

## 1.2 $n$ 阶行列式的性质

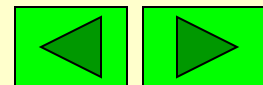
**定义** 设  $D = |a_{ij}|_n$ , 称

$$D^T = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix} \text{ 为 } D \text{ 的转置行列式.}$$

**性质1** (**转置**) 行列互换值不变, 即  $D = D^T$

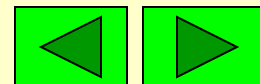
**例如**  $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = D^T$

性质1表明关于行的性质对列也成立.



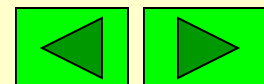
## 性质2 (换法) 换行(列)换号, 即

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$



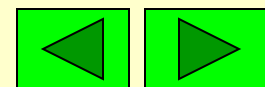
**推论** 两行(列)同值为零, 即

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0$$



**性质3** (**倍法**) 把行列式的某一行(列)的所有元素同乘以数 **$k$** , 等于用数 **$k$** 乘以这个行列式, 即

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = kD$$

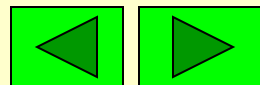


即:如果行列式某一行(列)有公因子 $k$ 时,  
则 $k$ 可以提到行列式符号的外面.

例如 
$$\begin{vmatrix} -15 & 45 \\ 2 & 3 \end{vmatrix} = 15 \times 3 \times \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} \\ = 45(-1-2) = -135$$

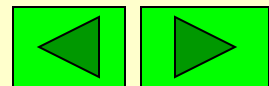
**推论** 两行(列)成比例, 值为零.

**性质4** (**分拆**) 如果行列式某行(列)的所有元素都是两数之和, 则该行列式为两个行列式之和, 即



$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} + b_{i1} & a_{i2} + b_{i2} & \cdots & a_{in} + b_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} =$$

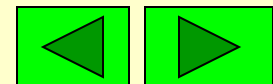
$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$



例如  $\begin{vmatrix} 4 & -1 \\ 202 & -99 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 200+2 & -100+1 \end{vmatrix}$

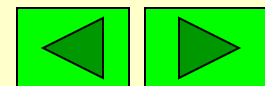
$$= \begin{vmatrix} 4 & -1 \\ 200 & -100 \end{vmatrix} + \begin{vmatrix} 4 & -1 \\ 2 & 1 \end{vmatrix} = 100 \begin{vmatrix} 4 & -1 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 4 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= -200 + 6 = -194$$



**性质5 (消法)** 将行列式的某一行(列)的各元素乘以常数加到另一行(列)的对应元素上去, 则行列式的值不变, 即

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} + ka_{j1} & a_{i2} + ka_{j2} & \cdots & a_{in} + ka_{jn} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$





# 总结行列式性质

性质1 (转置)  $D^T = D$ .

性质2 (换法) 换行(列)变号.

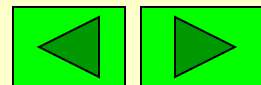
推论 两行(列)同, 值为零.

性质3 (倍法) 某行(列)乘数  $k=kD$ .

推论 两行(列)成比例, 值为零.

性质4  $D$ 可按某行(列)分拆成两行列式之和.

性质5 (消法)  $D$ 某行(列)乘数  $k$  加至另行(列),  
行列式值不变.



- 行列式的性质是有关行列式计算和推理的基础, 必须熟练掌握, 会灵活运用.

**注** 行列式变换的表示符号

行变换

列变换

换法

$$r_i \leftrightarrow r_j$$

$$c_i \leftrightarrow c_j$$

倍法

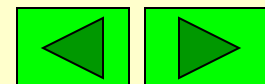
$$kr_i$$

$$kc_i$$

消法

$$kr_j + r_i$$

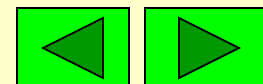
$$kc_j + c_i$$



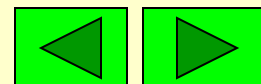
**例7** 计算  $D = \begin{vmatrix} 1 & 2 & -3 & 4 \\ 2 & 3 & -4 & 7 \\ -1 & -2 & 5 & -8 \\ 1 & 3 & -5 & 10 \end{vmatrix}$

**解** 通过行变换将 $D$ 化为上三角行列式

$$D \xrightarrow[\begin{smallmatrix} r_1 + r_3 \\ -r_1 + r_4 \end{smallmatrix}]{(-2) \times r_1 + r_2} \begin{vmatrix} 1 & 2 & -3 & 4 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 2 & -4 \\ 0 & 1 & -2 & 6 \end{vmatrix}$$



$$\underline{\underline{r_2 + r_4}} \begin{vmatrix} 1 & 2 & -3 & 4 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 5 \end{vmatrix} = -10$$



**例8** 设有四阶行列式:

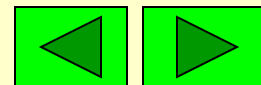
$$D = \begin{vmatrix} 2 & -1 & x & 2x \\ 1 & 1 & x & -1 \\ 0 & x & 2 & 0 \\ x & 0 & -1 & -x \end{vmatrix}$$

则展开式中 $x^4$ 的系数是( ).

