CS711008Z Algorithm Design and Analysis

Lecture 9. Algorithm design technique: Linear programming and duality [1]

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[1] The slides were made based on Ch 29 of Introduction to algorithms, Combinatorial optimization algorithm and complexity by C. H. Papadimitriou and K. Steiglitz.

Outline

- The first example: the dual of DIET problem;
- Understanding duality: Lagrangian duality explanation;
- Four properties of duality;
- Applications of duality: Farkas lemma and SHORTESTPATH problem;
- Dual simplex algorithm;
- PRIMAL_DUAL algorithm.

Remarks:

1. When minimizing a function $f(x)$, it is valuable to know a lower bound of $f(x)$ in advance.
2. Knowing lower bound is important to approximation algorithm, and *branch-and-bound* technique.
3. Duality and relaxation are powerful techniques to set a reasonable lower bound.
4. Linear programs come in primal/dual pairs.
5. It turns out that every feasible solution for one of these two problems provides a bound for the objective value for the other problem.

The first example: the dual of DIET problem.

Revisiting DIET problem

- A housewife wonders how much money she must spend on foods in order to get all the energy (2000 kcal), protein (55 g), and calcium (800 mg) that she needs every day.

Food	Energy	Protein	Calcium	Price	Quantity
Oatmeal	110	4	2	3	x_1
Whole milk	160	8	285	9	x_2
Cherry pie	420	4	22	20	x_3
Pork beans	260	14	80	19	x_4

- Linear program:

$$
\begin{array}{lrcrcrcrll}
\min & 3x_1 & + & 9x_2 & + & 20x_3 & + & 19x_4 & & \text{money} \\
s.t. & 110x_1 & + & 160x_2 & + & 420x_3 & + & 260x_4 & \geq 2000 & \text{energy} \\
 & 4x_1 & + & 8x_2 & + & 4x_3 & + & 14x_4 & \geq 55 & \text{protein} \\
 & 2x_1 & + & 285x_2 & + & 22x_3 & + & 80x_4 & \geq 800 & \text{calcium} \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq 0 &
\end{array}
$$

Dual of DIET problem: PRICING problem

- Consider a company producing protein powder, energy bar, and calcium tablet as substitution to foods.
- The company wants to design a reasonable pricing strategy to earn money as much as possible.
- However, the price cannot be arbitrarily high due to the following considerations:
  1. If the prices are competitive with foods, one might consider choosing a combination of the ingredients rather than foods;
  2. Otherwise, one will choose to buy foods directly.

LP model of PRICING problem

Food	Energy	Protein	Calcium	Price (cents)
Oatmeal	110	4	2	3
Whole milk	160	8	285	9
Cherry pie	420	4	22	20
Pork with beans	260	14	80	19
Price	y_1	y_2	y_3	

- Linear program:

$$
\begin{array}{rrcrcrcrl}
\max & 2000y_1 & + & 55y_2 & + & 800y_3 & & & \text{money} \\
s.t. & 110y_1 & + & 4y_2 & + & 2y_3 & \leq & 3 & \text{oatmeal} \\
 & 160y_1 & + & 8y_2 & + & 285y_3 & \leq & 9 & \text{milk} \\
 & 420y_1 & + & 4y_2 & + & 22y_3 & \leq & 20 & \text{pie} \\
 & 260y_1 & + & 14y_2 & + & 80y_3 & \leq & 19 & \text{pork\&beans} \\
 & y_1 & , & y_2 & , & y_3 & \geq & 0 &
\end{array}
$$

PRIMAL problem and DUAL problem

$$
\begin{array}{cccccc}
c_1 & c_2 & \dots & c_n & \\
a_{11} & a_{12} & \dots & a_{1n} & b_1 \\
a_{21} & a_{22} & \dots & a_{2n} & b_2 \\
 & & \dots & & \\
a_{m1} & a_{m2} & \dots & a_{mn} & b_m
\end{array}
$$

- PRIMAL problem and DUAL problem are two points of view of the coefficient matrix $\mathbf{A}$:
  - Primal problem: row point of view
  - Dual problem: column point of view

PRIMAL problem

$$
\begin{array}{lccccccccc}
\min & c_1 x_1 & + & c_2 x_2 & + & \dots & + & c_n x_n & & \\
& a_{11} x_1 & + & a_{12} x_2 & + & \dots & + & a_{1n} x_n & \geq & b_1 \\
& a_{21} x_1 & + & a_{22} x_2 & + & \dots & + & a_{2n} x_n & \geq & b_2 \\
& & & & & \dots & & & & \\
& a_{m1} x_1 & + & a_{m2} x_2 & + & \dots & + & a_{mn} x_n & \geq & b_m \\
& & & & & & & x_i & \geq & 0 \quad \text{for each } i
\end{array}
$$

- Primal problem: row point of view (in red);

$$
\begin{array}{rrl}
\min & \mathbf{c}^{\mathbf{T}}\mathbf{x} & \\
s.t. & \mathbf{A}\mathbf{x} & \geq \mathbf{b} \\
& \mathbf{x} & \geq \mathbf{0}
\end{array}
$$

Dual problem

$$
\begin{array}{ccccc}
c_1 & c_2 & \cdots & c_n & \\
\color{blue}{\vee|} & \color{blue}{\vee|} & & \color{blue}{\vee|} & \color{blue}{\max} \\
\color{blue}{y_1} a_{11} & \color{blue}{y_1} a_{12} & \cdots & \color{blue}{y_1} a_{1n} & \color{blue}{y_1} b_1 \\
+ & + & & + & + \\
\color{blue}{y_2} a_{21} & \color{blue}{y_2} a_{22} & \cdots & \color{blue}{y_2} a_{2n} & \color{blue}{y_2} b_2 \\
+ & + & & + & + \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
+ & + & & + & + \\
\color{blue}{y_m} a_{m1} & \color{blue}{y_m} a_{m2} & \cdots & \color{blue}{y_m} a_{mn} & \color{blue}{y_m} b_m \\
 & & & y_j \quad \geq & 0 \quad \text{for each } j
\end{array}
$$

- Dual problem: column point of view (in blue).

$$
\begin{array}{rrl}
\max & \mathbf{y^T b} & \\
s.t. & \mathbf{y} & \geq \mathbf{0} \\
 & \mathbf{y^T A} & \leq \mathbf{c^T}
\end{array}
$$

How to write DUAL problem? Case 1

- For each **constraint** in the PRIMAL problem, a **variable** is set in the DUAL problem.
- If the PRIMAL problem has **inequality constraints**, the DUAL problem is written as follows:

Primal:

$$\begin{array}{rr} \min & \mathbf{c^T x} \\ s.t. & \mathbf{Ax} \le \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0} \end{array}$$

Dual:

$$\begin{array}{rr} \max & \mathbf{y^T b} \\ s.t. & \mathbf{y} \le \mathbf{0} \\ & \mathbf{y^T A} \le \mathbf{c^T} \end{array}$$

same sign

change sign

How to write DUAL problem? Case 2

- For each **constraint** in the PRIMAL problem, a **variable** is set in the DUAL problem.
- If the PRIMAL problem has **equality constraints**, the DUAL problem is as follows:

Primal:

$$\begin{array}{lr} \min & \mathbf{c^T x} \\ s.t. & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

No Constraint

Dual:

$$\begin{array}{lr} \max & \mathbf{y^T b} \\ s.t. & \\ & \mathbf{y^T A} \leq \mathbf{c^T} \end{array}$$

Change Sign

Why can the DUAL problem be written as above?
— understanding duality from the Lagrangian dual point of view

A brief introduction to Lagrangian multiplier and **KKT** conditions

- It is relatively easy to optimize an objective function **without any constraint**, say:

$$\min \quad f(\mathbf{x})$$

- But how to optimize an objective function **with equality constraints**?

$$\begin{array}{llcll} \min & f(\mathbf{x}) & & & \\ s.t. & g_i(\mathbf{x}) & = & 0 & i = 1, 2, ..., m \end{array}$$

- And how to optimize an objective function **with inequality constraints**?

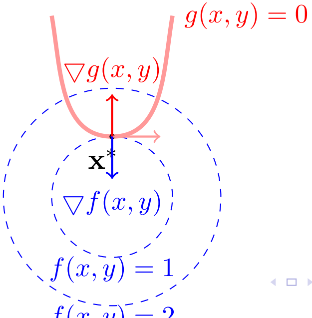
$$\begin{array}{llcll} \min & f(\mathbf{x}) & & & \\ s.t. & g_i(\mathbf{x}) & = & 0 & i = 1, 2, ..., m \\ & h_i(\mathbf{x}) & \leq & 0 & i = 1, 2, ..., p \end{array}$$

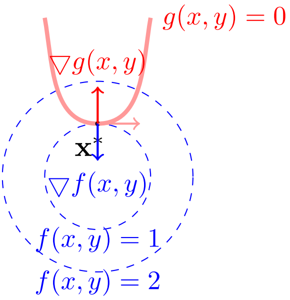
Lagrangian multiplier: under **equality constraints**

- Consider the following optimization problem:

$$\begin{array}{lll} \min & f(x,y) & \\ s.t. & g(x,y) & = & 0 \end{array}$$

- Intuition: suppose (x^*, y^*) is the optimum point. Thus at (x^*, y^*), $f(x,y)$ does not change when we walk along the curve $g(x,y) = 0$; otherwise, we can follow the curve to make $f(x,y)$ smaller, meaning that the starting point (x^*, y^*) is not optimum.





