

Targeted damage to interdependent networks

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(Received 14 February 2018; published 18 September 2018)

Complex systems consisting of interdependent subsystems may be represented by multilayer networks with interdependency links between layers. The giant mutually connected component (GMCC) of such an interdependent (or multiplex) network collapses with a discontinuous hybrid transition under random damage to the network. If the nodes to be damaged are selected in a targeted way, the collapse of the GMCC may occur significantly sooner. Understanding the limits of the resilience of such systems to targeted attacks is therefore an essential problem. Finding the minimal damage set which destroys the largest mutually connected component of a given interdependent network is a computationally prohibitive simultaneous optimization problem. We introduce a simple heuristic strategy—effective multiplex degree—for targeted attack on interdependent networks that leverages the indirect damage inherent in multiplex networks to achieve a damage set smaller than that found by any existing noncomputationally intensive algorithm. We show that the intuition from single-layer networks that decycling (damage of the 2-core) is the most effective way to destroy the giant component does not carry over to interdependent networks and in fact such approaches are worse than simply removing the highest degree nodes.

DOI: [10.1103/PhysRevE.98.032307](https://doi.org/10.1103/PhysRevE.98.032307)

I. INTRODUCTION

Interdependent networked systems are particularly vulnerable to damage, as damaged sites in one subsystem cause failures in another, which may in turn propagate damage back to the first [1]. Under sufficient damage, avalanches of propagating failures may eventually lead to the discontinuous collapse of the whole system [2]. Many essential natural, technological, and social systems [3–5] consist of fully or partially interdependent subsystems, allowing them to be represented by such interdependent or multiplex networks. Understanding the vulnerability of such systems to damage and cascading collapse is therefore of paramount importance.

If the nodes damaged are selected nonrandomly, that is, the damage is targeted, the amount of damage required for collapse may be dramatically reduced. In this paper, we consider the problem of finding the best approximation to the minimal damage set: A set of nodes whose removal leads to the collapse of the largest mutually connected component (LMCC). We show that the characteristics of highly effective strategies on interdependent networks are different from those on single-layer networks. We introduce a new heuristic targeted attack strategy which exploits the properties of interdependent or multiplex percolation to effectively destroy the LMCC more quickly than any previous noncomputationally intensive method, with an efficacy approaching that of simulated annealing, yet with a computation time for node ranking that is linear in system size.

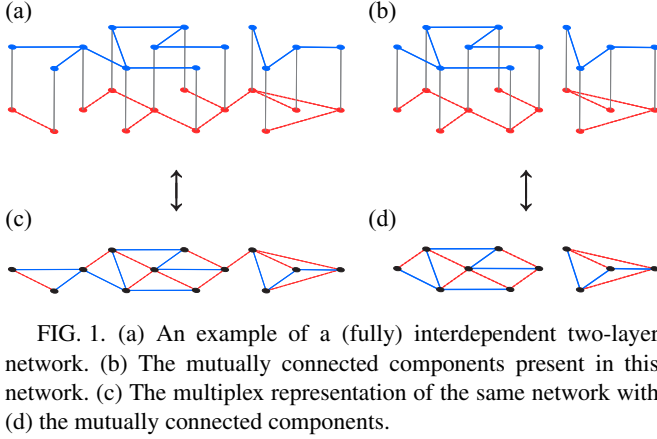
The quest to identify the minimal set of removals required to destroy the giant connected component in a single network has recently received renewed attention. The MinSum

algorithm [6] combines a message-passing approach with a statistical mechanics formulation, achieving results which outperform even stochastic search methods, such as simulated annealing. The approach is essentially similar to the MaxSum [7] method proposed for threshold problems. Although much more computationally efficient than simulated annealing, MinSum is nevertheless rather complex and opaque. Several more simple heuristic methods have also been proposed, such as collective influence (CI) [8,9] and CoreHD, a direct attack on the 2-core of the largest component [10]. Despite its simplicity, the CoreHD algorithm performs essentially as well as MinSum. These methods exploit the observation that the destruction of the largest connected component is in grand part a matter of decycling the network, i.e., removing all loops [6]. Once this is performed, the remaining tree can be broken down in very few steps. All loops in a network are contained within its 2-core, the subnetwork in which each member has at least two neighbors which are also members. Hence attacking the 2-core of a network is an effective strategy for dismantling a network.

Here we consider the equivalent problem for multiplex and interdependent networks where similar progress has not yet been achieved. An interdependent network consists of two or more network layers with nodes in different layers connected by interdependency links as shown in Fig. 1(a). If a node fails in (is removed from) one layer, its interdependent partner in another layer also fails. In the multiplex network representation, interdependent nodes are merged into a single node, and connections in different layers are considered different types of edges (represented by different colors) [11], Fig. 1(c). Mathematically the two representations are equivalent.

The generalization of a connected cluster to multiplex or interdependent networks is the mutually connected component (MCC). A node belongs to a MCC if at least one of

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its neighbors in the same layer belongs to the same MCC, and each of its interdependent partners in other layers also belongs to the same MCC [1] as shown in Fig. 1(b). In the multiplex representation (for fully interdependent networks) this corresponds to the rule that all pairs of nodes within a MCC must be connected by at least one path of each edge type (color) [2,11,12], Fig. 1(d). In the thermodynamic limit, the collapse of a giant mutually connected component (GMCC) under random damage occurs with a discontinuous hybrid phase transition [1,2]. The fraction of vertices that must be randomly removed to provoke the collapse of the system depends on the structure of the two layers.

