

REFLECTIONS ON THE ERDŐS DISCREPANCY PROBLEM

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ABSTRACT. We consider some coloring issues related to the famous Erdős Discrepancy Problem. A set of the form $A_{s,k} = \{s, 2s, \dots, ks\}$, with $s, k \in \mathbb{N}$, is called a *homogeneous arithmetic progression*. We prove that for every fixed k there exists a 2-coloring of \mathbb{N} such that every set $A_{s,k}$ is *perfectly balanced* (the numbers of red and blue elements in the set $A_{s,k}$ differ by at most one). This prompts reflection on various restricted versions of Erdős' problem, obtained by imposing diverse confinements on parameters s, k . In a slightly different direction, we discuss a *majority* variant of the problem, in which each set $A_{s,k}$ should have an excess of elements colored differently than the first element in the set. This problem leads, unexpectedly, to some deep questions concerning completely multiplicative functions with values in $\{+1, -1\}$. In particular, whether there is such a function with partial sums bounded from above.

1. INTRODUCTION

For a number $h \in \mathbb{N}$, a red-blue coloring of a finite set A is said to be *h-balanced* if the numbers of red and blue elements in A differ by at most h . If $h = 1$, then we call it *perfectly balanced*. For positive integers $s, k \in \mathbb{N}$, we denote by $A_{s,k} = \{s, 2s, \dots, ks\}$ the homogeneous arithmetic progression of length k and step s .

In 1932 Erdős [6] asked if there exists a constant h and a red-blue coloring of \mathbb{N} such that *every* homogeneous arithmetic progression is h -balanced. The property in question seems unbelievable, and in fact, he expressed a guess that there is no such constant. This was confirmed only recently by Tao [13], with a support of collective efforts in a Polymath Project [11].

We prove in this note that for every fixed $k \in \mathbb{N}$ there is a red-blue coloring of \mathbb{N} which is perfectly balanced on all sets $A_{s,k}$.

Theorem 1. *For every fixed $k \in \mathbb{N}$ there exists a red-blue coloring of \mathbb{N} such that every homogeneous arithmetic progression $A_{s,k}$ is perfectly balanced.*

The proof uses completely multiplicative functions with values in the set $\{+1, -1\}$ and some estimates for primes in arithmetic progressions. We give it in Section 2. In Section 3 we present another approach that did not appear to be successful, but leads to an intriguing open

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problem. In Section 4 we propose a new variant of the Erdős Discrepancy Problem, motivated by the *majority* coloring of graphs. This leads in turn to a question concerning completely multipliactive functions, resembling the famous conjecture of Pólya [10], concerning partial sums of the *Liouville function* $\lambda(n)$. Finally, in Section 5, we briefly describe our initial motivation that led us to Theorem 1, and discuss the related problem concerning *rainbow* homogeneous arithmetic progressions of fixed length k .

2. THE PROOF

Recall that an arithmetic function f is *completely multiplicative* if it satisfies $f(ab) = f(a)f(b)$ for every pair of positive integers $a, b \in \mathbb{N}$. Notice that this implies that $f(1) = 1$. Since we will consider only functions with two values $\{+1, -1\}$ (corresponding to colors red and blue), we will call them shortly *multiplicative colorings*.

We start with the following simple lemma.

Lemma 1. *Let k be a fixed positive integer. Suppose that c is a multiplicative coloring of the set $\{1, 2, \dots, k\}$ which is perfectly balanced. Then c can be extended to a multiplicative coloring of \mathbb{N} which is perfectly balanced on every set $A_{s,k}$.*

Proof. Let $C = (c(1), c(2), \dots, c(k))$ be the initial color sequence satisfying the assumptions of the lemma. In particular, it satisfies

$$(2.1) \quad c(1) + c(2) + \dots + c(k) \in \{+1, -1, 0\}.$$

We extend the coloring c to the whole of \mathbb{N} in a natural way. First, if $p \geq k+1$ is a prime number, then we may chose for $c(p)$ any value from $\{+1, -1\}$. If $n = p_1 p_2 \cdots p_r$ is a product of primes p_i , then we compute $c(n) = c(p_1)c(p_2)\cdots c(p_r)$. So, the coloring c is multiplicative. Consequently, the color sequence of every set $A_{s,k}$ satisfies

$$(2.2) \quad (c(s), c(2s), \dots, c(ks)) = c(s)(c(1), c(2), \dots, c(k)) = \pm C,$$

and is therefore perfectly balanced. \square

The next lemma comes from the paper by Borwein, Choi, and Coons [2].

