

On forward invariance in Lyapunov stability theorem for local stability

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Abstract

Forward invariance of a basin of attraction is often overlooked when using a Lyapunov stability theorem to prove local stability; even if the Lyapunov function decreases monotonically in a neighborhood of an equilibrium, the dynamic may escape from this neighborhood. In this note, we fix this gap by finding a smaller neighborhood that is forward invariant. This helps us to prove local stability more naturally without tracking each solution path. Similarly, we prove a transitivity theorem about basins of attractions without requiring forward invariance.

Keywords: Lyapunov function, local stability, forward invariance, evolutionary dynamics,

1 Introduction

The idea of Lyapunov stability theorem or Lyapunov's direct method is intuitive: if we find a mapping (*Lyapunov function*) from the current state of a dynamic to a real number such that i) the function attains a local minimum only at an equilibrium (possibly a set) and ii) its value decreases as long as the current state has not reached the equilibrium, then the equilibrium is stable under the dynamic. With this on hand, (we hope that) we do not have to identify a solution path; we just find a Lyapunov function and see how it behaves in the neighborhood—in particular, the value and first-order derivatives at each point in the state space. So, we typically find a neighborhood where the decrease in the Lyapunov function is guaranteed, which call here a monotone decrease neighborhood, and expect this neighborhood to be a basin of attraction.

However, a basin of attraction must be forward invariant. (This does not matter for global stability, of course.) Precisely, in known versions of Lyapunov stability theorem (e.g. Smirnov 2001), the monotone decrease must be assured to hold on each solution path. Even if we find a monotone decrease neighborhood, a solution path may escape from this neighborhood and eventually the Lyapunov function may not decrease after the escape. This imposes an additional burden of proof, losing an appeal of the theorem

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to intuition since we eventually need to identify a solution path. This is overlooked in practice; e.g. Sandholm (2010a) and Zusai (2018) on evolutionary dynamics in games, which we fix in this paper.

Similarly, we would expect transitivity of such basins of attractions. That is, if we find a Lyapunov function that decreases in X_1 and attains the minimum in X_2 and another that decreases in X_2 and attains the minimum in X^* , then we expect X^* to be stable in X_1 . Again, known versions of transitivity theorems as in Conley (1978) (see also (Oyama et al., 2015, Theorem 3)) require X_1 to be forward invariant and X_2 to be forward and also strongly negative (i.e., backward) invariant.¹

In applications to economics or game theory, we hope to find a Lyapunov function from economic intuition. Under an agent-based dynamic in a game or economic model, an aggregate of agents' possible gains from adjustment of their choices can be used as a candidate for a Lyapunov function once we find a neighborhood where an agent's revision of the choice incurs negative payoff externality to others' gains from further changes, as generally proven by Zusai (2020a). However, forward invariance needs more mathematical examination of the dynamic system, which may not be appealing to economic intuition.

In this paper, we reduce the burden of proof by showing that we can construct a forward invariant (smaller) neighborhood from a monotone decrease neighborhood. This fills the gap in applications, as in the papers mentioned above. Further, this helps us to establish a transitivity theorem without requiring forward or negative invariance.

We consider a differential inclusion (a set-valued differential equation) and also an equilibrium set, not necessarily a point. This generalization is needed to cover evolutionary dynamics in games, since Nash equilibrium may constitute a (connected) set and also a transition may not be uniquely specified when there are multiple best responses.

2 Definitions and theorems

We consider an autonomous differential inclusion \mathcal{V} such as

$$\dot{\mathbf{x}} \in \mathcal{V}(\mathbf{x})$$

on a compact metric A -dimensional real space $\mathcal{X} \subset \mathbb{R}^A$ with $A < \infty$. $T\mathcal{X}$ stands for the tangent space of \mathcal{X} .² As a solution concept for the differential inclusion, we adopt a Carathéodory solution; that is, a solution path $\{\mathbf{x}_t\}_{t \geq 0}$ must be Lipschitz continuous at every $t \geq 0$ and also differentiable with derivative $\dot{\mathbf{x}}_t \in \mathcal{V}(\mathbf{x}_t)$ at almost every t .

