

Preprint

CONNECTIONS BETWEEN COVERS OF \mathbb{Z} AND SUBSET SUMS

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ABSTRACT. In this paper we establish connections between covers of \mathbb{Z} by residue classes and subset sums in a field. Suppose that $\{a_s(n_s)\}_{s=0}^k$ covers each integer at least p times with the residue class $a_0(n_0)$ irredundant, where p is a prime not dividing any of n_1, \dots, n_k . Let $m_1, \dots, m_k \in \mathbb{Z}$ be relatively prime to n_1, \dots, n_k respectively. For any $c, c_1, \dots, c_k \in \mathbb{Z}/p\mathbb{Z}$ with $c_1 \cdots c_k \neq 0$, we show that the set

$$\left\{ \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} : I \subseteq \{1, \dots, k\} \text{ and } \sum_{s \in I} c_s = c \right\}$$

contains an arithmetic progression of length n_0 with common difference $1/n_0$, where $\{x\}$ denotes the fractional part of a real number x .

1. INTRODUCTION

For a finite set $S = \{a_1, \dots, a_k\}$ contained in the ring \mathbb{Z} or a field, sums in the form $\sum_{s \in I} a_s$ with $I \subseteq [1, k] = \{1, \dots, k\}$ are called *subset sums* of S . It is interesting to provide a lower bound for the cardinality of the set

$$\{a_1 x_1 + \cdots + a_k x_k : x_1, \dots, x_k \in \{0, 1\}\} = \left\{ \sum_{s \in I} a_s : I \subseteq [1, k] \right\}.$$

A more general problem is to study restricted sumsets in the form

$$\{x_1 + \cdots + x_k : x_1 \in X_1, \dots, x_k \in X_k, P(x_1, \dots, x_k) \neq 0\} \quad (1.1)$$

where X_1, \dots, X_k are subsets of a field and $P(x_1, \dots, x_k)$ is a polynomial with coefficients in the field.

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Let p be a prime. In 1964 Erdős and Heilbronn [EH] conjectured that if $\emptyset \neq X \subseteq \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ then

$$|\{x_1 + x_2 : x_1, x_2 \in X \text{ and } x_1 \neq x_2\}| \geq \min\{p, 2|X| - 3\}.$$

This conjecture was first confirmed by Dias da Silva and Hamidoune [DH] in 1994, who obtained a generalization which implies that if $S \subseteq \mathbb{Z}_p$ and $|S| > \sqrt{4p - 7}$ then any element of \mathbb{Z}_p is a subset sum of S . In this direction the most powerful tool is the following remarkable principle (see Alon [A99, A03]) rooted in Alon and Tarsi [AT] and applied in [AF], [ANR1, ANR2], [DKSS], [HS], [LS], [PS], [S03b], [S08] and [SZ].

Combinatorial Nullstellensatz (Alon [A99]). *Let X_1, \dots, X_k be finite subsets of a field F with $|X_s| > l_s$ for $s \in [1, k]$ where $l_1, \dots, l_k \in \mathbb{N} = \{0, 1, \dots\}$. If $f(x_1, \dots, x_k) \in F[x_1, \dots, x_k]$, and $[x_1^{l_1} \cdots x_k^{l_k}]f(x_1, \dots, x_k)$ (the coefficient of the monomial $\prod_{s=1}^k x_s^{l_s}$ in f) is nonzero and $\sum_{s=1}^k l_s$ is the total degree of f , then there are $x_1 \in X_1, \dots, x_k \in X_k$ such that $f(x_1, \dots, x_k) \neq 0$.*

One of many applications of the Combinatorial Nullstellensatz is the following result of [AT] concerning a conjecture of Jäger.

Alon-Tarsi Theorem. *Let F be a finite field with $|F|$ not a prime, and let M be a nonsingular $k \times k$ matrix over F . Then there exists a vector $\vec{x} = (x_1, \dots, x_k)^T$ with $x_1, \dots, x_k \in F$ such that neither \vec{x} nor $M\vec{x}$ has zero component.*

Now we turn to covers of \mathbb{Z} by finitely many residue classes.

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$, set

$$a(n) = a + n\mathbb{Z} = \{a + nx : x \in \mathbb{Z}\}$$

and call it a residue class with modulus n . For a finite system

$$A = \{a_s(n_s)\}_{s=1}^k \tag{1.2}$$

of residue classes, we define its *covering function* $w_A : \mathbb{Z} \rightarrow \mathbb{Z}$ by

$$w_A(x) = |\{1 \leq s \leq k : x \in a_s(n_s)\}|.$$

For properties of the covering function $w_A(x)$, one can consult [S03a, S04]. As in Sun [S97, S99], we call $m(A) = \min_{x \in \mathbb{Z}} w_A(x)$ the *covering multiplicity* of (1.2).

Erdős [E50] initiated the study of covers of \mathbb{Z} by residue classes. Zhang [Z89] showed that if (1.2) covers all the integers then $\sum_{s \in I} 1/n_s \in \mathbb{Z}^+$ for some $I \subseteq \{1, \dots, k\}$. Let $m \in \mathbb{Z}^+$. We call (1.2) an m -cover of \mathbb{Z} if $m(A) \geq m$. If (1.2) forms an m -cover of \mathbb{Z} but $A_t = \{a_s(n_s)\}_{s \neq t}$ does not, then we say that (1.2) is an m -cover of \mathbb{Z} with $a_t(n_t)$ *essential*.

The author [S99] established the following result.

Theorem (Sun [S99]). *Let $m \in \mathbb{Z}^+$ and let $A = \{a_s(n_s)\}_{s=0}^k$ be an m -cover of \mathbb{Z} with $a_0(n_0)$ essential. Let $m_1, \dots, m_k \in \mathbb{Z}^+$ be relatively prime to n_1, \dots, n_s respectively. Then the set*

$$\left\{ \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} : I \subseteq [1, k] \text{ and } \left\lfloor \sum_{s \in I} \frac{m_s}{n_s} \right\rfloor \geq m - 1 \right\}$$

contains an arithmetic progression of length n_0 with common difference $1/n_0$, where $\{x\}$ and $\lfloor x \rfloor$ are the fractional part and the integer part of a real number x .