**(A)** 2; (B) -2; (C) 1; (D) -1.

**解** 含 $x^4$ 的项只有一项

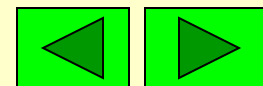
$$(-1)^{\tau(4321)} a_{14}a_{23}a_{32}a_{41} = 2x^4$$



**例9** 已知

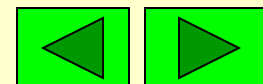
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a, \quad \begin{vmatrix} a_1' & c_1 & b_1 \\ a_2' & c_2 & b_2 \\ a_3' & c_3 & b_3 \end{vmatrix} = b$$

计算  $D = \begin{vmatrix} a_1 + 2a_1' & a_2 + 2a_2' & a_3 + 2a_3' \\ b_1 & b_2 & b_3 \\ c_1 + 3b_1 & c_2 + 3b_2 & c_3 + 3b_3 \end{vmatrix}$



**解** 由性质4  $D =$

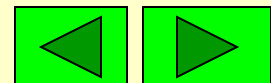
$$\begin{vmatrix} a_1 + 2a_1' & a_2 + 2a_2' & a_3 + 2a_3' \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 + 2a_1' & a_2 + 2a_2' & a_3 + 2a_3' \\ b_1 & b_2 & b_3 \\ 3b_1 & 3b_2 & 3b_3 \end{vmatrix}$$
$$= \begin{vmatrix} a_1 + 2a_1' & a_2 + 2a_2' & a_3 + 2a_3' \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} 2a_1' & 2a_2' & 2a_3' \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - 2 \begin{vmatrix} a_1' & c_1 & b_1 \\ a_2' & c_2 & b_2 \\ a_3' & c_3 & b_3 \end{vmatrix}$$

$$= a - 2b$$





## 二、行列式的计算

性质 3 为我们提供了使用矩阵初等变换计算行列式的简便方法，这种方法的计算工作量要比按定义展开的方法小得多。

利用初等变换计算行列式的一个基本程序：  
通过适当的初等变换把行列化为三角行列式。

行列式的“消元法”或“化零运算”。



例4. 设  $A = \begin{pmatrix} 1 & -3 & 7 \\ 2 & 4 & -3 \\ -3 & 7 & 2 \end{pmatrix}$ , 求  $\det A$ .

解一：化为“三角行列式”

$$\det A \xrightarrow[r_3+3r_1]{r_2-2r_1} \begin{vmatrix} 1 & -3 & 7 \\ 0 & 10 & -17 \\ 0 & -2 & 23 \end{vmatrix} \xrightarrow{r_3+r_2 \div 5} \begin{vmatrix} 1 & -3 & 7 \\ 0 & 10 & -17 \\ 0 & 0 & \frac{196}{10} \end{vmatrix} = 196$$



利用初等变换计算行列式的另一个基本程序：  
把行列式的某一行(列)的元素尽可能化为零，然后按该行(列)展开，降阶后再计算行列式的值。

解二：

$$\det A \xrightarrow[r_3+3r_1]{r_2-2r_1} \begin{vmatrix} 1 & -3 & 7 \\ 0 & 10 & -17 \\ 0 & -2 & 23 \end{vmatrix} \xrightarrow{\text{按} c_1 \text{展开}} \begin{vmatrix} 10 & -17 \\ -2 & 23 \end{vmatrix}$$

$$\xrightarrow{r_2+\frac{1}{5}r_1} \begin{vmatrix} 10 & -17 \\ 0 & \frac{196}{10} \end{vmatrix}$$

$$= 196$$



例5 计算  $D = \begin{vmatrix} 1 & 4 & -1 & 4 \\ 2 & 1 & 4 & 3 \\ 4 & 2 & 3 & 11 \\ 3 & 0 & 9 & 2 \end{vmatrix}$