Lemma 2 (Borwein, Choi, Coons, [2]). *Let b be a multiplicative coloring of \mathbb{N} defined by $b(3) = +1$ and $b(p) \equiv p(\text{mod } 3)$ for all other primes p . Then for every $k \geq 1$, the sum $\sum_{i=1}^k b(i)$ equals the number of 1's in base 3 expansion of k . In consequence, for every $k \geq 1$ we have*

$$(2.3) \quad 0 \leq \sum_{i=1}^k b(i) \leq \lceil \log_3 k \rceil + 1.$$

We will also need some estimates on the number of primes of the form $3m + 1$ in the interval $(N, 2N)$. Recall that the Chebyshev function $\theta(x; 3, 1)$ is defined by

$$(2.4) \quad \theta(x; 3, 1) = \sum_{p \equiv 1 \pmod{3}, p \leq x} \log p.$$

We will use the following result of McCurley [9].

Lemma 3 (McCurley [9]). *For $x \geq 17377$ we have*

$$(2.5) \quad 0.49x \leq \theta(x; 3, 1) \leq 0.51x.$$

Using this lemma we get a useful lower bound for the number of primes of the form $3m + 1$ between x and $2x$.

Lemma 4. *Let $f(x)$ denote the number of primes of the form $3m + 1$ in the interval $(x, 2x)$. Then for every $x \geq 17377$ we have*

$$(2.6) \quad f(x) \geq 0.47 \frac{x}{\log(2x)}.$$

Proof. Let $f(x) = r$, and let q_1, q_2, \dots, q_r be all the primes of the form $3m + 1$ in the interval $(x, 2x)$. Then we may write

$$(2.7) \quad \theta(2x; 3, 1) - \theta(x; 3, 1) = \sum_{i=1}^r \log q_i \leq \log((2x)^r) = r \log(2x).$$

Using Lemma 3 we get

$$(2.8) \quad \theta(2x; 3, 1) - \theta(x; 3, 1) \geq 0.98x - 0.51x = 0.47x.$$

Hence, we obtain $r \geq \frac{0.47x}{\log(2x)}$ for all $x \geq 17377$. □

Now we are ready to prove Theorem 1

Proof of Theorem 1. Let $k \in \mathbb{N}$ be fixed. By Lemma 1 it is sufficient to construct a multiplicative perfectly balanced coloring of $\{1, 2, \dots, k\}$. For small values of k , say for $k \leq 10^6$, this can be done by computer. So, assume that $k \geq 17377$, and let b be a multiplicative coloring from Lemma 2.

We will switch the colors of prime numbers of the form $p = 3m + 1$ lying in the range $(\frac{k}{2}, k)$, from plus to minus, so that the resulting coloring will be balanced. This will not affect multiplicativity. Moreover, there are sufficiently many such primes since by Lemma 2 and Lemma 4, their number satisfies

$$(2.9) \quad f\left(\frac{k}{2}\right) \geq 0.47 \frac{k}{2 \log k} \geq \lceil \log_3 k \rceil + 1 \geq \sum_{i=1}^k b(i).$$

This completes the proof. □

3. GREEDY MULTIPLICATIVE COLORING

To prove Theorem 1 we firstly considered a different approach proposed by Rejmer (personal communication). It is a simple algorithm producing a perfectly balanced multiplicative coloring of the set $\{1, 2, \dots, k\}$ in a *greedy* way.

