Brown Motion $B(t)$

① $B(0)=0$. ② Orthogonal. Increment

③ $B(t)-B(s) \sim N(0, \sigma^2(t-s))$

Random Walk $\quad X(nT) = \boxed{X_1 + \cdots + X_n}$ $\quad X_k$ i.i.d $\quad nT = t \Downarrow n = \frac{t}{T}$

$X_k \sim \text{Bernoulli} \begin{pmatrix} -S & S \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $\quad E(X(n)) = E(X_1 + \cdots + X_n)$

$= \sum_{k=1}^{n} E(X_k) = 0$. $\quad Var(X_k) = E(X_k^2) = \frac{1}{2}S^2 + \frac{1}{2}S^2 = S^2$

$Var(X(n)) = Var(X_1 + \cdots + X_n) = \sum_{k=1}^{n} Var(X_k)$.

Wiener

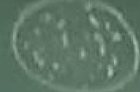
1930

$= nS^2 = \boxed{\frac{t}{T}} S^2$. $\quad S \to 0$, $T \to 0$. $\quad \boxed{\frac{S^2}{T} = \alpha}$

$\longrightarrow \alpha t \quad n \to \infty$

Central Limit Theorem $\quad N(0, \alpha t) \sim B(t)$

Einstein. (1905). Particle Movement. $S_1, S_2, \dots, S_n$.

 τ Distribution. $\phi(\Delta)$. Δ displac…

$$P(\Delta < S < \Delta + d\Delta) = \phi(\Delta)\,d\Delta.$$

$\phi(\Delta)$ density of S.

$$\phi(\Delta) = \phi(-\Delta)$$

Diffusion Processes $f(x,t)$

$$f(x, t+\tau) = \int_{-\infty}^{+\infty} f(x+\Delta, t)\,\phi(\Delta)\,d\Delta.$$

$$D = \int_{-\infty}^{+\infty} \Delta^2 \phi(\Delta)\,d\Delta$$

$$f(x, t+\tau) = f(x,t) + \left.\frac{\partial f}{\partial t}\right|_{(x,t)} \tau + \cdots$$

$$f(x+\Delta, t) = f(x,t) + \left.\frac{\partial f}{\partial x}\right|_{(x,t)} \cdot \Delta + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x,t)} \Delta^2 + \cdots$$

$$f(x,t) + \frac{\partial f}{\partial t}\cdot\tau = \int_{-\infty}^{+\infty}\left(f(x,t) + \frac{\partial f}{\partial x}\cdot\Delta + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\cdot\Delta^2\right)\phi(\Delta)\,d\Delta$$

$$f(x,t) + \frac{\partial f}{\partial t}\cdot\tau = f(x,t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\int_{-\infty}^{+\infty}\Delta^2\phi(\Delta)\,d\Delta$$

$$g(\boxed{B(t)}, t). \quad dg = \frac{\partial g}{\partial B}\underset{\sqrt{dt}}{dB} + \frac{\partial g}{\partial t}dt.$$

$$\mathrm{Var}(B(t)) \sim t \Rightarrow E(B^2(t)) \sim t \Rightarrow B^2(t) \sim t.$$

$$B(t) \sim \sqrt{t} \Rightarrow dB \sim \sqrt{dt}.$$

$$dg = \frac{\partial g}{\partial B}\cdot dB + \frac{\partial g}{\partial t}dt + \frac{1}{2}\frac{\partial^2 g}{\partial B^2}(dB)^2$$

$$= \frac{\partial g}{\partial B}\cdot dB + \frac{\partial g}{\partial t}dt + \frac{1}{2}\frac{\partial^2 g}{\partial B^2}dt \quad \text{(Ito Formula).}$$

$B(t)$.

$$\frac{\partial f}{\partial t}\cdot \tau = \frac{1}{2}D\cdot\frac{\partial^2 f}{\partial x^2}.$$

Ito. (1944). Calculus.

