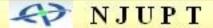
# 极限与连续习题课

- 一. 内容与要求
  - 1. 了解极限的两个存在准则并会应用
  - 2. 会用两个重要极限求极限
- 3. 掌握无穷小的比较与无穷小阶的估计,会利用等价无穷小替换求极限
- 4. 理解函数在一点连续、间断的概念,会判断间断点的类型
- 5. 了解初等函数的连续性,掌握闭区间上连续函数的性质



#### 知识要点:

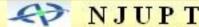
- 1. 两个准则:夹逼准则、单调有界必有极限准则
- 2. 两个重要极限(一般形式)

$$1^{0} \lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1; \qquad 2^{0} \lim_{\alpha \to 0} (1 + \alpha)^{\frac{1}{\alpha}} = e.$$

3.若:  $u(x) \rightarrow 1, v(x) \rightarrow \infty, (1^{\infty})$ 

$$:: \lim u(x)^{v(x)} = \lim \left[ (1 + u(x) - 1)^{\frac{1}{u(x) - 1}} \right]^{[u(x) - 1]v(x)}$$

$$\lim u(x)^{v(x)} = e^{\lim v(x)[u(x)-1]}$$



## 4. 常用的等价无穷小关系

$$x \to 0$$
 时,
 $\sin x \sim x$  ,  $\tan x \sim x$  ,  $1 - \cos x \sim \frac{1}{2}x^2$  ,
 $\arcsin x \sim x$  ,  $\arctan x \sim x$  ,
 $(1+x)^{\mu}-1 \sim \mu x$   $\sqrt[n]{1+x}-1 \sim \frac{1}{n}x$   $(n \in N^+)$   $\ln(1+x) \sim x$  ,  $\log_a(1+x) \sim \frac{x}{\ln a}$   $(a > 0, \neq 1)$   $e^x - 1 \sim x$  ,  $a^x - 1 \sim x \ln a$   $(a > 0, \neq 1)$ 

注: 以上各式中的x都可换成任意无穷小u(x).

### 5. 间段点的判别方法:

间断点存在:(1)函数无定义点(分母为零的点)

(2)分段函数的分段点可能是间断点

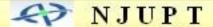
### 间断点的类型:

### 6. 求极限,常用方法如下:

- (1) 利用极限的运算性质
- (2) 利用函数的连续性
- (3) 利用极限存在两个准则
- (4) 利用两个重要极限
- (5) 利用等价无穷小代换
- (6) 利用左右极限
- (7) 利用变量代换

#### 7.判断极限不存在的方法:

(1). 子列(数列). (2).左、右极限.(3). 函数列.



#### 二.典型例题

#### 1. 求极限

(1). 
$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^6+n}} + \frac{2^2}{\sqrt{n^6+2n}} + \dots + \frac{n^2}{\sqrt{n^6+n^2}}\right)$$

(2). 
$$\lim_{n\to\infty} \sqrt[n]{1+x^n+(\frac{x^2}{2})^n} \ (x\geq 0)$$

$$(1) \Re \lim_{n \to \infty} \left( \frac{1}{\sqrt{n^6 + n}} + \frac{2^2}{\sqrt{n^6 + 2n}} + \dots + \frac{n^2}{\sqrt{n^6 + n^2}} \right)$$

$$\Re \because \frac{k^2}{\sqrt{n^6 + n^2}} \le \frac{k^2}{\sqrt{n^6 + kn}} \le \frac{k^2}{\sqrt{n^6 + n}}$$

$$\therefore \sum_{k=1}^n \frac{k^2}{\sqrt{n^6 + n^2}} \le \sum_{k=1}^n \frac{k^2}{\sqrt{n^6 + kn}} \le \sum_{k=1}^n \frac{k^2}{\sqrt{n^6 + n}}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{\sqrt{n^6 + n^2}} = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6\sqrt{n^6 + n^2}} = \frac{1}{3}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{\sqrt{n^6 + n}} = \frac{1}{3}$$
 由两边夹准则,原式 =  $\frac{1}{3}$ 

# 两个常用的极限

(1) 若 
$$|q| < 1$$
, 那么  $\lim_{n \to \infty} q^n = 0$ ;