解  $D \xrightarrow[r_3 - 2r_2]{r_1 - 4r_2} \begin{vmatrix} -7 & 0 & -17 & -8 \\ 2 & 1 & 4 & 3 \\ 0 & 0 & -5 & 5 \\ 3 & 0 & 9 & 2 \end{vmatrix} = (-1)^{2+2} \begin{vmatrix} -7 & -17 & -8 \\ 0 & -5 & 5 \\ 3 & 9 & 2 \end{vmatrix}$

$\xrightarrow{c_2 + c_3} \begin{vmatrix} -7 & -25 & -8 \\ 0 & 0 & 5 \\ 3 & 11 & 2 \end{vmatrix} = -5 \cdot \begin{vmatrix} -7 & -25 \\ 3 & 11 \end{vmatrix} = 10$



例6 计算  $D_n = \begin{vmatrix} x & y & \cdots & y \\ y & x & \cdots & y \\ \cdots & \cdots & \cdots & \cdots \\ y & y & \cdots & x \end{vmatrix}$

解

$$D_n \xrightarrow[\substack{c_1+c_j \\ j=2,\cdots,n}]{\quad} \begin{vmatrix} x+(n-1)y & y & \cdots & y \\ x+(n-1)y & x & \cdots & y \\ \cdots & \cdots & \cdots & \cdots \\ x+(n-1)y & y & \cdots & x \end{vmatrix}$$



$$= [x + (n-1)y](x-y)^{n-1}$$



## 例7 证明范德蒙行列式( $n \geq 2$ )

$$\begin{aligned} V_n &= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq j < i \leq n} (x_i - x_j), \\ &= (x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1) \\ &\quad (x_3 - x_2) \cdots (x_n - x_2) \\ &\quad \vdots \\ &\quad (x_n - x_{n-1}) \end{aligned}$$



## 例7 证明范德蒙行列式( $n \geq 2$ )

$$V_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq j < i \leq n} (x_i - x_j),$$

证  $n = 2$ :  $\begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1$ , 结论成立。

设对于 $n-1$ 阶结论成立, 对于 $n$ 阶:





$$V_n \begin{matrix} \text{按} c_1 \text{展} \\ \text{按} c_1 \text{展} \end{matrix} \begin{matrix} r_i - x_1 r_{i-1} \\ i = n, n-1, \dots, 2 \end{matrix} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 & \cdots & x_n - x_1 \\ 0 & x_2(x_2 - x_1) & x_3(x_3 - x_1) & \cdots & x_n(x_n - x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & x_2^{n-2}(x_2 - x_1) & x_3^{n-2}(x_3 - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix}$$

$$\begin{matrix} \text{按} c_1 \text{展} \\ \text{按} c_1 \text{展} \end{matrix} \begin{vmatrix} x_2 - x_1 & x_3 - x_1 & \cdots & x_n - x_1 \\ x_2(x_2 - x_1) & x_3(x_3 - x_1) & \cdots & x_n(x_n - x_1) \\ \cdots & \cdots & \cdots & \cdots \\ x_2^{n-2}(x_2 - x_1) & x_3^{n-2}(x_3 - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix}$$



$$= (x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & \cdots & x_n \\ \cdots & \cdots & \cdots & \cdots \\ x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \end{vmatrix}$$

**$n-1$ 阶范德蒙行列式**

$$\begin{aligned} V_n &= (x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1) \prod_{2 \leq j < i \leq n} (x_i - x_j) \\ &= \prod_{1 \leq j < i \leq n} (x_i - x_j) \end{aligned}$$



## 例8

$$\begin{aligned}
 D &= \begin{vmatrix} a & a^2 & a^3 & a^4 \\ b & b^2 & b^3 & b^4 \\ c & c^2 & c^3 & c^4 \\ d & d^2 & d^3 & d^4 \end{vmatrix} = abcd \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} \\
 &= abcd \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} \\
 &= abcd (d-c)(d-b)(d-a)(c-b)(c-a)(b-a)
 \end{aligned}$$



例9 计算

$$D_n = \begin{vmatrix} 1+a_1 & a_2 & \cdots & a_n \\ a_1 & 1+a_2 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & 1+a_n \end{vmatrix}$$

解

加边法

$$D_n = \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ 0 & 1+a_1 & a_2 & \cdots & a_n \\ 0 & a_1 & 1+a_2 & \cdots & a_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_1 & a_2 & \cdots & 1+a_n \end{vmatrix} \xrightarrow[r_i - r_1]{i=2, \dots, n+1} \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{vmatrix}$$



