$$n(0, t) = \begin{cases} k^2, & t = \text{left}, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad n(\ell, t) = \begin{cases} k^2, & t = \text{right}, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, for every  $i \in [\ell]$  and type  $t$ , since  $D_i$  arises from  $D_{i-1}$  by removing or adding a single vertex, we have

$$\left| n(i, t) - n(i-1, t) \right| \leq 1.$$

Since  $k$  is a multiple of 4,  $k^2/8$  is an integer. Hence, if  $j \in [\ell]_0$  is the smallest index such that

$$n(j, \text{left}) \leq \frac{7k^2}{8},$$

then

$$n(j, \text{left}) = \frac{7k^2}{8} \quad \text{and} \quad n(j, \text{right}) + n(j, 0) + n(j, 2) = k^2 - \frac{7k^2}{8} = \frac{k^2}{8}.$$

In order to complete the proof, we will show that the dominating set

$$D^* = D_j$$

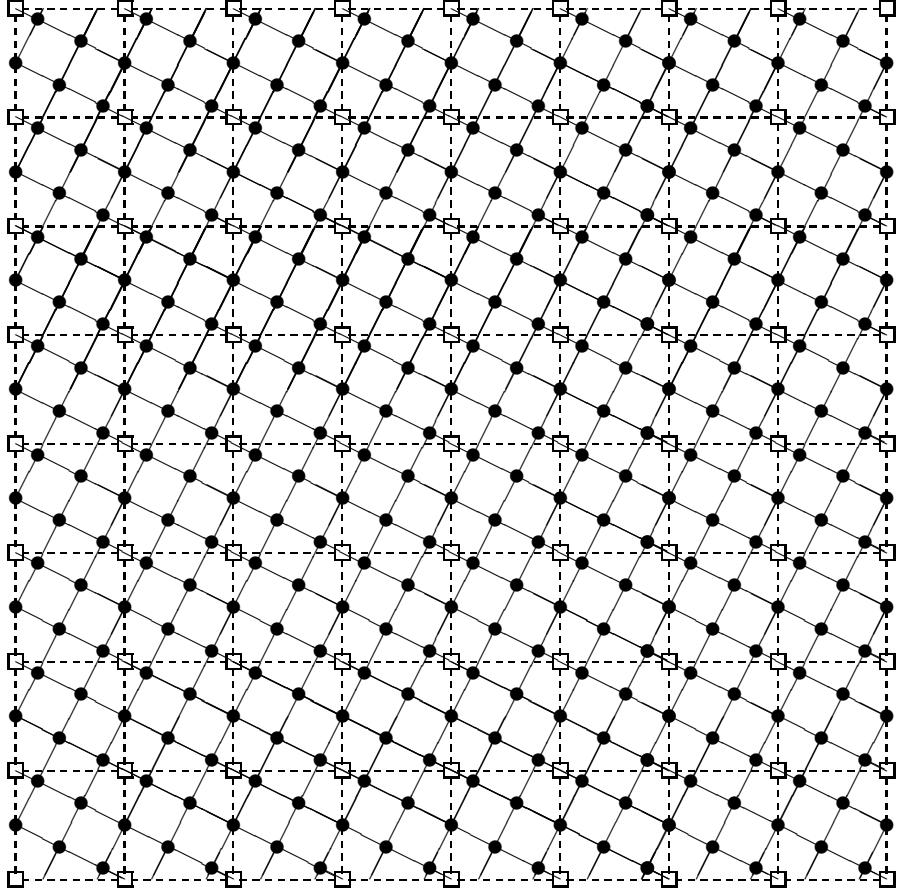


Figure 3: A  $9 \times 9$  grid subgraph of  $H_\infty$ . Identifying the left border with the right border both from top to bottom, and the top border with the bottom border both from left to right yields  $H$  isomorphic to  $\mathbb{Z}_8^2$  and a graph  $G$  of order  $5 \times 8^2$  embeddable on the torus for which  $D = V(G) \cap D_\square$  and  $D' = V(G) \cap D_\circlearrowright$  are two minimum dominating sets in  $G$  both of order  $8^2$ .