Let X^* be a nonempty closed set. We say X^* is **Lyapunov stable** under \mathcal{V} if for any open neighborhood O of X^* there exists a neighborhood O' of X^* such that *every* solution

¹Strong negative invariance of X means that, if a solution path (starting at time 0) visits X at any positive time, then it must have started from X at time 0.

²Below the statements of the definitions follow Sandholm (2010b), a canonical reference book on evolutionary dynamics in games.

path $\{\mathbf{x}^t\}_{t \geq 0}$ that starts from O' remains in O . X^* is **attracting** if there is a neighborhood O of X^* such that *every* solution that starts in O converges to X^* ; O is called a basin of attraction to X^* . If it is the entire space \mathcal{X} , then we say X^* is globally attracting. X^* is **asymptotically stable** if it is Lyapunov stable and attracting; it is globally asymptotically stable if it is Lyapunov stable and globally attracting.

Lyapunov stability theorem.

Theorem 1 (Lyapunov stability theorem). *Let X^* be a non-empty closed set in a compact metric space \mathcal{X} with tangent space $T\mathcal{X}$, and X' be a neighborhood of X^* . Suppose that continuous function $W : \mathcal{X} \rightarrow \mathbb{R}$ and lower semicontinuous function $\tilde{W} : \mathcal{X} \rightarrow \mathbb{R}$ satisfy (a) $W(\mathbf{x}) \geq 0$ and $\tilde{W}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in X'$ and (b) $\text{cl } X' \cap W^{-1}(0) = X' \cap \tilde{W}^{-1}(0) = X^*$. In addition, assume that W is Lipschitz continuous in $\mathbf{x} \in X'$. i) If a differential inclusion $\mathbf{V} : \mathcal{X} \rightarrow T\mathcal{X}$ satisfies³*

$$DW(\mathbf{x})\dot{\mathbf{x}} \leq \tilde{W}(\mathbf{x}) \quad \text{for any } \dot{\mathbf{x}} \in \mathbf{V}(\mathbf{x}) \quad (1)$$

whenever W is differentiable at $\mathbf{x} \in X'$, then X^ is asymptotically stable under \mathbf{V} . ii) If X' is forward invariant, i.e., every Carathéodory solution path $\{\mathbf{x}_t\}$ starting from X' at time 0 remains in X' for all moments of time $t \in [0, \infty)$, then X' is a basin of attraction to X^* .*

We call W a *Lyapunov function* and \tilde{W} a *decaying rate function*. Note that we allow for multiplicity of transition vectors, while requiring functions W and \tilde{W} to be well defined (the uniqueness of the values) as functions of state variable \mathbf{x} , independently of the choice of transition vector $\dot{\mathbf{x}}$ from $\mathbf{V}(\mathbf{x})$.

In a standard Lyapunov stability theorem (e.g. Robinson (1998, §5.5.3)) for a differential equation, a decaying rate function \tilde{W} is not explicitly required while \dot{W} is assumed to be (strictly) negative until \mathbf{x} reaches the limit set X^* . The most significant difference is the requirement of lower semicontinuity of \tilde{W} . This assures the existence of a lower bound on the decaying rate $\dot{W}(\mathbf{x}) \leq \bar{w} < 0$ in a hypothetical case in which \mathbf{x} remained out of an arbitrarily small neighborhood of X^* for an arbitrarily long period of time. This excludes the possibility that \mathbf{x} would stay there forever and guarantees convergence to X^* (not only Lyapunov stability, i.e., no asymptotic escape from X^*).

Zusai (2018, Theorem 7) modifies the Lyapunov stability theorem for a differential inclusion in Smirnov (2001, Theorem 8.2). While the latter is applicable to a singleton of an equilibrium point, the former allows convergence to a set of equilibria.