返回

$$\begin{array}{c} c_1 + c_j \\ \hline \hline j=2, \dots, n+1 \end{array} \left| \begin{array}{cccccc} 1 + \sum_{i=1}^n a_i & a_1 & a_2 & \cdots & a_n \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} \end{array} \right| = 1 + \sum_{i=1}^n a_i$$

(考虑：其他解法？)

(再考虑例6？)



解二

$$D_n = \begin{vmatrix} 1+a_1 & a_2 & \cdots & a_n \\ a_1 & 1+a_2 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & 1+a_n \end{vmatrix}$$

$$\begin{array}{c} c_1+c_j \\ \hline \hline j=2,\cdots,n \end{array} \begin{vmatrix} 1+\sum_{i=1}^n a_i & a_2 & \cdots & a_n \\ 1+\sum_{i=1}^n a_i & 1+a_2 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ 1+\sum_{i=1}^n a_i & a_2 & \cdots & 1+a_n \end{vmatrix}$$



返回

$$= \left( 1 + \sum_{i=1}^n a_i \right) \begin{vmatrix} 1 & a_2 & \cdots & a_n \\ 1 & 1+a_2 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ 1 & a_2 & \cdots & 1+a_n \end{vmatrix}$$

$$\begin{matrix} \mathbf{r}_i - \mathbf{r}_1 \\ \hline \hline \hline \hline \hline \\ i=2, \cdots, n \end{matrix} \left( 1 + \sum_{i=1}^n a_i \right) \begin{vmatrix} 1 & a_2 & \cdots & a_n \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= 1 + \sum_{i=1}^n a_i$$



# 计算行列式的方法归纳及综合运用：

- 1、依定义计算行列式
- 2、用对角线法则计算行列式，它只适用二阶、三阶行列式
- 3、用一些简单的、已知的行列式来计算行列式

三角行列式；

一行(列)元素全为零的行列式；

两行(列)元素对应成比例的行列式；

范德蒙行列式；

.....





4、用行列式性质对行列式进行变形，化成已知的或容易计算的行列式

5、利用行列式按行(列)展开的性质对行列式进行降阶来计算行列式

6、用数学归纳法证明行列式

7、综合运用上述方法计算行列式



# 例10 计算4阶行列式

$$D = \begin{vmatrix} a^2 + \frac{1}{a^2} & a & \frac{1}{a} & 1 \\ b^2 + \frac{1}{b^2} & b & \frac{1}{b} & 1 \\ c^2 + \frac{1}{c^2} & c & \frac{1}{c} & 1 \\ d^2 + \frac{1}{d^2} & d & \frac{1}{d} & 1 \end{vmatrix}$$

(已知  $abcd = 1$ )



解

$$D = \begin{vmatrix} a^2 & a & \frac{1}{a} & 1 \\ b^2 & b & \frac{1}{b} & 1 \\ c^2 & c & \frac{1}{c} & 1 \\ d^2 & d & \frac{1}{d} & 1 \end{vmatrix} + \begin{vmatrix} \frac{1}{a^2} & a & \frac{1}{a} & 1 \\ \frac{1}{b^2} & b & \frac{1}{b} & 1 \\ \frac{1}{c^2} & c & \frac{1}{c} & 1 \\ \frac{1}{d^2} & d & \frac{1}{d} & 1 \end{vmatrix}$$



$$= abcd \begin{vmatrix} a & 1 & \frac{1}{a^2} & \frac{1}{a} \\ b & 1 & \frac{1}{b^2} & \frac{1}{b} \\ c & 1 & \frac{1}{c^2} & \frac{1}{c} \\ d & 1 & \frac{1}{d^2} & \frac{1}{d} \end{vmatrix} + (-1)^3 \begin{vmatrix} a & 1 & \frac{1}{a^2} & \frac{1}{a} \\ b & 1 & \frac{1}{b^2} & \frac{1}{b} \\ c & 1 & \frac{1}{c^2} & \frac{1}{c} \\ d & 1 & \frac{1}{d^2} & \frac{1}{d} \end{vmatrix}$$

$$= 0.$$



# 思考题1

设 $n$ 阶行列式

$$D_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & n \end{vmatrix}$$

求第一行各元素的代数余子式之和：

$$A_{11} + A_{12} + \cdots + A_{1n}.$$



**解** 第一行各元素的代数余子式之和可以表示成

$$A_{11} + A_{12} + \cdots + A_{1n} = \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{3} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{n} \end{vmatrix} = n! \left( 1 - \sum_{j=2}^n \frac{1}{j} \right).$$