Huang *et al.* examined the effect of removing nodes preferentially according to their degree [13], and the results were extended to partially interdependent networks in Ref. [14]. In Ref. [15] the authors consider various existing strategies, finding the best of them is the limiting case of that in Ref. [13]: Remove the nodes with the highest degree sum or product. Similar cascade phenomena occur in single-layer networks under threshold criteria, such as the k -core or bootstrap percolation [16–19]. The MaxSum algorithm [7] provides a message-passing based method to find good solutions to the minimal seeding set problem in such threshold models. Such methods could, in principle, be applied to the multiplex percolation problem because a node's membership of the LMCC cannot be computed directly from the state of its neighbors but instead requires an exploration of the entire cluster [12]. In other words there is no local pruning rule that allows one to identify the LMCC.

II. TARGETED DAMAGE STRATEGIES

We consider a configuration model multilayer network with M layers. Each node $i, i \in \{1, 2, \dots, N\}$ in layer l is interdependent with a single node (for convenience also labeled i) in each of the other layers such that removal of i from layer l implies the failure (removal) of all counterpart nodes i in the other layers [1]. Alternatively, by considering all the interdependent nodes i as a single node spanning all layers, the system can be viewed as a multiplex network, defined by the joint degree distribution $P(q^{(1)}, q^{(2)}, \dots, q^{(M)}) \equiv P(\mathbf{q})$ such that node i has degree $q_i^{(l)}$ in layer l . These two formulations, multiplex and interdependent networks, are mathematically

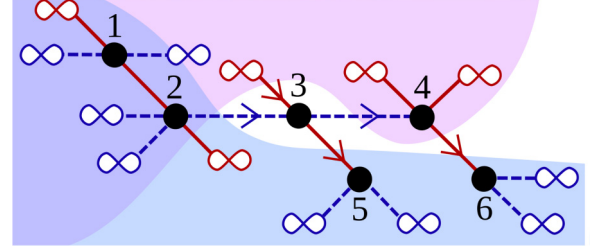


FIG. 2. A small grouping of nodes within the GMCC (LMCC) of a two-layer multiplex. Infinity symbols indicate a connection leading to an infinite GMCC subtree (or the LMCC 2-core in a finite system) in a given layer. Two or more such connections correspond to membership of the GMCC 2-core of that layer. Nodes 1, 2, 5, and 6 belong to the 2-core within the GMCC in the layer with dashed (blue) edges. Nodes 1, 2, and 4 belong to the 2-core of the solid (red) edged layer. Since nodes 1 and 2 belong to the 2-core of the GMCC in both layers they cannot be removed in an avalanche. Nodes 3–6 are outside one or both 2-cores and are therefore in a critical state, potentially being removed in an avalanche of propagating damage. For example, removal of node 2 will lead to the removal of all four of these nodes. Damage propagates along critical edges in the directions marked by arrows.

equivalent [11]. In Sec. V, below, we extend our analysis to partially interdependent networks in which some nodes do not have interdependent partners.

In the limit $N \rightarrow \infty$, the GMCC is extensive if it contains a finite fraction of the nodes in the multiplex. In large finite networks, we consider the LMCC to be extensive if its size S exceeds \sqrt{N} . Starting from a multiplex in which $S > \sqrt{N}$, our aim, then, is to select the smallest possible damage set s that destroys the LMCC.

A given attack strategy ranks nodes according to some metric, and nodes are removed in order starting from the highest ranked node until the LMCC collapses. The fraction $p_c \equiv |s|/N$ of removed nodes is the basis of comparison between different methods. An adaptive strategy may be used with ranks being recalculated after the network reaches equilibrium following each removal. This also avoids the possibility of redundancies caused by removing nodes that may have been removed anyway by avalanches. Only nodes belonging to the LMCC at the current step are considered for removal (including for random node removal). Note that the final result depends only on the set of nodes removed not on the order of removal.

In interdependent networks, critical nodes are nodes that depend on a single connection in one (or more) layers in order to remain within the LMCC. Damage propagates through clusters of such critical nodes as the removal of one node removes the support from neighboring critical nodes, and so on, as illustrated in Fig. 2. The avalanches of damage diverge in size approaching the critical point, leading to the discontinuous collapse of the GMCC. The 2-cores in each layer within the LMCC play an important role. Nodes within the LMCC 2-core in a given layer are not critical with respect to that layer so that nodes belonging to the 2-core in all layers are noncritical, whereas nodes failing to belong to one or more of these 2-cores are in a critical state and may be removed by avalanches, see Fig. 2. One might therefore imagine that, as

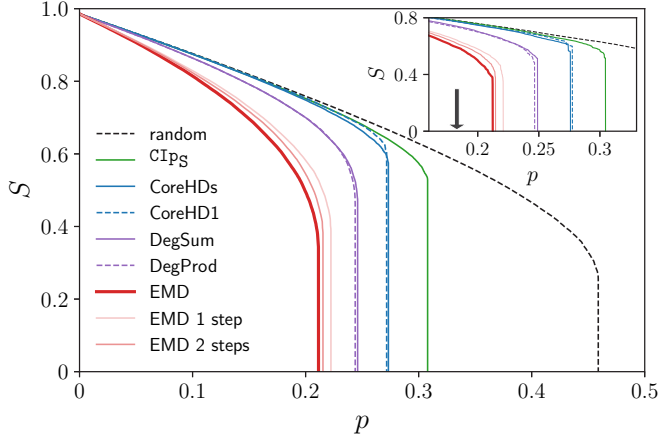


FIG. 3. Progression of the relative size S of the LMCC with a fraction p of the nodes removed under different targeted attack strategies for a duplex network of two uncorrelated Erdős-Rényi layers with mean degree $\mu = 5$ and $N = 10^5$. From right to left: random damage (dashed, black), collective influence CIPs (green), CoreHDs (solid blue), CoreHD1 (dashed blue), highest degree sum DegSum (purple), and product DegProd (dashed purple), Effective multiplex degree (EMD) (red). Also shown are the approximate effective multiplex degree results obtained using only one and two iterations of Eq. (2). The inset: Results for $N = 10^4$, allowing comparison with simulated annealing, marked by the arrow.

in single-layer networks, destruction of 2-cores might be an effective strategy in interdependent networks. We therefore generalized the leading single-layer strategies which target 2-cores, to see if they would be effective also in multiplex networks. Typical results are shown in Fig. 3. We found that the intuitions gained from studying single-layer networks do not carry over to multiplex networks.

