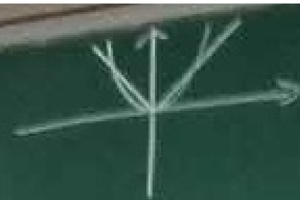


$X_1, X_2, \cdots, X_n$. $\quad X, Y$ r.v. $\in \mathbb{R}$.

Correlation. Binary Relation.

Distance (Metric). x, y const. $|x-y|$ Euclid. Dist

$|X-Y| \longrightarrow E|X-Y| \longrightarrow E|X-Y|^2$ $\quad E|X-Y|^2=0$

$\Downarrow$

$\longrightarrow (E|X-Y|^2)^{\frac{1}{2}} = d(X,Y)$. $\quad X=Y \approx P(X=Y)=1$

$d(x,y) : L \times L \longrightarrow \mathbb{R}_+$.

① $d(x,y) \geq 0$. $\quad d(x,y)=0 \Longleftrightarrow x=y$.

② $d(x,y)=d(y,x)$.

③ $d(x,y)+d(y,z) \geq d(x,z)$

Mean Square Distance $(E|X-Y|^2)^{\frac{1}{2}}$

$$E|X|^2+E|Z|^2 \geq E|X-Z|^2 = E|X|^2+E|Z|^2-2E(XZ)$$

$$(E|X|^2)^{\frac{1}{2}}+(E|Z|^2)^{\frac{1}{2}} \geq (E|X-Z|^2)^{\frac{1}{2}}$$

$$\left(E|\xi|^2 E|\zeta|^2\right)^{\frac{1}{2}} \geq |E(\xi\zeta)|. \quad \text{Cauchy-Schwarz}$$

$x_1, \cdots, x_n,\ y_1, \cdots, y_n \in \mathbb{R}$.

$$\left(\sum_{k=1}^{n}|\xi_k|^2 \sum_{k=1}^{n}|y_k|^2\right)^{\frac{1}{2}} \geq \left|\left(\sum_{k=1}^{n} x_k y_k\right)\right|$$

$$\left(\frac{x_1+\cdots+x_n}{n}\right)^2 \leq \frac{x_1^2+\cdots+x_n^2}{n}$$

$f(x),\ g(x),\ \mathbb{R}\to\mathbb{R}$.

$$\left(\int_{\mathbb{R}} f^2(x)dx \int_{\mathbb{R}} g^2(x)dx\right)^{\frac{1}{2}} \geq \left|\int_{\mathbb{R}} f(x)g(x)dx\right|$$

Inner Product

$x, y \in L.\ \langle x, y\rangle$

① $\langle x, x\rangle \geq 0$ $\quad \langle x, x\rangle = 0 \Leftrightarrow x = 0$

② Bilinear: $\langle \alpha x+\beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$.

$\langle x, \alpha y+\beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$.

$$\langle (x_1, \cdots, x_n), (y_1, \cdots, y_n)\rangle = \sum_{k=1}^{n} x_k y_k \qquad \sum_{k=1}^{n} x_k^2 \geq 0.$$

$$y_k \to (\alpha z_k + \beta h_k)$$

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)dx.$$

$$\langle \xi, Y\rangle = E(\xi Y)$$

$$(E|X-Y|^2)^{\frac{1}{2}} = (\underbrace{E|X|^2}_{1} + \underbrace{E|Y|^2}_{1} - 2\boxed{E(XY)})^{\frac{1}{2}}.$$

$E(XY)$. Correlation. Geometry

Angle. $\frac{\langle x, y\rangle}{|\langle x, x\rangle|^{\frac{1}{2}} \, |\langle y, y\rangle|^{\frac{1}{2}}} = \cos\angle(x, y)$.

