

Flexibility of planar graphs without C_4 and C_5 ^{*}

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Abstract

Let G be a $\{C_4, C_5\}$ -free planar graph with a list assignment L . Suppose a preferred color is given for some of the vertices. We prove that if all lists have size at least four, then there exists an L -coloring respecting at least a constant fraction of the preferences.

Key words: Planar graph, List coloring, Flexibility.

1 Introduction

In what follows, all graphs considered are simple, finite, and undirected, and we follow [1] for the terminologies and notation not defined here. A plane graph is a particular drawing of a planar graph in the Euclidean plane. Given a plane graph G , we denote its vertex set, edge set, face set and minimum degree by $V(G)$, $E(G)$, $F(G)$ and $\delta(G)$, respectively. The degree $d(v)$ is the number of edges incident with v . A vertex v is called a k -vertex (k^+ -vertex, or k^- -vertex) if $d(v) = k$ ($d(v) \geq k$, or $d(v) \leq k$, resp.). For any face $f \in F(G)$, the degree of f , denoted by $d(f)$, is the length of the shortest boundary walk of f , where each cut edge is counted twice. Analogously, a k -face (k^+ -face, or k^- -face) is a face of degree k (at least k , or at most k , resp.). We write $f = u_1u_2\dots u_nu_1$ if u_1, u_2, \dots, u_n are the boundary vertices of f in the clockwise order, and for integers d_1, \dots, d_n , we say that f is a (d_1, \dots, d_n) -face if $d(u_i) = d_i$ for all $i \in \{1, 2, \dots, n\}$. We say that f is a (d_1^+, \dots, d_n) -face if $d(v_1) \geq d_1$ and $d(v_i) = d_i$ for all $i \in \{2, \dots, n\}$; and similarly for other combinations. Let $f_k(v)$, $n_k(v)$ and $n_k(f)$ denote the number of k -faces incident with the vertex v , the number of k -vertices adjacent to the vertex v , and the number of k -vertices incident with the face f , respectively. Moreover, we use $\delta(f)$ to refer to the minimum degree of vertices incident with f .

A *list assignment* L for a graph G is a function that to each vertex $v \in V(G)$ assigns a set $L(v)$ of colors, and an *L -coloring* is a proper coloring ϕ such that $\phi(v) \in L(v)$ for

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all $v \in V(G)$. If G has an L -coloring, then we say that G is *L -colorable*. Initiated by Dvořák, Norin, Postle [4], a *request* for a graph G with a list assignment L is a function r with $\text{dom}(r) \subseteq V(G)$ such that $r(v) \in L(v)$ for all $v \in \text{dom}(r)$. For $\varepsilon > 0$, a request r is ε -*satisfiable* if there exists an L -coloring ϕ of G such that $\phi(v) = r(v)$ for at least $\varepsilon|\text{dom}(r)|$ vertices $v \in \text{dom}(r)$. We say that a graph G with the list assignment L is ε -*flexible* if every request is ε -satisfiable. Furthermore, we emphasize a stronger weighted form. A *weighted request* is a function w that to each pair (v, c) with $v \in V(G)$ and $c \in L(v)$ assigns a nonnegative real number. Let $w(G, L) = \sum_{v \in V(G), c \in L(v)} w(v, c)$. For $\varepsilon > 0$, we say that w is ε -*satisfiable* if there exists an L -coloring ϕ of G such that

$$\sum_{v \in V(G)} w(v, \phi(v)) \geq \varepsilon w(G, L).$$

We say that G with the list assignment L is *weighted ε -flexible* if every weighted request is ε -satisfiable.

It is worth pointing out that a request r is 1-satisfiable if and only if the precoloring given by r can be extended to an L -coloring of G . It is easy to deduce that weighted ε -flexibility implies ε -flexibility when we set there is only at most one color is requested at each vertex of G and all such colors have the same weight 1.

Dvořák, Norin and Postle [4] proposed this topic and studied some natural questions in the context. They obtained some elementary results as follows, we say a list assignment L is an f -assignment if $|L(v)| \geq f(v)$ for all $v \in V(H)$.

- There exists $\varepsilon > 0$ such that every planar graph with a 6-assignment is ε -flexible.
- There exists $\varepsilon > 0$ such that every planar graph of girth at least five with a 4-assignment is ε -flexible.
- For every integer $d \geq 2$, there exists $\varepsilon > 0$ such that every graph of maximum average degree at most d and choosability at most $d - 1$ with a $(d + 2)$ -assignment is weighted ε -flexible.

Especially, in [2], Dvořák et al. studied the 4-weighted flexibility of triangle-free planar graphs, and the result coincides with the choosability of triangle-free planar graphs. Later on, an interesting question is raised by Masařík [6] in the following.

Question 1. *Does there exist $\varepsilon > 0$ such that every C_4 -free planar graph G with a 4-assignment is weighted ε -flexible?*

If it is true, this would be optimal in terms of choosability [5]. However, it might be difficult to obtain such a result since even the procedure for getting the corresponding result of triangle-free is very involved. So far, triangle-free planar graphs are the only known result for ε -weighted flexibility with 4-assignment.

Based on the above results, we proceed a step towards Question 1 by proving the following theorem.

Theorem 1. *There exists $\varepsilon > 0$ such that every $\{C_4, C_5\}$ -free planar graph with a 4-assignment is weighted ε -flexible.*

In 2007, Voigt [9] proved that there exists $\{C_4, C_5\}$ -free planar graphs which are not 3-choosable. Hence, our result is the best possible up to the list size.

The rest of the paper is organized as follows. In order to prove Theorem 1, in Section 2, we introduce some basic notation and essential tools used in list coloring settings. Using discharging method, we first find some necessary reducible configurations in Section 3, and then present our discharging rules and final analysis in Section 4.

2 Preliminaries

We use P_k to denote a path of order k . Let H be a graph, $S \subseteq V(H)$, S is called a $(P_3 + P_4)$ -independent set if H contains neither P_3 nor P_4 connecting two vertices in S . Let 1_S denote the characteristic function of S , i.e., $1_S(v) = 1$ if $v \in S$ and $1_S(v) = 0$ otherwise. For functions that assign integers to vertices of H , we define addition and subtraction in the natural way, adding/subtracting their values at each vertex independently. For a function $f : V(H) \rightarrow \mathbb{Z}$ and a vertex $v \in V(H)$, let $f \downarrow v$ denote the function such that $(f \downarrow v)(w) = f(w)$ for $w \neq v$ and $(f \downarrow v)v = 1$.

Let G be a graph and H be an induced subgraph of another graph G . For an integer $k \geq 3$, let $\delta_{G,k} : V(H) \rightarrow \mathbb{Z}$ be defined by $\delta_{G,k}(v) = k - \deg_G(v)$ for each $v \in V(H)$. Then H is said to be a $(P_3 + P_4, k)$ -reducible induced subgraph of G if

(FIX) for every $v \in V(H)$, H is L -colorable for every $((\deg_H + \delta_{G,k}) \downarrow v)$ -assignment L , and

(FORB) for every $(P_3 + P_4, k)$ -independent set S in H of size at most $k-2$, H is L -colorable for every $(\deg_H + \delta_{G,k} - 1_S)$ -assignment L .

Note that **(FORB)** implies that $\deg_H(v) + \delta_{G,k}(v) \geq 2$ for all $v \in V(H)$.

To prove weighted ε -flexibility, we use the following observation made by Dvořák et al.[4]

Lemma 2.1 ([4]). *Let G be a graph and let L be a list assignment for G . Suppose G is L -colorable and there exists a probability distribution on L -colorings φ of G such that for every $v \in V(G)$ and $c \in L(v)$, $\Pr[\varphi(v) = c] \geq \varepsilon$. Then G with L is weighted ε -flexible.*

Moreover, we use the following well-known result due to Thomassen [7].

Lemma 2.2 ([7]). *Let G be a connected graph and L a list assignment such that $|L(u)| \geq \deg(u)$ for all $u \in V(G)$. If either there exists a vertex $u \in V(G)$ such that $|L(u)| > \deg(u)$, or some 2-connected component of G is neither complete nor an odd cycle, then G is L -colorable.*

The following lemma provides an essential technique to deal with the weighted flexibility of graphs.

Lemma 2.3. *For all integers $k \geq 3$, $b \geq 1$, there exists a positive constant ε such that the following holds. Let G be a $\{C_4, C_5\}$ -free graph. If for every $Z \subseteq V(G)$, the graph $G[Z]$ contains an induced $(P_3 + P_4)$ -reducible subgraph with at most b vertices, then G with any assignment of list of size k is weighted ε -flexible.*

Proof. Before proving the Lemma, we give the following claim at first.

Claim 2.4. *For every integer $k \geq 3$, there exist $\varepsilon, \delta > 0$ as follows. Let G be a $\{C_4, C_5\}$ -free graph and L be an assignment of lists of size k to vertices of G . Then there exists a probability distribution on L -colorings ϕ of G such that*

- (i) *for all $v \in V(G)$ and $c \in L(v)$, we have $\Pr[\phi(v) = c] \geq \varepsilon$, and*
- (ii) *for any $P_3 + P_4$ -independent subset S with $|S| \leq k - 2$, $\Pr[\phi(v) \neq c \text{ for all } v \in S] \geq \delta^{|S|}$.*

Proof. Let $\delta = (\frac{1}{k})^b$, $\varepsilon = (\frac{1}{k})^{b+k-2}$. Assume G is a $\{C_4, C_5\}$ -free graph, $Z \subseteq V(G)$, there exists $Y \subseteq Z$ of size at most b such that $G[Y] \subseteq G[Z]$ and $G[Y]$ is $(P_3 + P_4, k)$ -reducible. We prove the claim by induction on $|V(G)|$. A random L -coloring ϕ of G is chosen as follows: we choose an L -coloring ϕ_1 of $G - Y$ at random from the probability distribution obtained by the induction hypothesis. Let L' be the list assignment for $G[Y]$ defined by

$$L'(v) = L(v) \setminus \{\phi_1(u) : uv \in E(G), u \notin Y\}$$

for all $v \in Y$. Note that $|L'(v)| \geq \deg_{G[Y]}(v) + \delta_{G,k}(v)$ for all $v \in Y$, and thus $G[Y]$ has an L' -coloring by **(FORB)** applied with $S = \emptyset$. We choose an L' -coloring ϕ_0 uniformly at random among all L' -colorings of $G[Y]$, and let ϕ be the union of the colorings ϕ_1 and ϕ_0 . Note that $|L'(v)| \geq \deg_{G[Y]}(v) + \delta_{G,k}(v)$ for all $v \in Y$.

Firstly, we prove that (ii) holds. Let S be a (P_3+P_4) -independent subset in G , $S = S_1 \cup S_2$, where $S_1 = S \cap (G - Y)$, $S_2 = S \cap Y$. Obviously, both S_1 and S_2 are $P_3 + P_4$ -independent subsets in $G - Y$ and Y , respectively. By the induction hypothesis for $G - Y$, we obtain $\Pr[(\forall v \in S_1)\phi(v) \neq c] = \Pr[(\forall v \in S_1)\phi_1(v) \neq c] \geq \delta^{|S_1|}$. If $S_1 = S$, then (ii) holds. Therefore, suppose that $|S_1| \leq |S| - 1$. Fix ϕ_1 , then consider the probability that ϕ_0 gives all vertices of S_2 colors different from c . For $v \in S_2$, let $L_c(v) = L'(v) \setminus \{c\}$, and for $v \in Y \setminus S_2$, let $L_c(v) = L'(v)$. Note that $|L_c(v)| \geq \deg_{G[Y]}(v) + \delta_{G,k}(v) - 1_S(v)$ for all $v \in Y$. By **(FORB)**, there exists an L_c -coloring of $G[Y]$ in that no vertex of S_2 is assigned color c . Since ϕ_0 is chosen uniformly among the at most $k^b L'$ -colorings of $G[Y]$, we conclude that the probability that no vertex of S_2 is assigned color c by ϕ_0 is at least $(\frac{1}{k})^b = \delta$. Consequently, under the assumption that ϕ_1 does not assign color c to any vertex of S_1 , the probability that ϕ_0 does not assign color c to any vertex of S_2 is at least δ . Hence,

$$\begin{aligned} & \Pr[\phi(v) \neq c, \forall v \in S] \\ &= \Pr[\phi_1(u) \neq c, \forall u \in S_1; \phi_0(w) \neq c, \forall w \in S_2] \\ &\geq (\frac{1}{k})^b \delta^{|S_1|} \\ &\geq \delta^{|S_1|+1} \\ &\geq \delta^{|S|}. \end{aligned}$$

as required.

