PCA $X=(X_1,\cdots,X_n)$ $\quad E(X_i X_j)\neq 0$. Dispersion.

Second Order Moment (Statistics). Linear Relation

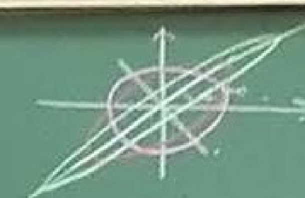
$\alpha\in\mathbb{R}^n$, $X\in\mathbb{R}^n$, $E|\alpha^T X|^2 = g(\alpha)$ $\quad \underline{\max_{\alpha} g(\alpha)}$, s.t $\|\alpha\|=1$.

$\alpha=U_1$ $\quad E(XX^T)=R_X=\sum_{k=1}^{n}\lambda_k U_k U_k^T$, $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$

$X=(X_1,X_2)$ $\quad E(X_1)=E(X_2)=0$, $Var(X_1)=Var(X_2)=1$. $E(X_1X_2)=\rho$

$$R_X=\begin{pmatrix} EX_1^2 & EX_1X_2 \\ EX_2X_1 & EX_2^2 \end{pmatrix}=\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \qquad (\rho>0).$$

$$p(\lambda)=\det(\lambda I-R_X)=\det\begin{pmatrix} \lambda-1 & -\rho \\ -\rho & \lambda-1 \end{pmatrix}.$$



$$=(\lambda-1)^2-\rho^2=0 \Rightarrow \lambda_1=1+\rho,\ \lambda_2=1-\rho.$$

$$(\lambda_1 I-R_X)y_1=0 \Rightarrow \begin{pmatrix} \rho & -\rho \\ -\rho & \rho \end{pmatrix} y_1=0 \Rightarrow y_1=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(\lambda_2 I-R_X)y_2=0 \Rightarrow \begin{pmatrix} -\rho & -\rho \\ -\rho & -\rho \end{pmatrix} y_2=0 \Rightarrow y_2=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

K-L Expansion. $X(t) = \sum_k \alpha_k \phi_k(t)$. ϕ_k : Basis Function

Biorthogonal. $E(\alpha_k \alpha_m) = \delta_{km}$. $\langle \phi_k(t), \phi_m(t) \rangle = \delta_{km}$. α_k : Coefficient

$[0, T]$. $\alpha_k = \frac{1}{T}\int_0^T X(t)\phi_k(t)\,dt$.

$$E(\alpha_k \alpha_m) = \frac{1}{T^2}\int_0^T\int_0^T E(X(t)X(s))\phi_k(t)\phi_m(s)\,dt\,ds.$$

$$= \frac{1}{T^2}\int_0^T\int_0^T \boxed{R_X(t,s)}\phi_k(t)\phi_m(s)\,dt\,ds = \delta_{km}$$

$$\boxed{\int_0^T R_X(t,s)\,\phi_k(s)\,ds = \lambda_k \phi_k(t)} \Longleftrightarrow \underline{R_X \cdot U_k = \lambda_k U_k}.$$

$R_X(t,s) = R_X(s,t)$. $\sum_s R_X(t,s)\,U_k(s) = \lambda_k U_k(t)$

$\lambda_k, \lambda_m, U_k, U_m$. $R_X \cdot U_k = \lambda_k U_k$. $R_X U_m = \lambda_m U_m$.

$$U_m^T R_X U_k = \underline{\lambda_k U_m^T U_k} = U_k^T R_X^T U_m = U_k^T R_X U_m = \underline{\lambda_m U_m^T U_k}$$

$$(\lambda_k - \lambda_m) U_k^T U_m = 0 \Rightarrow U_k^T U_m = 0,\ \lambda_k \neq \lambda_m$$

② $\cos(\omega_0 t) = \frac{1}{2}\left(\exp(j\omega_0 t) + \exp(-j\omega_0 t)\right)$.

③ $\underline{g(t)}$ is p.d $\Rightarrow$ $\underline{g(t)g(s)}$. is p.d.