Theorem 1 in this paper relaxes assumptions in Zusai (2018, Theorem 7). The previous version imposes a stronger assumption than (1): every Carathéodory solution $\{\mathbf{x}_t\}$ starting from X' should satisfy

$$\dot{W}(\mathbf{x}_t) \leq \tilde{W}(\mathbf{x}_t) \quad \text{for almost all } t \in [0, \infty). \quad (2)$$

³ D denotes differentiation, so $DW(\mathbf{x}) = dW/d\mathbf{x}(\mathbf{x}) = [\partial W/\partial x_1(\mathbf{x}), \dots, \partial W/\partial x_A(\mathbf{x})]$.

If X' is forward invariant, then (1) implies this condition; thus, Zusai (2018) is straightforwardly applied and we can conclude that X^* is asymptotically stable and X' is a basin of attraction, as restated in part ii) of Theorem 1.⁴ In part i) of Theorem 1 in the current version, we do not require forward invariance of X' ; a solution trajectory may escape from X' and thus (2) may not be maintained. Thus, the current version weakens the assumption.

Besides, (1) is assumed for every point in the entire space \mathcal{X} and \tilde{W} is assumed to be continuous. By checking the places in the proof where the definition of the domain for condition (i) in the theorem⁵ and continuity of \tilde{W} were used, one can easily find that it is innocuous to replace simplex $\Delta^A \subset \mathbb{R}^A$ with a closed subset X^* of a compact metric space $\mathcal{X} \subset \mathbb{R}^A$ and relax continuity of \tilde{W} to lower semicontinuity.⁶

Transitivity theorem.

Theorem 2 (Transitivity theorem). *Let $X_1 \supset X_2 \supset X^*$ be three non-empty subsets of a compact metric space \mathcal{X} ; assume that X^* is closed and X_1 is open. Suppose that two Lipschitz continuous functions $W_1, W_2 : X_1 \rightarrow \mathbb{R}$ and two lower semicontinuous functions $\tilde{W}_1, \tilde{W}_2 : X_1 \rightarrow \mathbb{R}$ satisfy the following assumptions: for any $\mathbf{x} \in X_1$,*

- a) i) $W_1(\mathbf{x}) \geq 0$, ii) $\tilde{W}_1(\mathbf{x}) \leq 0$, and iii) $\text{cl } X_1 \cap W_1^{-1}(0) = \text{cl } X_1 \cap \tilde{W}_1^{-1}(0) = \text{cl } X_2$;
- b) i) $W_2(\mathbf{x}) \geq 0$, ii) $[\mathbf{x} \in X_2 \Rightarrow \tilde{W}_2(\mathbf{x}) \leq 0]$, and iii) $\text{cl } X_2 \cap W_2^{-1}(0) = \text{cl } X_2 \cap \tilde{W}_2^{-1}(0) = X^*$;
- c) $\tilde{W}_1(\mathbf{x}) + \tilde{W}_2(\mathbf{x}) \leq 0$.

Furthermore, assume that

$$a\text{-iv)} DW_1(\mathbf{x})\dot{\mathbf{x}} \leq \tilde{W}_1(\mathbf{x}), \quad b\text{-iv)} DW_2(\mathbf{x})\dot{\mathbf{x}} \leq \tilde{W}_2(\mathbf{x}) \quad \text{for any } \dot{\mathbf{x}} \in \mathcal{V}(\mathbf{x}),$$

whenever W_1 and W_2 are differentiable at $\mathbf{x} \in X_1$. Then, X^* is asymptotically stable under \mathcal{V} .

Conditions a) imply that W_1 works as a Lyapunov function for local asymptotic stability of X_2 in (a subset of) X_1 and conditions b) imply that W_2 works as a Lyapunov function for local asymptotic stability of X^* in (a subset of) X_2 . So, we may jump to conclude that X^* is asymptotically stable in X_1 . If X_1 is indeed forward invariant and thus a basin of attraction to X_2 and X_2 is forward and also strongly negative invariant, then

⁴Once monotone decrease in W is confirmed for any solution path as in (2), convergence to $W^{-1}(0)$ is obtained simply by using Grönwall's inequality.

⁵In the notation of the current version, the condition reads as $W(\mathbf{x}) \geq 0$ and $\tilde{W}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \mathcal{X}$. Thus it corresponds with condition (a) in Theorem 1.