Hilbert Space

$$\frac{E(XY)}{|EX^2|^{\frac{1}{2}} \, |EY^2|^{\frac{1}{2}}} = \cos\angle(X, Y)$$

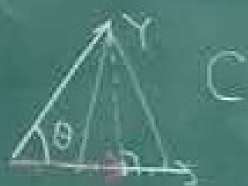
$$\alpha = \frac{E(XY)}{E(X^2)}.$$

$$E|X|^2 + E|Z|^2$$
$$\geq E|X - Z|^2.$$
$$\|$$
$$E|X|^2 + E|Z|^2$$
$$2E(XZ)$$
$$E|Z|^2)^{\frac{1}{2}}$$
$$Z|^2)^{\frac{1}{2}}$$

X, Y Orthogonal. $\angle(X, Y) = \frac{\pi}{2} \Leftrightarrow E(XY) = 0$.

$\|X\| = 1$.

Projection. $\mathrm{Proj}_X Y = C\frac{X}{\|X\|} = \frac{E(XY)}{E(X^2)} X$.



X

$$C = \|Y\| \cos\angle(X, Y) = \frac{E(XY)}{|EX^2|^{\frac{1}{2}} \, |EY^2|^{\frac{1}{2}}} \cdot |EY^2|^{\frac{1}{2}} = \frac{E(XY)}{(E|X|^2)^{\frac{1}{2}}}$$

$$\min_{\alpha} \|Y - \alpha X\| \longrightarrow \min_{\alpha} \|Y - \alpha X\|^2 = \min_{\alpha} \alpha^2 \|X\|^2 - 2\alpha E(XY) + \|Y\|^2$$

$R_X = E(X X^T)$. $R_X(i,j) = E(X_i X_j)$ $E(X_i X_j) = E(X_j X_i)$

$\frac{n(n+1)}{2}$, $\frac{n(n-1)}{2} = \binom{n}{2}$.   Correlation Matrix. $R_X(i,j)$

$\Downarrow$

$= R_X(j,i)$

$\Downarrow$

$R_X = R_X^T$

$$\nabla_\alpha E(Y - \alpha^T X)^2 = -r_{XY} - r_{XY} + (R_X + R_X^T)\cdot\alpha$$

$$= 2R_X\cdot\alpha - 2r_{XY} = 0$$

$$\Rightarrow R_X\cdot\alpha = r_{XY} \Rightarrow \underline{\alpha = R_X^{-1}\cdot r_{XY}} \quad \text{(Yule-Walker)}$$

$Y \in \mathbb{R}$. Y, $X_1, \cdots, X_n$. $(X_1, \cdots, X_n) \to Y$. Linear Approximation

$X \in \mathbb{R}^n$. $\alpha = (\alpha_1, \cdots, \alpha_n)^T$ $\min_\alpha \| Y - \sum_{k=1}^{n} \alpha_k X_k \|^2$ $X = (X_1, \cdots, X_n)^T$.

$$\min_\alpha E(Y - \alpha^T X)^2 = \min_\alpha E(Y - \alpha^T X)(Y - \alpha^T X)^T.$$

$$= \min_\alpha \left(E|Y|^2 - \alpha^T E(YX) - E(YX^T)\alpha + \alpha^T E(XX^T)\alpha \right)$$

$$= \min_\alpha (E|Y|^2 - \alpha^T r_{XY} - r_{XY}^T\cdot\alpha + \alpha^T R_X \alpha.$$

$\frac{\int_{\Omega} x\, g(x)\,dx}{1-\int_{\Omega} g(x)\,dx}$ — mass density

X. a determined constant.

$\min_a E(X-a)^2 \qquad H(a)=E(X-a)^2$

$\frac{d}{da}H(a) = -2E(X-a) = 0 \Rightarrow a = E(X)$

$E(X_1+\cdots+X_n) = E(X_1)+\cdots+E(X_n)$

Matching N persons, N hats.

$I_k = \begin{cases} 0 & k \text{ person matched} \\ 1 & k \text{ person mismatch} \end{cases} \qquad N = I_1+\cdots+I_n.$

Sample Space. $\{A_1 A_2 \cdots A_n, \cdots\} = \Omega$.