$\forall n.\ \forall \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}\quad \forall t_1, \cdots, t_n \in \mathbb{R}$.

$$\sum_{i=1}^{n}\sum_{j=1}^{n} g(t_i)g(t_j)\alpha_i\overline{\alpha_j} = \left|\sum_{i=1}^{n} g(t_i)\alpha_i\right|^2 \geq 0.$$

$b \in \mathbb{R}^n$. $\quad b \cdot b^T \geq 0 \qquad \int g(t)g(s)\phi_k(s)\,ds = \underline{g(t)}\boxed{\int g(s)\phi_k(s)\,ds}$

$$\frac{1}{T^2}\int_0^T \boxed{\int_0^T R_X(t,s)\,\Phi_m(s)\,ds}\,\Phi_k(t)\,dt.$$

$$\int_0^T \lambda_m \Phi_m(t)\,\Phi_k(t)\,dt = 0.$$

Period. T.

$$\boxed{\beta_1 \cos\frac{2\pi t + \tau}{T}\cos\frac{2\pi s + \tau}{T} + \beta_2\cos\frac{4\pi t}{T}\cos\frac{4\pi s}{T}}$$

① $\exp(j\omega_0 t) = \cos(\omega_0 t) + j\sin(\omega_0 t)$

$\exp(-j\omega_0 t)$ is positive definite

$\forall n\ \forall \alpha_1, \cdots, \alpha_n \in \mathbb{C}$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\exp(j\omega_0(t_i - t_j))\alpha_i\overline{\alpha_j}$$

$$= \left(\sum_{i=1}^{n}\exp(j\omega_0 t_i)\alpha_i\right)\left(\sum_{j=1}^{n}\exp(-j\omega_0 t_j)\overline{\alpha_j}\right)$$

$$= \left|\sum_{i=1}^{n}\exp(j\omega_0 t_i)\alpha_i\right|^2$$

$$\geq 0$$

$$Q = \begin{pmatrix} -\lambda & \frac{\lambda}{N-1} & \cdots & \frac{\lambda}{N-1} \\ \frac{\lambda}{N-1} & \ddots & & \frac{\lambda}{N-1} \\ \frac{\lambda}{N-1} & & & -\lambda \end{pmatrix}$$

$\boxed{\Phi_k(t)} = \boxed{\cos\left(\frac{2k\pi}{T}t\right)}$ $\quad k=1,2,3,\cdots,n,\cdots$

$$\int_0^T \cos\frac{2\pi t}{T}\cos\frac{2\pi s}{T}\cdot\cos\frac{2k\pi s}{T}\,ds = \cos\frac{2\pi t}{T}\int_0^T \cos\frac{2\pi s}{T}\cos\frac{2k\pi s}{T}\,ds$$

$k=1 \quad \neq 0$

$$\int_0^T \cos\frac{4\pi t}{T}\cos\frac{4\pi s}{T}\cos\frac{2k\pi s}{T}\,ds = \cos\frac{4\pi t}{T}\int_0^T \cos\frac{4\pi s}{T}\cos\frac{2k\pi s}{T}\,ds$$

$\underline{\gamma(t) = \alpha_1\cos\frac{2\pi t}{T} + \alpha_2\cos\frac{4\pi t}{T}}$ $\quad E(\alpha_1\alpha_2)=0 \quad E|\alpha_1|^2=E|\alpha_2|^2=1$

Spectral Analysis of Stochastic Processes.

Transformation Domain.