⁶Specifically, \tilde{A} in the proof (Zusai, 2018, p.25) should be defined as a subset of $\text{cl } X'$. The continuity of \tilde{W} was used to assure the existence of the minimum of \tilde{W} in \tilde{A} ; for this, lower semicontinuity is sufficient. Then, with the observation that forward invariance of X' implies (2), the proof for the previous version applies to part ii) of the current version.

the transitivity theorem as in Conley (1978, §II.5.3.D) and Oyama et al. (2015, Thorem 3) guarantees asymptotic stability of X^* with X_1 being a basin of attraction to X^* .

However, we do not assume invariance of these sets in our theorem. Even though Theorem 1 eventually assures the existence of some forward invariant subset of X_2 from conditions b), the Lyapunov function W_1 in conditions a) guarantees convergence only to X_2 , but not necessarily to this forward invariant subset. Furthermore, the standard transitivity theorem also requires strongly negative invariance of X_2 , which may not be satisfied by the basin of attraction that we could find using Theorem 1.

The above theorem avoids this issue by imposing condition c) to hold in the whole X_1 , which we use to construct a Lyapunov function in X_1 to X^* . Then, we apply Theorem 1 and thus the basin of attraction to X^* is smaller than X_1 . When current state \mathbf{x} is in the interim subset X_2 , conditions a) and b-ii) imply condition c). Condition c) deals with the case that \mathbf{x} is still out of X_2 and thus $\tilde{W}_2(\mathbf{x})$ may be positive (and thus W_2 may be increasing over time). Condition c) requires this to be suppressed by \tilde{W}_1 , which must be negative in the entire X_1 by condition a-iii).

Applications in game theory. Local stability of an equilibrium is one of the fundamental issues in game theory. A game may exhibit multiple Nash equilibria and thus each equilibrium may not be globally stable; thus, while we investigate global stability of the set of equilibria, we check local stability of each isolated equilibrium or each isolated connected set of equilibria.

A canonical condition to derive local stability under economically reasonable dynamics is negative payoff externality, specifically called *self-defeating externality* by Hofbauer and Sandholm (2009). We regard a (*population*) game as a mapping (payoff function) \mathbf{F} from a distribution of strategies among (continuously many) agents $\mathbf{x} \in \Delta^A$ to a payoff vector $\boldsymbol{\pi} = \mathbf{F}(\mathbf{x}) \in \mathbb{R}^A$. Self-defeating externality boils down to negative semidefiniteness of $\mathbf{z} \cdot D\mathbf{F}(\mathbf{x})\mathbf{z}$; a marginal deviation \mathbf{z} in the strategy distribution from \mathbf{x} triggers the change in the payoff vector, which is approximated as $D\mathbf{F}(\mathbf{x})\mathbf{z}$. Self-defeating externality imposes a negative correlation between \mathbf{z} and $D\mathbf{F}(\mathbf{x})\mathbf{z}$; a strategy whose share increases by this deviation should face a decrease in its payoff and thus becomes less disadvantageous.

From this condition on \mathbf{F} , economists naturally expect agents to return to the equilibrium, while it needs to formulate how agents revise their choices of strategies in response to payoff changes. An evolutionary dynamic is a dynamic of the strategy distribution, constructed as a mean dynamic of agents each of whom revises its own strategy upon a receipt of a revision opportunity following a Poisson process (Sandholm, 2010b). Sandholm (2010a) considers an equilibrium that is essentially characterized by self-defeating externality, called a regular evolutionary stable state and attempts to prove its local stability under several canonical classes of evolutionary dynamics, such as excess payoff dynamics and pairwise payoff comparison dynamics. Similarly, Zusai (2018) proves it for another class of dynamics, called tempered best response dynamics. These papers

refer to a standard version of Lyapunov stability theorem as mentioned after Theorem 1, which requires monotone decrease in the Lyapunov function along with each solution path. However, these papers confirm its decrease only at each point in a neighborhood of a regular ESS, where the self-defeating externality holds. It is left unchecked whether this neighborhood is forward invariant. Our Theorem 1 fixes this overlooked point.

