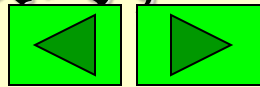


例1 $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, 求 A^n .

解
$$\begin{aligned} A^2 &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, A^3 = A \end{aligned}$$

当 n 为偶数时, $A^n = E$. 当 n 为奇数时, $A^n = A$.

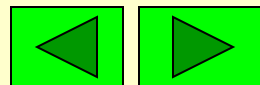


例2 求 $\begin{pmatrix} 3 & 4 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}^{2n}$.

解 原式 $= \begin{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}^{2n} & \\ & \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}^{2n} \end{pmatrix}$

$$\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}^2 = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 5^2 & \\ & 5^2 \end{pmatrix}, \cdots \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}^{2n} = \begin{pmatrix} 5^{2n} & \\ & 5^{2n} \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}^2 = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}, \cdots \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}^{2n} = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}^n$$

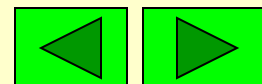


$$n=2, \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1+4 \\ 0 & 4^2 \end{pmatrix}, n=3, \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}^3 = \begin{pmatrix} 1 & 1+4+4^2 \\ 0 & 4^3 \end{pmatrix}$$

$$n=4, \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}^4 = \begin{pmatrix} 1 & 1+4+4^2+4^3 \\ 0 & 4^4 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}^n = \begin{pmatrix} 1 & 1+4+4^2+\dots+4^{n-1} \\ 0 & 4^n \end{pmatrix} = \begin{pmatrix} 1 & \frac{4^n-1}{3} \\ 0 & 4^n \end{pmatrix}$$

$$\therefore \begin{pmatrix} 3 & 4 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}^{2n} = \begin{pmatrix} 5^{2n} & & & \\ & 5^{2n} & & \\ & & 1 & \frac{4^n-1}{3} \\ & & & 4^n \end{pmatrix}.$$



例3 设 A 为 n 阶方阵,

$$f(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0, \quad a_0 \neq 0,$$

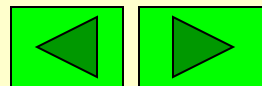
且 $f(A) = \mathbf{0}$. 试证 A 可逆, 并求出 A^{-1} 的表达式.

证 由 $f(A) = \mathbf{0}$, 得

$$f(A) = A^m + a_{m-1}A^{m-1} + \cdots + a_1A + a_0E = \mathbf{0}, \quad a_0 \neq 0$$

$$A \left[-\frac{1}{a_0} (A^{m-1} + a_{m-1}A^{m-2} + \cdots + a_1E) \right] = E, \text{ 知 } A \text{ 可逆.}$$

$$\therefore A^{-1} = -\frac{1}{a_0} (A^{m-1} + a_{m-1}A^{m-2} + \cdots + a_1E)$$



例4 $Q = \begin{pmatrix} A & B \\ B' & b \end{pmatrix}$, 其中 A 是 n 阶非奇异矩阵,
 B 是 $n \times 1$ 矩阵, b 是常数, 试证 Q 可逆的
 $\iff B'A^{-1}B \neq b$.

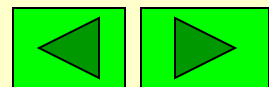
证 $Q = \begin{pmatrix} A & B \\ B' & b \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ \mathbf{0} & b - B'A^{-1}B \end{pmatrix}$

$$|Q| = \begin{vmatrix} A & B \\ B' & b \end{vmatrix} = \begin{vmatrix} A & B \\ \mathbf{0} & b - B'A^{-1}B \end{vmatrix} = |A| |b - B'A^{-1}B|$$

A 非奇异矩阵, $|A| \neq 0$,

$$\therefore |Q| \neq 0 \iff |b - B'A^{-1}B| \neq 0 \iff B'A^{-1}B \neq b$$

故 Q 可逆的 $\iff B'A^{-1}B \neq b$.

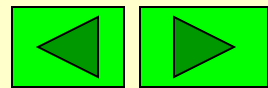


例5 设 A 为 $m \times n$ 矩阵, 若对任意 $n \times 1$ 矩阵 B 都有 $AB = 0$, 试证 $A = 0$.

证1 设 $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, $\because \forall B_{n \times 1}, AB = 0$

$$\text{取 } B_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, AB_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} = 0_{m \times 1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\text{有 } a_{11} = a_{21} = \cdots = a_{m1} = 0$$



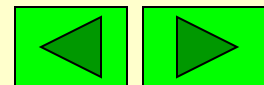
$$\text{取 } B_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, AB_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix} = \mathbf{0}_{m \times 1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\text{有 } a_{1j} = a_{2j} = \cdots = a_{mj} = 0, j = 1, 2, \cdots, n$$

$$\therefore A = \mathbf{0}$$

证2 反证, 若 $A \neq \mathbf{0}$, $\exists a_{ij} \neq 0$, 取 $B_0 = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$,

$$AB_0 = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix} \neq \mathbf{0}, \text{ 与题设矛盾, 所以 } A = \mathbf{0}.$$

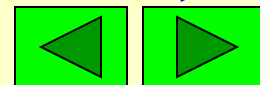


例6 设 A 为 n 阶实对称矩阵, 且 $A^2=0$, 试证
 $A=0$.