Subset sums seem to have nothing to do with covers of \mathbb{Z} . Before our work no one else has realized their close connections. Can you imagine that the Alon-Tarsi theorem are related to covers of \mathbb{Z} ?

The purpose of this paper is to present a surprising unified approach and embed the study of subset sums in the investigation of covers. The key point of our unification is to compare the following two sorts of quantities:

- (a) Degrees of multi-variable polynomials over rings or fields,
- (b) Covering multiplicities of covers of \mathbb{Z} by residue classes.

In Section 2 we will present a general unified theorem connecting subset sums with covers of \mathbb{Z} and derive from it some consequences.

In Section 3 we will pose a formula for polynomials over a ring.

On the basis of Section 3, the reader will understand quite well the technique in Section 4 used to prove Theorem 2.1 which connects covers of \mathbb{Z} with subset sums.

The author [S03c] announced a unified approach to covers of \mathbb{Z} , subset sums and zero-sum problems. The detailed connections between covers of \mathbb{Z} and zero-sum problems were published in [S09].

Let $m \in \mathbb{Z}^+$. The system (1.2) is called an m -system if $w_A(x) \leq m$ for all $x \in \mathbb{Z}$. One may wonder whether such systems are also related to subset sums. Let

$$A^* = \{a_s + r(n_s) : r \in [1, n_s - 1] \text{ and } s \in [1, k]\} \quad (1.3)$$

and call it the *dual system* of (1.2) as in [S10]. Then $w_A(x) + w_{A^*}(x) = k$ for all $x \in \mathbb{Z}$. Thus (1.2) is an m -system if and only if $m(A^*) \geq k - m$. In light of this, we can reformulate our results related to covers of \mathbb{Z} in terms of m -systems.

2. A GENERAL THEOREM AND ITS CONSEQUENCES

Now we state our general theorem connecting covers of \mathbb{Z} with subset sums.

Theorem 2.1. Let $A_0 = \{a_s(n_s)\}_{s=0}^k$ be a system of residue classes with $w_{A_0}(a_0) = m(A_0)$. Let $m_1, \dots, m_k \in \mathbb{Z}$ be relatively prime to n_1, \dots, n_k respectively. Let $J \subseteq \{1 \leq s \leq k : a_0 \in a_s(n_s)\}$ and $P(x_1, \dots, x_k) \in F[x_1, \dots, x_k]$ where F is a field with characteristic not dividing $N = [n_1, \dots, n_k]$. Assume that $0 \leq \deg P \leq |J|$ and

$$\left[\prod_{j \in J} x_j \right] P(x_1, \dots, x_k) (x_1 + \dots + x_k)^{|J| - \deg P} \neq 0. \quad (2.1)$$

Let $X_1 = \{b_1, c_1\}, \dots, X_k = \{b_k, c_k\}$ be subsets of F such that $b_s = c_s$ only if $a_0 \in a_s(n_s)$ and $s \notin J$. Then, for some $0 \leq \alpha < 1$, we have

$$|S_r| \geq |J| - \deg P + 1 > 0 \quad \text{for all } r = 0, 1, \dots, n_0 - 1, \quad (2.2)$$

where

$$S_r = \left\{ \sum_{s=1}^k x_s : x_s \in X_s, P(x_1, \dots, x_k) \neq 0, \left\{ \sum_{\substack{s=1 \\ x_s \neq b_s}}^k \frac{m_s}{n_s} \right\} = \frac{\alpha + r}{n_0} \right\}. \quad (2.3)$$

In the case $n_0 = n_1 = \dots = n_k = 1$, Theorem 2.1 yields the following basic lemma of the so-called polynomial method due to Alon, Nathanson and Ruzsa [ANR1, ANR2]: Let X_1, \dots, X_k be subsets of a field F with $|X_s| > l_s \in \{0, 1\}$ for $s \in [1, k]$. If $P(x_1, \dots, x_k) \in F[x_1, \dots, x_k] \setminus \{0\}$, $\deg P \leq \sum_{s=1}^k l_s$ and

$$[x_1^{l_1} \cdots x_k^{l_k}] P(x_1, \dots, x_k) (x_1 + \dots + x_k)^{\sum_{s=1}^k l_s - \deg P} \neq 0,$$

then

$$\left| \left\{ \sum_{s=1}^k x_s : x_s \in X_s \text{ and } P(x_1, \dots, x_k) \neq 0 \right\} \right| \geq \sum_{s=1}^k l_s - \deg P + 1.$$

Actually this remains valid even if l_s may be greater than one.

Corollary 2.1. Let $A_0 = \{a_s(n_s)\}_{s=0}^k$ be an m -cover of \mathbb{Z} with $a_0(n_0)$ essential. Let $m_1, \dots, m_k \in \mathbb{Z}$ be relatively prime to n_1, \dots, n_k respectively. Let F be a field with characteristic p not dividing $[n_1, \dots, n_k]$, and let $X_1 = \{b_1, c_1\}, \dots, X_k = \{b_k, c_k\}$ be any subsets of F with cardinality 2. Then, for some $0 \leq \alpha < 1$, we have

$$\left| \left\{ \sum_{s=1}^k x_s : x_s \in X_s, \left\{ \sum_{\substack{1 \leq s \leq k \\ x_s = c_s}} \frac{m_s}{n_s} \right\} = \frac{\alpha + r}{n_0} \right\} \right| \geq \min\{p', m\} \quad (2.4)$$

for all $r \in [0, n_0 - 1]$, where $p' = p$ if p is a prime, and $p' = +\infty$ if $p = 0$.

