Markov Chains

Recurrent or Non-Recurrent $\quad P_{ij}(n) \to 0. \ (n \to \infty)$

$\sum_{n=1}^{\infty} f_{ii}(n) = 1. \Longleftrightarrow \sum_{n=0}^{\infty} P_{ii}(n) = \infty.$ "常" $\quad f_{ii} = \sum_{n=1}^{\infty} f_{ii}(n).$

①. i. N. r.v. $E(N) = \sum_{k=1}^{\infty} k P(N=k) = \sum_{k=1}^{\infty} k (f_{ii})^k (1-f_{ii}).$

$$= f_{ii}(1-f_{ii}) \sum_{k=1}^{\infty} k f_{ii}^{k-1} = f_{ii}(1-f_{ii}) \left(\sum_{k=1}^{\infty} f_{ii}^{k}\right)' = f_{ii}(1-f_{ii}) \left(\frac{1}{1-f_{ii}} - 1\right)'$$

$$= f_{ii}(1-f_{ii}) \cdot \frac{1}{(1-f_{ii})^2} = \frac{f_{ii}}{1-f_{ii}} = \infty \ (\text{i is Recurrent})$$

② $P(N=\infty) = 1 \quad g_{ii}(m) = P(N \geq m \mid X_0 = i).$

$$g_{ii}(m+1) = P(N \geq m+1 \mid X_0 = i) = \sum_{k=1}^{\infty} P(X_k = i, X_{k-1} \neq i, \cdots, X_1 \neq i, N \geq m \mid X_0 = i)$$

$$= \sum_{k=1}^{\infty} P(N \geq m \mid X_k = i, X_{k-1} \neq i, \cdots, X_1 \neq i, X_0 = i) P(X_k = i, X_{k-1} \neq i, \cdots \mid X_0 = i)$$

$$= \sum_{k=1}^{\infty} P(N \geq m \mid X_k = i) f_{ii}(k) = \sum_{k=1}^{\infty} P(N \geq m \mid X_0 = i) f_{ii}(k)$$

$$= P(N \geq m \mid X_0 = i) \sum_{k=1}^{\infty} f_{ii}(k) = g_{ii}(m) f_{ii}$$

Zero-One Law

$$\frac{g_{ii}(m+1)}{g_{ii}(m)} = f_{ii} \qquad g_{ii}(m) = \overset{1}{\overset{\parallel}{g_{ii}(0)}}(f_{ii})^m = (f_{ii})^m .$$

$$g_{ii}(\infty) = P(N = \infty \mid X_0 = i) = \lim_{m \to \infty} (f_{ii})^m = \begin{cases} 1 & i \text{ Recurrent} \\ 0 & i \text{ Non-Recurrent} \end{cases}$$

i is Recurrent, $i \to j \Rightarrow j$ Recurrent

$\Rightarrow j \to i \qquad \exists m \quad P_{ij}(m) > 0$

$$\underset{\underset{1}{\parallel}}{g_{ii}(\infty)} = P(N = \infty \mid X_0 = i) = \sum_k P(N = \infty, X_m = k \mid X_0 = i).$$

$$= \sum_k P(N = \infty \mid X_m = k, X_0 = i)\, P(X_m = k \mid X_0 = i).$$

$$= \sum_k \underbrace{P_{ik}(m)}_{\leq 0} \cdot g_{ki}(\infty) = 1 = \sum_k P_{ik}(m) \qquad g_{ji}(\infty) = 1$$

$$\sum_k \underbrace{P_{ik}(m)}_{\geq 0}(\underbrace{g_{ki}(\infty) - 1}_{\leq 0}) = 0 \qquad \overset{>0}{P_{ij}(m)}(\overset{=0}{\underline{g_{ji}(\infty) - 1}}) = 0$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad P_{00}(n) = 0, 1, 0, 1, 0, 1, \cdots$$

$$P_{01}(n) = 1, 0, 1, 0, 1, 0, \cdots$$

Period: i. $d_i = \gcd\{k : P_{ii}(k) > 0\}$. (Greatest Common Divisor.)

$d_i = 1 \Rightarrow i$ is Non-Periodic $\quad d_0 = d_1 = 2$

$i \Leftrightarrow j \Rightarrow d_i = d_j$ ① $\exists\, m, n,\ P_{ij}(m) > 0,\ P_{ji}(n) > 0 \Rightarrow P_{ii}(m+n) > 0$

② $\forall k,\ P_{jj}(k) > 0 \Rightarrow P_{ii}(m+n+k) > 0 \Rightarrow \boxed{d_i \mid (m+n+k)},\ \boxed{d_i \mid (m+n)}$

③ $d_i \mid k \Rightarrow d_i = cd\{k : P_{jj}(k) > 0\}$.