(2) 若
$$a > 0$$
且 $a \neq 1$ ,那么 $\lim_{x \to 0} a^x = 1 = \lim_{n \to \infty} \sqrt[n]{a}$ 

例 设
$$x_n = (1 + 2^n + 3^n)^n$$
, 求 $\lim_{n \to \infty} x_n$ 

$$\mathfrak{M} : 3^n < 1 + 2^n + 3^n < 3 \cdot 3^n : 3 < x_n < 3^n \cdot 3,$$

$$\lim_{n\to\infty}3^{\frac{1}{n}}\cdot 3=3, \qquad \text{由两边夹准则, 原式=3}$$

思考:设a,b,c,d均为正数,则

$$\lim_{n \to \infty} \sqrt[n]{a^n + b^n + c^n + d^n} = \max\{a, b, c, d\}$$

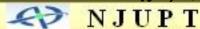
(2). 
$$\Re \lim_{n\to\infty} \sqrt[n]{1+x^n+(\frac{x^2}{2})^n} (x\geq 0)$$

$$\lim_{n\to\infty} \sqrt[n]{1+x^n+(\frac{x^2}{2})^n} = \max\{1, x, \frac{x^2}{2}\}$$

$$= \begin{cases} 1, & 0 \le x \le 1 \\ x, & 1 < x \le 2 \\ \frac{x^2}{2}, & x > 2 \end{cases}$$

注: 设
$$a_i > 0, (i = 1, 2, \dots m.),$$

求 
$$\lim_{n \to \infty} \sqrt[n]{a_1}^n + a_2^n + \dots + a_m^n = Max(a_1, a_2 + \dots + a_m)$$



试证 $\{a_n\}$ 收敛,并求 $\lim a_n$ .

证明 
$$a_{n+1} = \frac{1}{2}(a_n + \frac{a}{a_n}) \ge \sqrt{a} \implies \{a_n\}$$
有下界 
$$\frac{a_{n+1}}{a_n} = \frac{1}{2}(1 + \frac{a}{a_n^2}) \le 1 \qquad (a_n^2 \ge a)$$

 $\Rightarrow \{a_n\}$ 单调减少

$$\therefore a_n > 0, \qquad \therefore \lim_{n \to \infty} a_n = \sqrt{a}$$

### 3. 求极限

(1). 
$$\lim_{x\to a} \left(\frac{\sin x}{\sin a}\right)^{\frac{1}{x-a}} (a\neq n\pi)$$

(2). 
$$\lim_{x\to 0} \left(\frac{a^{x+1}+b^{x+1}+c^{x+1}+c^{x+1}}{a+b+c}\right)^{\frac{1}{x}}$$
  $(a>0,b>0,c>0)$ 

(3).  $\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}$ 

(1). 
$$\lim_{x \to a} \left(\frac{\sin x}{\sin a}\right)^{\frac{1}{x-a}} (a \neq n\pi) \quad (1^{\infty})$$
解: 原式 
$$= e^{\lim_{x \to a} \frac{1}{x-a} \cdot \left(\frac{\sin x}{\sin a} - 1\right)}$$

$$= e^{\lim_{x \to a} \frac{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a} \cdot \frac{1}{\sin a}} = e^{\cot a}$$

注: 
$$\lim u(x)^{v(x)} = \lim \left[ (1 + u(x) - 1)^{\frac{1}{u(x) - 1}} \right]^{[u(x) - 1]v(x)}$$

$$= e^{\lim v(x)[u(x) - 1]}$$

(2). 
$$\lim_{x\to 0} \left(\frac{a^{x+1}+b^{x+1}+c^{x+1}}{a+b+c}\right)^{\frac{1}{x}}$$
 (1<sup>\infty</sup>)

