

Incontestable Assignments*

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Abstract

In school districts where assignments are exclusively determined by a clearinghouse students can only appeal their assignment with a valid reason. An assignment is incontestable if it is appeal-proof. We study incontestability when students do not observe the other students' preferences and assignments. Incontestability is shown to be equivalent to individual rationality, non-wastefulness, and respect for top-priority sets (a weakening of justified envy). Stable mechanisms and those Pareto dominating them are incontestable, as well as the Top-Trading Cycle mechanism (but Boston is not). Under a mild consistency property, incontestable mechanisms are i -indistinguishable (Li, 2017), and share similar incentive properties. *JEL classification:* C78, D02

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1 Introduction

Stability is a desirable objective when designing matching or assignment mechanisms because it prevents eventual disruptions by blocking pairs, a cause of market failure—see Roth (1991) and Kagel and Roth (2000).^{1,2} While this criterion is unquestionably needed in many cases, we contend that stability may be unnecessarily demanding in settings in which assignments are exclusively determined by a central authority, like in most school districts. In such markets, blocking pairs are not free to form independently, but arise through the actions of individual students who can *appeal* their assignment.

Appealing one's assignment differ from the standard blocking concept because in practice most school districts only permit appeals by students who can provide justification for their complaint. For instance, a student could justify an appeal on the basis that given her priority ranking and school capacities, she should have gotten into, say, one of her top three schools, independently of the preferences of other students. In contrast, a student who is unhappy with her assignment but cannot provide a valid reason why her assignment should be different, has no recourse. What this means is that students may not have the information to identify every school with which they can form a blocking pair, so not all blocking pairs are a threat to an assignment. In other words, stability requires the absence of justified envy, but for a student to identify justified envy demands her to know the assignment of other students. This is usually not the case in school choice, especially in the first few weeks after notification of school assignment, when appeals must be made but students' assignments being not (yet) public information. This is the point of departure of our paper: *whether a school choice assignment is incontestable (appeal-proof) hinges on the information available to students.*

The objective of this paper is to extend the stability concept by incorporating students' ability to submit justified appeals. Specifically, we formalize the information students have

¹The rationale is easy to understand when considering a centralized job market, where workers can offer themselves to firms that rank higher in their preferences after being notified of their matches. If the matching were not stable, some firms might accept these ‘decentralized’ offers and thus reject candidates they have been assigned to by the clearinghouse. Incentives to participate in the matching mechanism are then weakened, which may end up in market failure. It should be noted, though, that stability does not always prevent market failure. See for instance McKinney et al. (2005).

²Another common justification for stability is fairness. However, in school choice problems this normative criterion comes at the cost of efficiency —see (Ergin, 2002).

access to when participating in a school choice mechanism, assuming that students only know their own assignment, and schools' priority rankings, capacities and enrollment sizes. That is, students do not have information about the preferences and assignments of other students. We say that a student has a *legitimate complaint* if the information she has shows that her assignment cannot be the result of a stable assignment. In other words, a student has a legitimate complaint if her assignment is inconsistent with a stable assignment for *any* preference profile of the other students. An assignment with no legitimate complaint is called *incontestable*, and the paper studies such assignments.

In principle, verifying incontestability of an assignment requires checking, for each student, all possible stable assignments given all possible preference profiles of the other students. Our first result (Theorem 1) shows that it can be verified much more easily: an assignment is incontestable if, and only if it is individually rational, non-wasteful, and satisfies a novel property, *respect for top-priority sets*. That latter requirement can be viewed as a non-justified envy condition adapted to the informational constraints of our setting. It differs from the standard non-justified envy in that it only applies to students for whom there exists a set of schools which they prefer to any other school and that cannot be filled only with students who have higher priority. We call such a set a *top priority set*, and an assignment respects top-priority sets if any student with such a set is assigned to a school in that set. When this property is violated for a student, one of the schools in her top-priority set must either be under capacity or have admitted a student with lower priority.

Theorem 1 implies that a student has a legitimate complaint whenever: 1) she is not assigned to a school in her top priority set (if she has one), or 2) she is assigned to a school she finds unacceptable (i.e., violating individual rationality) or 3) a school preferred to her assignment is not full. Under a mild consistency property, called *top-top consistency*, we obtain a counterpart characterization of incontestable mechanism outcomes that parallels Theorem 1 (Theorem 2). Top-top consistency is a weakening of the standard consistency property that applies only for student-school pairs that are mutually ranked first.³ Theorem 2 says that for students with a top-priority set, any school in that set can be the outcome of an incontestable and top-top consistent mechanism. That is, for any school in the set, and for any incontestable and top-top consistent mechanism, there exists a preference profile of other students under which she is assigned to that school by the mechanism. As for the

³We show in the Appendix that the most common mechanisms considered in the literature are top-top consistent.

students without a top-priority set, the set of possible outcomes is the set of all individually rational assignments. Remarkably, this characterization does not depend on the particular mechanism that is used. In other words, mechanisms that are incontestable and top-top consistent are all i -indistinguishable in the sense of Li (2017).

Students' rather limited information might make it reasonable to consider incontestable mechanisms rather than the more demanding stable mechanisms. A key question though, is what does this buy us? Crucially, incontestability is compatible with efficiency in a way that stability is not: while stable assignments may be Pareto dominated by unstable ones, any assignment that Pareto dominates an incontestable assignment is also incontestable. Since stable assignments are always incontestable, any mechanism that Pareto improves the student-optimal stable mechanism (SOSM) is also incontestable. Therefore, incontestable mechanisms can provide Pareto improvements on stable mechanisms. This has policy relevance, as our results provide an efficiency justification for using mechanisms that may not satisfy stability, but satisfy incontestability. For instance, the Boston mechanism does not satisfy incontestability (nor stability) but is used in school assignment due to its efficiency properties. Conversely, SOSM is stable but not efficient. Our result provides a rationale for incontestable mechanisms that can satisfy both efficiency and some weakened notion of stability, such as Kesten's (2010) Efficiency Adjusted Deferred Acceptance Mechanism (EADAM). Interestingly, there are also incontestable mechanisms that are not Pareto comparable to SOSM, like the Top Trading cycle mechanism and some of its variants.

Our characterization in Theorem 2 turns out to be extremely useful to study (and compare) the incentive properties of incontestable mechanisms. Of course, such mechanisms are not guaranteed to be strategyproof. For instance, EADAM is not strategyproof. Reny (2022) nevertheless finds that under EADAM, truth-telling is a maxmin optimal strategy. Theorem 2 makes it relatively easy to extend Reny's theorem to *any* incontestable and top-top consistent mechanism. We also show that we can easily get results that compare the manipulability of mechanisms, when varying the maximal number of schools students are permitted to include in their submitted preference lists.

Related literature Our paper intersects with several branches of the matching and assignment literature. The first of these branches deals with proposals to obtain assignments that Pareto dominate the student-optimal stable mechanism. In broad strokes, those mecha-

nisms allow for certain students to have justified envy.⁴ A key contribution in that direction is Kesten’s (2010)’s EADAM algorithm. His mechanism consists of identifying ‘interrupters,’ that is, students who will make the student-optimal assignment inefficient because they prefer certain schools to their assignment (but that they eventually do not get under SOSM). In Kesten’s algorithm, students are first asked whether they want to waive their priorities at the schools for which they are interrupters. Ehlers and Morrill (2020) and Reny (2022) propose different rationales than Kesten’s, but end up with a mechanism that is outcome equivalent to EADAM (when all students waive their priorities). Our approach differs from these contributions in that our notion of blocking is entirely based on the information available to the students. It is the lack of complete information on students’ side that allows us to implement assignments that Pareto dominates the student-optimal assignment, and not a tolerance for some level of justified envy.

We are not the first to propose a definition of stability that depends on the information available to participants. An early contribution in that respect is Chakraborty et al. (2010) who consider the problem where agents can infer the types of potential partners from the observation of the whole matching. Another contribution is Yenmez (2013), who considers ex-ante and interim stability conditions.⁵ A notable contribution is Liu et al. (2014), who extend Crawford and Knoer’s (1981) model where firms do not observe workers’ types. Our work is similar to Liu et al. in the sense that blocking and legitimate complaint are both based on a worst-case scenario principle. In their paper, a worker-firm pair blocks a matching if they can be both better off under any possible belief, and in our paper a legitimate complaint is a complaint that holds for any possible preference profile of the other students. However, our contribution is somehow orthogonal to Liu et al. (2014) or Chakraborty et al. (2010). They assume that firms observe the whole matching but not workers’ types, whereas students in our model observe schools’ priorities (i.e., their ‘types’) but not the complete assignment.

Our paper also deals with the question of agent’s information when participating in a mechanism, and Li’s (2017) contribution is obviously key to frame and interpret some of our results. Our results complement those of Möller (2022), who analyzes conditions under which students can be guaranteed that the central authority has used the mechanism it initially announced. Möller’s (2022) results and ours share some similarities but are obtained under different approaches. Möller’s contribution is motivated by questioning the

⁴See Dur et al. (2019) who propose a family of mechanisms based on this principle.

⁵See also Ehlers and Massó (2015).

central authority's credibility, while our initial motivation is to understand what stability entails in a low information environment.⁶

Finally, Our paper also contributes to the literature that compares the manipulability of school choice mechanisms (Pathak and Sönmez, 2013). Decerf and Van der Linden (2021) compare the occurrence of dominant strategies in various constrained mechanisms. Theorem 2 permits us to extend their comparisons results that involve (constrained) SOSM by identifying a class of mechanisms with the same occurrence of dominant strategies as (constrained) SOSM.

The paper is organized as follows. In Section 2 we present the standard school choice problem. The concepts of legitimate complaint and incontestability are defined in Section 3. In this section we also show that incontestability is equivalent to individual rationality, non-wastefulness and respect for top-priority sets. Incontestable mechanisms are analyzed in Section 4, where we present a full characterization of students' outcomes in such mechanisms. We also analyze how incontestability relates to stability and efficiency. Section 5 is devoted to incentives. We conclude in Section 6. Most proofs are relegated to the Appendices.

2 Preliminaries

2.1 School choice problems

We consider school choice problems with a set I of students and a set S of schools (both finite). Each student i has a strict preference ordering \succ_i over $S \cup \{i\}$. We say that student i prefers school s over school s' if $s \succ_i s'$. A school s that is such that $i \succ_i s$ is unacceptable for student i (acceptable otherwise). Given a set of students $J \subseteq I$, let $\succ_J = (\succ_i)_{i \in J}$. If $J = I$, the set of all students, we will generally omit the subscript and write \succ instead of \succ_I .

Each school $s \in S$ has a strict priority ordering r_s over the set of students and a capacity q_s that captures the maximum number of students that can be assigned at school s . Given a set of schools \widehat{S} , let $r_{\widehat{S}} = (r_s)_{s \in \widehat{S}}$ and $q_{\widehat{S}} = (q_s)_{s \in \widehat{S}}$. If $\widehat{S} = S$, the set of all schools, we will generally omit the subscript and write r and q instead of r_S and q_S , respectively. A priority

⁶Möller assumes that all students observe the complete assignments. See also for Hakimov and Raghavan (2022) for another contribution in a similar vein.

ordering r_s assigns *ranks* to the students, where $r_s(i) < r_s(j)$ means that student i has a higher priority (or lower rank) than student j at school s .

A (school choice) problem is a 5-tuple

$$\Gamma = (I, S, \succ_I, r_S, q_S).$$

We denote by $U_i(r_s)$ the set of students with a higher priority than i at school s given the priority ordering r_s , $U_i(r_s) = \{j \in I ; r_s(j) < r_s(i)\}$. For a set of school $\widehat{S} \subseteq S$, let $U_i(r_{\widehat{S}}) = \cup_{s \in \widehat{S}} U_i(r_s)$ denote the set of students with a higher priority than i at at least one school in S given $r_{\widehat{S}}$.

2.2 Assignments

An assignment is a bijection $\mu : I \cup S \rightarrow I \cup S$ such that for each student $i \in I$, $\mu(i) \in S \cup \{i\}$, for each school $s \in S$, $\mu(s) \in 2^I$, and for each student $i \in I$, $\mu(i) = s$ if, and only if $i \in \mu(s)$.

For a given set of students \widehat{I} , we will abuse notation and denote by $\mu(\widehat{I})$ the set of schools to which students in \widehat{I} are assigned to, that is, $\mu(\widehat{I}) = S \cap \cup_{i \in \widehat{I}} \mu(i)$. Similarly, we write $\mu(\widehat{S})$ to denote the set of students that are assigned to a set of schools \widehat{S} under assignment μ .

A student i 's preferences \succ_i over schools implicitly define a preference relation \succeq_i over assignments as follows: $\mu \succeq_i \mu'$ if, and only if $\mu(i) \succ_i \mu'(i)$ or $\mu(i) = \mu'(i)$. Abusing notation we write $\mu \succ_i \mu'$ when both $\mu \succeq_i \mu'$ and $\mu(i) \neq \mu'(i)$ hold.

2.3 Stable and efficient assignments

Stability is a standard concept in the school choice literature and is the conjunction of three conditions. Formally, given a problem (I, S, \succ, r, q) , an assignment μ is **stable** if

- (a) μ is **individually rational**: for each student $i \in I$, $\mu(i) \succ_i i$ or $\mu(i) = i$;
- (b) μ is **non-wasteful**: for each student $i \in I$, $s \succ_i \mu(i)$ implies $|\mu(s)| = q_s$; and
- (c) there is no **justified envy**: for all $i, j \in I$ with $\mu(j) = s$, $s \succ_i \mu(i)$ implies $r_s(j) < r_s(i)$.

Note that the definition of stability implicitly assumes that schools' priorities over sets of students are responsive —see (Roth, 1985). It is well known that for any problem (I, S, \succ, r, q) , the set of stable assignments is non-empty and forms a lattice. Two prominent stable assignments are the student-optimal and the student-pessimal assignments, denoted

μ_I and μ_S , respectively. Formally, μ_I and μ_S are the stable assignments such that, for any student i and any stable assignment μ , $\mu_I \succeq_i \mu$ and $\mu \succeq_i \mu_S$.

An assignment μ **Pareto dominates** an assignment μ' if for each student $i \in I$, $\mu \succeq_i \mu'$ and there exists at least one student i such that $\mu \succ_i \mu'$. An assignment is **efficient** if it is not Pareto dominated by any other assignment.

3 Incontestability

3.1 Information and complaints

When participating in a school choice mechanism each student has an information set that describes what is known to her. A typical information set for a student i is denoted by \mathcal{H}_i , and $\mathcal{H} = (\mathcal{H}_i)_{i \in I}$ denotes an information profile. For instance, $\mathcal{H}_i = \{S, \mu(i)\}$ denotes the fact that student i only knows her assignment and the set of schools. In the standard school choice literature, it is often assumed that student's information is complete, that is, $\mathcal{H}_i = \{I, S, \succ, r, q, \mu, \varphi\}$ for each student $i \in I$, where φ is the assignment mechanism used to assign students to schools. This is the case for instance when Haerlinger and Klijn (2009) analyze Nash equilibria under various mechanisms.

