FIBONACCI SEQUENCE

Group 8

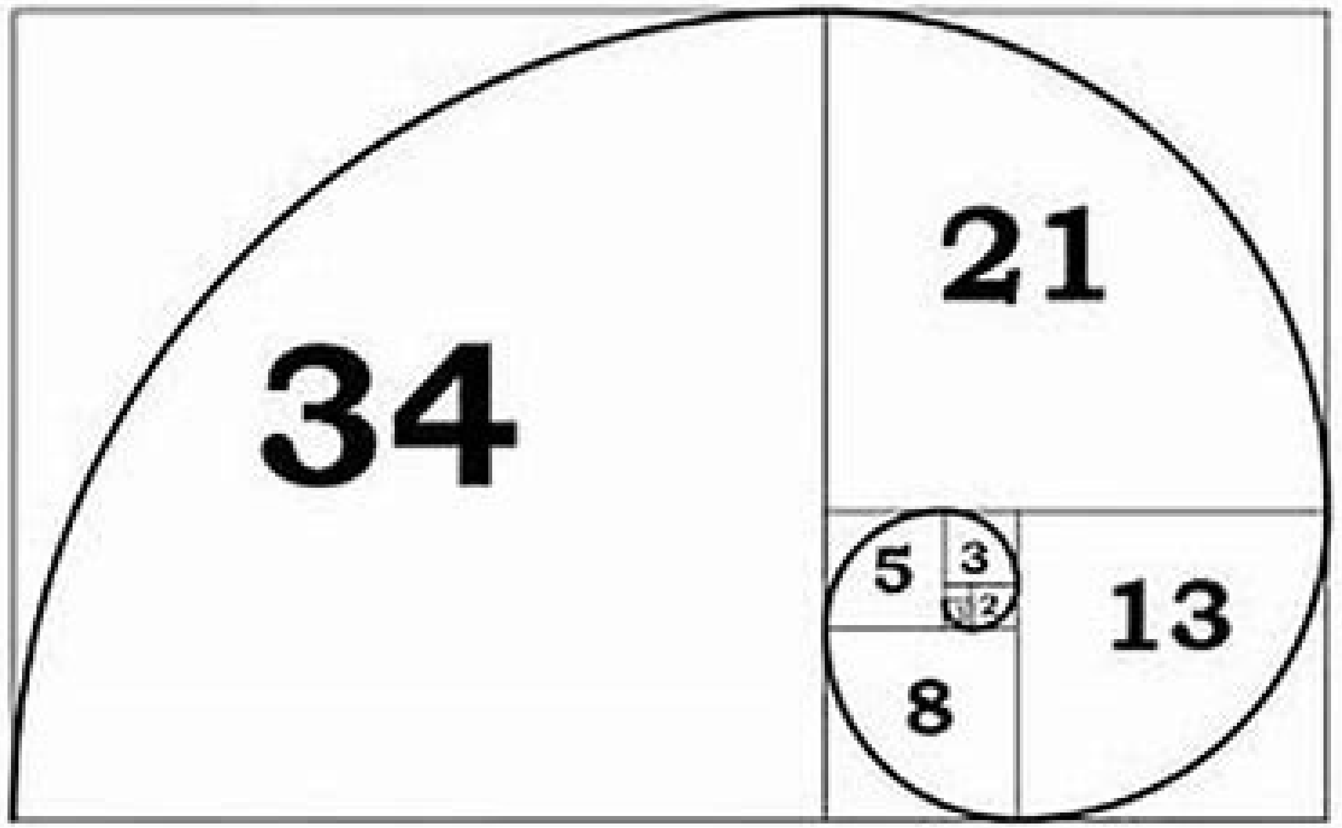
Introduction

THE FIBONACCI SEQUENCE IS A SERIES OF NUMBERS WHERE EACH NUMBER IS THE SUM OF THE TWO PRECEDING ONES.

MATHEMATICALLY:

F(N) = F(N-1) + F(N-2) WITH INITIAL VALUES F(0) =0 AND F(1) =1

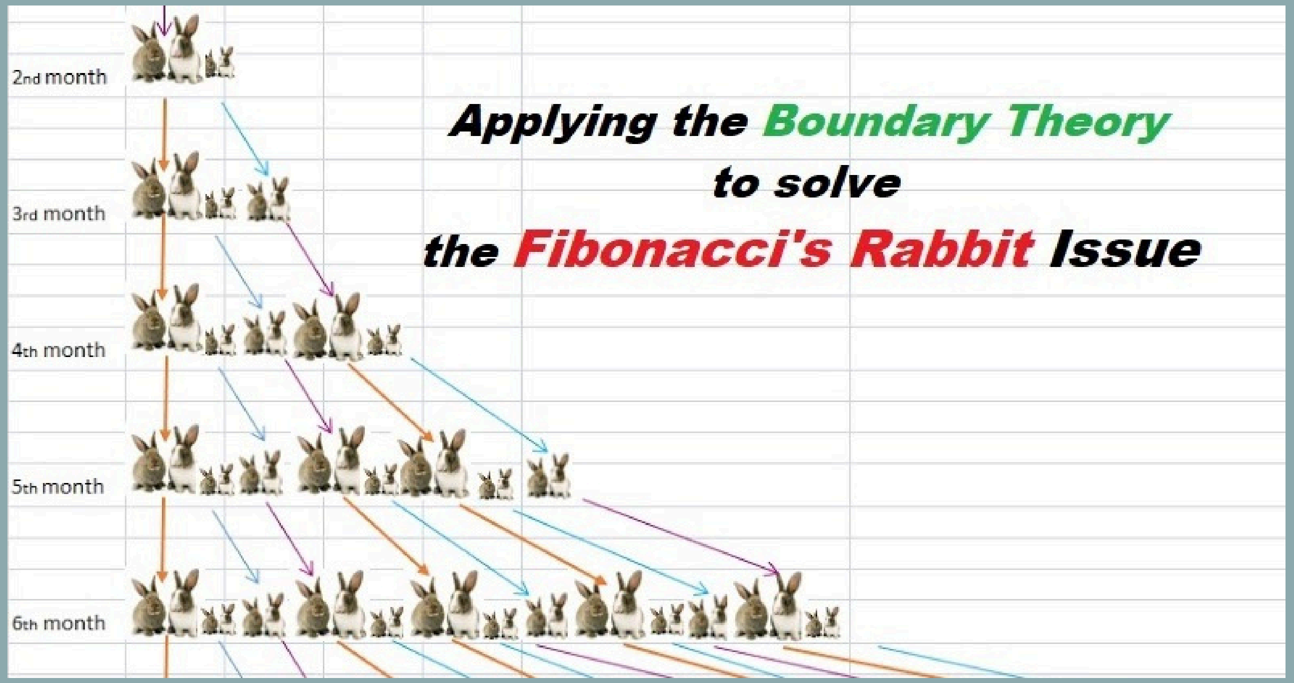
THE SEQUENCE BEGINS: 0, 1, 1, 2, 3, 5, 8, 13, ETC.



ORIGINS AND HISTORICAL BACKGROUND

FIBONACCI AND THE RABBIT PROBLEM

THE SEQUENCE IS NAMED AFTER LEONARDO OF PISA, OR FIBONACCI, WHO INTRODUCED IT IN HIS 1202 BOOK LIBER ABACI. HE USED IT TO MODEL RABBIT POPULATION GROWTH, LEADING TO THE DISCOVERY OF A RECURRING PATTERN IN NATURE.



SIGNIFICANCE IN MATHEMATICS

SINCE ITS DISCOVERY, THE FIBONACCI SEQUENCE HAS FASCINATED MATHEMATICIANS AND FOUND RELEVANCE ACROSS FIELDS LIKE BIOLOGY, ART, FINANCE, AND COMPUTING.

Notable Properties and Patterns in Fibonacci Numbers

- **PROPERTIES OF DIVISIBILITY**

EVERY THIRD FIBONACCI NUMBER IS EVEN, AND EVERY FOURTH NUMBER IS A MULTIPLE OF THREE, HIGHLIGHTING THE INHERENT STRUCTURE WITHIN THE SEQUENCE.

- **SUM OF FIBONACCI NUMBERS**

THE SUM OF THE FIRST N FIBONACCI NUMBERS IS EQUAL TO THE (N + 2)TH FIBONACCI NUMBER MINUS 1.

MATHEMATICAL REPRESENTATION

- **RECURSIVE FORMULA**

THE FIBONACCI SEQUENCE IS OFTEN DEFINED RECURSIVELY:

$$F(N) = F(N-1) + F(N-2)$$

- **CLOSED-FORM SOLUTION (BINET'S FORMULA)**

A NON-RECURSIVE FORMULA KNOWN AS BINET'S FORMULA APPROXIMATES THE NTH FIBONACCI NUMBER:

$$F(N) = (\Phi^{n}-(1-\Phi^{n}))/\sqrt{5}$$

WHERE Ø (PHI) REPRESENTS THE GOLDEN RATIO, APPROXIMATELY 1.618.

CASES

- **BASE CASE: F(0)=0,F(1)=1**
- **RECURSIVE CASE: F(N)=F(N-1)+F(N-2)**

RECURSIVE CASE

Proving By Induction:

Exponential growth. Since the Fibonacci numbers are designed to be a simple model of population growth, it is natural to ask how quickly they grow with n. We'll say they grow *exponentially* if we can find some real number $r > 1$ so that $f_n \geq r^n$ for all n.

The following claim shows that they indeed grow exponentially. We'll first present this with the value of r chosen as if "by magic," then come back to suggest how one might have come up with it.

- Claim: Let $r = \frac{1+\sqrt{5}}{2} \approx 1.62$, so that r satisfies $r^2 = r + 1$. Then $f_n \geq r^{n-2}$.

Given the fact that each Fibonacci number is defined in terms of smaller ones, it's a situation ideally designed for induction.

Proof of Claim: First, the statement is saying $\forall n \geq 1 : P(n)$, where $P(n)$ denotes "$f_n > r^{n-2}$." As with all uses of induction, our proof will have two parts.

RECURSIVE CASE

Proving By Induction:

STEP 1: BASE CASES

- First, the basis. $P(1)$ is true because $f_1 = 1$ while $r^{1-2} = r^{-1} \leq 1$. While we're at it, it turns out be convenient to handle Actually, we notice that f_2 is defined directly to be equal to 1, so it's natural to handle $P(2)$ directly here. $P(2)$ is true because $f_2 = 1$ and $r^{2-2} = r^0 = 1$.

