

ON BARYCENTRIC TRANSFORMATIONS OF FANO POLYTOPES

DONGSEON HWANG AND YEONSU KIM

ABSTRACT. We introduce the notion of barycentric transformation of Fano polytopes, from which we can assign a certain type to each Fano polytope. The type can be viewed as a measure of the extent to which the given Fano polytope is close to be Kähler-Einstein. In particular, we expect that every Kähler-Einstein or symmetric Fano polytope is of type B_∞ . We verify this expectation for some low dimensional cases. We emphasize that for a Fano polytope X of dimension 1, 3 or 5, X is Kähler-Einstein if and only if it is of type B_∞ .

1. INTRODUCTION

A *Fano polytope* of dimension n is a full dimensional convex lattice polytope in \mathbb{R}^n such that the vertices are primitive lattice points and the origin is an interior point. Note that the class of Fano polytopes of dimension n up to unimodular transformation has a one-to-one correspondence with the class of toric Fano varieties of dimension n up to isomorphism. A Fano polytope P is said to be *Kähler-Einstein* if the dual polytope of P has the origin as its barycenter. By Theorem [BB, Theorem 1.2], a Fano polytope P is Kähler-Einstein if and only if the associated toric Fano variety X_P admits a Kähler-Einstein metric.

Batyrev and Selivanova introduced the notion of symmetric Fano polytopes to study Kähler-Einstein polytopes. A Fano polytope P is said to be *symmetric* if the origin is the only lattice point fixed by every automorphisms of P onto itself.

Theorem 1.1. [BS, Theorem 1.1] *Every smooth symmetric Fano polytope is Kähler-Einstein.*

There had been interests on whether the converse statement holds([BS], [So, Remark in p.1257], [CL, Remark 4.3], [FOS, p.257]). For smooth Fano polytopes of dimension at most 8, an exhaustive investigation yields the following theorem.

Theorem 1.2. [NP, Proposition 2.1] *Let P be a smooth Kähler-Einstein Fano polytope of dimension at most 8, then P is not symmetric if and only if P is one of the three Fano polytopes Q_1, Q_2 or Q_3 in [NP] where Q_2 and Q_3 have dimension 8 and Q_1 has dimension 7.*

We remark that, for each integer $n \geq 9$, there exists a smooth Kähler-Einstein Fano polytope of dimension n that is not symmetric([N2, Corollary 5.3]). See also [HK] for the discussion in the singular setting.

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In this note, we shall describe a class of Fano polytopes that is expected to be very similar to the class of Kähler-Einstein Fano polytopes in the smooth case and have a close connection to symmetric or Kähler-Einstein Fano polytopes in general.

To be more precise, we shall introduce the notion of barycentric transformation, or B-transformation in short, of Fano polytopes. See Definition 2.1 for the precise definition. The B-transformation of a Fano polytope does not always produce a Fano polytope as we see in Example 2.3. By using this phenomenon of B-transformation, we define a *type* of Fano polytopes. Naively speaking, a Fano polytope is of *type* B_k if it is possible to take the B-transformation k times, and is of *strict type* B_k if we can take the B-transformation at most k times. See Definition 2.4 for the precise definition.

Theorem 1.3. *Let P be a smooth Fano polytope of dimension n .*

- (1) *Assume that n is odd and $n \leq 5$. Then, P is Kähler-Einstein if and only if P is of type B_∞ .*
- (2) *Assume that n is even and $n \leq 4$. If P is Kähler-Einstein, then P is of type B_∞ except possibly for one case with GRDB ID 54 ([GRDB] based on [O]).*
- (3) *If P is a Kähler-Einstein Fano polytope of dimension 7 or 8 that is not symmetric, i.e., $P = Q_1, Q_2, Q_3$ in [NP], then P is of type B_∞ .*

Moreover, the Kähler-Einstein property is preserved by the B-transformation in low dimension.

Theorem 1.4. *Let P be a smooth Fano polytope of dimension at most 6, or $P = Q_1, Q_2, Q_3$ in [NP]. If P is Kähler-Einstein, then so is $B(P)$.*

Based on these results, we boldly pose the following conjecture.

Conjecture 1.5. *Let P be a smooth Fano polytope.*

- (1) *If P has odd dimension, then P is Kähler-Einstein if and only if P is of type B_∞ .*
- (2) *If P is Kähler-Einstein, then $B(P)$ is also a Kähler-Einstein Fano polytope. In particular, P is of type B_∞ .*
- (3) *If P is symmetric, then $B(P)$ is a symmetric and Kähler-Einstein Fano polytope. In particular, P is of type B_∞ .*

See Question 3.1 for the discussion on the converse statement of Conjecture 1.5 (2) in the even dimensional case.

One can ask whether Conjecture 1.5 holds true for the singular varieties. Contrary to the smooth case, not every symmetric Fano polytope is Kähler-Einstein ([HK]) even for surfaces. But we still have the following result.

Theorem 1.6. *Let P be a Fano polygon.*

- (1) *If P is Kähler-Einstein, then P is of type B_1 . If, moreover, P is a triangle, then $B(P)$ is a Kähler-Einstein Fano triangle, hence P is of type B_∞ .*
- (2) *If P is symmetric, then $B(P)$ is a symmetric and Kähler-Einstein Fano polygon, hence P is of type B_∞ .*

Every known example of a Kähler-Einstein Fano polygon that is not symmetric is a triangle. For this reason, the following was suggested.