In this paper we only consider information about the set of students, the set of schools, the students' preferences, schools' priority rankings and capacities, and the assignment. For instance, we exclude from students' information sets any information about the properties of the assignment.

It will prove convenient to make the distinction between the information held at interim and ex-post stages of an assignment mechanism, that is, before or after the assignment mechanism has computed an assignment for all students.⁷ We thus say that an information set \mathcal{H}_i of a student i is **interim** if $\succ_i \in \mathcal{H}_i$ and \mathcal{H}_i does not contain any (partial) information about an assignment. The information set \mathcal{H}_i is **ex-post** if $\succ_i \in \mathcal{H}_i$ and \mathcal{H}_i contains some information about a realized assignment.⁸

⁷We will not consider in this paper the ex-ante stage (i.e., before students know their preferences over schools).

⁸Our approach to formalize students' information is similar to that of Fernandez (2018) or Chen and Möller (2023). However, these two papers only consider ex-post information structures.

Definition 1 A school choice problem $\Gamma = (I, S, \succ, r, q)$ is **compatible** with an interim information set \mathcal{H}_i if there is no element in Γ that contradicts an element in \mathcal{H}_i . A school choice problem $\Gamma = (I, S, \succ, r, q)$ and assignment μ are **compatible** with an ex-post information set \mathcal{H}_i if there is no element in Γ and μ that contradicts an element in \mathcal{H}_i .⁹

A problem $\Gamma = (I, S, \succ, r, q)$ is compatible with an interim information profile $\mathcal{H} = (\mathcal{H}_i)_{i \in I}$ if for each student i , Γ is compatible with \mathcal{H}_i . Similarly, the problem Γ and assignment μ are compatible with the ex-post information profile \mathcal{H} if for each $i \in I$, the pair Γ and μ are compatible with \mathcal{H}_i .

We say that a student i with an ex-post information set \mathcal{H}_i has a legitimate complaint at an assignment μ if \mathcal{H}_i is enough to show that if μ were a stable assignment then i should not be assigned to $\mu(i)$. To understand more formally what we mean by the fact that \mathcal{H}_i is ‘enough’, consider its negation. If the information set \mathcal{H}_i is ‘not enough’ for i ’s complaint then it must be that whether i can be assigned to $\mu(i)$ at a stable assignment depends on some elements not included in \mathcal{H}_i . That is, there are two problems, say, $\widehat{\Gamma}$ and $\widetilde{\Gamma}$, that are both compatible with \mathcal{H}_i , and there exists an assignment $\widehat{\mu}$ that is stable for $\widehat{\Gamma}$ such that $\widehat{\mu}(i) = \mu(i)$ but there is no assignment that is stable for $\widetilde{\Gamma}$ that assigns i to $\mu(i)$. In other words, a student i with information \mathcal{H}_i has a legitimate complaint at an assignment μ if it is not possible to ‘rationalize’ $\mu(i)$.

Definition 2 A student i with ex-post information set \mathcal{H}_i has a **legitimate complaint** at an assignment μ if there is no problem $\Gamma = (I, S, \succ, r, q)$ and assignment $\widehat{\mu}$ that are both compatible with \mathcal{H}_i such that $\widehat{\mu}$ is stable for Γ and $\widehat{\mu}(i) = \mu(i)$.

Definition 3 Given an ex-post information profile \mathcal{H} , a problem Γ and an assignment μ compatible with \mathcal{H} , μ is **incontestable** if no student has a legitimate complaint (and contestable otherwise).

3.2 Incontestable assignments

In this paper, we are interested in the case where each student does not know the other student’s preferences and assignments, but knows the rest of the problem (i.e., the set of

⁹We restrict in this paper to situations where each student is not informed about the other students’ preferences and assignments. So $\Gamma = (I, S, \succ', r, q)$ is compatible with $\mathcal{H}_i = \{I, S, \succ_i, r, q\}$ if $\succ'_i = \succ_i$. Similarly, $\Gamma = (I, S, \succ', r, q)$ and μ' are compatible with $\mathcal{H}_i = \{I, S, \succ_i, r, q, \mu(i), (|\mu(s)|)_{s \in S}\}$ if $\succ'_i = \succ_i$, $\mu'(i) = \mu(i)$, and $|\mu'(s)| = |\mu(s)|$ for each $s \in S$.

students, the schools' priority rankings and capacities). We call such information canonical. Thus, we refer to the information set $\mathcal{H}_i = \{I, S, \succ_i, r, q\}$ as the **interim canonical** information set of student i , and to $\mathcal{H}_i = \{I, S, \succ_i, r, q, \mu(i), (|\mu(s)|)_{s \in S}\}$ as the **ex-post canonical** information set of student i .

Our first question is to characterize incontestable assignments under ex-post canonical information. Clearly, incontestable assignments are necessarily individually rational and non-wasteful. Indeed, given an ex-post canonical information $\mathcal{H}_i = \{I, S, \succ_i, r, q, \mu(i), (|\mu(s)|)_{s \in S}\}$, for any compatible problem $\Gamma = (I, S, \widehat{\succ}, r, q)$ and assignment $\widehat{\mu}$ we have $\widehat{\succ}_i = \succ_i$, $\widehat{\mu}(i) = \mu(i)$, and $|\widehat{\mu}(s)| = |\mu(s)|$ for each $s \in S$. So, $s \succ_i \mu(i)$ implies $s \succ_i \widehat{\mu}(i)$. Also, $s \succ_i \mu(i)$ and $|\mu(s)| < q_s$ imply $s \widehat{\succ}_i \widehat{\mu}(i)$ and $|\widehat{\mu}(s)| < q_s$.

Our definition of incontestability suggests that checking whether a particular assignment is incontestable requires checking, for each student, whether there exists a preference profile of the other students under which the assignment of that student can be the result of some stable assignment. It is in fact much simpler. We show that incontestability is equivalent to three properties that define a weakening of the standard stability concept. To proceed, we borrow from Decerf and Van der Linden (2021) the concept of high-priority set for a student, which is a set of schools that cannot be filled only with students who have a higher priority than that student.¹⁰

Definition 4 Given a priority profile r and capacity vector q , a set of schools \widehat{S} is a **high-priority set** for a student i if there is no assignment μ such that for each $s \in \widehat{S}$, $|\mu(s)| = q_s$ and $\mu(s) \subseteq U_i(r_s)$.

It is straightforward to see that a set \widehat{S} is a high-priority set for a student i if there exists a set $\widetilde{S} \subseteq \widehat{S}$ such that Hall's marriage condition fails (Hall, 1935), that is, $|\cup_{s \in \widetilde{S}} U_i(r_s)| < \sum_{s \in \widetilde{S}} q_s$. Decerf and Van der Linden show that when a set \widehat{S} is a high-priority set for a student and she submits a preference list where all schools in \widehat{S} are preferred to any school not in \widehat{S} , the student is guaranteed to be assigned to one of those schools at the student-optimal assignment.¹¹ The case where a student lists all schools in a high-priority set as her most preferred schools is key to characterize incontestable assignments.¹²

¹⁰Decerf and Van der Linden (2021) call such a set *safe*.

¹¹Theorem 2 in Decerf and Van der Linden (2021).

¹²The conjunction of a “top” condition on a student's preferences and the existence of a high-priority set, which we call “top-priority set” can be seen as a generalization of the Niederle and Yariv's (2022) top-top property. See also Banerjee et al.'s (2001) top-coalition property.

Definition 5 Given a student's preferences \succ_i , a set of schools \widehat{S} is a **top-priority set** for i if \widehat{S} is a high-priority set and for each $s \in \widehat{S}$ and $s' \in S \setminus \widehat{S}$ it holds that $s \succ_i s'$; and all schools in \widehat{S} are acceptable for i .

Definition 6 Given a problem (I, S, \succ, r, q) , an assignment μ **respects top-priority sets** if for each student $i \in I$, $\mu(i) \in \widehat{S}$ whenever \widehat{S} is a top-priority set for i .

Several remarks are in order.

Remark 1 If \widehat{S} is a high-priority set, then any superset thereof, $\widetilde{S} \supset \widehat{S}$, is also a high-priority set. Similarly, if \widehat{S} is a top-priority set, then for any school s such that $s \notin \widehat{S}$ and $s \succeq_i i$, the set of schools more preferred to s , $\{s' \mid s' \succeq_i s\}$ is also a top-priority set for i . A student can thus have multiple top-priority sets, and those sets can always be ordered with the set inclusion (which is not the case for high-priority sets). It is thus straightforward to deduce that respect for top-priority sets necessarily implies that a student will be assigned to her smallest top-priority set.

Remark 2 If a student has a high priority set and finds all schools in that set acceptable, then that student has a top-priority set. To see this, let \widehat{S} be a high-priority set for student i given (r, q) such that i finds all schools in \widehat{S} acceptable. By Remark 1, $\widetilde{S} = \{s : s \succ_i i\}$ is a high-priority set because $\widehat{S} \subseteq \widetilde{S}$. By definition, \widetilde{S} is also a top-priority set.

Our first result shows that incontestability turns out to be akin to a weakening of stability, where absence of justified envy is replaced by respect for top-priority sets.

Theorem 1 Let μ be an assignment for a problem $\Gamma = (I, S, \succ, r, q)$ and \mathcal{H} the corresponding ex-post canonical information profile.¹³ Assignment μ is incontestable if, and only if it is individually rational, non-wasteful and respects top-priority sets.

Proof (Only if) Let μ be an incontestable assignment for a problem $\Gamma = (I, S, \succ, r, q)$. Clearly, if for some student i we have (a) $|\mu(s)| < q_s$ for some $s \succ_i \mu(i)$, or (b) $i \succ_i \mu(i)$, then (a) and (b) still hold for any problem compatible with \mathcal{H}_i . Therefore, we only have to consider an assignment that is individually rational, non-wasteful, but that does not respect top-priority sets. So, suppose there exists a student i such that $\mu(i) \notin \widehat{S}$ where \widehat{S} is a

¹³That is, for each student i , the problem Γ and the assignment μ are compatible with the ex-post canonical information set \mathcal{H}_i .

top-priority set for i . Since μ is incontestable, there exists a problem Γ' compatible with \mathcal{H}_i such that μ is stable for Γ' . Hence, in Γ' , i does not have justified envy against any other student. Since $s \succ_i \mu(i)$ for each $s \in \widehat{S}$, for each such school s it holds that $j \in \mu(s)$ implies $r_s(j) < r_s(i)$. Hence, $\mu(\widehat{S}) \subseteq U_i(r_{\widehat{S}})$. By non-wastefulness, $|\mu(s)| = q_s$ for each $s \in \widehat{S}$. However, since \widehat{S} is a top-priority set for i , it is a high-priority set given (r, q) , which in turn implies that there cannot be such an assignment. So, $\mu(i) \in \widehat{S}$, that is, μ respects top-priority sets.

(If) Let μ be an assignment that is individually rational, non-wasteful and that respects top-priority sets. Consider any student $i \in I$. We need to show that i cannot have a legitimate complaint, that is, there exists a problem $\Gamma = (I, S, \succ, r, q)$ and an assignment $\widehat{\mu}$ that are compatible with $\mathcal{H}_i = \{I, S, \succ_i, r, q, \mu(i), (|\mu(s)|)_{s \in S}\}$ and such that $\widehat{\mu}$ is stable for Γ . To this end, let $\widehat{S} = \{s : s \succ_i \mu(i)\}$.

Suppose first that $\widehat{S} = \emptyset$. If $\mu(i) = i$, then i finds all schools unacceptable and thus i is always assigned to herself at any stable assignment for any profile of the other students. So, assume that $\mu(i) \in S$. Let $\widehat{\mu}$ be any assignment such that $\widehat{\mu}(i) = \mu(i)$, and $|\widehat{\mu}(s)| = |\mu(s)|$. For each $j \neq i$, let \succ_j be such that $\widehat{\mu}(j)$ is j 's first choice (herself if $\widehat{\mu}(j) = j$ and a school otherwise) and $j \succeq_j s$ for each $s \neq \widehat{\mu}(j)$. For any stable assignment $\widehat{\mu}$ for the problem $(I, S, (\succ_i, \succ_{-i}), r, q)$ it must be that $\widehat{\mu}(i) = \mu(i)$. So, for the rest of the proof we assume that $\widehat{S} \neq \emptyset$.

Since μ respects top-priority sets, $\mu(i) \notin \widehat{S}$ implies that \widehat{S} is not a top-priority set for i . Therefore, \widehat{S} is not a high-priority set and thus there exists an assignment $\widehat{\mu}$ such that for each school $s \in \widehat{S}$, $\widehat{\mu}(s) \subseteq U_i(r_s)$ and $|\widehat{\mu}(s)| = q_s$. Note that since μ is non-wasteful, $|\mu(s)| = q_s$ for each $s \in \widehat{S}$. Define the assignment $\widetilde{\mu}$ as follows. For each $j \in \widehat{\mu}(\widehat{S})$, $\widetilde{\mu}(j) = \widehat{\mu}(j)$, $\widetilde{\mu}(i) = \mu(i)$, and assign each student $j \notin \widehat{\mu}(\widehat{S}) \cup \{i\}$ to schools in $S \setminus \widehat{S}$ until it holds that $|\widetilde{\mu}(s)| = |\mu(s)|$ for each $s \in S \setminus \widehat{S}$ (and any remaining student is assigned to herself).¹⁴

By construction, $\widetilde{\mu}$ is compatible with \mathcal{H}_i . For each student $j \neq i$, let \succ_j be such that $\widetilde{\mu}(j)$ is j 's top choice. Let $\Gamma = (I, S, (\succ_i, \succ_{-i}), r, q)$. This problem is obviously compatible with \mathcal{H}_i . Since each student $j \neq i$ is assigned to her first choice, $\widehat{\mu}$ is not stable for Γ only if i has justified envy against some student assigned to a school in \widehat{S} . By construction, all those schools are filled with students in $U_i(r_{\widehat{S}})$, so $\widetilde{\mu}$ is stable for Γ . ■

¹⁴Since $|\widehat{\mu}(s)| = |\mu(s)| = q_s$ for each $s \in \widehat{S}$, $|I| - \sum_{s \in \widehat{S}} q_s \geq \sum_{s \in S \setminus \widehat{S}} |\mu(s)|$, where the left-hand side is the number of students that can be assigned (under $\widehat{\mu}$) to a school in $S \setminus \widehat{S}$ and the right-hand side is the number of students who are assigned (under μ) to a school in $S \setminus \widehat{S}$.