Next, we prove that (i) holds. Consider any vertex $v \in V(G)$ and a color $c \in L(v)$. If $v \in V(G - Y)$, then $\Pr[\phi(v) = c] = \Pr[\phi_1(v) = c] \geq \varepsilon$ by the induction hypothesis for $G - Y$. Therefore, we assume that $v \in Y$. Let S be the set of neighbors of v in $V(G) \setminus Y$. Since G is $\{C_4, C_5\}$ -free and all vertices in S has a common neighbor, S is $P_3 + P_4$ -independent in $G - Y$. Furthermore, **(FORB)** implies $1 \leq \deg_{G[Y]}(v) + \delta_{G,k}(v) - 1_v(v) = \deg_{G[Y]}(v) + k - \deg_G(v) - 1 = k - 1 - |S|$, thus $|S| \leq k - 2$. By (ii) we obtain that no vertex of S is assigned color c by ϕ_1 with probability at least δ^{k-2} . **(FIX)** implies that there exists an L' -coloring of $G[Y]$ in which v is assigned color c . Since ϕ_0 is chosen uniformly among all L' -colorings of $G[Y]$, it follows that under the assumption that no vertex of S is colored by c , the probability that $\phi(v) = c$ is at least δ . Therefore,

$$\begin{aligned} & \Pr[\phi(v) = c \forall v \in V(G)] \\ &= \Pr[\phi_1(w) \neq c \text{ for } \forall w \in S; \phi_0(u) = c \forall u \in Y] \\ &\geq (\frac{1}{k})^b \delta^{|S|} \\ &\geq (\frac{1}{k})^{b+k-2}. \end{aligned}$$

This completes the proof of Claim 2.4. \square

Combining Claim 2.4 with Lemma 2.1, we obtain that Lemma 2.3 holds. \square

Recall that a *block* of H is a maximal connected subgraph without a cutvertex. The following lemma is easy to obtain by induction on the number of blocks in H . So we omit the proof.

Lemma 2.5. *Let L be an f -assignment for H . If every block B of H is L' -colorable for any $f \downarrow v$ -assignment L' with any fixed vertex $v \in V(B)$, then H satisfies **(FIX)**.*

3 Reducible subgraphs

Before proceeding, we introduce the following notation. A vertex v with $4 \leq d(v) \leq 12$ is *bad* if (i) the number of $(3, 4^-, v)$ -face is $\lfloor \frac{d(v)-2}{2} \rfloor$; (ii) $f_3(v) = \lfloor \frac{d(v)}{2} \rfloor$. A vertex v with $d(v) = 4$ is *vice* if $f_3(v) = 2$. And a vertex v with $5 \leq d(v) \leq 12$ is *dangerous* if $f_{3,3}(v) = \lfloor \frac{d(v)-3}{2} \rfloor$. On the other hand, each $(3, 4, 4)$ -face is called *poor*, each $(3, 4, v)$ -face is called *worse* and each $(3, 3, v)$ -face is called *worst*. Furthermore, denote by $f_{3,3}(v)$ (or $f_{3,4}(v)$), $f_{3b}(v)$ (or $f_{4b}(v)$) the number of worst 3-face (or worse 3-face incident with v), the number of $(3, w, v)$ -face (or $(4, w, v)$ -face) in which w is a bad vertex, respectively. In particular, let $f_{bb}(v)$ be the number of (v, w_1, w_2) -face in which both w_1 and w_2 are bad vertices. For each 6^+ -face f , denote by $\xi(f)$ the number of poor 3-face sharing an edge with f . A *nice* path connecting v and some vertex $u \in V(f)$ is a path of length at most two such that either

- (i) $d(u) = 3$ and all internal vertices have degree 3 in G' ; or
- (ii) $d(u) = 4$ and all internal vertices are vice 4-vertices, moreover, every two consecutive 4-vertices in the path are contained in a $(4, 4, 4)$ -face.

Note that in all figures of the paper, any vertex marked with \bullet has no edges of G incident with it other than those shown, any vertex marked with \blacksquare is bad unless stated otherwise.

Here, we describe a quite general case of $(P_3 + P_4, 4)$ -reducible configurations. Let G be a $\{C_4, C_5\}$ -free planar graph and v a vertex of G . A v -*stalk* is one of the following subgraphs:

- (a) An edge vu_1 ;
- (b) A path vu_1u_2 ;
- (c) A cycle vu_1u_2 ;

- (d) A cycle vu_1v_1 ;
- (e) A path vu_1u_2 and a path vv_1u_1 ;
- (f) A cycle vu_1w_1 ;
- (g) A path vu_1u_2 and a path vw_1u_1 ;
- (h) A cycle vw_1w_2 ;
- (i) A cycle vv_1w_1 ;
- (j) A path $vv_1v_2v_3$, a path vw_1v_1 and a path $v_2u_1v_3$ (v_1, v_2 may be the same vertex);
- (k) A path $vv_1v_3v_4$, a path vv_2v_1 and a path $v_3u_1v_4$ (v_1, v_3 may be the same vertex);
- (l) A path $vv_1v_3v_4$, a path $v_3u_1v_4$, a path $vv_2v_5v_6$, a path $v_5u_2v_6$ and an edge v_1v_2 (v_1, v_3 may be the same vertex, v_2, v_5 may be the same vertex);
- (m) A path $vv_1v_2v_3$ and a path $v_2u_1v_3$ (v_1, v_2 may be the same vertex),

where $d(u_i) = 3$, $d(v_i) = 4$, w_i is bad and $5 \leq d(w_i) \leq 12$ for each $i \in \{1, 2, \dots, 6\}$. It is worth noting that for each bad vertex w_i , the v -stalk includes all neighbors of w_i lying on the worse or worst 3-faces.

Lemma 3.1. *Assume that G is a $\{C_4, C_5\}$ -free planar graph and $v \in V(G)$ with $5 \leq d(v) \leq 12$. Let $A = \{(a), (b), (c), (d), (e), (f), (g), (h), (j), (l)\}$, $B = \{(a), (b), (c), (d), (e), (l)\}$, $C = \{(b), (c), (e), (l)\}$, $D = \{(a), (b), (c), (e), (f), (g), (h), (j), (l)\}$ and $K = \{(i), (k)\}$ be the sets of stalks of v . If there exists a induced subgraph H containing v such that one of the following holds:*

- (1) *$V(H)$ either consists of the vertices lying on any combination of elements in A , or any combination of elements in B together with one copy of (i), or any combination of elements in C together with one copy of (k), or any combination of elements in D together with one copy of (m) such that the resulting $(\deg_H + \delta_{G,4})$ -assignment L satisfies $|L(v)| \geq 3$.*
- (2) *$V(H)$ consists of the vertices lying on any combination of elements in A together with only one element in K , except the special case where (d) and (k) appear in the same combination, such that the resulting $(\deg_H + \delta_{G,4})$ -assignment L satisfies $|L(v)| \geq 4$.*

Then G contains a $(P_3 + P_4, 4)$ -reducible induced subgraph with at most 138 vertices.

Proof. For $x \in N(v)$, let T_x be a v -stalk witnessing this case. Let H be the subgraph of G induced by $\cup_{x \in N(v)} T_x$. Clearly, $|V(H)| \leq (2d(w) - 1) \times \lfloor \frac{d(v)}{2} \rfloor \leq 138$. Thus it suffices to show that H is $(P_3 + P_4, 4)$ -reducible.

By $\{C_4, C_5\}$ -freeness, it is easy to obtain that all stalks are pairwise vertex disjoint and there are no edges among them. Consider any vertex $z \in V(H)$ and a $(\deg_H + \delta_{G,4}) \downarrow z$ -assignment L for H , H is L -colorable by Lemma 2.5, implying **(FIX)**. Thus it only remains to show that H satisfies **(FORB)**. Let $S = \{s_1, s_2\}$ be a $(P_3 + P_4)$ -independent subset in G and L' a $(\deg_H + \delta - 1_S)$ -assignment for H . First, we performing the following operations:

- Arbitrarily choose any two v -stalks T_1 and T_2 which containing at least a bad vertex or a nice path randomly;
- Arbitrarily choose a vertex s_i in each T_i for $i \in \{1, 2\}$ with $\text{dist}(s_1, s_2) = 1$ or $\text{dist}(s_1, s_2) \geq 4$.

When $|L(v)| \geq 3$, if H is induced by the vertices of any combination of elements in A , let $A_0 = \{(a), (b), (c), (d)\}$. Obviously, the result holds for A_0 . Next, we add (e), (f), (g), (h), (j), (l) into A_0 in turn to obtain A and prove that the result also admits for A . Let $A_1 = A_0 \cup \{(e)\}$, we only need to consider whether **(FORB)** holds for the combination (b) and (e). In this situation, we can first L' -color s_1, s_2, v in order and then greedily L' -color the remaining vertices. Hence, A_1 admits the result. Let $A_2 = A_1 \cup \{(f)\}$, we perform the above operation to define S , after that we L' -color the blocks which containing s_1, s_2, v respectively, this is possible since $|L'| \geq 3$, and then greedily L' -color the remaining vertices. By the same argument, we eventually find that A admits the result.

If H is induced by the vertices of any combination of elements in B as well as only one copy of (i), then we first greedily L' -color the vertices around the bad vertex w_1 in (i) and then greedily L' -color the remaining vertices.

Similarly, if H is induced by the vertices of any combination of elements in C as well as only one copy of (k) or by the vertices of any combinations of elements in D as well as only one copy of (m), it is easy to verify the result also holds.

When $|L(v)| \geq 4$ and H is induced by the vertices of any combination of elements in A together with only one element in K . As usual, we perform the above operations. Then we first L' -color the vertices in the block which containing s_i , then greedily L' -color the remaining vertices. It is possible since $|L'(v)| \geq 4$. Hence, in all the cases, **(FORB)** holds.

In conclusion, H is $(P_3 + P_4, 4)$ -reducible. \square

Lemma 3.2. *Let G be a $\{C_4, C_5\}$ -free planar graph. If G contains one of the following configurations (see Figure 1). Here, $d(u_i) = 3$, $d(v_i) = 4$ and $5 \leq d(w_i) \leq 12$ for all i .*

- (1) A 4-vertex v with $f_{bb}(v) = 1$;
- (2) A 6-vertex v with $f_{3,3}(v) = 1, f_{3b} = 1$;
- (3) A $2t$ -vertex v with $f_{3,3}(v) = t - 1$, where $2 \leq t \leq 6$;
- (4) A $2t$ -vertex v with $f_{3,3}(v) = t - 2, f_{bb}(v) = 1$, where $2 \leq t \leq 6$;
- (5) A 5-vertex v with $f_{3b} = 1$, and a path vu_2u_3 ;
- (6) A 6-vertex v with $f_{3b} = 1$, and two paths vu_2u_3, vu_4u_5 ,

Then G contains a $(P_3 + P_4, 4)$ -reducible induced subgraph with at most 29 vertices.

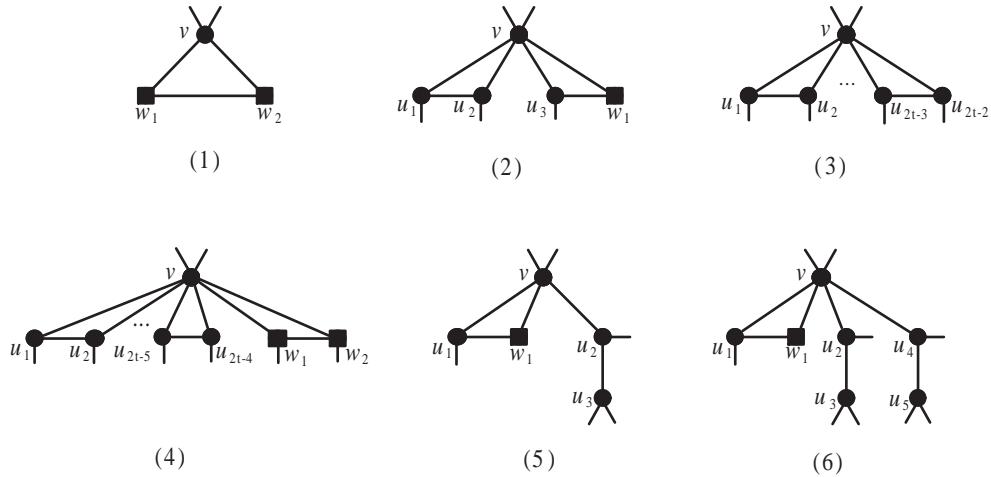


Figure 1: Situation in Lemma 3.2

Proof. Let H be the subgraph of G induced by $\{v, u_i, v_i, w_i\}$ for all i and all neighbors of bad vertices lying on any the worst or worse 3-faces. Note that $|V(H)| \leq 2d(w) - 1 + 6 \leq 29$. Consider any vertex $z \in V(H)$ and a $(\deg_H + \delta_{G,4}) \downarrow z$ -assignment L for H . Since each block B in H is a 3-face or an edge and then we can easily observe that B satisfies **(FIX)** under L , by Lemma 2.5, we know that H satisfies **(FIX)**.