- So at (x^*, y^*), the red line tangentially touches a blue contour, i.e. there exists a λ such that:

$$\triangledown f(x, y) = \lambda \triangledown g(x, y).$$

- Lagrange must have cleverly noticed that the equation above looks like partial derivatives of some larger scalar function:

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

- Necessary conditions of optimum point: If (x^*, y^*) is local optimum, then there exists λ such that $\triangledown L(x^*, y^*, \lambda) = 0$

Understanding Lagrangian function

- Lagrangian function: a combination of the original optimization objective function and constraints:
$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$
- Here $\lambda g(x, y)$ can be treated as **"penalty of violating constraints"** — $g(x, y)$ describes to what extent a constraint is violated, and λ (called Lagrangian multiplier) describes the penalty weight.
- Note: $\frac{\partial L(x,y,\lambda)}{\partial \lambda} = 0$ is essentially the equality constraint $g(x, y) = 0$.
- The critical point of Lagrangian $L(x, y, \lambda)$ occurs at saddle points rather than local minima (or maxima). To utilize numerical optimization techniques, we must first transform the problem such that the critical points lie at local minima. This is done by calculating the magnitude of the gradient of Lagrangian.

- Consider the following optimization problem:

$$\begin{array}{llll} \min & f(\mathbf{x}) & & \\ s.t. & g_i(\mathbf{x}) & \leq & 0 \quad i = 1, 2, ...m \end{array}$$

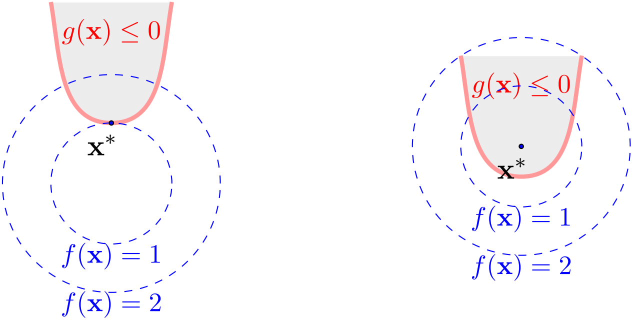


Figure: Case 1: the optimum point lies in the curve $g(\mathbf{x}) = 0$. Thus Lagrangian condition $\bigtriangledown L(\mathbf{x}^*, \lambda) = \mathbf{0}$ applies. Case 2: the optimum point lies within the region $g(\mathbf{x}) \leq 0$; thus we have $\bigtriangledown f(\mathbf{x}) = 0$ at $\mathbf{x}^*$

KKT conditions

- Lagrangian:

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x})$$

- Necessary conditions of optimum point:
  If $\mathbf{x}^*$ is local optimum satisfying some regularity conditions, then there exists λ such that:
  1. (Stationary point) $\bigtriangledown L(\mathbf{x}^*, \lambda) = \mathbf{0}$
  2. (Primal feasibility) $g(\mathbf{x}^*) \leq 0$
  3. (Dual feasibility) $\lambda_i \leq 0$, $i = 1, 2, ..., m$
  4. (Complementary slackness) $\lambda_i g_i(\mathbf{x}^*) = 0$, $i = 1, 2, ..., m$

- Note: KKT conditions are usually not solved directly in optimization; instead, iterative successive approximation is most often used. However, the final results should meet KKT conditions.

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[2] KKT conditions were named after William Karush, Harold W. Kuhn, and Albert W. Tucker.

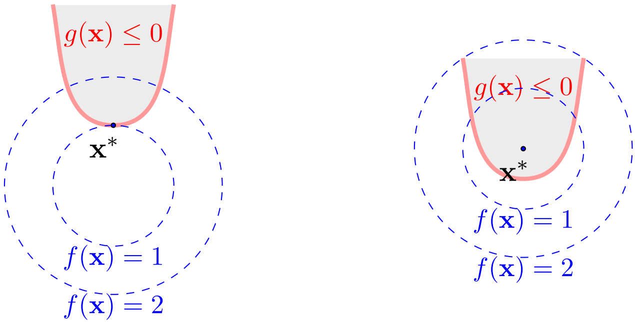


Figure: Case 1: the optimum point lies in the curve $g(\mathbf{x}) = 0$. Thus Lagrangian condition $\bigtriangledown L(\mathbf{x}^*, \lambda) = \mathbf{0}$ applies. Case 2: the optimum point lies within the region $g(\mathbf{x}) \leq 0$; thus, we have $\bigtriangledown f(\mathbf{x}) = 0$ at $\mathbf{x}^*$

These two cases were summarized as the following two conditions:

- (Stationary point) $\bigtriangledown L(\mathbf{x}^*, \lambda) = \mathbf{0}$
- (Complementary slackness) $\lambda_i g_i(\mathbf{x}^*) = 0,\ i = 1, 2, ..., m$

(Note that in the second case, $g(\mathbf{x}^*) < 0 \Rightarrow \lambda = 0 \Rightarrow \bigtriangledown f(\mathbf{x}^*) = 0$)

Applying Lagrangian dual to LP problem
— an explanation of the form of a dual LP

Lagrangian dual explanation of LP duality

- Consider a LP model:

$$\begin{array}{rrcl} \min & \mathbf{c}^{\mathbf{T}}\mathbf{x} & & \\ s.t. & \mathbf{A}\mathbf{x} & \geq & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

- Lagrangian:

$$L(\mathbf{x}, \lambda) = \mathbf{c}^{\mathbf{T}}\mathbf{x} - \sum_{i=1}^{m} \lambda_i (a_{i1}x_1 + a_{i2}x_2 + ... a_{in}x_n - b_i)$$

- Notice that **Lagrangian is a lower bound of the primal objective function**, i.e. $\mathbf{c}^{\mathbf{T}}\mathbf{x} \geq L(\mathbf{x}, \lambda)$, when $\lambda \geq \mathbf{0}$ and $\mathbf{x}$ is feasible.
- Furthermore we have

$$\mathbf{c}^{\mathbf{T}}\mathbf{x} \geq L(\mathbf{x}, \lambda) \geq \inf_{\mathbf{x}} L(\mathbf{x}, \lambda)$$

when $\lambda \leq \mathbf{0}$, and $\mathbf{x}$ is feasible.
- Denote **Lagrangian dual** $g(\lambda) = \inf_{\mathbf{x}} L(\mathbf{x}, \lambda)$. The above inequality can be rewritten as:

$$\mathbf{c}^{\mathbf{T}}\mathbf{x} \geq L(\mathbf{x}, \lambda) \geq g(\lambda)$$

- What is Lagrangian dual $g(\lambda)$?

$$
\begin{aligned}
g(\lambda) &= \inf_{\mathbf{x}} L(\mathbf{x}, \lambda) \\
&= \inf_{\mathbf{x}} (\mathbf{c}^{\mathbf{T}}\mathbf{x} - \sum_{i=1}^{m} \lambda_i (a_{i1}x_1 + a_{i2}x_2 + \ldots a_{in}x_n - b_i)) \\
&= \inf_{\mathbf{x}} (\lambda^{\mathbf{T}}\mathbf{b} + (\mathbf{c}^{\mathbf{T}} - \lambda^{\mathbf{T}}\mathbf{A})\mathbf{x}) \\
&= \begin{cases} \lambda^{\mathbf{T}}\mathbf{b} & \text{if } \mathbf{c}^{\mathbf{T}} \geq \lambda^{\mathbf{T}}\mathbf{A} \text{ and } \ \mathbf{x} \geq \mathbf{0} \\ -\infty & \text{otherwise} \end{cases}
\end{aligned}
$$

- Thus $\lambda^{\mathbf{T}}\mathbf{b}$ is a lower bound of $f(\mathbf{x})$ when $\mathbf{c}^{\mathbf{T}} \geq \lambda^{\mathbf{T}}\mathbf{A}$ and $\mathbf{x} \geq \mathbf{0}$.
- Note: here the constraints are linear inequalities. In fact, $g(\lambda)$ is always concave even if the constraints are not convex.

Find a tight bound

- Now let's try to find the **best** lower bound of $f(\mathbf{x})$.
- Thus the tight lower bound $\max g(\lambda)$ can be described as:

$$\begin{array}{rrcl} \max & \lambda^{\mathbf{T}}\mathbf{b} & & \\ s.t. & \lambda^{\mathbf{T}}\mathbf{A} & \leq & \mathbf{c}^{\mathbf{T}} \\ & \lambda & \geq & \mathbf{0} \end{array}$$

- Notes:
  1. This is actually the DUAL form of LP if replacing λ by $\mathbf{y}$.
  2. In addition, we have another explanation of DUAL variables $\mathbf{y}$: the Lagrangian multiplier.

An example

- Primal problem:

$$\begin{array}{lrcl} \min & x & & \\ s.t. & x & \geq & 2 \\ & x & \geq & 0 \end{array}$$

- Lagrangian:

$$L(x, y) = x - y * (x - 2) = 2y + x * (1 - y)$$

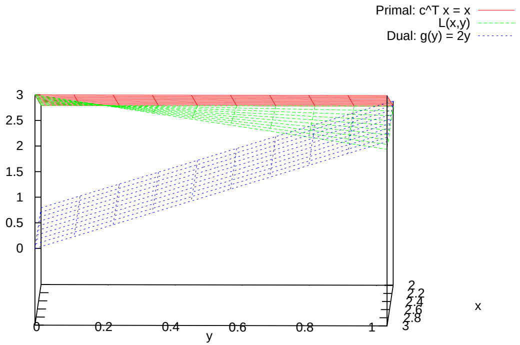
- Notice that when $y \geq 0$ and $x \geq 2$, $L(x, y)$ is a lower bound of x.
- Lagrangian dual function:

$$g(y) = \inf_x L(x, y) = \begin{cases} 2y & \text{if } x \geq 0 \text{ and } (1 - y) \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

- Dual problem:

$$\begin{array}{lr} \max & 2y \\ s.t. & y \leq 1 \\ & y \geq 0 \end{array}$$

Lagrangian connecting primal and dual



- Observation: PRIMAL objective function $x \geq$ Lagrangian $\geq$ DUAL objective function y in the feasible region.

