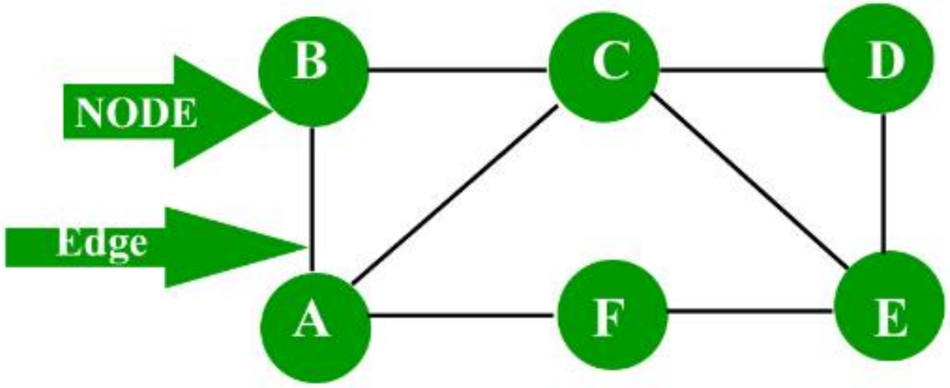
Graphs

Definition of Graph

Graph consists of two sets: a set of **vertices V** and a set of **edges E** obtained by joining certain vertices of V. It is denoted by G(V, E).

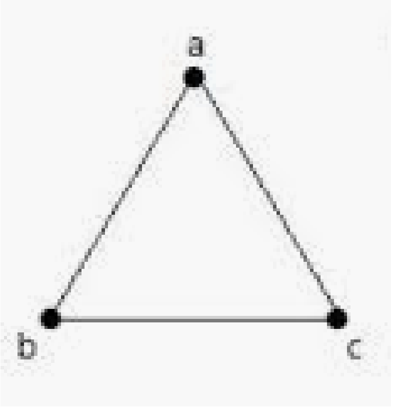


V = {A, B, C, D, E, F}

E = {(A, B), (A, C), (A, F), (B, C), (C, D), (C, E), (D, E), (F, E)}

Adjacent Vertices

Two vertices are said to be connected if they are connected by an edge.



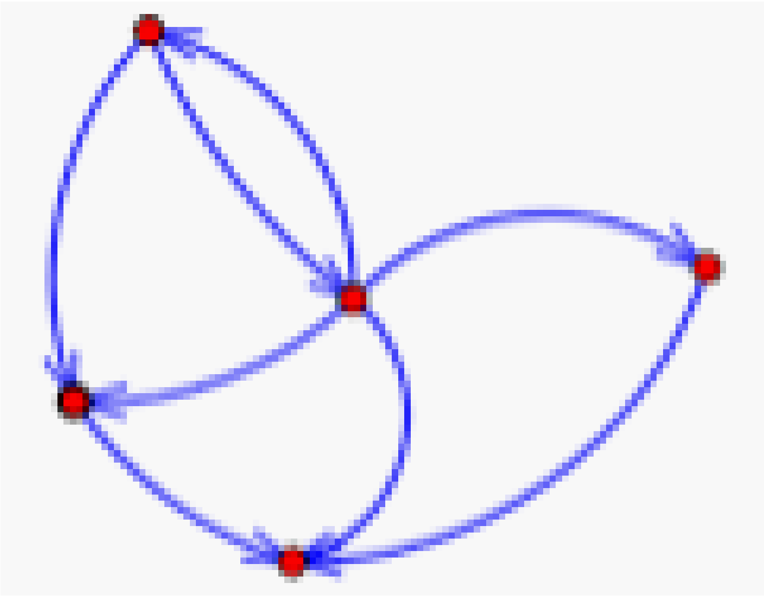
Types of Graphs

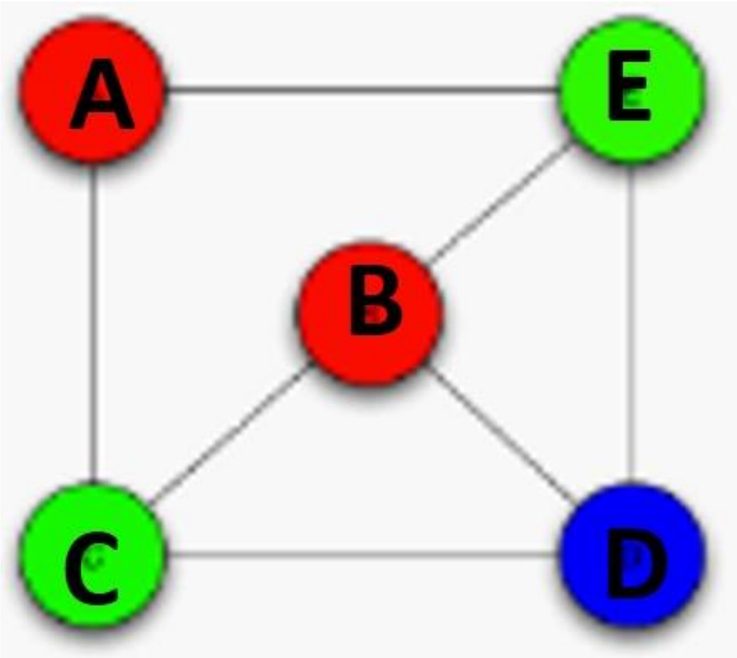
Types of graphs

Directed Graphs

A graph in which every edge is directed is called **directed graph** or **digraph**.

Every edge in a directed graph is directed.





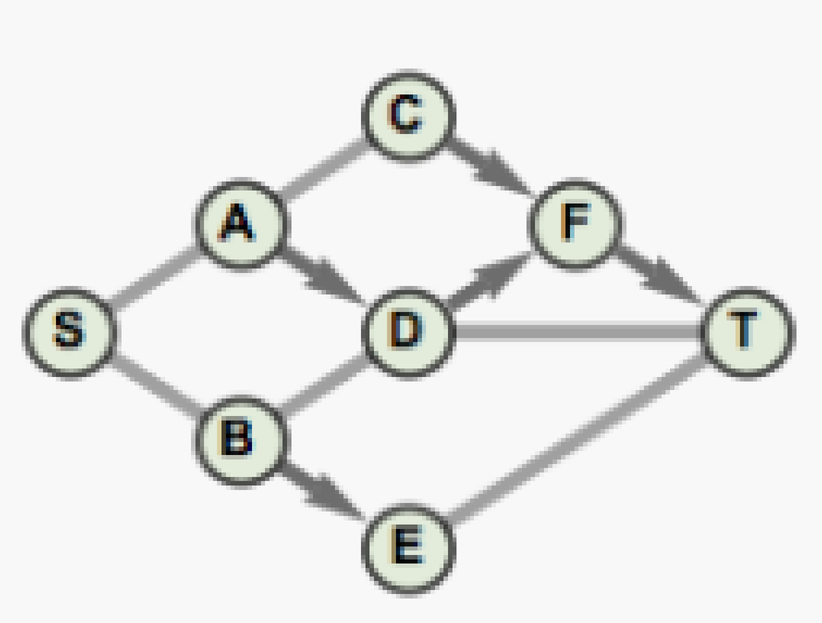
Undirected Graphs

A graph in which every edge is undirected is called **undirected graph**.

Types of graphs

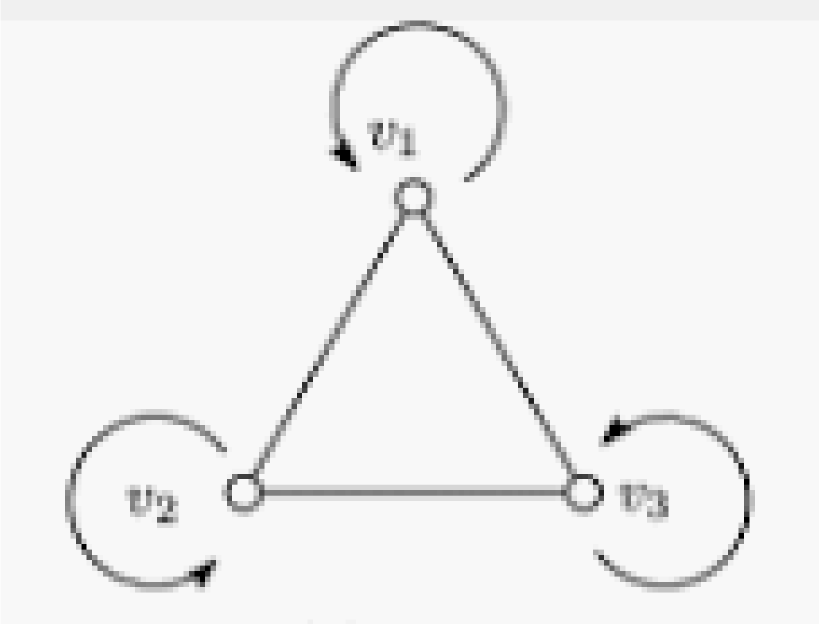
Mixed Graphs

A mixed graph in which both directed and undirected edges may exist.



Self Loops

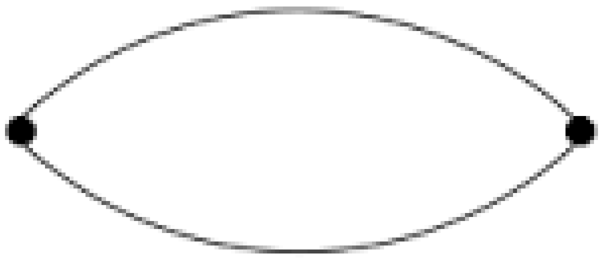
A self loop is an edge that connects a vertex to itself.



Types of graphs

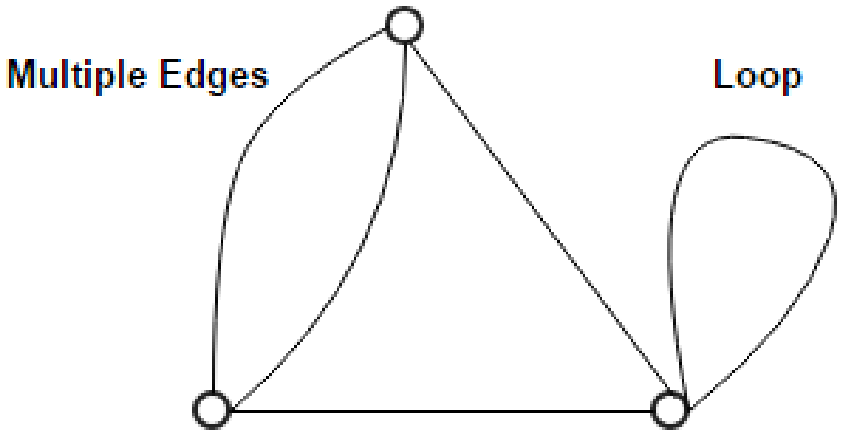
Parallel/ Multiple Edge

Multiple edges (also called **parallel edges** or a **multi-edge**), are two or more **edges** that are incident to the same two **vertices**.

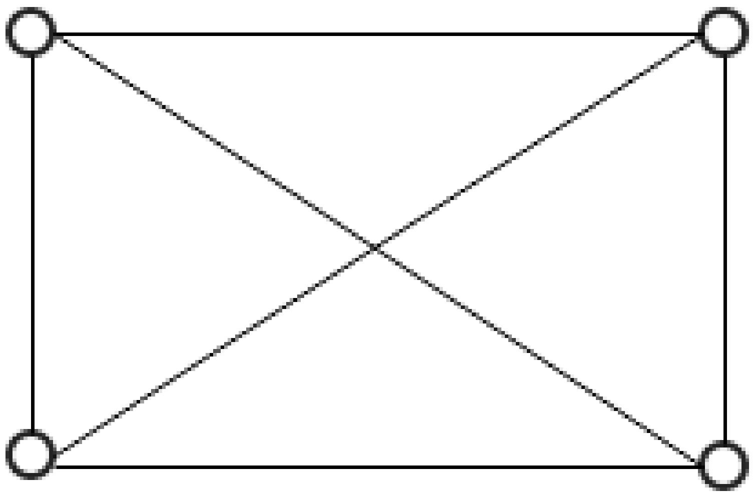


Simple graph

A graph without loops and parallel edges.



Not a Simple Graph

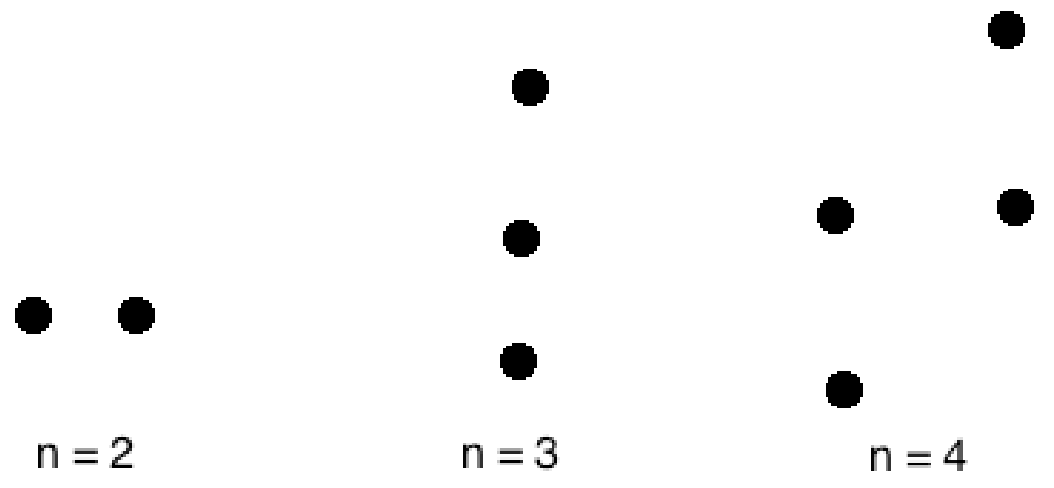


Simple Graph

Types of graphs

Null graph

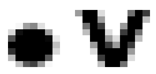
A **null graph** is a graph in which there are no edges between its vertices. A null graph is also called **empty graph**.



A null graph with n vertices is denoted by Nn.

Trivial graph

A **trivial graph** is the graph which has only one vertex.

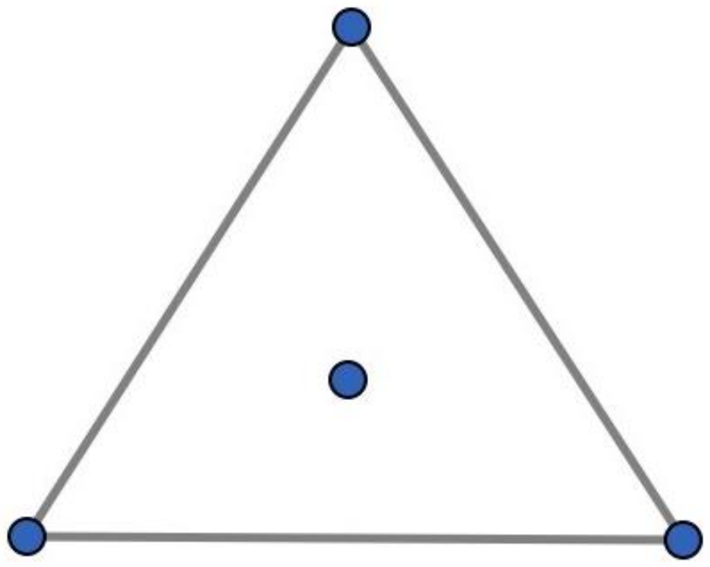


Order and Size
of
Graph

Order & size of graph

The number of vertices denoted by $|V(G)|$ is called order of G.

Order = 4



Size = 3

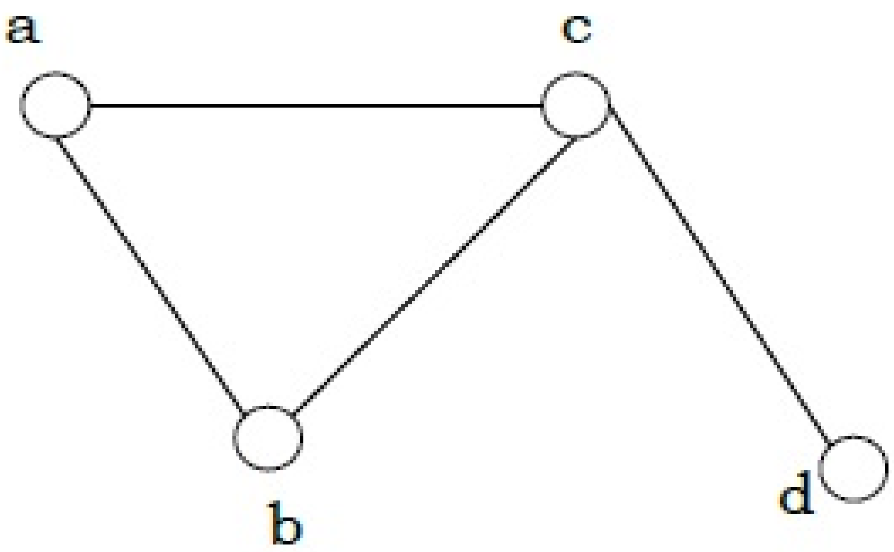
The number of edges denoted by $|E(G)|$ is called size of G.

Degree
of
Vertex

Degree of Vertex

It is the **number of edges incident** on a **vertex.**

The degree of vertex **a** is written as **deg(a)**



deg(a) = 2

deg(b) = 2

deg(c) = 3

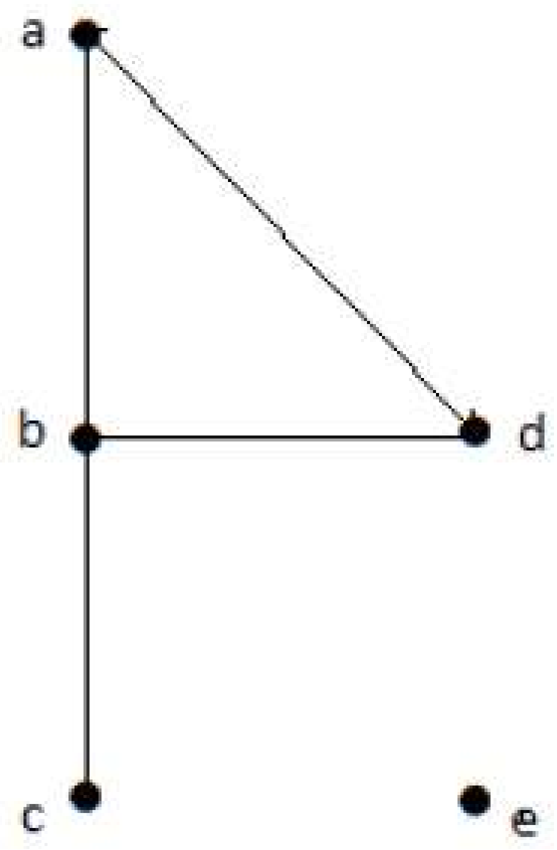
deg(d) = 1

If the degree of vertex a is 0 then it is called **isolated vertex.**

If the degree of vertex a is 1 then it is called **pendent vertex**.

Degree of Vertex in a Undirected graph

An **undirected graph** has **no directed edges**.



- deg(a) = 2, as there are 2 edges meeting at vertex 'a'.
- deg(b) = 3, as there are 3 edges meeting at vertex 'b'.
- deg(c) = 1, as there is 1 edge formed at vertex 'c'

  So 'c' is a **pendent vertex**.
- deg(d) = 2, as there are 2 edges meeting at vertex 'd'.
- deg(e) = 0, as there are 0 edges formed at vertex 'e'.

  So 'e' is an **isolated vertex**.

Degree of Vertex in a Directed graph

In a directed graph, each vertex has an **indegree** and an **outdegree**.

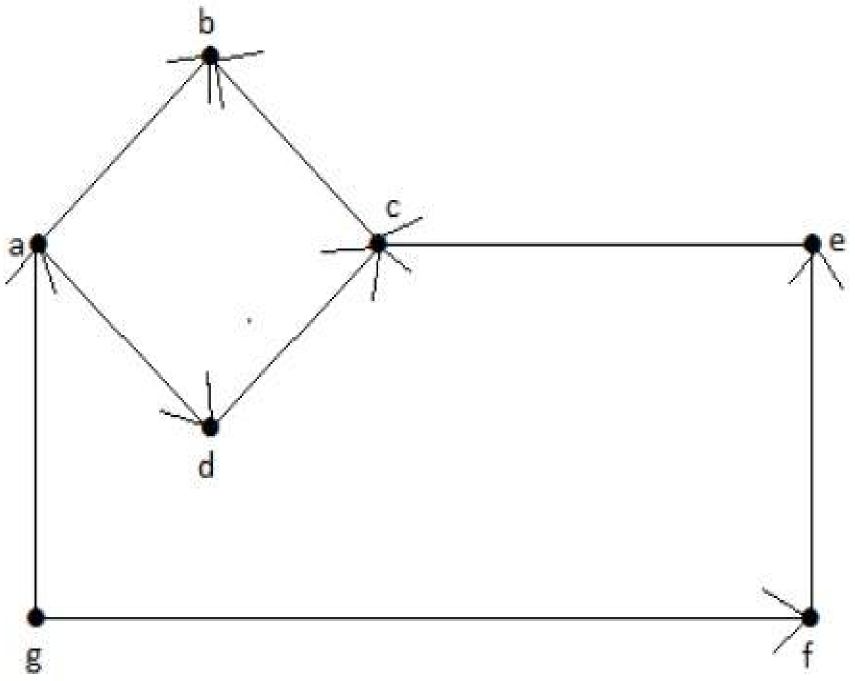
Indegree of a Graph

- Indegree of vertex V is the number of edges which are coming into the vertex V.
- **Notation** − $deg^{-}(V)$.

Outdegree of a Graph

- Outdegree of vertex V is the number of edges which are going out from the vertex V.
- **Notation** − $deg^{+}(V)$.

Degree of Vertex in a Directed graph



Vertex	Indegree	Outdegree
a	1	2
b	2	0
c	2	1
d	1	1
e	1	1
f	1	1
g	0	2

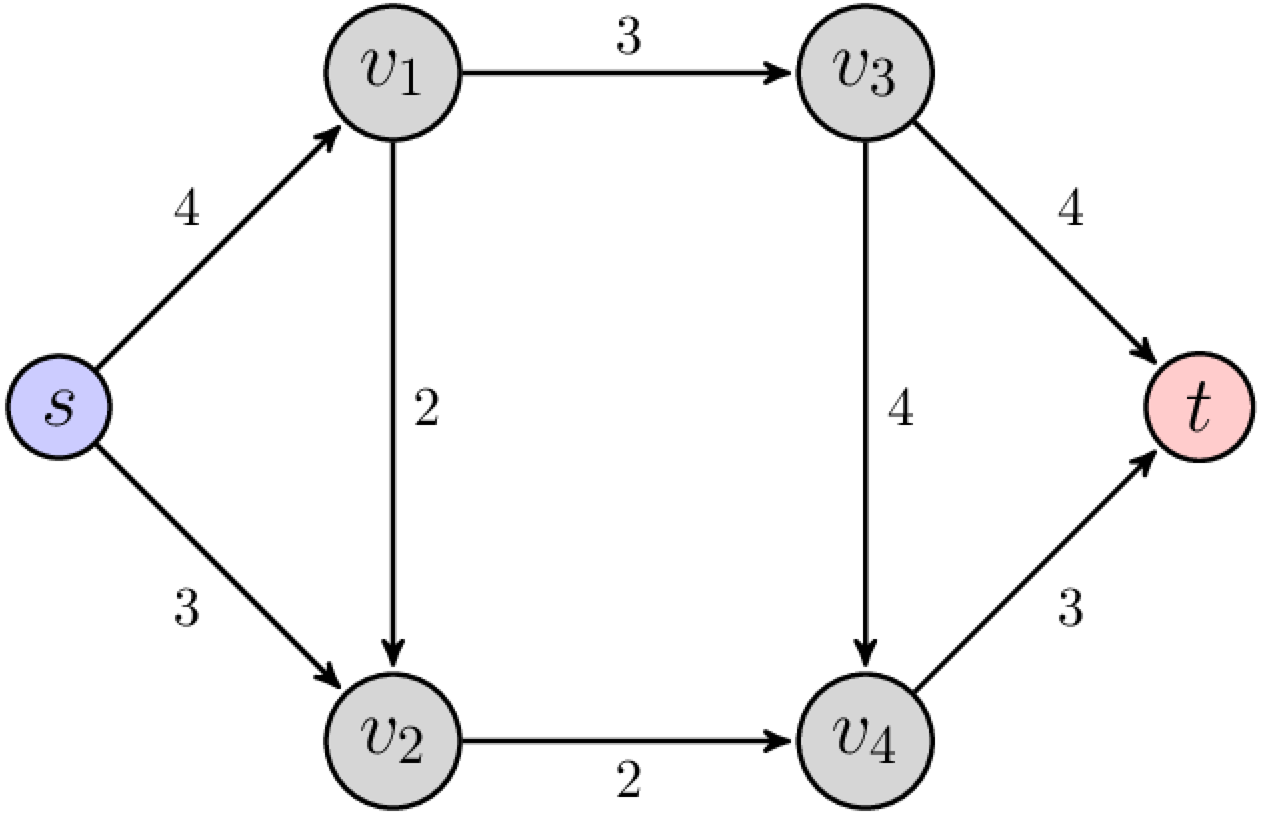
Degree of vertex V = Indegree of vertex V + Outdegree of vertex V

Sum of outdegree of vertices = Sum of indegree of vertices = Number of edges

Source
and
Sink

Source & Sink

A vertex V with zero indegree is called source



A vertex V with zero outdegree is called sink

Some results

RESULT 1

The sum of the degrees of the vertices of a graph G is equal to twice the number of edges in G.

$$\sum \mathbf{deg(v) = 2|E|}$$

This is also known as **Handshaking Lemma**

RESULT 2

In any graph, the number of vertices of odd degree is even.