In both the two papers, the Lyapunov function is decomposed to two parts; one is the aggregate of possible payoff gains for agents from revisions of strategies and another is the mass of agents who currently choose the strategies that are to be abandoned in the regular ESS. Thanks to this decomposition, we can apply Theorem 2. Zusai (2020b) uses it to generalize their results to a broader class of economically natural dynamics.

3 Proofs

Proof of Theorem 1

Proof. Here we prove the difference in part i) from Zusai (2018, Theorem 7). For this, we focus on the case of $X' \subsetneq \mathcal{X}$ and find a forward invariant subset of X' . Once we find it, any Carathéodory solution starting from the forward invariant subset remains there and thus satisfies (2) as (1) holds for $\mathbf{x} = \mathbf{x}_t$ at each time $t \in \mathbb{R}_+$. Then, Zusai (2018, Theorem 7) is applied and assures asymptotic stability of X^* while having the forward invariant subset as a basin of attraction.

First, construct a distance from point $\mathbf{x} \in \mathcal{X}$ to X^* based on the metric on \mathcal{X} , say $d : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$, by

$$d_*(\mathbf{x}) := \min_{\mathbf{x}^* \in X^*} d(\mathbf{x}, \mathbf{x}^*).$$

Since X^* is a non-empty compact set and $d(\mathbf{x}, \mathbf{x}^*)$ is continuous in \mathbf{x}^* when \mathbf{x} is fixed, Weierstrass theorem assures the existence of the minimum in the above definition of $d_*(\mathbf{x})$. This d_* satisfies

$$d_*(\mathbf{x}) \geq 0; \quad d_*(\mathbf{x}) = 0 \Leftrightarrow \mathbf{x} \in X^*.$$

Let \bar{d} be the shortest distance from the complement of X' to X^* :

$$\bar{d} := \min_{\mathbf{x} \in \mathcal{X} \setminus X'} d_*(\mathbf{x}). \tag{3}$$

Maximum theorem guarantees continuity of $d_* : \mathcal{X} \rightarrow \mathbb{R}_+$ by continuity of $d(\mathbf{x}, \mathbf{x}^*)$ in both \mathbf{x} and \mathbf{x}^* . Besides, $\mathcal{X} \setminus X'$ is a non-empty compact subset by $X' \subsetneq \mathcal{X}$ and the openness of X' . Hence, the minimum in (3) exists. It follows that

$$\bar{d} > 0; \quad d_*(\mathbf{x}) < \bar{d} \Rightarrow \mathbf{x} \in X'. \tag{4}$$

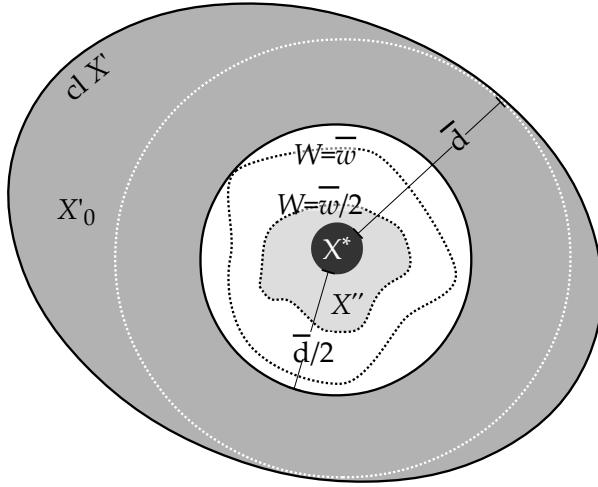


Figure 1: Sets in the proof of Theorem 1. X^* is the black area in the center and X'' is the light gray area. $\text{cl } X'$ is the entire oval, with the outermost outline. X'_0 is the dark gray area, including the both boundaries.