has cardinality at least  $k^2 + \Omega(k)$ . This is done by showing that  $D^*$  contains  $\Omega(k)$  so-called inefficient vertices, where a vertex  $u$  in  $D^*$  is *inefficient* if there is some vertex  $v$  in  $D^* \setminus \{u\}$  with  $N_G[u] \cap N_G[v] \neq \emptyset$ .

Let  $P(t) = P(j, t)$  and  $n(t) = n(j, t)$  for every type  $t$ . Let  $P'(\text{left})$  be the subset of  $P(\text{left})$  containing all vertices  $p = \{u_\square, u_\circlearrowright\}$  of  $H$  whose four neighbors in  $H$  are all not of type left. Double counting the edges of  $H$  between  $P'(\text{left})$  and  $P(\text{right}) \cup P(0) \cup P(2)$  implies

$$|P'(\text{left})| \leq |P(\text{right}) \cup P(0) \cup P(2)| = \frac{k^2}{8},$$

and, hence, the set  $P^* = P(\text{left}) \setminus P'(\text{left})$  satisfies

$$\frac{6k^2}{8} \leq |P^*| \leq \frac{7k^2}{8}. \quad (1)$$

Let  $P^{**}$  be the set of vertices in  $P^*$  that have a neighbor outside of  $P^*$ . Note that, by construction, every vertex in  $P^{**}$  belongs to  $P(\text{left})$ , has a neighbor in  $P(\text{left})$  as well as in  $P(\text{right}) \cup P(0) \cup P(2)$ , but has no neighbor in  $P'(\text{left})$ .

Our next goal is the following.

**Lemma 3.**  $|P^{**}| \geq \frac{k}{20} + O(1)$ .

*Proof.* For a set  $S$  of vertices of  $H$ , let  $N_H(S) = \{u \in V(H) \setminus S : N_H(u) \cap S \neq \emptyset\}$ . Let  $u$  be any

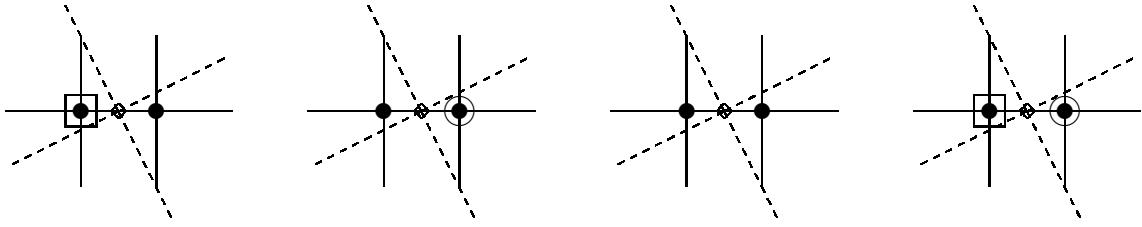


Figure 4: From left to right, a pair  $p = \{u_\square, u_\circ\}$  of type left, right, 0, and 2.

vertex of  $H$ , and, for a non-negative integer  $r$ , let  $B(r) = \{v \in V(H) : \text{dist}_H(u, v) \leq r\}$ . Since  $H$  is vertex-transitive, it follows that  $|B(r)|$  is independent of the choice of  $u$ . Bollobás and Leader [2] showed that, if  $|S| = |B(r)|$  for some non-negative integer  $r$ , then

$$|N_H(S)| \geq |N_H(B(r))| = |B(r+1)| - |B(r)|.$$

Note that their result only applies to sets whose cardinality is in  $\{|B(0)|, |B(1)|, |B(2)|, \dots\}$ , which causes some technicalities in our proof. Let the non-negative integer  $r^*$  be such that