Conjecture 1.7. [HK, Conjecture 1.6] Let P be a Kähler-Einstein Fano polygon. If P is not symmetric, then it is a triangle.

Assuming Conjecture 1.7, Theorem 1.6 proves Conjecture 1.5 for surfaces. Theorem 3.15 shows that Conjecture 1.7, hence Conjecture 1.5, holds true for Fano polygons of index at most 17 without any assumption.

Remark 1.8. Not every Fano polytope of type B_∞ becomes Kähler-Einstein after a suitable application of B-transformations. For example, consider a Fano polygon P with GRDB ID 13118 ([GRDB] based on [KKN]). Then, P is 2-periodic, i.e., $B^2(P) = P$. But both P and $B(P)$ are not Kähler-Einstein. Note that P is of index 3, which is the smallest possible index with this property.

Section 2 provides rigorous definitions and examples of barycentric transformations. Main theorems are proven in Section 3. Section 4 presents some interesting observation on the orbits of a Fano polygon under B-transformations and discussion on Fano polygons with zero barycenter.

2. B-TRANSFORMATION: DEFINITION AND EXAMPLES

2.1. Notation. For a lattice point $v = (x, y)$, we define the *primitive index* $I(v)$ of v by $I(v) = \gcd(x, y)$. A lattice point is called *primitive* if its primitive index is one. The *order* of the two dimensional cone C spanned by two lattice points $v_i = (x_i, y_i)$ and $v_{i+1} = (x_{i+1}, y_{i+1})$, denoted by $\text{ord}(v_i, v_{i+1})$, is defined by

$$\text{ord}(v_i, v_{i+1}) := \det \begin{pmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{pmatrix} = x_i y_{i+1} - y_i x_{i+1}.$$

2.2. B-transformations. To understand more about Kähler-Einstein Fano polytopes, we introduce the new notion called a barycentric transformation.

Definition 2.1. For a Fano polytope P , we can always obtain a lattice polytope by taking the convex hull of the barycenters of all maximal dimensional cones of P . See ([CLS, Exercise 11.1.10]) for the definition of the barycenter of a cone. This association is called a *barycentric transformation* of P or, in short, a *B-transformation* of P .

Note that the B-transformation is uniquely determined up to unimodular transformation.

We can easily compute the B-transformation of a Fano polygon using the following lemma, which immediately follows from definition.

Lemma 2.2. Let P be a Fano polygon with vertices v_1, \dots, v_n written in counter-clockwise order. Then,

$$B(P) = \text{conv} \left\{ \frac{v_1 + v_2}{I(v_1 + v_2)}, \dots, \frac{v_n + v_1}{I(v_n + v_1)} \right\}.$$

Example 2.3. For a Fano polytope P , $B(P)$ is not a Fano polytope in general.

(1) Take $P = \text{conv}\{(2, -1), (0, 1), (-1, 0)\}$. Then,

$$B(P) = \text{conv}\{(1, 0), (-1, 1), (1, -1)\}$$

is not a Fano polytope since it does not contain the origin as an interior point.

- (2) Take $P = \text{conv}\{(1, -2), (0, 1), (-1, -2)\}$. Then,

$$B(P) = \text{conv}\{(1, -1), (-1, -1)\}$$

is not a Fano polytope. Note that the dimension is decreased in this case.

Definition 2.4. Let P be a Fano polytope of dimension d .

- (1) P is said to be of type B_m if $B^m(P)$ is a Fano polytope of dimension d .
- (2) P is said to be of strict type B_m if it is of type B_m but not of type B_{m+1} .
- (3) P is said to be of type B_∞ if it is of type B_m for every positive integer m .

By convention, every Fano polytope P is of type B_0 . Note that if a Fano polytope P is type of B_s then it is of type B_t for every $t < s$.

We also introduce the following useful notion.

Definition 2.5. Let P be a Fano polytope.

- (1) P is said to be B -invariant if $B(P) = P$.
- (2) P is called k -periodic (or just periodic) if there exists an integer t such that $B^{k+t}(P) = B^t(P)$ for some integer $k \geq 1$.
- (3) P is called pseudo-periodic if the number of vertices of $B^k(P)$ is invariant under the B -transformation for some positive integer k .

Note that a periodic Fano polytope is of type B_∞ .

Example 2.6. Let P be a d -dimensional Fano cube. For example, if $d = 2$,

$$P = \text{conv}\{e_1 + e_2, -e_1 + e_2, -e_1 - e_2, e_1 - e_2\}$$

where e_1 and e_2 are standard bases of \mathbb{R}^2 . Then, P is of type B_∞ . Indeed, one can easily see that $B(P) = \text{conv}\{e_1, \dots, e_d, -e_1, \dots, -e_d\}$ is a d -dimensional Fano bipyramid and $B^2(P) = P$, i.e., P is 2-periodic. In this case, both P and $B(P)$ are Kähler-Einstein. Note that P is singular and $B(P)$ is smooth, from which we see that the B -transformation does not preserve the smoothness.

Example 2.7. Let $P = \text{conv}\{(0, 1), (3, -2), (-4, 1)\}$. It is easy to compute that $B(P) = \text{conv}\{(3, -1), (-1, -1), (-2, 1)\}$ and $B^2(P) = \text{conv}\{(-1, 0), (1, -1), (1, 0)\}$, from which we see that P is of strict type B_1 .