RECURSIVE CASE

Proving By Induction:

STEP 2: INDUCTIVE STEP

- Next, the induction step, for a fixed $n > 1$. (Actually, since we've already done $n = 2$, we can assume $n > 2$ from here on.) The induction hypothesis is that $P(1), P(2), \ldots, P(n)$ are all true. We assume this and try to show $P(n+1)$. That is, we want to show $f_{n+1} \geq r^{n-1}$.

  So consider f_{n+1} and write

  $$f_{n+1} = f_n + f_{n-1}. \tag{1}$$

  We now use the induction hypothesis, and particularly $f_n \geq r^{n-2}$ and $f_{n-1} \geq r^{n-3}$. Substituting these inequalities into line (1), we get

  $$f_{n+1} \geq r^{n-2} + r^{n-3} \tag{2}$$

  Factoring out a common term of r^{n-3} from line (2), we get

  $$f_{n+1} \geq r^{n-3}(r+1). \tag{3}$$

  Now we use the the fact that we've chosen r so that

  $$r^2 = r + 1. \tag{4}$$

  Plugging this into line (3), we get

  $$f_{n+1} \geq r^{n-3}(r+1) = r^{n-3} \cdot r^2 = r^{n-1}, \tag{5}$$

  which is exactly the statement of $P(n+1)$ that we wanted to prove. This concludes the proof.

An Exact Formula for the Fibonacci Numbers

WE'LL DISCUSS SOMETHING THAT'S A LITTLE MORE COMPLICATED, BUT IT SHOWS HOW REASONING INDUCTION CAN LEAD TO SOME NON-OBVIOUS DISCOVERIES. SPECIFICALLY, WE WILL USE IT TO COME UP WITH AN EXACT FORMULA FOR THE FIBONACCI NUMBERS, WRITING FN DIRECTLY IN TERMS OF N.

SO INSTEAD OF F(N) = R^N−2 , TRY PROVING FN = A(R^N) FOR SOME VALUE OF A YET TO BE DETERMINED. (NOTE THAT R^N−2 IS JUST AR^N FOR THE PARTICULAR CHOICE A = R^−2.) COULD THERE BE A VALUE OF A THAT WORKS? UNFORTUNATELY, NO. WE'D NEED TO HAVE 1 = F1 = AR AND 1 = F2 = AR^2. BUT BY THE DEFINING PROPERTY OF R, WE HAVE 1 = F2 = AR^2 = A(R +1) = AR +A. THUS WE HAVE:

$$\begin{aligned} 1 &= ar \\ 1 &= ar + a \end{aligned}$$

THESE 2 EQUATIONS CANNOT BE SATISFIED BY ANY VALUE OF A

But there's one more trick we can still deploy. $x^2 = x + 1$ is a quadratic equation, and it has *two* solutions: $r = \frac{1+\sqrt{5}}{2}$ and $s = \frac{1-\sqrt{5}}{2}$. Any statement $P(n)$ of the form $f_n = ar^n + bs^n$ (for any choices of a and b) would work just fine in the induction step: once we assume $P(1), \ldots, P(n)$ are all true, we can write

$$\begin{aligned}
f_{n+1} &= f_n + f_{n-1} \\
&= ar^n + bs^n + ar^{n-1} + bs^{n-1} \\
&= ar^n + ar^{n-1} + bs^n + bs^{n-1} \\
&= ar^{n-1}(r+1) + bs^{n-1}(s+1) \\
&= ar^{n-1} \cdot r^2 + bs^{n-1} \cdot s^2 \\
&= ar^{n+1} + bs^{n+1},
\end{aligned}$$

where we used the induction hypothesis to go from the first line to the second, and we used the property of r and s that $r^2 = r + 1$ and $s^2 = s + 1$ to go from the fourth line to the fifth. The last line is exactly the statement of $P(n+1)$.

So now we just need to see if the as-yet undetermined constants a and b can be chosen so that the base cases $P(1)$ and $P(2)$ work. To make these work, we need

$$\begin{aligned}
1 &= f_1 = ar + bs \\
1 &= f_2 = ar^2 + bs^2 = a(r+1) + b(s+1)
\end{aligned}$$

This is the Binet’s formula!

Solving these two equations in the two unknowns a and b, we find that there is a solution: $a = 1/\sqrt{5}$ and $b = -1/\sqrt{5}$. Thus, we arrive at a formula for the Fibonacci numbers:

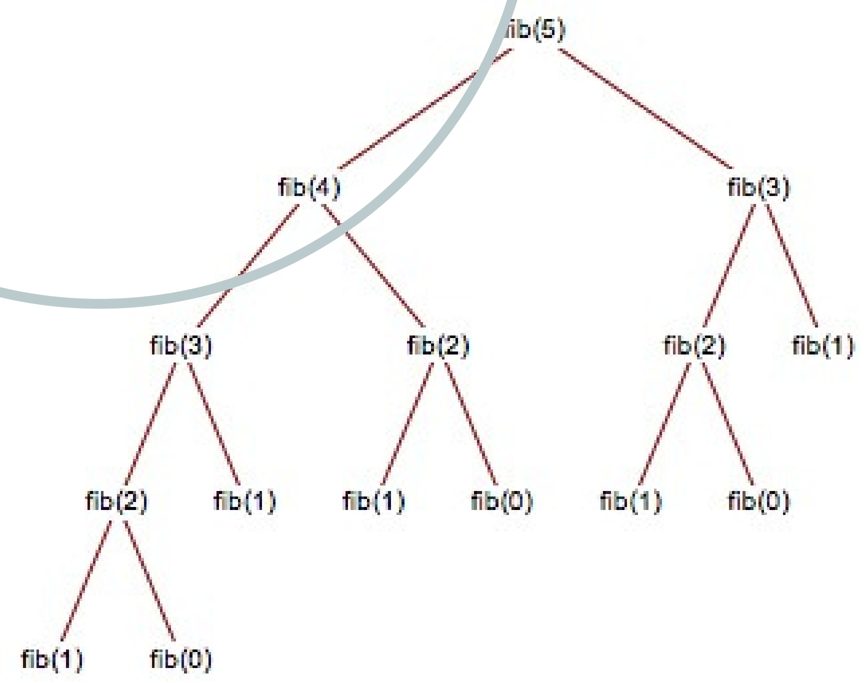
$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Fibonacci Binary Trees: Time Complexity and Levels