RESULT 3

The maximum degree of any vertex in a simple graph with n vertices is n-1.

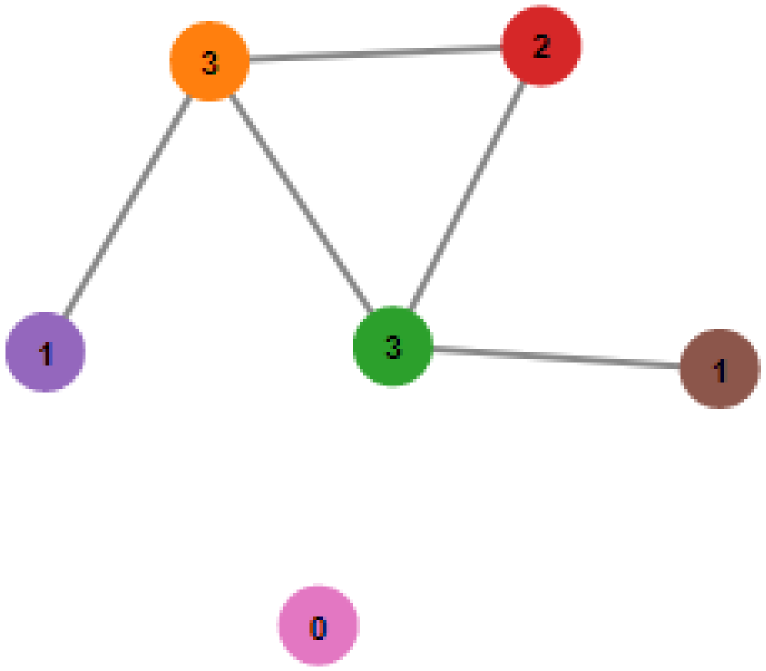
Some results

RESULT 4

The maximum number of edges in a graph with n vertices and no multiple edges are $\frac{n(n-1)}{2}$

Degree Sequence

Degree sequence of a graph is the list of degree of all the vertices of the graph. Usually we list the degrees in **non increasing order**, that is from largest degree to smallest degree.

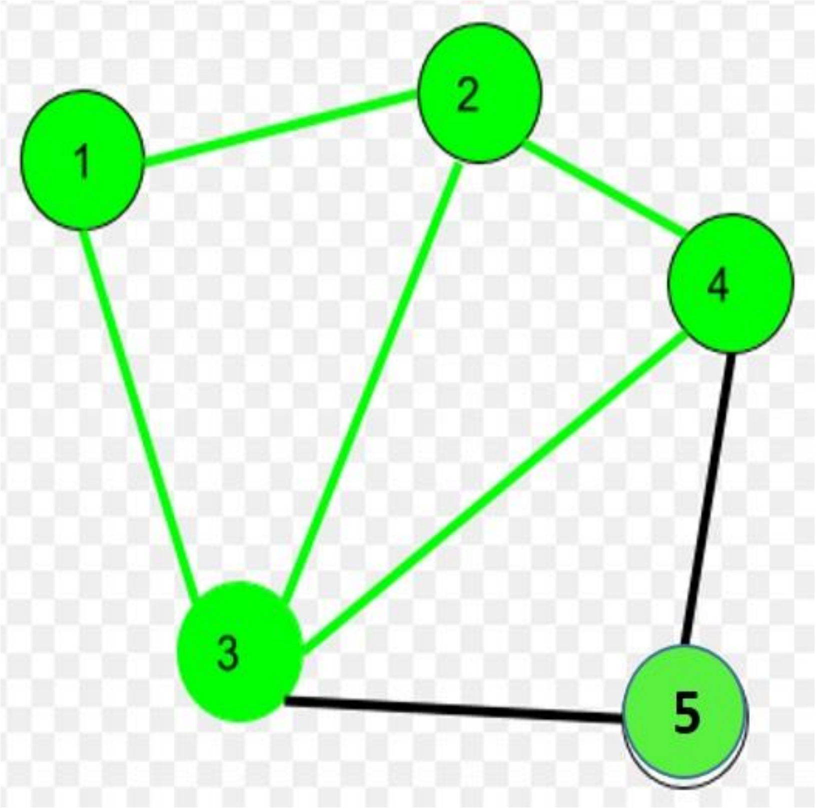


Degree Sequence $= (3, 3, 2, 1, 1, 0)$

Graph Theory
Part - II

Walk

Walk is sequence of adjacent vertices (or edges) in a graph.



1 – 2 – 4 – 5 – 3 is a walk from 1 to 3

1 – 2 – 3 -– 4 – 5 – 3 is a walk from 1 to 3

1 – 2 – 3 – 1 – 2 – 4 – 5 – 3 is a walk from 1 to 3

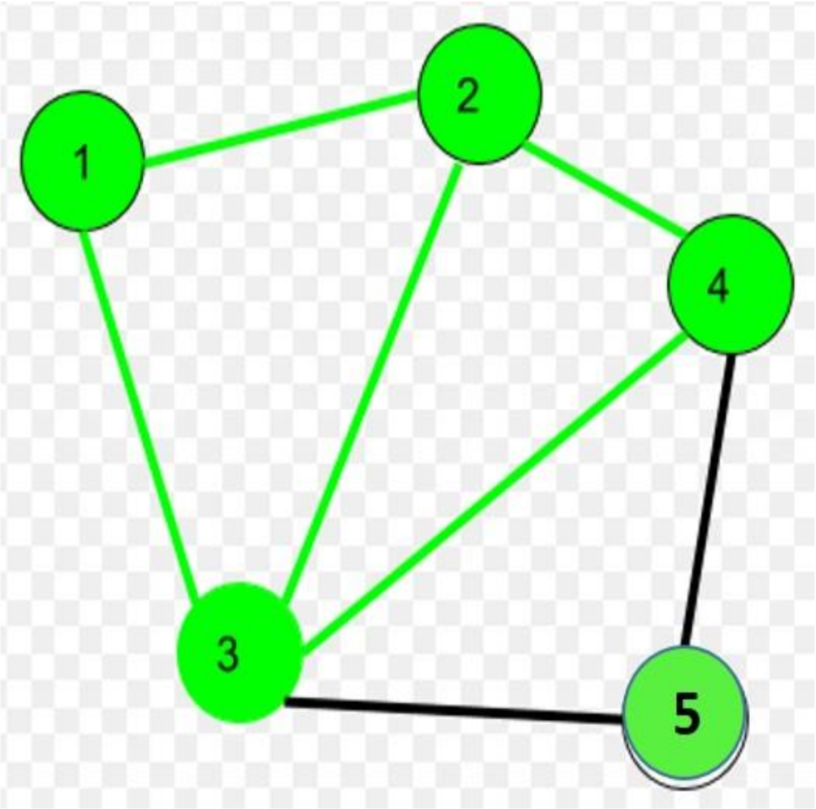
Note: A walk can contain vertices and edges multiple times.

Trail

Trail is an open walk where vertices can repeat, but not edges.

1 – 2 – 3 – 4 – 5 – 3 is a trail from 1 to 3.

1 – 2 – 3 – 4 – 5 – 3 – 1 is a closed trail.

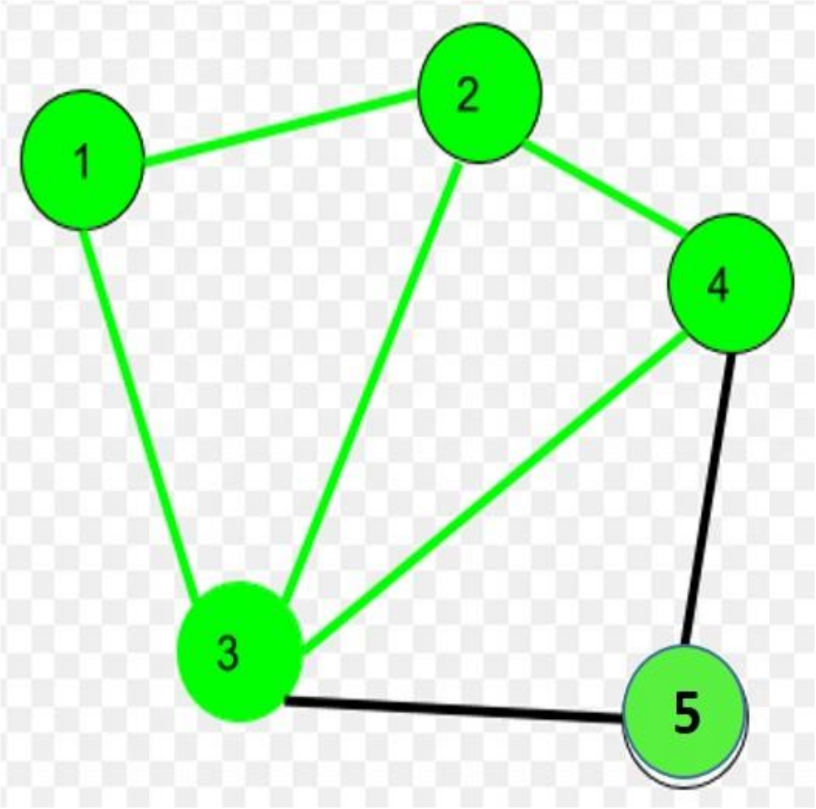


Path

Path is an open walk with no repetition of vertices and edges.

1 – 2 – 3 – 4 – 5 is a path from 1 to 5.

1 – 2 – 4 – 5 – 3 is a path from 1 to 3.

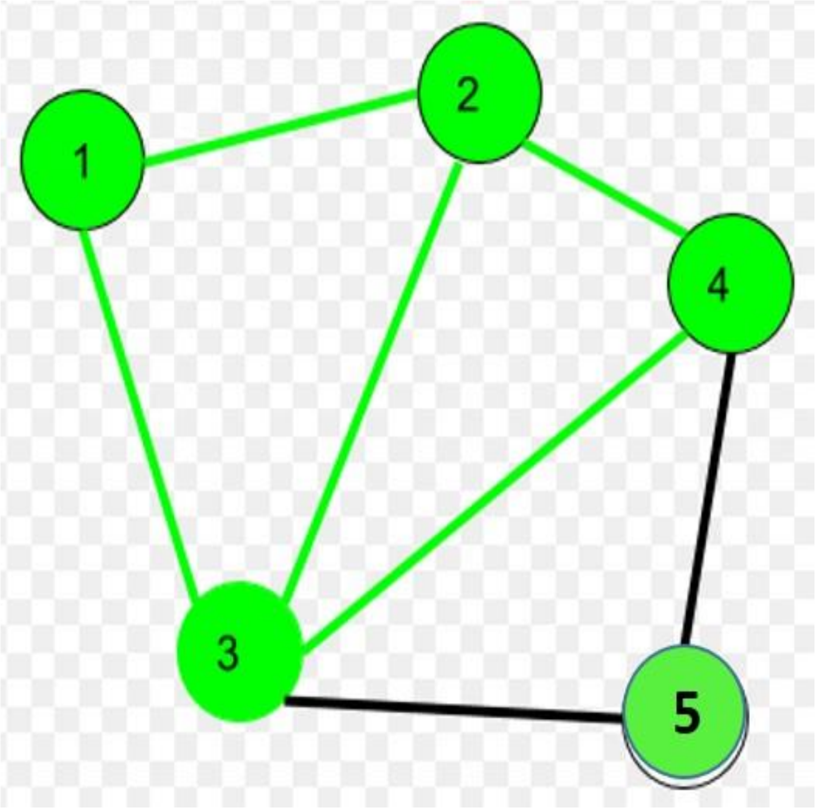


Circuit

Circuit is a closed walk where vertices can repeat, but not edges.

1 – 2 – 3 – 4 – 5 – 3 – 1 is a circuit.

1 – 3 – 5 – 4 – 3 – 2 – 1 is a circuit.

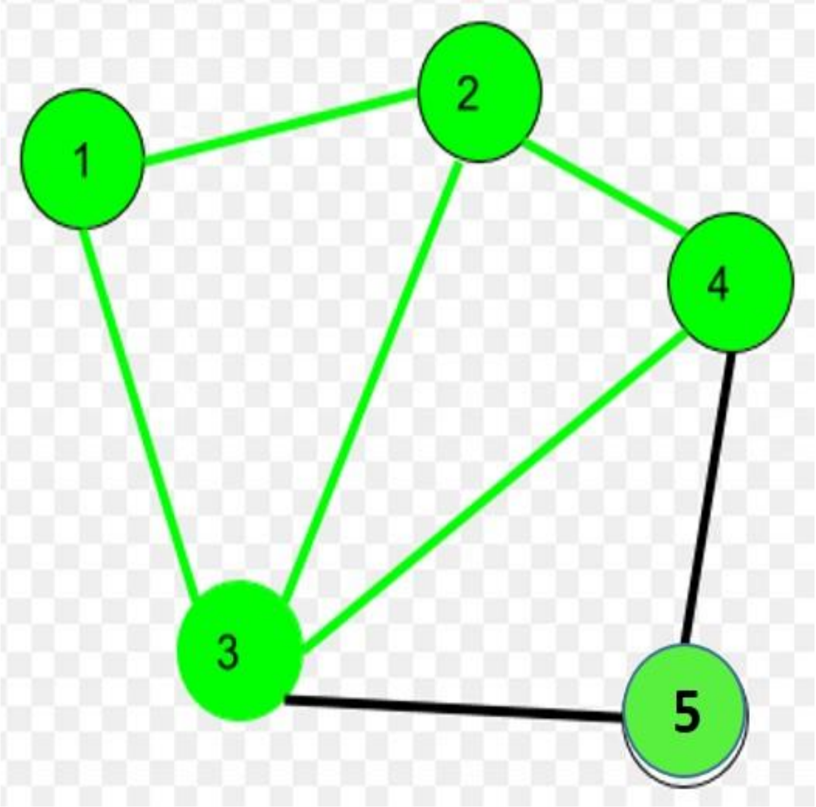


Cycle

Cycle is a closed walk where neither vertices nor edges can repeat. But since it is closed, the first and the last vertices are the same (one repetition).

1 – 2 – 4 – 5 – 3 – 1 is a cycle.

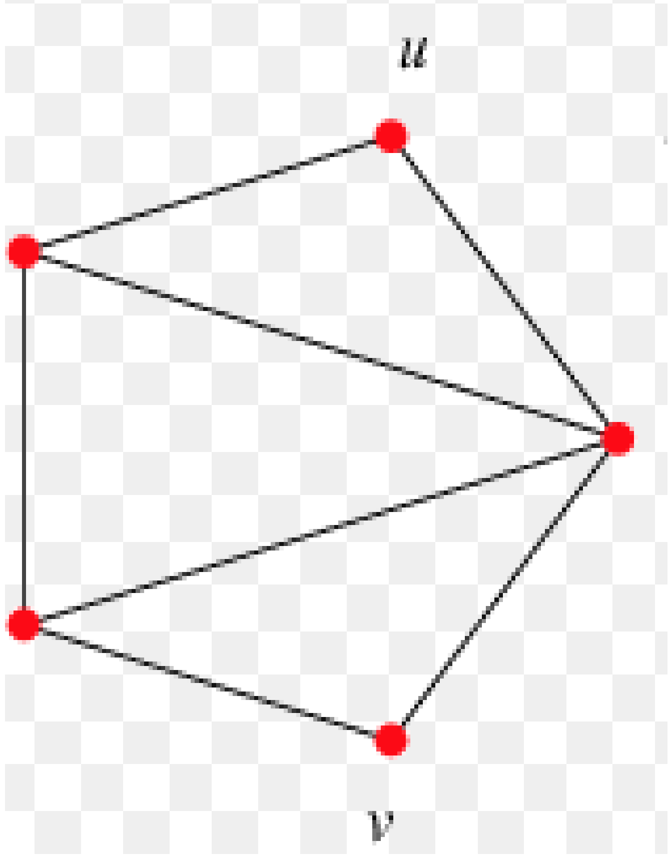
1 – 3 – 5 – 4 – 2 – 1 is a cycle.



Properties
of
Graphs

Distance

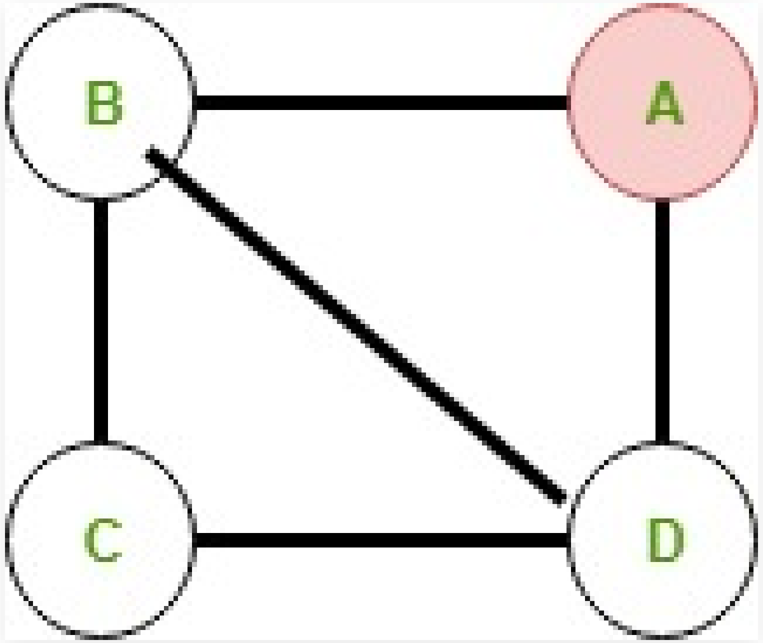
Distance between two vertices in a graph is the number of edges in a shortest path connecting them.



$$d(u, v) = 2$$

Eccentricity

Eccentricity It is defined as the maximum distance of one vertex from other vertex. It is denoted by **e(V)**.



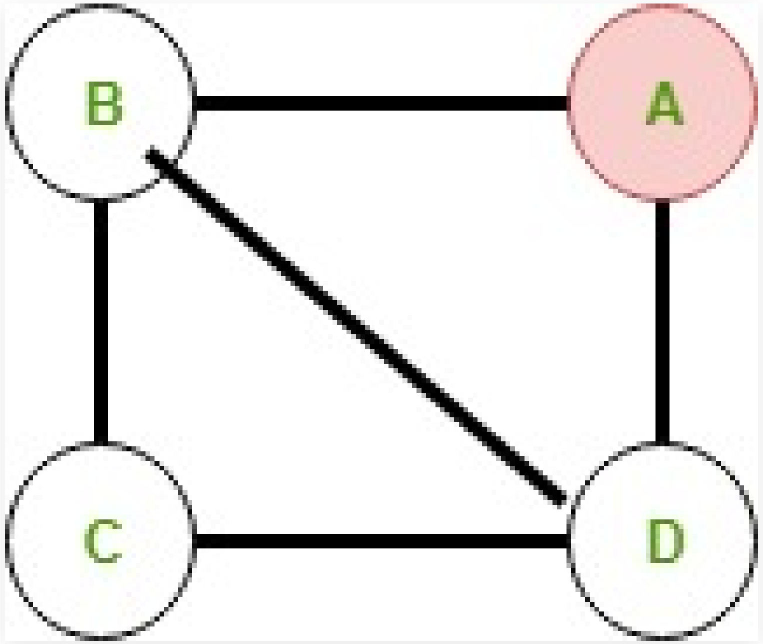

```
Eccentricity from:
(A, A) = 0
(A, B) = 1
(A, C) = 2
(A, D) = 1
Maximum value  is 2, So Eccentricity is 2
```

Diameter of Graph

The **diameter** of graph is the maximum distance between the pair of vertices.

Way to solve :

Find eccentricity of all vertices and then find maximum of all.




```
Eccentricity from:
(A, A) = 0
(A, B) = 1
(A, C) = 2
(A, D) = 1
Maximum value  is 2, So Eccentricity is 2
```

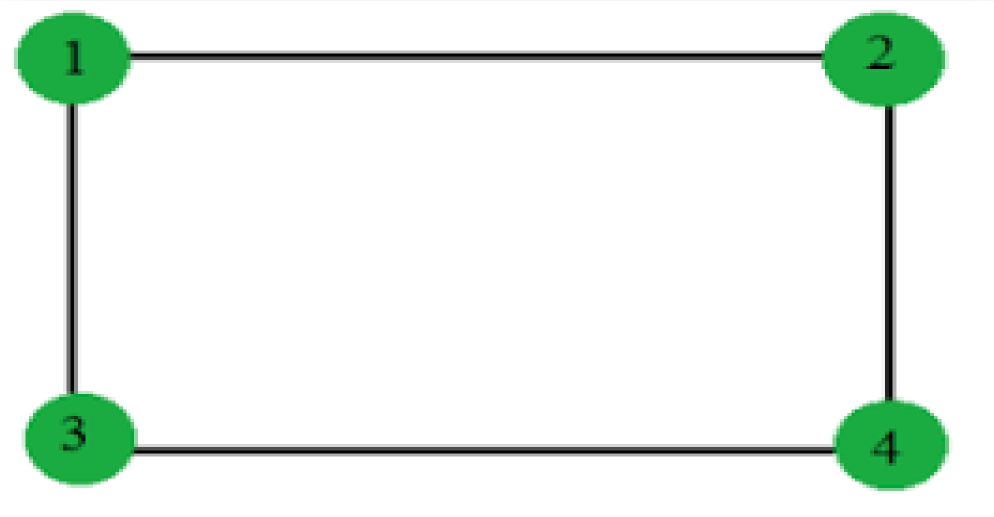
Special Graphs

Regular Graph

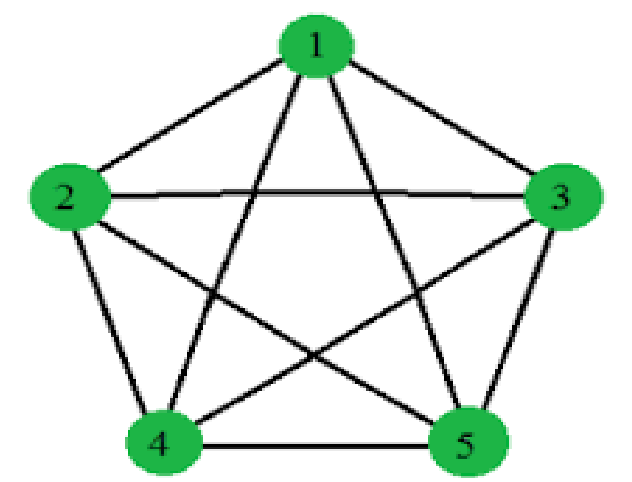
Regular graph

- If each vertex of a graph G has the **same degree** as every other vertex.
- A ***k* – regular** graph is a graph whose common degree is k.