Example 2.8. Let $P = \text{conv}\{(-25, -12), (-5, -6), (25, 14)\}$. One can compute that $B(P) = \text{conv}\{(-5, -3), (5, 2), (0, 1)\}$ and

$$B^2(P) = \text{conv}\{(0, -1), (5, 3), (-5, -2)\} = -B(P),$$

thus P is 2-periodic, and hence it is of type B_∞ . Now, it is easy to see that P is not Kähler-Einstein but so is $B(P)$. Note also that $B^n(P)$ is not symmetric for every integer $n \geq 0$.

Lemma 2.9. For a Fano polygon, the number of vertices is not increasing under B -transformations.

Proof. It immediately follows from the fact that, in dimension two, the number of maximal dimensional cones is equal to the number of vertices. \square

Example 2.10. The number of vertices is not invariant under B -transformation in general. Let

$$P = \text{conv}\{(3, -1), (3, 1), (1, 2), (-3, 1), (-3, -1), (-1, -2)\}.$$

Then, P is both symmetric and Kähler-Einstein but

$$B(P) = \text{conv}\{(4, 3), (-2, 3), (-4, -3), (2, -3)\}.$$

By using the argument in the proof of Lemma 3.4, it is easy to see that for a Kähler-Einstein Fano polygon with at most 4 vertices, the number of vertices is invariant under B -transformation. We do not know any example of such a Kähler-Einstein Fano polygon with 5 vertices.

Remark 2.11. Lemma 2.9 does not hold for higher dimensional Fano polytopes. See Example 2.6.

2.3. Condition to be of type B_1 . We provide a sufficient condition for a given Fano polygon P to be of type B_1 .

Lemma 2.12. *Let P be a convex lattice polygon with primitive vertices v_1, \dots, v_n written in counterclockwise order. Then, P contain the origin as an interior point if and only if $\text{ord}(v_i, v_{i+1}) > 0$ for all $i = 1, \dots, n$.*

Proof. If P contain the origin as an interior point, then P is a Fano polygon. By the choice of the orientation of vertices, we have $\text{ord}(v_i, v_{i+1}) > 0$ for every i . Conversely, suppose that $\text{ord}(v_i, v_{i+1}) > 0$ for all $i = 1, \dots, n$. Assume that P does not contain the origin as an interior point. Take a face F of P that is closest to the origin. Then, F has two vertices v_{i-1} and v_i of P . Since v_i is a primitive point, we can map v_i to $(0, 1)$ by an orientation-preserving unimodular transformation. Then, $v_{i-1} = (a, -b)$ and $v_{i+1} = (-c, d)$ for some positive integers a, b, c and d . Since the origin is not an interior point of P , v_{i+1} is above the line generated by v_{i-1} and v_i . Indeed, if otherwise, it is easy to see that there exists another face of P that is closer to the origin. Now, the three vertices v_{i-1}, v_i, v_{i+1} are in clockwise order, which is a contradiction. \square

Let P be a Fano polygon with vertices v_1, \dots, v_n written in counterclockwise order. Then, for each i with $1 \leq i \leq n$, we consider the following function

$$g(i) := \text{ord}(v_{i-1}, v_i) + \text{ord}(v_i, v_{i+1}) - \text{ord}(v_{i+1}, v_{i-1}).$$

Proposition 2.13. *Let P be a Fano polygon with vertices v_1, \dots, v_n written in counterclockwise order. If $g(i) > 0$ for all $i = 1, \dots, n$, then P is of type B_1 .*

Proof. It is enough to show that $B(P)$ contains the origin as an interior point. Since $g(i) > 0$ for all $i = 1, \dots, n$, it is easy to see that

$$g(i) = \text{ord}(v_{i-1}, v_i) + \text{ord}(v_i, v_{i+1}) - \text{ord}(v_{i+1}, v_{i-1}) = \text{ord}(v_{i-1} + v_i, v_i + v_{i+1}) > 0$$

Note that $\text{ord}(v_{i-1} + v_i, v_i + v_{i+1}) > 0$ if and only if $\text{ord}(\frac{v_{i-1} + v_i}{I_p(v_{i-1} + v_i)}, \frac{v_i + v_{i+1}}{I_p(v_i + v_{i+1})}) > 0$. Then, the origin is an interior point of $B(P)$ by Lemma 2.12. \square

Corollary 2.14. *Let P be a Fano triangle with $\text{ord}(P) = \{a, b, c\}$ with $a > b > c > 0$. Then, P is of type B_1 if $a < b + c$.*

Remark 2.15. If the number of vertices of P is equal to that of $B(P)$, then

$$\text{ord}(B(P)) = \left\{ \frac{g(i)}{I_p(v_{i-1} + v_i)I_p(v_i + v_{i+1})} \right\}.$$

3. SYMMETRIC OR KÄHLER-EINSTEIN FANO POLYTOPES UNDER B-TRANSFORMATION

3.1. Smooth case. We shall prove Theorem 1.3 and Theorem 1.4.

We deal with Fano polytopes for each fixed dimension. Note that there is only one Fano polytope $P = [-1, 1]$ of dimension one, whose associated toric variety is the projective line \mathbb{P}^1 , which is Kähler-Einstein. Note that P is B-invariant, hence it is of type B_∞ .

There are 5 smooth Fano polygons. (See [N1] and [KN].) It turns out that they are all of type B_∞ . More precisely, \mathbb{P}^2 is B-invariant and the other four smooth Fano polygons are 2-periodic. Among them, the three Fano polygons corresponding to the projective plane \mathbb{P}^2 , the Hirzebruch surface \mathbb{F}_0 of degree 0 and the del Pezzo surface of degree 5 are Kähler-Einstein.