# 思考题2

证明  $2n$  阶行列式

$$D_{2n} = \begin{vmatrix} a & & & & b \\ & \ddots & & & \\ & & a & b \\ & & c & d \\ & & & & \\ c & & & & d \end{vmatrix} = (ad - bc)^n$$



证一:

(1) 当  $a = 0$  时, 根据行列式的定义有

$$D_{2n} = (-1)^{n(2n-1)} (bc)^n = (-1)^n (bc)^n = (-bc)^n$$

(2) 当  $a \neq 0$  时, 利用行列式的初等变换有





$$D_{2n} = \begin{vmatrix} c_{2n-i+1} + c_i \times \left(-\frac{b}{a}\right) \\ i=1, 2, \dots, n \end{vmatrix}$$

$$= \begin{vmatrix} a & & & 0 \\ & \ddots & & \\ & & a & 0 \\ c & & d - \frac{bc}{a} & \\ & \ddots & & \\ & & c & d - \frac{bc}{a} \\ & & & \ddots \end{vmatrix}$$

$$= a^n \left(d - \frac{bc}{a}\right)^n = (ad - bc)^n$$



证二：将行列式第  $2n$  行依次与第  $2n-1$  行,  $\cdots$ , 第2行对调 (作  $2n-2$  次相邻对换), 再把第  $2n$  列依次与  $2n-1$  列,  $\dots$ , 第2列对调, 得

$$D_{2n} = \begin{vmatrix} a & b & & & \\ c & d & & & \\ & & a & & b \\ & & \ddots & & \ddots \\ & & & a & b \\ & & & c & d \\ & & & & \ddots \\ & & c & & d \end{vmatrix}$$

$$\begin{vmatrix} A & O \\ O & B \end{vmatrix} = |A||B|$$

$$= D_2 D_{2(n-1)}$$

$$= \dots$$

$$= (D_2)^n$$

$$= (ad - bc)^n$$



# 思考题3

计算

$$D_n =$$

$$\begin{vmatrix} x & -1 & 0 & \dots & 0 & 0 \\ 0 & x & -1 & \dots & 0 & 0 \\ 0 & 0 & x & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x & -1 \\ a_n & a_{n-1} & a_{n-2} & \dots & a_2 & x+a_1 \end{vmatrix}$$

$$D_n = xD_{n-1} + a_n$$

解

$$D_n$$

按第一列展开

$$=====$$

$$x$$

$$\begin{vmatrix} x & -1 & 0 & \dots & 0 & 0 \\ 0 & x & -1 & \dots & 0 & 0 \\ 0 & 0 & x & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x & -1 \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_2 & x+a_1 \end{vmatrix}$$

$$+ (-1)^{n+1} a_n (-1)^{n-1}$$



返回

$$D_n = xD_{n-1} + a_n$$

$$D_{n-1} = xD_{n-2} + a_{n-1}$$

$$D_{n-2} = xD_{n-3} + a_{n-2}$$

.....

$$D_2 = \begin{vmatrix} x & -1 \\ a_2 & x + a_1 \end{vmatrix} = xD_1 + a_2$$

$$D_1 = x + a_1$$

$$\therefore D_n = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$$



$$\begin{array}{c|cccccc|c|cccccc}
 (5) & 1 & 2 & 2 & \dots & 2 & 2 & & 1 & 2 & 2 & \dots & 2 & 2 \\
 & 2 & 2 & 2 & \dots & 2 & 2 & & 2 & 2 & 2 & \dots & 2 & 2 \\
 & 2 & 2 & 3 & \dots & 2 & 2 & \begin{array}{c} r_i - r_2 \\ \hline \hline \hline \end{array} & 0 & 0 & 1 & \dots & 0 & 0 \\
 & \dots & \dots & \dots & \dots & \dots & \dots & i=3, \dots, n & \dots & \dots & \dots & \dots & \dots & \dots \\
 & 2 & 2 & 2 & \dots & n-1 & 2 & & 0 & 0 & 0 & \dots & n-3 & 0 \\
 & 2 & 2 & 2 & \dots & 2 & n & & 0 & 0 & 0 & \dots & 0 & n-2
 \end{array}$$

$$\begin{array}{c|cccccc}
 \begin{array}{c} r_2 - 2r_1 \\ \hline \hline \hline \end{array} & 1 & 2 & 2 & \dots & 2 & 2 \\
 & 0 & -2 & -2 & \dots & -2 & -2 \\
 & 0 & 0 & 1 & \dots & 0 & 0 \\
 & \dots & \dots & \dots & \dots & \dots & \dots \\
 & 0 & 0 & 0 & \dots & n-3 & 0 \\
 & 0 & 0 & 0 & \dots & 0 & n-2
 \end{array}$$