$$|B(r^* - 1)| \leq |P^*| \leq |B(r^*)|.$$

Combining some simple geometric considerations illustrated in Figure 5 with (1) implies the existence of some  $c_1 \in [1/4, \sqrt{2}/4]$  such that

$$\begin{aligned} r^* &= (1 - c_1)k + O(1), \\ |N_H(B(r^*))| &= |B(r^* + 1)| - |B(r^*)| = 4c_1k + O(1), \text{ and} \\ |N_H(B(r^* - 1))| &= |B(r^*)| - |B(r^* - 1)| = 4c_1k + O(1). \end{aligned} \tag{2}$$

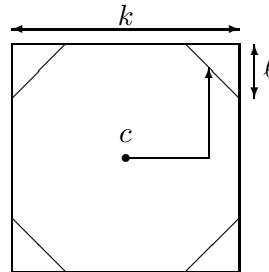


Figure 5: Within a  $k \times k$  square  $Q$  in  $\mathbb{R}^2$ , the set of points in  $Q$  at Manhattan distance at most  $k - \ell$  from the center point  $c$  of  $Q$  for some  $\ell < k/2$  has area  $k^2 - 2\ell^2$ . If  $k^2 - 2\ell^2 \in [6k^2/8, 7k^2/8]$ , then  $\ell/k \in [1/4, \sqrt{2}/4]$ .

If  $S$  is a set of vertices of  $H$ , and  $u$  is a vertex of  $H$  outside of  $S$ , then

$$|N_H(S)| + 4 \geq |N_H(S \cup \{u\})| \geq |N_H(S)| - 1. \tag{3}$$

If  $|B(r^*)| - |P^*| \leq \frac{4c_1k}{5}$ , then adding  $|B(r^*)| - |P^*|$  vertices to  $P^*$  yields a set  $P'$  of order  $|B(r^*)|$  with

$$|N_H(P^*)| \stackrel{(3)}{\geq} |N_H(P')| - 4(|B(r^*)| - |P^*|)$$

$$\begin{aligned}
&\geq |N_H(P')| - \frac{16c_1k}{5} \\
&\geq |N_H(B(r^*))| - \frac{16c_1k}{5} \\
&\stackrel{(2)}{\geq} 4c_1k \left(1 - \frac{4}{5}\right) + O(1) \\
&\geq \frac{4c_1k}{5} + O(1).
\end{aligned}$$

Conversely, if  $|B(r^*)| - |P^*| > \frac{4c_1k}{5}$ , then (2) implies  $|P^*| - |B(r^* - 1)| \leq \frac{16c_1k}{5} + O(1)$ , and removing  $|P^*| - |B(r^* - 1)|$  vertices from  $P^*$  yields a set  $P''$  of order  $|B(r^* - 1)|$  with

$$\begin{aligned}
|N_H(P^*)| &\stackrel{(3)}{\geq} |N_H(P'')| - \left(\frac{16c_1k}{5} + O(1)\right) \\
&\geq |N_H(P'')| - \frac{16c_1k}{5} + O(1) \\
&\geq |N_H(B(r^* - 1))| - \frac{16c_1k}{5} + O(1) \\
&\stackrel{(2)}{\geq} 4c_1k \left(1 - \frac{4}{5}\right) + O(1) \\
&\geq \frac{4c_1k}{5} + O(1).
\end{aligned}$$

Altogether, it follows that

$$|N_H(P^*)| \geq \frac{4c_1k}{5} + O(1) \geq \frac{k}{5} + O(1).$$

Since  $|P^{**}| \geq |N_H(P^*)|/4$ , the proof of Lemma 3 is complete.  $\square$

Now, let  $p = \{u_{\square}, u_{\circlearrowleft}\}$  be a vertex in  $P^{**}$ , illustrated as the pair  $\{(2, 2), (3, 2)\}$  in Figure 6.