Define set $X'_0 \subset \text{cl } X'$ by

$$X'_0 := \text{cl } X' \cap d_*^{-1}([\bar{d}/2, \infty)).$$

Since both $\text{cl } X'$ and $d_*^{-1}([\bar{d}/2, \infty))$ are closed, X'_0 is closed and thus compact in \mathcal{X} . It is not empty, as proven here. Suppose $X'_0 = \emptyset$; then, any $\mathbf{x} \in \mathcal{X}$ with $d_*(\mathbf{x}) \geq \bar{d}/2$ must be out of $\text{cl } X'$. On the other hand, since $\text{cl } X'$ is not empty, X' has at least one boundary point \mathbf{x}^0 ; then, $d_*(\mathbf{x}^0) \geq \bar{d}$.⁷ By the former statement, this implies $\mathbf{x}^0 \notin \text{cl } X'$ but it contradicts with \mathbf{x}^0 being on the boundary of X' ; hence, X'_0 cannot be empty.

Let \bar{w} be the minimum of W in X'_0 ;

$$\bar{w} := \min_{\mathbf{x} \in X'_0} W(\mathbf{x}).$$

Since X'_0 is compact and nonempty and W is (Lipschitz) continuous, the minimum exists. Furthermore, it is positive; we have $X'_0 \subset \text{cl } X'$ by construction and $W(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \text{cl } X'$ by condition (a) and continuity of W , while no element $\mathbf{x} \in X'_0$ belongs to X^* since $d_*(\mathbf{x}) \geq \bar{d} > 0$ for any $\mathbf{x} \in X'_0$. Because $X^* = \text{cl } X' \cap W^{-1}(0)$ by condition (b) and $X'_0 \subset \text{cl } X'$, it implies $\mathbf{x} \in X'_0 \Rightarrow W(\mathbf{x}) > 0$. Hence we have $\bar{w} > 0$; by the definition of \bar{w} , we have

$$\underbrace{[\mathbf{x} \in \text{cl } X' \text{ and } d_*(\mathbf{x}) \geq \bar{d}/2]}_{\text{i.e., } \mathbf{x} \in X'_0} \Rightarrow W(\mathbf{x}) \geq \bar{w}. \quad (5)$$

Define set $X'' \subset X'$ by

$$X'' = W^{-1}([0, \bar{w}/2]) \cap X'. \quad (6)$$

⁷We can make a sequence converging to \mathbf{x}^0 from elements out of X' , whose distance from X^* cannot be smaller than \bar{d} by (3).

This set is an (open) neighborhood of X^* by $X^* \subset X''$, since $W = 0$ at anywhere in X^* and $X^* \subset X'$. Now we prove that X'' is wholly contained in set $d_*^{-1}([0, \bar{d}/2])$. Assume that there exists $\mathbf{x} \in X''$ such that $d_*(\mathbf{x}) \geq \bar{d}/2$. These jointly imply $W(\mathbf{x}) \geq \bar{w}$ by (5) since $\mathbf{x} \in X'' \subset X' \subset \text{cl } X'$. However, this contradicts with $W(\mathbf{x}) \in [0, \bar{w}/2)$ for \mathbf{x} to belong to X'' . Hence, we have

$$\mathbf{x} \in X'' \Rightarrow d_*(\mathbf{x}) < \bar{d}/2. \quad (7)$$

Now we prove X'' is forward invariant. To verify it by contradiction, assume that there is a Carathéodory solution trajectory $\{\mathbf{x}_t\}$ starting from X'' but escaping X'' at some moment of time:

$$\mathbf{x}_0 \in X'', \quad \text{and } \mathbf{x}_T \notin X'' \text{ at some } T > 0. \quad (8)$$