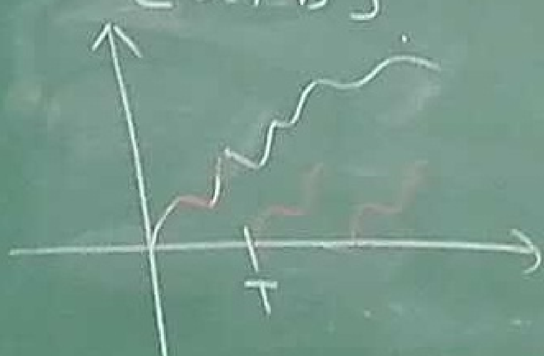
$[a, b]$

$X(t)$ is deterministic. <u>wave</u>

① Periodic T. $X(t+T)=X(t)$. $\frac{2\pi}{T}$

Fourier Series: $\underline{X(t)} \overset{?}{=} \sum_{k=-\infty}^{+\infty}\alpha_k\exp\left(j\frac{2k\pi}{T}t\right)$

$$\alpha_k = \frac{1}{T}\int_0^T X(t)\exp\left(-j\frac{2k\pi}{T}t\right)dt$$



② Non-Period. $\boxed{T \to \infty}$ Integral Sum $\frac{2\pi}{T}k$ ∞

$$x(t) = \sum_{k=-\infty}^{+\infty} \alpha_k \exp\left(j\frac{2k\pi}{T}t\right) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} \boxed{\int_{-\frac{T}{2}}^{\frac{T}{2}} x(s)\exp\left(-j\frac{2k\pi}{T}s\right)ds}$$

$x(s) \in L^1(\mathbb{R})$.

$\int_{-\infty}^{+\infty} |x(s)|ds < \infty$

$$\begin{cases} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{x}(\omega)\exp(j\omega t)\,d\omega & \cdot \exp\left(j\frac{2k\pi}{T}t\right)\cdot\frac{2\pi}{T} \\ \hat{x}(\omega) = \int_{-\infty}^{+\infty} x(s)\exp(-j\omega s)\,ds & \text{Fourier Transform} \end{cases}$$

$X(t)$

deterministic → Random ① Period. T. $E(X(t) - X(t+T))^2 \overset{0}{=}$

W.S.S. $R_X(\tau) = R_X(\tau + T)$. $R_X(\tau) = \sum_k \beta_k \exp\left(j\frac{2k\pi}{T}\tau\right)$

Wide-Sense Stationary

$$\int_0^T R_X(t-s)\phi_k(s)\,ds = \lambda_k\phi_k(t) \Rightarrow \phi_k(t) = \exp\left(j\frac{2k\pi}{T}t\right)$$

$$X(t) = \boxed{\sum_k \alpha_k \exp\left(j\frac{2k\pi}{T}t\right)} \qquad E(\alpha_k\alpha_m) = \delta_{km}.$$

$$0 \le \lim_{T\to\infty}\frac{1}{T}E\left|\int_{-\frac{T}{2}}^{\frac{T}{2}} X(t)\exp(-j\omega t)\,dt\right|^2 = \int_{-\infty}^{+\infty} R_X(\tau)\exp(-j\omega\tau)\,d\tau$$

Wiener-Khinchine Bochner-Khinchine

Agent $\begin{cases} S_X(\omega) = \int_{-\infty}^{+\infty} R_X(\tau)\exp(-j\omega\tau)\,d\tau. \\ R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_X(\omega)\exp(j\omega\tau)\,d\omega. \end{cases}$ Power Spectral Density (PSD)

$V^2 T = V^2/(1/T)$

$\int_{-\infty}^{+\infty} S_X(\omega)\,d\omega = 2\pi R_X(0)$

$S_X(\omega) \geq 0$, $S_{X+Y}(\omega) = S_X(\omega) + S_Y(\omega)$, $E(X(t)Y(s)) = 0$

$S_X(-\omega) = S_X(\omega)$

$$S_X(\omega) = \int_{-\infty}^{+\infty} R_X(\tau)\exp(-j\omega\tau)\,d\tau. \qquad R_X(\tau) = R_X(-\tau).$$

$$= \int_{-\infty}^{+\infty} R_X(\tau)\cos\omega\tau\,d\tau + j\int_{-\infty}^{+\infty} R_X(\tau)\sin(\omega\tau)\,d\tau.$$

$$= \int_{-\infty}^{+\infty} R_X(\tau)\cos(\omega\tau)\,d\tau.$$