Example: Consider $\boldsymbol{k_2}$ and $\boldsymbol{k_4}$. The degree of each vertex in $\boldsymbol{k_2}$ regular graph is 2 and $\boldsymbol{k_4}$ regular graph is 4. Hence $\boldsymbol{k_2}, \boldsymbol{k_4}$ are regular graphs.



2 Regular



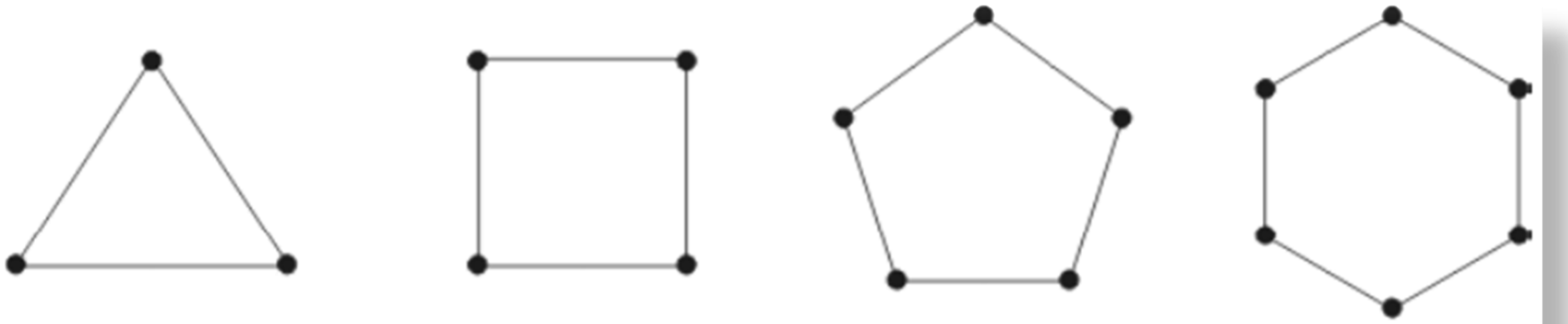
4 Regular

Regular Graph

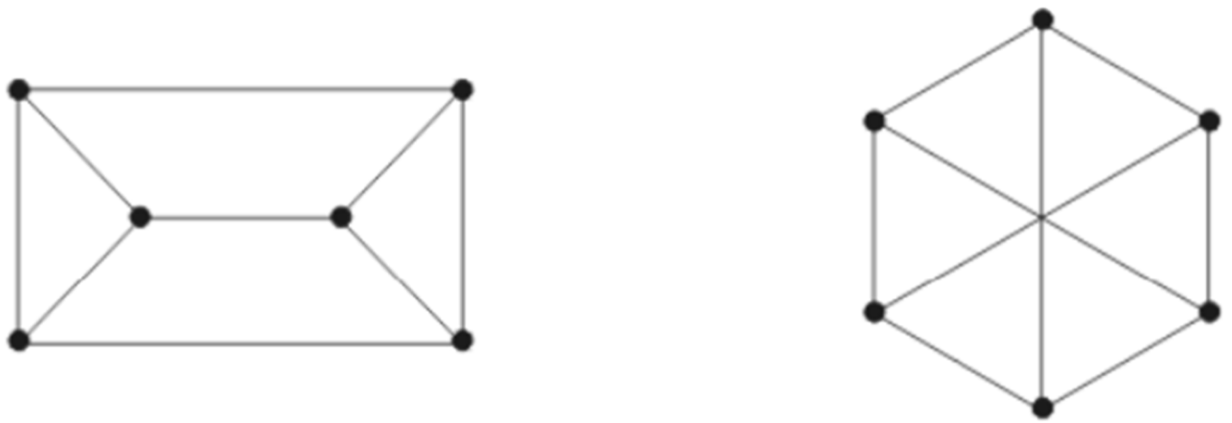
Regular graph

A ***k* – regular** graph is a graph whose common **degree is k**.

Example:



2-regular graphs



3-regular graphs

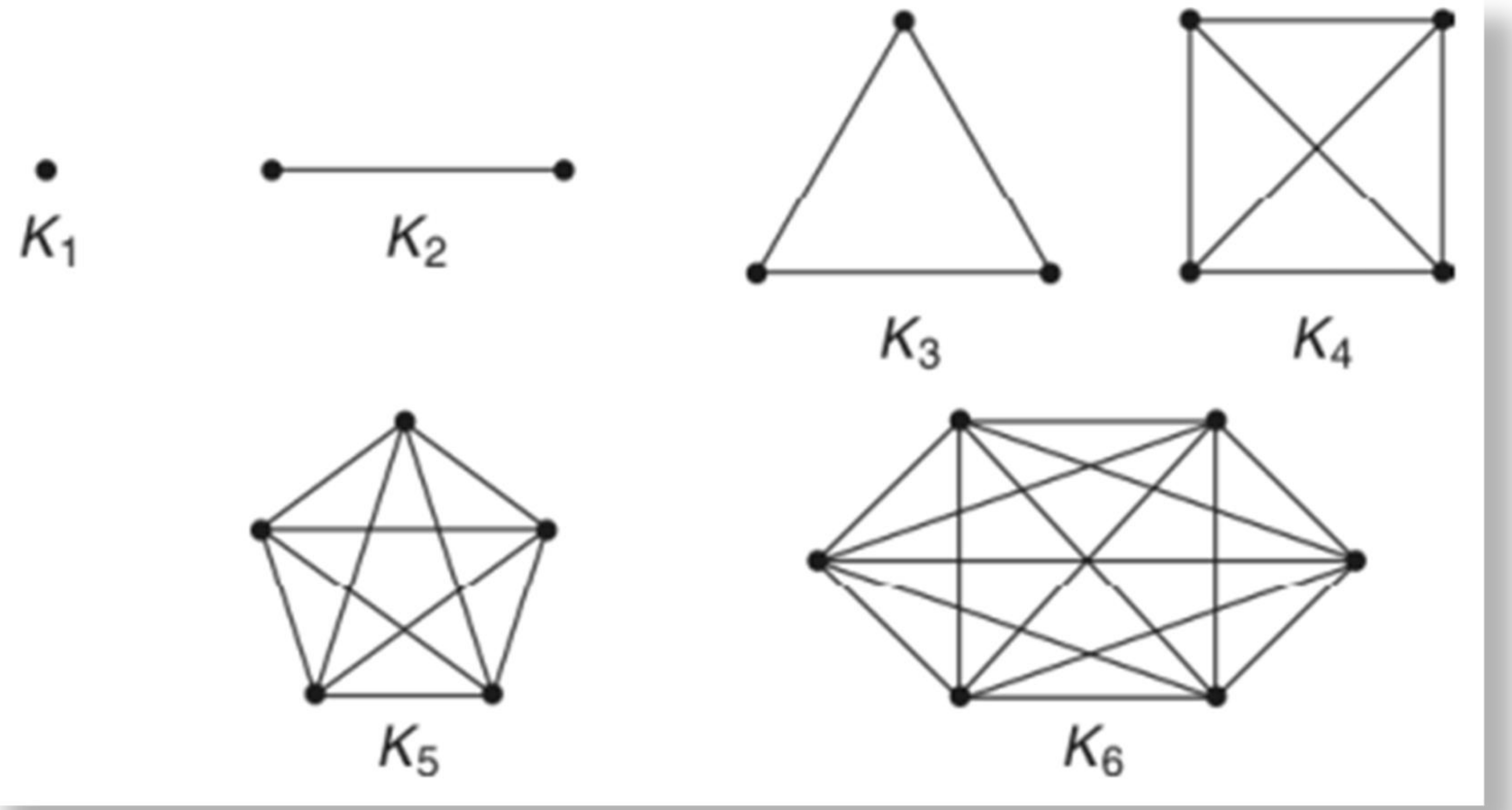
Complete Graph

Complete graph

- A simple graph in which there is **exactly one edge** between each pair of distinct vertices is called a complete graph. In a complete graph, every pair of vertices are adjacent.

Note:

The number of edges in K_n is n_{C_2} or $\frac{n(n-1)}{2}$. Hence, the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.



Complete Graph

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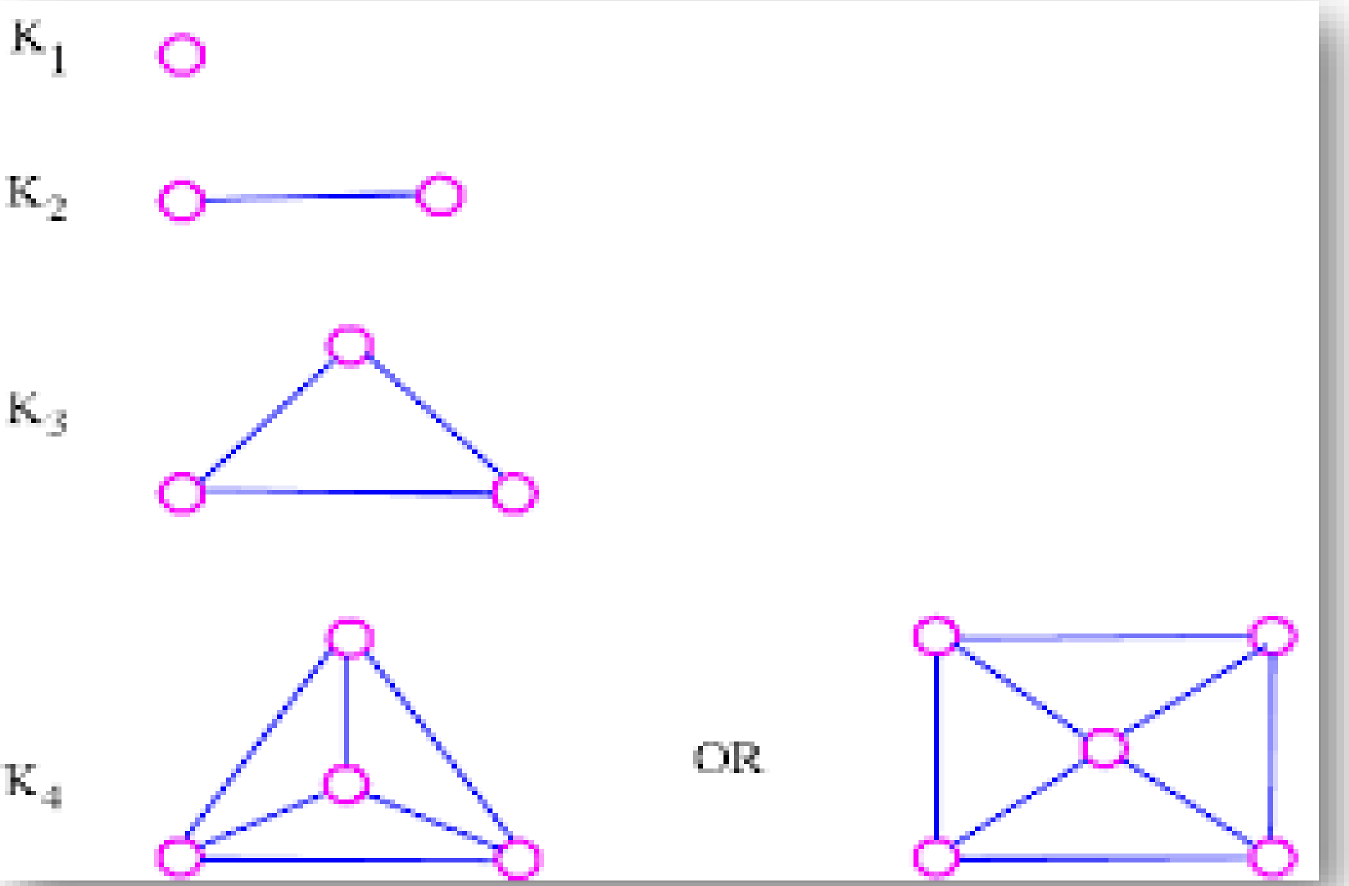


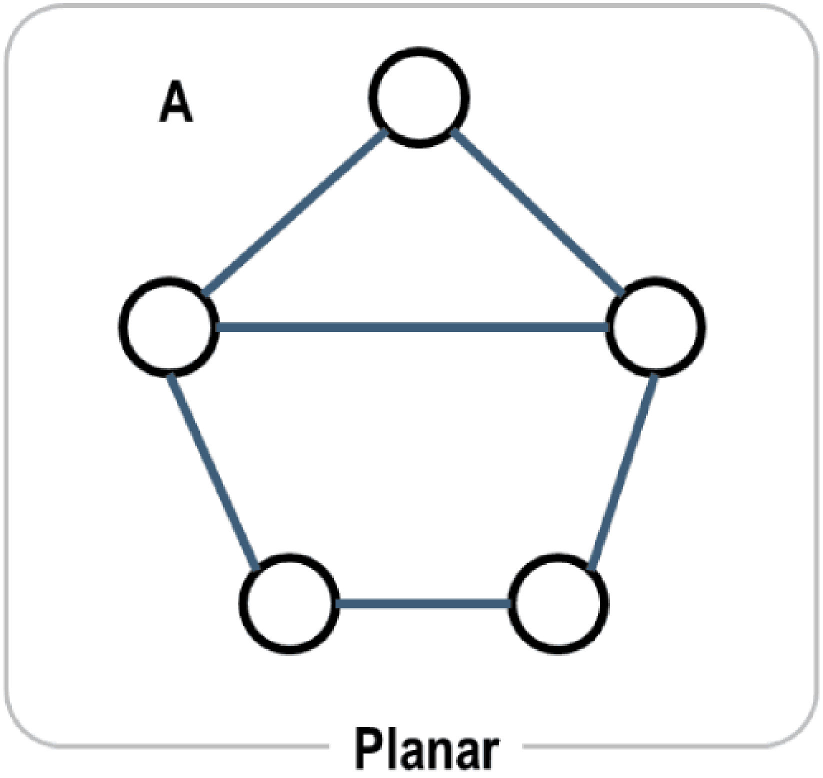
Planar/ Non-Planar Graphs

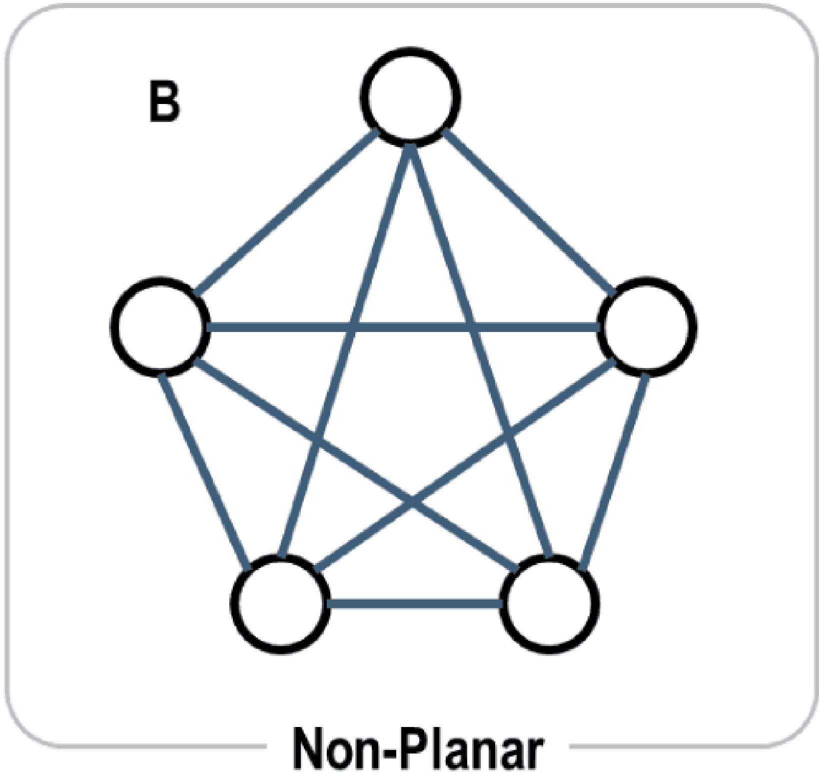
Planar Graphs

A graph is said to be **planar** if it can be drawn in a plane such that no two edges cross each other.

Non-Planar Graphs

A graph is said to be **non-planar** if it can be drawn in a plane such that two edges cross each other.





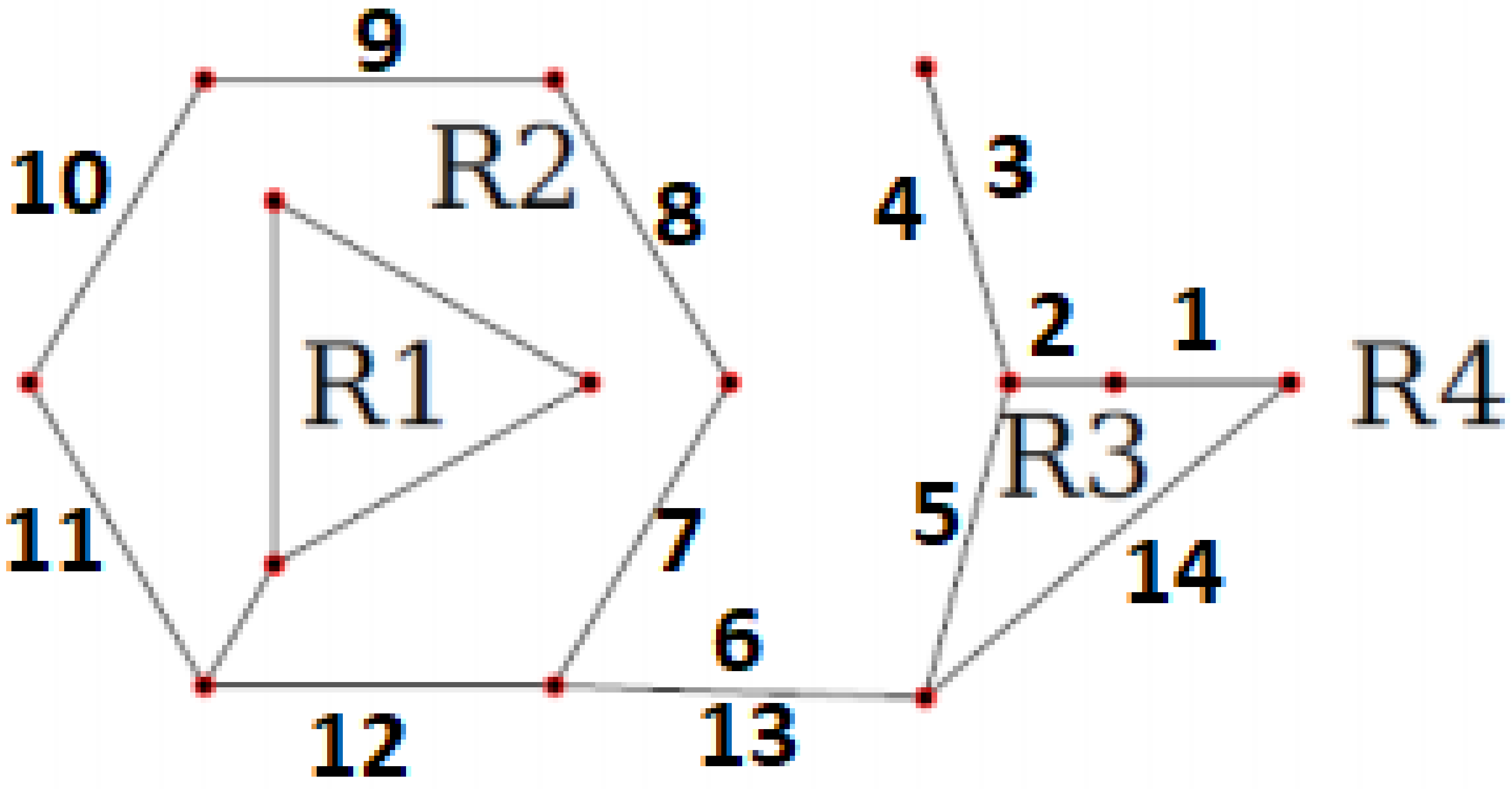
Maps/Regions

Maps

A particular planar representation of a finite planar multigraph is called a **Map**.

Region

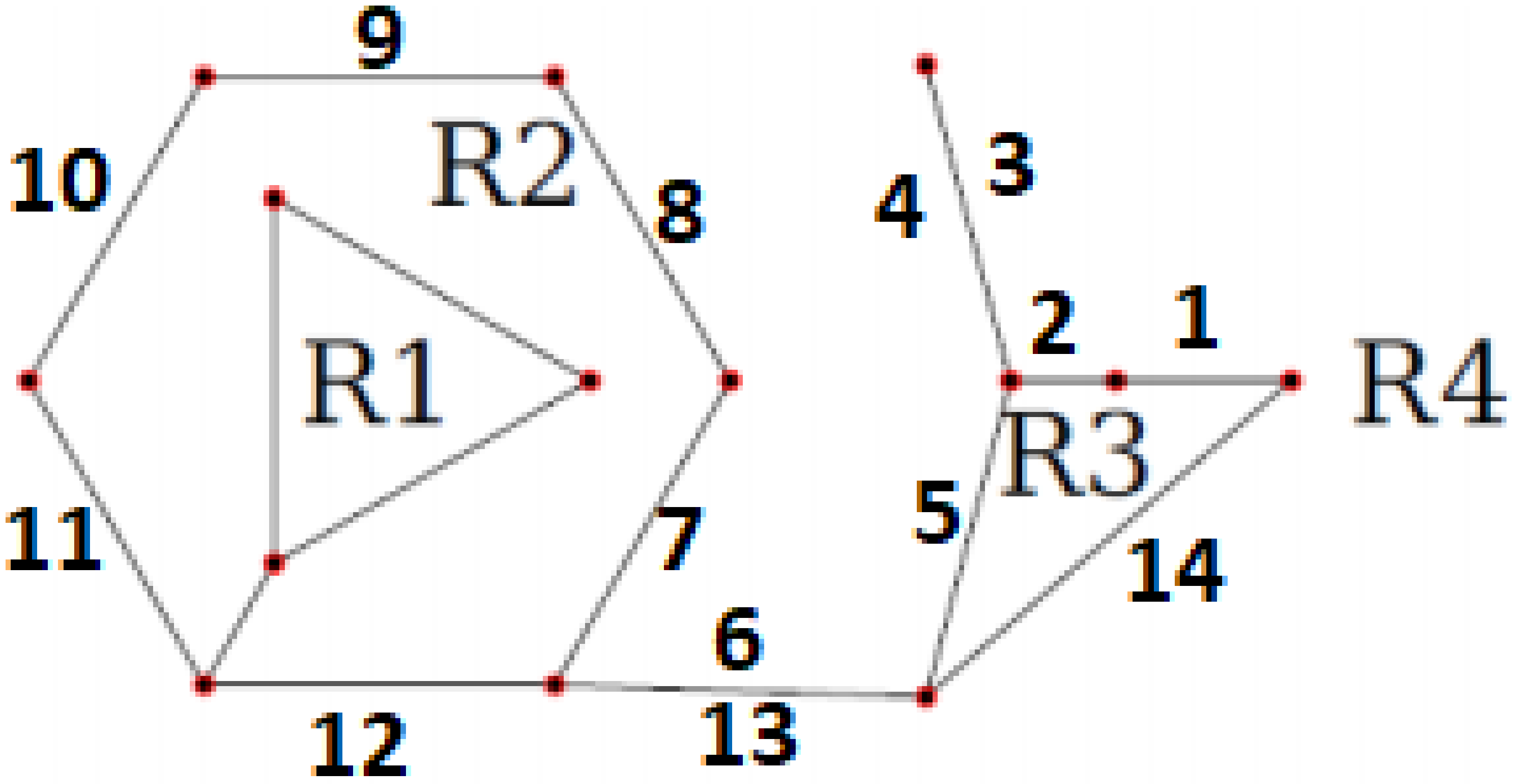
The map so drawn divides the plane into various areas bounded by edges which cannot be further subdivided.



Maps/Regions

i) Infinite Region

If the area of the region is infinite, then that region is called **infinite region**.