1. TIME COMPLEXITY AT EACH LEVEL:

AT EACH LEVEL OF A FIBONACCI BINARY TREE, THE NUMBER OF NODES DOUBLES:

- LEVEL 0: 1 NODE (ROOT, F(N)).
- LEVEL 1: 2 NODES (F(N-1), F(N-2)).
- LEVEL 2: 4 NODES (F(N-2), F(N-3), F(N-3), F(N-4)).
- LEVEL K: 2^K NODES.



WORK PER NODE IS CONSTANT, SO THE TOTAL WORK AT LEVEL K IS PROPORTIONAL TO THE NUMBER OF NODES:

WORK AT LEVEL K: O(2^K).

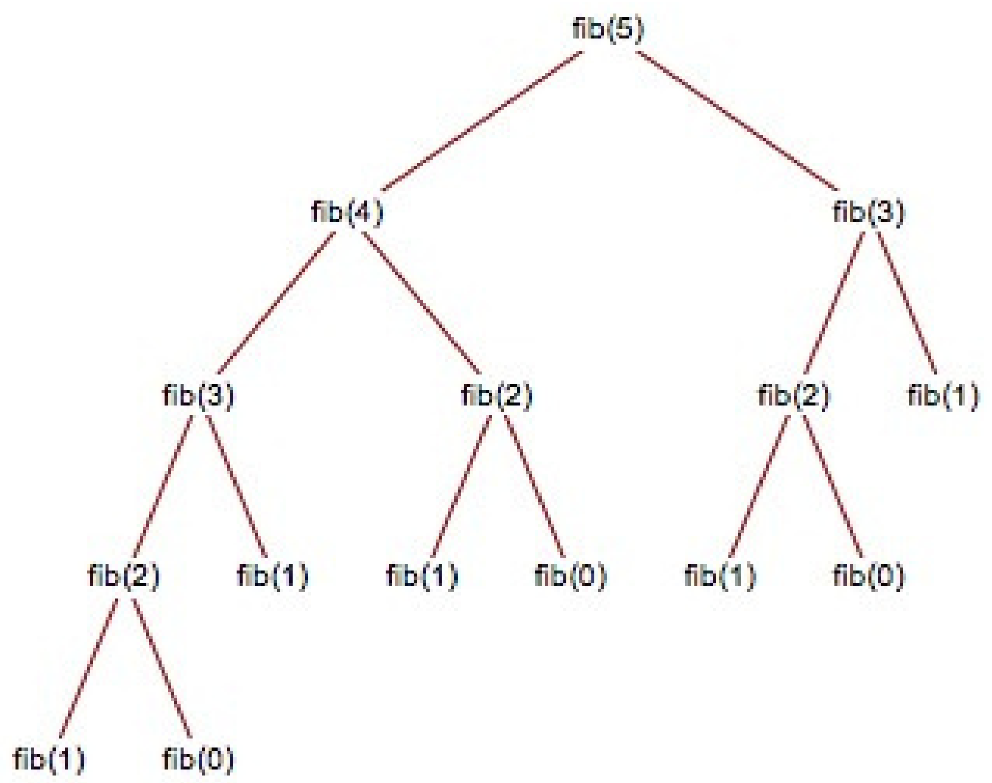
Fibonacci Binary Trees: Time Complexity and Levels

2. TOTAL TIME COMPLEXITY:

- **TOTAL WORK IS THE SUM OF WORK AT ALL LEVELS.**
- **THE FIBONACCI TREE OF ORDER N HAS EXACTLY F(N+2)−1 NODES.**

- SINCE THE HEIGHT OF THE TREE IS N (THE FIBONACCI INDEX), THE TOTAL TIME COMPLEXITY IS:

TOTAL TIME COMPLEXITY = O(2^N)