$$= -2(n-2)!$$



返回

$$\begin{array}{c} (5) \end{array} \left| \begin{array}{cccccc} 1 & 2 & 2 & \dots & 2 & 2 \\ 2 & 2 & 2 & \dots & 2 & 2 \\ 2 & 2 & 3 & \dots & 2 & 2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & 2 & \dots & n-1 & 2 \\ 2 & 2 & 2 & \dots & 2 & n \end{array} \right| \begin{array}{c} r_i - r_2 \\ \hline \hline \hline \hline \hline \\ i=1,3,L,n \end{array} \left| \begin{array}{cccccc} -1 & 0 & 0 & \dots & 0 & 0 \\ 2 & 2 & 2 & \dots & 2 & 2 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & n-3 & 0 \\ 0 & 0 & 0 & \dots & 0 & n-2 \end{array} \right|$$

$$\begin{array}{c} r_2 + 2r_1 \\ \hline \hline \hline \hline \hline \end{array} \left| \begin{array}{cccccc} -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 2 & \dots & 2 & 2 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & n-3 & 0 \\ 0 & 0 & 0 & \dots & 0 & n-2 \end{array} \right|$$

$$= -2(n-2)!$$



$$(6) \quad D = \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix}$$

	1	3	5	7
$c_4 - c_3$	4	5	7	9
$c_3 - c_2$	9	7	9	11
$c_2 - c_1$	16	9	11	13

	1	3	2	2
$c_4 - c_3$	4	5	2	2
$c_3 - c_2$	9	7	2	2
	16	9	2	2

$$= 0$$



$$(6) \ D = \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix} \begin{array}{c} \\ \underline{\underline{r_3 - r_2}} \\ \underline{\underline{r_4 - r_1}} \\ \end{array} \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 5 & 7 & 9 & 11 \\ 15 & 21 & 27 & 33 \end{vmatrix} = 0$$

两行对应成比例





推广  $D = \begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ (a+1)^2 & (a+2)^2 & (a+3)^2 & (a+4)^2 \\ (a+2)^2 & (a+3)^2 & (a+4)^2 & (a+5)^2 \\ (a+3)^2 & (a+4)^2 & (a+5)^2 & (a+6)^2 \end{vmatrix}$

**= 0**

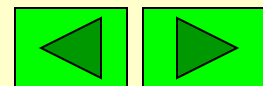


下面讨论将 $n$ 阶行列式转化为 $n-1$ 阶行列式计算的问题, 即

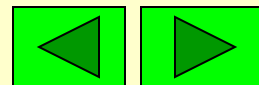
### 1.3 行列式展开定理

**定义** 在给定的 $n$ 阶行列式 $D = |a_{ij}|_n$ 中, 把元素 $a_{ij}$ 所在的 $i$ 行和 $j$ 列的元素划去, 剩余元素构成的 $n-1$ 阶行列式称为元素 $a_{ij}$ 的**余子式**, 记作 $M_{ij}$ ; 而元素 $a_{ij}$ 的**代数余子式**记作 $A_{ij}$

$$A_{ij} = (-1)^{i+j} M_{ij}$$



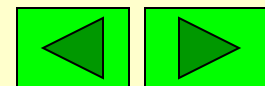
$$M_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}$$



**例10** 在行列式  $D = \begin{vmatrix} 1 & 2 & -3 & 4 \\ 2 & 3 & -4 & 7 \\ -1 & -2 & 5 & -8 \\ 1 & 3 & -5 & 10 \end{vmatrix}$  中

$$M_{11} = \begin{vmatrix} 3 & -4 & 7 \\ -2 & 5 & -8 \\ 3 & -5 & 10 \end{vmatrix} = 11, \quad A_{11} = (-1)^{1+1} M_{11} = 11$$

$$M_{21} = \begin{vmatrix} 2 & -3 & 4 \\ -2 & 5 & -8 \\ 3 & -5 & 10 \end{vmatrix} = 12, \quad A_{21} = (-1)^{2+1} M_{21} = -12$$

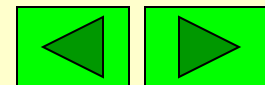


**引理** 若  $D$  的第  $i$  行元素除  $a_{ij}$  外都是零，  
则  $D = a_{ij}A_{ij}$

**定理3**  $n$ 阶行列式  $D = |a_{ij}|_n$  等于它的任意一  
行(列)的所有元素与其对应的代数  
余子式的乘积之和，即

$$D = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \quad (i = 1, 2, \cdots, n)$$

$$D = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \quad (j = 1, 2, \cdots, n)$$