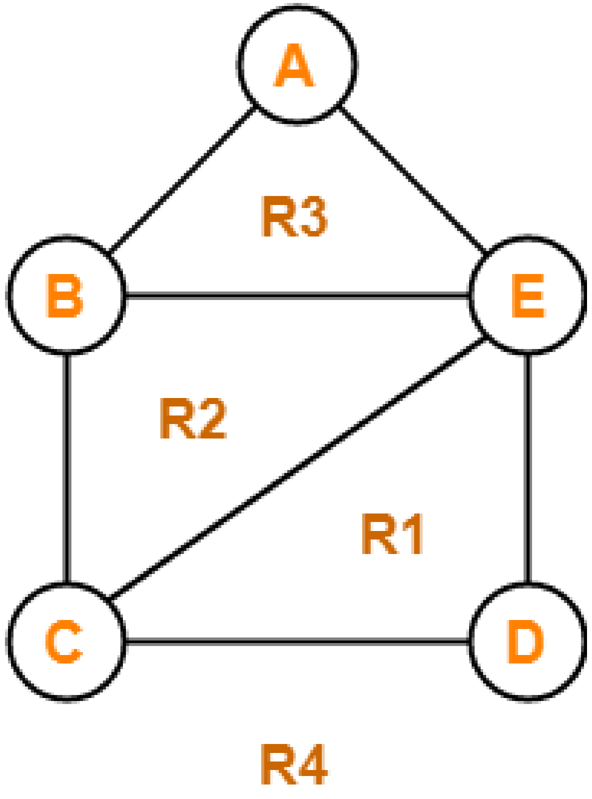


In the above figure, we have R4 to be the infinite region.

ii) Finite Region

If the area of the region is finite, then that region is called **finite region**.

In the figure, **R1, R2 & R3** are finite region and **R4** is infinite region.



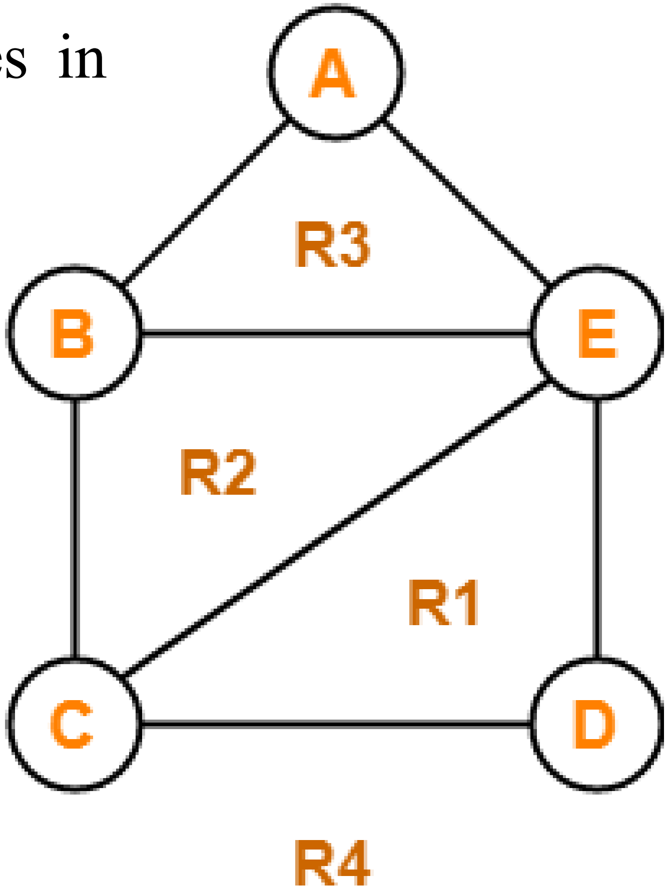
Regions of Plane

iii) Degree of Region

If G is a planar graph and R be its region, then number of edges in boundary of R is defined as degree of region R.

In the figure, $\boldsymbol{deg(R1) = 3}$, $\boldsymbol{deg(R2) = 3}$, $\boldsymbol{deg(R3) = 3}$ and $\boldsymbol{deg(R4) = 5}$.

Note: Degree of a cut edge is counted twice.



Regions of Plane

Planar Graph

Properties of Planar graphs

- If a connected planar graph G has E edges and R regions, then $\boldsymbol{R \leq \frac{2}{3}E}$.
- If a connected planar graph G has E edges, V vertices and R regions then $\boldsymbol{V - E + R = 2}$ (**Euler's Formula**)
- If a connected planar graph G has E edges and V vertices, then $\boldsymbol{3V - E \geq 6}$.
- A complete graph $\boldsymbol{K_n}$ is planar, if and only if $n < 5$.

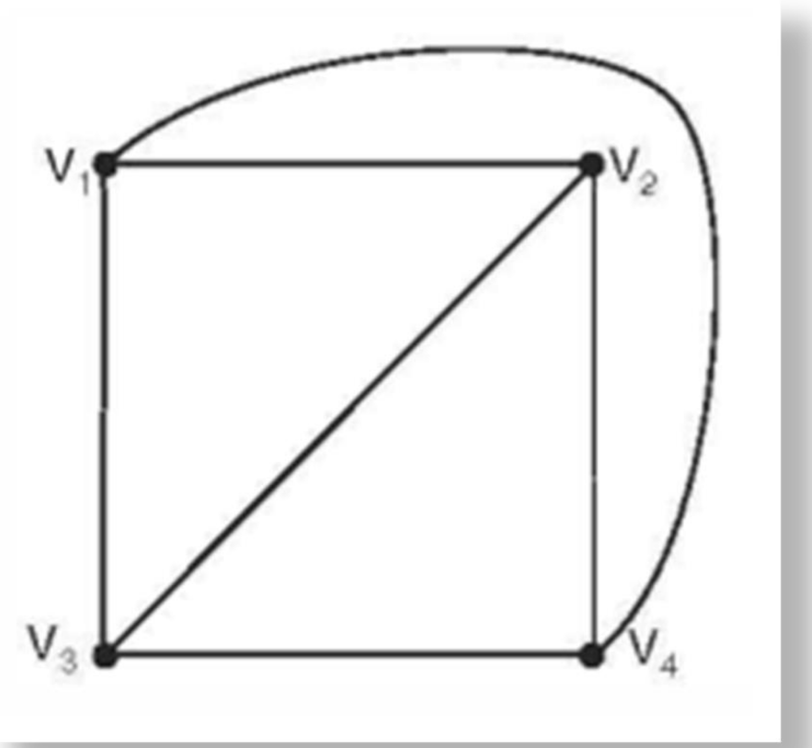
Planar Graph

Is the complete graph K_4 planar ?

The complete graph K_4 contains 4 vertices and 6 edges.

From the property, $3V - E \geq 6$, hence $3 * 4 - 6 = 6$, which satisfies the property of planar graphs.

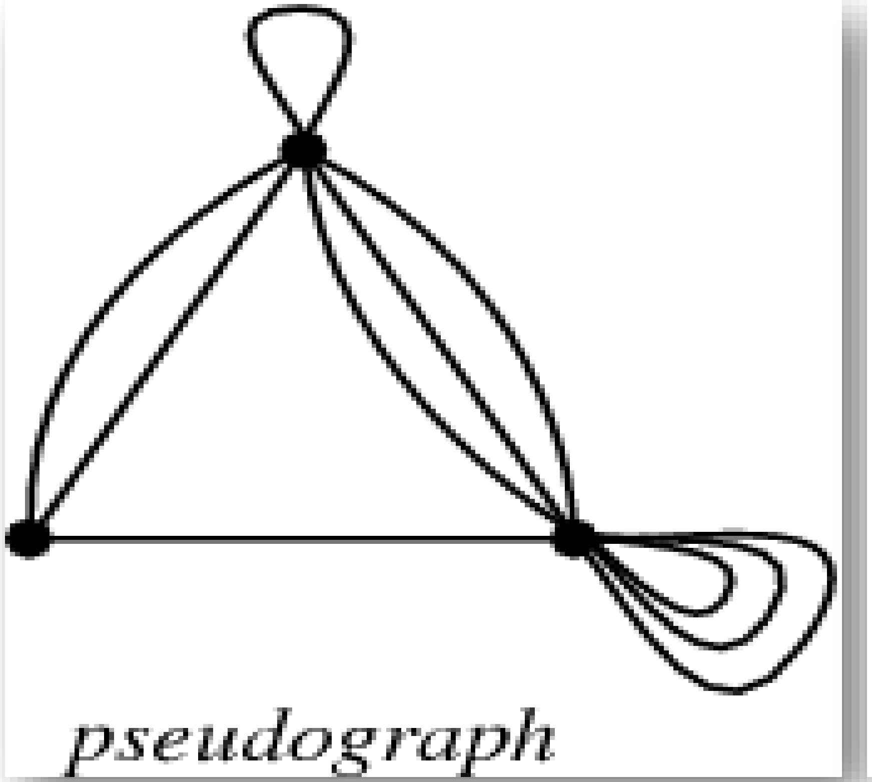
Therefore, K_4 is a planar graph.

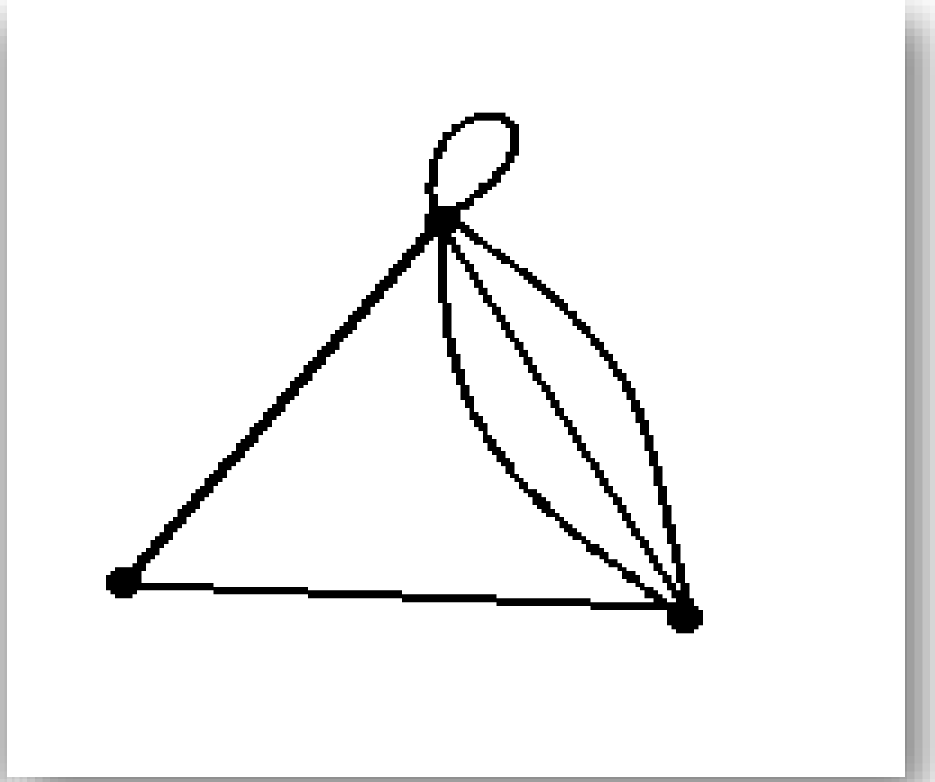


Pseudo Graph

Pseudo-graph

- The graphs in which loops and parallel edges are allowed.

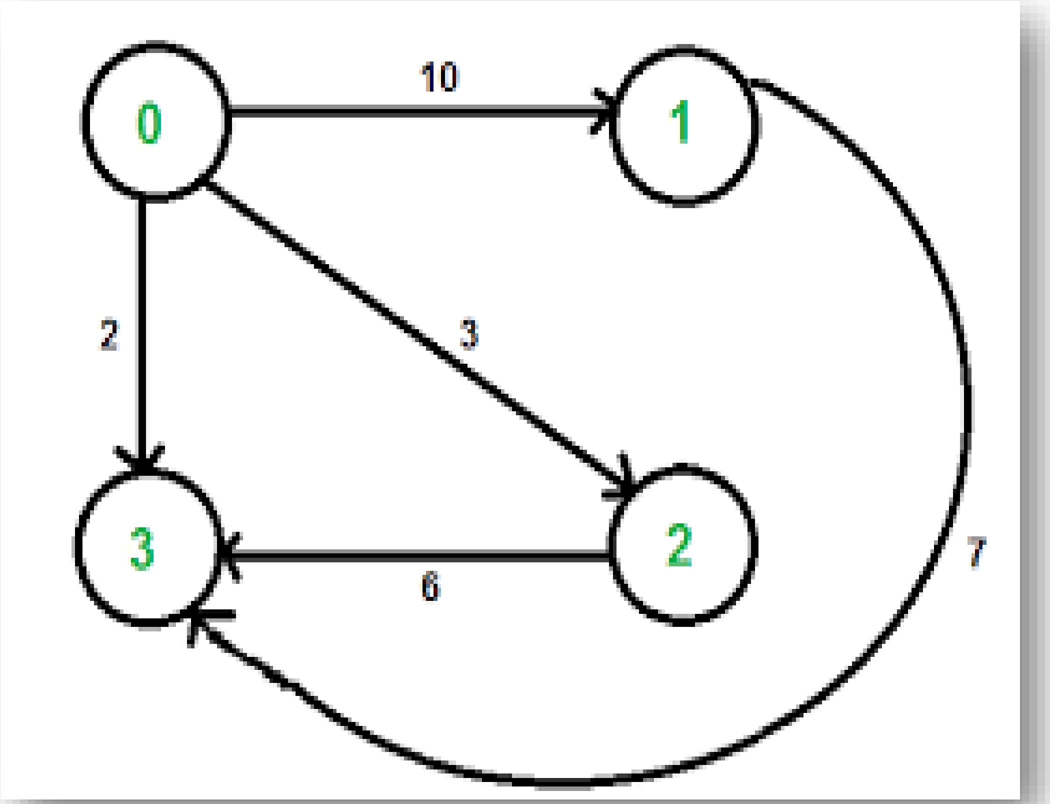


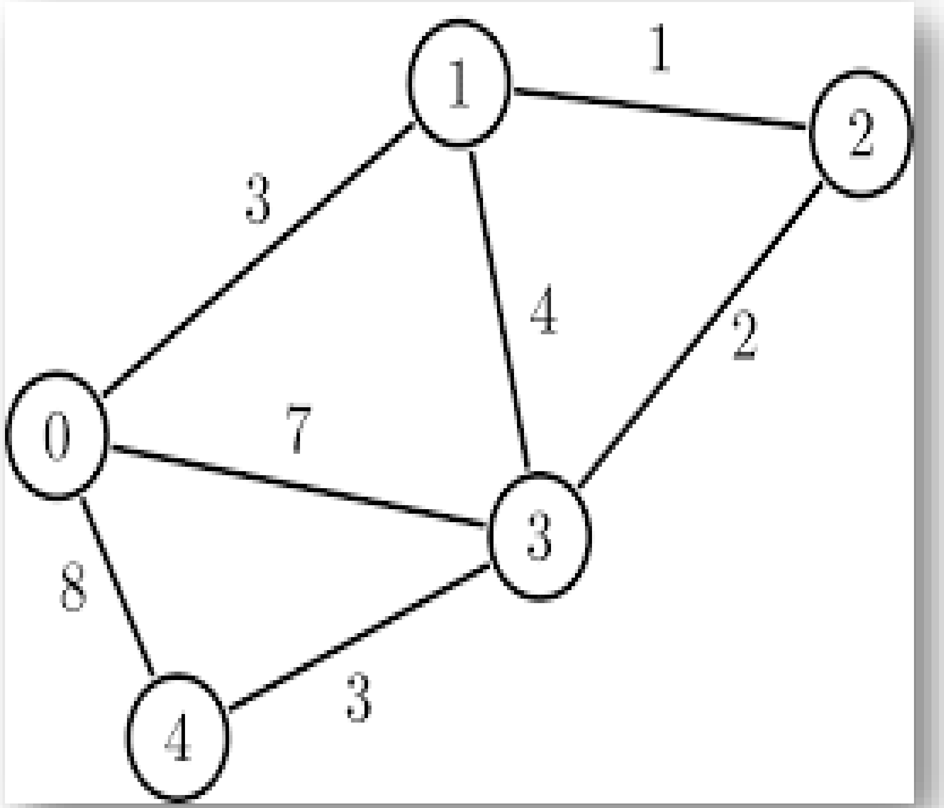


Weighted Graph

Weighted graph

- The graph in which weights are assigned to each edge.

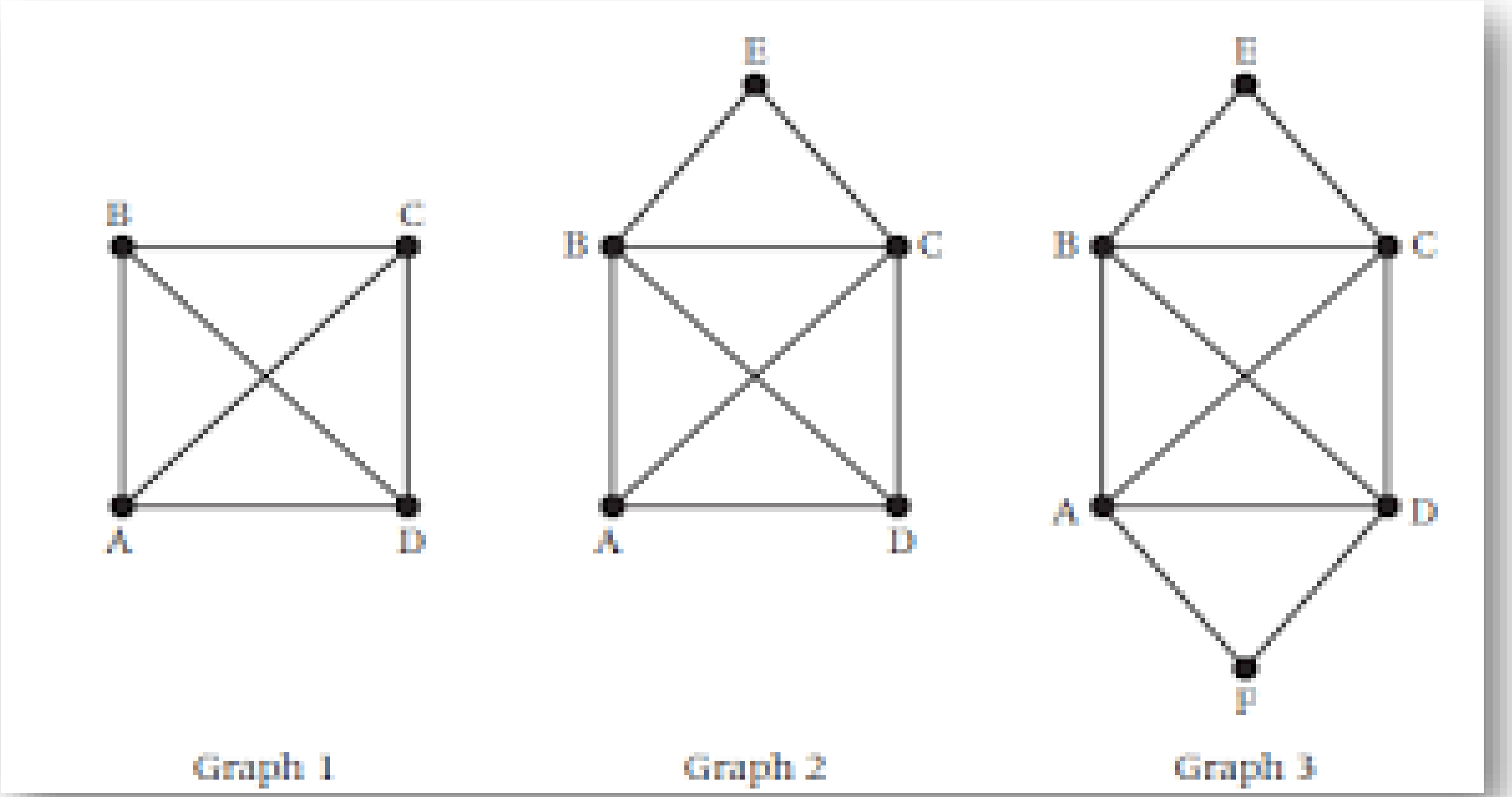




Traversable Graph

Traversable graphs

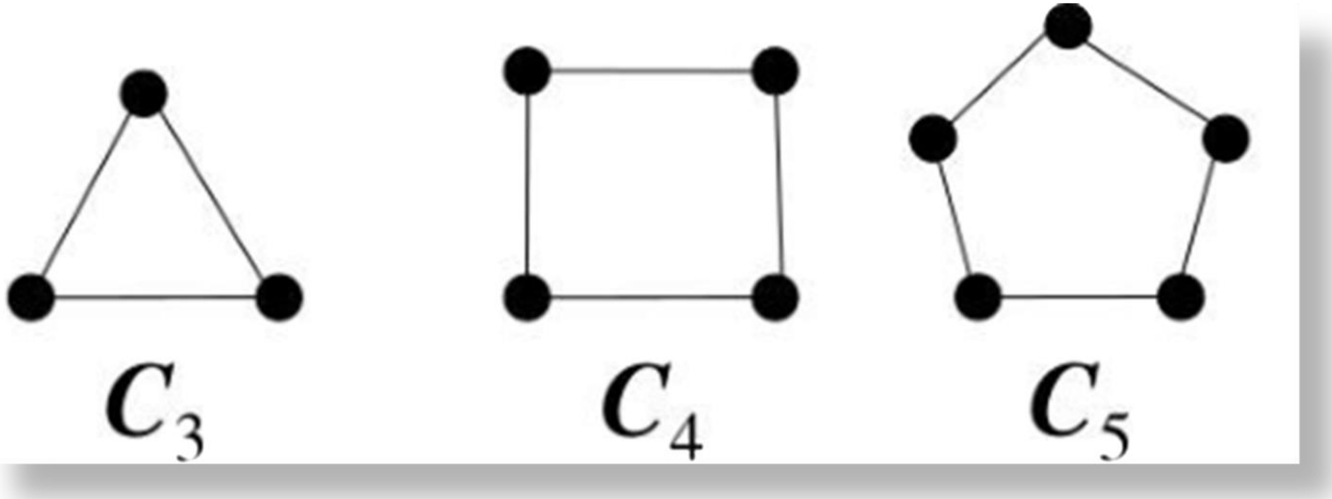
- A graph is traversable if you can draw a path between all the vertices without retracing the same path.



Cycle

Cycle:

- A **cycle C_n ,$n \geq 3$**, is a **2 −regular** graph consisting of n vertices and n edges.

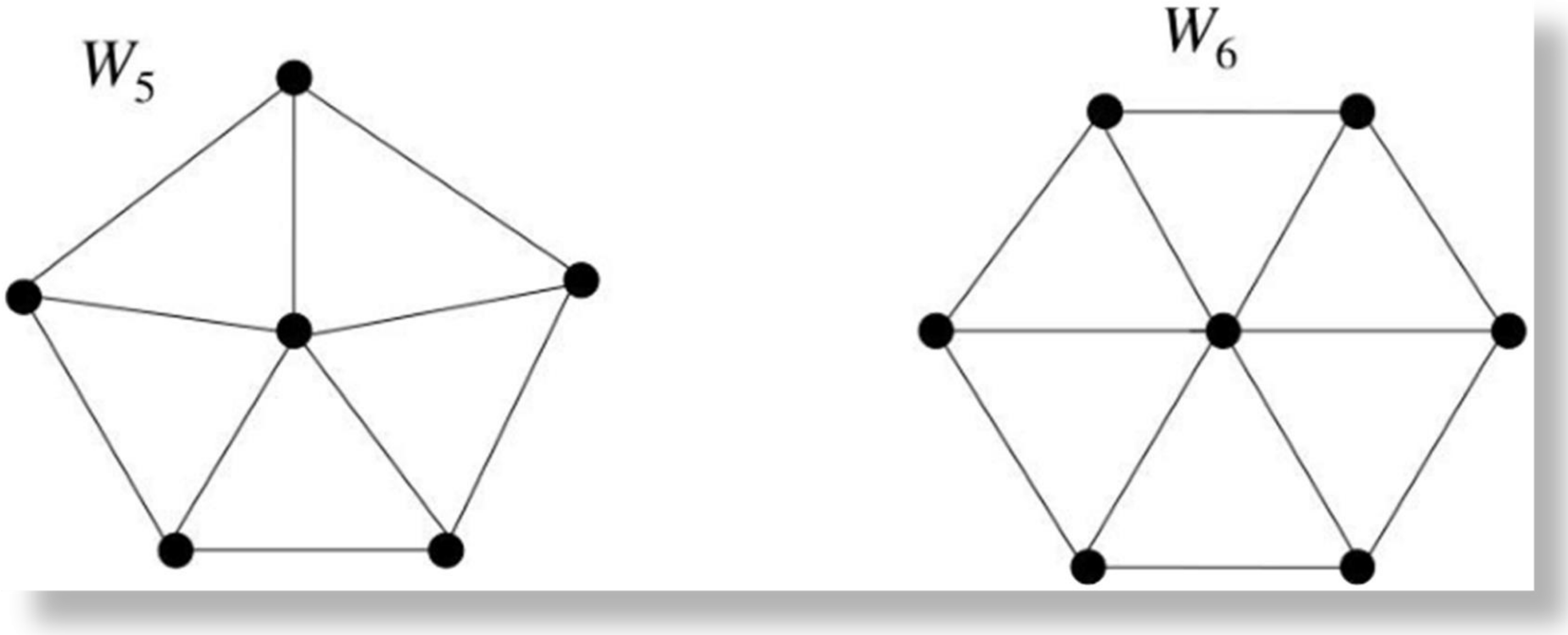


Note: Number of vertices $= n$ and number of edges $= n$.