Proof. Since $a_0(n_0)$ is essential, there is an $a \in a_0(n_0)$ such that $w_{A_0}(a) = m$. Note that $a_0(n_0) = a(n_0)$. Choose $J \subseteq \{1 \leq s \leq k : a \in a_s(n_s)\}$ with $|J| = \min\{p', m\} - 1$. Then

$$\left[\prod_{j \in J} x_j \right] (x_1 + \cdots + x_k)^{|J|} = \left[\prod_{j \in J} x_j \right] |J|! \prod_{j \in J} x_j \neq 0$$

since $|J| < p'$. Now it suffices to apply Theorem 2.1 with $P(x_1, \dots, x_k) = 1$. \square

Remark 2.1. Let $A_0 = \{a_s(n_s)\}_{s=0}^k$ be an m -cover of \mathbb{Z} with $a_0(n_0)$ essential. And let $m_1, \dots, m_k \in \mathbb{Z}^+$ be relatively prime to n_1, \dots, n_k respectively. By Corollary 2.1 in the case $F = \mathbb{Q}$ and $X_s = \{0, m_s/n_s\}$ ($1 \leq s \leq k$), for some $0 \leq \alpha < 1$ we have

$$\min_{r \in [0, n_0 - 1]} \left| \left\{ \sum_{s \in I} \frac{m_s}{n_s} : I \subseteq [1, k] \text{ and } \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} = \frac{\alpha + r}{n_0} \right\} \right| \geq m. \quad (2.5)$$

This implies that

$$\left\{ \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} : I \subseteq [1, k] \text{ and } \left\lfloor \sum_{s \in I} \frac{m_s}{n_s} \right\rfloor \geq m - 1 \right\}$$

contains an arithmetic progression of length n_0 with common difference $1/n_0$, which was first established by the author in [S99]. In 2007 the author [S07] showed that if the covering function $w_{A_0}(x)$ is periodic modulo n_0 then (2.5) holds with $m_1 = \cdots = m_k = 1$ and $\alpha = 0$.

Inspired by Corollary 2.2 (first announced in [S03c]) and an earlier paper [S97], the author [S10] proved that if $A_0 = \{a_s(n_s)\}_{s=0}^k$ forms an m -cover of \mathbb{Z} with $\sum_{s=1}^k 1/n_s < m$ then for any $a = 0, 1, 2, \dots$ we have

$$\left| \left\{ I \subseteq [1, k] : \sum_{s \in I} \frac{1}{n_s} = \frac{a}{n_0} \right\} \right| \geq \binom{m-1}{\lfloor a/n_0 \rfloor}.$$

Corollary 2.2. *Let $A_0 = \{a_s(n_s)\}_{s=0}^k$ be a p -cover of \mathbb{Z} with $a_0(n_0)$ essential, where p is a prime not dividing any of n_1, \dots, n_k . Let $m_1, \dots, m_k \in \mathbb{Z}$ be relatively prime to n_1, \dots, n_k respectively. Then, for any $c, c_1, \dots, c_k \in \mathbb{Z}_p$ with $c_1 \cdots c_k \neq 0$, the set*

$$\left\{ \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} : I \subseteq [1, k] \text{ and } \sum_{s \in I} c_s = c \right\} \quad (2.6)$$

contains an arithmetic progression of length n_0 with common difference $1/n_0$.

Proof. By Corollary 2.1 in the case $F = \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ and $X_s = \{0, c_s\}$ ($1 \leq s \leq k$), for some $0 \leq \alpha < 1$ we have $\{\sum_{s \in I} c_s : \{\sum_{s \in I} m_s/n_s\} = (\alpha + r)/n_0\} = \mathbb{Z}_p$ for every $r \in [0, n_0 - 1]$. So the desired result follows. \square

Remark 2.2. The author's colleague Z. Y. Wu once asked whether for any prime p and $c_1, \dots, c_{p-1} \in \mathbb{Z}_p \setminus \{0\}$ there is an $I \subseteq [1, p-1]$ such that $\sum_{s \in I} c_s = 1$. Corollary 2.2 in the case $n_0 = n_1 = \dots = n_k = 1$, provides an affirmative answer to this question.

Corollary 2.3. *Let $A_0 = \{a_s(n_s)\}_{s=0}^k$ be an $m+1$ -cover of \mathbb{Z} with $m \in \mathbb{N}$ and $w_{A_0}(a_0) = m+1$. Let F be a field with characteristic not dividing any of n_1, \dots, n_k , and let X_1, \dots, X_k be subsets of F with cardinality 2. Let $a_{ij}, b_i \in F$ and $c_i \in X_i$ for all $i \in [1, m]$ and $j \in [1, k]$. If $m_1, \dots, m_k \in \mathbb{Z}$ are relatively prime to n_1, \dots, n_k respectively, and*

$$\text{per}(a_{ij})_{i \in [1, m], j \in J} := \sum_{\{j_1, \dots, j_m\} = J} a_{1j_1} \cdots a_{mj_m} \neq 0$$

where $J = \{1 \leq s \leq k : a_0 \in a_s(n_s)\}$, then the set

$$\left\{ \left\{ \sum_{\substack{1 \leq s \leq k \\ x_s = c_s}} \frac{m_s}{n_s} \right\} : x_s \in X_s \text{ and } \sum_{j=1}^k a_{ij}x_j \neq b_i \text{ for all } i \in [1, m] \right\} \quad (2.7)$$

contains an arithmetic progression of length n_0 with common difference $1/n_0$.

Proof. Note that $|J| = m$. Set $P(x_1, \dots, x_k) = \prod_{i=1}^m (\sum_{j=1}^k a_{ij}x_j - b_i)$. Then

$$\left[\prod_{j \in J} x_j \right] P(x_1, \dots, x_k) = \left[\prod_{j \in J} x_j \right] \prod_{i=1}^m \sum_{j \in J} a_{ij}x_j = \text{per}(a_{ij})_{i \in [1, m], j \in J} \neq 0.$$

In view of Theorem 2.2, the set

$$\left\{ \left\{ \sum_{\substack{1 \leq s \leq k \\ x_s = c_s}} \frac{m_s}{n_s} \right\} : x_s \in X_s, P(x_1, \dots, x_k) \neq 0 \right\}$$

contains $\{(\alpha + r)/n_0 : r \in [0, n_0 - 1]\}$ for some $0 \leq \alpha < 1$. We are done. \square

Remark 2.3. When $n_0 = n_1 = \dots = n_k = 1$, Corollary 2.3 yields the useful permanent lemma of Alon [A99].