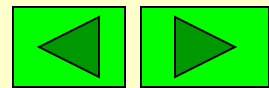
$$D = \begin{vmatrix} & a_{11} & & a_{12} & & \cdots & & a_{1n} \\ & \vdots & & \vdots & & & & \vdots \\ a_{i1} + 0 & + \cdots + 0 & 0 & + a_{i2} & + 0 & + \cdots + 0 & \cdots 0 & + \cdots + 0 & + a_{in} \\ & \vdots & & \vdots & & & & \vdots \\ & a_{n1} & & a_{n2} & & \cdots & & a_{nn} \end{vmatrix}$$

$$= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \quad (i = 1, 2, \cdots, n)$$

**定理4**  $n$ 阶行列式  $D = |a_{ij}|_n$ , 则

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} D & i = j \\ 0 & i \neq j \end{cases}$$

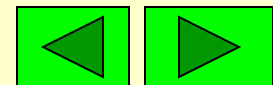
$$a_{1i}A_{1j} + a_{2i}A_{2j} + \cdots + a_{ni}A_{nj} = \begin{cases} D, & i = j \\ 0, & i \neq j \end{cases}$$



**证** 由  $G = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \begin{matrix} \text{---第 } i \text{ 行} \\ \\ \text{---第 } j \text{ 行} \end{matrix} = 0$

及降阶法将  $G$  按  $j$  行展开有

$$G = a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = 0$$



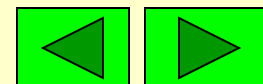


## 总结 $n$ 行列式的计算方法

1. 定义法——利用 $n$ 阶行列式的定义计算；
2. 三角形法——利用性质化为三角形行列式来计算；
3. 降阶法——利用行列式的按行(列)展开性质对行列式进行降阶计算；
4. 加边法(升阶法)；
5. 递推公式法；
6. 归纳法.

# 例1 计算 $n$ 阶行列式(行和相同)

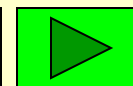
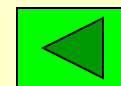
$$D_n = \begin{vmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ a & a & a & \cdots & x \end{vmatrix}$$



解

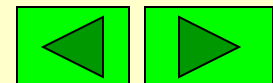
$$D_n \xrightarrow{c_1 + c_i (i=2,3,\dots,n)} \begin{vmatrix} x+(n-1)a & a & a & \cdots & a \\ x+(n-1)a & x & a & \cdots & a \\ x+(n-1)a & a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ x+(n-1)a & a & a & \cdots & x \end{vmatrix}$$

$$= [x+(n-1)a] \begin{vmatrix} 1 & a & a & \cdots & a \\ 1 & x & a & \cdots & a \\ 1 & a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a & a & \cdots & x \end{vmatrix}$$



$$r_j - r_1 \ (j=2,3,\cdots,n)$$

$$= [x + (n-1)a](x-a)^{n-1}$$

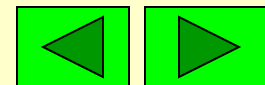


**例2** 计算  $n$  阶行列式 (两道一点)

$$D_n = \begin{vmatrix} a_1 & b_1 & & & \\ & a_2 & b_2 & & \\ & & \ddots & \ddots & \\ & & & a_{n-1} & b_{n-1} \\ b_n & & & & a_n \end{vmatrix}$$

**解**

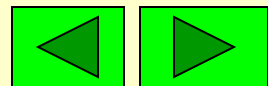
$$\begin{aligned} D_n &= a_1 a_2 \cdots a_n + (-1)^{n+1} b_n b_1 b_2 \cdots b_{n-1} \\ &= a_1 a_2 \cdots a_n + (-1)^{n+1} b_1 b_2 \cdots b_{n-1} b_n \end{aligned}$$



### 例3 计算 $n+1$ 阶行列式(爪形)

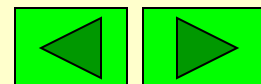
$$D = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ b_1 & d_1 & & & \\ b_2 & & d_2 & & \\ \vdots & & & \ddots & \\ b_n & & & & d_n \end{vmatrix}$$

