9、设函数 $f(x)$在$[0,a](a>0)$上有连续导数,且 $f(0)=0$ 证明：

$$\left|\int_0^a f(x)dx\right| \le \frac{Ma^2}{2},\quad 其中\ M=\max_{0\le x\le a}|f'(x)|\ 。$$

对 $\forall x\in[0,a]$，由微分中值 Th.

$f(x)-f(0)=f'(\xi)\cdot x$

$\therefore \left|\int_0^a f(x)dx\right| \le \int_0^a |f(x)|dx = \int_0^a |f(x)-f(0)|dx$

$= \int_0^a |f'(\xi)x|dx = |f'(\xi)|\int_0^a x\,dx \le M\cdot \frac{1}{2}x^2\Big|_0^a = \frac{Ma^2}{2}$

10、设函数 $f(x)$在$[0,1]$上单调减少，证明对任意 $a\in(0,1)$，都有

$\int_0^a f(x)dx \ge a\int_0^1 f(x)dx$ 。(提示：令$x=at$)

$\int_0^a f(x)dx \overset{令x=at}{=\!=\!=} \int_0^1 f(at)a\,dt = a\int_0^1 f(at)dt$

则 左边 − 右边 $= a\int_0^1 f(at)dt - a\int_0^1 f(t)dt$

$= a\int_0^1 (f(at)-f(t))dt\ (\because f(x)\downarrow)$

$\ge 0$

$\therefore \int_0^a f(x)dx \ge a\int_0^1 f(x)dx$

11、设 $f(x)$在$[0,1]$上可微，且 $x\in(0,1), 0<f'(x)<1, f(0)=0$,

证明：$[\int_0^1 f(x)dx]^2 > \int_0^1 f^3(x)dx$ 。

设 $F(x) = [\int_0^x f(t)dt]^2 - \int_0^x f^3(t)dt$ . $F(0)=0$

$F'(x) = 2f(x)[\int_0^x f(t)dt] - f^3(x)$

$= f(x)[2\int_0^x f(t)dt - f^2(x)]$

设 $g(x) = 2\int_0^x f(t)dt - f^2(x)$ . $g(0)=0$

$g'(x) = 2f(x) - 2f(x)f'(x) = 2f(x)[1-f'(x)]>0$, $g(x)$递增

$\therefore g(x)>g(0)=0$

$\therefore F'(x)>0$, $F(x)$递增. $F(x)>0$, $\therefore F(1)>F(0)=0$

$\therefore [\int_0^1 f(x)dx]^2 > \int_0^1 f^3(x)dx$

12、设 $f(x)=f(x-\pi)+\sin x$，且当 $x\in[0,\pi]$时，$f(x)=x$，

求 $\int_\pi^{3\pi} f(x)dx$ 。

$\int_\pi^{3\pi} f(x)dx = \int_\pi^{3\pi}[f(x-\pi)+\sin x]dx \overset{令x-\pi=u}{=\!=\!=} \int_0^{2\pi}[f(u)+\sin(u+\pi)]du$

$= \int_0^{2\pi}[f(u)-\sin u]du = \int_0^{2\pi} f(u)du + \cos u\Big|_0^{2\pi}$

$= \int_0^\pi u\,du + \int_\pi^{2\pi} f(u)du = \frac{1}{2}\pi^2 + \int_\pi^{2\pi}[f(u-\pi)+\sin u]du$

$\overset{令u-\pi=t}{=\!=\!=} \frac{1}{2}\pi^2 + \int_0^\pi f(t)+\sin(t+\pi)dt = \frac{1}{2}\pi^2 + \int_0^\pi (t-\sin t)dt$

$= \frac{1}{2}\pi^2 + \frac{1}{2}\pi^2 + \cos t\Big|_0^\pi = \pi^2 - 2$