Let k be a fixed positive integer. We start with putting $c(1) = +1$. In each consecutive step we color the next integer so that the new coloring is perfectly balanced and multiplicative. So, in the second step we put $c(2) = -1$. For a more precise description, suppose that after $j-1$ steps, $j \geq 2$, we obtained a perfectly balanced multiplicative coloring $(c(1), c(2), \dots, c(j-1))$. In the next step we distinguish two cases.

Case 1 ($j-1 = 2m$). In this case we must have $\sum_{i=1}^{2m} c(i) = 0$. Thus, any choice for $c(j)$ will not destroy the perfect balance. If j is composite, then $c(j)$ is determined by multiplicativity. If j is prime, then we may put $c(j) = -1$.

Case 2 ($j-1 = 2m-1$). Then j is even, so $c(j)$ is determined by multiplicativity. Since $\sum_{i=1}^{2m-1} c(i) = \pm 1$, we may have either $\sum_{i=1}^{2m} c(i) = 0$ or $\sum_{i=1}^{2m} c(i) = \pm 2$. In the former case we do nothing. In the later case, we find the largest prime $p > \frac{j}{2}$ such that $c(p)$ has “wrong” sign, and switch it. This makes the new coloring $(c(1), c(2), \dots, c(j))$ perfectly balanced.

Notice that by Bertrand’s Postulate, there is always a prime between $\frac{j}{2}$ and j . However, it is not clear that there will always be a prime whose sign-switching would improve balance. For instance, in the 16th step of the algorithm we get the following coloring:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
+	-	-	+	-	+	-	-	+	+	+	-	-	+	+	+

with 9 pluses and 7 minuses. To fix this imbalance, we go back to the first prime to the left, which is $p = 13$. However, $c(13) = -1$, so switching this color into $c(13) = +1$ would only increase imbalance. Fortunately, for the next prime $p = 11$ we have $c(11) = +1$, so we may switch it to $c(11) = -1$ and get in this way a perfectly balanced multiplicative coloring.

We do not know if this algorithm ever stops.

Conjecture 1. *Rejmer’s algorithm runs ad infinitum.*

Rejmer made some computational experiments with his algorithm. In particular, he ran it up to 10^9 steps producing in this way perfectly balanced multiplicative colorings of $A_{1,k}$ for all $k \leq 10^9$. Notice that the first half terms of the last coloring will not be changed in the future. Thus, assuming validity of Conjecture 1, the algorithm defines an intriguing recursive binary sequence $R(n)$ over $\{-1, +1\}$. Up to $n = 40$ Rejmer’s sequence coincides with the Liouville function $\lambda(n)$ (defined by $\lambda(p) = -1$ for all primes p), but $R(41) = +1$. The same

happens for many other primes, in particular $R(97) = R(101) = +1$. One may suspect that there will be infinitely many primes p with $R(p) = +1$.

Conjecture 2. *There exist infinitely many primes p for which $R(p) = +1$.*

4. MAJORITY VERSION OF THE ERDŐS DISCREPANCY PROBLEM

Let h be a positive integer and let c be a red-blue coloring of the set $A_{s,k}$. We say that c is a *majority* coloring if more than a half of the elements of $A_{s,k} - \{s\}$ have color different than the element s . Notice that a perfectly balanced coloring of $A_{s,k}$ satisfies the majority condition, while a majority coloring of $A_{s,k}$ can have all elements, except one, in the same color.

Is it possible that there is a red-blue coloring of \mathbb{N} which satisfies the majority condition on *every* homogeneous arithmetic progression $A_{s,k}$? Interpreting colors as numbers $\{+1, -1\}$, we may express the majority coloring of $A_{s,k}$ via the inequality:

$$(4.1) \quad c(a) \sum_{j=1}^k c(ja) \leq 0.$$

So, the answer to the above question would be positive if we could find a completely multiplicative function c satisfying the inequality

$$(4.2) \quad \sum_{j=1}^k c(j) \leq 0,$$

for every $k \geq 2$. A natural candidate for such negativity property is the Liouville function $\lambda(n)$. Actually in 1919 Pólya [10] conjectured that $\sum_{i=1}^n \lambda(i) \leq 0$ for all $n \geq 2$, and proved that this would imply the Riemann Hypothesis (see [3]). Unfortunately, this supposition occurred to be far from the truth, but the smallest counter-example is $n = 906150257$.