4 Incontestable mechanisms

We consider here school choice mechanisms (typically denoted φ), that is, mappings that compute an assignment for each possible school choice problem. We denote by $\varphi_i(\Gamma)$ the assignment of student i under mechanism φ for the school choice problem Γ . A mechanism φ is stable if for any school choice problem Γ , $\varphi(\Gamma)$ is a stable assignment. Efficient, incontestable or, for instance, non-wasteful mechanisms are defined similarly.

4.1 Incontestable outcomes

The objective here is to understand what are the set of schools that are attainable for a student when participating in an incontestable mechanism. We are thus considering here interim information sets. Characterizing the set of attainable schools will turn out to be key when comparing incontestable mechanisms and analyzing their strategic incentives.

From Theorem 1 we already know that students with a top-priority sets are necessarily assigned to a school in their top-priority set. Also, Theorem 1 (and its proof) show that for any student i and individually rational outcome $v \in S \cup \{i\}$ (in a top-priority set if the student has such a set) we can construct a preference profile for the other students such that i is assigned to v at a stable assignment under that profile. That is, for any student i , the assignments such that student i does not have a legitimate complaint are

- any individually rational outcome if i does not have a top-priority set;
- any school in the (smallest) top-priority set if i has a top-priority set.

We would like know if this is also the case for incontestable mechanisms, that is, if such mechanisms also span all the possible outcomes described by Theorem 1. The answer is negative. We will show shortly that there are incontestable mechanisms that never assign a student i to some school s even though i is assigned to s under some assignment that is stable for some preference profile of the other students. Why is that? If i can be assigned to s under a stable assignment, then it must be that the set of schools that i prefers to s is not a top-priority set for i (for otherwise it would contradict the fact that any stable assignment is incontestable). Therefore, it is possible to fill this set of schools with a set of students, say J , who have a higher priority than i . But for some incontestable mechanisms, that does not mean there exists some preference profile under which the mechanism assigns the students

in J to the set of schools that i reports above s . The crux of the problem comes from the fact that students with a higher priority than i may not have a top-priority set, and respect for top-priority sets does not make any prediction for the students without such a set. The following example helps understanding the problem.

Example 1 There are 3 schools, s_1 , s_2 , and s_3 (each with capacity one), and four students, i_h , $h \leq 4$. The schools' priorities are depicted in Table 1. With those priorities, only i_2 has high priority sets.

	s_1	s_2	s_3
i_2	i_2	i_2	
i_1	i_3	i_4	
i_3	i_1	i_1	
i_4	i_4	i_3	

Table 1: An incontestable mechanism

Let i_2 's preferences be such that s_1 is her most preferred school (and is acceptable), that i_3 and i_4 find all schools acceptable, and i_1 's preferences are $s_1 \succ_{i_1} s_2 \succ_{i_1} s_3 \succ_{i_1} i_1$. It is easy to check that under such a preference profile the assignment μ such that $\mu(i_1) = s_3$, $\mu(i_2) = s_1$, $\mu(i_3) = s_2$, and $\mu(i_4) = i_4$ is incontestable. (Only i_2 has a top-priority set.)

Consider the following mechanism φ . For any profile of preferences \succ , φ first assigns all students who have a top-priority set (to a school in their top-priority set). Then, φ consists of a serial dictatorship with the student i_1 choosing first if i_1 does not have a top-priority set. With the preferences we have specified for i_1 , there is no preference profile for students i_2 , i_3 , and i_4 such that φ assigns i_1 to s_3 . If i_2 has a top-priority set (which happens whenever she finds at least one school acceptable), i_1 ends up being assigned to either s_1 or s_2 . Otherwise, i_1 is assigned to s_1 . \square

The mechanism φ in Example 1 fails to assign i_1 to s_3 for any preference profile of the other students because it is not consistent. Suppose for instance that i_2 and i_3 's most preferred schools are s_1 and s_2 , respectively. So, $\{s_1\}$ is a top-priority set for i_2 . In the subproblem without i_2 and s_1 , $\{s_2\}$ becomes a top-priority set for i_3 . If φ were consistent i_3 's assignment should not depend on whether the pair (i_2, s_1) is present in the problem.

When imposing consistency, we obtain that i_1 is assigned to s_3 when i_1 finds all schools acceptable and i_4 finds s_3 unacceptable.

To ensure that a student can be assigned to any school that she can be assigned to under some stable assignment (for some preferences of the other students), we do not need to impose the standard consistency property. A weaker version is sufficient. This weakening, which we call top-top consistency, only considers students who are ranked top by their most preferred school. Top-top consistency requires that the assignment of the other students should remain the same whether or not we withdraw those students with a singleton top-priority set. We show in Appendix B that the most prominent assignment mechanisms do satisfy this consistency property.

Given a problem (I, S, \succ, r, q) , we say that (i, s) is a **top-top pair** if i 's most preferred acceptable school is s and i is the student with the highest priority at s . Formally, (i, s) is a top-top pair if $s \succ_i i$ and $s \succ_i s'$ for all $s' \neq s$, and for each $j \neq i$, $r_s(i) < r_s(j)$. Clearly, if (i, s) is a top-top pair, then $\{s\}$ is a top-priority set for i and thus i must be assigned to s under any incontestable assignment.

The problem (I, S, \succ, r, q) reduced by a pair $(i, s) \in I \times S$ is the problem (I', S', \succ', r, q') where, the set of students is $I' = I \setminus \{i\}$, the set of schools is $S' = S$ if $q_s > 1$ and $S' = S \setminus \{s\}$ otherwise, for each $j \in I'$, \succ'_j is a preference ordering over $S' \cup \{j\}$ that agrees with \succ_j , and for each school $s' \neq s$, $q'_{s'} = q_{s'}$, and $q'_s = q_s - 1$ if $q_s > 1$.^{15,16}

Definition 7 An assignment mechanism φ is **top-top consistent** if, for any problem Γ that has a top-top pair (i, s) ,

$$\varphi_i(\succ, r, q) = s \quad \Rightarrow \quad \varphi_j(\Gamma) = \varphi_j(\Gamma') \quad \text{for each } j \neq i \quad (1)$$

where Γ' is the problem Γ reduced by the pair (i, s) .

¹⁵That is, for any pair $v, v' \in S' \cup \{j\}$, $v \succ'_j v' \Leftrightarrow v \succ_j v'$.

¹⁶If $q_s = 1$ then school s is not part of the reduced problem and thus there is no need to define q'_s . Not including empty schools in the reduced problem is in line with the standard approach of consistency —see Ergin (2002) or Toda (2006). Another way to define consistency would be to require all schools to remain in the same problem (i.e., even if their capacity drops to 0). The proofs related to top-top consistency (in Appendix B) can be easily adapted for this other notion of consistency. It is easy to see that the Boston mechanism does satisfy this alternate notion of consistency (it is not the case with the notion of consistency we use in this paper). However, since the Boston mechanism does not satisfy respect for top-priority sets, excluding Boston with our notion of consistency is without loss.

Given an interim information set \mathcal{H}_i , let $\mathcal{O}_i(\mathcal{H}_i, \varphi)$ denote the set of possible assignments of student i when considering all possible problems that are compatible with \mathcal{H}_i under mechanism φ ,

$$\mathcal{O}_i(\mathcal{H}_i, \varphi) = \{v \in S \cup \{i\} \mid \varphi_i(\Gamma) = v \text{ for some } \Gamma \text{ compatible with } \mathcal{H}_i\}.$$

For a profile (r, q) and a preference ordering \succ_i , let $T^{\succ_i, r, q}$ denote the smallest top-priority set of student i if she has a top-priority set. If student i does not have any top-priority set, let $T^{\succ_i, r, q} = \{v \mid v \succeq_i i\}$. Note that in the latter case $i \in T^{\succ_i, r, q}$, that is, $T^{\succ_i, r, q}$ is the set of all individually rational outcomes under \succ_i .

Theorem 2 Let φ be an incontestable and top-top consistent mechanism. For any problem $\Gamma = (I, S, \succ, r, q)$, for any student i with the corresponding canonical interim information set \mathcal{H}_i ,

$$\mathcal{O}_i(\mathcal{H}_i, \varphi) = T^{\succ_i, r, q}. \quad (2)$$

Proof See Appendix A. ■

The key message of Theorem 2 is the right-hand part of Eq. (2), which does not depend on the particular mechanism that is used. It follows that for any student, the set of attainable schools is the same under any two incontestable and top-top consistent mechanisms. Any two such mechanisms are i -indistinguishable in the sense of Li (2017). In other words, knowledge of the exact mechanism being used does not affect students' ability to issue a legitimate complaint.

Theorem 2 also establishes an equivalence between the interim perspective and the ex-post perspective (given by the left-hand side and right-hand side of Eq. (2), respectively). For a student without any top-priority set, any individually rational outcome is a priori (interim) possible, and for any such outcome there exists a preference profile of the other students that rationalizes it (ex-post). Similarly, for a student with a top-priority set, any assignment in that set is a priori (interim) possible, and for any school in that set there exists a compatible problem that rationalizes it (ex-post).

4.2 Stability and efficiency

Stable assignments are necessarily incontestable because no student can have a legitimate complaint at a stable assignment. So, the existence of incontestable assignments follows immediately from the existence of stable assignments. Since both stability and incontestability

require individual rationality and non-wastefulness, the difference between the two concepts boils down to the difference between non-justified envy and respect of top-priority sets. The main novelty brought by the latter property is that it may not concern all students. That property is only relevant for the students who have a top-priority set. However, for the students who do have a top-priority set, a violation of respect for top-priority sets necessarily implies the existence of justified envy. The following lemma formalizes this.

Lemma 1 Let (I, S, \succ, r, q) be a problem such that for some student $i \in I$, \widehat{S} is a top-priority set and let μ be a non-wasteful and individually rational assignment with $\mu(i) \notin \widehat{S}$. Then there exists at least one student j and a school $s \in \widehat{S}$ such that i has justified envy against j at s .

Proof Let $i \in S$ be such that \widehat{S} is a top-priority set for a problem (I, S, \succ, r, q) , and let μ be a non-wasteful and individually rational assignment such that $\mu(i) \notin \widehat{S}$. Suppose by way of contradiction that i does not have any justified envy against any student in $\mu(\widehat{S})$. Hence, for each $s \in \widehat{S}$, $j \in \mu(s)$ implies $r_s(j) < r_s(i)$, and thus $\mu(\widehat{S}) \subseteq U_i(r_{\widehat{S}})$. Since $\mu(i) \notin \widehat{S}$, $s \succ_i \mu(i)$ for each $s \in \widehat{S}$. By non-wastefulness, $|\mu(s)| = q_s$ for each $s \in \widehat{S}$. Since \widehat{S} is a top-priority set for i , \widehat{S} is a high-priority set for i given (r, q) . Therefore, there does not exist any assignment μ such that $\mu(s) \subseteq U_i(r_s)$ and $|\mu(s)| = q_s$ for each $s \in \widehat{S}$, contradiction. ■

Lemma 1 highlights the relationship between justified envy and non-respect for top-priority sets. If an assignment does not respect top-priority sets, then the affected student has necessarily justified envy against another student, and thus the affected student is necessarily part of a blocking pair.¹⁷ That student can thus formulate a legitimate complaint about her assignment. There is, however, a fundamental difference between justified envy and non-respect for top priority sets in terms of the content of the complaint. In the case of a complaint based on justified envy, an affected student not only knows that she is part of a blocking pair, but also knows which school is part of that pair. In contrast, if a complaint is based on non-respect for top-priority sets, the student only knows that she is part of a blocking pair. She does not know with which school she can block the assignment. It is only

¹⁷It is easy to see that the converse is not true: some student may have justified envy even though the assignment respects top-priority sets. Consider a student i with $s_1 \succ_i s_2 \succ_i \dots$, and $\{s_1, s_2\}$ is a top-priority set. Let j and k be such that $r_{s_1}(j) < r_{s_1}(i) < r_{s_1}(k) < r_{s_1}(h)$ for any $h \neq i, j, k$. The assignment μ where $\mu(i) = s_2$ and $\mu(k) = s_1$ is such that i has justified envy against k at s_1 , but μ respects top-priority sets (if $\mu(j) \succeq_j s_1$).

when the ‘not respected’ top priority set is a singleton that a student can identify the school that is part of the blocking pair.

There is another information-related difference between justified-envy and respect for top-priority sets. A student who knows schools’ priorities but ignores the assignment of other students may ignore she has justified envy. However, when her top-priority set has been violated, the same student cannot ignore it. Hence, under ex-post canonical information, not all blocking pairs (in the sense of justified envy) lead to legitimate complaints. Only the blocking pairs associated with a violation of top priority sets lead to legitimate complaints. When none of the blocking pairs generate a violation of top priority sets, then the assignment is “appeal-proof”. Thus, some assignments are not stable even though they are appeal-proof. Proposition 1 provides a simple and useful way of finding some of these assignments.

Proposition 1 Any assignment that Pareto dominates an incontestable assignment is incontestable.

There are two immediate corollaries of Proposition 1. First, since there is always an efficient assignment that Pareto dominates the student-optimal assignment (if that latter is not already efficient), and since stable assignments are always incontestable, an efficient and incontestable assignment always exists. There is thus no tension between incontestability and Pareto efficiency. A more interesting corollary is the following.

Corollary 1 Any assignment that Pareto dominates a stable assignment is incontestable.

Corollary 1 implies that *any* mechanisms that Pareto improve the Student-Optimal Stable mechanism (e.g., Kesten’s (2010) efficiency adjusted mechanism) are incontestable. We provide more details on this later in this section.

The proof of Proposition 1 uses a generalization of the rural hospital theorem (Roth, 1986), which states that at any stable assignment the set of students assigned to a school is the same, and if a school does not fill its capacity at a stable assignment then it does not fill its capacity at any stable assignment. The generalization we offer (Theorem 4 in Appendix C) states that this result holds for any assignment that Pareto dominates an individually rational and non-wasteful assignment. That generalization was also proven by Alva and Manjunath (2019). The statement of the generalized rural hospital theorem and its proof can be found in the Appendix.

Proof of Proposition 1 Let μ Pareto dominate an incontestable assignment μ' . Since μ' is incontestable, it is individually rational, and since μ Pareto dominates μ' , μ is individually rational, too. Suppose that μ is wasteful. So, there exists $i \in I$ and $s \neq \mu(i)$ such that $s \succ_i \mu(i)$ and $|\mu(s)| < q_s$. By Theorem 4 (see Appendix C), we must have $|\mu'(s)| < q_s$. Since $\mu \succeq_i \mu'$, and therefore $s \succ_i \mu'(i)$, this implies that μ' is wasteful, a contradiction. Since μ' respects top-priority sets, if a set \widehat{S} is a top-priority set for some $i \in S$, $\mu'(i) \in \widehat{S}$. Since $\mu \succeq_i \mu'$, we also have $\mu(i) \in \widehat{S}$ because $s \succ_i s'$ for each $s \in \widehat{S}$ and $s' \in S \setminus \widehat{S}$. ■

Corollary 1 may not look too surprising given that incontestability is a weakening of stability. Since stable assignments are incontestable, at any Pareto improving assignment students with a top-priority set remain assigned to a school in their top-priority set. The only ‘difficulty’ in the proof of Proposition 1 is to ensure that non-wastefulness still holds at the Pareto improving assignment.