(See extra slides)

Explanation of dual variables $\mathbf{y}$

1. Price interpretation: constrained optimization plays an important role in economics. Dual variables are also called as shadow price, i.e. the instantaneous change in the optimization objective function when constraints are relaxed, or marginal cost when strengthening constraints.
2. Lagrangian multiplier: the penalty of violence of constraints. For example, when b_i increase to $b_i + \Delta b_i$, how much will the objective function value change. In fact, we have: $\frac{\partial L(\mathbf{x},\lambda)}{\partial b_i} = \lambda_i$.

Explanation of dual variables y: using DIET as an example

- Optimal solution to primal problem with $b_1 = 2000, b_2 = 55, b_3 = 800$:

$$\mathbf{x} = (14.24, 2.70, 0, 0),$$
$$\mathbf{c^T x} = 67.096.$$

- Optimal solution to dual problem:

$$\mathbf{y} = (0.0269, 0, 0.0164),$$
$$\mathbf{y^T b} = 67.096.$$

- Let's make a slight change on $\mathbf{b}$, and watch the effect on $\max \mathbf{c^T x}$.
  1. $b_1 = 2001$: $\max \mathbf{c^T x} = 67.123$ (Note that $\mathbf{y_1} = 0.0269 = 67.123 - 67.096$)
  2. $b_2 = \ \ 56$: $\max \mathbf{c^T x} = 67.096$ (Note that $\mathbf{y_2} = 0 = 67.096 - 67.096$)
  3. $b_3 = \ 801$: $\max \mathbf{c^T x} = 67.112$ (Note that $\mathbf{y_3} = 0.0164 = 67.112 - 67.096$)

(See extra slides)

Four properties of duality

Property 1: Primal is the dual of dual

Theorem

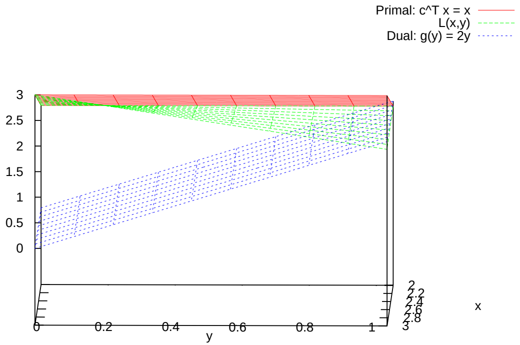
Primal is the dual of dual.

(see an extra slide)

Property 2: Weak duality

Theorem

(Weak duality) The objective value of **any** *feasible solution to the dual problem is always a lower bound of the objective value of primal problem.*



An example: DIET problem and its dual problem

- Primal problem P: a feasible solution
  $\mathbf{x^T} = [0, 8, 2, 0]^T \Rightarrow \mathbf{c^T x} = 112$

$$
\begin{array}{rrcrcrcrll}
\min & 3x_1 & + & 9x_2 & + & 20x_3 & + & 19x_4 & & \text{money} \\
s.t. & 110x_1 & + & 160x_2 & + & 420x_3 & + & 260x_4 & \geq 2000 & \text{energy} \\
 & 4x_1 & + & 8x_2 & + & 4x_3 & + & 14x_4 & \geq 55 & \text{protein} \\
 & 2x_1 & + & 285x_2 & + & 22x_3 & + & 80x_4 & \geq 800 & \text{calcium} \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq 0 &
\end{array}
$$

- Dual problem D: a feasible solution
  $\mathbf{y^T} = [0.0269, 0, 0.0164]^T \Rightarrow \mathbf{y^T b} = 67.096$

$$
\begin{array}{rrcrcrll}
\max & 2000y_1 & + & 55y_2 & + & 800y_3 & & \text{money} \\
s.t. & 110y_1 & + & 4y_2 & + & 2y_3 & \leq 3 & \text{oatmeal} \\
 & 160y_1 & + & 8y_2 & + & 285y_3 & \leq 9 & \text{milk} \\
 & 420y_1 & + & 4y_2 & + & 22y_3 & \leq 20 & \text{pie} \\
 & 260y_1 & + & 14y_2 & + & 80y_3 & \leq 19 & \text{pork\&beans} \\
 & y_1 & , & y_2 & , & y_3 & \geq 0 &
\end{array}
$$

- The theorem states that $\mathbf{c^T x} \geq \mathbf{y^T b}$ for **any** feasible solutions $\mathbf{x}$ and $\mathbf{y}$. The difference is called **"duality gap"**.

Proof.

- Consider the following PRIMAL problem:

$$\begin{array}{rrl} \min & \mathbf{c}^{\mathbf{T}}\mathbf{x} & \\ s.t. & \mathbf{A}\mathbf{x} & \geq \mathbf{b} \\ & \mathbf{x} & \geq \mathbf{0} \end{array}$$

and DUAL problem:

$$\begin{array}{rrl} \max & \mathbf{y}^{\mathbf{T}}\mathbf{b} & \\ s.t. & \mathbf{y} & \geq \mathbf{0} \\ & \mathbf{y}^{\mathbf{T}}\mathbf{A} & \leq \mathbf{c}^{\mathbf{T}} \end{array}$$

- Let $\mathbf{x}$ and $\mathbf{y}$ denote a feasible solution to primal and dual problems, respectively.
- We have $\mathbf{c}^{\mathbf{T}}\mathbf{x} \geq \mathbf{y}^{\mathbf{T}}\mathbf{A}\mathbf{x}$ (by the feasibility of dual problem, i.e., $\mathbf{y}^{\mathbf{T}}\mathbf{A} \leq \mathbf{c}^{\mathbf{T}}$, and $\mathbf{x}^{\mathbf{T}} \geq \mathbf{0}$)
- Therefore $\mathbf{c}^{\mathbf{T}}\mathbf{x} \geq \mathbf{y}^{\mathbf{T}}\mathbf{A}\mathbf{x} \geq \mathbf{y}^{\mathbf{T}}\mathbf{b}$ (by the feasibility of primal problem, i.e., $\mathbf{A}\mathbf{x} \geq \mathbf{b}$, and $\mathbf{y} \geq \mathbf{0}$)

$\square$

Property 3: Strong duality

Theorem

(Strong duality) If the primal problem has an optimal solution, then the dual problem also has an optimal solution with the same objective value.

Proof.

- Suppose $\mathbf{x}^* = \begin{bmatrix} \mathbf{B}^{-1}\mathbf{b} \\ 0 \end{bmatrix}$ be the optimal solution to the primal problem. We have $\mathbf{c}^{\mathbf{T}} - \mathbf{c}_{\mathbf{B}}^{\mathbf{T}}\mathbf{B}^{-1}\mathbf{A} \geq \mathbf{0}$.
- Define $\mathbf{y}^{*\mathbf{T}} = \mathbf{c_B}^{\mathbf{T}}\mathbf{B}^{-1}$. We will show that $\mathbf{y}^{*\mathbf{T}}$ is the optimal solution to the dual problem.
- In fact, we have $\mathbf{y}^{*\mathbf{T}}\mathbf{b} = \mathbf{c_B}^{\mathbf{T}}\mathbf{B}^{-1}\mathbf{b} = \mathbf{c}^{\mathbf{T}}\mathbf{x}^*$.
- That is, $\mathbf{y}^{*\mathbf{T}}\mathbf{b}$ reaches its upper bound. So $\mathbf{y}^{*\mathbf{T}}$ is an optimal solution to the dual problem.

$\square$

(See extra slides)

Property 4: Complementary slackness

Theorem

Let $\mathbf{x}$ and $\mathbf{y}$ denote feasible solutions to the primal and dual problems, respectively. Then $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions iff
$u_i = y_i(a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n - b_i) = 0$ *for any* $1 \leq i \leq m$, *and*
$v_j = (c_j - a_{1j}y_1 - a_{2j}y_2 - \ldots - a_{mj}y_m)x_j = 0$ *for any* $1 \leq j \leq n$.

- Intuition: a constraint of primal problem is loosely restricted $\Rightarrow$ the corresponding dual variable is tight.
- An example: the optimal solutions to DIET and its dual are $\mathbf{x} = (14.244, 2.707, 0, 0)$ and $\mathbf{y} = (0.0269, 0, 0.0164)$.

$$
\begin{array}{rcrcrcrll}
110x_1 & + & 160x_2 & + & 420x_3 & + & 260x_4 & =2000 & \\
4x_1 & + & 8x_2 & + & 4x_3 & + & 14x_4 & >55 & \Rightarrow y_2 = 0 \\
2x_1 & + & 285x_2 & + & 22x_3 & + & 80x_4 & =800 & \\
x_1 & , & x_2 & , & x_3 & , & x_4 & \geq 0 &
\end{array}
$$

Proof.

