**例1** $A=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&-1&1\\0&1&-1\\0&0&1\end{pmatrix}$, 求$A^n$.

**解** $$A^2=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\left[\begin{pmatrix}1&-1&1\\0&1&-1\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\right]\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&-1&1\\0&1&-1\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&-1&1\\0&1&-1\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&-1&1\\0&1&-1\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&-1&1\\0&1&-1\\0&0&1\end{pmatrix}=\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix},\quad A^3=A$$

当$n$为偶数时, $A^n=E$.当$n$为奇数时, $A^n=A$.

**例2** 求 $\begin{pmatrix} 3 & 4 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}^{2n}$.

**解** 原式 $= \begin{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}^{2n} & \\ & \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}^{2n} \end{pmatrix}$

$$\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}^2 = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 5^2 & \\ & 5^2 \end{pmatrix}, \therefore \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}^{2n} = \begin{pmatrix} 5^{2n} & \\ & 5^{2n} \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}^2 = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}, \therefore \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}^{2n} = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}^{n}$$

$$n=2,\begin{pmatrix}1&1\\0&4\end{pmatrix}^2=\begin{pmatrix}1&1+4\\0&4^2\end{pmatrix},\ n=3,\begin{pmatrix}1&1\\0&4\end{pmatrix}^3=\begin{pmatrix}1&1+4+4^2\\0&4^3\end{pmatrix}$$

$$n=4,\begin{pmatrix}1&1\\0&4\end{pmatrix}^4=\begin{pmatrix}1&1+4+4^2+4^3\\0&4^4\end{pmatrix}$$

$$\therefore\begin{pmatrix}1&1\\0&4\end{pmatrix}^n=\begin{pmatrix}1&1+4+4^2+\cdots+4^{n-1}\\0&4^n\end{pmatrix}=\begin{pmatrix}1&\dfrac{4^n-1}{3}\\0&4^n\end{pmatrix}$$

$$\therefore\begin{pmatrix}3&4&0&0\\4&-3&0&0\\0&0&-1&1\\0&0&0&2\end{pmatrix}^{2n}=\begin{pmatrix}5^{2n}&&&\\&5^{2n}&&\\&&1&\dfrac{4^n-1}{3}\\&&&4^n\end{pmatrix}.$$

**例3** 设$A$为$n$阶方阵,

$$f(x)=x^m+a_{m-1}x^{m-1}+\cdots+a_1x+a_0,\ a_0\neq 0,$$

且 $f(A)=0$. 试证$A$可逆, 并求出 $A^{-1}$ 的表达式.

**证** 由 $f(A)=0$, 得

$$f(A)=A^m+a_{m-1}A^{m-1}+\cdots+a_1A+a_0E=0,\ a_0\neq 0$$

$$A\left[-\frac{1}{a_0}\left(A^{m-1}+a_{m-1}A^{m-2}+\cdots+a_1E\right)\right]=E,$$ 知$A$可逆.

$$\therefore A^{-1}=-\frac{1}{a_0}\left(A^{m-1}+a_{m-1}A^{m-2}+\cdots+a_1E\right)$$

**例4** $Q=\begin{pmatrix} A & B \\ B' & b \end{pmatrix}$, 其中$A$是$n$阶非奇异矩阵，$B$是$n\times 1$矩阵, $b$是常数,试证$Q$可逆的 $\Longleftrightarrow B'A^{-1}B\neq b.$

**证** $Q=\begin{pmatrix} A & B \\ B' & b \end{pmatrix}\to\begin{pmatrix} A & B \\ 0 & b-B'A^{-1}B \end{pmatrix}$

$$|Q|=\begin{vmatrix} A & B \\ B' & b \end{vmatrix}=\begin{vmatrix} A & B \\ 0 & b-B'A^{-1}B \end{vmatrix}=|A||b-B'A^{-1}B|$$

$A$非奇异矩阵, $|A|\neq 0,$

$$\therefore |Q|\neq 0 \Leftrightarrow |b-B'A^{-1}B|\neq 0 \Leftrightarrow B'A^{-1}B\neq b$$

故$Q$可逆的 $\Longleftrightarrow$ $B'A^{-1}B\neq b.$

**例5** 设$A$为$m\times n$矩阵，若对任意$n\times 1$矩阵$B$都有$AB=0$，试证$A=0$.

**证1** 设
$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \because \forall B_{n\times 1}, AB=0$$

取
$$B_1=\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad AB_1=\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}=0_{m\times 1}=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

有
$$a_{11}=a_{21}=\cdots=a_{m1}=0$$

取 $B_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, $AB_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix} = \mathbf{0}_{m\times 1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

有 $a_{1j} = a_{2j} = \cdots = a_{mj} = 0,\ j = 1, 2, \cdots, n$

$$\therefore A = \mathbf{0}$$

**证2** 反证, 若 $A \neq \mathbf{0}$, $\exists a_{ij} \neq 0$, 取 $B_0 = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$,

$AB_0 = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix} \neq \mathbf{0}$, 与题设矛盾,所以 $A = \mathbf{0}$.

