

On The Log-Concavity of Polygonal Figurate Number Sequences

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Abstract: This paper presents the log-concavity of the m -gonal figurate number sequences. The author gives and proves the recurrence formula for m -gonal figurate number sequences and its corresponding quotient sequences which are found to be bounded. Finally, the author also shows that for $m \geq 3$, the sequence $\{S_n(m)\}_{n \geq 1}$ of m -gonal figurate numbers is a log-concave.

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1 Introduction

Figurate numbers, as well as a majority of classes of special numbers, have long and rich history. They were introduced in Pythagorean school as an attempt to connect Geometry and Arithmetic [1]. A figurate number is a number that can be represented by regular and discrete geometric pattern of equally spaced points [2]. It may be, say, a polygonal, polyhedral or polytopic number if the arrangement form a regular polygon, a regular polyhedron or a regular polytope, respectively. In particular, polygonal numbers generalize numbers which can be arranged as a triangle(triangular numbers), or a square (square numbers), to an m -gon for any integer $m \geq 3$ [3].

Some scholars have been studied the log-concavity(or log-convexity) of different numbers sequences such as Fibonacci and Hyperfibonacci numbers, Lucas and Hyperlucas numbers, Bell numbers, Hyperpell numbers, Motzkin numbers, Fine numbers, Franel numbers of order 3 and 4, Apéry numbers, Large Schröder numbers, Central Delannoy numbers, Catalan-Larcombe-French numbers sequences, and so on. See for instance [4, 5, 6, 7, 8, 9, 10, 11, 12].

To the best of the author's knowledge, among all the aforementioned works on the log-concavity and log-convexity of numbers sequences, no one has studied the log-concavity(or log-convexity) of m -gonal figurate number sequences. Hence this paper presents the log-concavity behavior of m -gonal figurate number sequences.

The paper is structured as follows. Definitions and mathematical formulations of figurate numbers are provided in Section 2. Section 3 focuses on the log-concavity of figurate number sequences, and Section 4 is about the conclusion.

2 Definitions and Formulas of Figurate Numbers

In [1, 2, 3], some properties of figurate numbers are given. In this paper we continue discussing the properties of m -gonal figurate numbers. Now we recall some definitions involved in this paper.

Definition 2.1. Let $\{s_n\}_{n \geq 0}$ be a sequence of positive numbers. If for all $j \geq 1$, $s_j^2 \geq s_{j-1}s_{j+1}$ ($s_j^2 \leq s_{j-1}s_{j+1}$), the sequence $\{s_n\}_{n \geq 0}$ is called a log-concave (or a log-convex).

Definition 2.2. Let $\{s_n\}_{n \geq 0}$ be a sequence of positive numbers. The sequence $\{s_n\}_{n \geq 0}$ is log-concave (log-convex) if and only if its quotient sequence $\left\{\frac{s_{n+1}}{s_n}\right\}_{n \geq 0}$ is non-increasing (non-decreasing).

Log-concavity and log-convexity are important properties of combinatorial sequences and they play a crucial role in many fields for instance economics, probability, mathematical biology, quantum physics and white noise theory [13, 4, 14, 15, 16, 17, 18, 19, 12].

Now we are going to consider the sets of points forming some geometrical figures on the plane. Starting from a point, add to it two points, so that to obtain an equilateral triangle. Six-points equilateral triangle can be obtained from three-points triangle by adding to it three points; adding to it four points gives ten-points triangle, etc.. Then organizing the points in the form of an equilateral triangle and counting the number of points in each such triangle, one can obtain the numbers 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ..., OEIS(Sloane's A000217), which are called *triangular numbers*, see [20, 21, 22, 23]. The n^{th} triangular number is given by the formula

$$S_n = \frac{n(n+1)}{2}, n \geq 1. \quad (1)$$

Following similar procedure, one can construct *square*, *pentagonal*, *hexagonal*, *heptagonal*, *octagonal*, *nonagonal*, *decagonal numbers*, ..., m -gonal numbers if the arrangement forms a regular m -gon [1]. The n^{th} term m -gonal number denoted by $S_n(m)$ is the sum of the first n elements of the arithmetic progression