Recall that smooth Fano polytopes of dimension at most 6 are completely classified in [GRDB] based on the algorithm in [O]. From this, we have a list of smooth Fano polytopes of dimension at most 6. By the help of computer (e.g., [Sa]), one can compute the number of smooth Fano polytopes of fixed dimension which is of strict type B_k for each given k . In particular, there are 18 smooth Fano 3-polytopes and we have the following table.

Strict type	B_0	B_1	B_2	B_3	B_4	B_∞	Total	KE
Number	2	3	7	0	1	5	18	5

We emphasize that the 5 Fano 3-polytopes of type B_∞ are precisely the 5 Kähler-Einstein smooth Fano 3-polytopes, and they are all 2-periodic, hence it is of type B_∞ . Indeed, for a smooth Fano 3-polytope P of type B_5 , we have $B^2(P) = P$ except for one case (GRDB ID = 18) in which case we have $B^3(P) = B(P)$.

Similarly, we compute the number of smooth Fano 4-polytopes of strict type B_k for each given k . Among 124 smooth Fano 4-polytopes, there are 14 Fano 4-polytope P of type B_4 , 11 of them being 2-periodic, hence of type B_∞ . The remaining three cases (GRDB ID: 53,54,55) are at least of type B_{20} . We expect that those three cases are of type B_∞ but we do not know the proof.

Strict type	B_0	B_1	B_2	B_3	B_4	\dots	B_{19}	B_{20}	B_∞	Total	KE
Number	28	33	47	2	0	\dots	0	≤ 3	≥ 11	124	12

There are precisely 12 smooth Kähler-Einstein Fano 4-polytopes where 11 of them are of B_∞ and the other one P has GRDB ID 54. We do not know whether P is periodic. But it seems that P is pseudo-periodic, which is not the case for the other two Fano polytopes (GRDB ID:53,55) of type B_{20} .

Question 3.1. Let P be a smooth 4-Fano polytope of type B_∞ . If P is pseudo-periodic, then is P Kähler-Einstein?

There are 866 smooth Fano 5-polytopes. We can similarly compute the number of smooth Fano 5-polytopes of strict type B_k for each given k . It is worthwhile to note that the 23 Fano polytopes of type B_∞ are precisely the 23 Kähler-Einstein smooth Fano 5-polytopes.

Strict type	B_0	B_1	B_2	B_3	B_4	B_∞	Total	KE
Number	342	278	215	4	4	23	866	23

There are 7622 smooth Fano 6-polytopes. In this case, there are 88 smooth Fano 6-polytopes of type B_3 , but we did not completely determine how many of them

are of strict type. See Remark 3.2. The 23 Fano 6-polytopes of type B_∞ are all 2-periodic. In fact, each P of them satisfies $B^2(P) = B^4(P)$.

Strict type	B_0	B_1	B_2	B_3	B_∞	Total	KE
Number	3884	2510	1140	≤ 65	≥ 23	7622	51

Remark 3.2. For a smooth Fano 6-polytope P , $B^3(P)$ has a huge number of vertices and facets. For example, the Fano 6-polytope P with GRDB ID 1787 has 12 vertices and 80 facets and is of type B_3 , but $B^3(P)$ has 2772 vertices and 1614 facets. According to our estimation, it takes more than one month to precisely determine the number of smooth Fano 6-polytopes of strict type B_3 , so we have stopped the computation.

Now, the following proposition will complete the proof of Theorem 1.3.

Proposition 3.3. Q_1 , Q_2 and Q_3 in Theorem 1.2 are all of type B_∞ .

Proof. By computation, we see that $B^2(Q_1) = B^4(Q_1)$, $B^2(Q_2) = B^4(Q_2)$ and $B^2(Q_3) = B^4(Q_3)$, so Q_1 , Q_2 and Q_3 are all periodic, hence they are of type B_∞ . \square

Since we have computed $B(P)$ for each Kähler-Einstein Fano polytope P of dimension at most 6 or Q_1 , Q_2 and Q_3 in Theorem 1.2, it is easy to compute the barycenter of $B(P)$. This proves Theorem 1.4.

3.2. Singular case. We shall prove Theorem 1.6.

3.2.1. Symmetric Fano polygons. Consider the Fano polygon

$$S_{m,n} = \text{conv}\{(m+1, -m), (-m, m+1), (-n-1, n), (n, -n-1)\},$$

which is symmetric by [HK, Proposition 3.2].

Lemma 3.4. For every non-negative integers m and n , the symmetric Fano polygon $S_{m,n}$ is of type B_∞ . Moreover, $B^k(S_{m,n})$ is Kähler-Einstein for every $k \geq 1$.

Proof. For any positive integer t , we have

$$B^{2t-1}(S_{m,n}) = \text{conv}\{(-1, 1), (-1, -1), (1, -1), (1, 1)\}$$

and

$$B^{2t}(S_{m,n}) = \text{conv}\{(1, 0), (0, 1), (-1, 0), (0, -1)\} = S_{0,0}.$$

Both of them are Kähler-Einstein, from which the result follows. \square

Remark 3.5. Note that $B(S_{m,n})$ always admits a rotation even though $S_{m,n}$ may not have one. It is easy to see that $B^t(S_{m,n})$ admits a reflection for every positive integer t .