解: 先求

$$\lim_{x\to 0} v(u-1) = \lim_{x\to 0} \frac{1}{x} \left( \frac{a^{x+1} + b^{x+1} + c^{x+1}}{a+b+c} - 1 \right)$$

$$= \lim_{x \to 0} \frac{1}{x} \left[ \frac{a^{x+1} - a + b^{x+1} - b + c^{x+1} - c}{a + b + c} \right]$$

$$= \frac{1}{a+b+c} \lim_{x\to 0} \left[ \frac{a(a^x-1)}{x} + \frac{b(b^x-1)}{x} + \frac{c(c^x-1)}{x} \right]$$

$$= \frac{1}{a+b+c}(a\ln a + b\ln b + c\ln c)$$

$$=\frac{\ln(a^ab^bc^c)}{a+b+c}$$

原式 = 
$$e^{\frac{\ln(a^ab^bc^c)}{a+b+c}}$$
 =  $(a^ab^bc^c)^{\frac{1}{a+b+c}}$ 

(3). 
$$\lim_{\pi} (\sin x)^{\tan x}$$
 (1<sup>\infty</sup>)

$$x \rightarrow \frac{x}{2}$$
 lim tan x

$$e^{\lim_{t\to 0}\frac{\cos t}{-\sin t}(\cos t-1)}$$

$$= e^{\lim_{t\to 0} \frac{\cos t}{-t}(-\frac{t^2}{2})} = e^0 = 1$$

4. 求下列极限

(1) 
$$\lim_{x \to 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x\sqrt{1 + \sin^2 x} - x}$$

(2) 
$$\lim_{x\to 0} \frac{\ln(e^x + \sin^2 x) - x}{\ln(e^{2x} + x^2) - 2x};$$

$$(3) \lim_{x \to \pi} \frac{\sin(mx)}{\sin(nx)} (m.n \in N^+)$$

$$(4)\lim_{x\to 0}\frac{\ln\cos\beta x}{\ln\cos\alpha x} \quad (\alpha,\beta\neq 0)$$

(5) 
$$\lim_{x\to 0} \left( \frac{2 + e^{\frac{1}{x}}}{1 + e^{\frac{4}{x}}} + \frac{\sin x}{|x|} \right)$$
.

(1) 
$$\lim_{x \to 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x\sqrt{1 + \sin^2 x} - x}$$

解.

(1) 
$$\lim_{x \to 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x\sqrt{1 + \sin^2 x} - x}$$

$$= \lim_{x \to 0} \frac{\tan x - \sin x}{x(\sqrt{1 + \sin^2 x} - 1)(\sqrt{1 + \tan x} + \sqrt{1 + \sin x})}$$

$$= \lim_{x \to 0} \frac{\tan x (1 - \cos x)}{x \cdot \frac{1}{2} \cdot \sin^2 x \cdot 2} = \lim_{x \to 0} \frac{x \cdot \frac{1}{2} x^2}{x \cdot x^2} = \frac{1}{2}$$

(2) 
$$\lim_{x\to 0} \frac{\ln(\sin^2 x + e^x) - x}{\ln(x^2 + e^{2x}) - 2x}$$

(2) 原式 = 
$$\lim_{x\to 0} \frac{\ln(\sin^2 x + e^x) - \ln e^x}{\ln(x^2 + e^{2x}) - \ln e^{2x}}$$

$$= \lim_{x \to 0} \frac{\ln(1 + e^{-x} \sin^2 x)}{\ln(1 + e^{-2x} x^2)} = \lim_{x \to 0} \frac{e^{-x} \sin^2 x}{e^{-2x} x^2} = 1$$

解法二: 
$$\ln(e^x + \sin^2 x) = \ln[e^x(1 + \frac{\sin^2 x}{e^x})]$$

$$= x + \ln(1 + \frac{\sin^2 x}{e^x}) = x + \ln(1 + e^{-x} \sin^2 x)$$

$$(3)\lim_{x\to\pi}\frac{\sin(mx)}{\sin(nx)}(m,n\in N^+)$$

解: 
$$\lim_{x \to \pi} \frac{\sin(mx)}{\sin(nx)} \stackrel{\text{\frac{a}}{x} - \pi = t}{= t} \lim_{t \to 0} \frac{\sin m(\pi + t)}{\sin n(\pi + t)}$$

$$= \lim_{t\to 0} \frac{(-1)^m \sin mt}{(-1)^n \sin nt}$$

$$= \lim_{t \to 0} \frac{(-1)^m mt}{(-1)^n nt} = \frac{(-1)^{m-n} m}{n}$$

$$注:\sin(x+n\pi)=(-1)^n\sin x$$

(4) 
$$\lim_{x\to 0} \frac{\ln \cos \beta x}{\ln \cos \alpha x}$$
  $(\alpha, \beta \neq 0)$   $(\frac{0}{0})$ 

$$= \lim_{x \to 0} \frac{\ln(1 + \cos \beta x - 1)}{\ln(1 + \cos \alpha x - 1)}$$

$$= \lim_{x \to 0} \frac{\cos \beta x - 1}{\cos \alpha x - 1}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{2}(\beta x)^{2}}{-\frac{1}{2}(\alpha x)^{2}} = \frac{\beta^{2}}{\alpha^{2}}$$