As in the original Erdős Discrepancy Problem, one may consider a relaxed version of majority coloring with some parameter h . Let us call a coloring c of $A_{s,k}$ an *h-majority* coloring if it satisfies:

$$(4.3) \quad c(s) \sum_{j=1}^k c(js) \leq h.$$

This leads to the following conjecture.

Conjecture 3. *There exists a constant h and a red-blue coloring of \mathbb{N} which is h -majority on every homogeneous arithmetic progression $A_{s,k}$.*

A natural first attempt is to look for an appropriate multiplicative coloring, which leads to the following problem.

Conjecture 4. *There exists a constant h and a completely multiplicative function c satisfying*

$$(4.4) \quad \sum_{j=1}^k c(j) \leq h,$$

for all $k \geq 1$.

We made some computer experiments with functions that are close to the Liouville function $\lambda(n)$ in the sense that only a few small primes have sign $+1$. For instance, switching only the sign of one small prime gives usually a function with much smaller partial sums. However, a standard argument using the Riemann Zeta function shows that in order to get a function satisfying Conjecture 4 one has to switch signs of infinitely many primes.

5. FINAL REMARKS

Curiously, our initial impulse for Theorem 1 came from a different direction and was related to the following problem posed independently by Pach and Pálvölgyi (see [5]).

Conjecture 5. *For every $k \in \mathbb{N}$ there is a k -coloring of \mathbb{N} such that every set $A_{s,k}$ is rainbow.*

Notice that the above statement easily implies the assertion of Theorem 1. Indeed, a desired red-blue coloring can be obtained by splitting the set of k colors into two subsets of (almost) the same cardinality.

Another consequence of Conjecture 5 is a positive answer to the following question of Graham [7]: Is it true that among any n distinct positive integers a_1, a_2, \dots, a_n there is always a pair a_i, a_j satisfying $\frac{a_i}{\gcd(a_i, a_j)} \geq n$? The problem was solved in the affirmative for sufficiently large n by Szegedy [12] and independently by Zaharescu [14]. Then Balasubramanian and Soundararajan [1] gave a complete solution by using methods of Analytic Number Theory.

To see a connection between these two problems, consider a graph G_k on positive integers in which two numbers r, s are joined by an edge if and only if $\frac{r}{\gcd(r,s)} \leq k$ and $\frac{s}{\gcd(r,s)} \leq k$. Let $\omega(G_k)$ and $\chi(G_k)$ denote the clique number and the chromatic number of the graph G_k , respectively. Then Graham's problem is equivalent to $\omega(G_k) = k$, while Conjecture 5 is equivalent to a much stronger statement that $\chi(G_k) = k$ (see [4], [5]).

Going back to the Erdős Discrepancy Problem, it is natural to wonder to what extent the original question can have a positive answer. Let us call a pair of sets (S, K) *cute* if there is a constant h and a red-blue coloring of \mathbb{N} such that every set $A_{s,k}$ is h -balanced, for all $s \in S$ and $k \in K$. So, the result of Tao [13] says that the pair (\mathbb{N}, \mathbb{N}) is not cute, while Theorem 1 asserts that (\mathbb{N}, K) is cute for every singleton $K = \{k\}$. It is not hard to prove that there are infinite sets K for which (\mathbb{N}, K) is still cute. For instance, one may use the multiplicative coloring b from Lemma 2 to infer that (\mathbb{N}, K) is cute if K is the set of positive

integers avoiding 1's in their base 3 expansion. Notice however, that this set K has density zero. Is there a cute pair (\mathbb{N}, K) with K of positive density?

On the other hand, there exist dense sets S for which the pair (S, \mathbb{N}) is cute. For instance, the alternating red-blue coloring of \mathbb{N} is perfectly balanced on all sets $A_{s,k}$ with s odd (see [8]). Is there a cute pair (S, K) with both sets of density one?

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