6.2.2 一阶线性微分方程

要求：熟练掌握一阶线性方程的解法，了解常数变易法，会解伯努利方程。

1、求下列一阶线性微分方程的通解。

(1) $y' + y = e^{-x}$

$P(x)=1 \quad Q(x)=e^{-x}$

$y = e^{-\int p(x)dx}\left[\int Q(x)e^{\int p(x)dx}dx + C\right] = e^{-x}\left[\int e^{-x}\cdot e^{x}dx + C\right]$

$= e^{-x}(x+C)$

(2) $(x-2xy-y^2)y' + y^2 = 0$

$\dfrac{dy}{dx} = -\dfrac{y^2}{x-2xy-y^2} \qquad \dfrac{dx}{dy} + \left(\dfrac{1}{y^2} - \dfrac{2}{y}\right)x = 1$

$x = e^{\int(\frac{2}{y}-\frac{1}{y^2})dy}\left[\int 1\cdot e^{\int(\frac{1}{y^2}-\frac{2}{y})dy}dy + C\right]$

$= e^{\ln y^2 + \frac{1}{y}}\left[\int e^{-(\ln y^2 + \frac{1}{y})}dy + C\right] = y^2 e^{\frac{1}{y}}\left[\int \frac{1}{y^2}e^{-\frac{1}{y}}dy + C\right]$

$= y^2 e^{\frac{1}{y}}\left[e^{-\frac{1}{y}} + C\right]$

$= y^2\left(1 + Ce^{\frac{1}{y}}\right)$

2、求下列微分方程满足初始条件的特解。

(1) $y' - \dfrac{1}{x}y = x^2, y(1) = 1$

$y' + \dfrac{x}{1-x^2}y = \dfrac{1}{1-x^2}$

$y = e^{-\int\frac{x}{1-x^2}dx}\left[\int \frac{1}{1-x^2}e^{\int\frac{x}{1-x^2}dx}dx + C\right]$

$= \sqrt{1-x^2}\left(\int \frac{1}{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}}dx + C\right]$ （作代换 $x=\sin t$）

$= \sqrt{1-x^2}\left(\frac{x}{\sqrt{1-x^2}} + C\right)$

$\because y(0)=1 \quad \therefore C=1$

$\therefore y = x + \sqrt{1-x^2}$

(2) $y' + y\cos x = \sin x\cos x, y(0) = 1$

$y = e^{-\int\cos x dx}\left[\int \sin x\cos x e^{\int\cos x dx}dx + C\right]$

$= e^{-\sin x}\left[\int \sin x e^{\sin x}d\sin x + C\right]$ （令 $\sin x = t$）

$= e^{-\sin x}\left[\sin x e^{\sin x} - e^{\sin x} + C\right]$

$\because y(0)=1 \quad \therefore C=2 \quad \therefore y = e^{-\sin x}\left(\sin x e^{\sin x} - e^{\sin x} + 2\right)$

$= \sin x - 1 + 2e^{-\sin x}$

3、求 $y' + \dfrac{y}{x} = x^2y^6$ 的通解。

令 $z = y^{-5}$，则 $\dfrac{dz}{dx} = -5y^{-6}\dfrac{dy}{dx}$

$\dfrac{dz}{dx} - 5\cdot\dfrac{z}{x} = -5x^2$

$\therefore z = e^{\int\frac{5}{x}dx}\left[\int -5x^2\cdot e^{\int -\frac{5}{x}dx}dx + C\right] = x^5\left[\int -5x^2\cdot x^{-5}dx + C\right]$

$= x^5\left[\frac{5}{2}x^{-2} + C\right] \qquad \therefore y^{-5} = \frac{5}{2}x^3 + Cx^5$

4、求一连续可导函数 $f(x)$，使其满足下列方程

$f(x) = \sin x - \int_0^x f(x-t)dt$ 。

$\int_0^x f(x-t)dt \xlongequal{令 x-t=u} \int_x^0 f(u)(-1)du = \int_0^x f(u)du$

$\therefore f(x) = \sin x - \int_0^x f(u)du$ 两边求导

$f'(x) = \cos x - f(x) \qquad f'(x) + f(x) = \cos x$

$f(x) = e^{-\int dx}\left[\int \cos x e^{\int dx}dx + C\right]$

$= e^{-x}\left[\int \cos x e^{x}dx + C\right]$ （用分部积分法）

$= e^{-x}\left[\frac{1}{2}e^{x}(\sin x + \cos x) + C\right]$

$\because f(0)=0 \quad \therefore C = -\frac{1}{2} \quad \therefore f(x) = \frac{1}{2}(\sin x + \cos x - e^{-x})$