The collective influence strategy may be adapted to multiplex networks by selecting nodes according to the sum or the product of their scores in each layer. We use the collective influence propagation (CIP) algorithm [9], and results shown are for the sum CIPs. Similarly, the CoreHD method may be generalized by removing the nodes with the highest sum of degrees within the union of the 2-cores of all layers (CoreHDs) or by removing the node with the highest degree within the 2-core of the layer with the (currently) lowest mean degree (CoreHD1). See Appendix B for more details. We found, however, that simply removing the node with the highest degree sum (DegSum) or product (DegProd) is more effective than any of these strategies, meaning that focusing on the 2-cores actually diminishes attack effectiveness. This further underlines the fundamentally different nature of the multiplex percolation transition.

III. EFFECTIVE MULTIPLEX DEGREE

Here we introduce a new strategy EMD, which gives a damage set significantly smaller than that found by any of the methods described above. The EMD algorithm leverages inhomogeneities between layer degrees and in the neighborhood of a node to maximize the effect of node removal.

We seek to ascribe weights w_i which represent the impact on the LMCC of removing node i . Consider a connection between nodes i and j in layer l . If i is removed, j loses an edge in layer l . If node j loses all of its $q_j^{(l)}$ connections within layer l , it is removed from the LMCC. A node is most vulnerable in the layer in which it has least connections. The smaller is $q_j^{(l)}$, the bigger is the impact of losing one of those connections. Suppose that node j has weight w_j , then we ascribe a value of $w_j/q_j^{(l)}$ to the impact of removing the edge from i to j in layer l . The weight of a node i is then the sum of all the weights ascribed to the edges emanating from it, a measure of the impact of removing i ,

$$w_i = \sum_{l=1}^M \sum_{j \in \mathcal{N}_i^{(l)}} \frac{1}{M_j} \frac{w_j}{q_j^{(l)}}, \quad (1)$$

where $\mathcal{N}_i^{(l)}$ is the set of neighbors of node i in layer l and M_j is the number of layers in which node j participates. This naturally accounts for the case of partial interdependence, see Sec. V below. The EMD weights are the self-consistent solutions of this set of equation. Larger weights are found in nodes with more connections and with connections to other highly weighted nodes in the layer in which they are weakest.

Equation (1) may be written more compactly as the matrix equation,

$$\mathbf{w} = \mathbf{R}\mathbf{w}, \quad (2)$$

with

$$R_{ij} = \sum_{l=1}^M \frac{a_{ij}^{(l)}}{M_j q_j^{(l)}}, \quad (3)$$

where $a_{ij}^{(l)}$ are the elements of the adjacency matrix in layer l . The matrix \mathbf{R} is a left stochastic matrix. Since the weights w_i are all positive, by the Perron-Frobenius theorem, the vector \mathbf{w} is the leading right eigenvector of \mathbf{R} , corresponding to the largest eigenvalue 1.

The weight w_i is equal to the equilibrium probability to find a random walker at node i when the random walker at each step chooses uniformly between layers and selects an edge uniformly within that layer to follow. In Ref. [20] a type of random walk is introduced which is a classical random walk within the layers and at each node the walker is allowed to randomly switch to the multiplex counterpart node in a negligible time. This process is equivalent to the one described by our matrix \mathbf{R} in a fully interdependent network of an arbitrary number of layers. Note, however, that the random walk corresponding to matrix \mathbf{R} is defined for an arbitrary network of networks in which any node can have an arbitrary number of interdependency neighbors in other layers.

In Ref. [21] a node centrality index, called functional multiplex PageRank, is introduced. This corresponds to a random walker on a general directed multiplex network with an arbitrary combination of overlapping links. Although the random walk transition probability from a node to its neighbor may depend on the type of link overlap, it is essentially determined by an “aggregated degree” of each node. The major advantage of EMD over this centrality definition is that it incorporates the degree heterogeneity of nodes over different

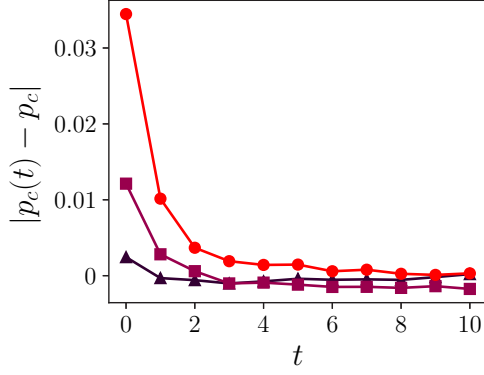


FIG. 4. The absolute difference between damaging fraction $p_c(t)$ obtained by limiting to k iterations of Eq. (2) and final value p_c , for two fully interdependent Erdős-Rényi layers with mean degree $\mu = 5$ and $N = 10^5$ nodes, averaged over ten realizations (circles, red). Initial value $t = 0$ corresponds to highest degree sum strategy. Also shown are progressions for partially interdependent networks, $f = 0.6$ (squares) and $f = 0.3$ (triangles).

layers. This degree heterogeneity is essentially important for the robustness of multiplex networks or the dynamical processes running on top of them. A number of other variations of random walks have been defined for interdependent networks [22,23] which treat interdependency links as qualitatively the same—from the point of view of the random walker—as intralayer connectivity links.