Corollary 2.4. *Let $A_0 = \{a_s(n_s)\}_{s=0}^k$ be an $m+1$ -cover of \mathbb{Z} with $a_0(n_0)$ essential. Let m_1, \dots, m_k be integers relatively prime to n_1, \dots, n_k respectively. Let F be a field of prime characteristic p , and let $a_{ij}, b_i \in F$ for all $i \in [1, m]$ and $j \in [1, k]$. Set*

$$X = \left\{ \sum_{j=1}^k x_j : x_j \in [0, p-1] \text{ and } \sum_{j=1}^k x_j a_{ij} \neq b_i \text{ for all } i \in [1, m] \right\}. \quad (2.8)$$

If p does not divide $N = [n_1, \dots, n_k]$ and the matrix $(a_{ij})_{1 \leq i \leq m, 1 \leq j \leq k}$ has rank m , then the set

$$S := \left\{ \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} : I \subseteq [1, k] \text{ and } |I| \in X \right\} \quad (2.9)$$

contains an arithmetic progression of length n_0 with common difference $1/n_0$; in particular, when $n_0 = N$ we have $S = \{r/N : r \in [0, N-1]\}$.

Proof. As $a_0(n_0)$ is essential, for some $a \in a_0(n_0)$ we have $w_{A_0}(a) = m+1$. Without loss of generality we assume that the matrix $M = (a_{ij})_{i,j \in [1,m]}$ is nonsingular, and that $\{1 \leq s \leq k : a \in a_s(n_s)\} = [1, m]$ (otherwise we can rearrange the k residue classes in a suitable order).

Since $\det M \neq 0$, by [AT] there are $l_1, \dots, l_m \in [0, p-1]$ with $l_1 + \dots + l_m = m$ such that $\text{per}(M^*) \neq 0$, where M^* is an $m \times m$ matrix whose columns consist of l_1 copies of the first column of M, \dots, l_m copies of the m th column of M . Let e denote the identity of the field F . By Corollary 2.5, there exists $0 \leq \alpha < 1$ such that for any $r \in [0, n_0 - 1]$ there are $\delta_1, \dots, \delta_k \in \{0, e\}$ for which $\{\sum_{\delta_s=e} m_s/n_s\} = (\alpha + r)/n_0$ and

$$\sum_{j=1}^k a_{ij}(\delta_{l_1+\dots+l_{j-1}+1} + \dots + \delta_{l_1+\dots+l_j}) \neq b_i \quad \text{for all } i \in [1, m],$$

where $l_j = 1$ for any $j \in [m+1, k]$. Observe that

$$x_j = |\{l_1 + \dots + l_{j-1} < s \leq l_1 + \dots + l_j : \delta_s = e\}| \leq l_j < p$$

and $|\{1 \leq s \leq k : \delta_s = e\}| = x_1 + \dots + x_k \in X$. So the set S given by (2.13) contains $\{(\alpha + r)/n_0 : r \in [0, n_0 - 1]\}$. As $T = \{r/N : r \in [0, N-1]\} \supseteq S$, we have $S = T$ if $n_0 = N$. This concludes the proof. \square

Remark 2.4. The Alon-Tarsi Theorem stated in Section 1 follows from Corollary 2.4 for the following reason: Let F be a field of prime characteristic p with identity e . If the matrix $(a_{ij})_{i,j \in [1,k]}$ over F is non-singular and $c \in F \setminus \{0, e, \dots, (p-1)e\}$, then by Corollary 2.4 in the

case $n_0 = n_1 = \cdots = n_k = 1$ there are $x_1, \dots, x_k \in [0, p - 1]$ with $x_1 + \cdots + x_k \leq k$ such that

$$\sum_{j=1}^k x_j a_{ij} \neq -c \sum_{j=1}^k a_{ij}, \quad \text{i.e. } \sum_{j=1}^k a_{ij} x_j^* \neq 0$$

for all $i \in [1, k]$ where $x_j^* = x_j e + c \neq 0$.

Corollary 2.5. *Let (1.2) be an m -cover of \mathbb{Z} with $a_k(n_k)$ essential and $n_k = N_A$. Let $m_1, \dots, m_{k-1} \in \mathbb{Z}$ be relatively prime to n_1, \dots, n_{k-1} respectively. Then, for any $J \subseteq K = \{1 \leq s < k : a_k \in a_s(n_s)\}$ and $r \in [0, N_A - 1]$, there exists an $I \subseteq [1, k - 1]$ such that $I \cap K = J$ and $\{\sum_{s \in I} m_s / n_s\} = r/N_A$.*

Proof. Clearly $w_A(a_k) = m$. Fix $J \subseteq K$. By Theorem 2.1 in the case $F = \mathbb{Q}$, the set

$$\begin{aligned} S &= \left\{ \left\{ \sum_{\substack{1 \leq s < k \\ x_s \neq 0}} \frac{m_s}{n_s} \right\} : x_s \in \{0, 1\}, \prod_{j \in J} x_j \neq 0, x_s \in \{0\} \text{ for } s \in K \setminus J \right\} \\ &= \left\{ \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} : I \subseteq [1, k - 1] \text{ and } I \cap K = J \right\} \end{aligned}$$

contains an arithmetic progression of length $n_k = N_A$ with common difference $1/n_k = 1/N_A$. Since $T = \{a/N_A : a \in [0, N_A - 1]\} \supseteq S$, we must have $S = T$ and thus $\{\sum_{s \in I} m_s / n_s\} = r/N_A$ for some $I \subseteq [1, k - 1]$ with $I \cap K = J$. \square

Remark 2.5. On the basis of the author's work [S95], his brother Z.-H. Sun pointed out that if (1.2) forms a cover of \mathbb{Z} with $a_k(n_k)$ essential and $n_k = N_A$, then $\{\{\sum_{s \in I} 1/n_s\} : I \subseteq [1, k - 1]\} = \{r/N_A : r \in [0, N_A - 1]\}$. This follows from Corollary 2.5 in the special case $m = m_1 = \cdots = m_{k-1} = 1$.

Let $n > 1$ be an integer, and let $m_1, \dots, m_{n-1} \in \mathbb{Z}$ be relatively prime to n . Applying Corollary 2.5 to the trivial cover $\{r(n)\}_{r=0}^{n-1}$, we find that the set $\{\sum_{s \in I} m_s : I \subseteq [1, n - 1]\}$ contains a complete system of residues modulo n . This is more general than the positive answer to Wu's question mentioned in Remark 2.2.