Lemma 3.6. Let P be a Fano polygon of type B_1 . If P admits a non-trivial automorphism σ , then $I(v_i + v_{i+1}) = I(\sigma(v_i) + \sigma(v_{i+1}))$ for all adjacent vertices v_i and v_{i+1} of P .

Proof. Since the lattice point $\frac{v_i + v_{i+1}}{I(v_i + v_{i+1})}$ is primitive, so is

$$\sigma\left(\frac{v_i + v_{i+1}}{I(v_i + v_{i+1})}\right) = \frac{\sigma(v_i) + \sigma(v_{i+1})}{I(v_i + v_{i+1})}$$

by Lemma [HK, Lemma 2.2], from which the result follows. \square

Proposition 3.7. *Let P be a symmetric Fano polygon of type B_1 . If P admits a non-trivial rotation, then so does $B(P)$.*

Proof. Let v_1, \dots, v_n be all vertices of P written in counterclockwise order and σ be a non-trivial rotation of P . By Lemma 3.6, $I_p(v_i + v_{i+1}) = I_p(\sigma(v_i) + \sigma(v_{i+1}))$ for all adjacent vertices v_i and v_{i+1} of P . Let

$$w_i = \frac{v_i + v_{i+1}}{I_p(v_i + v_{i+1})}$$

for every $i = 1, \dots, n$ and $W = \{w_1, \dots, w_n\}$ be an ordered set. Then, $\sigma(W) = W$ and $B(P) = \text{conv}(W)$. Note that a point of W need not be a vertex of $B(P)$ anymore in general. It is enough to show that σ preserves the vertex set of $B(P)$. Again, it is enough to show that if w_i is not a vertex of $B(P)$ then so is not $\sigma(w_i)$.

Suppose that w_i is not a vertex of $B(P)$. Let w_j and w_k be the vertices of $B(P)$ closest to w_i . Define

$$f(w_i) = \det(w_j, w_i) + \det(w_i, w_k) + \det(w_k, w_j)$$

where $j < i < k$. Then, $f(w_i) < 0$. Since σ is a rotation,

$$f(\sigma(w_i)) = \det(\sigma(w_j), \sigma(w_i)) + \det(\sigma(w_i), \sigma(w_k)) + \det(\sigma(w_k), \sigma(w_j)) < 0.$$

Thus, by [Su, Proposition 5], $\sigma(w_i)$ is not a vertex of W . \square

Corollary 3.8. *If a matrix σ induces an automorphism of a symmetric Fano polygon P , then it also induces an automorphism of $B(P)$.*

Proof. The result follows from Remark 3.5 and Proposition 3.7 by [HK, Theorem 3.3]. \square

3.2.2. *Kähler-Einstein Fano polygons.* We first consider the triangle case.

Theorem 3.9. *If P is a Kähler-Einstein Fano triangle, then P is of type B_∞ . Moreover, $B^{2s+1}(P) = -P$ and $B^{2s}(P) = P$ for all non-negative integer s . Hence, $B^s(P)$ is Kähler-Einstein for every non-negative integer s .*

Proof. By Proposition [HK, Proposition 3.10], we may assume that

$$P = \text{conv}\{(a, -b), (0, 1), (-a, b-1)\}.$$

Then, it is easy to see that $B(P) = \text{conv}\{(a, -b+1), (-a, b), (0, -1)\} = -P$. The rest follows immediately. \square

Example 3.10. The converse of Theorem 3.9 does not hold in general i.e., there exists a Fano triangle of type B_∞ that is not Kähler-Einstein. Let

$$P = \text{conv}\{(-25, -12), (-5, -6), (25, 14)\}.$$

Then, it is easy to see that P is not Kähler-Einstein. However, since

$$B(P) = \text{conv}\{(-5, -3), (5, 2), (0, 1)\}$$

is Kähler-Einstein, by Theorem 3.9, $B(P)$ is of type B_∞ , thus so is P .

We need the following technical lemma to prove Theorem 3.12.

Lemma 3.11. *Let P be a Fano triangle with vertices $(a, -b), (0, 1), (-c, d)$ written in counterclockwise order where a, b, c, d are positive integers. Let Q be the convex hull generated by P and v be a primitive lattice point that lies below the line spanned by $(a, -b)$ and $(c, -d)$. Then, we have the following.*

- (1) The dual triangle P^* of P has vertices w_1, w_2, w_3 written in counterclockwise order such that w_1 and w_2 are strictly below the x -axis and w_3 is strictly above the x -axis.
- (2) The dual polygon Q^* of Q has vertices w_1, w_2, w'_3, w'_4 written in counterclockwise order such that w'_3 lies on the line segment connecting w_2 and w_3 ; and w'_4 lies on the line segment connecting w_1 and w_3 .

Proof. Write $v_1 = (a, -b)$, $v_2 = (0, 1)$, $v_3 = (-c, d)$, and $v = (x, y)$. Then, we can compute that

$$P^* = \text{conv}\left\{\left(\frac{-b-1}{a}, -1\right), \left(\frac{1-d}{c}, -1\right), \left(\frac{b+d}{bc-ad}, \frac{a+c}{bc-ad}\right)\right\}$$

and

$$Q^* = \text{conv}\left\{\left(-\frac{b+1}{a}, -1\right), \left(\frac{1-d}{c}, -1\right), \left(\frac{y-d}{cy+dx}, -\frac{c+x}{cy+dx}\right), \left(-\frac{b+y}{bx+ay}, \frac{x-a}{bx+ay}\right)\right\}.$$

Take $w_1 = \left(-\frac{b-1}{a}, -1\right)$, $w_2 = \left(\frac{1-d}{c}, -1\right)$, $w_3 = \left(\frac{b+d}{bc-ad}, \frac{a+c}{bc-ad}\right)$, $w'_3 = \left(\frac{y-d}{cy+dx}, -\frac{c+x}{cy+dx}\right)$ and $w'_4 = \left(-\frac{b+y}{bx+ay}, \frac{x-a}{bx+ay}\right)$. Since v_1, v_2, v_3 are in counterclockwise order, we have $bc - ad > 0$. This proves (1).