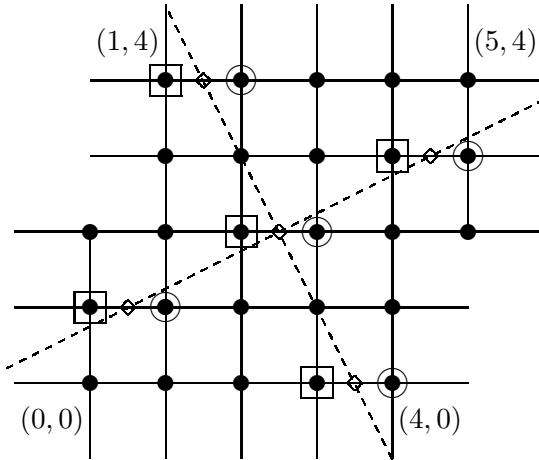


Figure 6: A vertex  $p = \{u_{\square}, u_{\circlearrowleft}\}$  from  $P^{**}$ , its four neighbors in  $H$ , and some part of  $G$ . For the illustrated vertices of  $G$ , we use the indicated coordinates, in particular,  $u_{\square}$  corresponds to  $(2, 2)$  and  $u_{\circlearrowleft}$  corresponds to  $(3, 2)$ .

Our next goal is the following.

**Lemma 4.**  $D^*$  contains an inefficient vertex  $u$  with  $\text{dist}_G(u, u_{\square}) \leq 3$ .

*Proof.* If  $p$  has a neighbor in  $P(2)$ , the statement is trivial. Hence, we may assume that no neighbor of  $p$  is in  $P(2)$ . Hence, by construction,  $p$  has a neighbor  $p' = \{u'_{\square}, u'_{\circlearrowleft}\}$  from  $P(\text{left})$  as well as a neighbor

$p'' = \{u''_{\square}, u''_{\circlearrowright}\}$  from  $P(\text{right}) \cup P(0)$ , in particular,  $u'_{\square} \in D^*$  and  $u''_{\square} \notin D^*$ . We consider several cases, where we denote the vertices using coordinates as explained in Figure 6.

**Case 1**  $(1, 4) \in D^*$  and  $(4, 3) \notin D^*$ .

$D^*$  contains a vertex  $u$  from  $N_G[(3, 3)] \setminus \{(4, 3)\}$ , which is necessarily inefficient.

**Case 2**  $(1, 4), (4, 3) \in D^*$  and  $(3, 0) \notin D^*$ .

$D^*$  contains a vertex  $u$  from  $N_G[(3, 1)] \setminus \{(3, 0)\}$ , which is necessarily inefficient.

**Case 3**  $(1, 4), (4, 3), (3, 0) \in D^*$  and  $(0, 1) \notin D^*$ .

$D^*$  contains a vertex  $u$  from  $N_G[(1, 1)] \setminus \{(0, 1)\}$ , which is necessarily inefficient.

**Case 4**  $(1, 4) \notin D^*$  and  $(4, 3) \in D^*$ .

$D^*$  contains a vertex from  $N_G[(1, 3)]$ . In view of the desired results, we may assume that  $D^*$  contains  $(0, 3)$ .  $D^*$  contains a vertex from  $N_G[(1, 4)]$ . In view of the desired results, we may assume that  $D^*$  contains  $(1, 5)$ .  $D^*$  contains a vertex  $u$  from  $N_G[(2, 4)]$ , which is now necessarily inefficient.

**Case 5**  $(1, 4), (4, 3) \notin D^*$  and  $(3, 0) \in D^*$ .

$D^*$  contains a vertex from  $N_G[(4, 2)]$ . In view of the desired results, we may assume that  $D^*$  contains  $(5, 2)$ .  $D^*$  contains a vertex from  $N_G[(4, 3)]$ . In view of the desired results, we may assume that  $D^*$  contains  $(4, 4)$ .  $D^*$  contains a vertex  $u$  from  $N_G[(3, 3)]$ , which is now necessarily inefficient.