One may wonder whether all unstable incontestable assignments Pareto dominate some stable assignment. This is not the case. Some unstable incontestable assignments are Pareto unrelated to stable assignments. In fact, the set incontestable assignments also contains assignments that are only ‘designed’ to be efficient. This is for instance the case of the top-trading assignment, as shown by the following result.

Proposition 2 The following mechanisms are incontestable and top-top consistent:

- The Student-Optimal Stable Mechanism;
- The Efficiency-Adjusted Deferred Acceptance mechanism (EADAM);
- The Top-Trading cycle mechanism;
- The Clinch and Trade (CT), and First Clean and Trade (FCT) mechanisms

Proof See Appendix B. ■

Note that Tang and Yu’s (2014) simplified efficiency adjusted mechanism, Ehlers and Morrill’s (2020) optimal legal assignment mechanism, and Reny’s (2022) priority efficient mechanism are also incontestable and top-top consistent since they are all equivalent to Kesten’s (2010) EADAM.

4.3 Contestable mechanisms

Because respect for top-priority sets has bite only for the students who have a top-priority set, the set of incontestable assignments is likely to be significantly larger than the set of stable assignments. However, incontestability is not an ‘anything goes’ concept: several well-known assignment mechanisms are contestable. We provide three examples.

4.3.1 The Boston mechanism

The easiest case is the Boston mechanism. This mechanism is individually rational and non-wasteful but does not respect top-priority sets. To see this, consider the example depicted in Table 2. In this example, each school has only one seat.

i_1	i_2	i_3	s_1	s_2	s_3
s_1	s_1	s_2	i_1	i_1	i_1
s_2	s_2	s_1	i_2	i_2	i_2
s_3	s_3	s_3	i_3	i_3	i_3

Table 2: The Boston mechanism is contestable

For student i_2 , $\{s_1, s_2\}$ is a top-priority set (it is the smallest). However, given students i_1 and i_3 ’s preferences student i_2 is assigned to s_3 under the Boston mechanism, a violation of respect for top-priority sets.

4.3.2 Application-Rejection mechanism

Another contestable mechanism is the Application-Rejection mechanism described by Chen and Kesten (2017). In broad strokes, this mechanism relies on a multi-round algorithm, where each round is parametrized by a *permanency-execution period* e . Within each round t , a deferred-acceptance algorithm is used by considering only the schools that rank between $(t-1)e+1$ and te in students’ preferences. At the end of each round, students who have been assigned are permanently assigned and removed from the problem, and schools’ capacities are reduced by the number of students permanently assigned.

It is relatively intuitive to see this mechanism does not respect top-priority sets. To see this, consider the example depicted in Table 3, where each school has only one seat.

i_1	i_2	i_3	i_4	s_1	s_2	s_3
s_1	s_1	s_1	s_3	i_1	i_1	i_1
s_2	s_2	s_2	s_1	i_2	i_2	i_2
s_3	s_3	s_3	s_2	i_3	i_3	i_3
				i_4	i_4	i_4

Table 3: The Application-Rejection mechanism is contestable

For student i_3 , $\{s_1, s_2, s_3\}$ is a top-priority set (and no subset thereof is a top-priority set). Let $e = 2$, so that the first round only considers the first two choices of each student. In the first round of the Application-Rejection algorithm, schools s_1 , s_2 , and s_3 are filled with students i_1 , i_2 , and i_4 , respectively. So i_3 cannot be assigned to any school in her top-priority set.

4.3.3 The Equitable Top Trading Cycles mechanism

The Equitable Top-Trading Cycles mechanism (ETTC) is a variant of the Top-Trading mechanism proposed by Hakimov and Kesten (2018) that aims at producing efficient assignment in the spirit of the TTC mechanism but that can generate fewer instances of justified envy. Like TTC, ETTC is efficient and group strategyproof. However, ETTC is a contestable mechanism. To see this, consider the example depicted in Table 4, where schools s_1 and s_2 have one seat and school s_3 has two seats.

i_1	i_2	i_3	i_4	s_1	s_2	s_3
s_1	s_2	s_3	s_1	i_3	i_3	i_1
			s_2	i_4	i_4	i_2
			s_3	i_1	i_1	i_3
				i_2	i_2	i_4

Table 4: The Equitable Top-Trading Cycle mechanisms is contestable

In the ETTC mechanism, seats at schools are first pre-assigned to the students, one by one, starting with the student with the highest priority ranking, until all seats have been pre-assigned. In our example student i_3 is pre-assigned the seats of schools s_1 and s_2 , and students i_1 and i_2 are pre-assigned the two seats at school s_3 .

Next, each student-school pair (i, s) points to the student-school pair (j, s') such that

- s' is student i 's most preferred school
- student j is the student with the highest priority at school s among the students who are pre-assigned a seat at school s' .

So, the pair (i_1, s_3) points to the pair (i_3, s_1) and the pair (i_2, s_3) points to the pair (i_3, s_2) . Student i_3 's most preferred school is s_3 . The two seats at s_3 are pre-assigned to i_1 and i_2 . At school s_1 , student i_1 has the highest priority, so (i_3, s_1) points to (i_1, s_3) . Similarly, (i_3, s_2) points to (i_2, s_3) .

We then have two cycles involving (i_3, s_1) and (i_1, s_3) , and (i_3, s_2) and (i_2, s_3) . Those pairs are removed. Notice that only school s_3 is left, so i_4 cannot be assigned to any school in her top-priority set $\{s_1, s_2\}$.

5 Incentives

The main take-away from Theorem 2 is that incontestable and top-top consistent mechanisms are indistinguishable. But that result also turns out to be a tool to analyze such mechanisms. This section aims at illustrating the power of the characterization provided by Theorem 2 to analyze various incentives properties. More precisely, we review in this section three types of standard incentive problems for assignment mechanisms: the existence of safe strategies, maxmin strategies, and manipulability comparisons. Such questions have already been addressed in the literature, but each time for specific mechanisms. We show in this section that thanks to Theorem 2, those properties can be analyzed at once for all incontestable and top-top consistent mechanisms.

5.1 Safe strategies

One issue that students face when participating in a school assignment mechanism is whether they will be assigned to a school. This is an important question when the mechanism is constrained, that is, when students cannot submit preference orderings that contain more than a certain number of schools. Following Haeringer and Klijn (2009), given a mechanism φ , we denote by φ^k the constrained version of φ where students cannot submit a preference list that contains more than k acceptable schools.

The following proposition is relatively straightforward, we leave the proof to the reader.¹⁸

Proposition 3 Let φ be an incontestable and top-top consistent mechanism. Then for any k , φ^k is incontestable and top-top consistent.

Decerf and Van der Linden (2021) identify a condition under which a student has a ‘safe’ strategy when the mechanism is SOSM. We show here that, thanks to Theorem 2, their result can be easily extended to any incontestable and top-top consistent mechanism. To this end, for a student i and a profile of priority rankings and capacities (r, q) , we say that a strategy \succ_i is **safe** under mechanism φ if student i is guaranteed to be assigned to a school for any preference profile of the other students. Formally, \succ_i is safe for student i if

$$\varphi_i(I, S, (\succ_i, \succ_{-i}), r, q) \in S, \quad \text{for all } \succ_{-i}. \quad (3)$$

Proposition 4 Let φ be an incontestable and top-top consistent mechanism. For any problem Γ , a student $i \in I$ has a safe strategy under φ^k if, and only if i has a high-priority set \widehat{S} such that $|\widehat{S}| \leq k$.

Proof Let \widehat{S} be a high-priority set for student i for some profile (r, q) , where $|\widehat{S}| \leq k$. So, with the preference ordering \succ_i such that for any $s \in \widehat{S}$ and $s' \notin \widehat{S}$, $s \succ_i s'$ and $s \succ_i i$, \widehat{S} is a top-priority set for i . By Theorem 2, we have $\mathcal{O}_i(\{I, S, \succ_i, r, q\}, \varphi) = \widehat{S}$ and thus for any preference profile \succ_{-i} we have $\varphi_i^k(I, S, (\succ_i, \succ_{-i}), r, q) \in \widehat{S}$.

Conversely, suppose that (r, q) is such that there is no set \widehat{S} with $|\widehat{S}| \leq k$ such that \widehat{S} is a high priority set. Take any set \widetilde{S} such that $|\widetilde{S}| \leq k$, and let \succ_i be such that for any $s \in \widetilde{S}$ and $s' \notin \widetilde{S}$, $s \succ_i i$ and $i \succ_i s'$. So, \succ_i has no top-priority set because \widetilde{S} is not a high-priority set. By Theorem 2, we have $\mathcal{O}_i(\{I, S, \succ_i, r, q\}, \varphi) = \widetilde{S} \cup \{i\}$ and thus for some preference profile \succ_{-i} we have $\varphi_i^k(I, S, (\succ_i, \succ_{-i}), r, q) = i$. ■

Proposition 4 has a useful implication, as it permits us to compare how safe a mechanism when varying the constraint imposed on students’ preference lists.

Definition 8 Let φ and ψ be two mechanisms.

¹⁸The argument of the proof is simple. The case when students cannot submit more than k acceptable school is equivalent to the case when students are not constrained but no student has more than k acceptable schools.

- (i) A mechanism φ is weakly safer than a mechanism ψ if for every problem Γ in which a student has a safe strategy with mechanism φ she also has a safe strategy with mechanism ψ .
- (ii) Mechanism φ is safer than mechanism ψ if φ is weakly safer and there exists a problem Γ for which a student has a safe strategy with mechanism φ and has no safe strategy with mechanism ψ .
- (iii) Mechanisms φ and ψ are equally safe if φ is weakly safer than ψ and ψ is weakly safer than φ .

Proposition 5 Let φ and ψ be two (unconstrained) incontestable and top-top consistent mechanisms. Then, for any $k \geq 2$,

- (i) φ^k and ψ^k are equally safe;
- (ii) φ^k is safer than φ^{k-1} ,

Proof (i). By Proposition 3, φ^k and ψ^k are incontestable and top-top consistent. The result readily follows from Proposition 4, which establishes that the existence of a safe strategy only depends on k and the profile (r, q) .

(ii). Clearly, if \succ_i is a safe strategy for φ^{k-1} then \succ_i is also a safe strategy for φ^k . It is easy to see that for any k there are problems where for some student i , the smallest high-priority set contains k schools.¹⁹ For such problems, under φ^{k-1} , i does not have any strategy with a top-priority set, and thus by Proposition 4 i does not have a safe strategy.

■

5.2 Maxmin strategies

Reny (2022) studies the incentive property of EADAM.²⁰ That mechanism is no strategyproof, but turns out to still carry some incentives. If students compare two strategies (i.e.,

¹⁹Such a problem can be constructed as follows. Let \widehat{S} be such that $|\widehat{S}| = k$, and let r and q be such that $q_s = 1$ for each $s \in \widehat{S}$, all schools have identical priority rankings, and for each school there are exactly $k - 1$ students with a higher priority than i .

²⁰Reny's (2022) proposes a different mechanism that EADAM's, but that turn out to be outcome equivalent to EADAM when all students waive their priorities.

preference lists over schools) and compare them with the worst possible outcome (according to the true preferences), then a student can do no better than being truthful. It is worth noting that this belief-free approach, which consists of looking at all possible preference lists of the other students, is shared with our definition of incontestability.

Given a mechanism φ and an interim canonical information set $\mathcal{H}_i = \{I, S, \succ_i, r, q\}$, let $w_i(\succ'_i | \succ_i, \varphi, \mathcal{H}_i)$ be the worst school according to \succ_i when student i submits the preference ordering \succ'_i when participating to the mechanism. Formally, $w_i(\cdot)$ is defined as follows,

$$w_i(\succ'_i | \succ_i, \varphi, \mathcal{H}_i) = \min_{\succ_{-i}} \varphi_i(I, S, (\succ'_i, \succ_{-i}), r, q). \quad (4)$$

where the min operator is with respect to the student's preferences \succ_i .

Theorem 5 of Reny (2022) shows that under the priority efficient mechanism (or EADAM) truth-telling is a maxmin optimal strategy for every student. We can easily generalize Reny's theorem to any incontestable and top-top consistent mechanism.

Theorem 3 Let $\Gamma = (I, S, \succ, r, q)$ and φ an incontestable and top-top consistent mechanism. For any student $i \in I$ with an ex-ante canonical information set $\mathcal{H}_i = \{I, S, \succ_i, r, q\}$, and for any preference ordering \succ'_i ,

$$w_i(\succ_i | \succ_i, \varphi, \mathcal{H}_i) \succeq_i w_i(\succ'_i | \succ_i, \varphi, \mathcal{H}_i). \quad (5)$$

Note that by Proposition 3, Theorem 3 also holds when considering constrained versions of strategy-proof incontestable and top-top consistent mechanisms.

Proof Let i be a student who does not have any top-priority set given (\succ_i, r, q) . So, by Theorem 2, $\mathcal{O}_i(\mathcal{H}_i, \varphi) = T^{\succ_i, r, q} = \{v \mid v \succeq_i i\}$ and thus $w_i(\succ_i | \succ_i, \varphi, \mathcal{H}_i) = i$. Let \succ'_i be any preference ordering, and suppose first that i has a top-priority set with \succ'_i . So, $T^{\succ'_i, r, q} \subseteq S$ (i.e., it does not include i), and thus there exists $s \in T^{\succ'_i, r, q}$ such that $s \notin T^{\succ_i, r, q}$. So, $i \succ_i s$. By Theorem 2, $s \in \mathcal{O}_i(\{I, S, \succ'_i, r, q\}, \varphi)$, and thus $s \succeq_i w_i(\succ'_i | \succ_i, \varphi, \mathcal{H}_i)$. Hence, $i = w_i(\succ_i | \succ_i, \varphi, \mathcal{H}_i) \succ_i w_i(\succ'_i | \succ_i, \varphi, \mathcal{H}_i)$. If i does not have any top-priority set under \succ'_i , then by Theorem 2, $\mathcal{O}_i(\{I, S, \succ'_i, r, q\}, \varphi) = \{v \mid v \succeq'_i i\}$, and thus $i \succeq_i w_i(\succ'_i | \succ_i, \varphi, \mathcal{H}_i)$, which implies that Eq. (5) holds.