$u_i = 0$ and $v_j = 0$ for any i and j
$\Leftrightarrow \sum_i u_i = 0$ and $\sum_j v_j = 0$ (since $u_i \geq 0, v_j \geq 0$)
$\Leftrightarrow \sum_i u_i + \sum_j v_j = 0$
$\Leftrightarrow (\mathbf{y}^\mathbf{T}\mathbf{A}\mathbf{x} - \mathbf{y}^\mathbf{T}\mathbf{b}) + (\mathbf{c}^\mathbf{T}\mathbf{x} - \mathbf{y}^\mathbf{T}\mathbf{A}\mathbf{x}) = \mathbf{0}$
$\Leftrightarrow \mathbf{y}^\mathbf{T}\mathbf{b} = \mathbf{c}^\mathbf{T}\mathbf{x}$
$\Leftrightarrow \mathbf{y}$ and $\mathbf{x}$ are optimal solutions (by strong duality property, i.e., both $\mathbf{y}^\mathbf{T}\mathbf{b}$ and $\mathbf{c}^\mathbf{T}\mathbf{x}$ reach its bound)

$\square$

Summary: 9 cases of primal and dual problems

Dual \ Primal	Bounded Optimal Objective Value	Unbounded Optimal Objective Value	Infeasible
Bounded Optimal Objective Value	Possible	Impossible	Impossible
Unbounded Optimal Objective Value	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

Example 1: PRIMAL has unbounded objective value and DUAL is infeasible

PRIMAL:

$$
\begin{array}{rrcrcr}
\min & -2x_1 & - & 2x_2 & & \\
s.t. & x_1 & - & x_2 & \leq & 1 \\
 & -x_1 & + & x_2 & \leq & 1 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

DUAL:

$$
\begin{array}{rrcrcr}
\max & y_1 & + & y_2 & & \\
s.t. & y_1 & & & \leq & 0 \\
 & & & y_2 & \leq & 0 \\
 & y_1 & - & y_2 & \leq & -2 \\
 & -y_1 & + & y_2 & \leq & -2
\end{array}
$$

Example 2: both PRIMAL and DUAL are infeasible

PRIMAL:

$$\begin{array}{rrcrcr} \min & x_1 & - & 2x_2 & & \\ s.t. & x_1 & - & x_2 & \geq & 2 \\ & -x_1 & + & x_2 & \geq & -1 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

DUAL:

$$\begin{array}{rrcrcr} \max & 2y_1 & - & y_2 & & \\ s.t. & y_1 & & & \geq & 0 \\ & & & y_2 & \geq & 0 \\ & y_1 & - & y_2 & \leq & 1 \\ & -y_1 & + & y_2 & \leq & -2 \end{array}$$

Application 1: A succinct proof of Farkas lemma [1894]

Theorem (Farkas lemma)

Given vectors $\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_m}, \mathbf{c} \in \mathbb{R}^n$. *Then either*

1. $\mathbf{c} \in \mathbf{C}(\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_m})$; *or*
2. *there is a vector* $\mathbf{y} \in \mathbb{R}^n$ *such that for all* i, $\mathbf{y^T a_i} \geq \mathbf{0}$ *but* $\mathbf{y^T c} < \mathbf{0}$.

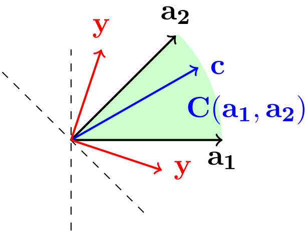


Figure: Case 1: $\mathbf{c} \in \mathbf{C}(\mathbf{a_1}, \mathbf{a_2})$

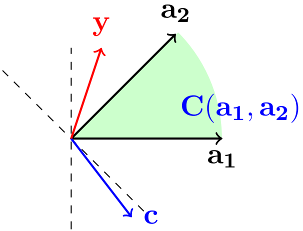


Figure: Case 2: $\mathbf{c} \notin \mathbf{C}(\mathbf{a_1}, \mathbf{a_2})$

- Here, $\mathbf{C}(\mathbf{a_1}, ..., \mathbf{a_m})$ denotes the cone spanned by $\mathbf{a_1}, ..., \mathbf{a_m}$, i.e. $\mathbf{C}(\mathbf{a_1}, ..., \mathbf{a_m}) = \{\mathbf{x} | \mathbf{x} = \sum_{i=1}^m \lambda_i \mathbf{a_i}, \lambda_i \geq 0\}$.

Proof.

- Suppose for any vector $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y^T a_i} \geq \mathbf{0}$ $(i = 1, 2, ..., m)$, we always have $\mathbf{y^T c} \geq \mathbf{0}$. We will show that $\mathbf{c}$ should lie within the cone $\mathbf{C}(\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_m})$.
- Consider the following PRIMAL problem:

$$\begin{array}{llll} \min & \mathbf{c^T y} & & \\ s.t. & \mathbf{a_i^T y} & \geq \mathbf{0} & i = 1, 2, ..., m \end{array}$$

- It is obvious that the PRIMAL problem has a feasible solution $\mathbf{y} = \mathbf{0}$, and is bounded since $\mathbf{c^T y} \geq \mathbf{0}$.
- Thus the DUAL problem also has a bounded optimal solution:

$$\begin{array}{lrcl} \max & 0 & & \\ s.t. & \mathbf{x^T A^T} & = & \mathbf{c^T} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

- In other words, there exists a vector $\mathbf{x}$ such that $\mathbf{c} = \sum_{i=1}^{m} x_i \mathbf{a_i}$.

$\square$

Farkas' lemma lies at the core of linear optimization. Using Farkas' lemma, we can prove SEPARATION theorem, and MINMAX theorem in the game theory.

Theorem

Let $\mathbf{A}$ be an $m \times n$ matrix, and $\mathbf{b} \in \mathbb{R}^m$. Then either

1. $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ *has a feasible solution; or*
2. *there is a vector* $\mathbf{y} \in \mathbb{R}^m$ *such that* $\mathbf{y}^\mathbf{T}\mathbf{A} \geq \mathbf{0}$ *but* $\mathbf{y}^\mathbf{T}\mathbf{b} < \mathbf{0}$.

Variants of Farkas' lemma

Theorem

Let $\mathbf{A}$ *be an* $m \times n$ *matrix, and* $\mathbf{b} \in \mathbb{R}^m$*. Then either*

1. $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ *has a feasible solution; or*
2. *there is a vector* $\mathbf{y} \in \mathbb{R}^m$ *such that* $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y}^{\mathbf{T}}\mathbf{A} \geq \mathbf{0}$ *but* $\mathbf{y}^{\mathbf{T}}\mathbf{b} < \mathbf{0}$.

Theorem

Given vectors $\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_m} \in \mathbb{R}^n$. *If* $\mathbf{x} \in \mathbf{C}(\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_m})$, *then there is a linearly independent vector set of* $\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_m}$, *say* $\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_r}$, *such that* $\mathbf{x} \in \mathbf{C}(\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_r})$.

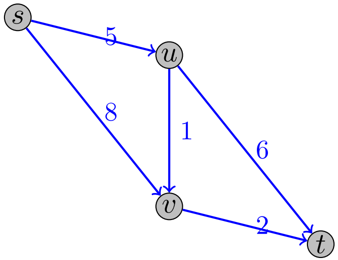
Theorem

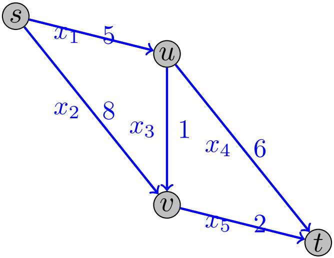
Let $\mathbf{C} \subset \mathbb{R}^n$ be a closed, convex set, and let $\mathbf{x} \in \mathbb{R}^n$. If $\mathbf{x} \notin \mathbf{C}$, then there exists a hyperplane separating $\mathbf{x}$ from $\mathbf{C}$.

Application 2: revisiting ShortestPath algorithm

ShortestPath problem

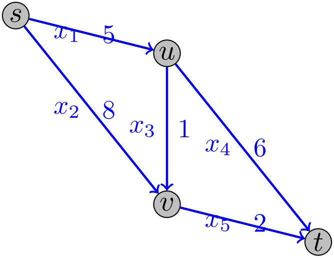
INPUT: n cities, and a collection of roads. A road from city i to j has a distance $d(i, j)$. Two specific cities: s and t.
OUTPUT: the shortest path from city s to t.





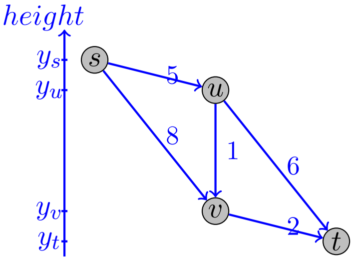
- Primal problem: set variables for **roads** (Intuition: $x_i = 0/1$ means whether edge i appears in the shortest path), and a constraint means that *"we enter a node through an edge and leaves it through another edge"*.

$$
\begin{array}{lrcrcrcrcrl l}
\min & 5x_1 & + & 8x_2 & + & 1x_3 & + & 6x_4 & + & 2x_5 & & \\
s.t. & x_1 & + & x_2 & & & & & & & = 1 & \text{vertex } s \\
 & & & & & & - & x_4 & - & x_5 & = -1 & \text{vertex } t \\
 & -x_1 & & & + & x_3 & + & x_4 & & & = 0 & \text{vertex } u \\
 & & - & x_2 & - & x_3 & & & + & x_5 & = 0 & \text{vertex } v \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & = 0/1 &
\end{array}
$$



- Primal problem: relax the 0/1 integer linear program into linear program by the **totally uni-modular** property.

$$
\begin{array}{lrcrcrcrcrll}
\min & 5x_1 & + & 8x_2 & + & 1x_3 & + & 6x_4 & + & 2x_5 & & \\
s.t. & x_1 & + & x_2 & & & & & & & = 1 & \text{vertex } s \\
 & & & & & & - & x_4 & - & x_5 & = -1 & \text{vertex } t \\
 & -x_1 & & & + & x_3 & + & x_4 & & & = 0 & \text{vertex } u \\
 & & - & x_2 & - & x_3 & & & + & x_5 & = 0 & \text{vertex } v \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq 0 & \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \leq 1 &
\end{array}
$$



- DUAL PROBLEM: set variables for **cities**. (Intuition: y_i means the height of city i; thus, $y_s - y_t$ denotes the height difference between s and t, providing a lower bound of the shortest path length.)