**Case 6**  $(1, 4), (4, 3), (3, 0) \notin D^*$  and  $(0, 1) \in D^*$ .

$D^*$  contains a vertex from  $N_G[(2, 0)]$ . In view of the desired results, we may assume that  $D^*$  contains  $(2, -1)$ .  $D^*$  contains a vertex  $u$  from  $N_G[(1, 0)]$ , which is now necessarily inefficient.

The considered cases exhaust all relevant situations, which completes the proof.  $\square$

Since there are 13 vertices at distance at most 2 as well as 25 vertices at distance at most 3 from every vertex of  $G$ , Lemma 3 and Lemma 4 imply that  $D^*$  contains at least  $\frac{k}{20 \cdot 25} + O(1) = \frac{k}{500} + O(1)$  inefficient vertices, and that there is a set  $I$  of at least  $\frac{k}{13 \cdot 500} + O(1) = \frac{k}{12500} + O(1)$  inefficient vertices from  $D^*$  such that every two vertices in  $I$  have pairwise distance at least 3. Now, double counting the number  $d$  of pairs  $(u, v)$  with  $u \in D^*$  and  $v \in N_G[u]$ , we obtain  $n(G) + |I| \leq d = 5|D^*|$ . This implies

$$|D^*| \geq \frac{n(G) + |I|}{5} = k^2 + \frac{|I|}{5} \geq k^2 + \frac{k}{62500} + O(1),$$

which completes the proof of Theorem 1.  $\square$

### 3 Proof of Theorem 2

Let  $\mathcal{G}$ ,  $\alpha$ ,  $G$ , as well as  $D$  and  $D'$  be as in the statement of the theorem. If  $D''$  is a dominating set in  $G$ ,  $D \xleftrightarrow{k} D''$ , and  $D' \xleftrightarrow{k} D''$ , then  $D \xleftrightarrow{k} D'$ . This implies that we may assume that  $D'$  is a minimum dominating set in  $G$ . Since  $\mathcal{G}$  is hereditary, there is a positive constant  $c_2$  such that every induced subgraph  $G'$  of  $G$  has a balanced separator of order  $c_2 n(G')^\alpha$ . Recursively removing balanced separators, it follows that there is a full binary tree  $T$  with root  $r$  as well as a set  $V(t)$  of vertices of  $G$  for every vertex  $t$  of  $T$  such that

- (i)  $(V(t))_{t \in V(T)}$  is a partition of  $V(G)$ ,

- (ii) if  $t$  is not a leaf and has depth  $d$ , then  $|V(t)| \leq c_2 \left(\frac{2}{3}\right)^{\alpha d} n^\alpha$ ,
- (iii) if  $t$  is a leaf, then  $|V(t)| \leq n^\alpha$ , and
- (iv) if  $t$  and  $t'$  are two distinct vertices of  $T$  such that  $G$  has an edge between  $V(t)$  and  $V(t')$ , then either  $t$  is an ancestor of  $t'$  or  $t'$  is an ancestor of  $t$ .

Allowing  $V(t)$  to be empty, we may assume that all leaves of  $T$  have the same depth. Note that (iii) implies that the depth of  $T$  is  $\Omega(\log n)$ , and that the important property (iv) is a consequence of the definition of (balanced) separators.

For every vertex  $t$  of  $T$  that is not a leaf, we label the two edges between  $t$  and its two children arbitrarily by 0 (corresponding to left) and 1 (corresponding to right). For a set  $S$  of vertices of  $T$  for which  $T[S]$  is connected, let  $L_T(S)$  be the set of vertices  $t$  of  $T$  such that there is a path  $s_0 s_1 \dots s_\ell$  in  $T$  with  $\ell \geq 1$ ,  $s_0 \in S$ ,  $s_i \notin S$  for every  $i \in [\ell]$ ,  $t = s_\ell$ ,  $s_i$  is a child of  $s_{i-1}$  for every  $i \in [\ell]$ , and the edge  $s_0 s_1$  has label 0. Define  $R_T(S)$  similarly requiring label 1 instead of 0 for the edge  $s_0 s_1$ . See Figure 7, where these definitions are illustrated for a root-to-leaf path in  $T$ .