3. A USEFUL POLYNOMIAL FORMULA AND ITS APPLICATIONS

For a predicate P , we define

$$[P] = \begin{cases} 1 & \text{if } P \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

The author [S03] first announced the following result in 2003.

Theorem 3.1. *Let R be a ring with identity, and let $f(x_1, \dots, x_k)$ be a polynomial over R . If $J \subseteq [1, k]$ and $|J| \geq \deg f$, then we have the formula*

$$\sum_{I \subseteq J} (-1)^{|J|-|I|} f(\llbracket 1 \in I \rrbracket, \dots, \llbracket k \in I \rrbracket) = \left[\prod_{j \in J} x_j \right] f(x_1, \dots, x_k). \quad (3.1)$$

Proof. Write $f(x_1, \dots, x_k) = \sum_{j_1, \dots, j_k \geq 0} c_{j_1, \dots, j_k} \prod_{s=1}^k x_s^{j_s}$, and observe that if $\emptyset \neq J' \subseteq [1, k]$ then $0 = \prod_{j \in J'} (1 - 1) = \sum_{I \subseteq J'} (-1)^{|I|}$. Therefore

$$\begin{aligned} & \sum_{I \subseteq J} (-1)^{|I|} f(\llbracket 1 \in I \rrbracket, \dots, \llbracket k \in I \rrbracket) \\ &= \sum_{I \subseteq J} (-1)^{|I|} \sum_{\substack{j_1, \dots, j_k \geq 0 \\ \{s: j_s \neq 0\} \subseteq I}} c_{j_1, \dots, j_k} \\ &= \sum_{\substack{j_1, \dots, j_k \geq 0 \\ \{s: j_s \neq 0\} \subseteq J}} \sum_{\substack{\{s: j_s \neq 0\} \subseteq I \subseteq J}} (-1)^{|I|} c_{j_1, \dots, j_k} \\ &= \sum_{\substack{j_1, \dots, j_k \geq 0 \\ \{s: j_s \neq 0\} \subseteq J}} \sum_{I' \subseteq J \setminus \{s: j_s \neq 0\}} (-1)^{|I'|} (-1)^{|\{s: j_s \neq 0\}|} c_{j_1, \dots, j_k} \\ &= \sum_{\substack{j_1, \dots, j_k \geq 0 \\ \{s: j_s \neq 0\} = J}} (-1)^{|J|} c_{j_1, \dots, j_k} = (-1)^{|J|} \left[\prod_{j \in J} x_j \right] f(x_1, \dots, x_k), \end{aligned}$$

where in the last step we note that if $\{s : j_s \neq 0\} = J$ and $j_s > 1$ for some s then $j_1 + \dots + j_k > |J| \geq \deg f$ and hence $c_{j_1, \dots, j_k} = 0$. This concludes the proof. \square

Remark 3.1. Let $f(x_1, \dots, x_k) \in R[x_1, \dots, x_k]$ where R is a ring with identity. It is easy to verify that for any $l_1, \dots, l_k \in \mathbb{N}$ we have

$$\left[\prod_{i=1}^k \prod_{j=1}^{l_i} x_{ij} \right] f \left(\sum_{j=1}^{l_1} x_{1j}, \dots, \sum_{j=1}^{l_k} x_{kj} \right) = l_1! \cdots l_k! [x_1^{l_1} \cdots x_k^{l_k}] f(x_1, \dots, x_k).$$

Thus, by Theorem 3.1, $l_1! \cdots l_k! [x_1^{l_1} \cdots x_k^{l_k}] f(x_1, \dots, x_k)$ is computable in terms of values of f provided that $\deg f \leq l_1 + \dots + l_k$.

Corollary 3.1 (Escott's identity). *Let R be a ring with identity. Given $c_1, \dots, c_k \in R$ we have*

$$\sum_{I \subseteq [1, k]} (-1)^{|I|} \left(\sum_{s \in I} c_s \right)^n = 0 \quad \text{for every } n = 0, 1, \dots, k-1. \quad (3.2)$$

Proof. Let $n \in [0, k - 1]$ and $f(x_1, \dots, x_k) = (\sum_{s=1}^k c_s x_s)^n$. By Theorem 3.1,

$$\sum_{I \subseteq [1, k]} (-1)^{k-|I|} f(\llbracket 1 \in I \rrbracket, \dots, \llbracket k \in I \rrbracket) = [x_1 \cdots x_k] f(x_1, \dots, x_k) = 0.$$

This yields the desired result. \square

Remark 3.2. Escott discovered (3.2) in the case $c_1, \dots, c_k \in \mathbb{C}$ (where \mathbb{C} is the complex field). Maltby [M] proved Corollary 3.1 in the case where R is commutative.

Corollary 3.2 ([Ro, Lemma 2.2]). *Let F be a field, and let V be the family of all functions from $\{0, 1\}^k$ to F . Then those functions $\chi_I \in V$ ($I \subseteq [1, k]$) given by $\chi_I(x_1, \dots, x_k) = \prod_{s \in I} x_s$ form a basis of the linear space V over F .*

Proof. For $f \in V$ and $x_1, \dots, x_k \in \{0, 1\}$, clearly

$$f(x_1, \dots, x_k) = \sum_{\delta_1, \dots, \delta_k \in \{0, 1\}} f(\delta_1, \dots, \delta_k) \prod_{s=1}^k \llbracket x_s = \delta_s \rrbracket.$$

So the dimension of V does not exceed 2^k .

Suppose that $f = \sum_{I \subseteq [1, k]} c_I \chi_I = 0$ where $c_I \in F$. If $J \subseteq [1, k]$, $c_J \neq 0$ and $\deg f(x_1, \dots, x_k) = |J|$, then

$$c_J = \sum_{I \subseteq J} (-1)^{|J|-|I|} f(\llbracket 1 \in I \rrbracket, \dots, \llbracket k \in I \rrbracket) = 0$$

by Theorem 3.1. Therefore those χ_I with $I \subseteq [1, k]$ are linearly independent over F . We are done. \square

Remark 3.3. Corollary 3.2 plays an important role in Rónyai's study of the Kemnitz conjecture (cf. [Ro]).