Let S be a $(P_3 + P_4)$ -independent subset in G with $|S| \leq 2$ and L' a $(\deg_H + \delta - 1_S)$ -assignment for H . In (1), (2), (3), (4), since every two vertices is connected by a path of length 2 or 3 in H , **(FORB)** is implied by **(FIX)**. In (5), if $S = \{v, u_2\}$ (or $S = \{u_2, u_3\}$), then we can first L' -color v, u_2, u_3 (or u_3, u_2, v) and then greedily L' -color the remaining vertices. If $S = \{s_1, u_3\}$, where s_1 lies on the worst or worse 3-face w_1s_1z incident with w_1 , then we can first L' -color u_3, s_1, z, v, u_2 in order and the greedily L' -color the remaining vertices. In (6), the result also holds by the same argument.

Hence, both **(FIX)** and **(FORB)** hold, and thus H is $(P_3 + P_4, 4)$ -reducible. \square

Lemma 3.3. Let G be a $\{C_4, C_5\}$ -free planar graph, there are three 3-faces $uvw, v_1v_2v_3, w_1w_2w_3$ that are incident with a 6-face $f = vwvw_1w_3v_3v_1v$ such that $d(u) = 3, d(v) = d(w) = d(v_1) = d(w_1) = 4$, and $d(v_i) \leq 12, d(w_i) \leq 12$ for all $i \in \{2, 3\}$. If each $x \in \{v_2, v_3, w_2, w_3\}$ satisfies that either $3 \leq d(x) \leq 4$ or x is dangerous, then G contains a $(P_3 + P_4, 4)$ -reducible induced subgraph with at most 49 vertices.

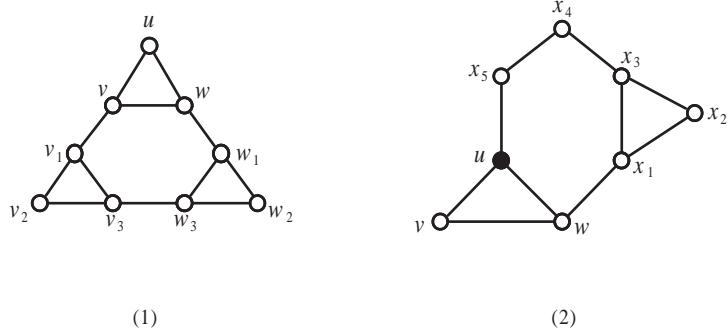


Figure 2: Situation in Lemma 3.3 and Lemma 3.4

Proof. Let H be the subgraph of G induced by $A = \{u, v, w, v_1, v_2, v_3, w_1, w_2, w_3\}$ and all possible 3-neighbors inside a worst triangle incident with each $x \in \{v_2, v_3, w_2, w_3\}$, which is denoted by B (possibly empty). Clearly, $V(H) = A \cup B$ and $|V(H)| \leq 49$. Now we are ready to prove that H is $(P_3 + P_4, 4)$ -reducible. It suffices to verify that H satisfies **(FIX)** and **(FORB)**. Firstly, consider an arbitrary $(\deg_H + \delta_{G,4})$ -assignment L for H . Then $|L(x)| > d_H(x)$ for each $x \in B \cup \{u\}$. In particular, $|L(x)| \geq d_{H'}(x)$ for each $x \in A$ and $|L(u)| > d_{H'}(u)$, where we denote $H' = H[A]$. It follows that **(FIX)** is immediately implied by Lemma 2.5.

It remains to verify that for all $\{P_3 + P_4\}$ -independent set S , H is L' -colorable for any $(\deg_H + \delta - 1_S)$ -assignment L' . By the assumption, we can easily observe that either $|S \cap A| \leq 1$ or $S \in \{\{v, v_1\}, \{w, w_1\}, \{v_3, w_3\}\}$. The case when S is a singleton is easily implied by **(FIX)**. If $|S| = 2$ and S contains at least one 3-vertex inside B , then we can first L' -color H' by **(FIX)** and then greedily L' -color all vertices in B . Hence it remains to consider the case when $S \in \{\{v, v_1\}, \{w, w_1\}, \{v_3, w_3\}\}$. In all cases, we firstly L' -color $H[S]$ and let $\{a, b\}$ be the resulting colors on S . Let $H^* = H' - S$ and define a list assignment L^* such that $L^*(z) = L'(z) \setminus \{a, b\}$ for each $z \in V(H') \cap N_H(S)$, while $L^*(z) = L'(z)$ for each $z \in V(H') \setminus N_H(S)$. It is easy to see that $|L^*(z)| \geq \deg_{H^*}(z)$ for all $z \in V(H^*)$. Since $|L^*(u)| > \deg_{H^*}(u)|$, by Lemma 2.2, we can always L^* -color all vertices in $A - S$ and then extend the coloring to B greedily. Hence H satisfies **(FORB)**.

This completes the proof of Lemma 3.3. \square

Lemma 3.4. Let G be a $\{C_4, C_5\}$ -free planar graph, if there are two 3-faces $uvw, x_1x_2x_3$ that are adjacent to a 6-face $uwx_1x_3x_4x_5u$ such that $d(u) = d(x_5) = 3$ and $d(v) = d(w) = d(x_1) = 4$. If each $x \in \{x_2, x_3, x_4\}$ satisfies that either $3 \leq d(x) \leq 4$ or x is dangerous, then G contains a $(P_3 + P_4, 4)$ -reducible induced subgraph with at most 37 vertices.

Proof. Let H be the subgraph of G induced by $A = \{u, v, w, x_1, x_2, x_3, x_4, x_5\}$ and all possible 3-neighbors inside a worst triangle incident with each x_i for all $i \in \{2, 3, 4\}$, which is denoted by B . Clearly, $V(H) = A \cup B$ and $|V(H)| \leq 37$. Now we are ready to prove that H is $(P_3 + P_4, 4)$ -reducible. It suffices to verify that H satisfies **(FIX)** and **(FORB)**. Consider an arbitrary $(\deg_H + \delta_{G,4})$ -assignment L of H , Then $|L(x)| > d_H(x)$ for each $x \in B \cup \{u, x_5\}$. In particular, $|L(x)| \geq d_{H'}(x)$ for each $x \in A$, where we denote $H' = H[A]$. It follows that **(FIX)** is immediately implied by Lemma 2.5.

It remains to verify that for all $\{P_3 + P_4\}$ -independent set S , H is L' -colorable for any $(\deg_H + \delta - 1_S)$ -assignment L' . By the assumption, we can easily observe that either $|S \cap A| \leq 1$ or $S \in \{\{w, x_1\}, \{x_3, x_4\}, \{x_4, x_5\}, \{u, x_5\}\}$. The case when S is a singleton is easily implied by **(FIX)**. If $|S| = 2$ and S contains at least one 3-vertex inside B , then we can first L' -color H' by **(FIX)** and then greedily L' -color all vertices in B . Hence it only remains to consider the case when $S \in \{\{w, x_1\}, \{x_3, x_4\}, \{x_4, x_5\}, \{u, x_5\}\}$.

If $S = \{u, x_5\}$, then consider the subgraph H' together with the list assignment L' , by Lemma 2.2, we can always L' -color all vertices in A , and then greedily L' -color all vertices in B . In the remaining cases, we firstly L' -color $H[S]$ and let $\{a, b\}$ be the resulting colors on S . Define a list assignment L^* on $V(H) - S$ such that $L^*(z) = L'(z) \setminus \{a, b\}$ for each $z \in N_H(S)$, while $L^*(z) = L'(z)$ for each $z \in V(H) \setminus N_H(S)$. Since $|L^*(u)| = 4, |L^*(x_5)| = 3$, by Lemma 2.2, we can always L^* -color all vertices in $A - S$ and then extend the coloring to B greedily. Hence H satisfies **(FORB)**. \square

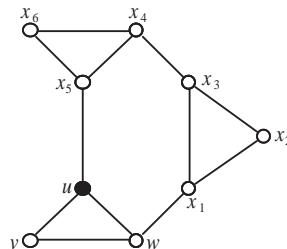


Figure 3: Situation in Lemma 3.5

Lemma 3.5. Let G be a $\{C_4, C_5\}$ -free planar graph, if there are three 3-faces $uvw, x_1x_2x_3, x_4x_5x_6$ that are adjacent to a 6-face $uwx_1x_3x_4x_5u$ such that $d(u) = 3$ and $d(v) = d(w) =$

$d(x_1) = d(x_6) = 4$ (see Figure 3). If x_5 is dangerous with $d(x_5)$ odd, while each $x \in \{x_2, x_3, x_4\}$ satisfies that either $3 \leq d(x) \leq 4$ or x is dangerous, then G contains a $(P_3+P_4, 4)$ -reducible induced subgraph with at most 45 vertices.

Proof. Let H be the subgraph of G induced by $A = \{u, v, w, x_1, x_2, x_3, x_4, x_5, x_6\}$ and all possible 3-neighbors inside a worst triangle incident with each x_i for all $i \in \{2, 3, 4, 5\}$, which is denoted by B . Clearly, $V(H) = A \cup B$ and $|V(H)| \leq 45$. Now we are ready to prove that H is $(P_3 + P_4, 4)$ -reducible. First, consider a $(\deg_H + \delta_{G,4})$ -assignment L for H , it is easy to observe that **(FIX)** is implied by Lemma 2.5.

It remains to verify that for all $\{P_3 + P_4\}$ -independent set S , H is L' -colorable for any $(\deg_H + \delta - 1_S)$ -assignment L' . The case when S is a singleton is easily implied by **(FIX)**, while the case when S contains a 3-vertex inside B is implied by **(FIX)** and Lemma 2.2. Hence it suffices to consider $S \in \{\{w, x_1\}, \{x_3, x_4\}, \{u, x_5\}\}$.

If $S = \{u, x_5\}$, then consider the subgraph $H[A]$ together with the list assignment L' , by Lemma 2.2, we can always L' -color all vertices in A , and then greedily L' -color all vertices in B . In the remaining cases, we firstly L' -color $H[S]$ and let $\{a, b\}$ be the resulting colors on S . Let $H^* = H - S$ and define a list assignment L^* such that $L^*(z) = L'(z) \setminus \{a, b\}$ for each $z \in V(H^*) \cap N_H(S)$, while $L^*(z) = L'(z)$ for each $z \in V(H^*) \setminus N_H(S)$. Since $|L^*(u)| = |L^*(x_5)| = 4$, by Lemma 2.2, we can always L^* -color all vertices in $A - S$ and then extend the coloring to B greedily. Hence H satisfies **(FORB)**.