$$1, 1 + (m - 2), 1 + 2(m - 2), 1 + 3(m - 2), \dots, m \geq 3. \quad (2)$$

Lemma 2.3 ([1]). For $m \geq 3$ and $n \geq 1$, the n^{th} term of m -gonal figurate number is given by

$$S_n(m) = \frac{n}{2}[(m - 2)n - m + 4]. \quad (3)$$

Proof. To prove (3), it suffices to find the sum of the first n elements of (2). Hence the first n elements of the arithmetic progression given in (2) is:

$$1, 1 + (m - 2), 1 + 2(m - 2), 1 + 3(m - 2), \dots, 1 + (n - 1)(m - 2), \forall m \geq 3.$$

Since the sum of the first n elements of an arithmetic progression $s_1, s_2, s_3, \dots, s_n$ is equal to

$\frac{n}{2}[s_1 + s_n]$, it follows that

$$\begin{aligned} S_n(m) &= \frac{n}{2}[s_1 + s_n] \\ &= \frac{n}{2}[1 + (1 + (n-1)(m-2))] \\ &= \frac{n}{2}[2 + (m-2)n - m + 2] \\ &= \frac{n}{2}[(m-2)n - m + 4] \quad \text{or} \\ S_n(m) &= \left(\frac{m-2}{2}\right)[n^2 - n] + n \end{aligned}$$

This completes the proof. \square

Lemma 2.4 ([1]). *For $m \geq 3$ and $n \geq 1$, the following recurrence formula for m -gonal numbers hold:*

$$S_{n+1}(m) = S_n(m) + (1 + (m-2)n), S_1(m) = 1. \quad (4)$$

Proof. By definition, we have

$$S_n(m) = 1 + (1 + (m-2)) + (1 + 2(m-2)) + \cdots + (1 + (m-2)(n-2)) + (1 + (m-2)(n-1))$$

It follows that

$$\begin{aligned} S_{n+1}(m) &= 1 + (1 + (m-2)) + (1 + 2(m-2)) + \cdots + (1 + (m-2)(n-1)) + (1 + (m-2)n) \\ S_{n+1}(m) &= S_n(m) + (1 + (m-2)n). \end{aligned}$$

Thus, for $m \geq 3$ and $n \geq 1$,

$$S_{n+1}(m) = S_n(m) + (1 + (m-2)n), S_1(m) = 1.$$

\square

3 Log-Concavity of m -gonal Figurate Number Sequences

In this section, we state and prove the main results of this paper.

Theorem 3.1. *For $m \geq 3$ and $n \geq 3$, the following recurrence formulas for m -gonal number sequences hold:*

$$S_n(m) = R(n)S_{n-1}(m) + T(n)S_{n-2}(m) \quad (5)$$

with the initial conditions $S_1(m) = 1$, $S_2(m) = m$ and the recurrence of its quotient sequence is given by

$$x_{n-1} = R(n) + \frac{T(n)}{x_{n-2}} \quad (6)$$

with the initial conditions $x_1 = m$, where

$$R(n) = \frac{m + 2(n-2)(m-2)}{1 + (n-2)(m-2)}$$

and

$$T(n) = -\frac{m-1+(n-2)(m-2)}{1+(n-2)(m-2)}.$$

Proof. By Lemma 2.4, we have

$$S_{n+1}(m) = S_n(m) + (1 + (m-2)n) \quad (7)$$

It follows that

$$S_{n+2}(m) = S_{n+1}(m) + (m-1 + (m-2)n) \quad (8)$$

Rewriting (7) and (8) for $n \geq 3$, we have

$$S_{n-1}(m) = S_{n-2}(m) + (1 + (m-2)(n-2)) \quad (9)$$

$$S_n(m) = S_{n-1}(m) + (m-1 + (m-2)(n-2)) \quad (10)$$

Multiplying (9) by $m-1 + (m-2)(n-2)$ and (10) by $1 + (m-2)(n-2)$, and subtracting as to cancel the non homogeneous part, one can obtain the homogeneous second-order linear recurrence for $S_n(m)$:

$$S_n(m) = \left[\frac{m+2(n-2)(m-2)}{1+(n-2)(m-2)} \right] S_{n-1}(m) - \left[\frac{m-1+(n-2)(m-2)}{1+(n-2)(m-2)} \right] S_{n-2}(m), \forall n, m \geq 3.$$

By denoting

$$\frac{m+2(n-2)(m-2)}{1+(n-2)(m-2)} = R(n)$$

and

$$-\frac{m-1+(n-2)(m-2)}{1+(n-2)(m-2)} = T(n),$$

one can obtain

$$S_n(m) = R(n)S_{n-1}(m) + T(n)S_{n-2}(m), \forall n, m \geq 3 \quad (11)$$

with given initial conditions $S_1(m) = 1$ and $S_2(m) = m$.

By dividing (11) through by $S_{n-1}(m)$, one can also get the recurrence of its quotient sequence x_{n-1} as

$$x_{n-1} = R(n) + \frac{T(n)}{x_{n-2}}, n \geq 3 \quad (12)$$

with initial condition $x_1 = m$. □

Lemma 3.2. For $m \geq 3$, the m -gonal figurate number sequence $\{S_n(m)\}_{n \geq 1}$, let $x_n = \frac{S_{n+1}(m)}{S_n(m)}$ for $n \geq 1$. Then we have $1 < x_n \leq m$ for $n \geq 1$.

Proof. It is clear that

$$x_1 = m, x_2 = 3 - \frac{3}{m}, x_3 = 2 - \frac{2}{3(m-1)} > 1, \text{ for } m \geq 3.$$

Assume that $x_n > 1$ for all $n \geq 3$. It follows from (12) that

$$x_n = \frac{m + 2(n-1)(m-2)}{1 + (n-1)(m-2)} - \frac{m-1+(n-1)(m-2)}{(1+(n-1)(m-2))x_{n-1}}, n \geq 2 \quad (13)$$

For $n \geq 3$, by (13), we have

$$x_{n+1} - 1 = \frac{m-1+n(m-2)}{1+n(m-2)} - \frac{m-1+n(m-2)}{1+n(m-2)x_n} \quad (14)$$

$$= \frac{(m-1+n(m-2))x_n - (n(m-2)+m-1)}{(1+n(m-2))x_n} \quad (15)$$

$$= \frac{(m-1+n(m-2))(x_n - 1)}{(1+n(m-2))x_n} \quad (16)$$

> 0 for $m \geq 3$.

Hence $x_n > 1$ for $n \geq 1$ and $m \geq 3$.

Similarly, it is known that

$$x_1 = m, x_2 = 3 - \frac{3}{m}, x_3 = 2 - \frac{2}{3(m-1)} < m, \text{ for } m \geq 3. \quad (17)$$

Assume that $x_n \leq m$ for all $n \geq 3$. It follows from (12) that

$$x_n = \frac{m + 2(n-1)(m-2)}{1 + (n-1)(m-2)} - \frac{m-1+(n-1)(m-2)}{(1+(n-1)(m-2))x_{n-1}}, n \geq 2 \quad (18)$$

For $n \geq 3$, by (18), we have

$$x_{n+1} - m = -\frac{n(m-2)^2}{1+n(m-2)} - \frac{m-1+n(m-2)}{1+n(m-2)x_n} \quad (19)$$

$$= -\frac{n(m-2)^2x_n + n(m-2) + m-1}{(1+n(m-2))x_n} \quad (20)$$

$$< -\frac{n(m-2)^2 + n(m-2) + m-1}{(1+n(m-2))x_n} \quad (21)$$

$$= -\frac{n(m-2)(2m-3)}{(1+n(m-2))x_n} \quad (22)$$

< 0 for $m \geq 3$.