A simplified non-self-referential weight may be obtained by using the degree sum $Q_j = \sum_{l=1}^M q_j^{(l)}$ in place of w_j in Eq. (1),

$$\tilde{w}_i = \sum_{l=1}^M \sum_{j \in \mathcal{N}_i^{(l)}} \frac{1}{M_j} \frac{Q_j}{q_j^{(l)}}. \quad (4)$$

This already gives significant improvement over DegSum and DegProd, see results EMD1 in Fig. 3. The convergence to \mathbf{w} can be examined by considering the sequence of weights obtained by iterating Eq. (2). We use as initial values the degree sum $w_i^{(0)} = Q_i$ and define $w_i^{(t+1)} = \sum_j R_{ij} w_j^{(t)}$ so that $w_i^{(1)} = \tilde{w}_i$ and $w_i = w_i^{(t \rightarrow \infty)}$. In Fig. 4 we plot the collapse threshold p_c obtained by restricting the number of iterations of Eq. (2). We see that the result converges rapidly after only a very few iterations. For a fixed number t of iterations, the calculation of $\mathbf{w}^{(t)}$ is linear in system size. See Appendix A for a more detailed discussion.

IV. RESULTS

To illustrate the results obtained by our method, in Fig. 3 we consider a multiplex consisting of two Erdős-Rényi layers with equal mean degree and plot the relative size S of the largest mutually connected component as a function of the fraction p of nodes removed, starting with the largest weight $\max\{w_i\}$, and recalculating the weights after each removal. The LMCC collapses at a fraction p_c that is significantly lower than for any of the other methods described. The result approaches that found by simulated annealing on a relatively small network ($N = 10^4$). The computational budget required

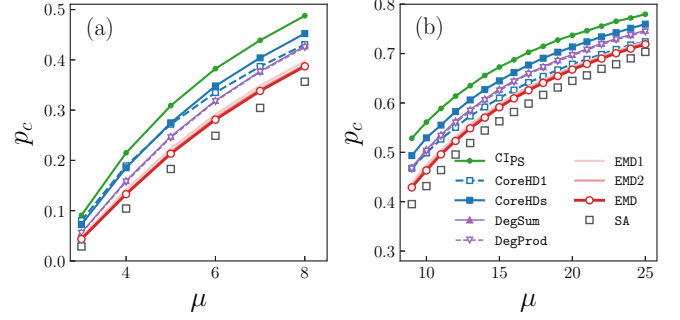


FIG. 5. Results for the relative number of nodes removed in destruction of the LMCC for two Erdős-Rényi layers, both with varying mean degree μ . (a) Values of μ from 3 to 8 and (b) μ from 9 to 25. All networks consist of $N = 10\,000$ nodes. The networks are fully interdependent, and the two layers are uncorrelated. All results were averaged over ten realizations, except for simulated annealing (SA), which was a single realization.

to perform simulated annealing grows rapidly with N , and already for $N = 10^5$, our method outperforms the simulated annealing results achievable in a reasonable computing time (see Appendix C). Furthermore, the surviving LMCC size S falls more rapidly than in other methods, being smaller at every value of p .

To understand the range of effectiveness of the method, we investigated several combinations of Erdős-Rényi and uncorrelated scale-free layers to compare the performance of our EMD algorithm with other attack strategies in different scenarios. In Fig. 5 we illustrate the dependence on network density. We plot the results for two Erdős-Rényi layers with equal mean degree μ . As μ increases the performance ranking of the different attack methods remains the same with the exception of CoreHD1, which begins to become relatively more effective. With very few vulnerable nodes in dense networks, reducing the density of one of the layers starts to become a more effective strategy. Our EMD algorithm remains the most effective across all the values of μ tested.

If one layer (of a two-layer network) is much more dense than the other, the membership in the LMCC of a pair of interdependent nodes is effectively decided by the node in the less dense layer, and the network behaves as a single-layer network. In this case, single-layer strategies may be more effective than multiplex ones. To explore this possibility, we considered a duplex network with the mean degree of one-layer μ_1 fixed and varied the mean degree μ_2 of the second layer, Fig. 6(a). For large μ_2 , the best single-layer strategy, CoreHD1 becomes the leading strategy, confirming the fact that beyond a certain density in layer 2 the destruction of the LMCC is equivalent to the destruction of the largest connected component in layer 1.

To examine the effect of heterogeneity in network degree, we considered two scale-free layers. When the two layers have the same power-law distribution with exponent γ , varying γ has no effect on the ranking of the algorithms with EMD always outperforming the other methods, Fig. 6(b). Similarly, if we fix the exponent for one layer and vary it for the other layer again the relative effectiveness of the different methods is unchanged, Fig. 7(a).

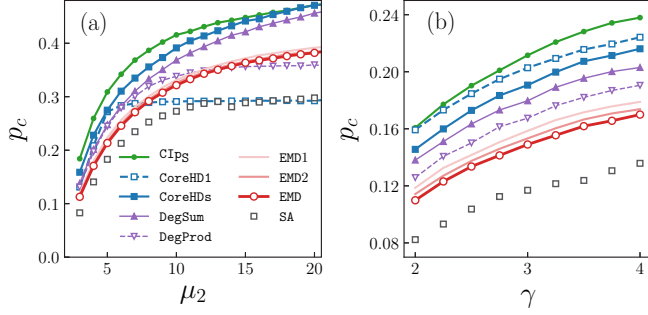


FIG. 6. Collapse threshold p_c for different attack strategies for (a) two uncorrelated Erdős-Rényi layers with different mean degrees. One layer has fixed mean degree $\mu_1 = 5$, whereas the other has mean degree μ_2 and (b) two power-law degree distributed layers $P(q) \propto q^{-\gamma}$ as a function of power-law exponent γ . Each layer has $N = 10\,000$, ten realizations, except simulated annealing, one realization.