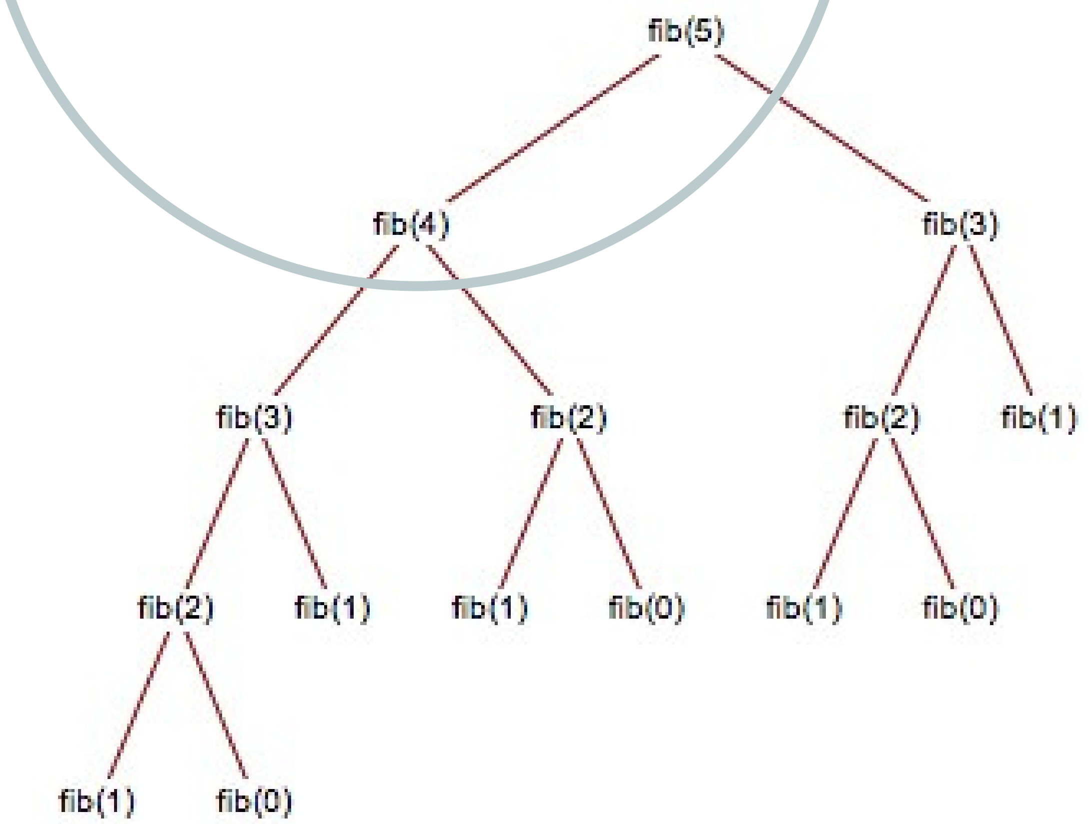


FIBONACCI BINARY TREES: TIME COMPLEXITY AND LEVELS

3. NUMBER OF LEVELS:

THE NUMBER OF LEVELS CORRESPONDS TO THE HEIGHT OF THE TREE, WHICH IS DETERMINED BY N:

- ROOT IS AT LEVEL 0.
- TREE HEIGHT: N.
- TOTAL LEVELS: N + 1



CODE

```cpp
#include<iostream>
using namespace std;
void print_Fibonacci(int num);
int main()
{

    int n;
    cout << "Enter how many terms do you want to print in Fibonacci sequence:";
    cin >> n;
    print_Fibonacci(n);
    return 0;
}
void print_Fibonacci(int num)
{
    int f1 = 0;
    int f2 = 1;
    cout << "fibonacci:" << f1 << "," << f2;
    int next;
    for (int i = 3; i <= num; i++)
    {
        next = f1 + f2;
        cout << "," << next;

        f1 = f2;
        f2 = next;
    }
    cout << endl;
}
```

OUTPUT:

```
Microsoft Visual Studio Debu

Enter how many terms do you want to print in Fibonacci sequence:10
fibonacci:0,1,1,2,3,5,8,13,21,34
```

GOLDEN RATIO

THE GOLDEN RATIO () IS A MATHEMATICAL CONSTANT OFTEN ASSOCIATED WITH AESTHETICS, DEFINED AS:

$$\Phi=(1+\sqrt{5})/2=1.618$$

CONVERGENCE TO THE GOLDEN RATIO

AS WE MOVE FURTHER ALONG THE FIBONACCI SEQUENCE, THE RATIO BETWEEN CONSECUTIVE TERMS APPROACHES THE GOLDEN RATIO, REFLECTING A NATURALLY RECURRING PATTERN THAT APPEARS IN ART, NATURE, AND ARCHITECTURE.

GOLDEN RATIO

3.1. **The Golden Ratio.** In calculating the ratio of two successive Fibonacci numbers, $\frac{u_{n+1}}{u_n}$, we find that as n increases without bound, the ratio approaches $\frac{1+\sqrt{5}}{2}$.

Theorem 2.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1+\sqrt{5}}{2}$$

Proof. Since

$$u_{n+1} = u_n + u_{n-1},$$

by definition, it follows that

$$\frac{u_{n+1}}{u_n} = 1 + \frac{u_{n-1}}{u_n}.$$

Now, let

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L.$$

We then see that

$$\lim_{n\to\infty} \frac{u_{n-1}}{u_n} = \frac{1}{L}.$$

We now have the statement

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1 + \lim_{n\to\infty} \frac{u_{n-1}}{u_n},$$