$$\frac{\partial f}{\partial t} = \frac{D}{2\tau}\frac{\partial^2 f}{\partial x^2} \quad \text{(Diffusion Equation).}$$

$$f(x,t) = C\cdot\frac{1}{\sqrt{t}}\exp\left(-\frac{x^2}{2t}\right)$$

$$g(x,t) \Rightarrow dg(x,t) = \frac{\partial g}{\partial x}\cdot dx + \frac{\partial g}{\partial t}\cdot dt \quad \text{(Linear)}$$

$$\int_0^1 B(t)\,dB(t) = \int_0^1 \frac{1}{2}\,dB^2(t) = \frac{1}{2}B^2(t)\Big|_0^1 = \frac{1}{2}\left(B^2(1)-B^2(0)\right)$$

$$\frac{1}{2}\,dB^2(t) = B(t)\,dB(t) \qquad g(B(t),t) = \frac{1}{2}B^2(t)$$

$$dg = B(t)\,dB(t) + \frac{1}{2}\,dt \Rightarrow B(t)\,dB(t) = \frac{1}{2}\,dB^2(t) - \frac{1}{2}\,dt$$

$$\int_0^1 B(t)\,dB(t) = \int_0^1 \frac{1}{2}\left(dB^2(t) - dt\right) = \frac{1}{2}\left(B^2(1) - 1\right)$$

$$g(x) = |x|. \quad x^2$$

Risk Bachelier. $B(t) \to$ Stock Price.

Control. Samuelson: $\exp(B(t)) \Rightarrow S(t) = \exp(\alpha t + \beta B(t))$.

(1940's). Geometrical Brown Motion.

Pricing Option! Financial Derivative

Portfolio. (Heritage) Arbitrage

$g(\boxed{B(t)}, t)$. $\quad dg = \frac{\partial g}{\partial B}\overset{\sqrt{dt}}{\underline{dB}} + \frac{\partial g}{\partial t} dt$.

$\mathrm{Var}(B(t)) \sim t \Rightarrow E(B^2(t)) \sim t \Rightarrow B^2(t) \sim t$.

$B(t) \sim \sqrt{t} \Rightarrow dB \sim \sqrt{dt}$

$$dg = \frac{\partial g}{\partial B} \cdot dB + \frac{\partial g}{\partial t} dt + \frac{1}{2}\frac{\partial^2 g}{\partial B^2}(dB)^2$$

$$= \boxed{\frac{\partial g}{\partial B} \cdot dB + \frac{\partial g}{\partial t} dt + \boxed{\frac{1}{2}\frac{\partial^2 g}{\partial B^2} dt}} \quad \text{(Ito Formula)}.$$

V : option. $\quad S$: Stock.

$V(S(t), t)$. $\quad S(B(t), t)$. $\quad$ 1+ interest

$P = V - \lambda S$. $\quad \boxed{\underline{dP = dV - \lambda dS} = r \cdot P dt}$

$$dS = d\exp(\alpha t + \beta B(t)) = \beta \cdot S \cdot dB(t) + \alpha S \cdot dt + \frac{1}{2}\beta^2 \cdot S \cdot dt$$

$$= \beta S dB(t) + (\alpha + \tfrac{1}{2}\beta^2) S dt$$

$$\frac{dS}{S} = \beta \cdot dB(t) + \left(\alpha + \frac{1}{2}\beta^2\right) dt.$$

$$dV(S(t), t) = \frac{\partial V}{\partial S} \cdot dS + \frac{\partial V}{\partial t} \cdot dt + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS)^2$$

$$(dS)^2 = \beta^2 S^2 \cdot (dB(t))^2 = \beta^2 \cdot S^2 \, dt.$$

$$dV = \frac{\partial V}{\partial S} \cdot \left(\beta S \, dB + \left(\alpha + \frac{1}{2}\beta^2\right) S \, dt\right) + \frac{\partial V}{\partial t} dt + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \beta^2 S^2 \, dt$$

$$= \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \beta^2 S^2 \, dt$$

$$dP = dV - \lambda \, dS = \left(\frac{\partial V}{\partial S} - \underline{\underline{\lambda}}\right) dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\beta^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt.$$

$S(0) = 2.5.$

$$= r(V - \lambda S) \, dt.$$

$S(T) = ?$

$K = 3.5$

$T \quad K$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\beta^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$

(Black-Scholes-Merton)

$$\underline{\underline{V(T)}} = \max(S(T) - K, 0) \Rightarrow V(0)$$