□

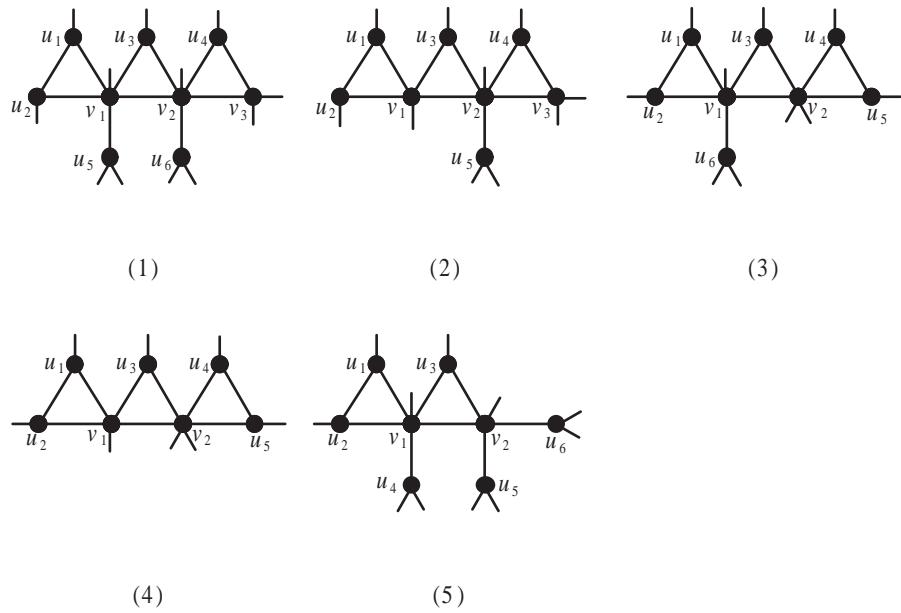


Figure 4: Situation in Lemma 3.6

Lemma 3.6. *If G has a subgraph isomorphic to one of the configurations in Figure 4, then G contains a $(P_3 + P_4, 4)$ -reducible induced subgraph with at most 9 vertices.*

Proof. Let H be the graph isomorphic one of the configurations in Figure 4. Note that **(FIX)** is easily obtained by Lemma 2.5. It remains to verify that for all $\{P_3 + P_4\}$ -independent set S , H is L' -colorable for any $(\deg_H + \delta - 1_S)$ -assignment L' . The case when S is a singleton is easily implied by **(FIX)**. Now we only consider $|S| = 2$. In all cases, S induces a pendant edge in H and it follows that we can first L' -color $H[S]$, and then greedily L' -color the remaining vertices in H . Hence, **(FORB)** holds. \square

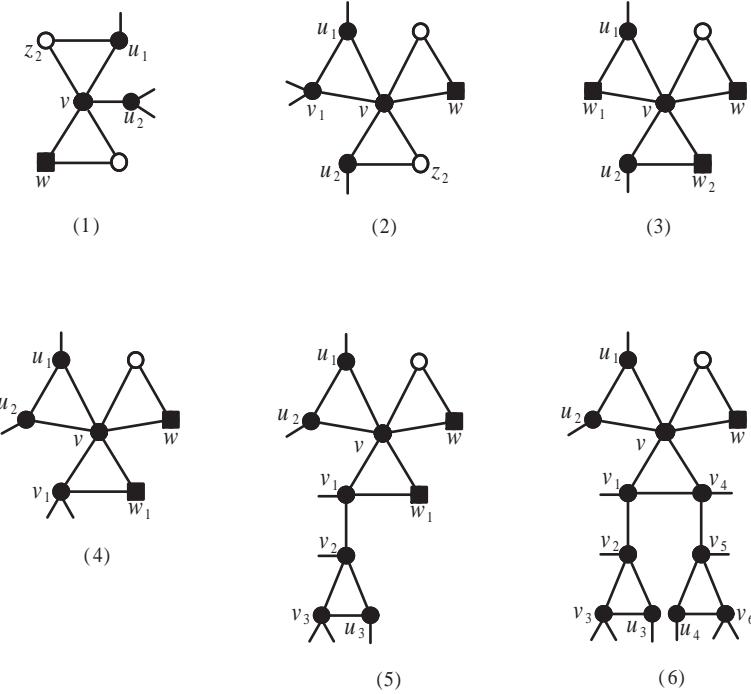


Figure 5: Situation in Lemma 3.7

Lemma 3.7. *Let G be a planar graph satisfying any one of the following conditions (see Figure 5). Here $d(u_i) = 3$, $d(v_i) = 4$, w_i is a bad 5^+ -vertex for all i , while w is both bad and dangerous.*

- (1) *A 5-vertex v with $f_3(v) = 2$, and vu_2 is a pendant edge, vu_1x is a 3-face, where z_1 is either a 4-vertex or bad;*
- (2) *A 6-vertex v with $f_3(v) = 3$, and vu_1v_1 , vu_2y are 3-faces, where z_2 is either a 4-vertex or bad;*
- (3) *A 6-vertex v with $f_3(v) = 3$, and $f_{3b}(v) = 2$;*

- (4) A 6-vertex v with $f_3(v) = 3$, and $f_{3,3}(v) = 1$, $f_{4b}(v) = 1$;
- (5) A 6-vertex v with $f_3(v) = 3$, and $f_{4b}(v) = 1$, two paths $v_1v_2v_3$, $v_2u_3v_3$ (v_1 , v_2 may be the same vertex);
- (6) A 6-vertex v with $f_3(v) = 3$, and $f_{3,3}(v) = 1$, four paths $vv_1v_2v_3$, $v_2u_3v_3$, $vv_4v_5v_6$, $v_5u_4v_6$ (v_1 , v_2 may be the same vertex, v_4 , v_5 may be the same vertex).

Then G contains a $(P_3 + P_4, 4)$ -reducible induced subgraph with at most 37 vertices.

Proof. We denote by $N^*(x)$ the set of neighbors of each bad vertex x lying on the worse or worst 3-faces. In all cases, if $d(w)$ is even, then by the assumption that $f_{3,3}(w) = \frac{d(w)-2}{2}$, Lemma 3.2(3) directly implies a $(P_3 + P_4, 4)$ -reducible induced subgraph on at most 37 vertices. Hence we may assume that $d(w)$ is odd.

Now consider the cases (1)-(4), let H be the subgraph induced by v, w , all u_i, v_i, w_i, z_i and all vertices in $N^*(x)$ for each bad vertex $x \in \{w, w_1, w_2, (z_i)\}$. Then we claim that H is a $(P_3 + P_4, 4)$ -reducible induced subgraph. In fact, for any $(\deg_H + \delta_{G,4})$ -assignment L of H , we have $|L(w)| = 2$, $|L(v)| \geq 3$ and $|L(w_i)| \geq 3$, $|L(v_i)| = 2$ for each $i \in [2]$. In addition, $|L(y)| = 3$ for all $y \in N^*(w)$, $|L(u_2)| = 2$ in configuration (1), while $|L(u_i)| = 3$ for each $i \in [2]$ in configurations (2)-(4). It follows from Lemma 2.5 that **(FIX)** holds. It remains to verify that for all $\{P_3 + P_4\}$ -independent set S , H is L' -colorable for any $(\deg_H + \delta - 1_S)$ -assignment L' . The case when S is a singleton is easily implied by **(FIX)**. Now we only consider $|S| = 2$. In all cases, we can first L' -color $H[S]$, and then greedily L' -color the remaining vertices in H . Hence, **(FORB)** holds.

Consider the configuration (5)-(6). Here, we just consider (6) since (5) can be solved by the same argument. Let H be the subgraph induced by v, w , all u_i, v_i and all vertices in $N^*(w)$. Now we claim that H is a $(P_3 + P_4, 4)$ -reducible induced subgraph. In fact, for any $(\deg_H + \delta_{G,4})$ -assignment L of H , we have $|L(x)| = 2$ for each $x \in \{w, v_3, v_6\}$, $|L(y)| = 3$ for each $y \in \{v, u_1, u_2, u_3, u_4, v_1, v_2, v_4, v_5\} \cup N^*(w)$. It follows from Lemma 2.5 that **(FIX)** holds. It remains to verify that for all $\{P_3 + P_4\}$ -independent set S , H is L' -colorable for any $(\deg_H + \delta - 1_S)$ -assignment L' . By similar arguments, in all cases, we can first L' -color $H[S]$, and then greedily L' -color the remaining vertices in H . Hence, **(FORB)** holds. \square

Lemma 3.8. *Let uv be an edge of G such that $d(u) = 3$ and $d(v) = 4$. If there exists a vice vertex v_2 such that both vv_1v_2 and $v_2v_3u_1$ are 3-faces incident with v_2 with $d(u_1) = 3$ and $d(v_i) = 4$ for each $i \in \{1, 2, 3\}$. Then G contains a $(P_3 + P_4, 4)$ -reducible induced subgraph with at most 6 vertices.*

Proof. Let H be the subgraph induced by $\{u, v, u_1, v_1, v_2, v_3\}$, consider any vertex $z \in V(H)$ and a $(\deg_H + \delta_{G,4}) \downarrow z$ -assignment L for H , $H - z$ is L -colorable by Lemma 2.2, implying **(FIX)**. Now we only consider $|S| = 2$. Since S induces a pendant edge in H and it follows that we can first L' -color $H[S]$, and then greedily L' -color the remaining vertices in H . Consequently, **(FORB)** admits. Hence, H is $(P_3 + P_4, 4)$ -reducible. \square

4 Discharging Process

To prove Theorem 1, the main idea is to apply Lemma 2.3. Let G be a $\{C_4, C_5\}$ -free planar graph. If G satisfies the assumption in Lemma 2.3, then we are done. Now we may assume that there exists $Z_0 \subseteq V(G)$ such that $G[Z_0]$ does not contain any $(P_3 + P_4, 4)$ -reducible subgraph on at most 100 vertices. Let $G' = G[Z_0]$. Since G' is also a plane graph, by Euler's Formula, we obtain

$$\sum_{v \in Z_0} (d(v) - 2) + \sum_{f \in F(G')} (-2) = -4.$$

Now we redistribute the charges of all vertices and faces as follows, where we use $c(x \rightarrow y)$ to denote the charge sent from an element x to another element y . Let f be a face of G' and v_1, v_2, v_3 be three vertices on f .

R0. If f is a 3-face and $d(v_1) \geq 13$, then $c(v_1 \rightarrow f) = \frac{4}{3}$, $c(v_2 \rightarrow f) = c(v_3 \rightarrow f) = \frac{1}{3}$;

R1. For each $v \in V(f)$, if either $d(v) = 3$, or $d(v) \geq 4$ and $d(f) \geq 6$, then $c(v \rightarrow f) = \frac{1}{3}$;

R2. Let $f = v_1v_2v_3$ with $d(v_1) = 4$.

R2.1. If $d(v_2) = 4$, $d(v_3) = 3$ and v_1 is not vice, then $c(v_1 \rightarrow f) = 1$, $c(v_2 \rightarrow f) = \frac{2}{3}$;

R2.2. If $d(v_2) = 4$, $5 \leq d(v_3) \leq 12$ and v_i is bad, v_{3-i} is not vice for some $i \in \{1, 2\}$, then $c(v_i \rightarrow f) = \frac{1}{2}$, $c(v_{3-i} \rightarrow f) = \frac{5}{6}$;

R2.3. Otherwise, let $c(v_1 \rightarrow f) = \frac{2}{3}$.

R3. If $f = v_1v_2v_3$ with $3 \leq d(v_1) \leq d(v_2) \leq 4 < d(v_3) \leq 12$, then $c(v_3 \rightarrow f) = 2 - c(v_1 \rightarrow f) - c(v_2 \rightarrow f)$.

R4. Let $f = v_1v_2v_3$ such that $d(v_1) = 3$, $5 \leq d(v_2) \leq 12$ and $5 \leq d(v_3) \leq 12$.

R4.1. If there exists a bad vertex $v_i \in \{v_2, v_3\}$, then $c(v_i \rightarrow f) = \frac{2}{3}$, $c(v_{5-i} \rightarrow f) = 1$;

R4.2. If there is no bad vertices on f , then $c(v_2 \rightarrow f) = c(v_3 \rightarrow f) = \frac{5}{6}$.

R5. Let $f = v_1v_2v_3$ such that $d(v_1) = 4$, $5 \leq d(v_2) \leq 12$, and $5 \leq d(v_3) \leq 12$.

R5.1. If there exists a bad vertex $v_i \in \{v_2, v_3\}$, then $c(v_i \rightarrow f) = \frac{1}{2}$, $c(v_{5-i} \rightarrow f) = \frac{5}{6}$;

R5.2. If neither v_2 nor v_3 is bad, then $c(v_2 \rightarrow f) = c(v_3 \rightarrow f) = \frac{2}{3}$.