which is equivalent to the the equation

$$L = 1 + \frac{1}{L}.$$

This equation can then be rewritten as

$$L^2 - L - 1 = 0,$$

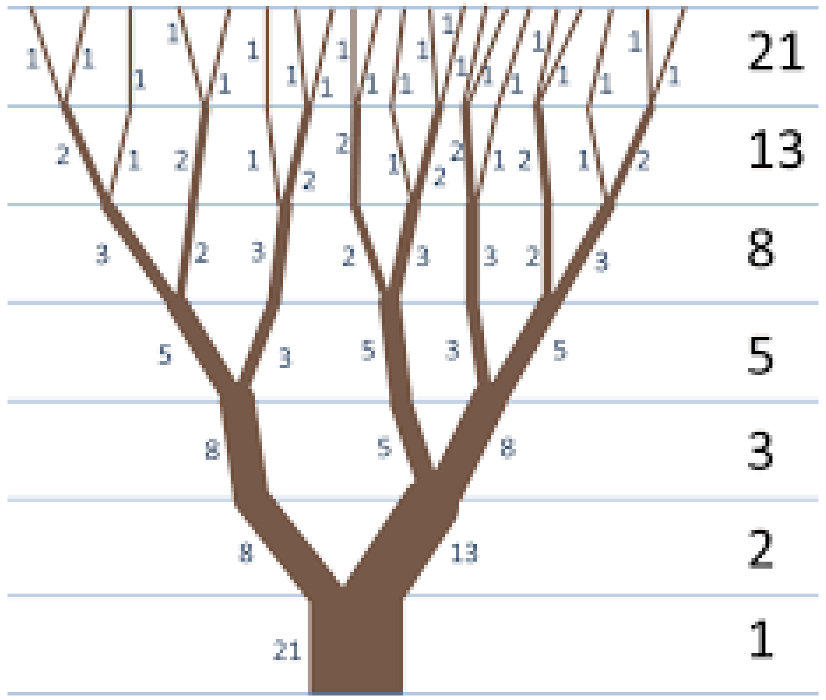
which is easily solved using the quadratic formula. By using the quadratic formula, we have

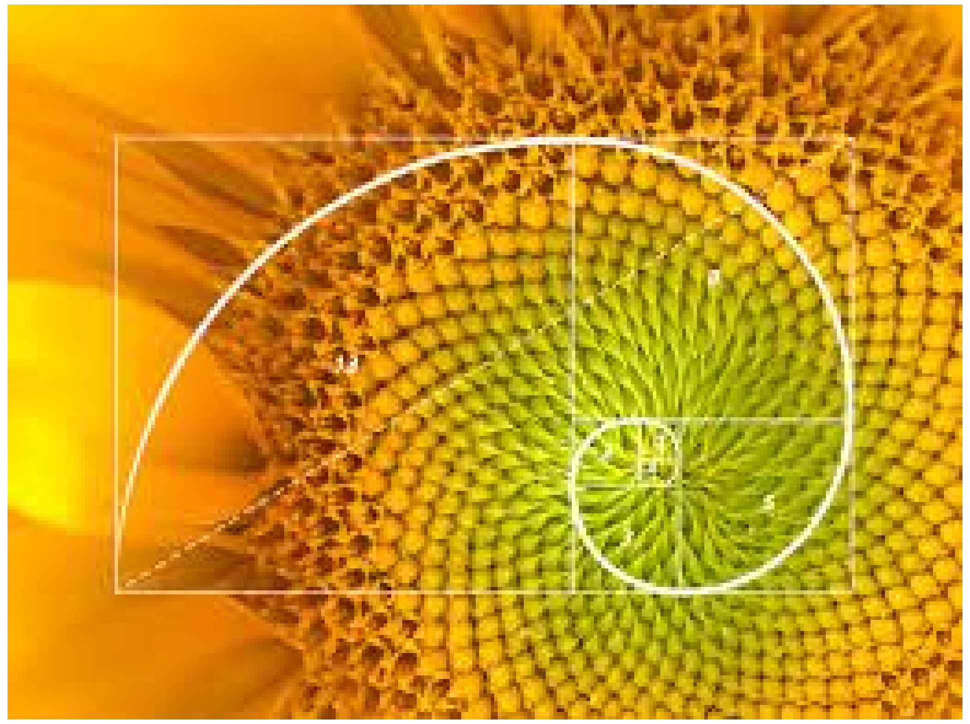
$$L = \frac{1 \pm \sqrt{5}}{2}.$$

APPLICATIONS

- ## PATTERNS IN NATURE

THE FIBONACCI SEQUENCE IS VISIBLE IN NATURAL PHENOMENA, INCLUDING THE ARRANGEMENT OF LEAVES, BRANCHING OF TREES, AND PATTERNS IN SHELLS AND FLOWERS.

