2.3 高阶导数及相关变化率

要求：了解高阶导数和相关变化率的概念，会求 n 阶导数。

1、填空题（其中 $f''(x)$ 存在）

(1) $(e^{-x}\sin x)'' = \underline{-2e^{-x}\cos x}$；(2) $(f(x^2))'' = \underline{2f'(x^2)+4x^2 f''(x^2)}$

(3) $(\cos^2 2x)^{(n)} = \underline{2^{2n-1}\cos(4x+\frac{n\pi}{2})}$。

2、计算题

(1) $y = 2x^2 + x|x|$，求 $\dfrac{d^2y}{dx^2}$。

$y(x)=\begin{cases}3x^2 & x\ge 0\\ x^2 & x<0\end{cases}$

$\lim\limits_{x\to 0^+}\dfrac{y(x)-y(0)}{x-0}=\lim\limits_{x\to 0^+}\dfrac{2x^2+x^2}{x}=0$

$\lim\limits_{x\to 0^-}\dfrac{y(x)-y(0)}{x-0}=\lim\limits_{x\to 0^-}\dfrac{2x^2-x^2}{x}=0$

$\therefore y'(0)=0 \quad \therefore y'(x)=\begin{cases}6x & x\ge 0\\ 2x & x<0\end{cases}$

$\lim\limits_{x\to 0^+}\dfrac{6x-0}{x}=6 \qquad \lim\limits_{x\to 0^-}\dfrac{2x-0}{x}=2$

$\therefore y'(x)$在0点不可导

$\therefore y''(x)=\begin{cases}6 & x>0\\ 2 & x<0\end{cases}$

(2) $\sin y + xe^y = 0$，求 $\left.\dfrac{d^2y}{dx^2}\right|_{\substack{x=0\\y=0}}$。

$y'\cos y + xe^y\cdot y' + e^y = 0 \qquad \therefore y' = \dfrac{-e^y}{\cos y + xe^y} = \dfrac{e^y}{\sin y-\cos y}$

$y'' = \dfrac{e^y\cdot y'(\sin y-\cos y)-e^y(\cos y+\sin y)\cdot y'}{(\sin y-\cos y)^2} = \dfrac{e^{2y}(\sin y-\cos y)-e^{2y}(\cos y+\sin y)}{(\sin y-\cos y)^3}$

$= \dfrac{-2e^{2y}\cos y}{(\sin y-\cos y)^3} = \dfrac{2e^{2y}\cos y}{(\cos y-\sin y)^3}$

(3) $y = x-\ln y$，求 $\dfrac{d^2y}{dx^2}$。

$y' = 1-\dfrac{y'}{y}$

$\therefore y' = \dfrac{1}{1+\frac{1}{y}} = \dfrac{y}{y+1}$

$y'' = \dfrac{y'(y+1)-y\cdot y'}{(y+1)^2} = \dfrac{y'}{(y+1)^2} = \dfrac{y}{(y+1)^3}$

(4) $\begin{cases}x=a(t-\sin t)\\ y=a(1-\cos t)\end{cases}$，求 $\dfrac{d^2y}{dx^2}$。

$\dfrac{dy}{dx}=\dfrac{y_t'}{x_t'}=\dfrac{a\sin t}{a(1-\cos t)}=\dfrac{\sin t}{1-\cos t}$

$\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=\dfrac{d}{dt}\left(\dfrac{\sin t}{1-\cos t}\right)\cdot\dfrac{dt}{dx}=\dfrac{\cos t(1-\cos t)-\sin^2 t}{(1-\cos t)^2}\cdot\dfrac{1}{a(1-\cos t)}$

$=\dfrac{\cos t-1}{(1-\cos t)^2}\cdot\dfrac{1}{a(1-\cos t)}$

$=-\dfrac{1}{a(1-\cos t)^2}$

(5) $\begin{cases}x=f'(t)\\ y=tf'(t)-f(t)\end{cases}$，其中 $f(t)$ 具有二阶导数且 $f''(t)\ne 0$，求 $\dfrac{d^2y}{dx^2}$。

$\dfrac{dy}{dx}=\dfrac{y_t'}{x_t'}=\dfrac{f'(t)+tf''(t)-f'(t)}{f''(t)}=t$

$\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=\dfrac{d}{dt}(t)\cdot\dfrac{dt}{dx}=\dfrac{1}{f''(t)}$

(6) $y=\dfrac{1}{x^2-3x+2}$，求 $y^{(n)}$。

$y=\dfrac{1}{x-2}-\dfrac{1}{x-1}$

$y^{(n)}=(-1)^n\dfrac{n!}{(x-2)^{n+1}}-(-1)^n\dfrac{n!}{(x-1)^{n+1}}$

3、用莱布尼兹公式计算 $(x^2\sin 2x)^{(50)}$。

$(x^2\sin 2x)^{(50)}=2^{50}\sin(2x+25\pi)x^2+50\times 2^{49}\sin(2x+\frac{49\pi}{2})\cdot 2x$

$+\dfrac{50\times 49}{2}\times 2^{48}\sin(2x+24\pi)\cdot 2$

$=-2^{50}x^2\sin 2x+50\times 2^{50}\cdot x\cos 2x+\dfrac{25\times 49}{2}\times 2^{50}\sin 2x$

$=2^{50}(50x\cos 2x+\dfrac{1225}{2}\sin 2x-x^2\sin 2x)$