Finally we considered what happens when one layer is of a significantly different character than the other by pairing an Erdős-Rényi layer with a scale-free layer, each with the same mean degree, Fig. 7(b). DegProd improved relative to the other methods. We modified the CoreHD1 method to attack only the scale-free layer as highly heterogeneous networks are more fragile under targeted attack and found that this algorithm outperformed even EMD, reinforcing our results for layers of different densities that when one layer is significantly more fragile than the other single-layer strategies are more effective.

V. PARTIAL INTERDEPENDENCE

The condition that every node is interdependent with nodes in other layers is a strong one. This may be relaxed to give a more realistic model by considering partially interdependent two-layer networks [24], parametrized by the fraction of nodes f within a given layer which are interdependent. When $f = 1$, the network is fully interdependent. When $f = 0$ the layers are not interdependent at all, and the system behaves as a pair of single-layer networks. In the multiplex

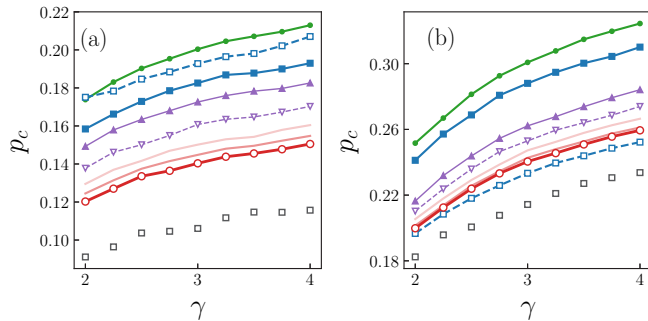


FIG. 7. Results for the relative number of nodes removed in destruction of the LMCC for (a) two scale-free layers, both with mean degree 10, one with exponent $\gamma_1 = 2.5$, the other with varying exponent γ_2 and (b) An Erdős-Rényi layer and a scale-free layer both with mean degree 10 and varying degree distribution exponent γ . In this case the CoreHD1 algorithm was modified to always attack the scale-free layer.

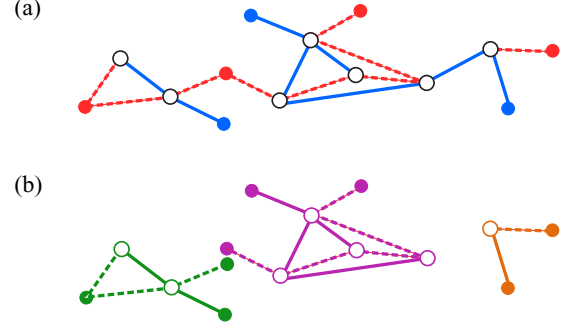


FIG. 8. (a) The multiplex representation of a partially interdependent two-layer network. Interdependent nodes are shown as open circles, and noninterdependent nodes are shown as closed circles. In (b) we show the mutually connected components present in this network. Note that noninterdependent nodes may belong to more than one MCC.

representation, the fraction of interdependent nodes in the entire system is $\tilde{f} = f/(2 - f)$. A simple example of such a network is illustrated in Fig. 8(a). Under random damage, the collapse of the GMCC transitions from a hybrid discontinuous transition to a second order percolation transition for sufficiently small f [24].

The definition of mutually connected components in this case is not as obvious as for fully interdependent networks. Usually a recursive definition of the GMCC is used whereby a node belongs to the GMCC if it has at least one link to the GMCC in its own layer and all of the nodes with which it is interdependent also belong to the GMCC. In terms of multiplex networks, this corresponds to the condition that a node has at least one link to the GMCC in each of the layers in which it participates. This definition applies both to fully and partially interdependent networks. However, we are not aware of a strict definition of *finite* mutually connected components in partially interdependent networks.

In this paper we define a MCC in a multiplex to be a maximal set of nodes that are connected on all the layers they participate on. In other words, for interdependent networks: For any layer, all the nodes of the MCC that are present on the given layer must form a connected cluster on this layer. This definition is a straightforward interpretation of the widely accepted recursive definition. Note that according to this definition it is possible for MCCs to overlap: Nodes that only participate on one layer may belong to more than one MCC [see the example in Fig. 8(b)]. However, interdependent nodes (nodes that are present on all layers) can only be members of one MCC. Therefore interdependent nodes form the disjunct cores of MCCs in partially interdependent networks. In actual simulations one needs to identify all MCCs and choose the largest of them to identify the LMCC.

We compared the results of our effective multiplex degree method against other methods as a function of f , Fig. 9. Note that the EMD algorithm naturally generalizes to partially interdependent networks. When $f = 0$, in the absence of any interdependence, naturally the single-layer methods CoreHDs and CIPs perform best. In this case EMD simply returns the node degree. As f increases, the performance of EMD steadily improves relative to the single-layer methods. Beyond

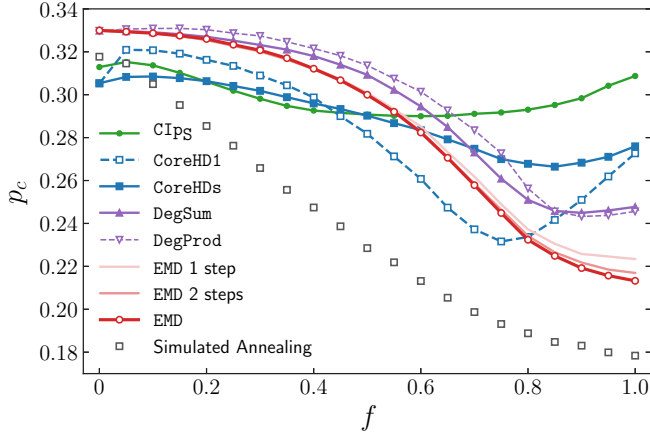


FIG. 9. Comparison of collapse threshold p_c as a function of interdependent node fraction f for different targeted attack strategies for a duplex network of two uncorrelated Erdős-Rényi layers with mean degree $\mu = 5$ and $N = 10\,000$, averaged over 50 realizations. Simulated annealing results were averaged over five realization.

approximately $f = 1/2$ it outperforms all other methods except CoreHD1. The relatively strong performance of CoreHD1 for moderately large values of f is intriguing, but we do not know of a simple explanation. Our EMD method outperforms all other studied methods beyond the point where approximately one third of the nodes are noninterdependent ($\tilde{f} = 2/3$, $f = 0.8$). In this case this point coincides with the value of f below which the noninterdependent nodes form a giant cluster by themselves.