$$
\begin{array}{rcccccccl}
\max & y_s & - & y_t & & & & & \\
s.t. & y_s & & & - & y_u & & & \leq 5 \quad x_1 : \text{edge } (s,u) \\
 & y_s & & & & & - & y_v & \leq 8 \quad x_2 : \text{edge } (s,v) \\
 & & & & & y_u & - & y_v & \leq 1 \quad x_3 : \text{edge } (u,v) \\
 & & - & y_t & + & y_u & & & \leq 6 \quad x_4 : \text{edge } (u,t) \\
 & & - & y_t & & & + & y_v & \leq 2 \quad x_5 : \text{edge } (v,t)
\end{array}
$$

Dual simplex method

Revisiting PRIMAL SIMPLEX algorithm

- Consider the following PRIMAL problem **P**:

$$
\begin{array}{rrrrrrrr}
\min & x_1 & + & 14x_2 & + & 6x_3 & & \\
s.t. & x_1 & + & x_2 & + & x_3 & \leq & 4 \\
 & x_1 & & & & & \leq & 2 \\
 & & & & & x_3 & \leq & 3 \\
 & & & 3x_2 & + & x_3 & \leq & 6 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
$$

- PRIMAL simplex tabular:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
-z= 0	$\overline{c_1}$= 1	$\overline{c_2}$=14	$\overline{c_3}$=6	$\overline{c_4}$=0	$\overline{c_5}$=0	$\overline{c_6}$=0	$\overline{c_7}$=0
$\mathbf{x_{B1}} = b'_1$=4	1	1	1	1	0	0	0
$\mathbf{x_{B2}} = b'_2$=2	1	0	0	0	1	0	0
$\mathbf{x_{B3}} = b'_3$=3	0	0	1	0	0	1	0
$\mathbf{x_{B4}} = b'_4$=6	0	3	1	0	0	0	1

- Primal variables: $\mathbf{x}$; Feasible: $\mathbf{B^{-1}b} \geq \mathbf{0}$.
- A basis $\mathbf{B}$ is called **primal feasible** if all elements in $\mathbf{B^{-1}b}$ (the first column except for $-z$) are non-negative.

Revisiting Primal Simplex algorithm cont'd

- Now let's consider the Dual problem **D**:

$$
\begin{array}{rccccccccc}
\max & 4y_1 & + & 2y_2 & + & 3y_3 & + & 6y_4 & & \\
s.t. & y_1 & + & y_2 & & & & & \leq & 1 \\
 & y_1 & & & & & + & 3y_4 & \leq & 14 \\
 & y_1 & & & + & y_3 & + & y_4 & \leq & 6 \\
 & y_1 & , & y_2 & , & y_3 & , & y_4 & \leq & 0
\end{array}
$$

- Primal simplex tabular:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
-z= 0	$\overline{c_1}=1$	$\overline{c_2}=14$	$\overline{c_3}=6$	$\overline{c_4}=0$	$\overline{c_5}=0$	$\overline{c_6}=0$	$\overline{c_7}=0$
$\mathbf{x_{B1}} = b'_1=4$	1	1	1	1	0	0	0
$\mathbf{x_{B2}} = b'_2=2$	1	0	0	0	1	0	0
$\mathbf{x_{B3}} = b'_3=3$	0	0	1	0	0	1	0
$\mathbf{x_{B4}} = b'_4=6$	0	3	1	0	0	0	1

- Dual variables: $\mathbf{y^T} = \mathbf{c_B^T B^{-1}}$; Feasible: $\mathbf{y^T A} \leq \mathbf{c^T}$.
- A basis $\mathbf{B}$ is called **dual feasible** if all elements in $\overline{\mathbf{c^T}} = \mathbf{c} - \mathbf{c_B^T B^{-1} A} = \mathbf{c^T} - \mathbf{y^T A}$ (the first row except for $-z$) are non-negative.

- Thus another view point of the PRIMAL SIMPLEX algorithm can be described as:
  1. **Starting point:** The PRIMAL SIMPLEX algorithm starts with a primal feasible solution ($\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$);
  2. **Maintenance:** Throughout the process we maintain the primal feasibility and move towards the dual feasibility;
  3. **Stopping criteria:** $\overline{\mathbf{c}}^{\mathbf{T}} = \mathbf{c}^{\mathbf{T}} - \mathbf{c}_{\mathbf{B}}^{\mathbf{T}}\mathbf{B}^{-1}\mathbf{A} \geq \mathbf{0}$, i.e., $\mathbf{y}^{\mathbf{T}}\mathbf{A} \leq \mathbf{c}^{\mathbf{T}}$. In other words, the iteration process ends when the basis is both primal feasible and dual feasible.

- DUAL SIMPLEX:
  1. **Starting point:** The DUAL SIMPLEX algorithm starts with a dual feasible solution ($\overline{\mathbf{c}}^{\mathbf{T}} \geq \mathbf{0}$);
  2. **Maintenance:** Throughout the process we maintain the dual feasibility and move towards the dual feasibility;
  3. **Stopping criteria:** $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$. In other words, the iteration process ends when the basis is both primal feasible and dual feasible.

Primal simplex vs. Dual simplex

- Both primal simplex and dual simplex terminate at the same condition, i.e. the basis is both primal feasible and dual feasible.
- However, the final objective is achieved in totally opposite fashions— the primal simplex method keeps the primal feasibility while the dual simplex method keeps the dual feasibility during the pivoting process.
- The primal simplex algorithm *first selects an entering variable and then determines the leaving variable*.
- In contrast, the dual simplex method does the opposite; it *first selects a leaving variable and then determines an entering variable*.

DUAL SIMPLEX($B_I, z, \mathbf{A}, \mathbf{b}, \mathbf{c}$)

```
//DUAL SIMPLEX starts with a dual feasible basis. Here, B_I contains the indices of the basic variables.
while TRUE do
    if there is no index l (1 ≤ l ≤ m) has b_l < 0 then
        x = CALCULATEX(B_I, A, b, c);
        return (x, z);
    end if;
    choose an index l having b_l < 0 according to a certain rule;
    for each index j (1 ≤ i ≤ n) do
        if a_lj < 0 then
            Δ_j = − c_j / a_lj;
        else
            Δ_j = ∞;
        end if
    end for
    choose an index e that minimizes Δ_j;
    if Δ_e = ∞ then
        return ''no feasible solution'';
    end if
    (B_I, A, b, c, z) = PIVOT(B_I, A, b, c, z, e, l);
end while
```

An example

- Standard form:

$$\begin{array}{rrcrcrcr} \min & 5x_1 & + & 35x_2 & + & 20x_3 & & \\ s.t. & x_1 & - & x_2 & - & x_3 & \leq & -2 \\ & -x_1 & - & 3x_2 & & & \leq & -3 \\ & x_1 & , & x_2 & , & x_3 & \geq & 0 \end{array}$$

- Slack form:

$$\begin{array}{rrcrcrcrcrcr} \min & 5x_1 & + & 35x_2 & + & 20x_3 & & & & & & \\ s.t. & x_1 & - & x_2 & - & x_3 & + & x_4 & & & = & -2 \\ & -x_1 & - & 3x_2 & & & & & + & x_5 & = & -3 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0 \end{array}$$

Step 1

	x_1	x_2	x_3	x_4	x_5
-z= 0	$\overline{c_1}$= 5	$\overline{c_2}$=35	$\overline{c_3}$=20	$\overline{c_4}$=0	$\overline{c_5}$=0
$\mathbf{x_{B1}} = b'_1$=-2	1	-1	-1	1	0
$\mathbf{x_{B2}} = b'_2$=-3	-1	-3	0	0	1

- Basis (in blue): $\mathbf{B} = \{\mathbf{a_4}, \mathbf{a_5}\}$
- Solution: $\mathbf{x} = \begin{bmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix} = [0, 0, 0, -2, -3]^T$.
- Pivoting: choose $\mathbf{a_5}$ to leave basis since $b'_2 = -3 < 0$; choose $\mathbf{a_1}$ to enter basis since $\min_{j, a_{2j}<0} \frac{\bar{c}_j}{-a_{2j}} = \frac{\bar{c}_1}{-a_{21}}$.

Step 2

	x_1	x_2	x_3	x_4	x_5
-z= -15	$\overline{c_1}$= 0	$\overline{c_2}$=20	$\overline{c_3}$=20	$\overline{c_4}$=0	$\overline{c_5}$=5
$\mathbf{x_{B1}} = b'_1$=-5	0	-4	-1	1	1
$\mathbf{x_{B2}} = b'_2$=3	1	3	0	0	-1

- Basis (in blue): $\mathbf{B} = \{\mathbf{a_1}, \mathbf{a_4}\}$
- Solution: $\mathbf{x} = \begin{bmatrix} \mathbf{B^{-1}b} \\ \mathbf{0} \end{bmatrix} = [3, 0, 0, -5, 0]^T$.
- Pivoting: choose $\mathbf{a_4}$ to leave basis since $b'_1 = -5 < 0$; choose $\mathbf{a_2}$ to enter basis since $\min_{j, a_{1j}<0} \frac{\bar{c}_j}{-a_{1j}} = \frac{\bar{c}_2}{-a_{12}}$.

Step 3

	x_1	x_2	x_3	x_4	x_5
-z= -40	$\overline{c_1}$= 0	$\overline{c_2}$=0	$\overline{c_3}$=15	$\overline{c_4}$=5	$\overline{c_5}$=10
$\mathbf{x_{B1}} = b'_1 = \frac{5}{4}$	0	1	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
$\mathbf{x_{B2}} = b'_2 = -\frac{3}{4}$	1	0	$-\frac{3}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$

- Basis (in blue): $\mathbf{B} = \{\mathbf{a_1}, \mathbf{a_2}\}$
- Solution: $\mathbf{x} = \begin{bmatrix} \mathbf{B^{-1}b} \\ \mathbf{0} \end{bmatrix} = [\frac{5}{4}, -\frac{3}{4}, 0, 0, 0]^T$.
- Pivoting: choose $\mathbf{a_1}$ to leave basis since $b'_2 = -\frac{3}{4} < 0$; choose $\mathbf{a_3}$ to enter basis since $\min_{j, a_{2j}<0} \frac{\bar{c}_j}{-a_{2j}} = \frac{\bar{c}_3}{-a_{23}}$.