R6. If $f = v_1v_2v_3$ such that $5 \leq d(v_1) \leq d(v_2) \leq d(v_3) \leq 12$,

R6.1. If there exists exactly one vertex which is both bad and dangerous, say v_1 , then

$$c(v_1 \rightarrow f) = \frac{1}{2}, c(v_2 \rightarrow f) = c(v_3 \rightarrow f) = \frac{3}{4};$$

R6.2. If there are two vertices, say v_1, v_2 , where each v_i is bad and dangerous, then

$$c(v_1 \rightarrow f) = c(v_2 \rightarrow f) = \frac{1}{2}, c(v_3 \rightarrow f) = 1.$$

R6.3. Otherwise, let $c(v_i \rightarrow f) = \frac{2}{3}$ for each $i \in \{1, 2, 3\}$.

Let $ch_1(x)$ be the new charge for each $x \in V(G') \cup F(G')$ after applying rules R1-R6. It is easy to see that for each vertex $v \in V(G')$, $f_3(v) \leq \lfloor \frac{d(v)}{2} \rfloor$. Since G' does not contain any $(P_3 + P_4, 4)$ -reducible subgraph, Lemma 3.1 and Lemma 3.2 immediately imply the following corollary.

Corollary 4.1. *Each 3^+ -vertex $v \in V(G')$ satisfies the following:*

$$(1) \quad f_{3,3}(v) + f_{3,4}(v) + f_{3b}(v) + f_{bb}(v) \leq \lfloor \frac{d(v)}{2} \rfloor - 1;$$

$$(2) \quad \text{If } v \text{ is dangerous, then } f_{3b}(v) = f_{bb}(v) = 0. \text{ Furthermore, if } d(v) \text{ is odd, then } f_{3,4}(v) = f_{4b}(v) = 0$$

Moreover, we consider all 5^+ -vertices that satisfy certain properties as follows.

Claim 4.2. *For each vertex v of G' . If $d(v) \geq 5$ and v is not dangerous, then $ch_1(v) \geq \frac{1}{6}$.*

Proof. By Corollary 4.1(1), $f_{3,3}(v) + f_{3,4}(v) + f_{3b}(v) + f_{bb}(v) \leq \lfloor \frac{d(v)}{2} \rfloor - 1$. If $d(v) \geq 5$ and v is not dangerous, then $f_{3,3}(v) \leq \max\{0, \frac{d(v)+1}{2} - 3\}$. Hence, when $d(v) = 5$, $ch_1(v) \geq 5 - 2 - 1 - \frac{5}{6} - \frac{3}{3} \geq \frac{1}{6}$. When $d(v) = 6$, $ch_1(v) \geq 6 - 2 - 1 - 1 - \frac{5}{6} - \frac{3}{3} \geq \frac{1}{6}$. When $d(v) \geq 7$,

$$\begin{aligned} ch_1(v) &\geq d(v) - 2 - \frac{4}{3} \times (\frac{d(v)+1}{2} - 3) - 2 - \frac{1}{3} \times \frac{d(v)+1}{2} \\ &\geq d(v) - \frac{5d(v)+5}{6} \\ &= \frac{d(v)-5}{6} \\ &\geq \frac{1}{3}. \end{aligned}$$

□

Claim 4.3. For each 5^+ -vertex v of G' . If v is incident with a face $f = (u, v, w)$ in which u is a bad 4-vertex and $d(w) \geq 4$, then $ch_1(v) \geq \frac{1}{6}$.

Proof. If $d(v) \geq 13$, then $ch_1(v) \geq d(v) - 2 - \frac{4}{3} \times (\lfloor \frac{d(v)}{2} \rfloor - 1) - \frac{5}{6} - \frac{1}{3} \times (d(v) - \lfloor \frac{d(v)}{2} \rfloor) = \frac{2}{3}d(v) - \lfloor \frac{d(v)}{2} \rfloor - \frac{3}{2} \geq \frac{d(v)-9}{6} \geq \frac{1}{3}$ by R1 and R6. Next it suffices to consider the case when $d(v) \leq 12$. Moreover, by Claim 4.2, it suffices to consider the case when v is dangerous.

By Corollary 4.1(2), since $f_{3,3}(v) = \lfloor \frac{d(v)-3}{2} \rfloor$, so $f_{bb}(v) = 0$ and w is not bad. Then $c(v \rightarrow f) \leq \frac{2}{3}$. If $d(v) = 5$, then v is bad, and by Lemma 3.2(1), we know that $d(w) \geq 5$. Hence, by R5.1, $ch_1(v) \geq 5 - 2 - \frac{4}{3} - \frac{1}{2} - \frac{3}{3} \geq \frac{1}{6}$. When $d(v) = 6$ and $f_{3,3}(v) = 1$, if v is bad, then $d(w) \geq 5$ and by R5.1, $c(v \rightarrow f) = \frac{1}{2}$ and it follows that $ch_1(v) \geq 6 - 2 - \frac{4}{3} - 1 - \frac{1}{2} - \frac{3}{3} \geq \frac{1}{6}$. Otherwise, since $f_{3,4}(v) + f_{3b}(v) + f_{bb}(v) = 0$, $ch_1(v) \geq 6 - 2 - \frac{4}{3} - \frac{5}{6} - \frac{2}{3} - \frac{3}{3} \geq \frac{1}{6}$. When $d(v) \geq 7$ is odd, by Corollary 4.1(2), we have $f_{3b}(v) + f_{3,4}(v) + f_{bb}(v) = 0$ and

$$\begin{aligned} ch_1(v) &\geq d(v) - 2 - \frac{4}{3} \times \left(\frac{d(v)+1}{2} - 2\right) - \frac{2}{3} - \frac{1}{3} \times \frac{d(v)+1}{2} \\ &\geq d(v) - \frac{5d(v)+5}{6} \\ &= \frac{d(v)-5}{6} \\ &\geq \frac{1}{3}. \end{aligned}$$

When $d(v) \geq 8$ is even, we have $f_{3,3}(v) = \frac{d(v)}{2} - 2$, $f_{3,4}(v) \leq 1$. Hence,

$$\begin{aligned} ch_1(v) &\geq d(v) - 2 - \frac{4}{3} \times \left(\frac{d(v)}{2} - 2\right) - 1 - \frac{2}{3} - \frac{1}{3} \times \frac{d(v)}{2} \\ &\geq d(v) - \frac{5d(v)}{6} - 1 \\ &= \frac{d(v)-6}{6} \\ &\geq \frac{1}{3}. \end{aligned}$$

□

A vertex v in G' is called *well* when $ch_1(v) \geq \frac{1}{12}$. Given a poor face $f = (3, 4, 4)$ and a well vertex v . From now on, let $\varpi(v)$ be the number of nice paths starting at v . For each poor face f , we apply the following rules.

R7. If f is poor and g is a 7^+ -face sharing an edge with f , then f receives $(\frac{d(g)}{3} - 2)/\xi(g)$ from g .

R8. If f receives less than $\frac{1}{3}$ by **R7** and v_1, \dots, v_t ($1 \leq t \leq 2$) are the well vertices such that all nice paths connecting each v_i with f has the same internal vertices, then f receives $\frac{1}{6t}$ from each v_i ($i \in \{1, \dots, t\}$).

Let $ch_2(x)$ be the final charge for each element $x \in V(G') \cup F(G')$ after applying R7 and R8.

4.1 Each poor face f satisfies $ch_2(f) \geq 0$.

Let $f = (u, v, w)$ be a poor face such that $d(u) = 3, d(v) = 4, d(w) = 4$ and f_1, f_2, f_3 be three adjacent faces sharing edge vw, uw, uv with f , respectively. In addition, let x_5 be the neighbor of u outside f . If v is not vice, then by **R1-R2**, $ch_2(f) \geq 0$. Hence by symmetry, we may assume that both v and w are vice. Let w_1, x_1 and v_1, y_1 be the other two neighbors of w and v outside f , respectively. Then they are all 4^+ -vertices.

If there exists either a 5^+ -vertex or a 4-vertex that is not vice, among $\{v_1, y_1, x_1, w_1\}$, say x_1 , then by Claim 4.3, $ch_1(x_1) \geq \frac{1}{6}$. In particular, if there are at least two such vertices as x_1 among $\{v_1, y_1, x_1, w_1\}$, then by R7-R8, we have $ch_2(f) \geq 0$. Hence, without loss of generality, we further assume that at least three among them, say x_1, w_1, v_1 , are vice 4-vertices. Note that the vertex sets $\{v_2, v_3, w_2, w_3\}$ and $\{x_1, x_2, x_3, x_4\}$ do not overlap.

Claim 4.4. *If $d(f_1) = 6$ and x_1, y_1, w_1, v_1 are vice 4-vertices, then f receives at least $\frac{1}{6}$ from a vertex in $\{v_2, v_3, w_2, w_3\}$.*

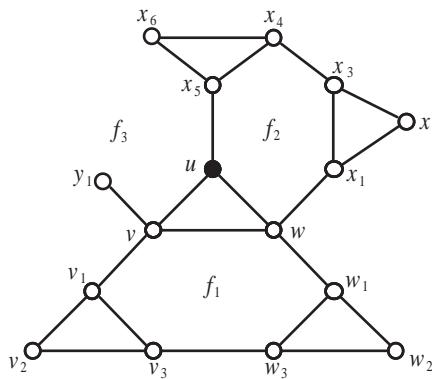


Figure 6

Proof. We may assume that each vertex in $\{v_2, v_3, w_2, w_3\}$ has degree at most 12. Otherwise, by R8, f receives at least $\frac{1}{6}$ in all from all 13^+ -vertices in $\{v_2, v_3, w_2, w_3\}$, and we are done. It is easily derived from Lemma 3.3 that there exists a vertex in $\{v_2, v_3, w_2, w_3\}$, say v_2 , such that $d(v_2) \geq 5$ and v_2 is not dangerous. By Claim 4.2, $ch_1(v_2) \geq \frac{1}{6}$. Since there is a nice path connecting v_2 with v , by R8, f receives $\frac{1}{6}$ from v_2, v_3 altogether. \square

Claim 4.5. *If $d(f_2) = 6$, then f receives at least $\frac{1}{6}$ from x_2, x_3, x_4, x_5 altogether.*

Proof. Suppose it is not true. We may similarly assume that each vertex in $\{x_2, x_3, x_5\}$ has degree at most 12.

If $d(x_5) = 3$ and $d(x_4) \geq 13$, then by R8, f receives $\frac{1}{6}$ from x_4 , a contradiction. If $d(x_5) = 3$ and $d(x_4) \leq 12$, then by Lemma 3.4, there exists a 5^+ -vertex in $\{x_2, x_3, x_4\}$ that is not dangerous. It follows from Claim 4.2 and R8 that f receives $\frac{1}{6}$ from x_2, x_3, x_4 altogether.

If $d(x_5) = 4$, then x_5 is not vice. Moreover, by Corollary 4.1(1), $f_{3,4}(v) = 0$, and it is easy to see that x_5 is well. So f receives $\frac{1}{6}$ from x_5 .

If $5 \leq d(x_5) \leq 12$, then by Claim 4.2, it remains to consider the case when x_5 is dangerous. Now we have the following observations:

- (1) $f_{3b}(x_5) = f_{bb}(x_5) = 0$;
- (2) x_4x_5 is contained in a 3-face, say $x_4x_5x_6$;
- (3) x_4, x_6 are 4^+ -vertices;
- (4) $d(x_5)$ is odd.

In fact, (1) and (3) are immediately derived from Corollary 4.1. For (2), if x_4x_5 is not contained in a 3-face, then x_5 is incident with at most $\frac{d(v)-2}{2}$ 3-faces. Since x_5 is dangerous, it is easy to see that if $d(x_5)$ is odd, then by Corollary 4.1(2), $f_{3,4}(x_5) = 0$ and $ch_1(x_5) \geq d(x_5) - 2 - \frac{4}{3} \times \frac{d(x_5)-3}{2} - \frac{1}{3} \times \lceil \frac{d(x_5)+1}{2} \rceil \geq \frac{1}{6}$, a contradiction. If $d(x_5)$ is even, then $ch_1(x_5) \geq d(x_5) - 2 - \frac{4}{3} \times \frac{d(x_5)-4}{2} - 1 - \frac{1}{3} \times \frac{d(x_5)+2}{2} \geq \frac{1}{3}$, a contradiction. For (4), If $d(x_5)$ is even, then since ux_5 is not contained in a 3-face, we know x_5 is incident with at most $\frac{d(v)-2}{2}$ 3-faces. Applying similar arguments as above, we know that $ch_1(x_5) \geq d(x_5) - 2 - \frac{4}{3} \times \frac{d(x_5)-4}{2} - 1 - \frac{1}{3} \times \frac{d(x_5)+2}{2} \geq \frac{1}{3}$. In all those cases, f receives $\frac{1}{6}$ from x_5 , a contradiction.