VI. CONCLUSIONS

The problem of finding the minimal damaging set of even a single-layer network is already a difficult problem, although recent effort has produced simple algorithms that produce highly effective approximations of the minimal set. Here our aim is to extend this effort to interdependent and multiplex networks. Due to the nonlocal definition of mutually connected clusters (the commonly used generalization of connected cluster to multilayer systems)—there is no local pruning algorithm that can give the LMCC—this task is even more difficult. We show that reasonable generalizations of single network methods, whereas more effective than random attacks, are far from optimal. Rather surprisingly, direct attacks based solely on node degrees are more effective.

We have developed a simple heuristic algorithm that exploits the heterogeneity between nodes' degrees in different layers and the vulnerability of nodes in the neighborhood to give a damaging set significantly smaller than that found by any other computationally nonintensive methods and approaching the damaging set size found by computationally costly simulated annealing. The advantage of our method is that, due to its low computational overhead, it may be applied to very large systems where Monte Carlo methods, such as simulated annealing would be prohibitively expensive (in time and computational power) to carry out.

Our method naturally generalizes to partially interdependent networks and remains more effective than other methods

so long as interdependent nodes remain in a significant majority. When there is a significant difference in density between two layers, the connectivity of the system is controlled by the less dense layer, and single-layer methods again become the most effective. Our algorithm also naturally extends to more general formulations of dependency within networks, for example, the case of networks of connectivity links and dependency links [25]. A study of the effectiveness of effective multiplex degree in such networks is left for future work.

ACKNOWLEDGMENTS

This work was supported by National Funds through FCT—Portuguese Foundation for Science and Technology, I.P. Project No. IF/00726/2015. G.T. was partially supported by FCT project No. BPD-35/I3N/SET2016.

APPENDIX A: COMPUTATION OF THE EFFECTIVE MULTIPLEX DEGREE ALGORITHM

In this appendix we discuss the computation of the EMDs on an arbitrary multiplex network. We show that this algorithm is computationally efficient, essentially preserving the complexity of existing popular strategies while providing considerably better results.

The EMD values are computed iteratively according to

$$w_i^{(t+1)} = \sum_{l=1}^M \sum_{j \in \mathcal{N}_i^{(l)}} \frac{1}{M_j} \frac{w_j^{(t)}}{q_j^{(l)}}, \quad (\text{A1})$$

where $\mathcal{N}_i^{(l)}$ is the set of neighbors of node i on layer l and $q_j^{(l)}$ is the degree of node j on layer l . Note that Eq. (A1) can also be used for partially interdependent networks. In this case some of the $\mathcal{N}_i^{(l)}$'s will be empty sets, and some of the $q_j^{(l)}$'s will be 0. We set the initial values $w_i^{(t=0)}$ to the sum of the degrees of nodes i over all the layers [see Eq. (4)]. In most cases this is a good initial guess, especially in partially interdependent networks for small values of f (fraction of interdependent nodes). For $f = 0$ this initial guess coincides with the actual solution. In Eq. (A1) we only consider nodes and links inside the LMCC. In this case the matrix involved in Eq. (A1)—Eq. (3) in the main text—is a left stochastic matrix (i.e., has a largest eigenvalue 1), the weights w_i converge to the components of the principal right eigenvector.

The number of operations required in computing one iteration of Eq. (A1) is $\sim L$, the number of links in the network as we have to sum over all the neighbors on all the layers for every node. The overall time complexity of our algorithm is therefore,

$$T \sim n_{\text{iter}} L, \quad (\text{A2})$$

where n_{iter} is the number of iterations to convergence. Assuming n_{iter} to be a constant number, the EMD algorithm runs in time linear in system size, achieving essentially the same efficiency as that of DegSum and CoreHD. (Of course, considering the complete problem of destroying the largest mutually connected component, we must repeat the identification of the LMCC and the node selection step for every node in the

damaging set. This applies to all of the adaptive strategies considered.)

As shown in Fig. 4(b), typically only a few (three to five) iterations of Eq. (A1) are needed to achieve practically the same result as for the fully converged solutions. (Our criterion for the solutions to have fully converged is that the largest relative difference in all the w_i values in an iteration be less than 10^{-7} .) This proved to be the case for a range of f values and different types of network architectures, so indeed it seems reasonable to consider n_{iter} to be a small constant.

Method to identify the components

To identify MCCs in a partially interdependent network, one can do the following. First identify all the connected components on layer 1 and call these “candidate components.” (These candidate components may contain interdependent nodes and free nodes on layer 1.) Then consider the next layer, check if the interdependent nodes of the existing candidate components form connected components on layer 2 (also considering free nodes on layer 2). If any of the interdependent nodes of existing candidate components are disconnected on layer 2, then subdivide them into connected components (connected on layer 2, considering also free nodes on layer 2), and mark them as new candidate components. Repeating this procedure on every layer, perform as many subdivisions as necessary until the point when a full iteration (checking all the layers sequentially) can be performed without further subdivisions. At this point the interdependent nodes in the existing candidate components are the cores of MCCs in the network. As mentioned earlier, MCCs in partially interdependent networks may overlap, but the interdependent cores of MCCs are disjoint.