We mention that the Combinatorial Nullstellensatz (as stated in Section 1) in the important case $l_1, \dots, l_k \in \{0, 1\}$ also follows from Theorem 3.1. Let $b_1 \in X_1, \dots, b_k \in X_k$ and $c_j \in X_j \setminus \{b_j\}$ for $j \in J = \{1 \leq s \leq k : l_s = 1\}$. Set

$$\bar{f}(x_1, \dots, x_k) = f(b_1 + (c_1 - b_1)x_1, \dots, b_k + (c_k - b_k)x_k)$$

where $c_s = b_s$ for $s \in [1, k] \setminus J$. Then $|J| = \deg f \geq \deg \bar{f}$ and

$$\left[\prod_{j \in J} x_j \right] \bar{f}(x_1, \dots, x_k) = \prod_{j \in J} (c_j - b_j) \times \left[\prod_{j \in J} x_j \right] f(x_1, \dots, x_k) \neq 0.$$

By Theorem 3.1, for some $I \subseteq J$ we have $\bar{f}(\llbracket 1 \in I \rrbracket, \dots, \llbracket k \in I \rrbracket) \neq 0$ and hence $f(a_1, \dots, a_k) \neq 0$ where $a_s = b_s + (c_s - b_s)\llbracket s \in I \rrbracket \in X_s$ for $s \in [1, k]$.

Lemma 3.1 (Sun [S09]). *Let p be a prime, and let $h \in \mathbb{N}$ and $a \in \mathbb{Z}$. Then we have the following congruence*

$$\binom{a-1}{p^h-1} \equiv [\![p^h \mid a]\!] \pmod{p}. \quad (3.3)$$

Our next theorem is related to zero-sum problems on a general abelian p -group $\mathbb{Z}_{p^{h_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{h_l}}$.

Theorem 3.2. *Let $k, h_1, \dots, h_l \in \mathbb{Z}^+$ and $k \geq \sum_{t=1}^l (p^{h_t} - 1)$ where p is a prime. Let $c_{st}, c_t \in \mathbb{Z}$ for all $s \in [1, k]$ and $t \in [1, l]$. Then*

$$\begin{aligned} & \sum_{\substack{I \subseteq [1, k] \\ p^{h_t} \mid \sum_{s \in I} c_{st} - c_t \text{ for } t \in [1, l]}} (-1)^{|I|} \\ & \equiv \sum_{\substack{I_1 \cup \dots \cup I_l = [1, k] \\ |I_t| = p^{h_t} - 1 \text{ for } t \in [1, l]}} \prod_{t=1}^l \prod_{s \in I_t} c_{st} \pmod{p}. \end{aligned} \quad (3.4)$$

Proof. Set

$$f(x_1, \dots, x_k) = \prod_{t=1}^l \left(\frac{\sum_{s=1}^k c_{st} x_s - c_t - 1}{p^{h_t} - 1} \right).$$

Then $\deg f \leq \sum_{t=1}^l (p^{h_t} - 1) \leq k$. Whether $n = k - \sum_{t=1}^l (p^{h_t} - 1)$ is zero or not, $[x_1 \cdots x_k] f(x_1, \dots, x_k)$ always coincides with

$$[x_1 \cdots x_k] \prod_{t=1}^l \frac{(\sum_{s=1}^k c_{st} x_s)^{p^{h_t}-1}}{(p^{h_t} - 1)!} = \sum_{\substack{I_1 \cup \dots \cup I_l = [1, k] \\ |I_t| = p^{h_t} - 1 \text{ for } t \in [1, l]}} \prod_{t=1}^l \prod_{s \in I_t} c_{st}.$$

On the other hand, by Theorem 3.1 and Lemma 3.1 we have

$$\begin{aligned} & (-1)^n [x_1 \cdots x_k] f(x_1, \dots, x_k) = [x_1 \cdots x_k] f(x_1, \dots, x_k) \\ & = \sum_{I \subseteq [1, k]} (-1)^{k-|I|} f([\![1 \in I]\!], \dots, [\![k \in I]\!]) \\ & \equiv (-1)^n \sum_{I \subseteq [1, k]} (-1)^{|I|} \prod_{t=1}^l \left[\left[p^{h_t} \mid \sum_{s \in I} c_{st} - c_t \right] \right] \pmod{p}. \end{aligned}$$

(Note that $(-1)^{k-n} \equiv 1 \pmod{p}$.) Therefore (3.4) holds. \square

Remark 3.4. In the case $k > \sum_{t=1}^l (p^{h_t} - 1)$, Theorem 3.2 yields a theorem of Olson [O] on Davenport constants of abelian p -groups because the right hand side of the congruence (3.4) vanishes. In the same spirit, we can easily prove Theorem 2 of Baker and Schmidt [BS] whose original proof is very deep and complicated.

Corollary 3.3. *Let p be a prime and let $h \in \mathbb{Z}^+$.*

(i) *If $c, c_1, \dots, c_{p^h-1} \in \mathbb{Z}$, then*

$$\sum_{\substack{I \subseteq [1, p^h-1] \\ p^h \mid \sum_{s \in I} c_s - c}} (-1)^{|I|} \equiv c_1 \cdots c_{p^h-1} \pmod{p}. \quad (3.5)$$

(ii) *For $c, c_1, \dots, c_{2p^h-2} \in \mathbb{Z}$ we have*

$$\begin{aligned} & \left| \left\{ I \subseteq [1, 2p^h-2] : |I| = p^h - 1 \text{ and } p^h \mid \sum_{s \in I} c_s - c \right\} \right| \\ & \equiv [x^{p^h-1}] \prod_{s=1}^{2p^h-2} (x - c_s) \pmod{p}. \end{aligned} \quad (3.6)$$

Proof. (i) Simply apply Theorem 3.2 with $l = 1$.