Consider now a student i who has a top-priority set. So, $T^{\succ_i, r, q} \subseteq S$. Let s be the least preferred school in $T^{\succ_i, r, q}$ (according to \succ_i). By Theorem 2, $s \in \mathcal{O}_i(\mathcal{H}_i, \varphi)$, and thus $s = w_i(\succ_i | \succ_i, \varphi, \mathcal{H}_i)$. Since $T^{\succ_i, r, q}$ is a top priority set, $s \succ_i i$. Suppose first that i does not have a top-priority set under \succ'_i . Then, from the previous paragraph, we know

that $i \succeq_i w_i(\succ'_i \mid \succ_i, \varphi, \mathcal{H}_i)$, and thus $w_i(\succ_i \mid \succ_i, \varphi, \mathcal{H}_i) \succ_i w_i(\succ'_i \mid \succ_i, \varphi, \mathcal{H}_i)$. Suppose now that i has a top-priority set under \succ'_i , say, \widehat{S} . If $\widehat{S} = T^{\succ_i, r, q}$ then by Theorem 2 we have $\mathcal{O}_i(\{I, S, \succ'_i, r, q\}, \varphi) = \mathcal{O}_i(\{I, S, \succ_i, r, q\}, \varphi)$, and thus $w_i(\succ'_i \mid \succ_i, \varphi, \mathcal{H}_i) = w_i(\succ_i \mid \succ_i, \varphi, \mathcal{H}_i)$. If $\widehat{S} \neq T^{\succ_i, r, q}$, then there exists a school s' such that $s' \in \widehat{S}$ and $s' \notin T^{\succ_i, r, q}$. Hence, $s'' \succ_i s'$ for any $s'' \in T^{\succ_i, r, q}$. Let \widehat{s} be the least preferred school in \widehat{S} (according to \succ_i). So, by Theorem 2, $\widehat{s} \in \mathcal{O}_i(\{I, S, \succ'_i, r, q\}, \varphi)$, and thus $\widehat{s} = w_i(\succ'_i \mid \succ_i, \varphi, \mathcal{H}_i)$. Hence, $s' \succeq_i \widehat{s}$. We thus have $s = w_i(\succ_i \mid \succ_i, \varphi, \mathcal{H}_i) \succ_i s' \succeq_i w_i(\succ'_i \mid \succ_i, \varphi, \mathcal{H}_i) = \widehat{s}$. ■

Together, Theorem 2 and Theorem 3 imply that the worst assignment student i can secure does not depend on the incontestable and top-top consistent mechanism used. If i has a top-priority set given her preference \succ_i , then the worst school she can secure is the school she likes the least in her smallest top-priority set. If i does not have a top-priority set given \succ_i , then i 's worst assignment is to be unassigned.

5.3 Dominant strategies

Comparisons in terms of dominant strategies inform about the extent to which mechanisms are manipulable. Over the past decades, some school districts reformed their assignment procedures in the objective to reduce their vulnerability to manipulation (Pathak and Sönmez, 2013). A mechanism that is less vulnerable to manipulations ‘levels the playing field’ because it should a priori penalizes less students who do not strategize well. Understanding to what degree a mechanism is vulnerable to strategic manipulations is of interest for several reasons because strategyproof mechanisms do not necessarily eliminate the incentive to misrepresent one’s preferences when students are constrained, a common practice —see Haerlinger and Klijn (2009). Comparing the manipulability of mechanisms is now a standard exercise. However, this is often done by comparing two explicit mechanisms. We show here that Theorem 2 permits us to compare wholly different mechanisms, for instance, TTC when students are constrained to list at most k schools and SOSM when students are constrained to list at most $k + 1$ schools.

Following Arribillaga and Massó (2016), we say that a mechanism φ is **less manipulable** than a mechanism ψ if every time a student has a dominant strategy in ψ she also has a dominant strategy in φ (or, conversely, every time a student has profitable misrepresentation φ , she also has a profitable misrepresentation in ψ).

We can extend this comparison relation to a strict comparison and an equivalence relation.

A mechanism φ is strictly less manipulable than a mechanism ψ if φ is less manipulable than ψ and there exists a problem Γ for which truthful preference revelation is a dominant strategy for a student under φ but not under ψ . Finally, a mechanism φ is equally manipulable as a mechanism ψ if φ (resp. ψ) is less manipulable than ψ (resp. φ).²¹

Proposition 6 Let φ be a (unconstrained) strategyproof, incontestable and top-top consistent mechanisms. For any k , the following are equivalent.

- (i) Student i has a dominant strategy in φ^k ;
- (ii) Submitting her k most preferred school (in the same order as in her preferences) is a dominant strategy in φ^k for student i ;
- (iii) Either student i has no more than k acceptable schools or her $k' \leq k$ most-preferred schools are all acceptable and form a top-priority set.

Proof Clearly, (ii) \Rightarrow (i).

(iii) \Rightarrow (ii) Let i be a student with preferences \succ_i , and denote by \succ_i^k the truncation of i 's preferences after the k -th most preferred school. If there are at most k schools acceptable under \succ_i , then \succ_i remains a dominant strategy in φ^k .²² Suppose now that \succ_i contains $k+1$ or more acceptable schools, and that i 's k most preferred schools is a top-priority set for i . Denote by \widehat{S} that set of schools. By Proposition 3, φ^k is incontestable and top-top consistent. Hence, for any \succ_{-i} , $\varphi_i(\succ_i, \succ_{-i}) \in \widehat{S}$. Since φ is strategyproof,

$$\varphi_i^k(\succ_i^k, \succ_{-i}) \succeq_i^k \varphi_i^k(\succ'_i, \succ_{-i}) \quad (6)$$

for any \succ'_i that contains at most k acceptable schools and any profile \succ_{-i} where for each student j there are at most k acceptable schools. Since $\varphi_i^k(\succ_i^k, \succ_{-i}) \in \widehat{S}$ for all \succ_{-i} and \succ_i^k and \succ_i are identical over \widehat{S} , (6) implies

$$\varphi_i^k(\succ_i^k, \succ_{-i}) \succeq_i \varphi_i^k(\succ'_i, \succ_{-i})$$

²¹This is the criterion used by Decerf and Van der Linden (2021) to compare the manipulability of several school choice mechanisms.

²²For any \succ_{-i} (and thus for any \succ_{-i} where each student $j \neq i$ reports at most k schools acceptable), φ being strategyproof implies $\varphi_i(\succ_i, \succ_{-i}) \succeq_i \varphi_i(\succ'_i, \succ_{-i})$ for any $\succ'_i \neq \succ_i$. Hence, $\varphi_i(\succ_i, \succ_{-i}) \succeq_i \varphi_i(\succ'_i, \succ_{-i})$ for any $\succ'_i \neq \succ_i$ that contains at most k acceptable schools and for any \succ_{-i} where each student $j \neq i$ reports at most k schools acceptable). For profiles (\succ_i, \succ_{-i}) and (\succ'_i, \succ_{-i}) , $\varphi^k = \varphi$. So, $\varphi_i^k(\succ_i, \succ_{-i}) \succeq_i \varphi_i^k(\succ'_i, \succ_{-i})$.

for any \succ'_i that contains at most k acceptable schools and any profile \succ_{-i} where for each student j there are at most k acceptable schools. So, \succ_i^k is a dominant strategy in φ^k .

(i) \Rightarrow (iii) Let \succ_i denote the preferences of student i . Suppose by way of contradiction that \succ_i has $k+1$ or more acceptable schools and that the k most preferred schools in \succ_i , which we denote \widehat{S} , is not a top-priority set. We show that i does not have a dominant strategy. Let \succ'_i be any preference ordering that contains at most k acceptable schools.

Suppose first that \widehat{S} is precisely the set of acceptable schools in \succ'_i . Since φ^k is incontestable and top-top consistent, by Theorem 2 there exists a profile \succ_{-i} (where each student $j \neq i$ has at most k acceptable schools) such that $\varphi_i^k(\succ'_i, \succ_{-i}) = i \notin \widehat{S}$. Note this also holds if for each student $j \neq i$ all of j 's acceptable schools are in \widehat{S} .²³ Let $s \in S \setminus \widehat{S}$ be a school acceptable to i under \succ_i , and let \succ''_i be such that only s is acceptable. Since $s \notin \widehat{S}$, under the profile (\succ''_i, \succ_{-i}) only student i lists s as an acceptable school. Since φ^k is individually rational, no student $j \neq i$ can be assigned to s under (\succ''_i, \succ_{-i}) . So, since φ^k is non-wasteful, $\varphi_i^k(\succ''_i, \succ_{-i}) = s$. To sum up, we have $\varphi_i^k(\succ''_i, \succ_{-i}) = s \succ_i i = \varphi_i(\succ'_i, \succ_{-i})$. So, \succ'_i is not a dominant strategy. Since \succ'_i is any feasible strategy for φ^k , student i does not have a dominant strategy in φ^k .

Suppose now that there exists a school $s' \notin \widehat{S}$ that is reported as acceptable in \succ'_i . This implies that there must be a school $s \in \widehat{S}$ that is not reported as acceptable by strategy \succ'_i . We have $s \succ_i s'$ because $s \in \widehat{S}$ and $s' \notin \widehat{S}$. By Theorem 2, there exists a profile \succ_{-i} (where each student $j \neq i$ has at most k acceptable schools) such that $\varphi_i^k(\succ'_i, \succ_{-i}) = s'$. Again, profile \succ_{-i} can be constructed such that no student $j \neq i$ reports school s and the strategy \succ''_i for which i only reports school s is such that $\varphi_i^k(\succ''_i, \succ_{-i}) = s$. Again, the generic strategy \succ'_i is not dominant. Finally, suppose that only a strict subset of schools in \widehat{S} are reported as acceptable by strategy \succ'_i (all $s' \notin \widehat{S}$ are reported unacceptable by \succ'_i). This case is such that \succ'_i has no top-priority set. Also, there is a school $s \in \widehat{S}$ that is not reported as acceptable by strategy \succ'_i . By Theorem 2, there exists a profile \succ_{-i} (where each student $j \neq i$ has at most k acceptable schools) such that $\varphi_i^k(\succ'_i, \succ_{-i}) = i$. Again, profile \succ_{-i} can be constructed such that no student $j \neq i$ reports school s and the strategy \succ''_i for which i only reports school s is such that $\varphi_i^k(\succ''_i, \succ_{-i}) = s$. Again, the generic strategy \succ'_i is not dominant. ■

²³In the proof of Theorem 2 we only consider the most preferred schools of each student $j \neq i$. That is, the proof of Theorem 2 is unchanged if each student $j \neq i$ has only one acceptable school.

Proposition 7 Let φ and ψ be two (unconstrained) strategyproof, incontestable and top-top consistent mechanisms. Then, for any $k \geq 2$,

- (i) φ^k and ψ^k are equally manipulable;
- (ii) φ^k is strictly less manipulable than φ^{k-1} .

Proof (i) The characterization of dominant strategies provided in part (iii) of Proposition 6 does not depend on the specific mechanism. So a student has a dominant strategy in φ^k if, and only if she has a dominant strategy in ψ^k .

(ii) We first show that φ^k is less manipulable than φ^{k-1} . To this end, suppose that student i has a dominant strategy with φ^{k-1} . By Proposition 6, either i has no more than $k-1$ acceptable schools or i 's $k-1$ most preferred schools constitute a top-priority set. These two properties still hold under φ^k , so by Proposition 6 i has a dominant strategy with φ^k .

Consider now the problem (I, S, \succ, r, q) , where $q_s = 1$ for each $s \in S$, and for any two schools $s, s' \in S$ and any student $i \in I$, $r_s(i) = r_{s'}(i)$. Let i be the student for whom exactly $k-1$ other students have higher priority than i , i.e., $r_s(i) = k$ for each $s \in S$. Let \succ_i be any preference ordering such that i has at least k schools acceptable. It is easy to see that i 's k most preferred schools constitute a top-priority set and that this is i 's smallest top-priority set. By Proposition 6, i has a dominant strategy with φ^k , but not with φ^{k-1} . ■

6 Conclusion

There are two main arguments to advocate for stability as a condition to prevent market failure. First, it is traditionally understood that blocking pairs will eventually block (unstable) assignments. Second, if blocking pairs are not allowed to block, the expectation of having an unjustified assignment can undermine market participation. Our paper is motivated by the fact that those rationales do not apply in many assignment markets like school choice. However, in such markets participants are still allowed to submit an appeal, on the condition that it is motivated. Requiring students to justify their appeals boils down to the standard notion of blocking as long as students can observe each other's assignments. However, if

students do not observe the other students' assignments and their preferences, appeals are hard to establish.

We proposed in this paper a notion of appeal-proof assignments, incontestability, when students have limited information and showed that this is equivalent to a weakening of the stability concept. Interestingly, unlike the standard notion of stability, there is no tradeoff between incontestability and efficiency. Although incontestability can enlarge the set of mechanisms that can be considered (e.g., TTC or EADAM), not all efficient, non-wasteful and individually rational mechanisms are incontestable (e.g., Boston is contestable).

One of the main results of this paper is that incontestable and top-top consistent mechanisms are *i*-indistinguishable. That is, from a student's perspective there is no way to know whether her assignment has been determined by, say, the student-optimal mechanism, EADAM, TTC, or any other incontestable and top-top consistency. This result implies that taking into account student's information about the participants, the preferences and/or priorities, or the assignment that obtains should be part of the fields' research agenda.

The Venn diagram in Figure 1 summarizes the main properties of the well-known mechanisms discussed in this paper. The diagram reveals that some mechanisms are simultaneously Pareto efficient, strategy-proof and incontestable. It is well-known that there is only one mechanism (TTC) that is individually rational, efficient, and strategyproof for one-to-one assignment problems (Ma, 1994). That result breaks down, however, in the many-to-one case: Morrill's (2015) 'clinching' algorithms, or Hakimov and Kesten's (2018) equitable version, or the 'standard' version of Abdulkadiroğlu and Sönmez (2003) are all individually rational, efficient, and strategyproof mechanisms that do not always coincide. Incontestability may thus be used as an additional criterion when considering efficient, and strategyproof mechanisms in many-to-one settings.

We considered in this paper the polar case when students know each school's complete priority ordering. It is relatively intuitive to see that any mechanism that is incontestable with canonical information sets remains incontestable with coarser information sets. This is so because any problem that is compatible for a canonical information set is also compatible for a coarser information set. However, the characterization result of Theorem 1 may no longer be valid. Weakenings of the non-wastefulness and/or respect for top-priority sets may be needed for coarser information sets. We leave this question open for future research.