④ $d_i \mid d_j$. $d_j \mid d_i \Rightarrow d_i = d_j$.

$$\begin{cases} P(n) = P \cdot P(n-1) \\ P(n) = P(n-1) \cdot P \end{cases}$$

$X_n = g(X_{n-1}) \longrightarrow X = g(X)$

$\boxed{\text{Non-Periodic}}$ + Irreducible $\Longrightarrow \lim\limits_{n\to\infty} P_{ij}(n)$ Exists!

$\lim\limits_{n\to\infty} P_{ij}(n) = \pi_j \qquad P(n) \longrightarrow \begin{pmatrix} \pi_0, \pi_1, \pi_2, \cdots \\ \pi_0, \pi_1, \pi_2, \cdots \\ \pi_0, \pi_1, \pi_2, \cdots \\ \cdots \end{pmatrix} = \Pi$

$$\begin{pmatrix} \pi_k \\ \vdots \\ \pi_k \end{pmatrix} = P \cdot \Pi = P \cdot \begin{pmatrix} \pi_k \\ \vdots \\ \pi_k \end{pmatrix}$$

$$P(X_{k+1}=j \mid X_k=i) = \begin{cases} \frac{N-i}{N} & j=i+1. \\ \frac{i}{N} & j=i-1. \\ 0. & \text{others.} \end{cases}$$

$(\pi_0\ \pi_1 \cdots)$

$$\begin{pmatrix} 0 & 1 & & & & \\ \frac{1}{N} & 0 & \frac{N-1}{N} & & & \\ & \frac{2}{N} & 0 & \frac{N-2}{N} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{N-1}{N} & 0 & \frac{1}{N} \\ & & & & 1 & 0 \end{pmatrix} = P$$

$$\pi = \pi \cdot P$$

$$\pi_1 = N\pi_0 = \binom{N}{1}\pi_0$$

$$\pi_2 = \frac{N(N-1)}{2}\pi_0 = \binom{N}{2}\pi_0$$

$$\begin{cases} \pi_0 = \frac{1}{N}\pi_1 \\ \pi_1 = \pi_0 + \frac{2}{N}\pi_2 \\ \pi_2 = \frac{N-1}{N}\pi_1 + \frac{3}{N}\pi_3 \\ \pi_k = \frac{N-k+1}{N}\pi_{k-1} + \frac{k+1}{N}\pi_{k+1} \end{cases}$$

$\boxed{\pi = \pi \cdot P}$. $\pi = (\pi_0, \pi_1, \cdots, \pi_n, \cdots)$.

$Px = x$. $x = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$\Updownarrow$

$\det(P - I) = 0$

$\Updownarrow$

$\pi = \pi \cdot P \Leftrightarrow P^T \pi^T = \pi^T \Leftrightarrow (P^T - I)\pi^T = 0 \Leftrightarrow \det(P^T - I) = 0$

Ehrenfest Model. ① Time Discretization $\{0, 1, 2, 3, \cdots\}$.

N

② . Diffusion $\Leftrightarrow$ Left $\rightleftarrows$ Right

③ . Randomly Selected $\{0, \cdots, N\}$

X_k : Number of Particles in Left, Time k.

$$\begin{cases} \pi_0 = \underset{p\pi_1}{p}\cdot\pi_0 + p\pi_1 . \Rightarrow q\pi_0 = p\pi_1 \Rightarrow \pi_1 = \frac{q}{p}\pi_0 . \\ \pi_1 = q\pi_0 + p\pi_2 \Rightarrow q\pi_1 = p\pi_2 \Rightarrow \pi_2 = \left(\frac{q}{p}\right)^2 \pi_0 . \\ \pi_2 = q\pi_1 + p\cdot\pi_3 \Rightarrow q\pi_2 = p\pi_3 \Rightarrow \pi_3 = \left(\frac{q}{p}\right)^3 \pi_0 \end{cases}$$

$$\pi_k = \left(\frac{q}{p}\right)^k \pi_0 . \Rightarrow \left(\sum_{k=0}^{\infty} \left(\frac{q}{p}\right)^k\right)\pi_0 = 1 \qquad p > q \quad \ldots$$

$$\pi_0 = 1 - \frac{q}{p}$$

$$\pi_k = \left(\frac{q}{p}\right)^k \left(1 - \frac{q}{p}\right)$$

Local $\Rightarrow$ Global.

$$\sum_{k=0}^{N} \pi_k = 1 . \Rightarrow \sum_{k=0}^{N} \binom{N}{k} \pi_0 = 1 \Rightarrow 2^N \cdot \pi_0 = 1 \Rightarrow \pi_0 = 2^{-N} .$$

$$\pi_k = \binom{N}{k} \cdot \frac{1}{2^N}$$

$M/G/\infty$. Filtered Poisson.

① Time Discretization ② Queue Length $-1, 0, +1$

$$\begin{pmatrix} p & q & & & & \\ p & & q & & & \\ & p & & q & & \\ & & p & & q & \\ & & & \ddots & & \ddots \end{pmatrix}$$

Random Walk with one elastic wall

$p < q$ Non-Recurrent

$p > q$ Recurrent $\qquad \pi = \pi \cdot P .$