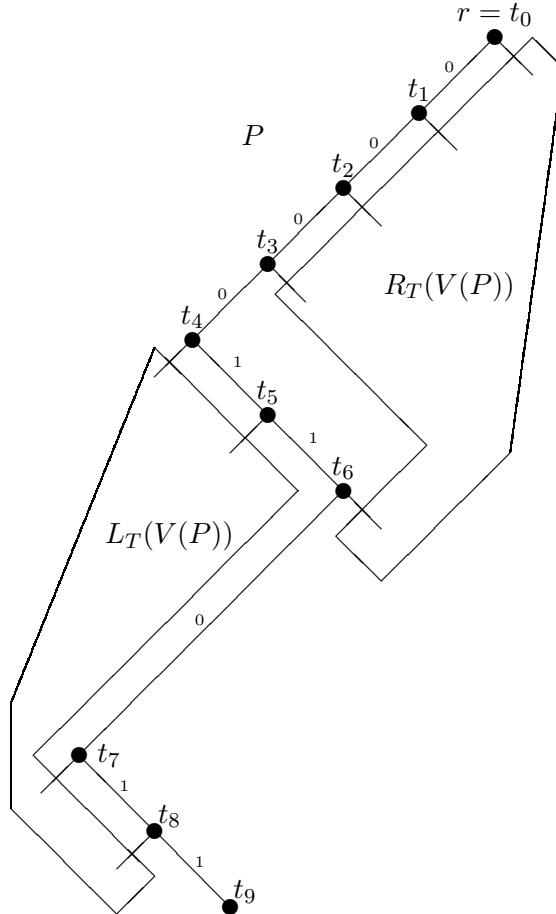


Figure 7: A root-to-leaf path  $P : t_0 t_1 \dots t_9$  and the two sets  $L_T(V(P))$  and  $R_T(V(P))$ .

Let

$$V_G(S) = \bigcup_{t \in S} V(t), \quad L_G(S) = \bigcup_{t \in L_T(S)} V(t), \quad \text{and} \quad R_G(S) = \bigcup_{t \in R_T(S)} V(t).$$

Note that, if  $S$  contains the root of  $T$ , then

- the sets  $S$ ,  $L_T(S)$ , and  $R_T(S)$  partition  $V(T)$ , and
- the sets  $V_G(S)$ ,  $L_G(S)$ , and  $R_G(S)$  partition  $V(G)$ , and there is no edge between  $L_G(S)$  and  $R_G(S)$ .

If  $P : t_0 t_1 \dots t_d$  is a root-to-leaf path in  $T$  with  $r = t_0$ , then the above properties imply that the set

$$D(P) = V_G(V(P)) \cup (L_G(V(P)) \cap D') \cup (R_G(V(P)) \cap D)$$

is a dominating set in  $G$ . Note, for instance, that all neighbors of vertices from  $L_G(V(P))$  outside of  $L_G(V(P))$  lie in  $V_G(V(P))$ . We call  $D(P)$  the *special dominating set associated to P*. Note that

$$\begin{aligned} |V_G(V(P))| &\stackrel{(ii),(iii)}{\leq} \sum_{i \in [\ell-1]_0} c_2 \left(\frac{2}{3}\right)^{\alpha i} n^\alpha + n^\alpha \\ &\leq \left( \frac{c_2}{1 - (\frac{2}{3})^\alpha} + 1 \right) n^\alpha = c_3 n^\alpha, \quad \text{where } c_3 = \left( \frac{c_2}{1 - (\frac{2}{3})^\alpha} + 1 \right). \end{aligned} \quad (4)$$

The following lemma contains the important observation that  $D(P)$  is not much larger than  $D$ .