Wheel

Wheel:

- A **wheel** W_n, is obtained from C_n by adding an extra vertex at centre and connecting this new vertex to each vertex of C_n.

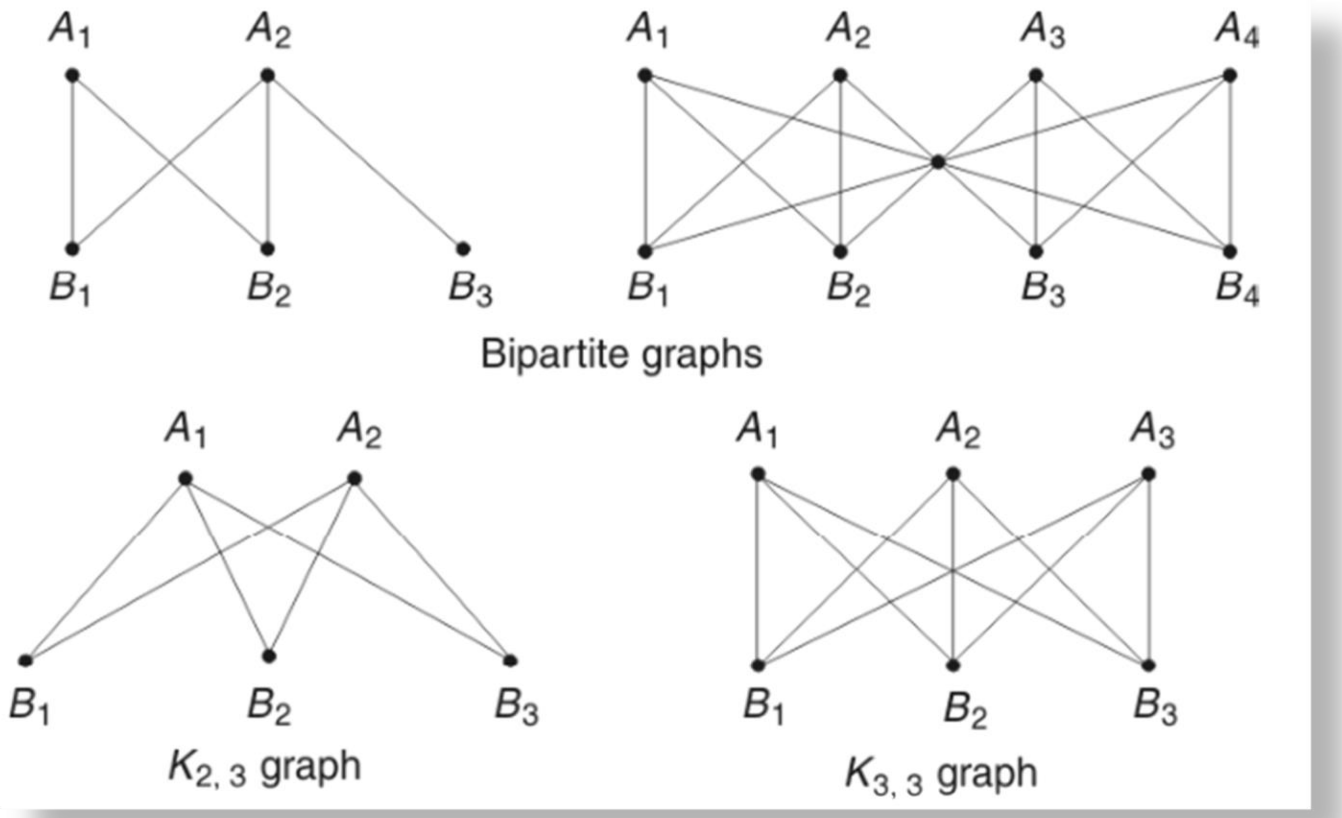


Note: Number of vertices $= n + 1$ and number of edges $= 2n$. W_n is not a regular graph for $n \neq 3$.

Bipartite graph

Bipartite:

- If the vertex set V of a simple graph $G = (V, E)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2), then G is called a **bipartite graph**.

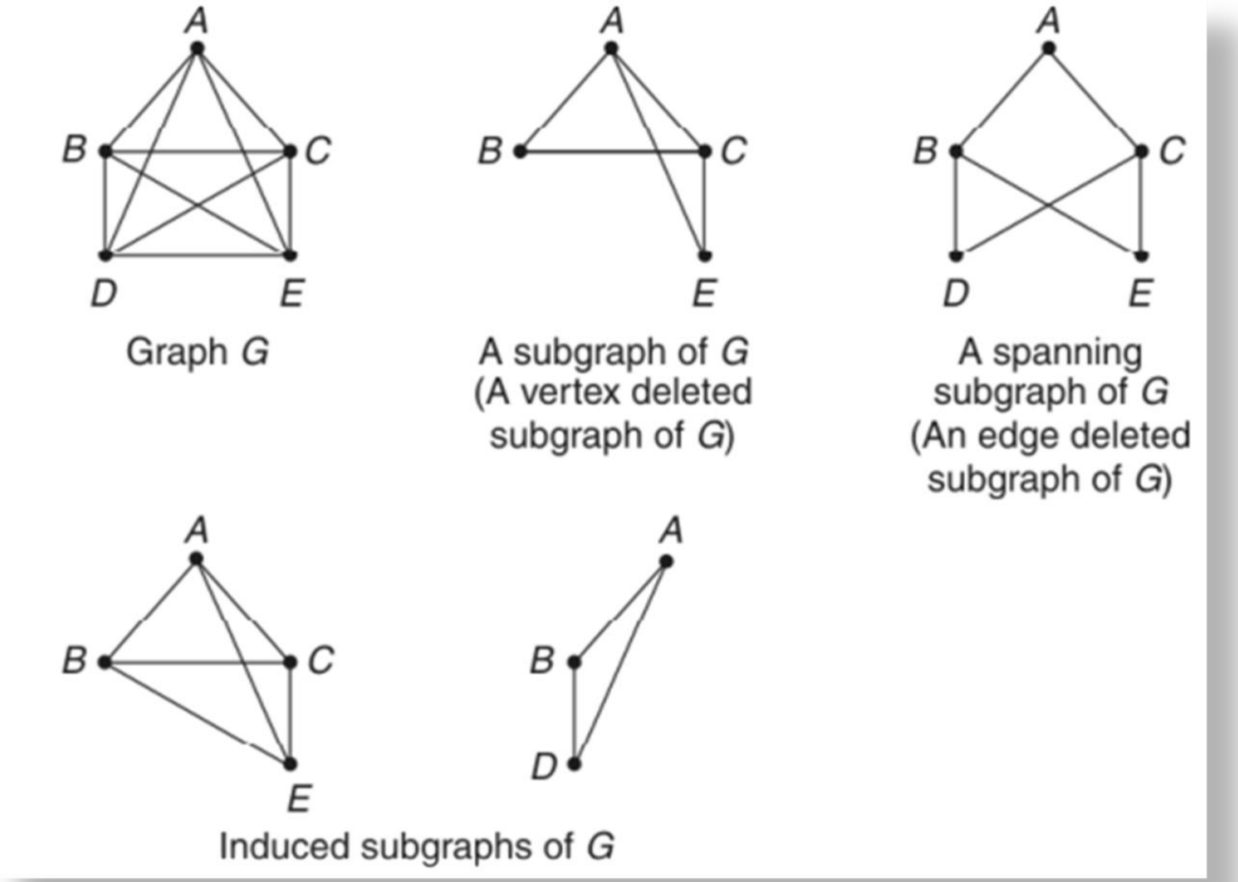


Note: If each vertex of V_1 is connected with every vertex of V_2 by an edge, then G is called a **completely bipartite graph**. $K_{m.n}$.

Subgraphs

Subgraph:

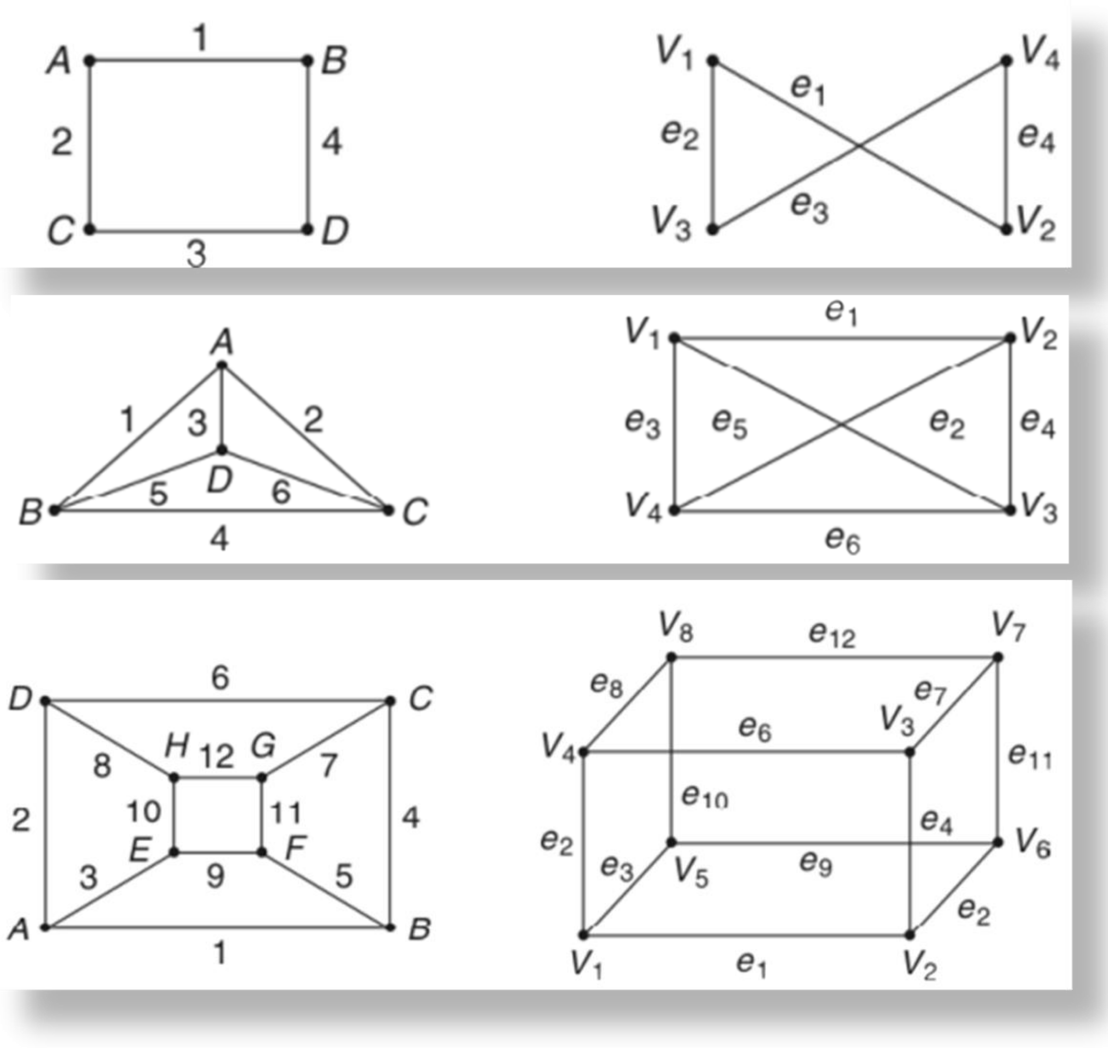
- A graph $H = (V', E')$ is called a **subgraph** of $G = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$.
- If $V' \subset V$ and $E' \subset E$, then H is called a **proper subgraph** of G.
- If $V' = V$, then H is called a **spanning subgraph** of G. A spanning subgraph of G **need not** contain **all its edges**.



Isomorphic Graphs

Isomorphic Graphs:

- Two graphs G_1 and G_2 are said to be isomorphic to each other, if there exists a **one-to-one correspondence** between the **vertex sets** which **preserves adjacency** of the vertices.



Isomorphic graphs have

(i) the same number of **vertices**,

(ii) the same number of **edges** and

(iii) the corresponding vertices with the **same degree**.

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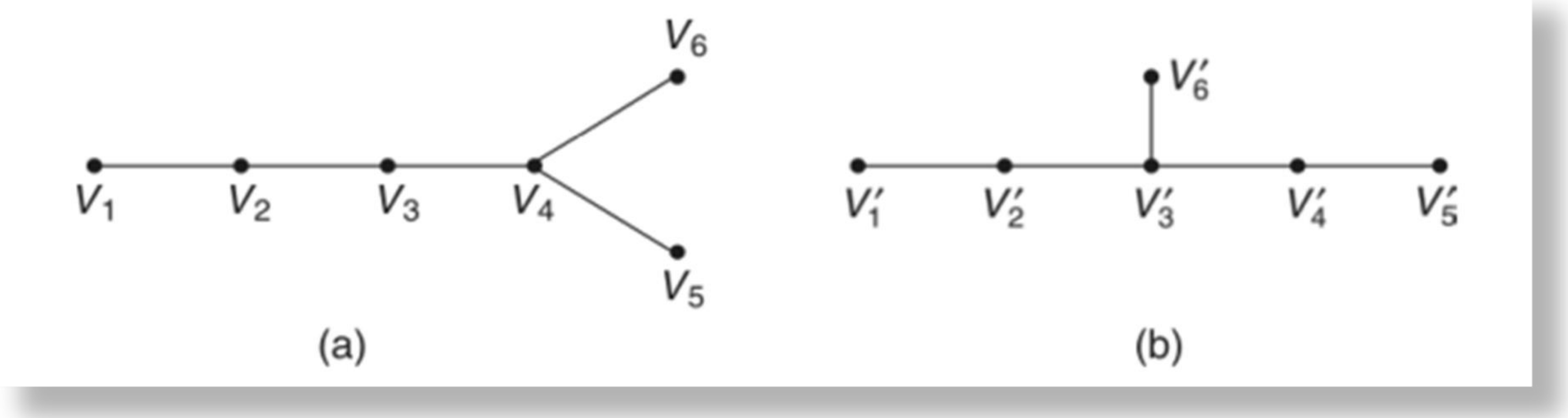
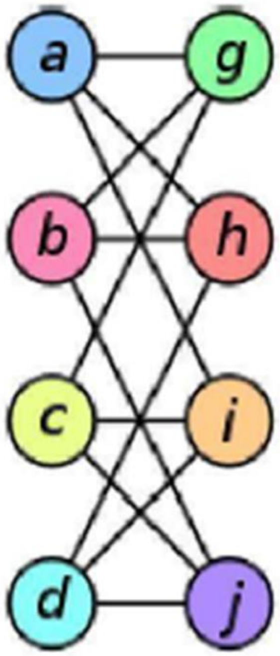
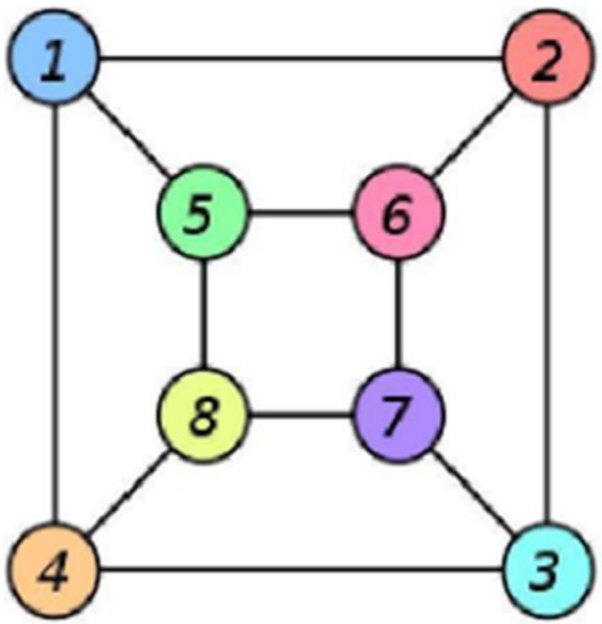


Isomorphic graphs have
(i) the same number of **vertices**,
(ii) the same number of **edges** and
(iii) the corresponding vertices with the **same degree**.

Adjacency of the vertices is not **preserved**.
As V_2 and V_3 are adjacent in (a) whereas the corresponding vertices V'_2 and V'_4 are not adjacent.
Hence **not Isomorphic**.

Isomorphic Graphs

Isomorphic Graphs:

Graph G	Graph H	An isomorphism between G and H
		$f(a) = 1$ $f(b) = 6$ $f(c) = 8$ $f(d) = 3$ $f(g) = 5$ $f(h) = 2$ $f(i) = 4$ $f(j) = 7$

Matrix representation of graphs

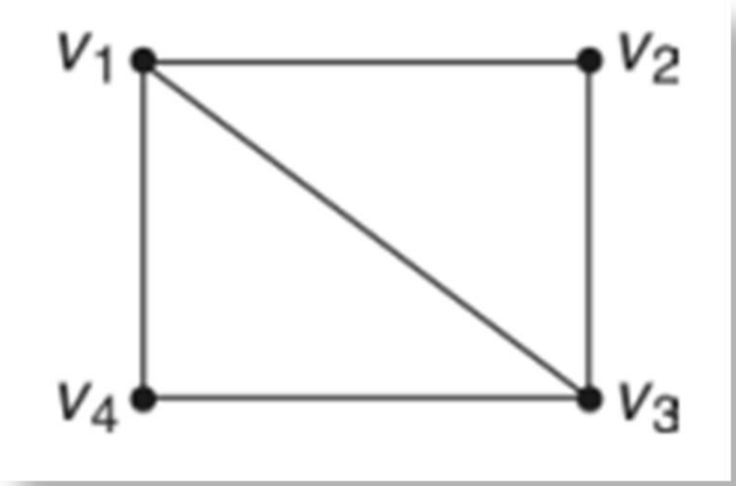
Adjacency Matrix :

- When **G** is a **simple graph** with n vertices $v_1, v_2, \ldots v_n$, the matrix **A(or A_G)** $[\boldsymbol{a_{ij}}]$,

$$\text{where, } a_{ij} = \begin{cases} 1, & \text{if } \boldsymbol{v_i v_j} \text{ is an edge of } \boldsymbol{G} \\ 0, & \text{otherwise} \end{cases}$$

Adjacency Matrix

- Since a simple graph has no loops, each diagonal entry of A, $\boldsymbol{a_{ii} = 0}$, for $i = 1, 2, \ldots n$.
- The adjacency matrix of simple graph is symmetric, $\boldsymbol{a_{ij} = a_{ji}}$.
- $\boldsymbol{deg}(\boldsymbol{v_i})$ is equal to the number of 1's in the i^{th} row or i^{th} column.



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Matrix representation of graphs

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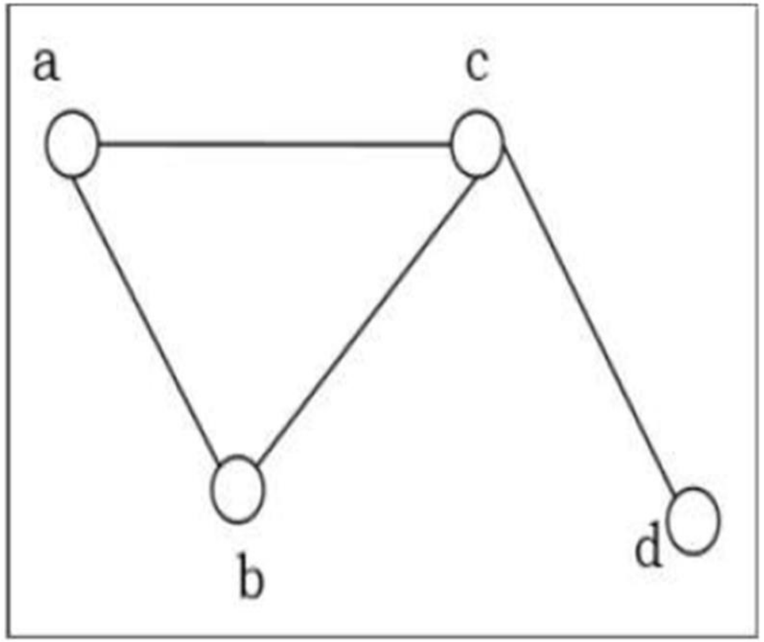

	a	b	c	d
a	0	1	1	0
b	1	0	1	0
c	1	1	0	1
d	0	0	1	0

Matrix representation of graphs

Adjacency Matrix :

- When **G** is a graph with n vertices $v_1, v_2, \dots v_n$, the matrix **A(or A_G) $[a_{ij}]$,**

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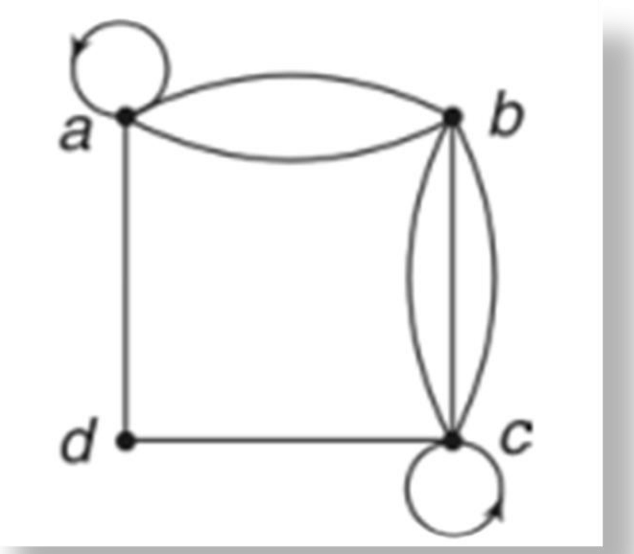
Adjacency Matrix

- Pseudo graph has loops/multi-edges. If v_i has a loop then the diagonal entry of $\boldsymbol{a_{ii}} = \mathbf{1}$.



$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Matrix representation of graphs

Adjacency Matrix :

- When **G** is a graph with n vertices $v_1, v_2, \ldots v_n$, the matrix **A(or A_G) [a_{ij}],**

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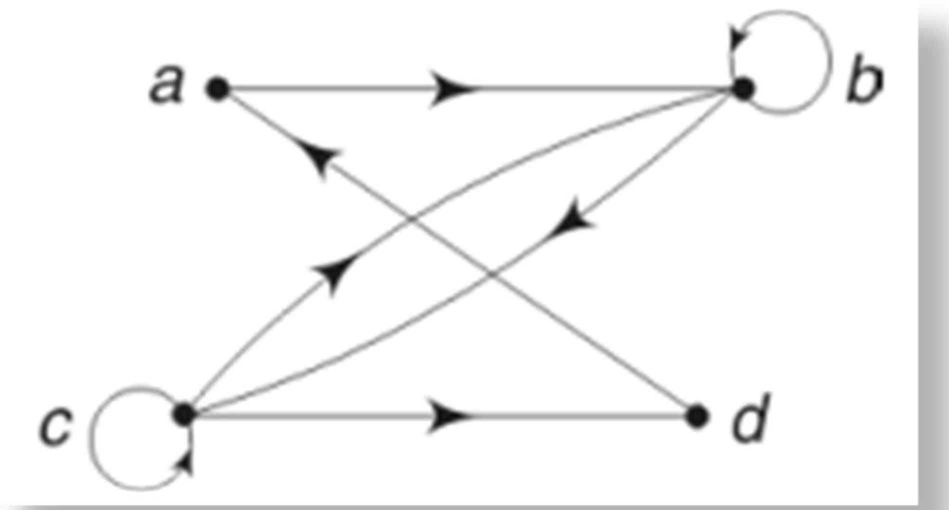
Adjacency Matrix

- Directed simple or multigraphs can also be represented by adjacency matrices.
- May not be symmetric.