Step 4

	x_1	x_2	x_3	x_4	x_5
-z= -55	$\overline{c_1}$= 20	$\overline{c_2}$=0	$\overline{c_3}$=0	$\overline{c_4}$=20	$\overline{c_5}$=5
$\mathbf{x_{B1}} = b'_1 = 1$	$\frac{1}{3}$	1	0	0	$-\frac{1}{3}$
$\mathbf{x_{B2}} = b'_2 = 1$	$-\frac{4}{3}$	0	1	-1	$\frac{1}{3}$

- Basis (in blue): $\mathbf{B} = \{\mathbf{a_2}, \mathbf{a_3}\}$
- Solution: $\mathbf{x} = \begin{bmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix} = [0, 1, 1, 0, 0]^T.$
- Done!

When dual simplex method is useful?

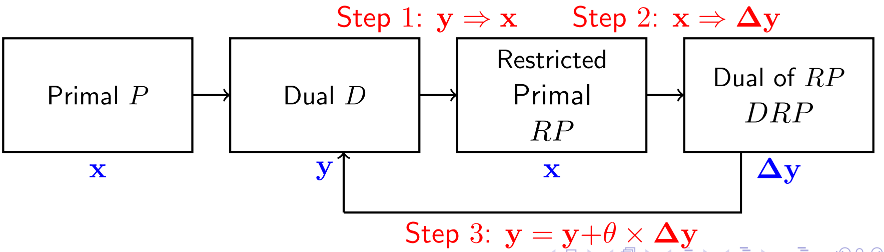
1. The dual simplex algorithm is most suited for problems for which an INITIAL DUAL FEASIBLE SOLUTION is easily available.
2. It is particularly useful for **reoptimizing** a problem after a constraint has been added or some parameters have been changed so that the previously optimal basis is no longer feasible.
3. Dual simplex algorithm might be useful when the number of constraints is much larger than the number of variables.
4. Trying dual simplex is particularly useful if your LP appears to be highly degenerate, i.e. there are many vertices of the feasible region for which the associated basis is degenerate. We may find that a large number of iterations (moves between adjacent vertices) occur with little or no improvement.[3]

[3] References: Operations Research Models and Methods, Paul A. Jensen and Jonathan F. Bard; OR-Notes, J. E. Beasley

Primal_Dual: another IMPROVMENT approach

Primal_Dual Method

- Primal_Dual method is a dual method, which exploits the lower bound information in subsequent linear programming operations.
- Advantages:
  1. Unlike dual simplex starting from a **dual basic feasible solution**, primal_dual method requires only a **dual feasible solution**.
  2. An optimal solution to DRP usually has combinatorial explanation, especially for graph-theory problems.



Basic idea of primal_dual method

- Primal P:

$$
\begin{array}{lccccccccccc}
\min & c_1x_1 & + & c_2x_2 & + & \ldots & + & c_nx_n & & & \\
s.t. & a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 & (y_1) \\
 & a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 & (y_2) \\
 & & & & & \ldots & & & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m & (y_m) \\
 & x_1 & , & x_2 & , & \ldots & , & x_n & \geq & 0 &
\end{array}
$$

- Dual D:

$$
\begin{array}{lccccccccc}
\max & b_1y_1 & + & b_2y_2 & + & \ldots & + & b_my_m & & \\
s.t. & a_{11}y_1 & + & a_{21}y_2 & + & \ldots & + & a_{m1}y_m & \leq & c_1 \\
 & a_{12}y_1 & + & a_{22}y_2 & + & \ldots & + & a_{m2}y_m & \leq & c_2 \\
 & & & & & \ldots & & & & \\
 & a_{1n}y_1 & + & a_{2n}y_2 & + & \ldots & + & a_{mn}y_m & \leq & c_n
\end{array}
$$

- Basic idea: Suppose we are given a dual feasible solution $\mathbf{y}$. Let's verify whether $\mathbf{y}$ is an optimal solution or not:
  1. If $\mathbf{y}$ is an optimal solution to the dual problem D, then the corresponding primal variables $\mathbf{x}$ should satisfy a restricted primal problem called RP;
  2. Furthermore, even if $\mathbf{y}$ is not optimal, the solution to the dual of RP (called DRP) still provide invaluable information — it can be used to improve $\mathbf{y}$.

- Dual problem D:

$$
\begin{array}{lcccccccccc}
\max & b_1 y_1 & + & b_2 y_2 & + & \dots & + & b_m y_m & & & \\
s.t. & a_{11} y_1 & + & a_{21} y_2 & + & \dots & + & a_{m1} y_m & \leq & c_1 & ('=' \Rightarrow x_1 \geq 0) \\
 & & & & & \dots & & & & & \\
 & a_{1n} y_1 & + & a_{2n} y_2 & + & \dots & + & a_{mn} y_m & \leq & c_n & ('<' \Rightarrow x_n = 0)
\end{array}
$$

- $\mathbf{y}$ provides information of the corresponding primal variables $\mathbf{x}$:
  1. Given a dual feasible solution $\mathbf{y}$. Let's check whether $\mathbf{y}$ is optimal solution or not.
  2. If $\mathbf{y}$ is optimal, we have the following restrictions on $\mathbf{x}$: $a_{1i} y_1 + a_{2i} y_2 + \dots + a_{mi} y_m < c_i \Rightarrow x_i = 0$ (Reason: complement slackness. An optimal solution $\mathbf{y}$ satisfies $(a_{1i} y_1 + a_{2i} y_2 + \dots + a_{mi} y_m - c_i) \times x_i = 0$)
  3. Let's use J to record the index of **tight constraints** where "$=$" holds.

$$
\begin{array}{cccc}
 & J & & x_n = 0 \\
 & & & \Uparrow \\
c_1 & c_2 & \cdots & c_n \\
\| & \| & & \vee \\
y_1 a_{11} & y_1 a_{12} & \cdots & y_1 a_{1n} \\
+ & + & & + \\
y_2 a_{21} & y_2 a_{22} & \cdots & y_2 a_{2n} \\
+ & + & & + \\
\vdots & \vdots & \cdots & \vdots \\
+ & + & & + \\
y_m a_{m1} & y_m a_{m2} & \cdots & y_m a_{mn}
\end{array}
$$

4. Thus the corresponding primal solution $\mathbf{x}$ should satisfy the following restricted primal (RP):
5. RP:

$$
\begin{array}{rcrcccrcll}
a_{11}x_1 & + & a_{12}x_2 & + & ... & + & a_{1n}x_n & = & b_1 & \\
a_{21}x_1 & + & a_{22}x_2 & + & ... & + & a_{2n}x_n & = & b_2 & \\
 & & & & ... & & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & ... & + & a_{mn}x_n & = & b_m & \\
 & & & & & & x_i & = & 0 & i \notin J \\
 & & & & & & x_i & \geq & 0 & i \in J
\end{array}
$$

6. In other words, the optimality of $\mathbf{y}$ is determined via solving RP.

- RP:

$$\begin{array}{rcrcccrcll}
a_{11}x_1 & + & a_{12}x_2 & + & ... & + & a_{1n}x_n & = & b_1 & \\
a_{21}x_1 & + & a_{22}x_2 & + & ... & + & a_{2n}x_n & = & b_2 & \\
 & & & & ... & & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & ... & + & a_{mn}x_n & = & b_m & \\
 & & & & & & x_i & = & 0 & i \notin J \\
 & & & & & & x_i & \geq & 0 & i \in J
\end{array}$$

- How to solve RP? Recall that $\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$ can be solved via solving an extended LP.

But how to solve RP? II

- RP (extended through introducing slack variables):

$$
\begin{array}{lllllllllll}
\min & \epsilon = & s_1 & +s_2 & \dots & +s_m & & & & & \\
s.t. & & s_1 & & & & +a_{11}x_1 & \dots & +a_{1n}x_n & = b_1 & \\
 & & & s_2 & & & +a_{21}x_1 & \dots & +a_{2n}x_n & = b_2 & \\
 & & & & \dots & & & \dots & & & \\
 & & & & & s_m & +a_{m1}x_1 & \dots & +a_{mn}x_n & = b_m & \\
 & & & & & & & & \color{blue}{x_i} & \color{blue}{= 0} & \color{blue}{i} \\
 & & & & & & & & \color{red}{x_i} & \color{red}{\geq 0} & \color{red}{i} \\
 & & & & & & & & s_i & \geq 0 & \forall
\end{array}
$$

1. If $\epsilon_{OPT} = 0$, then we find a feasible solution to RP, implying that $\mathbf{y}$ is an optimal solution;
2. If $\epsilon_{OPT} > 0$, $\mathbf{y}$ is not an optimal solution.

- Alternatively, we can solve the dual of RP, called DRP:

$$
\begin{array}{rrrrrrrrrrr}
\max & w & = & b_1 y_1 & + & b_2 y_2 & + & \dots & + & b_m y_m & & \\
s.t. & & & a_{11} y_1 & + & a_{21} y_2 & + & \dots & + & a_{m1} y_m & \leq & 0 \\
 & & & a_{12} y_1 & + & a_{22} y_2 & + & \dots & + & a_{m2} y_m & \leq & 0 \\
 & & & & & & & \dots & & & & \\
 & & & a_{1|J|} y_1 & + & a_{2|J|} y_2 & + & \dots & + & a_{m|J|} y_m & \leq & 0 \\
 & & & y_1 & , & y_2 & , & & , & y_m & \leq & 1
\end{array}
$$

1. If $w_{OPT} = 0$, $\mathbf{y}$ is an optimal solution
2. If $w_{OPT} > 0$, $\mathbf{y}$ is not an optimal solution. However, the optimal solution still provides useful information — the optimal solution to DRP can be used to improve $\mathbf{y}$.