**Lemma 5.**  $|D(P)| \leq |D| + 2c_3 n^\alpha$ .

*Proof.* Suppose, for a contradiction, that  $|D(P)| > |D| + 2c_3 n^\alpha$ .

By the definition of  $D(P)$ , we obtain

$$\begin{aligned} c_3 n^\alpha + |L_G(V(P)) \cap D'| + |R_G(V(P)) \cap D| &\geq |D(P)| \\ &> |D| + 2c_3 n^\alpha \\ &\geq |L_G(V(P)) \cap D| + |R_G(V(P)) \cap D| + 2c_3 n^\alpha, \end{aligned}$$

and, hence,

$$|L_G(V(P)) \cap D'| > |L_G(V(P)) \cap D| + c_3 n^\alpha.$$

Now,

$$D'' = V_G(V(P)) \cup (L_G(V(P)) \cap D) \cup (R_G(V(P)) \cap D')$$

is a dominating set in  $G$  with

$$\begin{aligned} |D''| &\leq c_3 n^\alpha + |L_G(V(P)) \cap D| + |R_G(V(P)) \cap D'| \\ &< |L_G(V(P)) \cap D'| + |R_G(V(P)) \cap D'| \\ &\leq |D'|, \end{aligned}$$

contradicting the fact that  $D'$  is a minimum dominating set. This completes the proof.  $\square$

In order to prove the theorem, we will construct a sequence  $D_0, D_1, \dots, D_\ell$  of dominating sets in  $G$  such that  $D = D_0$ ,  $D' = D_\ell$ ,  $D_{i-1}$  is adjacent to  $D_i$  for every  $i \in [\ell]$ , and, for every  $i \in [\ell]_0$ , there is some  $j \in [\ell]_0$  with

- $|j - i| \leq 2c_3 n^\alpha$  and

- $D_j$  is a special dominating set associated to some root-to-leaf path in  $T$ .

Together with Lemma 5, this implies

$$|D_i| \leq |D| + 4c_3 n^\alpha \text{ for every } i \in [\ell]_0,$$

which completes the proof.

For an edge  $e$  of  $T$ , let  $\sigma(e) \in \{0, 1\}$  be its label. For a root-to-leaf path  $P : t_0 t_1 \dots t_d$  in  $T$  with  $r = t_0$ , let

$$\sigma(P) = (\sigma(t_0 t_1), \sigma(t_1 t_2), \dots, \sigma(t_{d-1} t_d)).$$

We consider the root-to-leaf paths  $P$  according to the lexicographic order of the  $\sigma(P)$ 's, that is,  $P$  comes immediately before  $P'$  if there is some  $i \in [d-1]$  such that

$$\begin{aligned} \sigma(P) &= (\sigma(t_0 t_1), \dots, \sigma(t_{i-1} t_i), 0, 1, \dots, 1) \text{ and} \\ \sigma(P') &= (\sigma(t_0 t_1), \dots, \sigma(t_{i-1} t_i), 1, 0, \dots, 0). \end{aligned}$$

See Figure 8 for an illustration.