To find all the members of a MCC that a given core belongs to, simply search for all the nodes reachable from the interdependent members of each core on each layer. This step may involve the addition of a large number of free nodes to the MCC. Because MCCs can overlap—and the overlap may be large—it is generally not efficient to store all the members of all MCCs. A good strategy is to just store the interdependent cores of MCCs (these cannot overlap), then the actual MCC that a given core belongs to can be quickly reproduced in a simple search on all the layers.

Because MCCs can overlap in partially interdependent networks, their sizes do not obey any obvious normalization condition, and their size distribution, to our knowledge, has not yet been investigated. Such studies are outside the scope of this paper where we only concentrated on identifying the largest of the MCCs, i.e., the one that has the most members (free or interdependent).

APPENDIX B: GENERALIZATION OF EXISTING STRATEGIES TO MULTIPLEX NETWORKS

In this appendix we discuss the multiplex generalizations of some of the most efficient single-layer damage schemes. Of the great number of possibilities we concentrate on some simple and meaningful generalizations that formed the basis for our comparison of the EMD strategy with other efficient algorithms. All of the algorithms considered in this

work—including EMD—are adaptive, meaning that after removing a node, all of the weights of the remaining nodes are recalculated.

1. Highest degree

The multiplex generalization of the highest degree strategy is simple and straightforward. Each node has a well-defined degree on each of the layers in which it participates. We considered two ways of combining these degrees. We define the DegSum score of node i as the sum of the degrees of node i over all the layers in which it participates. Similarly, let the DegProd score be the product of the degrees of node i over all the layers in which it participates. These two strategies provide similar results, which are—somewhat surprisingly—significantly better than results from the multiplex versions of the more sophisticated CoreHD and CIP algorithms.

2. CoreHD

Many possibilities arise when aiming to generalize the CoreHD algorithm to multiplex networks. First, one may consider the intersection of the 2-cores of the separate layers or the union of them. We have tried variations of both and found that strategies where only the intersection of the 2-cores was considered always performed noticeably worse than methods dealing with the union of the 2-cores. It appears that considering only the intersection is not a good strategy because a node that is the member of a 2-core only on one layer may still play a more important role—by virtue of this one layer—than some other less connected nodes that happen to be in the intersection of the 2-cores. Due to these considerations we focus on generalizations of CoreHD that attack the union of the 2-cores of the separate layers.

A second question that arises is how to combine the CoreHD scores of different layers. We considered both a sum and a product version, CoreHD_S and CoreHD_P, respectively. For a given node i these methods simply compute the sum or the product of the CoreHD scores for each layer of node i . If node i is in the intersection of the 2-cores, then it has a well-defined CoreHD score (≥ 2) on each layer. If on any of the layers node i does not belong to the 2-core, we assign it a score 0 for the given layer in the case of CoreHD_S and 1 in the case of CoreHD_P. This scheme applies also to partially interdependent networks: A node certainly does not belong to the 2-core of a layer in which it does not participate, and so it is assigned a score as explained above. When none of the layers contain a 2-core, i.e., all layers are trees, we proceed with the DegSum algorithm to break the MCCs up into subextensive sizes. This generally only takes a small number of steps and the actual method of tree breaking does not play a relevant role. For the majority of networks considered, CoreHD_S proved to be the better strategy, therefore this was chosen in the main text for comparison with other methods. (In most cases CoreHD_P produced very similar slightly worse results.)

As pointed out in the main text, when one of the layers in the multiplex network is much sparser than the others, then this layer may essentially determine connectivity in the network. Destroying the LMCC in this case corresponds closely to destroying the largest component in the weakest

(most sparse) layer, therefore single-layer attack strategies can be expected to work very well. We considered an algorithm—CoreHD1—where, in every step, we find the layer on which the LMCC has the smallest mean degree. Then we assign weights to nodes using the original CoreHD method on this layer. As can be seen in Fig. 4 of the main text, CoreHD1 is a very good candidate for multiplex networks with highly different layer densities but also performs well in symmetric networks with a moderate fraction of interdependent nodes.

3. Collective influence propagation

We considered a multiplex generalization of the CIP algorithm [9]. For simplex networks this algorithm was shown to work better than the previous collective influence algorithm, so we chose only to focus on a generalization of CIP. This algorithm assigns a number to all nodes in a single layer: For nodes in the 2-core the score is positive, and for nodes outside the 2-core the score is 0. Again we have a choice of how to combine the scores of nodes on different layers. Unlike in the case of simplex networks where only the relative value of scores matters, here the normalization of the CIP scores becomes relevant because the scores of nodes on different layers must be comparable in a meaningful way. (Note that this issue does not arise in the case of CoreHD where the scores are simply the degrees—inside the 2-core—of nodes in the 2-core.) An intuitively appropriate normalization scheme is to have the CIP scores on each layer sum to N_m and the number of nodes on layer m . Then, we define the values $\text{CIP}_s(i)$ for each node i as the sum of the (correctly normalized) $\text{CIP}(i)$ values over all the layers in which node i participates.

We did not consider the product version of the CIP algorithm because in this case the following problem arises. Consider two nodes on a given layer that have the same CIP values (on this layer). Assume that one of the nodes is present only on this layer, but the other one participates on all other layers. If the latter (interdependent) node has CIP values less than 1 on the other layers, it will receive a score that is less than the score of the first (free) node if the scores of different layers are combined as a product. This is rather counterintuitive. There are certainly ways of circumventing this problem, but such methods would look rather contrived and arbitrary, therefore we chose to consider only the sum version CIP_s .