(ii) In view of Theorem 3.2 in the case $l = 2$,

$$\sum_{\substack{I \subseteq [1, 2p^h-2] \\ p^h \mid \sum_{s \in I} c_s - c \\ p^h \mid \sum_{s \in I} 1+1}} (-1)^{|I|} \equiv \sum_{\substack{I \subseteq [1, 2p^h-2] \\ |I| = p^h - 1}} \prod_{s \in I} c_s \times \prod_{s \notin I} 1 \pmod{p}.$$

This is equivalent to (3.6) and we are done. \square

Let $q > 1$ be a power of a prime p , and let $c_1, \dots, c_{4q-2} \in \mathbb{Z}_q^2$. Using Lemma 3.1 and Theorem 3.1 we can prove that

$$\begin{aligned} & \left| \left\{ I \subseteq [1, 4q-2] : |I| = q \text{ and } \sum_{s \in I} c_s = 0 \right\} \right| \\ & \equiv \left| \left\{ I \subseteq [1, 4q-2] : |I| = 3q \text{ and } \sum_{s \in I} c_s = 0 \right\} \right| + 2 \pmod{p}. \end{aligned}$$

This is helpful to understand the full proof of the Kemnitz conjecture given by Reiher [Re].

4. PROOF OF THEOREM 2.1

In this section we fix a finite system (1.2) of residue classes, and set $I_z = \{1 \leq s \leq k : z \in a_s(n_s)\}$ for $z \in \mathbb{Z}$. We first extend [S09, Lemma 4.1] to any field containing an element of (multiplicative) order N_A , where N_A is the least common multiple of the moduli n_1, \dots, n_k in (1.2).

Lemma 4.1. *Let A be as in (1.2) and let $m_1, \dots, m_k \in \mathbb{Z}$. Let F be a field containing an element ζ of (multiplicative) order N_A , and let $f(x_1, \dots, x_k)$ be a polynomial over F with $\deg f \leq m(A)$. If $[\prod_{s \in I_z} x_s]f(x_1, \dots, x_k) = 0$ for all $z \in \mathbb{Z}$, then we have $\psi(\theta) = 0$ for any $0 \leq \theta < 1$, where*

$$\psi(\theta) := \sum_{\substack{I \subseteq [1, k] \\ \{\sum_{s \in I} m_s/n_s\} = \theta}} (-1)^{|I|} f([\![1 \in I]\!], \dots, [\![k \in I]\!]) \zeta^{N_A \sum_{s \in I} a_s m_s/n_s}.$$

The converse holds when m_1, \dots, m_k are relatively prime to n_1, \dots, n_k respectively.

Proof. Let $z \in \mathbb{Z}$ and $J \subseteq [1, k]$. Clearly

$$\begin{aligned} & [\![J \supseteq I_z]\!] \prod_{s=1}^k \left([\![s \notin J]\!] - \zeta^{N_A(a_s-z)m_s/n_s} \right) \\ &= \sum_{I \subseteq [1, k]} \prod_{\substack{s=1 \\ s \notin I}}^k [\![s \notin J]\!] \times (-1)^{|I|} \zeta^{N_A \sum_{s \in I} a_s m_s/n_s} \zeta^{-z N_A \sum_{s \in I} m_s/n_s} \\ &= \sum_{\theta \in S} \zeta^{-z N_A \theta} \sum_{\substack{J \subseteq I \subseteq [1, k] \\ \{\sum_{s \in I} m_s/n_s\} = \theta}} (-1)^{|I|} \zeta^{N_A \sum_{s \in I} a_s m_s/n_s} \end{aligned}$$

where

$$S = \left\{ \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} : I \subseteq [1, k] \right\}. \quad (4.1)$$

Write $f(x_1, \dots, x_k) = \sum_{j_1, \dots, j_k \geq 0} c_{j_1, \dots, j_k} x_1^{j_1} \cdots x_k^{j_k}$. Obviously

$$f([\![1 \in I]\!], \dots, [\![k \in I]\!]) = \sum_{\substack{j_1, \dots, j_k \geq 0 \\ \{1 \leq s \leq k : j_s \neq 0\} \subseteq I}} c_{j_1, \dots, j_k} \quad \text{for all } I \subseteq [1, k].$$

If $c_{j_1, \dots, j_k} \neq 0$ and $J = \{1 \leq s \leq k : j_s \neq 0\} \supseteq I_z$, then $\deg f \geq |J| \geq |I_z| = w_A(z) \geq \deg f$; hence $w_A(z) = \deg f$, $J = I_z$ and $j_s = 1$ for $s \in J$.

In view of the above, for any $z \in \mathbb{Z}$ the sum $\sum_{\theta \in S} \zeta^{-z N_A \theta} \psi(\theta)$ coincides

with

$$\begin{aligned}
& \sum_{\theta \in S} \zeta^{-zN_A \theta} \sum_{\substack{I \subseteq [1, k] \\ \{\sum_{s \in I} m_s/n_s\} = \theta}} (-1)^{|I|} \sum_{\substack{j_1, \dots, j_k \geq 0 \\ \{s: j_s \neq 0\} \subseteq I}} c_{j_1, \dots, j_k} \zeta^{N_A \sum_{s \in I} a_s m_s/n_s} \\
&= \sum_{j_1, \dots, j_k \geq 0} c_{j_1, \dots, j_k} \sum_{\theta \in S} \zeta^{-zN_A \theta} \sum_{\substack{\{s: j_s \neq 0\} \subseteq I \subseteq [1, k] \\ \{\sum_{s \in I} m_s/n_s\} = \theta}} (-1)^{|I|} \zeta^{N_A \sum_{s \in I} a_s m_s/n_s} \\
&= \sum_{\substack{j_1, \dots, j_k \geq 0 \\ J = \{s: j_s \neq 0\} \supseteq I_z}} c_{j_1, \dots, j_k} \prod_{s=1}^k \left([\![s \notin J]\!] - \zeta^{N_A (a_s - z)m_s/n_s} \right) \\
&= c(I_z) \prod_{s=1}^k \left([s \notin I_z] - \zeta^{N_A (a_s - z)m_s/n_s} \right),
\end{aligned}$$

where $c(I_z) = [\prod_{s \in I_z} x_s] f(x_1, \dots, x_k)$. Therefore

$$\sum_{\theta \in S} \zeta^{-zN_A \theta} \psi(\theta) = (-1)^k c(I_z) \prod_{\substack{s=1 \\ s \notin I_z}}^k \left(\zeta^{N_A (a_s - z)m_s/n_s} - 1 \right). \quad (4.2)$$

When $n_1 = \dots = n_k = 1$, this yields the equality

$$\sum_{I \subseteq [1, k]} (-1)^{|I|} f([\![1 \in I]\!], \dots, [\![k \in I]\!]) = (-1)^k [x_1 \cdots x_k] f(x_1, \dots, x_k)$$

as asserted by Theorem 3.1.