The difference between DRP and D

- Dual problem D:

$$
\begin{array}{rcccccccccc}
\max & b_1y_1 & + & b_2y_2 & + & \ldots & + & b_my_m & & \\
s.t. & a_{11}y_1 & + & a_{21}y_2 & + & \ldots & + & a_{m1}y_m & \leq & c_1 \\
 & a_{12}y_1 & + & a_{22}y_2 & + & \ldots & + & a_{m2}y_m & \leq & c_2 \\
 & & & & & \ldots & & & & \\
 & a_{1n}y_1 & + & a_{2n}y_2 & + & \ldots & + & a_{mn}y_m & \leq & c_n
\end{array}
$$

- DRP:

$$
\begin{array}{rccccccccccc}
\max & w & = & b_1y_1 & + & b_2y_2 & + & \ldots & + & b_my_m & & \\
s.t. & & & a_{11}y_1 & + & a_{21}y_2 & + & \ldots & + & a_{m1}y_m & \leq & 0 \\
 & & & a_{12}y_1 & + & a_{22}y_2 & + & \ldots & + & a_{m2}y_m & \leq & 0 \\
 & & & & & & & \ldots & & & & \\
 & & & a_{1|J|}y_1 & + & a_{2|J|}y_2 & + & \ldots & + & a_{m|J|}y_m & \leq & 0 \\
 & & & y_1 & , & y_2 & , & & , & y_m & \leq & 1
\end{array}
$$

- How to write DRP from D?
  - Replacing c_i with 0;
  - Only $|J|$ restrictions in DRP;
  - An additional restriction: $y_1, y_2, ..., y_m \leq 1$;

Why $\mathbf{\Delta y}$ can be used to improve $\mathbf{y}$? Consider an improved dual solution $\mathbf{y}' = \mathbf{y} + \theta\mathbf{\Delta y}, \theta > 0$. We have:

- **Objective function:** Since $\mathbf{\Delta y}^\mathbf{T}\mathbf{b} = w_{OPT} > 0$, $\mathbf{y}'^\mathbf{T}\mathbf{b} = \mathbf{y}^\mathbf{T}\mathbf{b} + \theta w_{OPT} > \mathbf{y}^\mathbf{T}\mathbf{b}$. In other words, $(\mathbf{y} + \theta\mathbf{\Delta y})$ is better than $\mathbf{y}$.
- **Constraints:** The dual feasibility requires that:
  - For any $j \in J$, $a_{1j}\Delta y_1 + a_{2j}\Delta y_2 + ... + a_{mj}\Delta y_m \leq 0$. Thus we have $\mathbf{y}'^\mathbf{T}\mathbf{a_j} = \mathbf{y}^\mathbf{T}\mathbf{a_j} + \theta\mathbf{\Delta y}^\mathbf{T}\mathbf{a_j} \leq \mathbf{c_j}$ for any $\theta > 0$.
  - For any $j \notin J$, there are two cases:

1. $\forall j \notin J, a_{1j}\Delta y_1 + a_{2j}\Delta y_2 + ... + a_{mj}\Delta y_m \leq 0$:
   Thus $\mathbf{y}'$ is feasible for any $\theta > 0$ since for $\forall 1 \leq j \leq n$,

$$\begin{aligned}
& a_{1j}y_1' + a_{2j}y_2' + ... + a_{mj}y_m' && (1)\\
= \quad & a_{1j}y_1 + a_{2j}y_2 + ... + a_{mj}y_m && (2)\\
+ \quad & \theta(a_{1j}\Delta y_1 + a_{2j}\Delta y_2 + ... + a_{mj}\Delta y_m) && (3)\\
\leq \quad & c_j && (4)
\end{aligned}$$

   Hence dual problem D is unbounded and the primal problem P is infeasible.
2. $\exists j \notin J, a_{1j}\Delta y_1 + a_{2j}\Delta y_2 + ... + a_{mj}\Delta y_m > 0$:
   We can safely set $\theta \leq \frac{c_j - (a_{1j}y_1 + a_{2j}y_2 + ... + a_{mj}y_m)}{a_{1j}\Delta y_1 + a_{2j}\Delta y_2 + ... + a_{mj}\Delta y_m} = \frac{\mathbf{c_j} - \mathbf{y}^\mathbf{T}\mathbf{a_j}}{\mathbf{\Delta y}^\mathbf{T}\mathbf{a_j}}$
   to guarantee that $\mathbf{y}'^\mathbf{T}\mathbf{a_j} = \mathbf{y}^\mathbf{T}\mathbf{a_j} + \theta\mathbf{\Delta y}^\mathbf{T}\mathbf{a_j} \leq \mathbf{c_j}$.

Primal_Dual algorithm

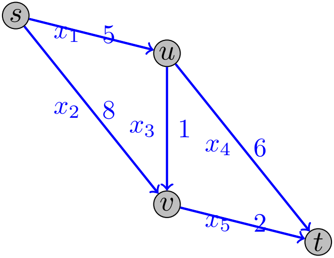
Infeasible = "No"
Optimal = "No"
$\mathbf{y} = \mathbf{y_0}$; $//\mathbf{y_0}$ is a feasible solution to the dual problem D
while TRUE **do**
Finding tight constraints index J, and set corresponding $x_j = 0$ for $j \notin J$.
Thus we have a smaller RP.
Solve DRP. Denote the solution as $\Delta \mathbf{y}$.
if DRP objective function $w_{OPT} = 0$ **then**
Optimal="Yes"
return y;
end if
if $\mathbf{\Delta y^T a_j} \leq \mathbf{0}$ (for all $j \notin J$) **then**
Infeasible = "Yes";
return ;
end if
Set $\theta = \min \frac{\mathbf{c_j - y^T a_j}}{\mathbf{\Delta y^T a_j}}$ for $\mathbf{\Delta y^T a_j} > 0$, $j \notin J$.
Update $\mathbf{y}$ as $\mathbf{y} = \mathbf{y} + \theta \mathbf{\Delta y}$;
end while

Advantages of Primal_Dual algorithm

- Some facts:
  - Primal_dual algorithm ends if using anti-cycling rule. (Reason: the objective value $\mathbf{y}^{\mathbf{T}}\mathbf{b}$ increases if there is no degeneracy.)
  - Both RP and DRP do not explicitly rely on $\mathbf{c}$. In fact, the information of $\mathbf{c}$ is represented in J.
  - This leads to another advantage of primal_dual technique, i.e., RP is usually a purely combinatorial problem. Take ShortestPath as an example. RP corresponds to a "connection" problem.
  - More and more constraints become tight in the primal_dual process.

(See Lecture 10 for a primal_dual algorithm for MaximumFlow problem.)

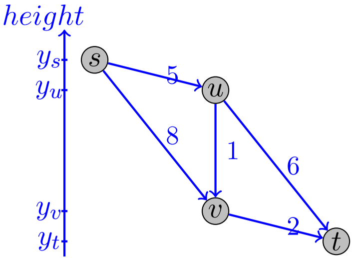
SHORTESTPATH: Dijkstra's algorithm is essentially Primal_Dual algorithm



- PRIMAL problem: relax the 0/1 integer linear program into linear program by the **totally uni-modular** property.

$$
\begin{array}{rrrrrrrrrrrll}
\min & 5x_1 & + & 8x_2 & + & 1x_3 & + & 6x_4 & + & 2x_5 & & & \\
s.t. & x_1 & + & x_2 & & & & & & & =1 & \text{vertex } s \\
 & & & & & & - & x_4 & - & x_5 & =-1 & \text{vertex } t \\
 & -x_1 & & & + & x_3 & + & x_4 & & & =0 & \text{vertex } u \\
 & & - & x_2 & - & x_3 & & & + & x_5 & =0 & \text{vertex } v \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq 0 & \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \leq 1 &
\end{array}
$$

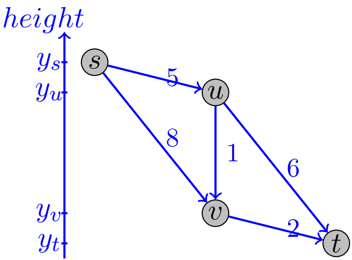
Dual of ShortestPath problem



- Dual problem: set variables for **cities**. (Intuition: y_i means the height of city i; thus, $y_s - y_t$ denotes the height difference between s and t, providing a lower bound of the shortest path length.)

$$
\begin{array}{rrrrrrrrrll}
\max & y_s & - & y_t & & & & & & & \\
s.t. & y_s & & & - & y_u & & & \le 5 & x_1 : \text{edge } (s,u) \\
 & y_s & & & & & - & y_v & \le 8 & x_2 : \text{edge } (s,v) \\
 & & & & & y_u & - & y_v & \le 1 & x_3 : \text{edge } (u,v) \\
 & & - & y_t & + & y_u & & & \le 6 & x_4 : \text{edge } (u,t) \\
 & & - & y_t & & & + & y_v & \le 2 & x_5 : \text{edge } (v,t)
\end{array}
$$

A simplified version



- Dual problem: simplify by setting $y_t = 0$ (and remove the 2nd constraint in the primal problem P, accordingly)

$$
\begin{array}{rllllll}
\max & y_s & & & & & \\
s.t. & y_s & - & y_u & & \leq 5 & x_1 : \text{edge } (s,u) \\
 & y_s & & & - \; y_v & \leq 8 & x_2 : \text{edge } (s,v) \\
 & & & y_u & - \; y_v & \leq 1 & x_3 : \text{edge } (u,v) \\
 & & & y_u & & \leq 6 & x_4 : \text{edge } (u,t) \\
 & & & & y_v & \leq 2 & x_5 : \text{edge } (v,t)
\end{array}
$$