证 设 $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \because A^T = A$

$$A^2 = AA^T =$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} =$$



$$= \begin{pmatrix} \sum_{j=1}^n a_{1j}^2 & * & * & * \\ * & \sum_{j=1}^n a_{2j}^2 & * & * \\ * & * & \ddots & * \\ * & * & * & \sum_{j=1}^n a_{nj}^2 \end{pmatrix} = \mathbf{0}$$

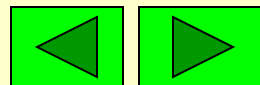
期中 a_{ij} 为实数

有 $\sum_{j=1}^n a_{1j}^2 = 0, \therefore a_{1j}^2 = 0, a_{1j} = 0, j = 1, 2, \dots, n$

$$\sum_{j=1}^n a_{ij}^2 = 0, \therefore a_{ij}^2 = 0, a_{ij} = 0, j = 1, 2, \dots, n$$

得 $a_{ij} = 0, i = 1, 2, \dots, n, j = 1, 2, \dots, n.$

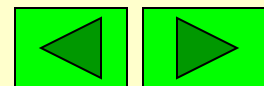
$$\therefore A = \mathbf{0}$$



例7 设 A, B 都是 $m \times n$ 矩阵, A 经过初等行变换可以化成 B , 若记 α_j 为 A 的第 j 列, β_j 为 B 的第 j 列, 即 $A = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $B = (\beta_1, \beta_2, \dots, \beta_n)$, 则当 $\beta_i = \sum_{\substack{j=1 \\ j \neq i}}^n k_j \beta_j$ 时, 有 $\alpha_i = \sum_{\substack{j=1 \\ j \neq i}}^n k_j \alpha_j$.

证 因为 A 经过初等行变换可以化成 B , 所以 \exists 可逆阵 P 使 $PA=B$, 即

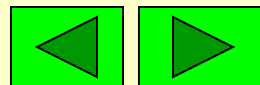
$$\begin{aligned} PA &= P(\alpha_1, \alpha_2, \dots, \alpha_n) = (P\alpha_1, P\alpha_2, \dots, P\alpha_n) \\ &= (\beta_1, \beta_2, \dots, \beta_n) = B, P\alpha_j = \beta_j, j = 1, 2, \dots, n. \end{aligned}$$



$$\therefore P\alpha_i = \beta_i = \sum_{\substack{j=1 \\ j \neq i}}^n k_j \beta_j = \sum_{\substack{j=1 \\ j \neq i}}^n k_j P\alpha_j = P \sum_{\substack{j=1 \\ j \neq i}}^n k_j \alpha_j$$

两边左乘 P^{-1} ,

得
$$\alpha_i = \sum_{\substack{j=1 \\ j \neq i}}^n k_j \alpha_j.$$



例8 设 A 是 n 阶方阵, 则 $r(A) \leq 1 \iff$

\exists 两个 $n \times 1$ 的矩阵 U, V 使 $A = UV^T$.

证 $\longleftarrow \because A = UV^T$

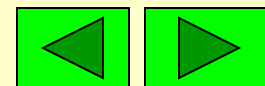
$$\therefore r(A) = r(UV^T) \leq \min(r(U), r(V)) \leq 1.$$

\longrightarrow (1) $r(A) = 0 \Rightarrow A = 0$

令 $U = V = (0, 0, \dots, 0)^T,$

$$\therefore A = UV^T.$$

(2) $r(A) = 1$, 则 $A \xrightarrow{\text{初}} \begin{pmatrix} 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix},$



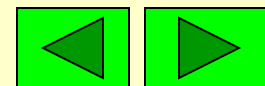
\exists 可逆阵 P, Q 使

$$PAQ = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} (1 \ 0 \ \cdots \ 0)$$

$$A = P^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} (1 \ 0 \ \cdots \ 0) Q^{-1}$$

令 $U = P^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad V^T = (1 \ 0 \ \cdots \ 0) Q^{-1}$

$$\therefore A = UV^T.$$



例9 设 A 为 $m \times n$ 矩阵, B 为 $n \times P$ 阶矩阵,
 $r(A) = n$, 则 $r(AB) = r(B)$.

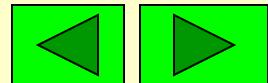
证1 $\because r(B) = r(A) + r(B) - n \leq r(AB) \leq r(B)$
 $\therefore r(AB) = r(B)$

证2 A 为 $m \times n$ 矩阵, 且 $r(A) = n$, 则

\exists 可逆阵 P, Q 使 $A = P \begin{pmatrix} E_n \\ 0 \end{pmatrix} Q$,

$$AB = P \begin{pmatrix} E_n \\ 0 \end{pmatrix} QB = P \begin{pmatrix} QB \\ 0 \end{pmatrix}$$

$$\therefore r(AB) = r \left(P \begin{pmatrix} QB \\ 0 \end{pmatrix} \right) = r \begin{pmatrix} QB \\ 0 \end{pmatrix} = r(QB) = r(B)$$



例10 设 A 为 n 阶方阵, n 是奇数, 且

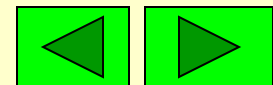
$$AA^T = E_n, |A| = 1. \text{ 证明 } |E_n - A| = 0.$$

证 $\because |E_n - A| = |AA^T - A| = |A||A^T - E|$
 $= |A^T - E| = |A - E| = (-1)^n |E - A|$
 $= -|E - A|$

$$\therefore |A - E| = 0.$$

预习第三章3.1-3.2

(^_^) Bye!



例3 若 A 为 $m \times n$ 矩阵 $r(A) = m < n$, B 是 n 阶矩阵, 以下哪些结论成立?

- (A) A 的任意一个 m 阶子式 $\neq 0$;
- (B) A 的任意 m 列线性无关;
- (C) $|A^T A| \neq 0$;
- (D) A 的 m 行线性无关;
- (E) 若 $AB = 0$, 则 $B = 0$;
- (F) 若 $r(B) = n$, 则 $r(AB) = m$.

[(A),(B),(C), (E), 不正确; (D) (F)正确.]