Since Q is a Fano polygon,

$$\text{ord}(v_3, v) + \text{ord}(v, v_1) + \text{ord}(v_1, v_3) = -cy - dx - bx - ay - (bc - ad) > 0.$$

Thus,

$$\text{ord}(w'_3, w_3) = \frac{-cy - dx - bx - ay - (bc - ad)}{(-cy - dx)(bc - ad)} > 0$$

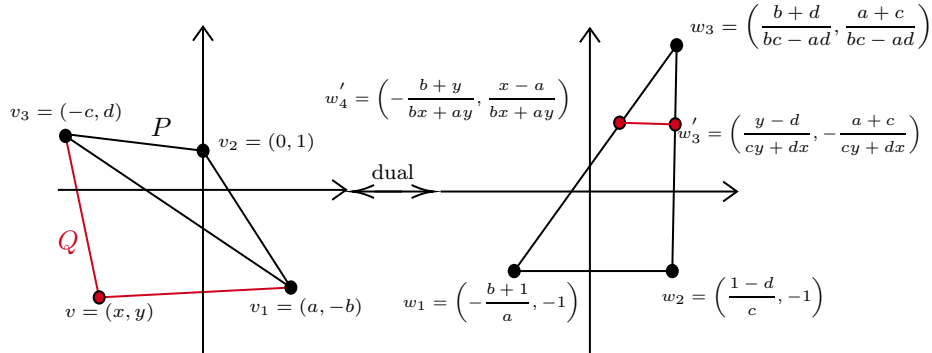
and

$$\text{ord}(w_3, w'_4) = \frac{-cy - dx - bx - ay - (bc - ad)}{(bc - ad)(-bx - ay)} > 0.$$

So, w'_3, w_3, w'_4 are written in counterclockwise order. Let $S(w_i, w_j)$ be the slope between w_i and w_j . Then,

$$S(w_2, w_3) = S(w_2, w'_3) = \frac{c}{d} \quad \text{and} \quad S(w_1, w'_4) = S(w_1, w_3) = \frac{a}{b}.$$

Thus, we see that w'_3 lies on the line segment connecting w_2 and w_3 , and w'_4 lies on the line segment connecting w_1 and w_3 .



□

It follows from Lemma 3.11 that the y -coordinate of the barycenter of P' is less than that of P since w_1 and w_2 does not depend on the choice of v .

Theorem 3.12. *If P is a Kähler-Einstein Fano polygon, then P is of type B_1 .*

Proof. Let P be a Kähler-Einstein Fano polygon. Assume that P is not of type B_1 . Then, by Proposition 2.13, there exists a vertex v_i of P such that $g(i) \leq 0$. Up to a unimodular transformation, we may assume that $v_{i-1} = (a, -b)$, $v_i = (0, 1)$ and $v_{i+1} = (-c, d)$ for some positive integers a, b, c, d where v_{i-1}, v_i, v_{i+1} are adjacent vertices of P in counterclockwise order.

Let $P' = \text{conv}\{v_{i-1}, v_i, v_{i+1}\}$ and $(P')^*$ be its dual polygon. Note that the y -coordinate $y_{(P')^*}$ of the barycenter of $(P')^*$ is

$$y_{(P')^*} = \frac{1}{3} \left(\frac{a+c}{bc-ad} - 2 \right) = \frac{(a+c-(bc-ad)) - (bc-ad)}{3(bc-ad)}.$$

Since $g(i) = a+c-(bc-ad) \leq 0$, we have $y_{(P')^*} < 0$. By Lemma 3.11, if P has another vertex v other than v_{i-1}, v_i and v_{i+1} , then $y_{P^*} \leq y_{(P')^*} < 0$. So, the barycenter of P^* is not the origin. Thus, P is not Kähler-Einstein, a contradiction. \square

Theorem 3.13. *Let P be a Fano polygon. If P is symmetric, then P is of type B_∞ and $B(P)$ is both symmetric and Kähler-Einstein.*

Proof. Let P be a symmetric Fano polygon. If P does not admit a rotation, then $P = S_{m,n}$ with $m \neq n$ by [HK, Theorem 3.3], thus the result follows from Lemma 3.4. Now, we may assume that P admits a rotation. Then, P is Kähler-Einstein, and hence it is of type B_1 by Theorem 3.12. By Proposition 3.7, $B(P)$ has an induced non-trivial rotation. Thus, $B(P)$ is also a symmetric Fano polygon of type B_1 . This argument shows that P is of type B_∞ . Moreover, $B^s(P)$ is Kähler-Einstein for every $s \geq 1$ by [HK, Corollary 3.5]. \square

Now, Theorem 3.12, Theorem 3.9 and Theorem 3.13 prove Theorem 1.6.