(5). 
$$\Re \lim_{x \to 0} \left( \frac{2 + e^{\frac{1}{x}}}{1 + e^{\frac{4}{x}}} + \frac{\sin x}{|x|} \right)$$
.

解:

$$\lim_{x \to 0^{+}} \left( \frac{2 + e^{\frac{1}{x}}}{1 + e^{\frac{4}{x}}} + \frac{\sin x}{|x|} \right) = \lim_{x \to 0^{+}} \left( \frac{2 e^{-\frac{4}{x}} + e^{-\frac{3}{x}}}{e^{-\frac{4}{x}} + 1} + \frac{\sin x}{x} \right) = 1$$

$$\lim_{x \to 0^{-}} \left( \frac{2 + e^{\frac{1}{x}}}{1 + e^{\frac{4}{x}}} + \frac{\sin x}{|x|} \right) = \lim_{x \to 0^{-}} \left( \frac{2 + e^{\frac{1}{x}}}{1 + e^{\frac{4}{x}}} - \frac{\sin x}{x} \right) = 1$$

5. 已知 
$$\lim_{x\to 0} \frac{\sqrt{1+f(x)\sin 2x-1}}{e^{3x}-1} = 2$$
, 求  $\lim_{x\to 0} f(x)$ 。

解: 由题设知 
$$\lim_{x\to 0} (\sqrt{1+f(x)\sin 2x}-1)=0$$

进而知  $\lim_{x\to 0} f(x)\sin 2x = 0$ 

于是有 
$$\lim_{x \to 0} \frac{\sqrt{1 + f(x)\sin 2x} - 1}{e^{3x} - 1} = \lim_{x \to 0} \frac{\frac{1}{2}f(x)\sin 2x}{3x}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2} f(x) \cdot 2x}{3x} = \lim_{x \to 0} \frac{f(x)}{3} = 2$$

所以 
$$\lim_{x\to 0} f(x) = 6$$

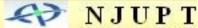
6.(1).求
$$x \to 1^+$$
时,  $f(x) = \sqrt{3x^2 - 2x - 1 \cdot \ln x}$ 是 $x - 1$ 的几阶无穷小?

 $(2)x \to 0$ 时 $f(x) = e^{x^2} - \cos x$ 是x的几阶无穷小? 1.10总习题 1(8)

解(1). : 
$$f(x) = \sqrt{3x+1} \cdot \sqrt{x-1} \cdot \ln[1+(x-1)]$$
  
:  $\lim_{x \to 1^+} \frac{f(x)}{(x-1)^{\frac{3}{2}}} = 2$ 

∴ 当 $x \to 1$ 时, f(x)是x - 1的 $\frac{3}{2}$ 阶无穷小

$$(2) f(x) = e^{x^2} - \cos x = e^{x^2} - 1 + 1 - \cos x$$



$$\lim_{x \to 0} \frac{e^{x^2} - \cos x}{x^2} = \lim_{x \to 0} \frac{e^{x^2} - 1 + 1 - \cos x}{x^2}$$

$$= \lim_{x \to 0} \frac{e^{x^2} - 1}{x^2} + \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{3}{2}$$

$$\mathbb{P}f(x) = e^{x^2} - \cos x = e^{x^2} - 1 + 1 - \cos x$$

$$\sim x^2 + \frac{1}{2}x^2 = \frac{3}{2}x^2$$

$$\therefore x \to 0$$
时 $f(x) = e^{x^2} - \cos x = 2$ 的2阶无穷小。

注: 若
$$\alpha \sim \alpha', \beta \sim \beta', 且 \lim \frac{\alpha}{\beta} \neq -1, 则$$
  $\alpha + \beta \sim \alpha' + \beta'$ 

设
$$f(x) = \frac{x}{\frac{x}{1-x}}$$
,讨论间断点及类型

1.10 总习题1(7).