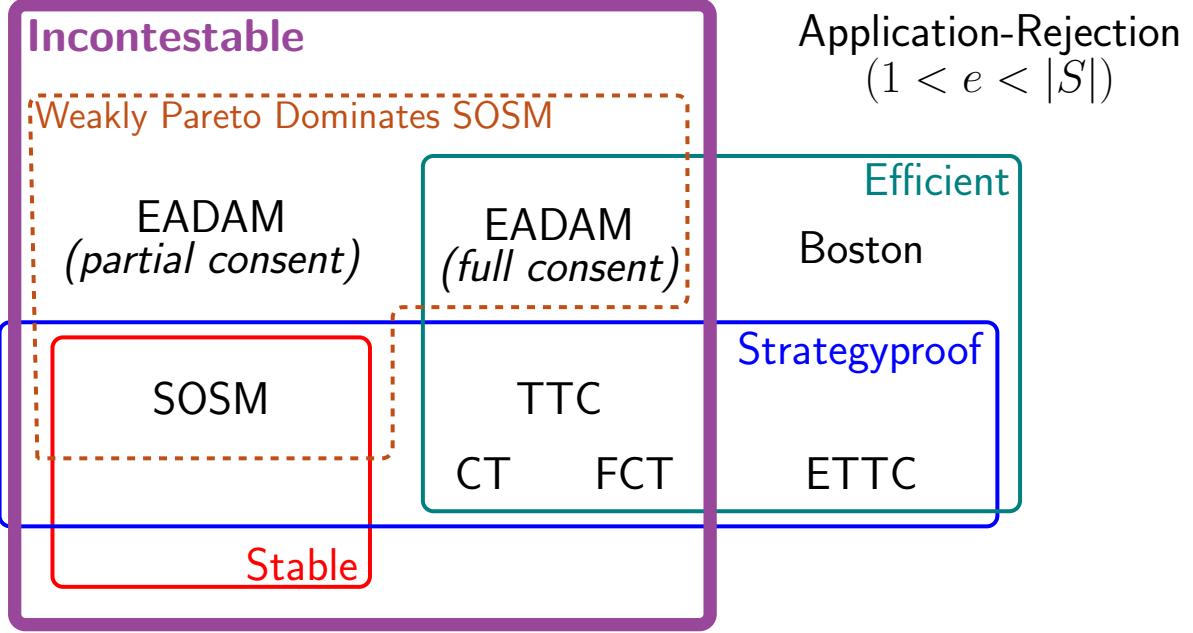


Figure 1: Comparison of mechanisms

A Proof of Theorem 2

The proof of Theorem 2 relies on the following key lemma.²⁴

Proposition 8 Let (I, S, r, q) be such that a set of schools \widehat{S} is not a high-priority set for a student i . For any incontestable and top-top consistent mechanism φ , and for any preference ordering \succ_i , there exists \succ_{-i} for which $j \succ_j s$ for all $s \notin \widehat{S}$ and $j \neq i$ and for which the problem $\Gamma = (I, S, (\succ_i, \succ_{-i}), r, q)$ is such that $\varphi_i(\Gamma) \notin \widehat{S}$.

The proof of Proposition 8 uses the following result. Let r be a priority profile and q a capacity vector. Following Haerlinger and Iehl   (2019), we call an assignment μ **comprehensive** for a set of schools S' if whenever a student i is assigned to a school in S' under μ , all students with a higher priority than i at $\mu(i)$ are also assigned to a school in S' . Formally, an assignment μ is comprehensive for a set of schools S' if for each school $s \in S'$, $i \in \mu(s)$

²⁴Proposition 8 bears some resemblance with the *only if* part of the proof of Theorem 1. In both proofs, we show that there exists a problem such that a student cannot be assigned to a school in some set \widehat{S} whenever \widehat{S} is not a high-priority set. The main difference between these two results is that in Theorem 1 we need to construct a stable assignment, whereas in Proposition 8 we show that this holds for an incontestable and top-top consistent mechanism.

implies $\mu(U_i(r_s)) \subseteq S'$. Given a problem (I, S, \succ, r, q) , an assignment μ is **maximum** if there is no assignment μ' such that $\sum_{s \in S} |\mu'(s)| > \sum_{s \in S} |\mu(s)|$.²⁵ Note that if at an assignment all schools fill their capacity then the assignment is necessarily maximum.

Lemma 2 Let (I, S, \succ, r, q) be a problem such that \widehat{S} is not a high-priority set for some student i . Then there exists an assignment μ , comprehensive for \widehat{S} , such that $|\mu(s)| = q_s$ for each $s \in \widehat{S}$ and $\mu(s) \subseteq U_i(r_s)$ for each $s \in \widehat{S}$.

Note that if μ is such that $\mu(s) \subseteq U_i(r_s)$ and $|\mu(s)| = q_s$ for each school $s \in \widehat{S}$, then we necessarily have $\mu(i) \notin \widehat{S}$.

Proof Let $\Gamma = (I, S, \succ, r, q)$ such that for some $i \in I$ a non-empty set $\widehat{S} \subseteq S$ is not a high-priority set for i . Consider the following restricted problem $\widehat{\Gamma} = (\widehat{I}, \widehat{S}, \widehat{\succ}, \widehat{r}, (q_s)_{s \in \widehat{S}})$, where the set of schools is \widehat{S} , the set of students is $\widehat{I} = \cup_{s \in \widehat{S}} U_i(r_s)$, and $\widehat{\succ}_{\widehat{I}}$ agrees with \succ_I on \widehat{S} , and for each school $s \in \widehat{S}$, \widehat{r}_s is a priority ranking over $U_i(r_s)$ such that for each $i \in U_i(r_s)$, $\widehat{r}_s(i) = r_s(i)$.²⁶

Since \widehat{S} is not a high-priority set for i , there exists an assignment for the problem Γ , say, $\widehat{\mu}$, such that $|\widehat{\mu}(s)| = q_s$ and $\widehat{\mu}(s) \subseteq U_i(\widehat{r}_s)$ for each $s \in \widehat{S}$. Hence, $\widehat{\mu}$ is a well-defined assignment for the problem $\widehat{\Gamma}$. Since $|\widehat{\mu}(s)| = q_s$ for each $s \in \widehat{S}$, the assignment $\widehat{\mu}$ is maximum for the problem $\widehat{\Gamma}$. Therefore, any maximum assignment $\widetilde{\mu}$ for that problem must be such that $|\widetilde{\mu}(s)| = q_s$ for each $s \in \widehat{S}$.

Consider the empty assignment, μ^0 (i.e., $\mu(i) = i$ for each $i \in \widehat{I}$). This assignment is trivially comprehensive and not maximum (since $q_s > 0$ for each $s \in \widehat{S}$, $\widehat{S} \neq \emptyset$). By Lemma 1 in Haeringer and Iehl   (2019), there exists a comprehensive assignment, say, μ^1 such that $|\mu^1(s)| = |\mu^0(s)| + 1$ for one school $s \in \widehat{S}$.²⁷ If μ^1 is not maximum, again by that Lemma there exists an assignment, say, μ^2 , that assigns one additional student. It suffices to repeat the calling on that Lemma until we obtain an assignment, say, μ^k , that is maximum and comprehensive. By maximality, we have $|\mu^k(s)| = q_s$ for each $s \in \widehat{S}$. By construction of $\widehat{\Gamma}$,

²⁵Note that this is equivalent to require that there is no assignment μ' such that $|\{i \in I : \mu'(i) \in S\}| > |\{i \in I : \mu(i) \in S\}|$.

²⁶In the problem $\widehat{\Gamma}$ some students in \widehat{I} may not be ‘acceptable’ for some schools in \widehat{S} . So, strictly speaking, $\widehat{\Gamma}$ is not a school choice problem as defined in Section 2. This does not affect the proof. Lemma 1 from Haeringer and Iehl   (2019) that is invoked later in the proof holds for many-to-one matching problems where schools (*departments* in Haeringer and Iehl  ’s paper) find some students unacceptable.

²⁷Lemma 1 in Haeringer and Iehl   (2019) states that if an assignment μ is comprehensive but not maximum then there exists a comprehensive assignment μ' that assigns one additional student.

we have $\mu^k(\widehat{S}) \subseteq U_i(r_{\widehat{S}})$ and thus $i \notin \mu^k(\widehat{S})$. By comprehensiveness, the fact that $i \notin \mu^k(\widehat{S})$ implies $\mu^k(s) \subseteq U_i(r_s)$ for each $s \in \widehat{S}$, the desired result. ■

Proof of Lemma 8 Let (I, S, r, q) , $i \in I$, and $\widehat{S} \subseteq S$ satisfy the conditions of the lemma. Since \widehat{S} is not a high-priority set for i , by Lemma 2 there exists an assignment μ^0 comprehensive for \widehat{S} such that $\mu^0(\widehat{S}) \subseteq U_i(r_{\widehat{S}})$ and $|\mu^0(s)| = q_s$ for each $s \in \widehat{S}$. Hence, $\mu^0(i) \notin \widehat{S}$.

Consider now a TTC algorithm restricted to the schools in \widehat{S} and all students in I , where schools point to students in I according to their priority rankings and each student $i \in I$ points to $\mu^0(i)$. However, unlike the standard TTC algorithm, here we assign students to the school that points to them. Let μ^1 be the assignment we obtain with this algorithm.

Claim: $\mu^1(\widehat{S}) = \mu^0(\widehat{S})$ and, for each student $j \in \mu^1(\widehat{S})$, there is no student $j' \notin \mu^1(\widehat{S})$ such that $r_{\mu^1(j)}(j') < r_{\mu^1(j)}(j)$.²⁸

To see this, note that if at some step of the TTC algorithm school $s \in \widehat{S}$ points to a student $j \in \mu^0(s)$, then we have a cycle between s and j and thus $\mu^1(j) = s$. Therefore, if there is a student $j \in \mu^0(s)$ for whom $\mu^1(j) \neq s$, then j was part of some larger cycle and school $\mu^1(j)$ was pointing to j . Since j points to s (because $j \in \mu^0(s)$), it must be that in that larger cycle s points to another student, say, j' , and thus we necessarily have $r_s(j') < r_s(j)$. Since μ^0 is comprehensive for \widehat{S} , $j' \in \mu^0(\widehat{S})$. More generally, the comprehensiveness of μ^0 implies that all students in that larger cycle belong to $\mu^0(\widehat{S})$. Hence, $\mu^1(\widehat{S}) = \mu^0(\widehat{S})$. Also, for each student $j \in \mu^0(s)$ for whom $j \notin \mu^1(s)$, student j is “replaced” at s by a student with a higher priority than j , for otherwise s would point to j before that student and thus we would have $\mu^0(j) = \mu^1(j)$, a contradiction. That is, since μ^0 is comprehensive for \widehat{S} , there cannot be any student, say, j , such that $j \notin \mu^1(\widehat{S})$ and $r_s(j) < r_s(j')$ for some $s \in \widehat{S}$ and $j' \in \mu^1(\widehat{S})$. □

Let Z_1, Z_2, \dots, Z_ℓ be the set of students who are assigned at step 1, 2, ..., ℓ , of the TTC algorithm defined above. Let \succ_{-i} be such that for each student $j \in \mu^1(\widehat{S})$, $\mu^1(j) \succ_j j$ and $\mu^1(j) \succ_j s$ for each school $s \neq \mu^1(j)$. Let \succ_{-i} be also such that for each student $j \notin \mu^1(\widehat{S}) \cup \{i\}$ we have $j \succ_j s$ for each $s \notin \widehat{S}$. For $\Gamma = (I, S, (\succ_i, \succ_{-i}), r, q)$, let $\mu = \varphi(\Gamma)$,

²⁸That is, μ^1 is such that, for each school $s \in \widehat{S}$, there is no student not in $\mu^1(\widehat{S})$ who has a higher priority at s than a student in $\mu^1(s)$.

where φ is an incontestable and top-top consistent mechanism. By construction of μ^1 and \succ_{-i} , for each student $j \in Z_1$, $\{\mu^1(j)\}$ is a top-priority set, and thus, since μ respects top-priority sets we must have $\mu(j) = \mu^1(j)$ for each $j \in Z_1$. Consider any student $j_1 \in Z_1$. Clearly, $(j_1, \mu^1(j_1))$ is a top-top pair. Accordingly, consider the problem Γ_1 , which is the problem Γ reduced by $(j_1, \mu^1(j_1))$. Since φ is top-top consistent, we have $\varphi_k(\Gamma_1) = \varphi_k(\Gamma)$ for each $k \neq j_1$. If there is another student, say, $j_2 \in Z_1$, then $(j_2, \mu^1(j_2))$ is also a top-top pair in Γ and Γ_1 . We can then consider the problem Γ_2 , which is the problem Γ_1 reduced by $(j_2, \mu^1(j_2))$. Continuing this way with all students in Z_1 we eventually get the problem $\Gamma^1 = (I^1, S^1, \succ^1, r^1, q^1)$, which is the problem Γ reduced by $(Z_1, \mu^1(Z_1))$, and we must have $\varphi_j(\Gamma^1) = \varphi_j(\Gamma)$ for each $j \notin Z_1$.

In the problem Γ^1 , $(j, \mu^1(j))$ is a top-top pair for each $j \in Z_2$. Reducing Γ^1 by removing successively all pairs in Z_2 yields the problem Γ^2 , which is the problem Γ^1 reduced by $(Z_2, \mu^1(Z_2))$. Top-top consistency after each removal implies that $\varphi_j(\Gamma^2) = \varphi_j(\Gamma)$ for each $j \in I \setminus (Z_1 \cup Z_2)$. Continuing this way with the sets Z_3, \dots, Z_ℓ , we eventually deduce that $\varphi_j(\Gamma) = \mu^1(j)$ for each $j \in \mu^1(\widehat{S})$. All schools in \widehat{S} are filled with students in $\mu^1(\widehat{S})$ because $\mu^1(\widehat{S}) = \mu^0(\widehat{S})$ and $|\mu^0(s)| = q_s$ for each $s \in \widehat{S}$. Since $i \notin \mu^0(\widehat{S})$, we have $i \notin \mu^1(\widehat{S})$ and we therefore have $\varphi_i(\Gamma) \notin \widehat{S}$, the desired result. \blacksquare

Proof of Theorem 2 Let i be a student such that $T^{\succ_i, r, q} \subseteq S$. So, $i \notin T^{\succ_i, r, q}$, and thus $T^{\succ_i, r, q}$ is a top-priority set (and the smallest one). So, since φ respects top-priority sets, $\mathcal{O}_i(\mathcal{H}_i, \varphi) \subseteq T^{\succ_i, r, q}$. Consider any school $s \in T^{\succ_i, r, q}$, and let $\widehat{S} = T^{\succ_i, r, q} \setminus \{s\}$. So, \widehat{S} is not a high-priority set, and thus by Proposition 8 there exists a problem Γ compatible with \mathcal{H}_i such that $\varphi_i(\Gamma) \notin \widehat{S}$. Since $T^{\succ_i, r, q}$ is a top-priority set and φ is incontestable, respect for top-priority sets implies $\varphi_i(\Gamma) \in T^{\succ_i, r, q}$. Since $\varphi_i(\Gamma) \notin \widehat{S}$, we have $\varphi_i(\Gamma) = s$. This conclusion is valid for any $s \in T^{\succ_i, r, q}$, so we have $T^{\succ_i, r, q} \subseteq \mathcal{O}_i(\mathcal{H}_i, \varphi)$ and thus $\mathcal{O}_i(\mathcal{H}_i, \varphi) = T^{\succ_i, r, q}$.