Hence $x_n \leq m$ for $n \geq 1$ and $m \geq 3$.

Thus, in general, from the above two cases it follows that $1 < x_n \leq m$ for $n \geq 1$ and $m \geq 3$. \square

Lemma 3.3 ([15]). Let $\{A_n\}_{n \geq 0}$ be a sequence of positive real numbers given by the recurrence

$$A_n = R(n)A_{n-1} + T(n)A_{n-2}, n \geq 2$$

with given initial conditions A_0, A_1 and $\{x_n\}_{n \geq 1}$ its quotient sequence, given by

$$x_n = R(n) + \frac{T(n)}{x_{n-1}}, n \geq 2$$

with initial condition $x_1 = \frac{A_1}{A_0}$. If there is $n_0 \in \mathbb{N}$ such that $x_{n_0} \geq x_{n_0+1}$, $R(n) \geq 0$, $T(n) \leq 0$, and

$$\Delta R(n)x_{n-1} + \Delta T(n) \leq 0$$

for all $n \geq n_0$, then the sequence $\{A_n\}_{n \geq n_0}$ is a log-concave.

Theorem 3.4. For all $m \geq 3$, the sequence $\{S_n(m)\}_{n \geq 1}$ of m -gonal figurate numbers is a log-concave.

Proof. Let $\{S_n(m)\}_{n \geq 1}$ be a sequence of m -gonal figurate numbers given by the recurrence (5) and let $\{x_n\}_{n \geq 1}$ be its quotient sequence given by (6).

In order to prove the log-concavity of $\{S_n(m)\}_{n \geq 1}$ for all $m \geq 3$, by Lemma 3.3, we only need to show that $\{x_n\}_{n \geq 1}$ is non-increasing, $R(n) \geq 0$, $T(n) \leq 0$, and

$$\Delta R(n)x_{n-2} + \Delta T(n) \leq 0$$

for all $n \geq 3$.

By (11), since $R(n) \geq 0$ and $T(n) \leq 0$, for $m, n \geq 3$, assume, inductively that $x_1 \geq x_2 \geq x_3 \geq \dots \geq x_{n-2} \geq x_{n-1}$.

Expressing x_n from (6) and taking in to account that $\frac{T(n+1)}{x_{n-1}} \leq \frac{T(n+1)}{x_{n-2}}$, one can obtain

$$x_n = R(n+1) + \frac{T(n+1)}{x_{n-1}} \leq R(n+1) + \frac{T(n+1)}{x_{n-2}} \quad (23)$$

Now, we need to show that $x_n \leq x_{n-1}$. To show this, consider

$$R(n+1) + \frac{T(n+1)}{x_{n-2}} \leq R(n) + \frac{T(n)}{x_{n-2}} = x_{n-1} \quad (24)$$

Hence from (23) and (24), we can conclude that the quotient sequence $\{x_n\}_{n \geq 1}$ is non-increasing. It follows from (24) that

$$[R(n+1) - R(n)]x_{n-2} + T(n+1) - T(n) \leq 0 \quad (25)$$

By denoting $R(n+1) - R(n) = \Delta R(n)$ and $T(n+1) - T(n) = \Delta T(n)$, we get the compact expression for (25) as:

$$\Delta R(n)x_{n-2} + \Delta T(n) \leq 0, \forall n \geq 3.$$

Thus, by Lemma 3.3, the sequence $\{S_n(m)\}_{n \geq 1}$ of m -gonal figurate numbers is a log-concave for $m \geq 3$.

This completes the proof of the theorem. \square

4 Conclusion

In this paper, we have discussed the log-behavior of m -gonal figurate number sequences. We have also proved that for $m \geq 3$, the sequence $\{S_n(m)\}_{n \geq 1}$ of m -gonal figurate numbers is a log-concave.

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