解

间断点:  $x = 1, 1 - e^{\overline{x-1}} = 0 \Rightarrow x = 0$ .

$$x = 0, \lim_{x \to 0} \frac{x}{1 - e^{\frac{x}{1 - x}}} = \lim_{x \to 0} \frac{x}{-\frac{x}{1 - x}} = -1,$$

$$\therefore x = 0$$
为可去间断点。

$$x = 1, \lim_{x \to 1^{+}} \frac{x}{1 - e^{\frac{x}{1 - x}}} = 1, (\frac{x}{1 - x} \to -\infty, e^{\frac{x}{1 - x}} \to 0)$$

$$\lim_{x \to 1^{-}} \frac{x}{1 - e^{\frac{x}{1 - x}}} = 0, (\frac{x}{1 - x} \to +\infty, e^{\frac{x}{1 - x}} \to +\infty).$$

x = 1为跳跃间段点。

8.(1) 讨论
$$f(x) = \lim_{n \to \infty} \frac{x^n}{1 + x^n}$$
 ( $x \ge 0$ )的连续性.

解 当 
$$x \in [0,1)$$
 时,  $f(x) = 0$ ;  
当  $x = 1$  时,  $f(x) = \frac{1}{2}$ ; 即  $f(x) = \begin{cases} 0, & 0 \le x < 1 \\ \frac{1}{2}, & x = 1 \end{cases}$   
当  $x > 1$ 时,  $f(x) = \lim_{n \to \infty} \frac{1}{(\frac{1}{x})^n + 1} = 1$ 

由初等函数连续性知:

f(x)在[0,1)及(1,+∞) 内连续, x=1是f(x)的跳跃间断点.

(2).设
$$f(x) = \lim_{n \to \infty} \frac{x^{2n-1} + ax + b}{1 + x^{2n}}$$
为连续函数,求 $a,b$ 

$$f(x) = \begin{cases} ax + b & |x| < 1\\ \frac{1}{x} + \frac{a}{x^{2n-1}} + \frac{b}{x^{2n}} \\ \frac{1}{x^{2n}} + 1 \\ \frac{1}{2}(1+a+b) & x = 1\\ \frac{1}{2}(-1-a+b) & x = -1\\ \Rightarrow 1 = a+b, -a+b = -1 & x = 1\\ 1 & x = 1 \end{cases}$$

$$\Rightarrow 1 = a + b, -a + b = -1$$

$$\Rightarrow a = 1, b = 0$$



证明
$$\exists \xi \in (a,b)$$
 使得 $f(\xi) = \frac{1}{n} [f(x_1) + f(x_2) + \dots + f(x_n)]$ 

证: 当
$$n=1$$
时,取 $\xi=x_1$ 即可.

$$\therefore \exists M = \max_{x \in [x_1, x_n]} f(x), \quad m = \min_{x \in [x_1, x_n]} f(x)$$

$$\therefore m \leq \frac{1}{n} \Big[ f(x_1) + f(x_2) + \dots + f(x_n) \Big] \leq M$$

∴ 
$$\exists \xi \in [x_1, x_n] \subseteq (a,b)$$
 使得

$$f(\xi) = \frac{1}{n} [f(x_1) + f(x_2) + \dots + f(x_n)]$$

综上,结论成立.27

10.设 $f(x) \in C_{[0,2a]}$ ,且f(0) = f(2a),证明:日 $\xi \in [0,a]$ ,

使得 $f(\xi) = f(a + \xi)$ . 1.10 总习题 第10题

证明 
$$\diamondsuit F(x) = f(x) - f(a+x) \in C_{[0,a]}$$

$$F(0) = f(0) - f(a),$$

$$F(a) = f(a) - f(2a) = f(a) - f(0)$$

(1)若 $f(a) \neq f(0)$ ,则 $F(0) \cdot F(a) < 0$  根据零点定理:

$$\exists \xi \in [0,a]$$
, 使得 $F(\xi) = 0$ , 即 $f(\xi) = f(a+\xi)$ .

$$(2)$$
若 $f(a) = f(0)$ , 取 $\xi = 0$  即得 $f(\xi) = f(a + \xi)$ .