APPENDIX C: SIMULATED ANNEALING

As a benchmark, against which all other strategies can be compared, we used a SA algorithm to identify a close upper bound on the size of the minimal damaging set that makes the LMCC subextensive. Our SA method employs a standard Metropolis Monte Carlo algorithm as follows. We consider the configuration space where all nodes in the network (free or interdependent) can be in one of two states: *present* or *removed*. We exclude all configurations where there is an extensive LMCC in the network when only nodes present are considered. (Our criterion for extensive is to have a size larger than \sqrt{N} , the square root of the network size.) We define the energy of a configuration simply as the number R of nodes removed. We simulate this system in thermal equilibrium at temperature T by applying the following steps repeatedly:

(1) We choose a node in the network uniformly at random and make the trial change in switching its state to the opposite. (If it was *removed*, switch to *present*, and if it was *present*, switch to *removed*.)

(2) Find the size of the LMCC in the network, and if it is extensive ($> \sqrt{N}$), reverse the trial change, and go to Step 1.

(3) If the LMCC after the trial change is still subextensive, accept the trial change with probability $p = \min(1, e^{-\Delta R/T})$, and go to Step 1.

As initial condition, we set all nodes to *removed*, i.e., $R = N$. The initial temperature for the simulations was $T_{\max} = 1$, and in every step it was reduced by δT until $T_{\min} = 0.01$ was reached. $\delta T = 10^{-7}$ was used for networks of size $N = 10^4$, and in a reasonable running time, results were all considerably better than those provided by faster attack strategies. For network size $N = 10^5$, it was necessary to increase δT to 10^{-6} in order to maintain a feasible running time. As a result, SA already performed much worse than some of the best heuristic strategies.

Simulated annealing is a widely used technique, that is, assumed to approach the optimal solution when given sufficient time to thoroughly probe the configuration space. It is therefore a good choice as a benchmark for comparisons, but it is prohibitively slow for large system sizes, and so it cannot be considered a viable candidate for practical purposes. Also, being a purely stochastic method, it does not give us any insight as to the structure of a good damage set.

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- [1] S. V. Buldyrev, R. Parshani, G. P. H. E. Stanley, and S. Havlin, *Nature (London)* **464**, 1025 (2010).
 - [2] G. J. Baxter, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, *Phys. Rev. Lett.* **109**, 248701 (2012).
 - [3] M. J. O. Pocock, D. M. Evans, and J. Memmott, *Science* **335**, 973 (2012).
 - [4] S. M. Rinaldi, J. P. Peerenboom, and T. K. Kelly, *IEEE Control Syst. Mag.* **21**, 11 (2001).
 - [5] M. Szell, R. Lambiotte, and S. Thurner, *Proc. Natl. Acad. Sci. USA* **107**, 13636 (2010).
 - [6] A. Braunstein, L. Dall'Asta, G. Semerjian, and L. Zdeborová, *Proc. Natl. Acad. Sci. USA* **113**, 12368 (2016).
 - [7] F. Altarelli, A. Braunstein, L. Dall'Asta, and R. Zecchina, *J. Stat. Mech.: Theory Exp.* (2013) P09011.
 - [8] F. Morone and H. A. Makse, *Nature (London)* **524**, 65 (2015).
 - [9] F. Morone, B. Min, L. Bo, R. Mari, and H. A. Makse, *Sci. Rep.* **6**, 30062 (2016).
 - [10] L. Zdeborová, P. Zhang, and H.-J. Zhou, *Sci. Rep.* **6**, 37954 (2016).
 - [11] S.-W. Son, G. Bizhani, C. Christensen, P. Grassberger, and M. Paczuski, *Europhys. Lett.* **97**, 16006 (2012).
 - [12] G. J. Baxter, D. Cellai, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, *Interconnected Networks* (Springer, New York, 2016), pp. 101–123.

- [13] X. Huang, J. Gao, S. V. Buldyrev, S. Havlin, and H. E. Stanley, [Phys. Rev. E **83**, 065101 \(2011\)](#).
- [14] G. Dong, J. Gao, L. Tian, R. Du, and Y. He, [Phys. Rev. E **85**, 016112 \(2012\)](#).
- [15] S. Osat, A. Fagheh, and F. Radicchi, [Nat. Commun. **8**, 1540 \(2017\)](#).
- [16] B. Pittel, J. Spencer, and N. Wormald, [J. Comb. Theory, Ser. B **67**, 111 \(1996\)](#).
- [17] J. Balogh and B. G. Pittel, [Random Struct. Alg. **30**, 257 \(2007\)](#).
- [18] G. J. Baxter, S. N. Dorogovtsev, K.-E. Lee, J. F. F. Mendes, and A. V. Goltsev, [Phys. Rev. X **5**, 031017 \(2015\)](#).
- [19] G. J. Baxter, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, [Phys. Rev. E **83**, 051134 \(2011\)](#).
- [20] M. De Domenico, A. Solé-Ribalta, S. Gómez, and A. Arenas, [Proc. Natl. Acad. Sci. USA **111**, 8351 \(2014\)](#).
- [21] J. Iacovacci, C. Rahmede, A. Arenas, and G. Bianconi, [Europhys. Lett. **116**, 28004 \(2016\)](#).
- [22] A. Solé-Ribalta, M. De Domenico, S. Gómez, and A. Arenas, [Physica D \(Amsterdam\) **323-324**, 73 \(2016\)](#).
- [23] Z. Kuncheva and G. Montana, in *Proceedings of the 2015 IEEE/ACM International Conference on Advances in Social Networks Analysis and Mining (ASONAM), 2015*, edited by J. Pei, F. Silvestri, and J. Tang (IEEE, Piscataway, NJ, 2015), pp. 1308–1315.
- [24] R. Parshani, S. V. Buldyrev, and S. Havlin, [Phys. Rev. Lett. **105**, 048701 \(2010\)](#).
- [25] R. Parshani, S. V. Buldyrev, and S. Havlin, [Proc. Natl. Acad. Sci. USA **108**, 1007 \(2011\)](#).