The statement $\mathbf{x}_T \notin X''$ means $\mathbf{x}_T \notin X'$ or $W(\mathbf{x}_T) > \bar{w}/2$ by (6). In the former case, we have $d_*(\mathbf{x}_T) \geq \bar{d}$ by (3) while $d_*(\mathbf{x}_0) < \bar{d}/2$ by (7). By continuity of $d_*(\mathbf{x})$ in \mathbf{x} and of \mathbf{x}_t in t on a Carathéodory solution trajectory $\{\mathbf{x}_t\}$, $d_*(\mathbf{x}_t)$ is continuous in t ; hence, there exists a moment of time $T' \in (0, T)$ such that $d_*(\mathbf{x}_{T'}) = 0.9\bar{d} \in (0.5\bar{d}, \bar{d}) \subset (d_*(\mathbf{x}_0), d_*(\mathbf{x}_T))$. At this point, $\mathbf{x}_{T'} \notin X''$ by (7) while $\mathbf{x}_{T'} \in X'$ by (4); thus, $W(\mathbf{x}_{T'}) \geq \bar{w}/2$ by (6). Hence, the first case of escaping X'' implies the existence of $T' > 0$ such that

$$W(\mathbf{x}_{T'}) \geq \bar{w}/2 \quad \text{and} \quad \mathbf{x}_{T'} \in X'.$$

In the second (but not the first) case, we have $W(\mathbf{x}_T) \geq \bar{w}/2$ but $\mathbf{x}_T \in X'$; that is, the above statement holds with $T' = T$.

This implies the existence of $\bar{T} \in (0, T']$ such that

$$W(\mathbf{x}_{\bar{T}}) \geq \bar{w}/2, \quad \text{and} \quad [\mathbf{x}_t \in X' \text{ for all } t < \bar{T}]. \quad (9)$$

To prove it, assume $\mathbf{x}_{t'} \notin X'$ at some $t' < T'$, i.e., the negation of the latter condition with $\bar{T} = T'$; if there is no such $t' \leq T'$, then it suggests that the claim (9) holds at $\bar{T} = T'$ by the fact $W(\mathbf{x}_{T'}) \geq \bar{w}/2$. By (4), the hypothesis $\mathbf{x}_{t'} \notin X'$ implies $d_*(\mathbf{x}_{t'}) \geq \bar{d}$. Again, by continuity of $d_*(\mathbf{x}_t)$ in t , the set $\{t \leq t' \mid d_*(\mathbf{x}_t) \geq \bar{d}\}$ is closed and thus compact; by the fact $d_*(\mathbf{x}_0) < \bar{d}/2$, this implies the existence of the minimum \bar{T} in this set and $\bar{T} > 0$. That is, we have $d_*(\mathbf{x}_t) < \bar{d}$ for all $t < \bar{T}$ while $d_*(\mathbf{x}_{\bar{T}}) = \bar{d}$. The former implies $\mathbf{x}_t \in X'$ for all $t < \bar{T}$ by (4) and the latter implies $W(\mathbf{x}_{\bar{T}}) \geq \bar{w}$ by $\mathbf{x}_{\bar{T}} = \lim_{t \rightarrow \bar{T}} \mathbf{x}_t \in \text{cl } X'$ and (5). Thus, the above claim (9) holds at this $\bar{T} \in (0, T']$. Since condition (a) and (1) hold almost everywhere in X' , we have $\dot{W}(\mathbf{x}_\tau) \leq \tilde{W}(\mathbf{x}_\tau) \leq 0$ at almost all $\tau < \bar{T}$ ⁸; thus, we have

$$W(\mathbf{x}_{\bar{T}}) \leq W(\mathbf{x}_0) + \int_0^{\bar{T}} \tilde{W}(\mathbf{x}_\tau) d\tau \leq W(\mathbf{x}_0).$$

⁸A Carathéodory solution trajectory is differentiable at almost all moments of time, though it may not be so at all moments.

Since $W(\mathbf{x}_0) < \bar{w}/2$ by $\mathbf{x}_0 \in X''$, we have $W(\mathbf{x}_{\bar{T}}) < \bar{w}/2$ in (9). This contradicts with $W(\mathbf{x}_{\bar{T}}) \geq \bar{w}/2$.