Consider now the case of \succ_i such that $T^{\succ_i, r, q} = \{v : v \succeq_i i\}$. So, $\widehat{S} = \{s : s \succ_i i\}$ is not a high-priority set for i . By Proposition 8, there exists a problem Γ compatible with \mathcal{H}_i such that $\varphi_i(\Gamma) \notin \widehat{S}$. Since φ is incontestable, it is individually rational, and thus $\varphi_i(\Gamma) = i$. Consider now any school $s \in \widehat{S}$. We need to show that there exists Γ , compatible with \mathcal{H}_i , such that $\varphi_i(\Gamma) = s$. Let $\widetilde{S} = \{s' : s' \succ_i s\}$. Since $\widetilde{S} \subset \widehat{S}$, \widetilde{S} is not a high-priority set. By Proposition 8, there exists a problem $\Gamma = (I, S, (\succ_i, \succ_{-i}), r, q)$ compatible with \mathcal{H}_i such that $\varphi_i(\Gamma) \notin \widetilde{S}$ and where \succ_{-i} is such that $j \succ_j s'$ for each $s' \notin \widetilde{S}$ and $j \neq i$. Since φ is incontestable, it is individually rational and non-wasteful. By individual rationality,

$\varphi_j(\Gamma) \neq s$ for each $j \neq i$. By non-wastefulness we thus have $\varphi_i(\Gamma) \succeq_i s$, and $\varphi_i(\Gamma) \neq \tilde{S}$ thus implies $\varphi_i(\Gamma) = s$. ■

For students without a top-priority set, the proof of Theorem 2 utilizes respect for top-priority only for top-top pairs. It may thus seem natural to ask whether the theorem still holds without requiring respect for top-priority set and augmenting our top-top consistency condition by also requiring that top-top pairs are assigned together. The result of theorem would still hold in that case for students who do not have a top-priority set (the proof would not be affected). However, it would no longer guarantee that students with a top-priority set are necessarily assigned in a school in that set. To see this, consider a student i whose smallest top-priority set has two schools (so there is no top-top pair involving i), and let the profile \succ_{-i} be such that no student in $I \setminus \{i\}$ has a top-priority set.²⁹ Under such a profile, there is no top-top pair and thus we are left with non-wastefulness and individual rationality, thereby not being able to guarantee that i is assigned to a school in her top-priority set.

²⁹If such students have a high-priority set then it suffices to have at least one school in that set to be unacceptable.

B Proof of Proposition 2

B.1 Student-Optimal Stable mechanism

We know since Section 3.2 that the Student-Optimal Stable mechanism (SOSM) is incontestable. We only have to show that it is top-top consistent.

Lemma 3 The Student-Optimal Stable mechanism is top-top consistent.

Proof Let Γ be a problem and Γ' the problem Γ reduced by a top-top pair (i, s) . Let μ and μ' be the student-optimal assignment for Γ and Γ' , respectively. We need to show that for each $j \neq i$, $\mu(j) = \mu'(j)$. Define $\widehat{\mu}$ as $\widehat{\mu}(j) = \mu'(j)$ for each $j \neq i$, and $\widehat{\mu}(i) = s$. It is easy to see that since μ' is stable for Γ' , $\widehat{\mu}$ is individually rational and non-wasteful for Γ .³⁰ We claim that $\widehat{\mu}$ is stable for the problem Γ . To see this, note that since i is assigned to her most preferred school under $\widehat{\mu}$, i cannot be part of a blocking pair. So, if $\widehat{\mu}$ is not stable for the problem Γ , there must be two students, say j and k , such that $\widehat{\mu}(k) \succ_j \widehat{\mu}(j)$ and $r_{\widehat{\mu}(k)}(j) < r_{\widehat{\mu}(k)}(k)$. Since s is i 's most preferred school, $i \neq j$. Since i has the highest priority at s , $i \neq k$. Hence, $\widehat{\mu}(j) = \mu'(j)$ and $\widehat{\mu}(k) = \mu'(k)$. Thus, j has also justified envy against k under μ' , a contradiction with μ' being stable for Γ' .

Suppose that $\widehat{\mu} \neq \mu$. Note that we must have $\mu(i) = s$, for otherwise μ would not be stable for Γ . Therefore, if $\mu(j) \neq \widehat{\mu}(j)$ then $j \neq i$. Hence, $\widetilde{\mu}$ defined as $\widetilde{\mu}(j) = \mu(j)$ for each $j \neq i$ is a stable assignment for Γ' that Pareto dominates μ' , which contradicts the assumption that μ' is the student-optimal assignment for Γ' . ■

B.2 Top-Trading Cycle mechanism

Lemma 4 The Top-Trading Cycle (TTC) mechanism is incontestable.

Proof Since TTC is individually rational and non-wasteful (Abdulkadiroğlu and Sönmez, 2003), we only have to show that TTC respects top-priority sets.

Let i be a student with a top-priority set, denoted \widehat{S} , and let μ be the assignment obtained when running the TTC algorithm. Suppose by way of contradiction that $\mu(i) \notin \widehat{S}$. Let k_i

³⁰By definition, (i, s) being a top-top pair implies that s is acceptable for i . Also, note that school s has an additional seat in Γ compared to Γ' .

and $k_{\widehat{S}}$ denote the steps of the TTC algorithm when i is assigned and the last school in \widehat{S} has filled capacity, respectively.³¹ If $\mu(i) \notin \widehat{S}$ then it must be that all schools in \widehat{S} have filled capacity when i points to a school less preferred to any school in \widehat{S} . Hence, $k_i > k_{\widehat{S}}$. Recall that in the TTC algorithm schools point to students following the order given by their priority rankings. So, a school $s \in \widehat{S}$ points to a student j such that $r_s(i) < r_s(j)$ only after i has been assigned. Since μ is non-wasteful and individually rational, Lemma 1 implies that there must be at least one student j such that $\mu(j) \in \widehat{S}$ and $r_{\mu(j)}(i) < r_{\mu(j)}(j)$. This implies that $k_i < k_{\widehat{S}}$, a contradiction. ■

Lemma 5 The Top-Trading Cycle mechanism is top-top consistent.

To prove Lemma 5 we use the following concept developed by (Dur and Paiement, 2022). Given a problem Γ , an assignment μ , and a set $\widehat{I} \subset I$ of students, the problem Γ reduced by $\mu(\widehat{I})$ is the problem obtained from Γ by removing the students in \widehat{I} and adjusting the schools' capacity by reducing for each school the number of seats corresponding to the students in \widehat{I} assigned to it.³² For each student $i \in I$, let S^i be the set of schools for which student i is the student with the highest priority, that is, $S^i = \{s \in S : r_s(i) < r_s(j) \text{ for all } j \neq i\}$. Let φ be a mechanism, and let \widehat{I} be a set of students such that for each $i \in \widehat{I}$, $S^i \cap \varphi_{\widehat{I}}(\Gamma) \neq \emptyset$.³³ A mechanism φ is **weakly consistent** if for each $i \notin \widehat{I}$,

$$\varphi_i(\Gamma') = \varphi_i(\Gamma),$$

where Γ' is the problem Γ reduced by $\varphi_{\widehat{I}}(\Gamma)$.

Proof Let φ denote the top-trading cycle mechanism. Clearly, if (i, s) is a top-top pair then $\varphi_i(\Gamma) = s$. Hence, $S^i \cap \varphi_i(\Gamma) \neq \emptyset$. Since TTC is weakly consistent (Dur and Paiement, 2022), $\varphi_j(\Gamma') = \varphi_j(\Gamma)$ for each $j \neq i$, where Γ' is the problem Γ reduced by $\varphi_i(\Gamma)$. Note that Γ' is also the problem Γ reduced by the top-top pair (i, s) . So φ is top-top consistent. ■

³¹That is, at the beginning of step $k_{\widehat{S}}$ there is a school in \widehat{S} that has still an available seat, and at step $k_{\widehat{S}} + 1$ all schools in \widehat{S} have filled their capacities.

³²That is, for each school s , its capacity in the reduced problem is $q_s - |\mu(s) \cap \widehat{I}|$, where q_s is the capacity of school s in the problem Γ .

³³That is, \widehat{I} is such that for each student $i \in \widehat{I}$, there is at least one student in \widehat{I} (not necessarily i) who is assigned under φ to one of the schools for which i is the student with the highest priority.

B.3 Clinch and Trade and First Clinch and Trade mechanisms

Both Clinch and Trade (CT) and First Clinch and Trade (FCT) are individually rational and non-wasteful —see (Morrill, 2015). Hence, to prove that they are incontestable it suffices to show that they satisfy respect for top-priority sets.

The Clinch and Trade mechanism uses the multi-round following algorithm. Each round consists of two steps, and each step has multiple rounds.

Step 1 Let $q_s^0 = q_s$. At each round k , a student i clinches a school s if s is i 's preferred school and i is among the q_s^k students with the highest priority. If a school s is assigned to a student at round k , let $q_s^{k+1} = q_s^k - 1$, and $q_s^{k+1} = q_s^k$ otherwise.

Step 1 ends at round ℓ if there is no student who has one of the q_s^ℓ highest priority at her most preferred school s .

Step 2 Each student points to her most preferred school, and each school points to the student with the highest priority. There must be at least one cycle. If a student is in a cycle, assign her to the school she is pointing to. Remove from the problem all assigned students and reduce the capacity of each school in a cycle by one.

Lemma 6 The Clinch and Trade mechanism is incontestable.

Proof Let \widehat{S} be a top-priority set for a student i . We consider two cases.

Case 1. There is at least one school $s^ \in \widehat{S}$ such that $|U_i(r_{s^*})| < q_{s^*}$.*

Consider the first round of the CT algorithm. If i clinches a school in Step 1 that must be to her most preferred school, which is necessarily in \widehat{S} . So, suppose that i does not clinch a school in Step 1 and let $s^* \in \widehat{S}$ be such that $|U_i(r_{s^*})| < q_{s^*}$. Since at any round h of Step 1 only the $q_{s^*}^h$ students with a highest priority can clinch s^* , i not clinching any school in Step 1 implies that at any round h of Step 1 we have

$$|\{j : r_{s^*}(j) < r_{s^*}(i) \text{ and } j \text{ not assigned yet}\}| < q_{s^*}^h. \quad (7)$$

Hence, if i does not clinch a school in Step 1, school s^* cannot fill its capacity in Step 1.

Let $\bar{\Gamma} = (\bar{I}, \bar{S}, \bar{\succ}, \bar{r}, \bar{q})$ be the residual problem at the beginning of Step 2. So, $i \in \bar{I}$ and $s^* \in \bar{S}$, and from Eq. (7) we have $|U_i(\bar{r}_{s^*})| < \bar{q}_{s^*}$. In Step 2, students are assigned as in the first round of TTC. If i is assigned to a school in that step it must to her most preferred school (among the schools still available), say s' . So $s' \bar{\succeq}_i s^* \Leftrightarrow s' \succeq_i s^*$, and thus, since

$s^* \in \widehat{S}$, we have $s' \in \widehat{S}$. We claim that if i is not assigned Step 2, then s^* cannot fill its capacity. To see this, suppose by way of contradiction that s^* fill its capacity in Step 2 but i is not assigned to a school. So, we have $\bar{q}_{s^*} = 1$, and from $|U_i(\bar{r}_{s^*})| < \bar{q}_{s^*}$ it must be that i is the student with the highest priority at s^* in $\bar{\Gamma}$. Hence, s^* points to i and is part of a cycle. So i is part of a cycle and is thus assigned, a contradiction.

To sum up, if i is not assigned in the first round of the CT algorithm then s^* has still some available capacity at the beginning of the second round, say, $q_{s^*}^2$, and there are at most $q_{s^*}^2 - 1$ students (among the students not unassigned yet) with a higher priority than i at s^* . For the next rounds of the CT algorithm it suffices to repeat the above reasoning, which yields that i either ends up being assigned to $s^* \in \widehat{S}$, or to a school preferred to s^* .

Case 2: For each school $s \in \widehat{S}$, $|U_i(r_s)| \geq q_s$.

Consider the first round of the CT algorithm. We show that if i is assigned in Step 1 or in Step 2 then i is assigned a school in \widehat{S} , and if i is not assigned in either Step 1 or Step 2 of the first round then in the second round i still has a top-priority set, denoted, \widehat{S}_2 , and $\widehat{S}_2 \subseteq \widehat{S}$.

If i is assigned to a school in Step 1 it must be her most preferred school, which implies that i is assigned to a school in \widehat{S} and we are done. So, suppose that at the end of Step 1 student i is not assigned to a school yet. For our purposes, it suffices to assume that, for each $s \in \widehat{S}$, we have

$$|\{j : r_{s^*}(j) < r_{s^*}(i) \text{ and } j \text{ not assigned yet}\}| \geq q_{s^*}^h. \quad (8)$$

Indeed, if there is a round h such that Eq. (8) does not hold then one can use the arguments developed in *Case 1*. Let $\bar{\Gamma} = (\bar{I}, \bar{S}, \bar{\succ}, \bar{r}, \bar{q})$ be the problem at the beginning of Step 2. So, $i \in \bar{I}$ and since Eq. (8) holds for each round of Step 1, $\widehat{S} \cap \bar{S} \neq \emptyset$.

We claim that i has a top-priority set in $\bar{\Gamma}$ that is a subset of \widehat{S} . Since \widehat{S} is a top-priority set in Γ , there exists $\widetilde{S} \subseteq \widehat{S}$ such that

$$|U_i(r_{\widetilde{S}})| < \sum_{s \in \widetilde{S}} q_s. \quad (9)$$

Let $\tilde{\mu}$ be the assignment at the end of Step 1. So, for each $j \in \bar{I}$, $\tilde{\mu}(j) = j$. Since $|U_i(r_s)| > q_s$ for each $s \in \widehat{S}$, we must have $\tilde{\mu}(\widetilde{S}) \subseteq U_i(r_{\widetilde{S}})$.³⁴ Let $\widetilde{U} = \cup_{s \in \widetilde{S}} \{j \in \bar{I} : r_j(s) < r_i(s)\}$. So, $\widetilde{U} = U_i(r_{\widetilde{S}}) \setminus (\tilde{\mu}(\widetilde{S}) \cup \tilde{\mu}(S \setminus \widetilde{S}))$, and thus $\widetilde{U} \subseteq U_i(r_{\widetilde{S}}) \setminus \tilde{\mu}(\widetilde{S})$. Therefore,

³⁴That set inclusion is key for the rest of the proof and does not necessarily hold in Case 1.