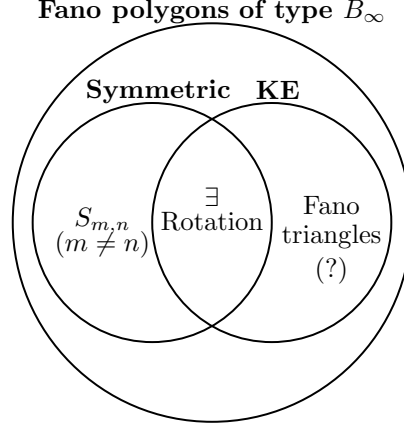


FIGURE 1. Symmetric/Kähler-Einstein polygons are of type B_∞ assuming Conjecture 1.7.

Question 3.14. Let P be a Kähler-Einstein Fano polygon. Is $B(P)$ Kähler-Einstein?

If Question 3.14 is true, then Conjecture 1.5 is true for surfaces by Theorem 1.6. Figure 1 depicts Conjecture 1.5 for surfaces.

3.2.3. *Fano polygons of index at most 17.* We determine the number of Fano polygons of given strict type B_k using the database [GRDB] based on [KKN] as we did in the smooth case.

Index	B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_∞	Total	KE
1	3	0	0	1	0	0	0	0	0	12	16	5
2	11	6	0	2	3	0	0	0	0	8	30	1
3	35	20	10	2	4	2	0	0	0	26	99	4
4	35	25	10	2	3	0	0	0	0	16	91	2
5	100	63	35	10	6	2	0	0	0	34	250	7
6	126	93	49	25	8	8	1	0	0	69	379	4
7	186	100	73	9	7	1	0	0	0	53	429	10
8	145	78	41	3	3	2	0	0	0	35	307	3
9	319	165	112	14	5	5	0	0	0	70	690	7
10	384	262	142	39	15	3	1	0	1	69	916	4
11	472	190	155	28	6	0	0	0	0	88	939	12
12	535	325	185	74	26	9	2	0	0	123	1279	6
13	563	227	205	27	11	2	0	0	0	107	1142	17
14	725	423	261	34	20	2	1	0	0	79	1545	4
15	1711	1119	784	305	129	24	5	1	0	234	4312	14
16	564	237	137	9	9	5	0	0	0	69	1030	5
17	1007	353	330	53	9	1	0	0	0	139	1892	19

A direct computation based on the above proves Conjecture 1.7 and Conjecture 1.5 for Fano polygons of index at most 17.

Theorem 3.15. *Let P be a Kähler-Einstein Fano polygon of index at most 17. Then,*

- (1) *P is of type B_∞ and $B(P)$ is again Kähler-Einstein.*
- (2) *If P is not symmetric, then P is a triangle.*

4. FURTHER DISCUSSIONS

In this section, we discuss the number of Fano polytopes that are obtained by a sequence of B-transformations of a given Fano polytope, and study the relations between the classes in Figure 3.2.2 and the class of Fano polygons with barycenter zero.

4.1. Orbits of a Fano polytope under B-transformation.

Definition 4.1. Let P be a Fano polytope. Then, a Fano polytope Q is called an *orbit* of P under B-transformation if it is obtained by applying a sequence of B-transformations of P , i.e., $Q = B^n(P)$ for some non-negative integer n .

Theorem 4.2. *Let P be a symmetric Fano polygon that is not Kähler-Einstein or a Kähler-Einstein Fano triangle. Then, the number of orbits of P under B-transformation is at most two.*

Proof. It immediately follows from the proof of Lemma 3.4 and Theorem 3.9. \square

Question 4.3. Let P be a Fano polygon of type B_∞ . Then, is the number of orbits of P under B-transformation at most two?

Conjecture 1.7 gives a positive answer to Question 4.3.

4.2. Fano polygons with barycenter zero. We discuss the generalization of the following proposition for Kähler-Einstein Fano triangles to arbitrary Kähler-Einstein Fano polygons.

Proposition 4.4. [HK, Proposition 3.10] *Let P be a Kähler-Einstein Fano triangle. Then, P has the origin as its barycenter.*

Theorem 4.5. *Let P be a Fano polygon. If P admits a non-trivial rotation, then its barycenter is the origin.*

Proof. Let v be the barycenter of P . Since the set of vertices of P is invariant under the action of rotations, it follows that $\sigma(v) = v$. But, since the origin is the only fixed point of a rotation, we conclude that v is the origin. \square

Remark 4.6. Theorem 4.5 does not hold if P admits no rotation. Let $P = S_{m,n}$ for some non-negative integers m and n . Then, the barycenter of P is

$$\left(\frac{m-n}{6(m+n+1)}, \frac{m-n}{6(m+n+1)} \right),$$

which is not the origin if $m \neq n$. Recall that $S_{m,n}$ is symmetric of type B_∞ by Lemma 3.4.

Theorem 4.7. *Assume that Conjecture 1.7 holds. Then, every Kähler-Einstein Fano polygon has the origin as its barycenter.*

Proof. If P has a non-trivial rotation, then P is symmetric, so the barycenter of P is the origin by Theorem 4.5. If otherwise, P is a Kähler-Einstein Fano triangle by Conjecture 1.7. Now the result immediately follows from [HK, Proposition 3.10]. \square

Question 4.8. Is it possible to remove the assumption in Theorem 4.7?

Thanks to [GRDB] based on [KKN], one can check that Kähler-Einstein Fano polygons are precisely the Fano polygons with barycenter zero if the index is at most 17. But there exist examples of Fano polygons with barycenter zero that are not Kähler-Einstein in higher index.