By the above observations, we know x_5 is also a bad vertex. Consider the 3-face $f' = x_4x_5x_6$, if $d(x_i) \geq 5$ for all $i \in \{4, 6\}$, then by R6, $c(x_5 \rightarrow f') = \frac{1}{2}$ and it follows that $ch_1(x_5) \geq d(x_5) - 2 - \frac{4}{3} \times \frac{d(x_5)-3}{2} - \frac{1}{2} - \frac{1}{3} \times \frac{d(x_5)+1}{2} \geq \frac{1}{6}$. If $d(x_i) = 4$ for some $i \in \{4, 6\}$, then x_{10-i} is not bad. If $d(x_{10-i}) \geq 13$, then by R0, $c(x_5 \rightarrow f') = \frac{1}{3}$. We know that $ch_1(x_5) \geq d(x_5) - 2 - \frac{4}{3} \times \frac{d(x_5)-3}{2} - \frac{1}{3} - \frac{1}{3} \times \frac{d(x_5)+1}{2} \geq \frac{1}{3}$, a contradiction. If $5 \leq d(x_{10-i}) \leq 12$, then by R5.1, $c(x_5 \rightarrow f') = \frac{1}{2}$, and it follows that $ch_1(x_5) \geq d(x_5) - 2 - \frac{4}{3} \times \frac{d(x_5)-3}{2} - \frac{1}{2} - \frac{1}{3} \times \frac{d(x_5)+1}{2} \geq \frac{1}{6}$, a contradiction. Now it remains to consider the case when $d(x_4) = d(x_6) = 4$. By Lemma 3.5, there is a 5^+ -vertex inside $\{x_2, x_3\}$ that is not dangerous, say x_2 , and by Claim 4.2, f receives $\frac{1}{6}$ from x_2, x_3 altogether, a contradiction. \square

Now we proceed to verify that $ch_2(f) \geq 0$. If there are at least two 7^+ -faces among $\{f_1, f_2, f_3\}$, then $ch_2(f) \geq 0$ by R7. Hence we assume at least two of them are 6-face. By

the assumption that x_1, w_1, v_1 are vice 4-vertices, If y_1 is not vice and f_2 is 7^+ -face, then we are done. Otherwise, if y_1 is vice, then by Claim 4.4, f receives at least $\frac{1}{6}$ from a vertex in $\{v_2, v_3, w_2, w_3\}$. Meanwhile, if f_2 is a 6-face, then by Claim 4.5, f receives at least $\frac{1}{6}$ from a vertex in $\{x_2, x_3, x_4, x_5\}$. This completes the proof.

4.2 Final analysis

Now we shall verify that $ch_2(x) \geq 0$ for all $x \in V(G') \cup F(G')$. It is easy to observe that for each $v \in V(G')$, if $f_3(v) < \lfloor \frac{d(v)}{2} \rfloor$, then $ch_2(v) \geq 0$. From now on, it suffices to only care the case $f_3(v) = \lfloor \frac{d(v)}{2} \rfloor$. We assume that each bad vertex satisfies $d(v) \geq 5$. For $v \in V(G')$, denote by $N_b(v)$ the set of bad neighbors of v and let $|N_b(v)| = n_b(v)$.

Claim 4.6. *Let v be a vertex of G' with $5 \leq d(v) \leq 12$. If v is not dangerous, then $ch_2(v) \geq 0$.*

Proof. If v is not dangerous, it follows that $f_{3,3}(v) \leq \lfloor \frac{d(v)-3}{2} \rfloor - 1$. Accordingly,

$$\begin{aligned} ch_2(v) &\geq d(v) - 2 - \frac{4}{3}f_{3,3}(v) - f_{3,4}(v) - f_{3b}(v) - \frac{2}{3}f_{4,4}(v) - \frac{5}{6}f_{4b}(v) - f_{bb}(v) \\ &\quad - \frac{1}{6}f_{3,4}(v) - \frac{1}{6}f_{3b}(v) - 2 \times \frac{1}{6}f_{4,4}(v) - \frac{1}{6}f_{4b}(v) - \frac{1}{3}(d(v) - f_3(v)) \\ &\geq \frac{2}{3}d(v) - \frac{2}{3}f_3(v) - \frac{1}{3}f_{3,3}(v) - \frac{1}{6}(f_3(v) - f_{3,3}(v) - 1) - 2 \\ &\geq \frac{d(v) - 8}{6}. \end{aligned} \tag{*}$$

Thus, $ch_2(v) \geq 0$ when $d(v) \geq 8$.

In particular, when $d(v) = 7$, we get $ch_2(v) \geq \frac{2}{3} \times 7 - \frac{2}{3} \times 3 - \frac{1}{3} \times 1 - \frac{1}{6} \times 1 - 2 = \frac{1}{6} > 0$.

When $d(v) = 6$. Let $f_1 = v_1v_2v$, $f_2 = v_3v_4v$ and $f_3 = v_5v_6v$ be three 3-faces incident with v , respectively. We discuss the following two cases depending on whether v is bad:

- v is bad but not dangerous in G' ;

We may assume that f_2 and f_3 are worse, and it follows from <{(a), (d), (d)} that $d(v_1) \geq 4$, $d(v_2) \geq 4$. If $d(v_1) = d(v_2) = 4$, then $\varpi(v) \leq 2$ by <{(e), (e), (k)} and Lemma 3.8, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 2 \times 1 - \frac{2}{3} - 2 \times \frac{1}{6} = 0$ by R3 and R8. If $d(v_1) = 4$, $d(v_2) \geq 5$, then by <{(d), (d), (i)}, we get $n_b(v) = 0$ and $\varpi(v) \leq 3$ by Lemma 3.8. Thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 2 \times 1 - \frac{1}{2} - 3 \times \frac{1}{6} = 0$ by R5.1. If $d(v_1) \geq 5$, $d(v_2) \geq 5$, then $n_b(v) \leq 1$ by <{(d), (d), (h)}. If there exists a bad vertex, then it can not be dangerous by Lemma 3.7(2), thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 2 \times 1 - \frac{2}{3} - 2 \times \frac{1}{6} = 0$ by R6.3.

- v is neither bad nor dangerous in G' ;

Case 1. v is incident with a worse face;

W.l.o.g, let f_3 be worse and $N_1(v) = N(v) \setminus \{v_5, v_6\}$. Suppose there are two 3-vertices in $N_1(v)$, say v_1 and v_3 , then $d(v_2) \geq 5$, $d(v_4) \geq 5$. By $\{(a), (d), (f)\}$, we obtain $n_b(v) = 0$, by Lemma 3.6, we get that both v_2 and v_4 are well vertices, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 1 - 2 \times \frac{5}{6} - \frac{1}{6} - 2 \times \frac{1}{12} = 0$ by R8. Suppose there are only one 3-vertex in $N_1(v)$, say $d(v_2) = 3$, then $d(v_1) \geq 5$. If $d(v_3) = d(v_4) = 4$, we first consider v_1 is bad, then $\varpi(v) \leq 2$ by $\{(e), (g), (k)\}, \{(e), (f), (l)\}, \{(d), (g), (l)\}$, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 1 - 1 - \frac{2}{3} - 2 \times \frac{1}{6} = 0$ by R3 and R4. Otherwise, v_1 is not bad and it follows from $\{(b), (d), (l)\}$ that $\varpi(v) \leq 3$, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 1 - \frac{5}{6} - \frac{2}{3} - 3 \times \frac{1}{6} = 0$ by R4. If there exists a 5^+ -vertex in $\{v_3, v_4\}$, say v_3 , it follows from $\{(d), (f), (i)\}$ that $n_b(v) \leq 1$. If v_1 is bad, then $\varpi(v) \leq 2$ by $\{(e), (g), (m)\}$. On the other hand, if v_3 is bad, we can also get $\varpi(v) \leq 2$ by $\{(b), (d), (j)\}$. If there is no bad vertex in $N_1(v)$, then $\varpi(v) \leq 3$ by Lemma 3.8. Thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 1 - \max\{1 + \frac{2}{3} + 2 \times \frac{1}{6}, 2 \times \frac{5}{6} + 2 \times \frac{1}{6}, \frac{5}{6} + \frac{2}{3} + 3 \times \frac{1}{6}\} = 0$ by R4 and R5. If $d(v_3) \geq 5$, $d(v_4) \geq 5$, then it follows from $\{(a), (d), (h)\}$ that v_3 and v_4 are not bad at the same time, thus $n_b(v) \leq 2$. If $n_b(v) = 2$, then by Lemma 3.7(2), some v_i for $i \in \{3, 4\}$ is bad but not dangerous, and it follows that $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 1 - 1 - \frac{2}{3} - 2 \times \frac{1}{6} = 0$ by R4 and R6. If $n_b(v) \leq 1$, then $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 1 - \max\{1 + \frac{2}{3} + 2 \times \frac{1}{6}, \frac{5}{6} + \frac{3}{4} + 2 \times \frac{1}{6}\} = 0$ by R4, R6, and R8. Next, we consider $d(z) \geq 4$ for all $z \in N_1(v)$. If $d(z) = 4$ for all $z \in N_1(v)$, then $\varpi(v) \leq 4$ by $\{(d), (l), (l)\}$, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 1 - 2 \times \frac{2}{3} - 4 \times \frac{1}{6} = 0$ by R3. Suppose there exists a 5^+ -vertex in $N_1(v)$, say v_1 . If v_1 is bad, then $\varpi(v) \leq 3$ by $\{(d), (j), (l)\}$, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 1 - \frac{2}{3} - \frac{5}{6} - 3 \times \frac{1}{6} = 0$ by R5.1. If v_1 is not bad, then $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 1 - 2 \times \frac{2}{3} - 4 \times \frac{1}{6} = 0$ by R5.2. Suppose there exists two 5^+ -vertices in $N_1(v)$, say v_1 and v_2 or v_1 and v_3 . If $n_b(v) = 2$, then $\varpi(v) \leq 2$ by $\{(d), (j), (j)\}$ or $\{(d), (h), (l)\}$, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 1 - \max\{2 \times \frac{5}{6}, 1 + \frac{2}{3}\} - 2 \times \frac{1}{6} = 0$ by R3, R5 and R6. If $n_b(v) \leq 1$, then $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 1 - \max\{\frac{3}{4} + \frac{2}{3}, \frac{2}{3} + \frac{5}{6}\} - 3 \times \frac{1}{6} = 0$ by R5 and R6. If there exists three 5^+ -vertices in $N_1(v)$, then it follows from $\{(d), (h), (i)\}$ that $n_b(v) \leq 2$ and $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 1 - \max\{1 + \frac{2}{3}, \frac{5}{6} + \frac{3}{4}\} - 2 \times \frac{1}{6} = 0$ by R5 and R6. If $d(z) \geq 5$ for all $z \in N_1(v)$, then it follows from $\{(d), (h), (h)\}$ that $n_b(v) \leq 3$ and $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 1 - \frac{3}{4} - 1 - \frac{1}{6} = \frac{1}{12} > 0$ by R6.