Therefore, the hypothesis (8) cannot hold: any Carathéodory solution trajectory $\{\mathbf{x}_t\}$ starting from X'' cannot escape X'' at any moment of time. That is, X'' is forward invariant. \square

Proof of Theorem 2

Proof. Define a Lyapunov function $W : X_1 \rightarrow \mathbb{R}$ and a decaying rate function $\tilde{W} : X_1 \rightarrow \mathbb{R}$ by

$$W(\mathbf{x}) := 2W_1(\mathbf{x}) + W_2(\mathbf{x}), \quad \tilde{W}(\mathbf{x}) := 2\tilde{W}_1(\mathbf{x}) + \tilde{W}_2(\mathbf{x}) \quad \text{for each } \mathbf{x} \in X_1.$$

Lipschitz continuity of W_1 and W_2 and lower semicontinuity of \tilde{W}_1 and \tilde{W}_2 are succeeded to those of W and \tilde{W} , respectively. It is immediate from assumptions a-i,iv), b-i,iv) and c) to see that

$$\begin{aligned} W(\mathbf{x}) &= 2W_1(\mathbf{x}) + W_2(\mathbf{x}) \geq 0, \\ \tilde{W}(\mathbf{x}) &= \tilde{W}_1(\mathbf{x}) + \{\tilde{W}_1(\mathbf{x}) + \tilde{W}_2(\mathbf{x})\} \leq 0, \\ DW(\mathbf{x})\dot{\mathbf{x}} &= 2DW_1(\mathbf{x})\dot{\mathbf{x}} + DW_2(\mathbf{x})\dot{\mathbf{x}} \leq 2\tilde{W}_1(\mathbf{x}) + \tilde{W}_2(\mathbf{x}) = \tilde{W}(\mathbf{x}) \end{aligned} \quad (10)$$

for any $\mathbf{x} \in X_1, \dot{\mathbf{x}} \in \mathcal{V}(\mathbf{x})$ (for the last equation assuming that W_1 and W_2 are differentiable at \mathbf{x}).

Further, since $X^* \subset X_2$, it follows assumptions a-iii) and b-iii) that $W(\mathbf{x}) = \tilde{W}(\mathbf{x}) = 0$ if $\mathbf{x} \in X^*$; thus X^* is contained in $\text{cl } X_1 \cap W^{-1}(0)$ and $\text{cl } X_1 \cap \tilde{W}^{-1}(0)$ by $X^* \subset X_2 \subset X_1 \subset \text{cl } X_1$. In contrary, assume $W(\mathbf{x}) = 0$ at $\mathbf{x} \in \text{cl } X_1$ first. By assumptions a-i) and b-i), it must be the case that $W_1(\mathbf{x}) = 0$ and $W_2(\mathbf{x}) = 0$. The former implies $\mathbf{x} \in \text{cl } X_2$ by assumption a-iii). Together with this, the latter implies $\mathbf{x} \in X^*$ by assumption b-iii). Separately from this, now assume $\tilde{W}(\mathbf{x}) = 0$ at $\mathbf{x} \in \text{cl } X_1$. By assumptions a-ii) and c), it must be the case that $\tilde{W}_1(\mathbf{x}) = 0$ and $\tilde{W}_1(\mathbf{x}) + \tilde{W}_2(\mathbf{x}) = 0$.⁹ The former implies $\mathbf{x} \in \text{cl } X_2$ by assumption a-iii); besides, by plugging the former into the latter, we have $\tilde{W}_2(\mathbf{x}) = 0$. These two statements jointly imply $\mathbf{x} \in X^*$ by assumption b-iii). In sum, we have verified

$$\text{cl } X_1 \cap W^{-1}(0) = \text{cl } X_1 \cap \tilde{W}^{-1}(0) = X^*. \quad (11)$$

Note that the first equality is due to the fact that $X^* \subset X_1$ and thus $X^* \cap \text{bd } X_1 = \emptyset$ since X_1 is open.

We have verified all the assumptions in Theorem 1; therefore, X^* is asymptotically stable. Notice that X_1 may not be forward invariant, but part i) of Theorem 1 assures that we can make some subset of X_1 as a basin of attraction to X^* . \square

⁹Note that the latter condition alone cannot assure $\tilde{W}_2(\mathbf{x}) = 0$, since $\tilde{W}_2(\mathbf{x})$ could take a positive value unless \mathbf{x} is in X_2 .

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