$A_k \in \{H_1, \cdots, H_n\}$.

$\frac{(n-1)!}{n!} = \frac{1}{n}$ $\quad B_1 \subseteq \Omega$. 1 person matched

$P(B_1 \cap B_2)$ $\quad B_2 \subseteq \Omega$. 2 person matched

$= \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$ $\quad \cdots$ $\quad B_n \subseteq \Omega$, n person matched

$P(\overline{B_1} \cap \overline{B_2} \cap \cdots \cap \overline{B_n})$

$= P(\overline{B_1 \cup B_2 \cup \cdots \cup B_n})$

$= 1 - P(B_1 \cup B_2 \cup \cdots \cup B_n)$

$P(B_1 \cup B_2 \cup \cdots \cup B_n)$. (Inclusion–Exclusion).

$= P(B_1) + \cdots + P(B_n)$

$- \sum_{i<j} P(B_i \cap B_j)$

$+ \sum_{i<j<k} P(B_i \cap B_j \cap B_k) - \cdots$

$$= n \cdot \frac{1}{n} - \frac{n(n-1)}{2} \cdot \frac{1}{n(n-1)} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n(n-1)(n-2)} + \cdots$$

$$\exp(-x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$$

$P(\bar{B}_1 \cap \cdots \cap \bar{B}_n)$

$$= 1 - \left(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}\right.$$

$$= \exp(-1)$$

$E(N) = E(I_1 + \cdots + I_n) = E(I_1) + \cdots + E(I_n)$.

$E(I_k) = 1 \cdot P(I_k = 1) + 0 \cdot P(I_k = 0)$.

$$= 1 \cdot \frac{1}{n} = \frac{1}{n}$$

$$E(N) = n \cdot \frac{1}{n} = 1.$$

$X(t) = X(\omega, t)$ Sample Path.

$X(t)$, $X(s)$. $E(X(t)X(s)) = R_X(t, s)$. Binary Function.

(Auto-)Correlation Function. $R_X(t, t) \geq 0$. $E|X(t)|^2 \geq 0$.

Cross-Correlation $E(X(t)Y(s))$ $R_X(t, s) = R_X(s, t)$

$$|R_X(t, s)| \leq (R_X(t, t) R_X(s, s))^{\frac{1}{2}}$$

Positive-Definite. $g(x,y)$ is P.d.

$\Longleftrightarrow \forall n$, $\forall t_1,\cdots,t_n$, t.s. $(g(t_i,t_j))_{ij} = G$, G is P.d.

$\in \mathbb{R}$

$G \in \mathbb{R}^{n\times n}$ is P.d $\Longleftrightarrow$ $\underline{\forall \alpha \in \mathbb{R}^n}$ $\underline{\alpha^T G \alpha} \geq 0$.

$R_X(t_i,t_j) = E(X(t_i)X(t_j))$ Let $X = (X(t_1),\cdots,X(t_n))^T$

$G_X = E(\underline{X}\cdot \underline{X}^T)$ $\alpha^T G_X \alpha = \alpha^T E(XX^T)\alpha = E(\alpha^T X X^T \alpha) = E|\alpha^T X|^2 \geq 0$

$E(N) = E(I_1 + \cdots + I_n) = E(I_1) + \cdots + E(I_n)$

$E(I_k) = 1\cdot P(I_k = 1)$ Characteristic Property

$\underline{\frac{1}{2}(A+A^T)}$ $= 1\cdot \frac{1}{n} = \frac{1}{n}$ Correlation $\Longleftrightarrow$ Positive Definite

$\alpha^T A \alpha = \underline{\alpha}^T\left(\frac{1}{2}(\underline{A}+\underline{A^T})\right)\underline{\alpha}$ $R_X(t,t) \geq 0$

$\frac{1}{2}(R_X(t,s) + R_X(s,t))$ $\begin{pmatrix} R_X(t,t) & R_X(t,s) \\ R_X(s,t) & R_X(s,s) \end{pmatrix}$