$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Matrix representation of graphs

Adjacency Matrix :

- When **G** is a graph with n vertices $v_1, v_2, \dots v_n$, the matrix **A(or A_G) [a_{ij}],**

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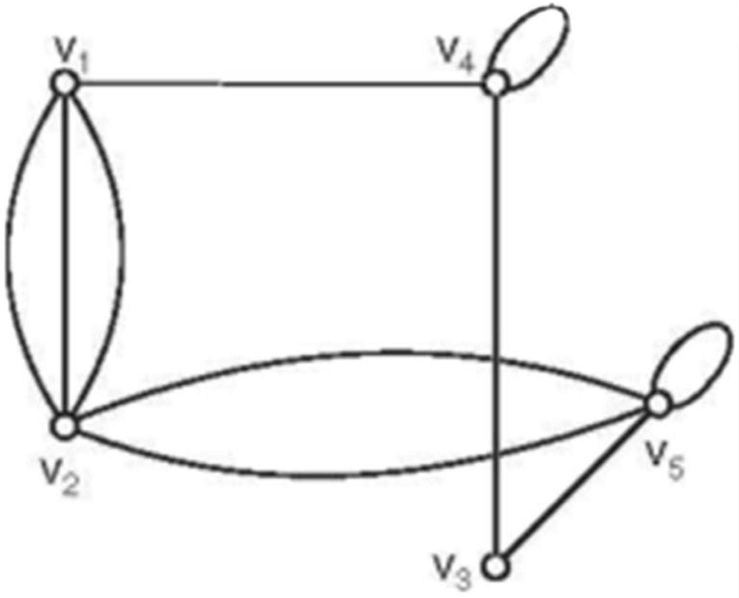

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Matrix representation of graphs

Adjacency Matrix

- Directed simple or multigraphs can also be represented by adjacency matrices.
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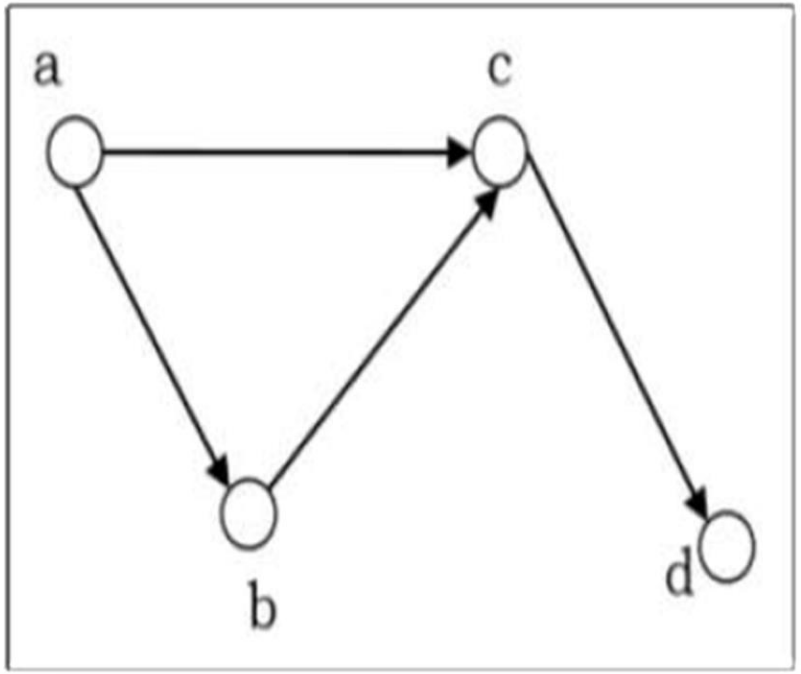


	a	b	c	d
a	0	1	1	0
b	0	0	1	0
c	0	0	0	1
d	0	0	0	0

Matrix representation of graphs

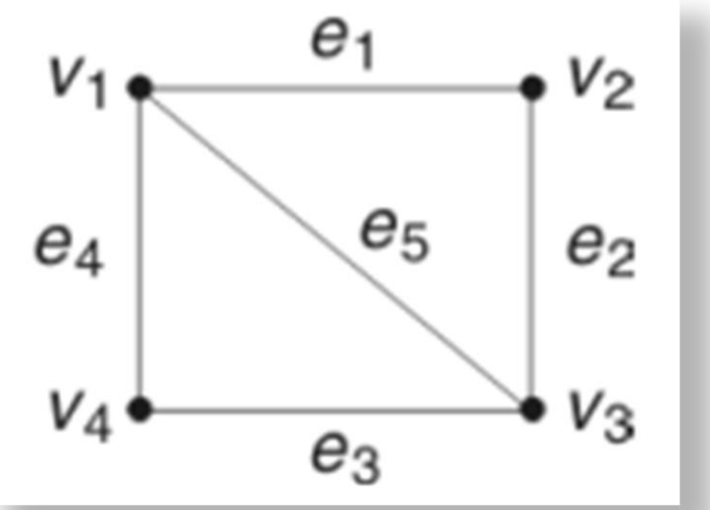
Incidence Matrix :

- When **G** is an undirected graph with n vertices $v_1, v_2, \dots v_n$ and m edges $e_1, e_2, \dots e_m$, the matrix $\mathbf{B} = [\boldsymbol{b_{ij}}]$,

$$\text{where, } b_{ij} = \begin{cases} 1, & \text{when } edge\ \boldsymbol{e_i} \text{ is incident } on\ \boldsymbol{v_i} \\ 0, & \text{otherwise} \end{cases}$$

Incidence matrix

- Each column of **B** contains exactly **two unit** entries.
- A row with all 0 entries corresponds to an **isolated vertex**.
- A row with a **single unit** entry corresponds to a **pendant vertex**.
- $\boldsymbol{deg}(\boldsymbol{v_i})$ is equal to the number of $1's$ in the i^{th} row.



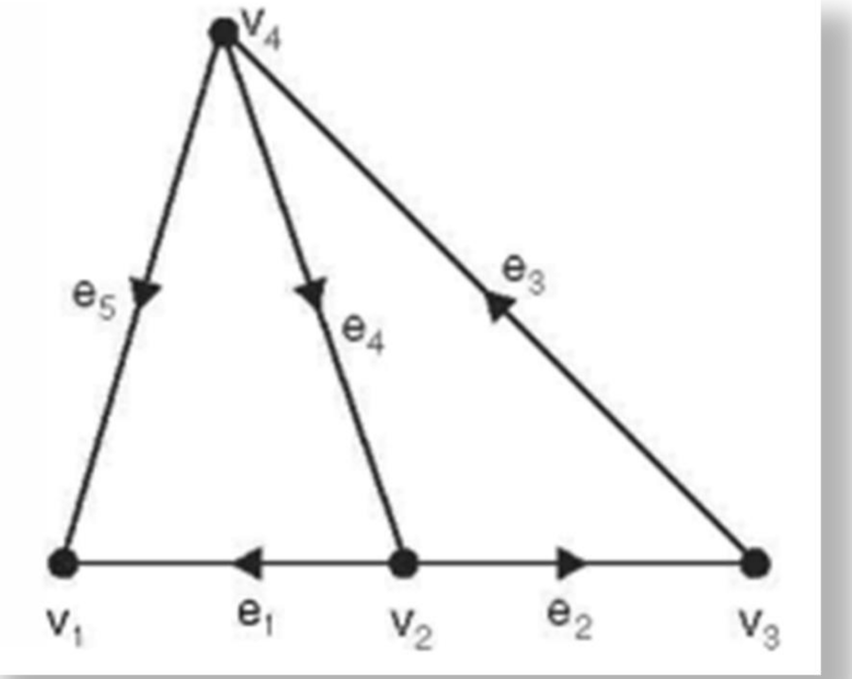
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Matrix representation of graphs

Incidence Matrix :

- When **G** is an Directed graph with n vertices $v_1, v_2, \ldots v_n$ and m edges $e_1, e_2, \ldots e_m$, the matrix $\mathbf{B} = [\boldsymbol{b_{ij}}]$,

$$\text{where, } b_{ij} = \begin{cases} 1, & \text{if } \boldsymbol{v_i} \textit{ is initial vertex of edge } \boldsymbol{e_j} \\ -1, & \text{if } \boldsymbol{v_i} \textit{ is final vertex of edge } \boldsymbol{e_j} \\ 0, & \text{if } \boldsymbol{v_i} \textit{ is not incident on edge } \boldsymbol{e_j} \end{cases}$$



$$M_I = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \end{array}$$

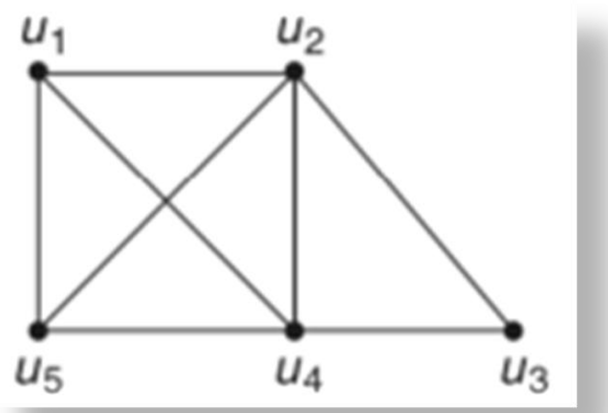
Isomorphism and Adjacency Matrices

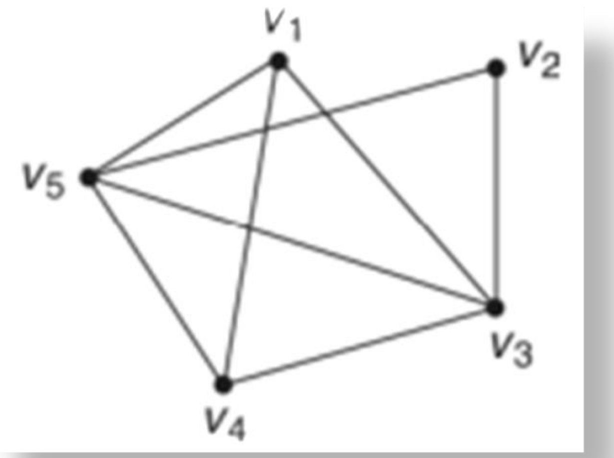
Result:

Two graphs are **isomorphic**, if and only if their vertices can be labeled in such a way that the corresponding **adjacency matrices are equal**.

Example:

Let us consider the two graphs





$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ v_1 \\ v_5 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_5 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

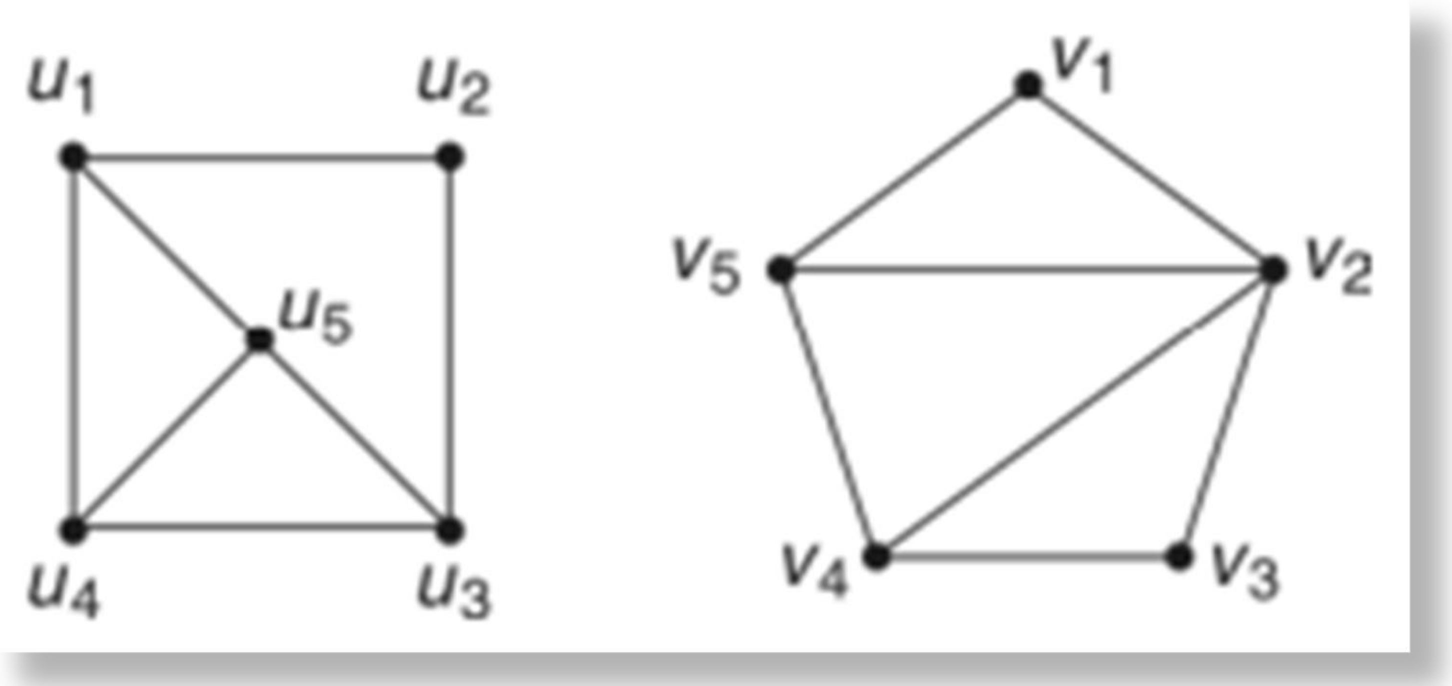
Isomorphism and Adjacency Matrices

Result:

Two graphs are **isomorphic**, if and only if their vertices can be labeled in such a way that the corresponding **adjacency matrices are equal**.

Example:

Though, there are equal number of vertices and equal number of edges in the two graphs, the degrees of vertices are not equal. Hence not isomorphic.



Paths, Cycles and Connectivity

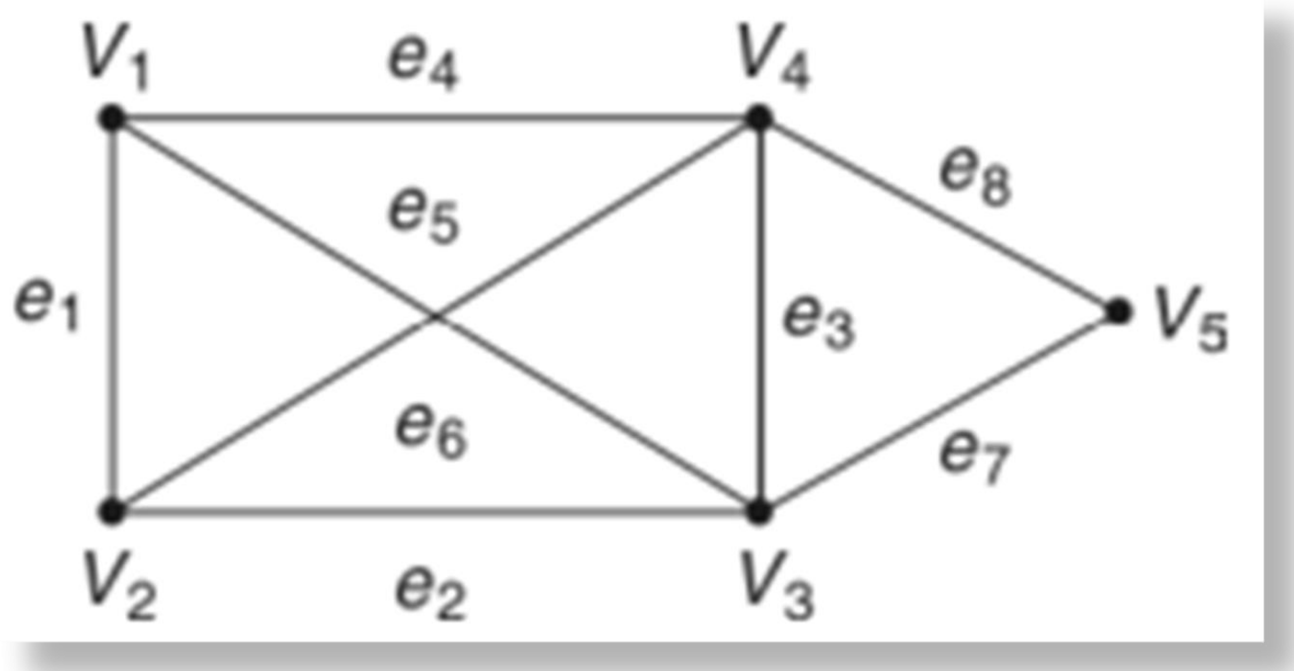
Path:

A path in a graph is a **finite alternating sequence of vertices and edges**, beginning and ending with vertices, such that each edge is incident on the vertices preceding and following it.

It the edges in a path are distinct, it is called a **simple path**.

Example:

- $V_1 e_1 V_2 e_2 V_3\ e_5\ V_1\ e_1 V_2$ is a path, since it contains the e_1 twice.
- $V_1\ e_4\ V_4\ e_6\ V_2\ e_2\ V_3\ e_7 V_5$ is a **simple path**, as no edge appears more than once. The **number of edges** in a path (simple or general) is called the **length** of the path.



Euler Path/Circuit

Euler Path:

An **Euler path** (or chain) through a graph is a path whose edge list contains **each edge** of the graph **exactly once**.

Result:

If a graph G has **more than two vertices of odd degree**, then there can be **no Euler path in G**.

Euler circuit:

An **Euler circuit (or cycle)** is a path through a graph, in which the initial vertex appears second time as the final vertex.

Euler Graph:

An **Euler graph** is a graph that possesses **an Euler circuit**. An Euler circuit uses **every edge exactly once** but **vertices may be repeated**.

Euler Path/Circuit

Euler circuit:

An **Euler circuit (or cycle)** is a path through a graph, in which the initial vertex appears second time as the final vertex.

Example:

- $V_1, V_2, V_3, V_5, V_2, V_4, V_7, V_{10}, V_6, V_3, V_9, V_6, V_4, V_{10}, V_8, V_5, V_9, V_8, V_1,$ is an Euler circuit.

Result:

An undirected graph possesses an Eulerian path iff it is connected and has **either zero** or **two vertices of odd degree**.

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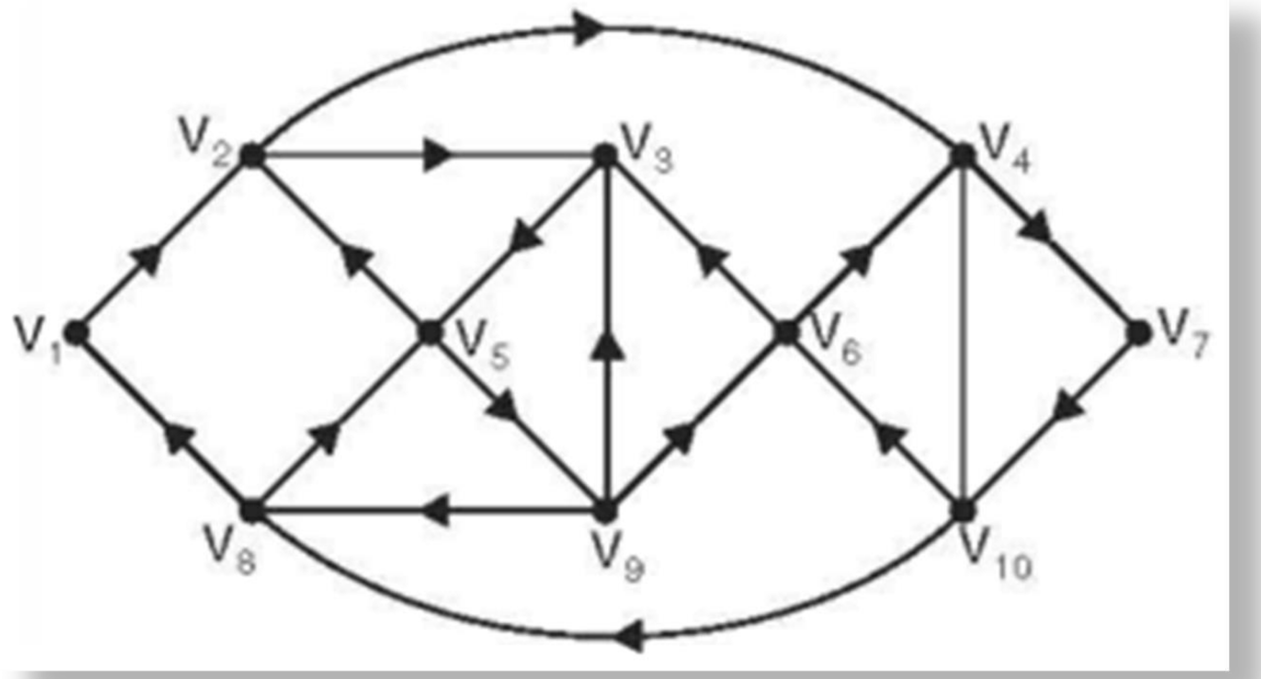
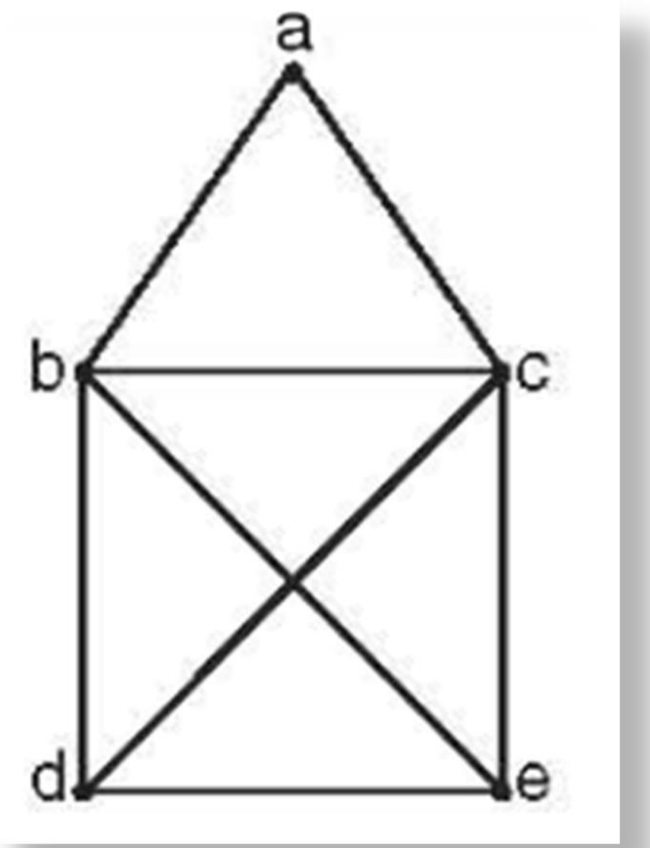
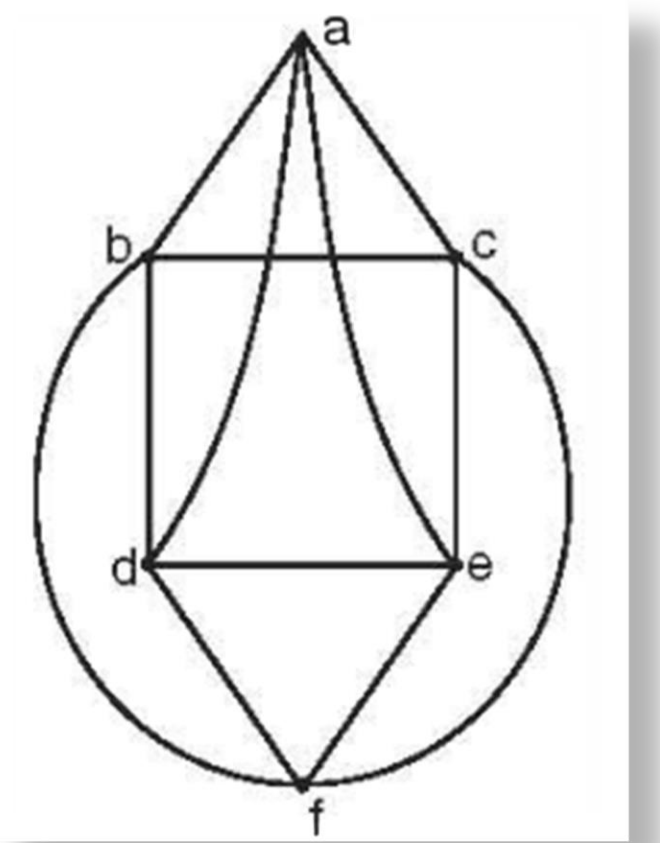
An undirected graph possesses an Eulerian path iff it is connected and has **either zero** or **two vertices of odd degree**.