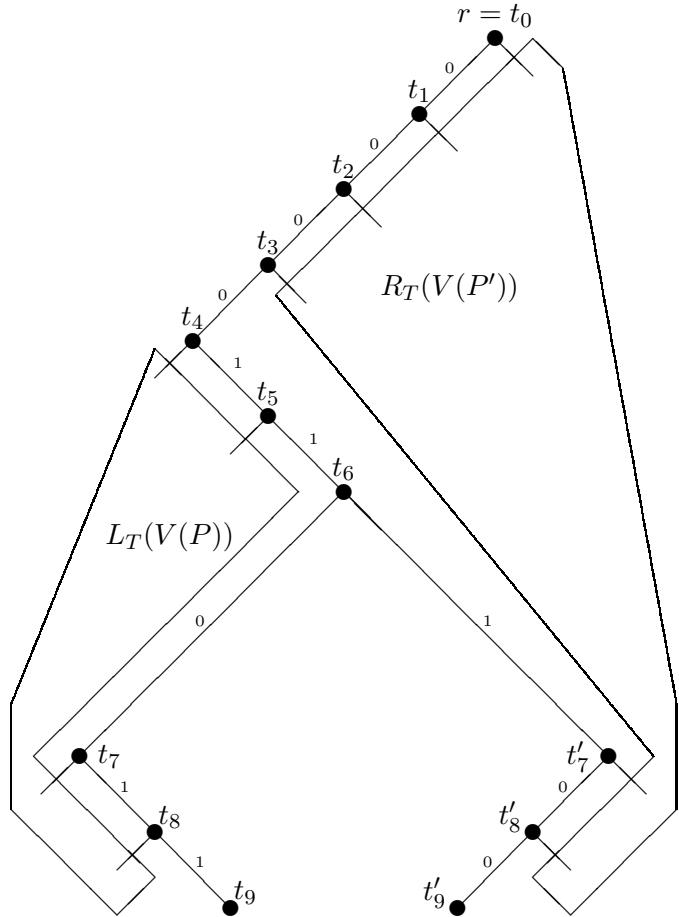


Figure 8: The two paths  $P : t_0 t_1 \dots t_6 t_7 t_8 t_9$  and  $P' : t_0 t_1 \dots t_6 t_7' t_8' t_9'$  are lexicographically consecutive.

Let

$$P(0, \dots, 0, 0), P(0, \dots, 0, 1), P(0, \dots, 1, 0), P(0, \dots, 1, 1), \dots, P(1, \dots, 1, 0), P(1, \dots, 1, 1)$$

be this order of the root-to-leaf paths in  $T$ .

Now, we describe how to construct the sequence  $D_0, \dots, D_\ell$  mentioned after Lemma 5. Starting with  $D_0 = D$ , we first transform  $D$  into  $D(P(0, \dots, 0, 0))$ , where  $P(0, \dots, 0, 0) : t_0 t_1 \dots t_d$  as follows:

```

for  $i = 0$  to  $d$  do
| Add the vertices from  $V(t_i) \setminus D$  one by one to the current dominating set;
end
```

Since  $|V_G(V(P))| \leq c_3 n^\alpha$ , this requires at most  $c_3 n^\alpha$  steps. Symmetrically, the set  $D(P(1, \dots, 1, 1))$  can be transformed into  $D'$  by a sequence of at most  $c_3 n^\alpha$  steps. In order to complete the proof, we explain how to transform  $D(P)$  into  $D(P')$ , where  $P : t_0 t_1 \dots t_i t_{i+1} \dots t_d$ ,  $P' : t_0 t_1 \dots t_i t'_{i+1} \dots t'_d$ , and  $P$  comes immediately before  $P'$  in the lexicographic order:

```

for  $j = d$  to  $i + 1$  do
| Remove from the current dominating set one by one the vertices from  $V(t_j) \setminus D'$ ;
end

for  $j = i + 1$  to  $d$  do
| Add the vertices from  $V(t'_j) \setminus D$  one by one to the current dominating set;
end
```

Since

$$\left| \bigcup_{j \in \{i+1, \dots, d\}} V(t_j) \right| + \left| \bigcup_{j \in \{i+1, \dots, d\}} V(t'_j) \right| \leq |V_G(V(P))| + |V_G(V(P'))| \stackrel{(4)}{\leq} 2c_3 n^\alpha,$$

this requires at most  $2c_3 n^\alpha$  steps. It is easy to see that all intermediate sets during these transformations are dominating sets in  $G$ , which completes the proof of Theorem 2.  $\square$

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