其中  $d_i \neq 0, \quad i = 1, 2, \cdots, n$



解

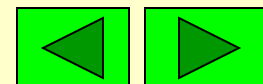
$$D = \left| \begin{array}{cccccc} a_0 & a_1 & a_2 & \cdots & a_n \\ b_1 & d_1 & & & \\ b_2 & & d_2 & & \\ \vdots & & & \ddots & \\ b_n & & & & d_n \end{array} \right|$$



当  $d_1, d_2, \dots, d_n$  全不为零时

$$D \begin{array}{c} c_1 - \frac{b_{j-1}}{d_{j-1}} c_j \\ \hline j = 2, 3, \dots, n \end{array} \left| \begin{array}{cccccc} a_0 - \sum_{k=1}^n \frac{a_k b_k}{d_k} & a_1 & a_2 & \cdots & a_n \\ 0 & d_1 & & & \\ 0 & & d_2 & & \\ \vdots & & & \ddots & \\ 0 & & & & d_n \end{array} \right|$$

$$= (a_0 - \sum_{k=1}^n \frac{a_k b_k}{d_k}) d_1 d_2 \cdots d_n$$

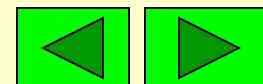




# 例4 证明 $n$ 阶 (三对角) 行列式

$$D_n = \begin{vmatrix} \alpha + \beta & \alpha\beta & & & \\ 1 & \alpha + \beta & \alpha\beta & & \\ & 1 & \alpha + \beta & \alpha\beta & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & \alpha + \beta & \alpha\beta \\ & & & & 1 & \alpha + \beta \end{vmatrix}$$

$$= \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} \quad \text{其中 } \alpha \neq \beta$$



**证** 对行列式阶数 $n$ 用数学归纳法证明

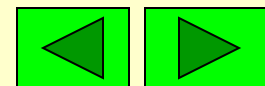
$$1^\circ n=1 \text{ 时, } D_1 = |\alpha + \beta| = \alpha + \beta = \frac{\beta^2 - \alpha^2}{\beta - \alpha}$$

结论成立.

$$n=2 \text{ 时, } D_2 = \begin{vmatrix} \alpha + \beta & \alpha\beta \\ 1 & \alpha + \beta \end{vmatrix}$$

$$= (\alpha + \beta)^2 - \alpha\beta = \frac{\beta^3 - \alpha^3}{\beta - \alpha}$$

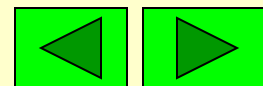
结论成立.



2° 设 $n-1$ ,  $n-2$ 时结论成立,

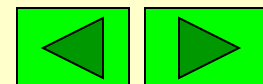
则对于 $n$ 阶行列式  $D_n$  按第一行展开有

$$\begin{aligned} D_n &= (\alpha + \beta)D_{n-1} - \alpha\beta D_{n-2} \\ &= (\alpha + \beta) \frac{\beta^n - \alpha^n}{\beta - \alpha} - \alpha\beta \frac{\beta^{n-1} - \alpha^{n-1}}{\beta - \alpha} \\ &= \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} \end{aligned}$$



## 例5 证明范德蒙(Vandermonde)行列式

$$V_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq j < i \leq n} (x_i - x_j) \quad (n \geq 2)$$



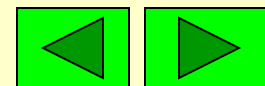
**证** 用数学归纳法证明

$$n=2 \text{ 时, } V_2 = \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1 = \prod_{1 \leq j < i \leq 2} (x_i - x_j)$$

结论成立.

假设对  $n-1$  阶行列式结论成立, 下证  $n$  阶成立.

从第  $n$  行开始, 每一行减去前一行的  $x_1$  倍, 目的是把第一列除 1 以外的元素都化为零. 然后按第一列展开, 并提取各列的公因子, 可以得到:

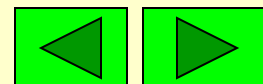


$$V_n = (x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \end{vmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1) V_{n-1}$$

$$= (x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1) \prod_{2 \leq j < i \leq n} (x_i - x_j)$$

$$= \prod_{1 \leq j < i \leq n} (x_i - x_j)$$



或者利用递推公式

$$V_n = (x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1)V_{n-1}$$

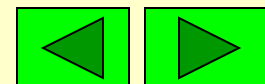
$$V_{n-1} = (x_3 - x_2)(x_4 - x_2) \cdots (x_n - x_2)V_{n-2}$$

... ..

$$V_3 = (x_n - x_{n-2})(x_{n-1} - x_{n-2})V_2$$

$$V_2 = x_n - x_{n-1}$$

由上述递推结果即可得到结论.



# 预 习 1.4-- 2.2