Example 4.9. The following examples have the origin as their barycenters but are not Kähler-Einstein.

- (1) Let $P_1 = \text{conv}\{(-2, -1), (-1, 3), (1, 2), (2, -3)\}$. Then, the barycenter of P_1 is the origin but P_1 is not Kähler-Einstein. Moreover, P_1 is of type B_∞ and the index of P_1 is 280.
- (2) Let $P_2 = \text{conv}\{(-5, -4), (-5, 8), (5, 1), (8, -5)\}$. Then, the barycenter of P_2 is the origin but P_2 is not Kähler-Einstein. Moreover,

$$B^5(P_2) = \text{conv}\{(-1, 0), (1, -1), (5, -3)\}, \text{ and}$$

$$B^6(P_2) = \text{conv}\{(0, -1), (3, -2), (4, -3)\}.$$

Thus, P_2 is of type B_5 . Note that the index of P_2 is 270180.

Remark 4.10. The Fano polygon P in Example 2.8 supports the example of a Fano polygon of type B_∞ with nonzero barycenter that is neither symmetric nor Kähler-Einstein. The Fano polygon P_1 in Example 4.9 (1) supports the example of a Fano polygon of type B_∞ with barycenter zero that is neither symmetric nor Kähler-Einstein. The Fano polygon P_2 in Example 4.9 (2) supports the example

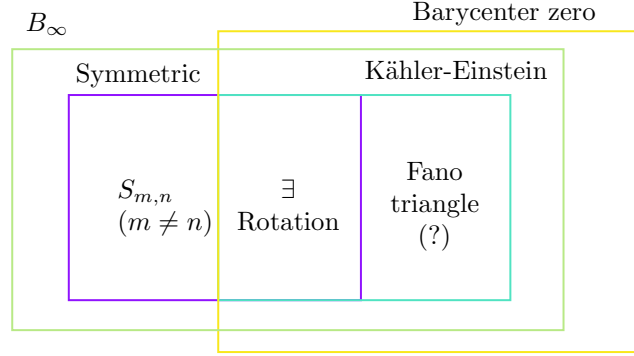


FIGURE 2. Hierarchy of Fano polygons assuming Conjecture 1.7

of a Fano polygon, not of type B_∞ , with barycenter zero that is neither symmetric nor Kähler-Einstein.

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REFERENCES

- [BB] R. J. Berman and B. Berndtsson, *Real Monge-Ampère equations and Kähler-Ricci solitons on toric log Fano varieties*, Ann. Fac. Sci. Toulouse Math. (6) **22** (2013), no. 4, 649–711.
- [BS] V. V. Batyrev and E. N. Selivanova, *Einstein-Kähler Metrics on Symmetric Toric Fano Manifolds*, J. Reine Angew. Math., **512** (1999), 225–236.
- [CL] K. Chan, N.C. Leung, *Miyaoka-Yau-type inequalities for Kähler-Einstein manifolds*, Commun. Anal. Geom. **15** (2007), 359–379.
- [CLS] D. Cox, J. Little, and H. Schenck, *Toric Varieties*, (2011), 379–386.
- [FOS] A. Futaki, H. Ono and Y. Sano, *Hilbert series and obstructions to asymptotic semistability*, Adv. in Math., **226** (2011), 254–284.
- [GRDB] Graded Ring Database, <http://www.grdb.co.uk/>
- [HK] D. Hwang and Y. Kim, *Symmetric and Kähler-Einstein toric log del Pezzo surfaces*, preprint, submitted for publication.
- [KKN] A. M. Kasprzyk, M. Kreuzer and B. Nill, *On the combinatorial classification of toric log del Pezzo surfaces*, LMS Journal of Computation and Mathematics **13** (2010), 33–46.
- [KN] A. M. Kasprzyk, B. Nill, *Fano polytopes*, Strings, gauge fields, and the geometry behind, 349–364, World Sci. Publ., Hackensack, NJ, 2013.
- [N1] Y. Nakagawa, *Combinatorial Formulae for Futaki characters and generalized Killing forms of toric Fano orbifolds*, The Third Pacific Rim Geometry Conference (Seoul, 1996), 223–260, Monogr. Geom. Topology, 25, Int. Press, Cambridge, MA, 1998.
- [N2] Y. Nakagawa, *On the examples of Nill and Paffenholz*, Int. J. Math., **26** (2015), 1540007 (15 pages).
- [NP] B. Nill and A. Paffenholz, *Examples of Kähler-Einstein toric Fano manifolds associated to non-symmetric reflexive polytopes*, Beitr. Algebra Geom. **52** (2011), 297–304.
- [O] M. Øbro, *An algorithm for the classification of smooth Fano polytopes*, arXiv:0704.0049.
- [Sa] SageMath, the Sage Mathematics Software System (Version 9.1), The Sage Developers, 2020, <https://www.sagemath.org>.
- [So] J. Song, *The α -invariant on toric Fano manifolds*, Am. J. Math. **127** (2005), 1247–1259.
- [Su] Y. Suyama, *Classification of Toric log del Pezzo surfaces with few singular points*, arXiv:1910.00206v1.

DEPARTMENT OF MATHEMATICS, AJOU UNIVERSITY, SUWON 16499, REPUBLIC OF KOREA

Email address: `dshwang@ajou.ac.kr`

Email address: `kys3129@ajou.ac.kr`