Case 2. v is not incident with a worse face;

If $n_3(v) = 3$, then $n_b(v) \leq 1$ by $\{(a), (f), (f)\}$. In particular, $\varpi(v) \leq 2$ when $n_b(v) = 1$ by Lemma 3.2(6), thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \max\{1 + 2 \times \frac{5}{6} + 2 \times \frac{1}{6}, 3 \times \frac{5}{6} + 3 \times \frac{1}{6}\} = 0$ by R4. If $n_3(v) = 2$, w.l.o.g, we say $d(v_1) = d(v_3) = 3$, then $d(v_2) \geq 5$, $d(v_4) \geq 5$. Suppose $d(v_5) = d(v_6) = 4$, if $n_b(v) = 2$, then $\varpi(v) \leq 2$ by $\{(g), (g), (k)\}$, and it follows that $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 2 \times 1 - \frac{2}{3} - 2 \times \frac{1}{6} = 0$ by R3 and R4. If $n_b(v) \leq 1$, then

$ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \max\{1 + \frac{5}{6} + \frac{2}{3} + 3 \times \frac{1}{6}, 2 \times \frac{5}{6} + \frac{2}{3} + 4 \times \frac{1}{6}\} = 0$. If there is a 5⁺-vertex in $\{v_5, v_6\}$, say v_5 , then $n_b(v) \leq 2$ by <{(f), (f), (i)}. If $n_b(v) = 2$, then $\varpi(v) \leq 2$ by <{(g), (g), (m)} or <{(b), (g), (j)}, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \max\{2 \times 1 + \frac{2}{3}, 1 + \frac{5}{6} + \frac{5}{6}\} - 2 \times \frac{1}{6} = 0$ by R4 and R5. Otherwise $n_b(v) \leq 1$, we get $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \max\{1 + \frac{5}{6} + \frac{2}{3}, 3 \times \frac{5}{6}\} - 3 \times \frac{1}{6} = 0$ by R4 and R5. If both of v_5 and v_6 are 5⁺-vertex, then $n_b(v) \leq 3$ by <{(f), (f), (h)}. If $n_3(b) = 3$, then v_i ($i \in \{5, 6\}$) is bad but not dangerous by Lemma 3.7(3), thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 2 \times 1 - \frac{2}{3} - 2 \times \frac{1}{6} = 0$ by R6. Otherwise $n_3(b) \leq 2$, we have $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \max\{1 + \frac{3}{4} + \frac{5}{6}, 1 + 2 \times \frac{5}{6}, 1 \times 2 + \frac{2}{3}\} - 2 \times \frac{1}{6} = 0$ by R4 and R6. If $n_3(v) = 1$, w.l.o.g, say $d(v_1) = 3$, then $d(v_2) \geq 5$, we denote $N_2(v) = N(v) \setminus \{v_1, v_2\}$. If $d(z) = 4$ for all $z \in N_2(v)$. Suppose $n_b(v) = 1$, then $\varpi(v) \leq 3$ by <{(f), (k), (l)}, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 1 - 2 \times \frac{2}{3} - 3 \times \frac{1}{6} = \frac{1}{6} > 0$ by R3 and R4. Otherwise $n_b(v) = 0$, it follows that $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{5}{6} - 2 \times \frac{2}{3} - 5 \times \frac{1}{6} = 0$. If there exists an 5⁺-vertex in $N_2(v)$, say v_3 . If $n_b(v) = 2$, then $\varpi(v) \leq 3$ by <{(g), (j), (l)}, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 1 - \frac{5}{6} - \frac{2}{3} - 3 \times \frac{1}{6} = 0$ by R4 and R5. Otherwise $n_b(v) \leq 1$, we get $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \max\{1 + 2 \times \frac{2}{3}, 2 \times \frac{5}{6} + \frac{2}{3}\} - 4 \times \frac{1}{6} = 0$ by R4 and R6. If there are two 5⁺-vertices in $N_2(v)$, say v_3 and v_4 , or v_3 and v_5 . In the former case, if $n_b(v) = 3$, then $\varpi(v) \leq 1$ by <{(f), (h), (k)}, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 2 \times 1 - \frac{2}{3} - \frac{1}{6} = \frac{1}{6} > 0$. If $n_b(v) \leq 2$, then $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \max\{1 + \frac{5}{6} + \frac{2}{3}, 1 + \frac{3}{4} + \frac{2}{3}\} - 3 \times \frac{1}{6} = 0$. In the latter case, if $n_b(v) = 3$, then $\varpi(v) \leq 2$ by <{(g), (j), (j)}, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 1 - 2 \times \frac{5}{6} - 2 \times \frac{1}{6} = 0$. If $n_b(v) \leq 2$, then $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \max\{1 + \frac{5}{6} + \frac{2}{3}, 3 \times \frac{5}{6}\} - 3 \times \frac{1}{6} = 0$. If there are three 5⁺-vertices in $N_2(v)$, then $n_b(v) \leq 3$ by <{(f), (h), (i)}, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \max\{1 + 2 \times \frac{5}{6}, 2 \times 1 + \frac{2}{3}, 1 + \frac{5}{6} + \frac{3}{4}\} - 2 \times \frac{1}{6} = 0$. If all vertices are 5⁺-vertices in $N_2(v)$, then $n_b(v) \leq 4$ by <{(f), (h), (h)}, it follows that $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \max\{2 \times 1 + \frac{5}{6}, 2 \times 1 + \frac{3}{4}\} - \frac{1}{6} = 0$ by R4 and R6. Eventually, we consider $n_3(v) = 0$. If $n_{5^+}(v) = 0$, i.e. $d(z) = 4$ for all $z \in N(v)$, then $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 3 \times \frac{2}{3} - 6 \times \frac{1}{6} = 0$ by R3. If $n_{5^+}(v) = 1$, it follows that $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 2 \times \frac{2}{3} - \frac{5}{6} - 5 \times \frac{1}{6} = 0$ by R5. If $n_{5^+}(v) = 2$, then $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \max\{2 \times \frac{2}{3} + 1, 2 \times \frac{5}{6} + \frac{2}{3}\} - 4 \times \frac{1}{6} = 0$ by R5 and R6. If $n_{5^+}(v) = 3$, then $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \max\{1 + \frac{5}{6} + \frac{2}{3}, 3 \times \frac{5}{6}\} - 3 \times \frac{1}{6} = 0$ by R5 and R6. If $n_{5^+}(v) = 4$, then $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \max\{2 \times 1 + \frac{2}{3}, 2 \times \frac{5}{6} + 1\} - 2 \times \frac{1}{6} = 0$ by R5 and R6. If $n_{5^+}(v) = 5$, then $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 2 \times 1 - \frac{5}{6} - \frac{1}{6} = 0$ by R5 and R6. If $n_{5^+}(v) = 6$, then $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - 3 \times 1 = 0$ by R6.

When $d(v) = 5$. Let $f_1 = v_1v_2v$, $f_2 = v_3v_4v$ be two 3-faces incident with v . Similarly, we consider whether v is bad.

- v is bad but not dangerous in G' ;

Assume f_2 is worse, if $d(v_1) = 3$, then $d(v_2) \geq 5$, and note that v_2 is not bad by <{(d), (f)}, it follows from <{(a), (b), (d)} that $\varpi(v) \leq 2$, thus $ch_2(v) = 5 - 2 - 3 \times \frac{1}{3} - 1 - \frac{2}{3} - 2 \times \frac{1}{6} = 0$

by R3 and R4. If $d(v_1) = d(v_2) = 4$, then $\varpi(v) \leq 2$ by $\{(a), (d), (k)\}$, thus $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - 1 - \frac{2}{3} - 2 \times \frac{1}{6} = 0$ by R3. If $d(v_1) = 4$ and $d(v_2) \geq 5$, then v_2 is not bad by $\{(d), (i)\}$ and $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - 1 - \frac{1}{2} - 3 \times \frac{1}{6} = 0$ by R3 and R5. If $d(v_1) \geq 5$ and $d(v_2) \geq 5$, then it follows that $n_b(v) \leq 1$ by $\{(d), (h)\}$. If $n_b(v) = 1$, w.l.o.g., let v_1 be a bad vertex. If v_1 is also dangerous, then $d(v_5) \geq 4$ by Lemma 3.7(1), and it follows that $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - 1 - \frac{3}{4} - \frac{1}{6} = \frac{1}{12}$ by R3 and R6. If v_1 is not dangerous, then $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - 1 - \frac{2}{3} - 2 \times \frac{1}{6} = 0$. If $n_b(v) = 0$, then $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - 1 - \frac{2}{3} - 2 \times \frac{1}{6} = 0$.

- v is neither bad nor dangerous in G' ;

Let $N_3(v) = \{v_1, v_2, v_3, v_4\}$, we denote $N_{3b}(v)$, $N_3^*(v)$ the set of bad vertices and 3-vertices in $N_3(v)$ respectively. For simplicity, let $n_{3b}(v) = |N_{3b}(v)|$, $n_3^*(v) = |N_3^*|$. If $n_3^*(v) = 2$, say v_1 and v_3 , then $d(v_2) \geq 5$, $d(v_4) \geq 5$. We get $n_{3b}(v) \leq 1$ by $\{(f), (f)\}$. If $n_{3b}(v) = 1$, then $\varpi(v) \leq 1$ by Lemma 3.2(5), thus $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - 1 - \frac{5}{6} - \frac{1}{6} = 0$. Otherwise, $n_{3b}(v) = 0$, if $d(v_5) \geq 4$, then $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - 2 \times \frac{5}{6} - 2 \times \frac{1}{6} = 0$. If $d(v_5) = 3$, it follows from Lemma 3.6 that both v_2 and v_4 are well vertices, thus $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - 2 \times \frac{5}{6} - 2 \times \frac{1}{12} - \frac{1}{6} = 0$. If $n_3^*(v) = 1$, say v_1 , then $d(v_2) \geq 5$. When $d(v_3) = d(v_4) = 4$, if v_1 is bad, then $\varpi(v) \leq 2$ by $\{(a), (g), (k)\}$, thus $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - 1 - \frac{2}{3} - 2 \times \frac{1}{6} = 0$ by R3 and R4. Otherwise, v_1 is not bad, then $\varpi(v) \leq 3$ by $\{(a), (b), (l)\}$, thus $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - \frac{5}{6} - \frac{2}{3} - 3 \times \frac{1}{6} = 0$. When $d(v_i) \geq 5$ for some $i \in \{3, 4\}$, if $n_{3b}(v) = 2$, then $d(v_5) \geq 4$ by $\{(a), (f), (i)\}$. Moreover, $\varpi(v) \leq 1$ by $\{(g), (j)\}$, thus $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - 1 - \frac{5}{6} - \frac{1}{6} = 0$. If $n_{3b}(v) = 1$, first, suppose v_i is bad, then $\varpi(v) \leq 2$ by $\{(a), (b), (i)\}$, thus $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - 2 \times \frac{5}{6} - 2 \times \frac{1}{6} = 0$. Second, suppose v_2 is bad, it follows from $\{(a), (g), (m)\}$ that $\varpi(v) \leq 2$, thus $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - 1 - \frac{2}{3} - 2 \times \frac{1}{6} = 0$. If neither v_2 nor v_i is bad, then $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - \frac{5}{6} - \frac{2}{3} - 3 \times \frac{1}{6} = 0$. When $d(v_3) \geq 5$, $d(v_4) \geq 5$, $n_{3b}(v) \leq 2$ by $\{(f), (h)\}$. If $n_{3b}(v) = 2$, we get $\varpi(v) \leq 1$ by Lemma 3.7(1), $\{(a), (b), (h)\}$, thus $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - \max\{1 + \frac{5}{6} + \frac{1}{6}, 1 + \frac{3}{4} + \frac{1}{6}, 1 + \frac{2}{3} + 2 \times \frac{1}{6}\} = 0$ by R5-R6. Otherwise, $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - \max\{1 + \frac{2}{3}, \frac{5}{6} + \frac{3}{4}\} - 2 \times \frac{1}{6} = 0$. Next, we consider $n_3^*(v) = 0$, which means $d(z) \geq 4$ for all $z \in N_3(v)$. If $d(z) = 4$ for all $z \in N_3(v)$, then $\varpi(v) \leq 4$ by $\{(a), (l), (l)\}$, it follows that $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - 2 \times \frac{2}{3} - 4 \times \frac{1}{6} = 0$ by R3. If there exist a 5^+ vertex in $N_3(v)$, say v_1 , if v_1 is bad, then $\varpi(v) \leq 3$ by $\{(a), (j), (l)\}$, thus $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - \frac{2}{3} - \frac{5}{6} - 3 \times \frac{1}{6} = 0$. Otherwise, v_1 is not bad, it follows that $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - 2 \times \frac{2}{3} - 4 \times \frac{1}{6} = 0$ by R3 and R5. If there are two 5^+ -vertices in $N_3(v)$, say v_1 and v_2 or v_1 and v_3 , if $n_{3b}(v) = 2$, then $\varpi(v) \leq 2$ by $\{(a), (h), (l)\}$, $\{(a), (j), (j)\}$, then $ch_2(v) = 5 - 2 - 3 \times \frac{1}{3} - \max\{2 \times \frac{5}{6}, 1 + \frac{2}{3}\} - 2 \times \frac{1}{6} = 0$. Otherwise if $n_{3b}(v) \leq 1$, then $ch_2(v) = 5 - 2 - 3 \times \frac{1}{3} - \max\{\frac{5}{6} + \frac{2}{3}, \frac{3}{4} + \frac{2}{3}\} - 3 \times \frac{1}{6} = 0$. If there are three 5^+ -vertices in $N_3(v)$, and if $n_{3b}(v) = 3$, then $\varpi(v) \leq 1$ by $\{(a), (h), (j)\}$, thus

$ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - \frac{5}{6} - 1 - \frac{1}{6} = 0$ by R5 and R6. Otherwise $n_{3b}(v) \leq 2$, then $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - \max\{\frac{2}{3} + 1, \frac{3}{4} + \frac{5}{6}\} - 2 \times \frac{1}{6} = 0$. If $d(z) \geq 5$ for all $z \in N_3(v)$, then $n_{3b}(v) \leq 3$ by {(h), (h)}, thus $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - 1 - \frac{3}{4} - \frac{1}{6} = \frac{1}{12} > 0$.