$|\tilde{U}| \leq |U_i(r_{\tilde{S}}) \setminus \tilde{\mu}(\tilde{S})| = |U_i(r_{\tilde{S}})| - |\tilde{\mu}(\tilde{S})|$, where the equality comes from the fact that $\tilde{\mu}(\tilde{S}) \subseteq U_i(r_{\tilde{S}})$. Subtracting $|\tilde{\mu}(\tilde{S})|$ from both side of Eq. (9) we get

$$|\tilde{U}| \leq |U_i(r_{\tilde{S}})| - |\tilde{\mu}(\tilde{S})| < \sum_{s \in \tilde{S}} q_s - |\tilde{\mu}(\tilde{S})| = \sum_{s \in \tilde{S}} \bar{q}_s. \quad (10)$$

Hence, Hall's marriage condition does not hold for the set of students \tilde{U} and the set of schools \tilde{S} , which implies that \tilde{S} is a high-priority set for i in $\bar{\Gamma}$. Let $S^* = \{s \in \tilde{S} : s \succeq_i s' \text{ for some } s' \in \tilde{S}\}$. Since \tilde{S} is a high-priority set for i , S^* is a top-priority set for i in $\bar{\Gamma}$ (see Remark 2), and $\tilde{S} \subseteq \hat{S}$ implies $S^* \subseteq \hat{S}$. Without loss of generality, we assume hereafter that S^* is i 's smallest top-priority set (see Remark 1).

Consider now Step 2, which is equivalent to the first round TTC (applied to the reduced problem). If i is assigned to a school in that Step, it is to her preferred school in S^* and we are done. Suppose then that i is not part of any cycle at the end of Step 2. Let $\Gamma^1 = (I^1, S^1, \succ^1, r, q^1)$ be the problem at the end of Step 2 (and thus $i \in I^1$). We claim that i has a top-priority set in Γ^1 that is a subset of S^* . Since S^* is a top-priority set in $\bar{\Gamma}$, there exists $\tilde{S} \subseteq S^* \subseteq \hat{S}$ such that

$$|\bar{U}_i(r_{\tilde{S}})| < \sum_{s \in \tilde{S}} \bar{q}_s. \quad (11)$$

where $\bar{U}_i(r_{\tilde{S}}) = \cup_{s \in \tilde{S}} \{j \in \bar{I} : r_s(j) < r_s(i)\}$. Any school $s \in \tilde{S}$ that is part of a cycle in Step 2 will have its capacity reduced by one in Γ^1 . Note that s can be assigned to a student j such that $r_s(j) > r_s(i)$. However, since any school in \tilde{S} points to a student in $\bar{U}_i(r_{\tilde{S}})$, for any seat in a school in \tilde{S} assigned through a cycle in that Step —affecting the right-hand side of Eq. (11), there is one student in $\bar{U}_i(r_{\tilde{S}})$ who is assigned —affecting the left-hand side of Eq. (11).³⁵ So at the end of Step 2 we must have

$$|U_i^1(r_{\tilde{S}})| < \sum_{s \in \tilde{S}} q_s^1, \quad (12)$$

where $U_i^1(r_{\tilde{S}}) = \cup_{s \in \tilde{S}} \{j \in I^1 : r_s(j) < r_s(i)\}$. Note that this implies that there is at least one school in \tilde{S} that still has some available capacity in Γ^1 because there are fewer students in $\bar{U}_i(r_{\tilde{S}})$ than seats in \tilde{S} (\tilde{S} is high-priority for i in $\bar{\Gamma}$). Eq. (12) implies that \tilde{S} is a high-priority set for i in Γ^1 , and thus like after Eq. (10) we can deduce that i has a top-priority set in Γ^1 , say, S^{**} , with $S^{**} \subseteq \tilde{S}$.

³⁵Note that some students in $\bar{U}_i(r_{\tilde{S}})$ may be assigned to schools not in \tilde{S} , which would reduce further the left-hand side of Eq. (11).

For the next rounds of the CT algorithm it suffices to repeat the argument developed for Case 1 or Case 2 (whichever applies in Γ^1). ■

The First Clinch and Trade (FCT) works as follows. At each round students points to their most preferred school and each school points to the students with the highest priority among the students who are not assigned yet. The assignment in each round is obtained as follows:

Step 1 If a student i is pointing to a school s and she is among the q_s students with the highest priority, assign i to s (so i clinches school s). Once there is no more student who can clinch, reduce each school's capacity by the number of students who clinched that school.

Step 2 For the remaining (i.e., non-clinching students), if there is a cycle assign each student in the cycle to the school she is pointing to. Reduce the capacity of each school in a cycle by one.

Lemma 7 The First Clinch and Trade mechanism is incontestable.

Proof Let \widehat{S} be a top-priority set for a student i , and consider the first round of the FCT algorithm. If i is assigned to a school s in Step 1 (the clinching step), then s must be i 's most preferred school, and thus $s \in \widehat{S}$. Using the same argument as in the proof of Lemma 6, we can deduce that if Γ^1 is the residual problem at the end of Step 1 (i.e., once we have removed the assigned students and adjusted schools' capacities), then i has a top-priority set in Γ^1 , say, S^1 , with $S^1 \subseteq \widehat{S}$. For Step 2, it suffices to reproduce the argument used in the proof of Lemma 6 to deduce that if i is not assigned to a school in Step 2 then in the problem at the beginning of the second round i has a top-priority set, S^2 , with $S^2 \subseteq S^1 \subseteq \widehat{S}$. It suffices to repeating the argument for each additional round and deduce that if i is not assigned to a school in her top-priority set in a round she still has a top-priority set (a subset of \widehat{S}) in the next round. Since the number of students is finite i must eventually be assigned at some round k to a school in \widehat{S} . ■

Lemma 8 The Clinch and Trade mechanism and the First Clinch and Trade mechanisms are top-top consistent.

The proof considers the Clinch and Trade mechanism. The proof for the First Clinch and Trade mechanism is identical.

Proof Let (i, s) be a top-top pair and let φ denote the Clinch and Trade (CT) mechanism. So, i clinches s in the Step 1 of the first round of the CT algorithm, that is, $\varphi_i(\Gamma) = s$. Let $\widehat{\Gamma}$ be the problem Γ reduced by (i, s) . Clearly, a student $j \neq i$ clinches a school s' in the first step of the first round of CT under Γ if, and only if j clinches school s' in the Step 1 of the first round of CT under $\widehat{\Gamma}$. To see this, let i be the first student to clinch under Γ . The set of students who can clinch after i is the same as the set of students who can clinch in $\widehat{\Gamma}$. Therefore, the problem at the end of Step 1 of the first round under Γ is the same as the problem at the end of the Step 1 of the first round under $\widehat{\Gamma}$. It follows that a student $j \neq i$ is assigned to a school s' after the Step 1 of the first round under Γ if, and only if j is assigned to school s' after Step 1 of the first round under $\widehat{\Gamma}$. To sum up, for each $j \neq i$, we have $\varphi_j(\Gamma') = \varphi_j(\Gamma)$, that is, CT is top-top consistent. ■

B.4 Efficiency Adjusted Deferred Acceptance mechanism

We consider Tang and Yu's (2014) simplified Efficiency Adjusted Deferred Acceptance mechanism (SEADM), which yields the same assignment as the Efficiency Adjusted Deferred Acceptance mechanism, or the optimal legal mechanism (Ehlers and Morrill, 2020), or the priority efficient mechanism (Reny, 2022).

For any assignment μ , a school s is under-demanded if there is no student i such that $s \succ_i \mu(i)$. SEADM is a multi-round algorithm that works as follows. Let Γ be a problem, and let $\Gamma^1 = \Gamma$. For each round $k \geq 1$, let μ^k be the student-optimal assignment for Γ^k . Let S^k be the set of under-demanded schools at μ^k . If $S^k \neq \emptyset$, for each student i , let $\mu^*(i) = \mu^k(i)$ if $\mu^k(i)$ is under-demanded. Let Γ^{k+1} be the problem Γ^k reduced by $\mu^k(S^k)$ and proceed to round $k + 1$. If $S^k = \emptyset$, then $\mu^*(i) = \mu^k$ for each student in the problem Γ^k .

It is well known that EADAM Pareto dominates SOSM, so by Corollary 1, EADAM is incontestable (and thus so are the priority efficient and legal optimal mechanisms).

Lemma 9 The Efficiency Adjusted Deferred Acceptance mechanism is top-top consistent.

Proof Let Γ be a problem, with (i, s) being a top-top pair, and let φ denote the student-optimal stable mechanism. Note that $\varphi_i(\Gamma) = s$. Let $\widehat{\Gamma}$ be the problem Γ reduced by (i, s) .

Let Γ^k and $\widehat{\Gamma}^k$ denote the problems at round k of SEADAM when the initial problem is Γ and $\widehat{\Gamma}$, respectively.

Consider any round $k \geq 1$ and assume that i is not assigned yet at the beginning of round k . Clearly, we must have $\varphi_i(\Gamma^{k'}) = s$ for each $k' \leq k$. Repeated use of Lemma 3 also implies that $\varphi_j(\widehat{\Gamma}^{k'}) = \varphi_j(\Gamma^{k'})$ for each $j \neq i$ and each $k' \leq k$. If s is an under-demanded school at $\varphi(\Gamma^k)$, then i 's assignment under SEADAM is finalized at round k and we have $\Gamma^{k+1} = \widehat{\Gamma}^{k+1}$. So, at each round $k' > k$ of SEADAM the assignments we have $\varphi(\Gamma^{k'}) = \varphi(\widehat{\Gamma}^{k'})$. If s is not an under-demanded school, then $\varphi_i(\Gamma^{k+1}) = s$ and by Lemma 3, $\varphi_j(\widehat{\Gamma}^{k+1}) = \varphi_j(\Gamma^{k+1})$ for each $j \neq i$. Repeating the argument for steps $k' \leq k + 1$ yields the desired result. ■

C Generalized rural hospital theorem

Theorem 4 Let μ be a matching that Pareto dominates an individually rational and non-wasteful assignment μ' . Then, the following holds true:

- (i) for each school $s \in S$, $|\mu(s)| = |\mu'(s)|$.
- (ii) If $|\mu'(s)| < q_s$ then $\mu(s) = \mu'(s)$.
- (iii) for each student $i \in I$, $\mu(i) \in S$ if, and only if $\mu'(i) \in S$.

Theorem 4 is a generalization of the Rural Hospital Theorem, which requires the assignments μ and μ' to be stable.³⁶ Given two matchings μ and μ' , let $L_s^{\mu,\mu'}$ and $C_s^{\mu,\mu'}$ denote the students who “leave” and “come to” school s when changing the matching from μ' to μ , respectively. Formally,

$$\begin{aligned} C_s^{\mu,\mu'} &= \mu(s) \setminus \mu'(s) \\ L_s^{\mu,\mu'} &= \mu'(s) \setminus \mu(s). \end{aligned}$$

The proof consists of showing that for each school $s \in S$, $|C_s^{\mu,\mu'}| = |L_s^{\mu,\mu'}|$.

Proof Let μ be a matching that Pareto dominates μ' . Let $I^{ex} = \{i \in I : \mu(i) \neq \mu'(i)\}$ and $\bar{S} = \{s \in S : |\mu'(s)| = q_s\}$.

It is easy to see that $C_s^{\mu,\mu'} \neq \emptyset$ implies $s \in \bar{S}$. To see this, take any school s such that $C_s^{\mu,\mu'} \neq \emptyset$ and let $i \in C_s^{\mu,\mu'}$. So, $\mu'(i) \neq \mu(i) = s$, which implies that, since preferences are strict, $s \succ_i \mu'(i)$. Since μ' is non-wasteful, we have $|\mu'(s)| = q_s$ and thus $s \in \bar{S}$. By contraposition, $s \notin \bar{S}$ implies $C_s^{\mu,\mu'} = \emptyset$.

Since μ' is individually rational, $i \in I^{ex}$ implies $\mu(i) \in S$. Hence, $\bigcup_{s \in S} C_s^{\mu,\mu'} = I^{ex}$. We thus have

$$\bigcup_{s \in \bar{S}} C_s^{\mu,\mu'} = \bigcup_{s \in S} C_s^{\mu,\mu'} = I^{ex} \quad \Rightarrow \quad \sum_{s \in \bar{S}} |C_s^{\mu,\mu'}| = |I^{ex}|. \quad (13)$$

Consider now $L_s^{\mu,\mu'}$. By definition, $\sum_{s \in S} |L_s^{\mu,\mu'}| \leq |I^{ex}|$.³⁷ Therefore,

$$\sum_{s \in S} |L_s^{\mu,\mu'}| \leq \sum_{s \in \bar{S}} |C_s^{\mu,\mu'}| = \sum_{s \in S} |C_s^{\mu,\mu'}|. \quad (14)$$

³⁶See Alva and Manjunath (2019) for a similar result.

³⁷The inequality can be strict if there is i such that $\mu'(i) = i$ and $\mu(i) \in S$.

If $s \in \overline{S}$, then $|C_s^{\mu,\mu'}| \leq |L_s^{\mu,\mu'}|$, which implies from Eq. (13) that $|I^{ex}| = \sum_{s \in \overline{S}} |C_s^{\mu,\mu'}| \leq \sum_{s \in \overline{S}} |L_s^{\mu,\mu'}|$. Combining with Eq. (14) this yields

$$\sum_{s \in \overline{S}} |C_s^{\mu,\mu'}| = \sum_{s \in \overline{S}} |L_s^{\mu,\mu'}|. \quad (15)$$

Hence, $|C_s^{\mu,\mu'}| \leq |L_s^{\mu,\mu'}|$ for each $s \in \overline{S}$, which then implies $|C_s^{\mu,\mu'}| = |L_s^{\mu,\mu'}|$ for each $s \in \overline{S}$. Note that, by definition, $|\mu(s)| = q_s$ if $s \in \overline{S}$. So, we have $|\mu'(s)| = |\mu(s)| - |L_s^{\mu,\mu'}| + |C_s^{\mu,\mu'}| = |\mu(s)| = q_s$ for each $s \in \overline{S}$.

It remains to show that $|C_s^{\mu,\mu'}| = |L_s^{\mu,\mu'}|$ for each school $s \notin \overline{S}$. Note that $s \notin \overline{S}$ implies $C_s^{\mu,\mu'} = \emptyset$. So, we need to show $L_s^{\mu,\mu'} = \emptyset$. Since

$$|I^{ex}| = \sum_{s \in \overline{S}} |C_s^{\mu,\mu'}| + \sum_{s \notin \overline{S}} |C_s^{\mu,\mu'}| = \sum_{s \in \overline{S}} |L_s^{\mu,\mu'}| + \sum_{s \notin \overline{S}} |L_s^{\mu,\mu'}|,$$

$|C_s^{\mu,\mu'}| = |L_s^{\mu,\mu'}|$ for each school $s \in \overline{S}$ and $|C_s^{\mu,\mu'}| = 0$ for each school $s \notin \overline{S}$ imply $|L_s^{\mu,\mu'}| = 0$ for each $s \notin \overline{S}$, the desired result. \blacksquare

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