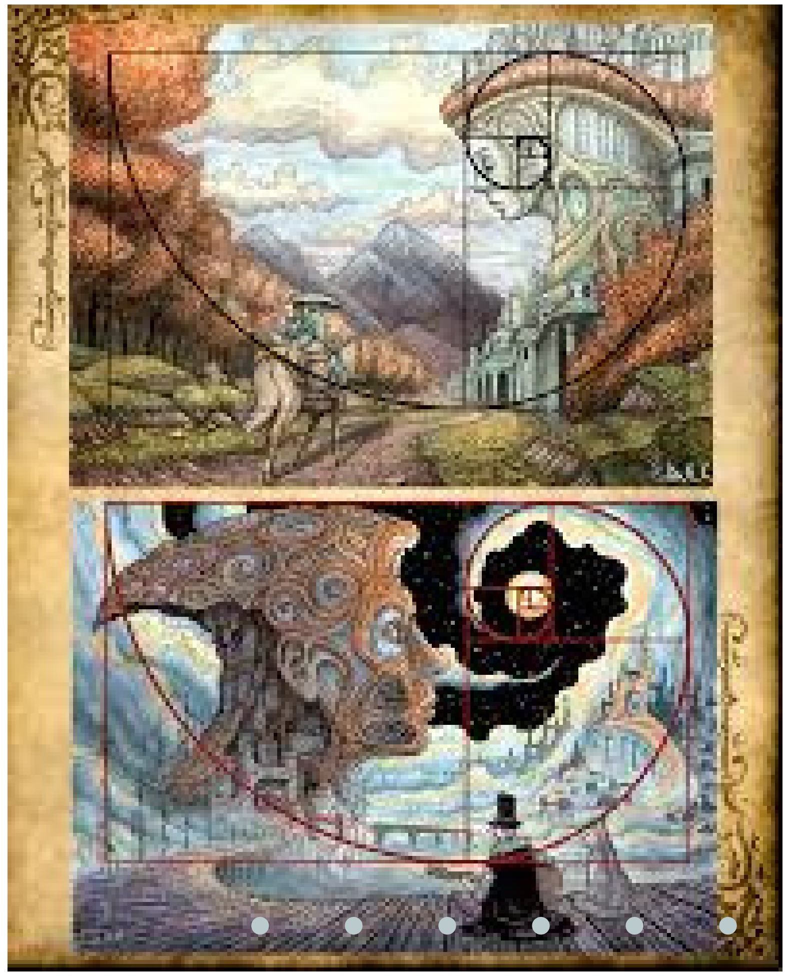


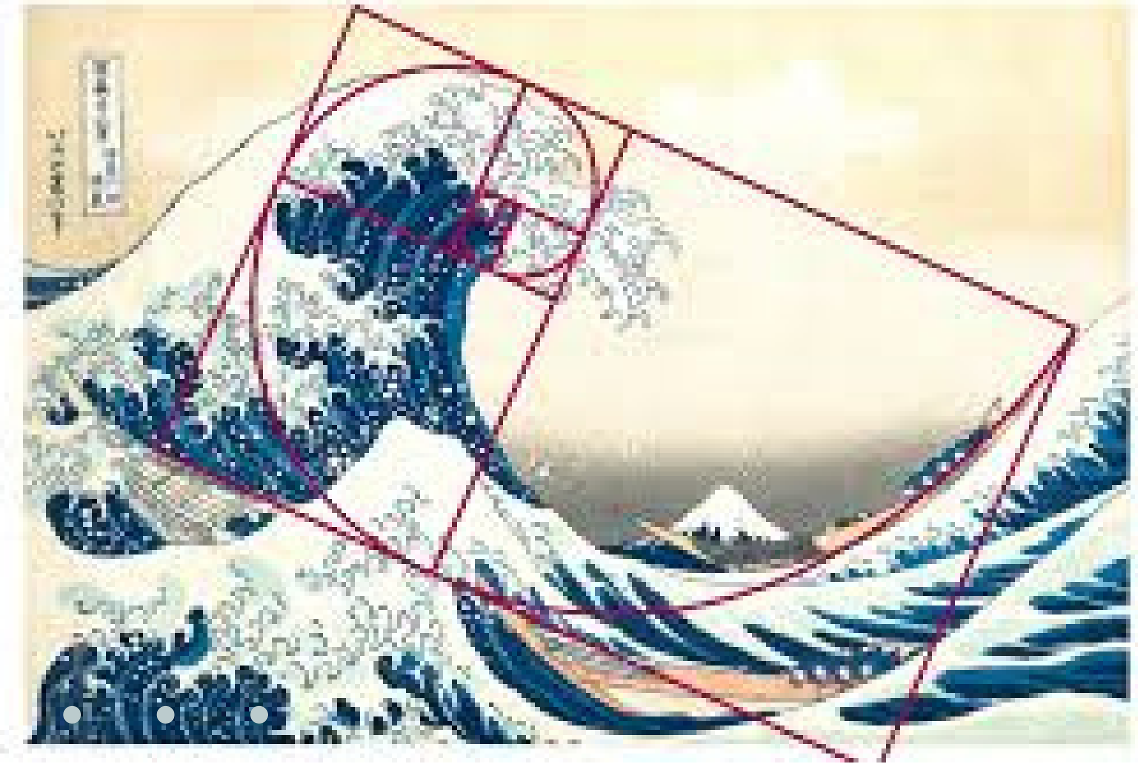


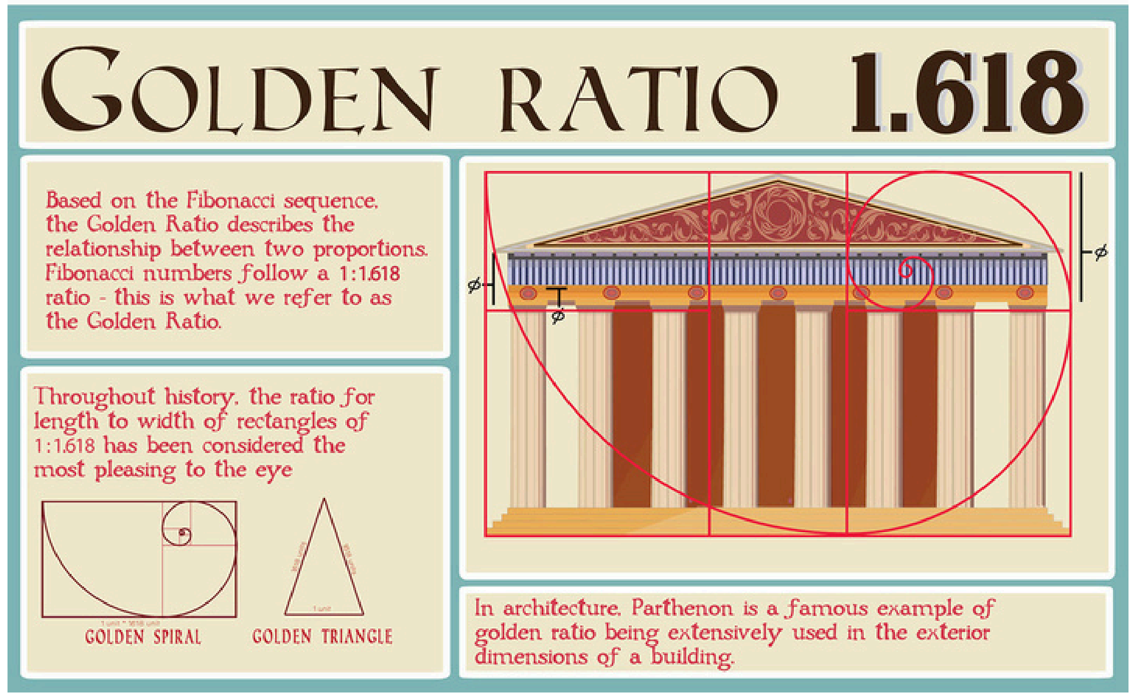
APPLICATIONS

- ## ART AND AESTHETIC APPEAL

ARTISTS AND ARCHITECTS USE THE GOLDEN RATIO TO ACHIEVE HARMONY IN THEIR WORK. THE PARTHENON IN GREECE AND PAINTINGS BY LEONARDO DA VINCI ILLUSTRATE THE PLEASING PROPORTIONS DERIVED FROM FIBONACCI NUMBERS.





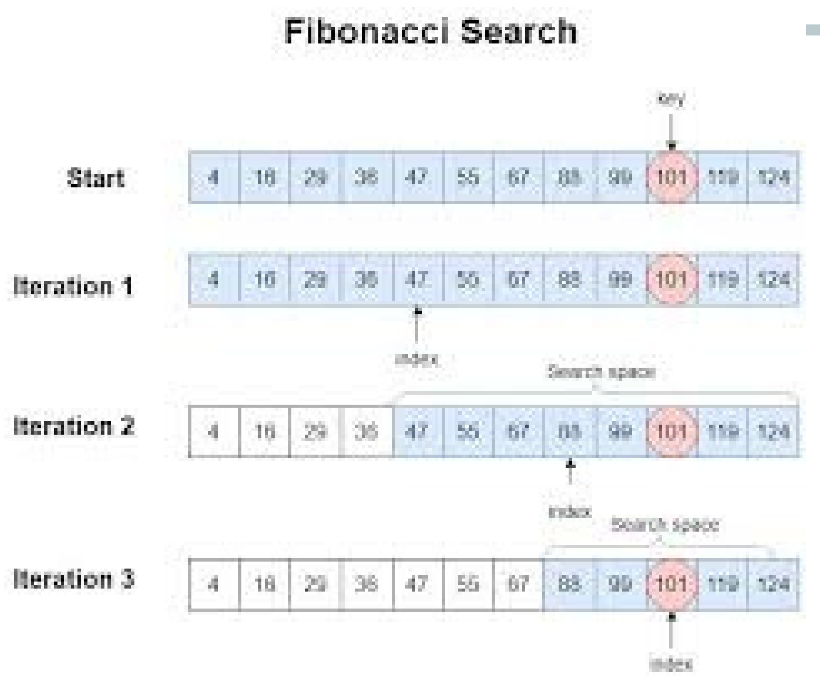


APPLICATIONS

- ## COMPUTER SCIENCE

1. FIBONACCI SEARCH ALGORITHM:

THE FIBONACCI SEQUENCE IS USED IN SEARCH ALGORITHMS, PROVIDING AN EFFICIENT METHOD FOR SEARCHING SORTED ARRAYS.



2. DYNAMIC PROGRAMMING EXAMPLE:

THE SEQUENCE OFTEN APPEARS IN DYNAMIC PROGRAMMING PROBLEMS, WHERE RECURSIVE SOLUTIONS CAN BE OPTIMIZED THROUGH MEMOIZATION, ILLUSTRATING EFFICIENT PROBLEM-SOLVING TECHNIQUES.

APPLICATIONS

- ## FINANCIAL ANALYSIS

1. FIBONACCI RATIOS AND TRADING

IN TECHNICAL ANALYSIS, FIBONACCI RATIOS (SUCH AS 23.6%, 38.2%, 61.8%) ARE USED TO PREDICT POTENTIAL REVERSAL POINTS AND IDENTIFY PRICE TRENDS, MAKING THEM A VALUABLE TOOL FOR TRADERS.

APPLICATIONS

- CRYPTOGRAPHY:

1. PSEUDORANDOM NUMBER GENERATORS:

   FIBONACCI SEQUENCES CAN SEED RANDOM NUMBER GENERATORS FOR CRYPTOGRAPHIC APPLICATIONS.

2. STEGANOGRAPHY:

   FIBONACCI-BASED ENCODING SCHEMES HIDE DATA WITHIN OTHER MEDIA

3. NETWORK THEORY:
   - ROUTING ALGORITHMS:

     FIBONACCI NUMBERS OPTIMIZE PACKET ROUTING AND LOAD BALANCING.
   - DISTRIBUTED SYSTEMS:

     NODES FOLLOW FIBONACCI RULES FOR FAILOVER OR TASK REASSIGNMENT.