This completes the proof of Claim 4.6. \square

Now, we are ready to verify all vertices in G' satisfying $ch_2(v) \geq 0$.

Let v be a 3-vertex in G' . Then $ch_2(v) = ch_1(v) = 1 - 3 \times \frac{1}{3} = 0$ by R1.

Let v be a 4-vertex in G' . Then $ch_2(v) = ch_1(v) \geq 2 - \max\{1 + 3 \times \frac{1}{3}, \frac{1}{2} + \frac{2}{3} + 2 \times \frac{1}{3}, \frac{5}{6} + 3 \times \frac{1}{3}\} = 0$ by R2.

Let v be a 5^+ -vertex in G' . Suppose v is not dangerous, then $ch_2(v) \geq 0$ by Claim 4.6. Next, we consider the case v is dangerous.

If $d(v)$ is even, then $f_{3,3}(v) = \frac{d(v)}{2} - 2$, thus we obtain that $ch_2(v) = d(v) - 2 - \frac{4}{3}(\frac{d(v)}{2} - 2) - \frac{1}{3}(d(v) - f_3(v)) - 1 - \frac{7}{6} = \frac{d(v)-9}{6} \geq 0$ when $d(v) \geq 9$.

Let v be a 8-vertex in G' . Note that $ch_2(v) = 2 - \frac{7}{6}f_{3,4}(v) - f_{4,4}(v) - f_{4b}(v)$. Let f_3 as well as f_4 be worst and f_1 and f_2 be the rest two 3-faces. If v is bad, then $\varpi(v) \leq 2$ by {(c), (c), (e), (l)}, and it follows that $ch_2(v) \geq 8 - 2 - 4 \times \frac{1}{3} - 2 \times \frac{4}{3} - 1 - \frac{2}{3} - 2 \times \frac{1}{6} = 0$ by R3 and R5. Otherwise, $ch_2(v) \geq 8 - 2 - 4 \times \frac{1}{3} - 2 \times \frac{4}{3} - \max\{2 \times \frac{2}{3} + 4 \times \frac{1}{6}, 2 \times \frac{5}{6} + 2 \times \frac{1}{6}\} = 0$ by R3-R5.

Let v be a 6-vertex in G' . We assume that $f_1 = v_1v_2v$, $f_2 = v_3v_4v$ and f_3 is worst.

• v is bad and dangerous in G' ;

W.l.o.g, assume f_2 is worse. If $d(v_1) = d(v_2) = 4$, then $ch_2(v) = ch_1(v) = 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - 1 - \frac{2}{3} = 0$. If $d(v_i) \geq 5$ for some $i \in \{1, 2\}$, then v_i can not be bad by {(c), (d), (i)}, and $\varpi(v) \leq 1$ by {(c), (e), (m)}, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - 1 - \frac{1}{2} - \frac{1}{6} = 0$ by R4 and R5. If $d(v_i) \geq 5$ for all $i \in \{1, 2\}$, then $n_b(v) \leq 1$ by {(c), (d), (h)}, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - 1 - \frac{1}{2} - \frac{1}{6} = 0$ by R3 and R6.

• v is not bad but dangerous in G' ;

Case 1. $n_3(v) = 4$;

Then $n_{5+}(v) = 2$. It follows that $ch_2(v) = ch_1(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - 2 \times \frac{5}{6} = 0$.

Case 2. $n_3(v) = 3$;

Let v_1 be another 3-vertex, then $d(v_2) \geq 5$. If $d(v_3) = d(v_4) = 4$, then $n_b(v) = 0$ by Lemma 3.2(2), and $\varpi(v) \leq 1$ by {(a) or (b), (c), (k)}, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - \frac{5}{6} - \frac{2}{3} - \frac{1}{6} = 0$ by R3 and R4. If $d(v_i) \geq 5$ for some $i \in \{3, 4\}$, then $n_b(v) = 0$ by {(a), (c), (i)} and Lemma 3.2. Moreover, both v_2 and v_i are well vertices by Lemma 3.6, thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - \frac{5}{6} - \frac{2}{3} - 2 \times \frac{1}{12} = 0$ by R8. If $d(v_i) \geq 5$ for all $i \in \{3, 4\}$, note that $n_b(v) = 1$ by {(a), (c), (h)}

and v_2 must be well vertices by Lemma 3.6, then $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - \frac{5}{6} - \frac{3}{4} - \frac{1}{12} = 0$ by R8.

Case 3. $n_3(v) = 2$;

If $n_4(v) = 4$, then $\varpi(v) \leq 2$ by <{(c), (k), (l)}, it follows that $ch_2(v) = 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - 2 \times \frac{2}{3} - 2 \times \frac{1}{6} = 0$ by R3. We denote $N_4(v) = N(v) \setminus \{v_5, v_6\}$, suppose $n_{5^+}(v) = 1$. If $n_b(v) = 1$, then $\varpi(v) \leq 1$ by <{(c), (i), (l)}, <{(c), (j), (k)}, it follows that $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - \frac{2}{3} - \frac{5}{6} - \frac{1}{6} = 0$ by R3 and R5. Otherwise if $n_b(v) = 0$, then $\varpi(v) \leq 2$ by <{(c), (l), (m)}, it follows that $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - 2 \times \frac{2}{3} - 2 \times \frac{1}{6} = 0$ by R3 and R5. If $n_{5^+}(v) = 2$, say v_1, v_2 or v_1, v_3 . In the former case, by Lemma 3.2(4), we get that $n_b(v) \leq 1$. Suppose $n_b(v) = 1$, say v_1 , if v_1 is also dangerous, then $\varpi(v) \leq 1$ by Lemma 3.7(6), then $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - \frac{2}{3} - \frac{3}{4} - \frac{1}{6} = \frac{1}{12} > 0$. Otherwise, we get $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - \frac{2}{3} - \frac{2}{3} - 2 \times \frac{1}{6} = 0$ by R6. Suppose $n_b(v) = 0$, then $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - 2 \times \frac{2}{3} - 2 \times \frac{1}{6} = 0$ by R3 and R6. In the latter case, suppose $n_b(v) = 2$, then $ch_2(v) = ch_1(v) = 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - 2 \times \frac{5}{6} = 0$. If $n_b(v) = 1$, then $\varpi(v) \leq 1$ by <{(c), (j), (m)}, it follows that $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - \frac{2}{3} - \frac{5}{6} - \frac{1}{6} = 0$ by R5. Otherwise if $n_b(v) = 0$, it follows that $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - 2 \times \frac{2}{3} - 2 \times \frac{1}{6} = 0$ by R5. If $n_{5^+}(v) = 3$, then $n_b(v) \leq 2$ by <{(c), (h), (i)}. If $n_b(v) = 2$, by the same argument, we have $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - \max\{\frac{5}{6} + \frac{3}{4}, \frac{5}{6} + \frac{2}{3} + \frac{1}{6}\} = 0$ by R5 and R6. If $n_b(v) \leq 1$, then $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - \max\{\frac{5}{6} + \frac{2}{3}, \frac{2}{3} + \frac{3}{4}\} - \frac{1}{6} = 0$ by R5 and R6. If $n_{5^+}(v) = 4$, then there are at most two bad vertices by Lemma 3.2(4), thus $ch_2(v) \geq 6 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - 2 \times \frac{3}{4} = \frac{1}{6} > 0$ by R3 and R6.

If $d(v)$ is odd, note that v is also bad, it follows that $f_{3,4}(v) = f_{3b}(v) = f_{4b}(v) = f_{bb}(v) = 0$. Then $ch_2(v) \geq d(v) - 2 - \frac{4}{3}f_{3,3}(v) - 1 - \frac{1}{3}(d(v) - f_3(v)) = \frac{d(v)-7}{6} \geq 0$ when $d(v) \geq 7$.

Let v be a 5-vertex in G' . Let $f_1 = v_1v_2v$ and assume that f_2 is worst. If there exists a 3-vertex lying on f_1 , say v_1 , then $d(v_2) \geq 5$ and v_2 is not bad by <{(c), (f)}. It follows that $ch_2(v) = ch_1(v) \geq 5 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - \frac{2}{3} = 0$ by R3 and R4. If $d(v_1) = d(v_2) = 4$, then $ch_2(v) = ch_1(v) = 5 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - \frac{2}{3} = 0$ by R3. Otherwise, there exists a 5⁺-vertex lying on f_1 which is not bad by <{(c), (i)}, then $\varpi(v) \leq 1$ by <{(a) or (b), (c), (m)}. It follows that $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - \frac{1}{2} - \frac{1}{6} = 0$ by R3 and R5. If there are two 5⁺-vertices lying on f_1 , it follows from <{(c), (h)} that $n_b(v) \leq 1$, thus $ch_2(v) \geq 5 - 2 - 3 \times \frac{1}{3} - \frac{4}{3} - \frac{1}{2} - \frac{1}{6} = 0$ by R3 and R5.

Let f be a 6⁺-face in G' . Then $ch_2(f) \geq 0$ by R1 and R7.

Let f be a 3-face in G' . Let $f = v_1v_2v_3$, we next consider different cases corresponding to the shape of f . If f is poor, it follows that $ch_2(f) \geq -2 + \frac{1}{3} + 2 \times \frac{2}{3} + \min\{\frac{1}{3}, 2 \times \frac{1}{6}\} = 0$ by R1, R7 and R8. In particular, if there exists at least one 4-vertex which is not vice on f , then $ch_2(f) \geq -2 + 1 + \frac{1}{3} + \frac{2}{3} = 0$ by R2.1. If $d(v_1) = 3$, $3 \leq d(v_2) \leq 4$, $d(v_3) \geq 5$, note that v_2

cannot be bad, then $ch_2(f) \geq -2 + \frac{1}{3} + \min\{\frac{1}{3} + \frac{4}{3}, \frac{2}{3} + 1\} = 0$ by R3. If $d(v_1) = 3$, $d(v_2) \geq 5$, $d(v_3) \geq 5$, by Lemma 3.2, there is at most one bad vertex contained in $\{v_2, v_3\}$. It follows that $ch_2(f) \geq -2 + \frac{1}{3} + \min\{1 + \frac{2}{3}, 2 \times \frac{5}{6}\} = 0$ by R4. If $d(v_i) = 4$ for each $i \in \{1, 2, 3\}$, then $ch_2(f) = -2 + 3 \times \frac{2}{3} = 0$ by R2.3. If $d(v_1) = 4$, $d(v_2) = 4$, $d(v_3) \geq 5$, it follows that there is at most one bad vertex contained in $\{v_1, v_2, v_3\}$, then $ch_2(f) \geq -2 + \min\{\frac{2}{3} + \frac{1}{2} + \frac{5}{6}, 3 \times \frac{2}{3}\} = 0$. If $d(v_1) = 4$ and $d(v_i) \geq 5$ for each $i \in \{2, 3\}$, then v_2 and v_3 are not bad at the same time, it follows that $ch_2(f) \geq -2 + \frac{2}{3} + \min\{\frac{1}{2} + \frac{5}{6}, 2 \times \frac{2}{3}\} = 0$. If $d(v_i) \geq 5$ for all $i \in \{1, 2, 3\}$, then $ch_2(f) \geq -2 + \min\{3 \times \frac{2}{3}, \frac{1}{2} + 2 \times \frac{3}{4}, 2 \times \frac{1}{2} + 1\} = 0$ by R6.

Hence, $ch_2(x) \geq 0$ for all $x \in V(G') \cup F(G')$, this contradiction completes the proof of Theorem 1.

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