**例6** 设$A$为$n$阶实对称矩阵，且$A^2=0$，试证 $A=0$.

**证** 设 $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$，$\because A^{\mathrm{T}}=A$

$$A^2=AA^{\mathrm{T}}=$$

$$=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}=$$

$$= \begin{pmatrix} \sum_{j=1}^{n} a_{1j}^2 & * & * & * \\ * & \sum_{j=1}^{n} a_{2j}^2 & * & * \\ * & * & \ddots & * \\ * & * & * & \sum_{j=1}^{n} a_{nj}^2 \end{pmatrix} = \mathbf{0}$$

期中 $a_{ij}$ 为实数

有 $\sum_{j=1}^{n} a_{1j}^2 = 0, \therefore a_{1j}^2 = 0, a_{1j} = 0, j = 1, 2, \cdots, n$

$\sum_{j=1}^{n} a_{ij}^2 = 0, \therefore a_{ij}^2 = 0, a_{ij} = 0, j = 1, 2, \cdots, n$

得 $a_{ij} = 0, i = 1, 2, \cdots, n. j = 1, 2, \cdots, n.$

$$\therefore A = \mathbf{0}$$

**例7** 设$A,B$ 都是$m\times n$矩阵, $A$经过初等**行**变换可以化成$B$, 若记$\alpha_j$为$A$的第$j$ 列,$\beta_j$ 为$B$的第$j$ 列, 即 $A=(\alpha_1,\alpha_2,\cdots,\alpha_n)$, $B=(\beta_1,\beta_2,\cdots,\beta_n)$, 则当$\beta_i=\sum\limits_{\substack{j=1\\ j\neq i}}^{n}k_j\beta_j$时, 有 $\alpha_i=\sum\limits_{\substack{j=1\\ j\neq i}}^{n}k_j\alpha_j$.

**证** 因为$A$经过初等行变换可以化成$B$, 所以$\exists$可逆阵$P$ 使$PA=B$, 即

$$PA=P(\alpha_1,\alpha_2,\cdots,\alpha_n)=(P\alpha_1,P\alpha_2,\cdots,P\alpha_n)$$

$$=(\beta_1,\beta_2,\cdots,\beta_n)=B,\ P\alpha_j=\beta_j,\ j=1,2,\cdots,n.$$

$$\therefore \boldsymbol{P\alpha}_i = \boldsymbol{\beta}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} k_j \boldsymbol{\beta}_j = \sum_{\substack{j=1 \\ j\neq i}}^{n} k_j \boldsymbol{P\alpha}_j = \boldsymbol{P}\sum_{\substack{j=1 \\ j\neq i}}^{n} k_j \boldsymbol{\alpha}_j$$

两边左乘 $\boldsymbol{P}^{-1}$,

得
$$\boldsymbol{\alpha}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} k_j \boldsymbol{\alpha}_j.$$

**例8** 设$A$是$n$阶方阵，则$r(A)\le 1 \Longleftrightarrow$ $\exists$两个$n\times 1$的矩阵$U, V$使 $A=UV^{\mathrm{T}}$.

**证** $\Longleftarrow$ $\because A=UV^{\mathrm{T}}$

$$\therefore r(A)=r(UV^{\mathrm{T}})\le \min(r(U),r(V))\le 1.$$

$\Longrightarrow$ $(1) r(A)=0 \Rightarrow A=\mathbf{0}$

令 $U=V=(0,0,\cdots,0)^{\mathrm{T}}$,

$$\therefore A=UV^{\mathrm{T}}.$$

$(2) r(A)=1$, 则 $A \xrightarrow{初} \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$,

$\exists$可逆阵$P, Q$ 使

$$PAQ=\begin{pmatrix}1 & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\end{pmatrix}=\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}\begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}$$

$$A=P^{-1}\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}\begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}Q^{-1}$$

令 $$U=P^{-1}\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix},\ V^{T}=\begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}Q^{-1}$$

$$\therefore A=UV^{T}.$$

**例9** 设$A$为$m\times n$矩阵，$B$为$n\times P$阶矩阵，

$r(A)=n$，则$r(AB)=r(B)$.

**证1** $\because r(B)=r(A)+r(B)-n\le r(AB)\le r(B)$

$$\therefore r(AB)=r(B)$$

**证2** $A$为$m\times n$矩阵，且$r(A)=n$，则

$\exists$可逆阵$P,Q$使$A=P\begin{pmatrix}E_n\\0\end{pmatrix}Q,$

$$AB=P\begin{pmatrix}E_n\\0\end{pmatrix}QB=P\begin{pmatrix}QB\\0\end{pmatrix}$$

$$\therefore r(AB)=r\left(P\begin{pmatrix}QB\\0\end{pmatrix}\right)=r\begin{pmatrix}QB\\0\end{pmatrix}=r(QB)=r(B)$$

**例10** 设$A$为$n$阶方阵, $n$是奇数, 且

$$AA^{\mathrm{T}}=E_n, |A|=1.$$ 证明 $$|E_n-A|=0.$$

**证** $$\because |E_n-A|=|AA^{\mathrm{T}}-A|=|A||A^{\mathrm{T}}-E|$$

$$=|A^{\mathrm{T}}-E|=|A-E|=(-1)^n|E-A|$$

$$=-|E-A|$$

$$\therefore |A-E|=0.$$

# 预习第三章3.1-3.2

**(^-^)  Bye!**

**例3** 若$A$为$m\times n$矩阵$r(A)=m<n$, $B$是$n$阶矩阵，以下哪些结论成立？

(A) $A$的任意一个$m$阶子式$\neq 0$;

(B) $A$的任意$m$列线性无关;

(C) $|A^{\mathrm{T}}A|\neq 0$;

(D) $A$的$m$行线性无关;

(E) 若$AB=0$，则$B=0$;

(F) 若$r(B)=n$，则$r(AB)=m$.

[(A),(B),(C), (E),不正确；(D) (F)正确.]