Observe that $c(I_z) = 0$ if $\psi(\theta) = 0$ for all $0 \leq \theta < 1$ and each m_s is relatively prime to n_s .

Suppose that $c(I_z) = 0$ for all $z \in \mathbb{Z}$. Then $\sum_{\theta \in S} \zeta^{-nN_A \theta} \psi(\theta) = 0$ for all $n \in [0, |S| - 1]$. As the Vandermonde-type determinant

$$\det[(\zeta^{-N_A \theta})^n]_{n \in [0, |S| - 1], \theta \in S}$$

is nonzero, we have $\psi(\theta) = 0$ for all $\theta \in S$. If $0 \leq \theta < 1$ and $\theta \notin S$, then $\psi(\theta) = 0$ holds trivially.

In view of the above, we have completed the proof of Lemma 4.1. \square

Lemma 4.2. *Let F be a field of characteristic p , and let n be a positive integer. Then $p \nmid n$ if and only if there is an extension field of F containing an element of (multiplicative) order n .*

Proof. (i) Suppose that $p \mid n$ and E/F is a field extension. If $\zeta \in E$ and $\zeta^n = 1$, then $(\zeta^{n/p} - 1)^p = (\zeta^{n/p})^p - 1 = \zeta^n - 1 = 0$ and hence $\zeta^{n/p} - 1 = 0$. So E contains no element of order n .

(ii) Now assume that $p \nmid n$. Let E be the splitting field of the polynomial $f(x) = x^n - 1$ over F . Then $G = \{\zeta \in E : \zeta^n = 1\}$ is a finite subgroup of the multiplicative group $E^* = E \setminus \{0\}$, therefore it is cyclic by field theory. Since $p \nmid n$, $f'(\zeta) = n\zeta^{n-1} \neq 0$ for any $\zeta \in G$. So the equation $f(x) = 0$ has no repeated roots in E and hence $|G| = n$. Any generator of the cyclic group G has order n .

Combining the above we obtain the desired result. \square

Proof of Theorem 2.1. For convenience we set $h = |J| - \deg P$, $A = \{a_s(n_s)\}_{s=1}^k$, $J^* = \{1 \leq s \leq k : a_0 \in a_s(n_s)\}$ and $J' = J^* \setminus J$.

Let d_1, \dots, d_h be any elements of F and define

$$\begin{aligned} f(x_1, \dots, x_k) &= P(b_1 + (c_1 - b_1)x_1, \dots, b_k + (c_k - b_k)x_k) \\ &\times \prod_{j=1}^h \left(\sum_{s=1}^k (b_s + (c_s - b_s)x_s) - d_j \right) \times \prod_{s \in J'} (x_s - 1). \end{aligned}$$

Then $\deg f \leq \deg P + |J'| + h = |J^*| = m(A)$. As $[\prod_{s \in J^*} x_s]f(x_1, \dots, x_k)$ equals

$$\prod_{j \in J} (c_j - b_j) \times \left[\prod_{j \in J} x_j \right] P(x_1, \dots, x_k) \left(\sum_{s=1}^k x_s \right)^h \neq 0,$$

we have $\deg f = m(A)$. Recall that m_s is relatively prime to n_s for each $s \in [1, k]$. In light of Lemma 4.1,

$$\psi(\theta) = \sum_{\substack{I \subseteq [1, k] \\ \{\sum_{s \in I} m_s/n_s\} = \theta}} (-1)^{|I|} f([\![1 \in I]\!], \dots, [\![k \in I]\!]) \zeta^{N_A \sum_{s \in I} a_s m_s / n_s} \neq 0$$

for some $0 \leq \theta < 1$, where ζ is an element of order $N = N_A$ in an extension field of F (whose existence follows from Lemma 4.2).

Let $\alpha = \{n_0\theta\}$ and $r \in [0, n_0 - 1]$. Then $(\alpha + r)/n_0 = \theta + \bar{r}/n_0$ where $\bar{r} = r - \lfloor n_0\theta \rfloor$. As $w_A(a_0 + N) = w_A(a_0) < w_{A_0}(a_0) = m(A_0)$, we must have $a_0 + N \in a_0(n_0)$ and hence $n_0 \mid N$. Note that $\deg f = m(A) < m(A_0)$. Applying Lemma 4.1 to the system A_0 we find that

$$\psi(\theta) + \sum_{\substack{I \subseteq [1, k] \\ \{\sum_{s \in I} m_s/n_s + (-\bar{r})/n_0\} = \theta}} (-1)^{|I|+1} f([\![1 \in I]\!], \dots, [\![k \in I]\!]) \zeta^{N \beta(I)} = 0$$

where $\beta(I) = \sum_{s \in I} a_s m_s / n_s + a_0(-\bar{r})/n_0$. It follows that

$$\psi\left(\frac{\alpha + r}{n_0}\right) = \psi\left(\theta + \frac{\bar{r}}{n_0}\right) = \zeta^{N a_0 \bar{r} / n_0} \psi(\theta) \neq 0.$$

So, there exists an $I \subseteq [1, k]$ with $\{\sum_{s \in I} m_s/n_s\} = (\alpha + r)/n_0$ such that $f(\llbracket 1 \in I \rrbracket, \dots, \llbracket k \in I \rrbracket) \neq 0$. Note that $x_s = b_s + (c_s - b_s)\llbracket s \in I \rrbracket \in X_s$ for all $s \in [1, k]$. Also, $P(x_1, \dots, x_k) \neq 0$, $I \cap J' = \emptyset$ and $I = \{1 \leq s \leq k : x_s \neq b_s\}$. Thus S_r contains $x_1 + \dots + x_k$ which is different from d_1, \dots, d_h .

If $|S_r| \leq h$, then we can select $d_1, \dots, d_h \in F$ such that $\{d_1, \dots, d_h\} = S_r$, hence we get a contradiction from the above. Therefore $|S_r| \geq h + 1$ and we are done. \square

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