Euler Path/Circuit

Result:
An undirected graph possesses an Eulerian path iff it is connected and has **either zero** or **two vertices of odd degree**.





Result:
An undirected graph possesses an Eulerian path iff it is connected and has **either zero** or **two vertices of odd degree**.

Hamiltonian path (or chain)

Hamiltonian path :

A **Hamiltonian path (or chain)** through a graph is a path whose vertex list contain **each vertex of the graph exactly once**, except if path is a circuit.

Hamiltonian circuit (or cycle) :

A **Hamiltonian circuit (or cycle)** is a path in which the initial vertex appears a second time as the final vertex.

Hamiltonian graph:

A Hamiltonian graph is a graph that possesses a **Hamiltonian path**. A Hamiltonian path uses **each vertex exactly** once but **edges may not be included**.

Result:

A graph G has a Hamilton circuit if $e \geq \frac{n^2 - 3n+6}{2}$, where n is the number of vertices and e the number of edges in G.

Hamiltonian path (or chain)

Results:

- A graph **G** has a Hamilton circuit if $e \geq \frac{n^2-3n+6}{2}$, where n is the number of vertices and e the number of edges in **G**.
- If a graph **G** has n vertices, then a Hamilton path in **G** must contain exactly $(n-1)$ edges and a Hamilton circuit in **G** must contain exactly n edges.
- In a Hamilton circuit, there cannot be more than three or more edges incident with one vertex. i.e., every vertex **v** in a Hamilton circuit will contain exactly **2** edges incident on **v**.

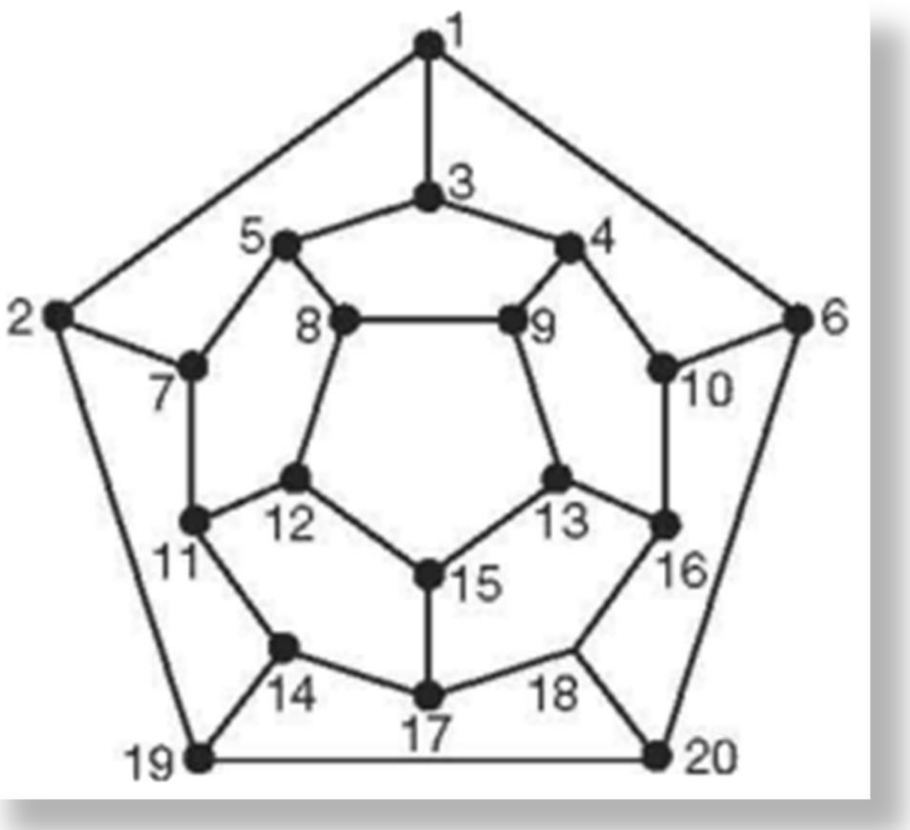
Remarks:

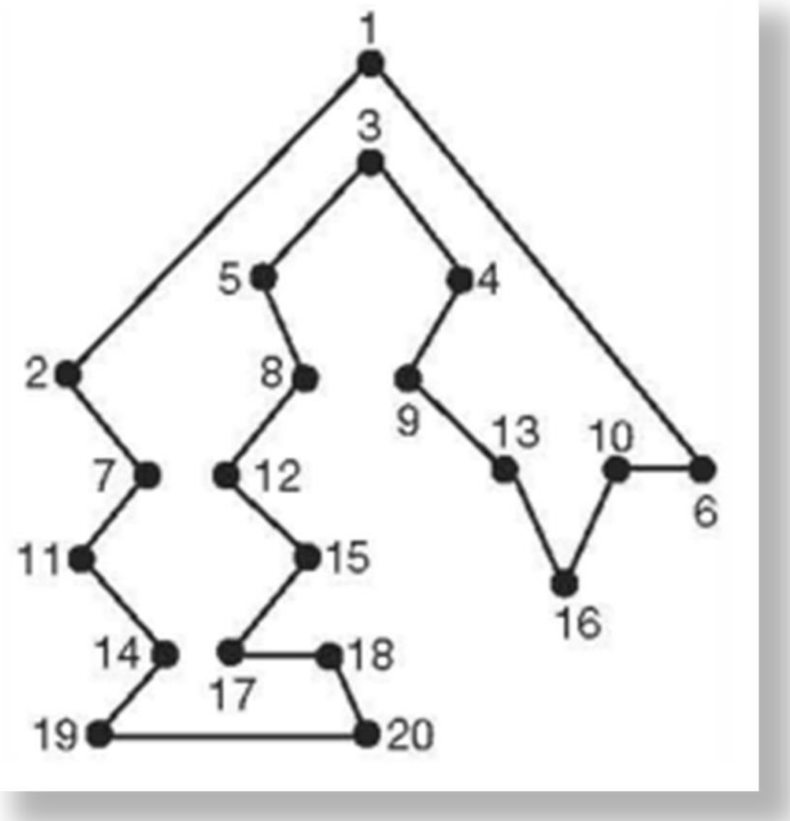
1) Multigraph **cannot** have Hamilton circuit.
2) Hamilton path, if it exists, is the **longest simple path** in a graph.
3) Each graph which has Hamilton cycle will have Hamilton path, the converse, however, is not true.
4) Every complete graph K_n is Hamilton for $n \geq 3$.
5) A Hamilton graph with n vertices must have atleast n edges.

Hamiltonian path (or chain)

Example:

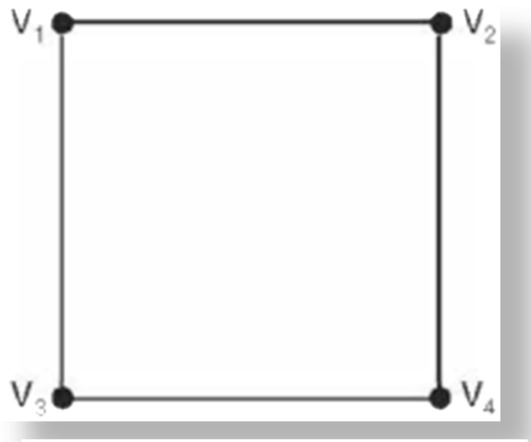
The given graph is a Hamiltonian graph. Determine Hamiltonian circuit for this graph.



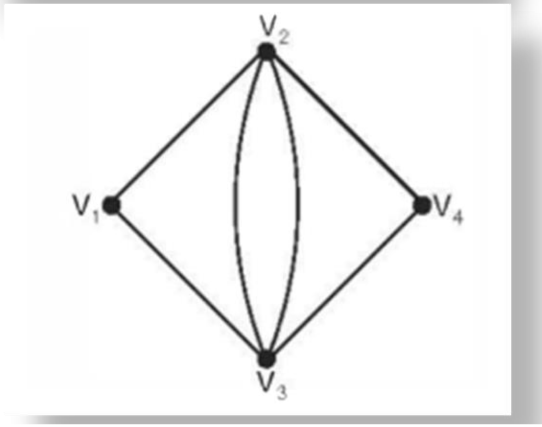


Hamiltonian/Eulerian Graphs

An example of a graph that has an Euler circuit which is also a Hamiltonian circuit.



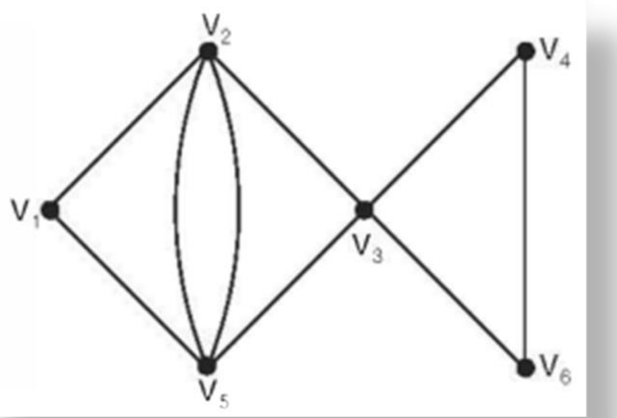
An example of a graph that has an Euler circuit and a Hamiltonian circuit, which are distinct.



An example of a graph which has an Euler circuit but not a Hamiltonian circuit.
The Euler circuit is:

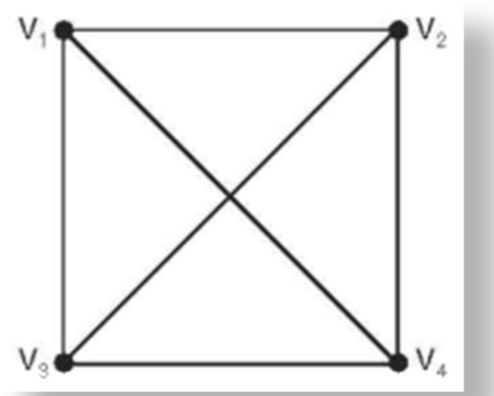
V1 V5 V2 V5 V3 V4 V6 V3 V2 V1

There is no Hamiltonian circuit.



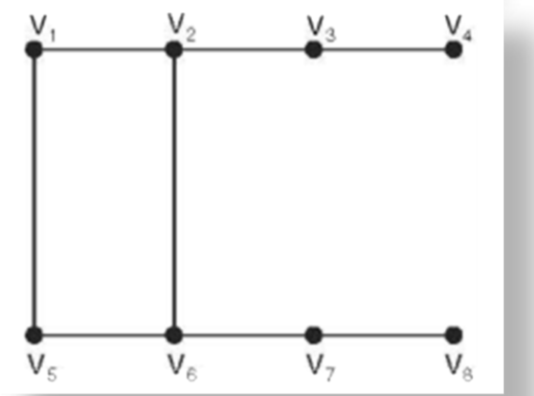
Hamiltonian/Eulerian Graphs

An example of a graph which has a Hamiltonian circuit but not an Euler circuit.

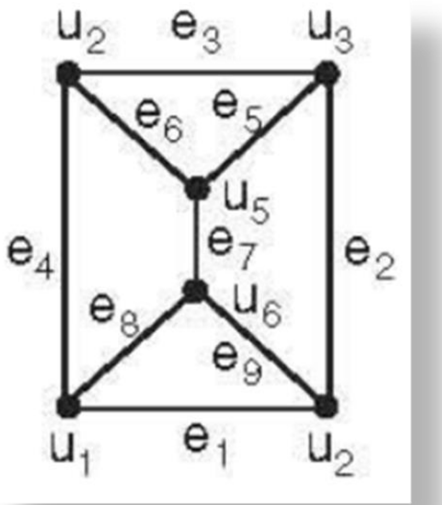


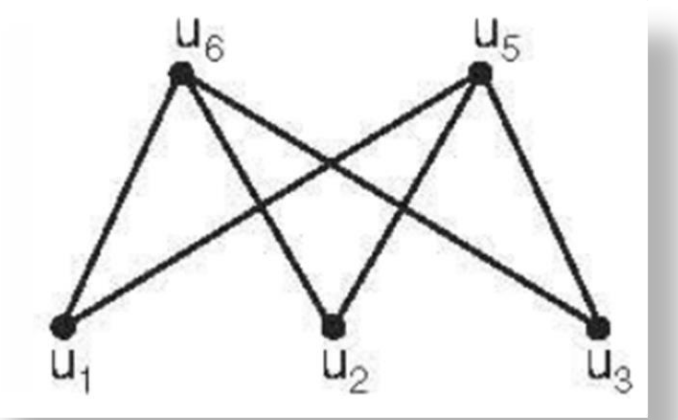
Hamiltonian circuit: V1 V2 V4 V3 V1
There is no Euler circuit.

An example of a graph that has neither an Euler circuit nor a Hamiltonian circuit.



Hamiltonian circuit and no Hamiltonian circuit

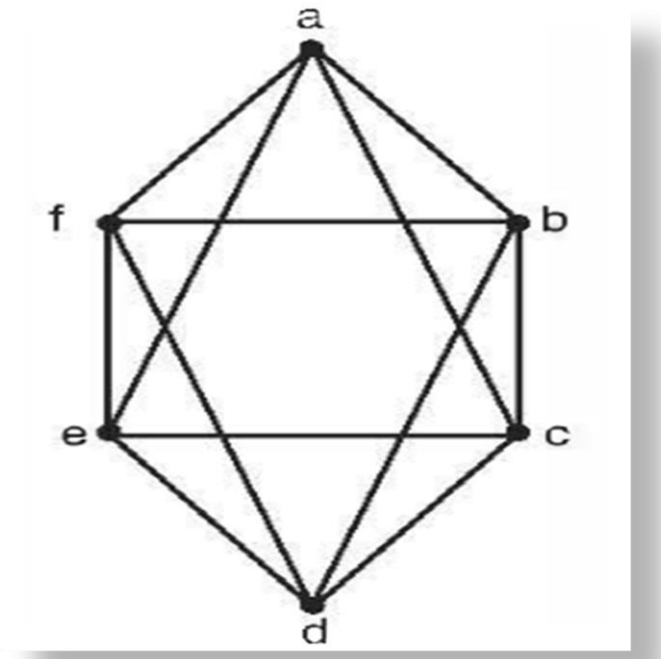




Hamiltonian/Eulerian Graphs

Consider the graph G

(a) Is it a complete graph?
(b) Is G connected and regular?
(c) Is it a planar graph? If so, find the number of regions.
(d) Is G Eulerian?



(a) Since the edge between a and d is not present in the given graph. It cannot be complete graph. (Every pair of vertices must be joined by an edge.)

(b) Graph is a **4-regular graph**.

Also it is **connected** because there is a path from every vertex to other.

(c) Graph is **planar** graph as it can be redrawn as which no two edges cross.

Here **V = 6, E = 12**

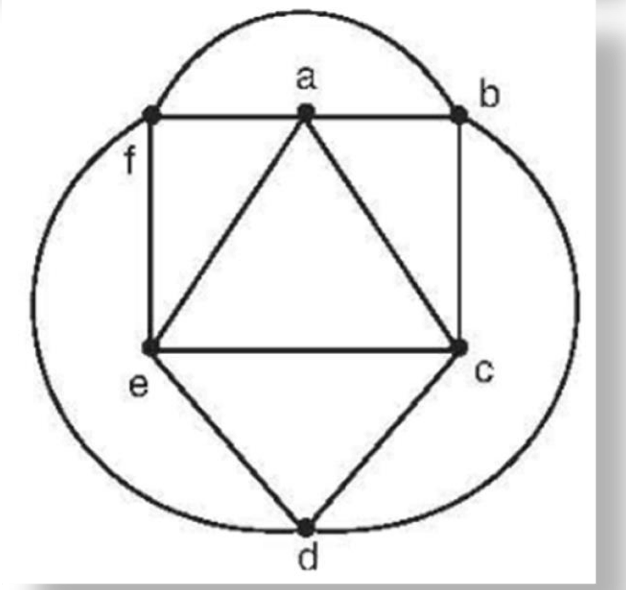
By Euler ' s theorem, V -E + R = 2

=> 6 - 12 + R = 2

⇒ **R= 8**

(d) Graph is 4-regular. every vertex is of degree 4 (even)

G has an Euler's circuit and hence G is an Eulerian graph.

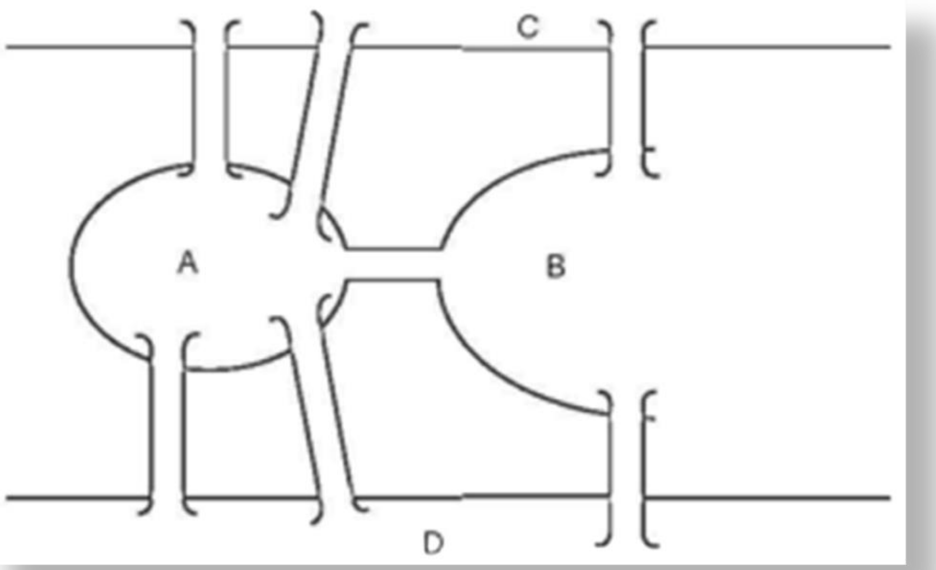


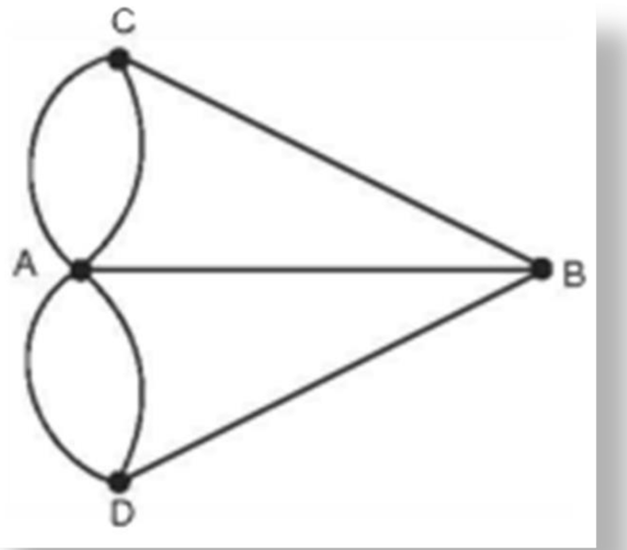
Konigsberg's Bridge Problem

It was not possible to cross each of the seven bridges once and only once in a walking tour.

A map of the Konigsberg is

Multigraph corresponding to the Konigsberg bridge problem has **four odd vertices**. Thus, **one cannot** walk through Konigsberg so that **each bridge is crossed exactly once**.





Euler Theorem:

A finite connected graph is Eulerian iff each vertex has even degree.

Planar Graphs

Planar Graphs:

A graph is said to be planar if it can be drawn in a plane so that no edges cross.

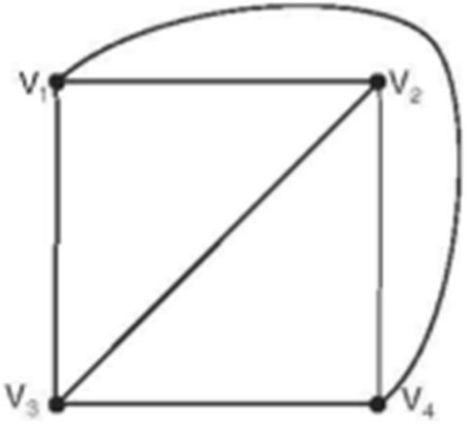
Result:

I. If a connected planar graph **G** has **e** edges and **R** regions, then $r \leq \frac{2}{3}e$.

II. If a connected planar graph **G** has **e** edges and **v** vertices, then $3v - e \geq 6$.

III. A complete graph K_n is planar if and only if $n < 5$.

IV. A complete **bipartite graph** $K_{m,n}$ is planar if and only if **m** < 3 or **n** > 3.



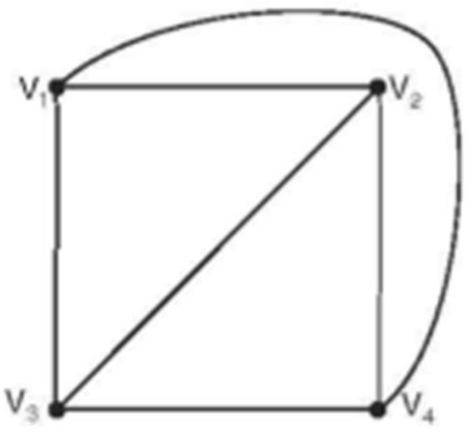
Planar Graphs

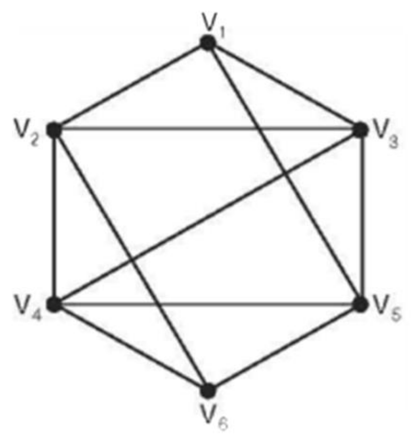
Planar Graphs:

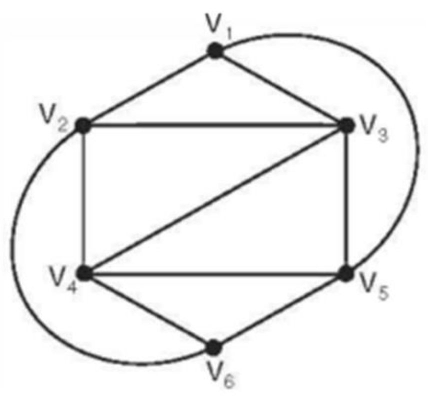
A graph is said to be planar if it can be drawn in a plane so that no edges cross.

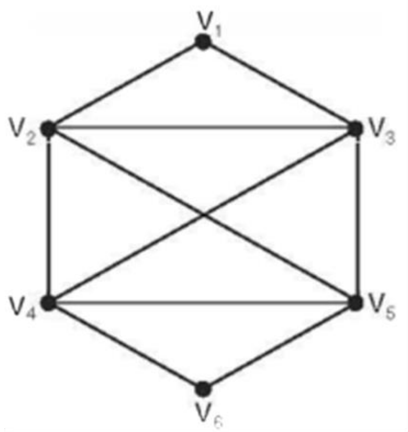
Result:

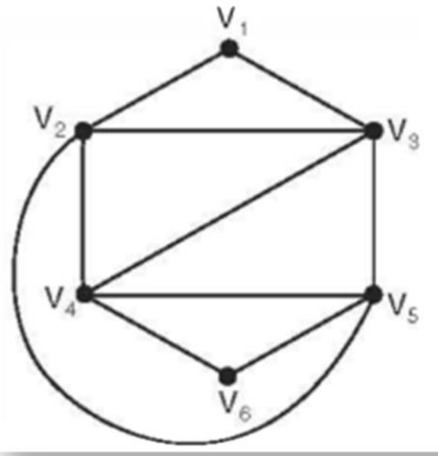
A planar and connected graph has a vertex of
degree less than or equal
to 5.











Travelling Salesman Problem

Problem:

A traveler needs to visit all the cities from a list, where distances between all the cities are known and each city should be visited just once. What is the shortest possible route that he visits each city exactly once and returns to the origin city?

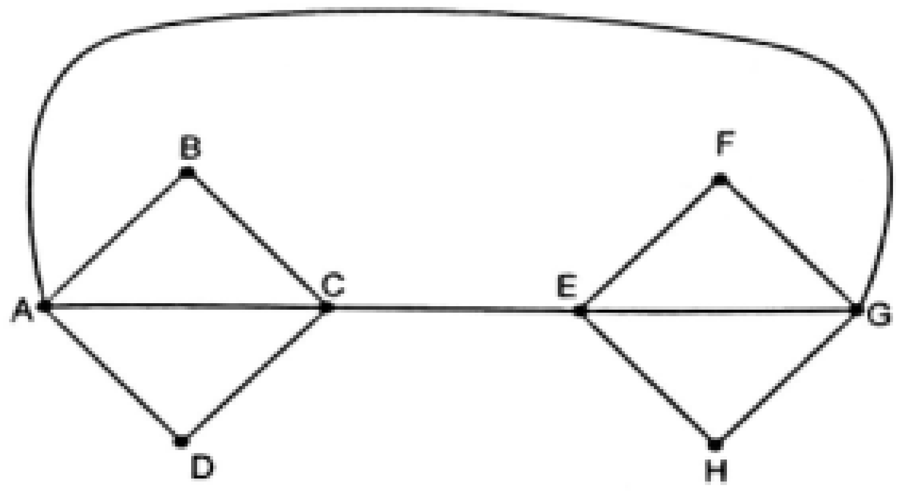
Result:

If there are "n" cities, then there will be $\frac{1}{2}(\mathbf{n}-\mathbf{1})!$ **Hamiltonian circuits**. However, the salesman problem is to determine that **Hamiltonian circuits** that has least sum of distance.

Fleury's algorithm for Euler path/circuit

Algorithm:

- The graph must have either zero or exactly 2 vertices of odd degree.
- Keep in mind **Don't burn bridges.**



Sr. No.	Current Path (P)	Next Edge	Description
1.	A	{A, B}	No edge is a cut edge choose any.
2.	A,B	{B, C}	Only one edge from B remains.
3.	A,B,C	{C, D}	No edge from C is a cut edge. So choose any.
4.	A,B,C,D	{D, A}	Only one edge from D remains.
5.	A,B,C,D,A	{A, C}	Neither AG or AC is cut edge. Chose any.
6.	A,B,C,D,A,C	{C, E}	Only one edge from C remains.
7.	A,B,C,D,A,C,E	{E, F}	No edge from E is cut edge So choose any.
8.	A,B,C,D,A,C,E,F	{F, G}	Only one edge from F remains.
9.	A,B,C,D,A,C,E,F,G	{G, H}	GA is cut edge so choose GH or GE.
10.	A,B,C,D,A,C,E,F,G,H	{H, E}	Only one edge from H remains
11.	A,B,C,D,A,C,E,F,G,H,E	{E, G}	Only one edge from E remains
12.	A,B,C,D,A,C,E,F,G,H,E,G	{G, A}	Only one edge from G remains
13.	A,B,C,D,A,C,E,F,G,H,E,G,A	No edge	

The required Euler circuit is

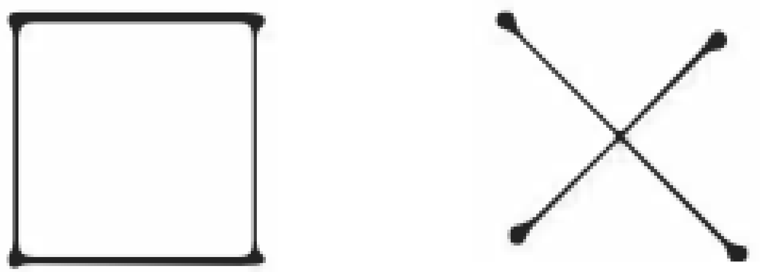
A, B, C, D, A, C, E, F, G, H, E, G, A.

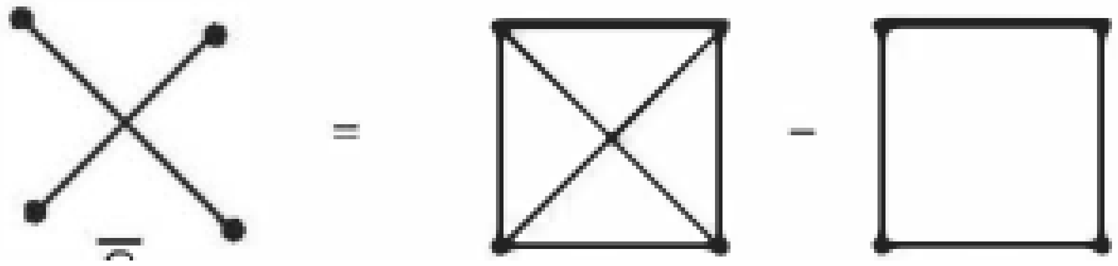
Complement of a graph

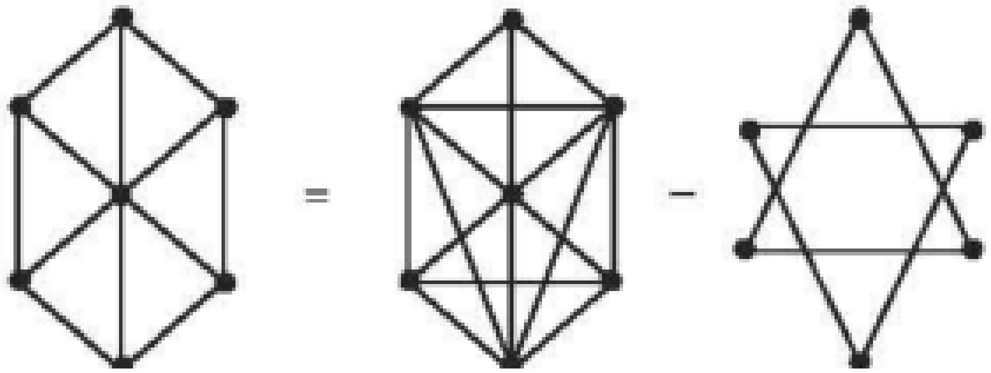
Complement of a graph:

Let G = (V, E) be a given graph. A graph $G' = (V', E')$ is said to be complement of G = (V, E). If $V' = V$ and E' does not contain edges of of E. i.e., edges in E are join of those pairs of vertices which are not joined in G.

Example:





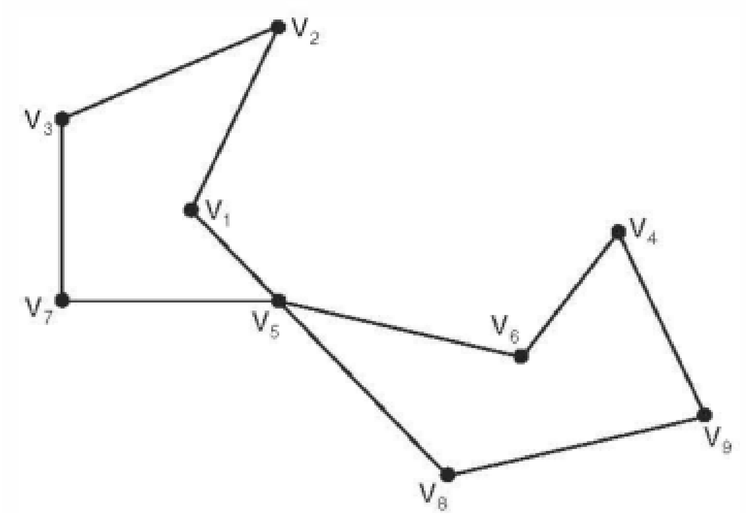


Cut points/Cut vertices

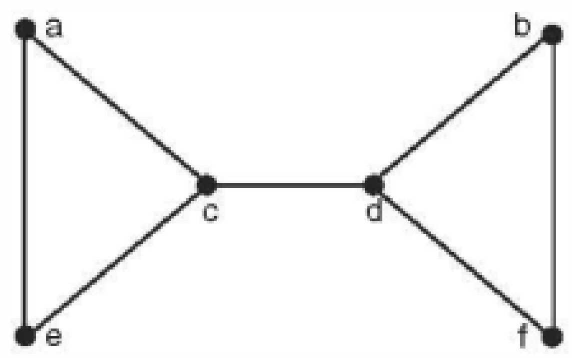
Cut Vertex:

Let **G = (V, E)** be a given graph. A cut point for a graph **G**, is a vertex v such that **G**$-v$ has more connected components than **G** or disconnected.

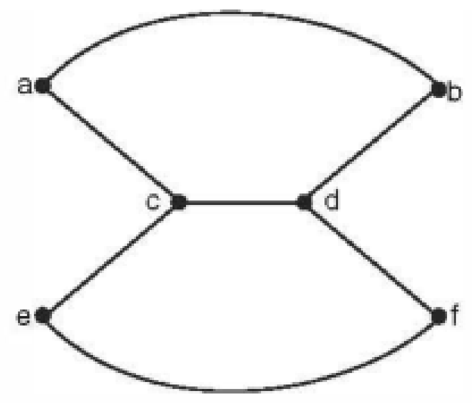
Example:



V_5 is a cut vertex



2 cut vertex



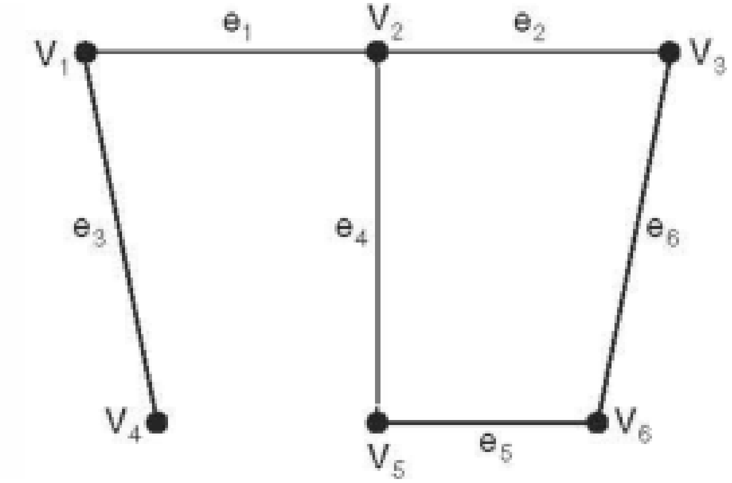
No cut vertex

Bridge (Cut edges)

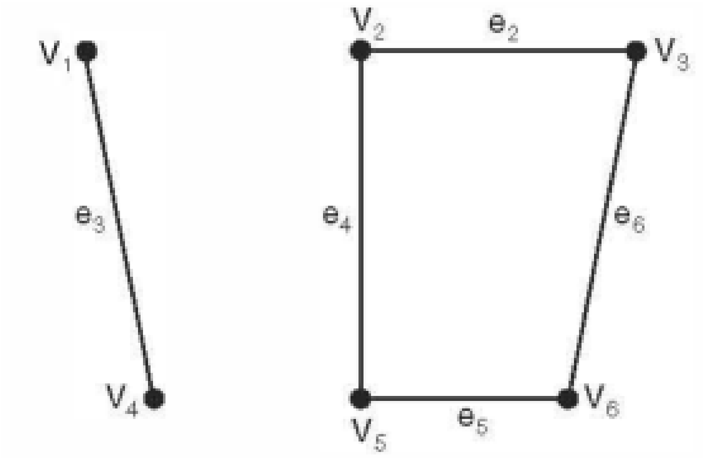
Cut Edge:

Let **G = (V, E)** be a given graph. A bridge for a graph **G**, is an edge **e** such that **G**$-e$ has more connected components than **G** or disconnected.

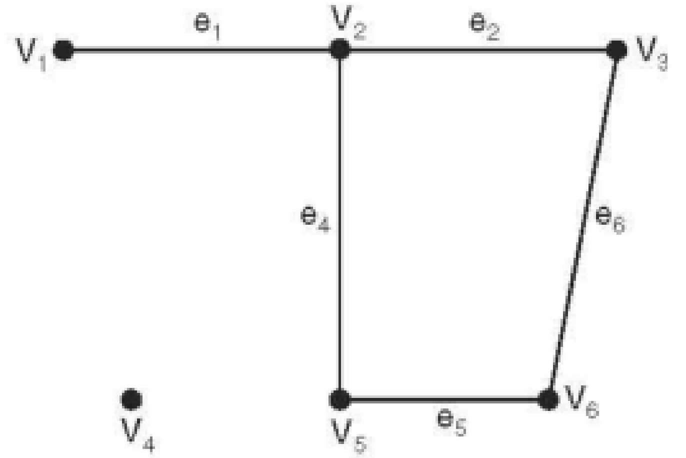
Example:



G has **1** connected component



G$-e_1$
has **2** connected component



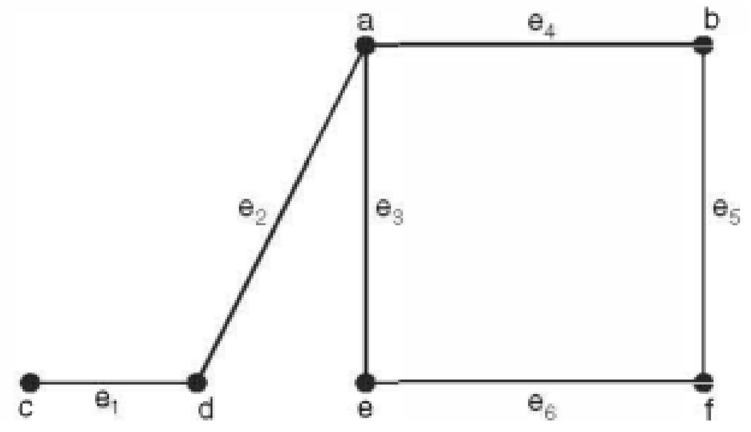
G$-e_3$
has **2** connected component

Bridge (Cut edges)

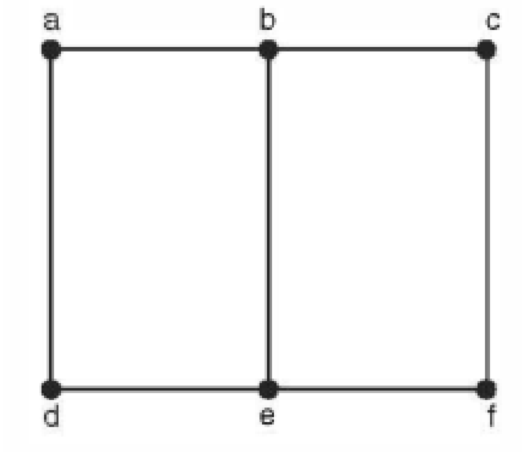
Cut Edge:

Let **G = (V, E)** be a given graph. A bridge for a graph **G**, is an edge **e** such that **G**$-e$ has more connected components than **G** or disconnected.

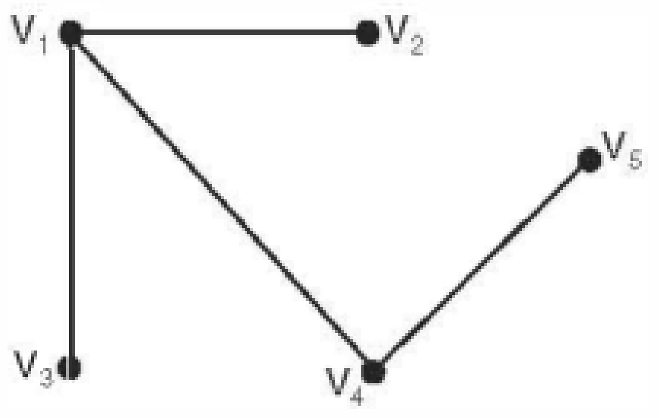
Example:



G has **2** Bridges



G has ***no*** Bridge



G every edge is a Bridge

Edges and Vertices

Question:

A graph G has 5 vertices, 2 of degree 3 and 3 of degree 2. Find the number of edges.

Solution:

$$\sum \deg(v_i) = 2(\textit{number of edges})$$
$$2 \times 3 + 3 \times 2 = 12$$
$$2e = 12 \Rightarrow e = 6.$$

Question:

How many nodes (vertices) are required to construct a graph with exactly 6 edges in which each node is of degree 2?

Solution:

$$\sum \deg(v_i) = 2(\textit{number of edges})$$
$$2 \times n = 2 \times 6$$
$$n = 6$$

Edges and Vertices

Question:

A graph G has 16 edges and all vertices of G are of degree 2. Find the number of vertices.

Solution:

$$\sum \mathbf{deg}(\boldsymbol{v_i}) = \mathbf{2}(\boldsymbol{number\ of\ edges})$$
$$\sum deg(v_i) = 2 \times 16$$
$$2 \times \mathrm{n} = 32$$
$$\mathrm{n} = 16$$

Question:

A graph G has 21 edges, 3 vertices of degree 4 and other vertices are of degree 3. Find the number of vertices in G.

Solution:

$$\sum \mathbf{deg}(\boldsymbol{v_i}) = \mathbf{2}(\boldsymbol{number\ of\ edges})$$
$$4 + 4 + 4 + 3(n - 3) = 2 \times 21$$
$$3n + 3 = 42$$
$$\mathrm{n} = 13$$

Edges and Vertices

Question:

Show that there does not exist a graph with 5 vertices with degrees 1, 3, 4, 2, 3 respectively.

Solution:

$$\sum \deg(v_i) = 2(\boldsymbol{number\ of\ edges})$$

$$1 + 3 + 4 + 2 + 3 = 2 \times e$$

$$e = \frac{13}{2}$$

The number of edges should be whole numbers.

Question:

Can there be a graph with 8 vertices and 29 edges?

Solution:

$$\textbf{Maximum number of edges} = \frac{n(n-1)}{2}$$

$$= \frac{8 \times 7}{2}$$

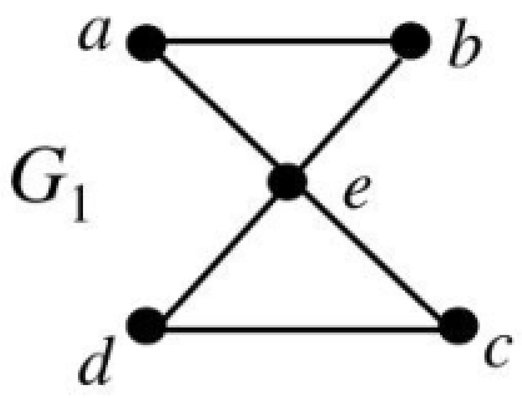
$$e = 28$$

Hence it is possible.

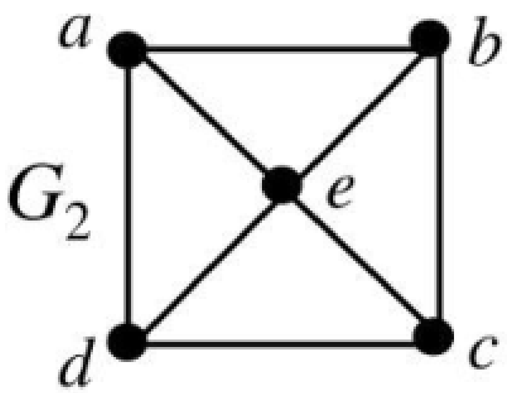
Euler Path/ Circuit

Question:

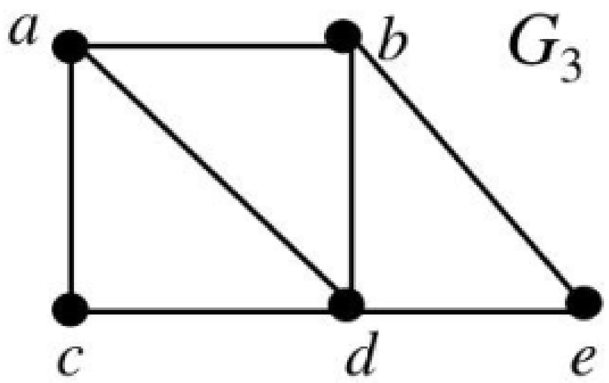
Which of the following have a Euler Path/ Circuit?



Euler circuit



none

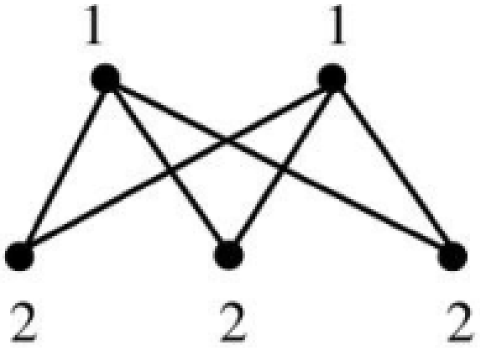


Euler path

Chromatic number

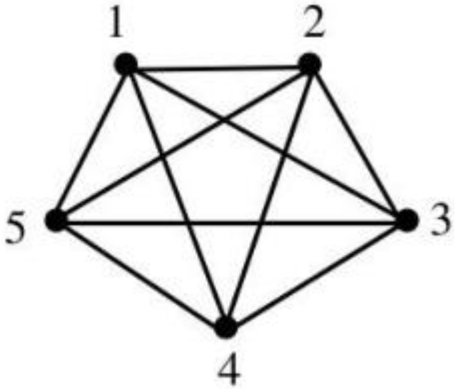
Chromatic number:

The Charomatic number of a graph is the least number of colors needed for coloring of this graph $\boldsymbol{\chi(G)}$.



$\chi(K_{2,3}) = 2.$

$\chi(K_{m,n}) = 2$



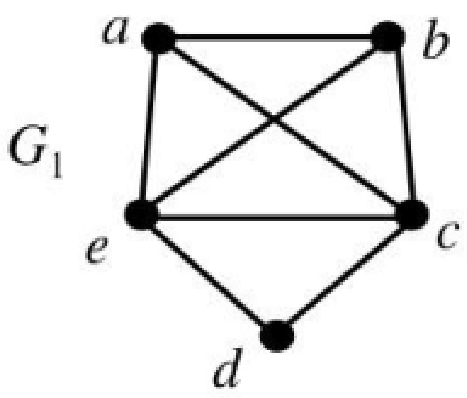
$\chi(K_5)=5$

$\chi(K_n)=n$

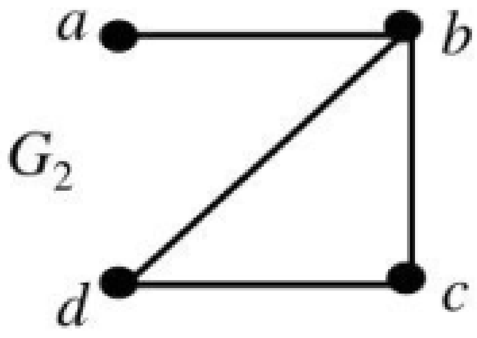
Hamiltonian Path/ Circuit

Question:

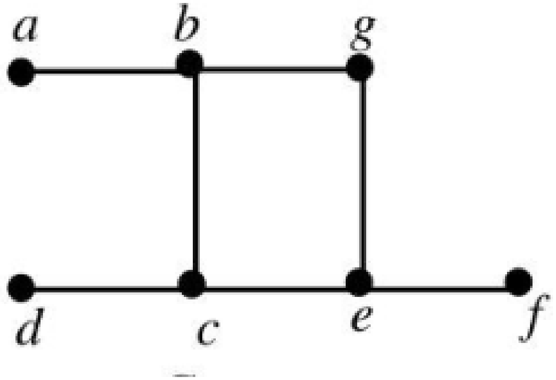
Which of the following have a Hamiltonian Path/ Circuit?



Hamilton circuit



Hamilton path



none