- Dual feasible solution: $\mathbf{y^T} = (\mathbf{0}, \mathbf{0}, \mathbf{0})$. Let's check the constraints in D:

$$
\begin{array}{ccccccccc}
y_s & - & y_u & & & < & 5 & \Rightarrow x_1 = 0 \\
y_s & & & - & y_v & < & 8 & \Rightarrow x_2 = 0 \\
& & y_u & - & y_v & < & 1 & \Rightarrow x_3 = 0 \\
& & y_u & & & < & 6 & \Rightarrow x_4 = 0 \\
& & & & y_v & < & 2 & \Rightarrow x_5 = 0
\end{array}
$$

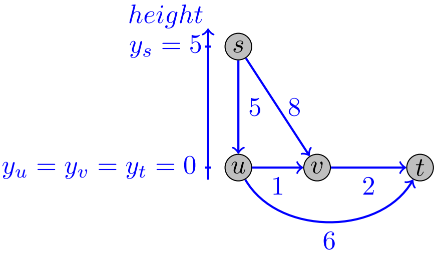
- Identifying tight constraints in D: $J = \Phi$, implying that $x_1, x_2, x_3, x_4, x_5 = 0$.
- RP:

$$
\begin{array}{lccccccccccl}
\min & s_1 & +s_2 & +s_3 & & & & & & & & \\
s.t. & s_1 & & & +x_1 & +x_2 & & & & =1 & \text{node } s \\
 & & s_2 & & -x_1 & & +x_3 & +x_4 & & =0 & \text{node } u \\
 & & & s_3 & & -x_2 & -x_3 & & +x_5 & =0 & \text{node } v \\
 & s_1, & s_2, & s_3, & & & & & & \geq 0 & \\
 & & & & x_1, & x_2, & x_3, & x_4, & x_5 & =0 &
\end{array}
$$

- DRP:

$$\begin{array}{rllll} \max & y_s & & & \\ s.t. & y_s & & & \leq 1 \\ & & y_u & & \leq 1 \\ & & & y_v & \leq 1 \end{array}$$

- Solve DRP using combinatorial technique: optimal solution $\Delta \mathbf{y^T} = (1, 0, 0)$. *Note: the optimal solution is not unique*
- Step length θ: $\theta = \min\{\frac{\mathbf{c_1} - \mathbf{y^T a_1}}{\mathbf{\Delta y^T a_1}}, \frac{\mathbf{c_2} - \mathbf{y^T a_2}}{\mathbf{\Delta y^T a_2}}\} = \min\{5, 8\} = 5$
- Update $\mathbf{y}$: $\mathbf{y^T} = \mathbf{y^T} + \theta \Delta \mathbf{y^T} = (5, 0, 0)$.



- From the point of view of Dijkstra's algorithm:
  - Optimal solution to DRP is $\Delta\mathbf{y}^{\mathbf{T}} = (1, 0, 0)$: the explored vertex set $S = \{s\}$ in Dijkstra's algorithm. In fact, DRP is solved via identifying the nodes reachable from s.
  - Step length $\theta = \min\{\frac{\mathbf{c_1} - \mathbf{y}^{\mathbf{T}}\mathbf{a_1}}{\mathbf{\Delta y^T a_1}}, \frac{\mathbf{c_2} - \mathbf{y}^{\mathbf{T}}\mathbf{a_2}}{\mathbf{\Delta y^T a_2}}\} = \min\{5, 8\} = 5$: finding the closest vertex to the nodes in S via comparing all edges going out from S.

Iteration 2 I

- Dual feasible solution: $\mathbf{y^T} = (\mathbf{5}, \mathbf{0}, \mathbf{0})$. Let's check the constraints in D:

$$
\begin{array}{cccccccl}
y_s & - & y_u & & & = & 5 & \\
y_s & & & - & y_v & < & 8 & \Rightarrow x_2 = 0 \\
 & & y_u & - & y_v & < & 1 & \Rightarrow x_3 = 0 \\
 & & y_u & & & < & 6 & \Rightarrow x_4 = 0 \\
 & & & & y_v & < & 2 & \Rightarrow x_5 = 0
\end{array}
$$

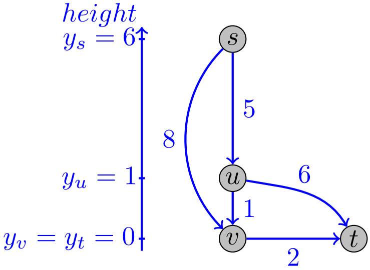
- Identifying tight constraints in D: $J = \{1\}$, implying that $x_2, x_3, x_4, x_5 = 0$.
- RP:

$$
\begin{array}{llllllllll}
\min & s_1 & +s_2 & +s_3 & & & & & & \\
s.t. & s_1 & & & +x_1 & +x_2 & & & & = 1 \quad \text{node } s \\
 & & s_2 & & -x_1 & & +x_3 & +x_4 & & = 0 \quad \text{node } u \\
 & & & s_3 & & -x_2 & -x_3 & & +x_5 & = 0 \quad \text{node } v \\
 & s_1, & s_2, & s_3, & & & & & & \geq 0 \\
 & & & & & x_2, & x_3, & x_4, & x_5 & = 0
\end{array}
$$

- DRP:

$$\begin{array}{rcccccl} \max & y_s & & & & & \\ s.t. & y_s & - & y_u & & & \leq 0 \\ & y_s & , & y_u & , & y_v & \leq 1 \end{array}$$

- Solve DRP using combinatorial technique: optimal solution $\Delta\mathbf{y^T} = (1, 1, 0)$. *Note: the optimal solution is not unique*
- Step length θ:
$\theta = \min\{\frac{\mathbf{c_2 - y^T a_2}}{\mathbf{\Delta y^T a_2}}, \frac{\mathbf{c_3 - y^T a_3}}{\mathbf{\Delta y^T a_3}}, \frac{\mathbf{c_4 - y^T a_4}}{\mathbf{\Delta y^T a_4}}\} = \min\{3, 1, 6\} = 1$
- Update $\mathbf{y}$: $\mathbf{y^T} = \mathbf{y^T} + \theta\Delta\mathbf{y^T} = (6, 1, 0)$.



- From the point of view of Dijkstra's algorithm:
  - Optimal solution to DRP is $\Delta\mathbf{y}^{\mathbf{T}} = (1, 1, 0)$: the explored vertex set $S = \{s, u\}$ in Dijkstra's algorithm. In fact, DRP is solved via identifying the nodes reachable from s.
  - Step length $\theta = \min\{\frac{\mathbf{c_2} - \mathbf{y}^{\mathbf{T}}\mathbf{a_2}}{\Delta\mathbf{y}^{\mathbf{T}}\mathbf{a_2}}, \frac{\mathbf{c_3} - \mathbf{y}^{\mathbf{T}}\mathbf{a_3}}{\Delta\mathbf{y}^{\mathbf{T}}\mathbf{a_3}}, \frac{\mathbf{c_4} - \mathbf{y}^{\mathbf{T}}\mathbf{a_4}}{\Delta\mathbf{y}^{\mathbf{T}}\mathbf{a_4}}\} = \min\{3, 1, 6\} = 1$: finding the closest vertex to the nodes in S via comparing all edges going out from S.

- Dual feasible solution: $\mathbf{y^T} = (\mathbf{6}, \mathbf{1}, \mathbf{0})$. Let's check the constraints in D:

$$\begin{array}{cccccccl}
y_s & - & y_u & & & = & 5 & \\
y_s & & & - & y_v & < & 8 & \Rightarrow x_2 = 0 \\
& & y_u & - & y_v & = & 1 & \\
& & y_u & & & < & 6 & \Rightarrow x_4 = 0 \\
& & & & y_v & < & 2 & \Rightarrow x_5 = 0
\end{array}$$

- Identifying tight constraints in D: $J = \{1, 3\}$, implying that $x_2, x_4, x_5 = 0$.
- RP:

$$\begin{array}{llllllllll}
\min & s_1 & +s_2 & +s_3 & & & & & & \\
s.t. & s_1 & & & +x_1 & +x_2 & & & & = 1 \quad \text{node } s \\
& & s_2 & & -x_1 & & +x_3 & +x_4 & & = 0 \quad \text{node } u \\
& & & s_3 & & -x_2 & -x_3 & & +x_5 & = 0 \quad \text{node } v \\
& s_1, & s_2, & s_3, & & & & & & \geq 0 \\
& & & & & x_2, & & x_4, & x_5 & = 0
\end{array}$$

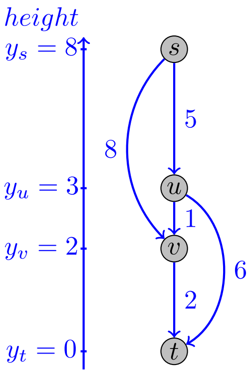
Iteration 3 II

- DRP:

$$\begin{array}{rccccccl} \max & y_s & & & & & & \\ s.t. & y_s & - & y_u & & & & \leq 0 \\ & & & y_u & - & y_v & & \leq 0 \\ & y_s & , & y_u & , & y_v & & \leq 1 \end{array}$$

- Solve DRP using combinatorial technique: optimal solution $\Delta\mathbf{y}^{\mathbf{T}} = (1,1,1)$. *Note: the optimal solution is not unique*
- Step length θ: $\theta = \min\{\frac{\mathbf{c_4} - \mathbf{y}^{\mathbf{T}}\mathbf{a_4}}{\Delta\mathbf{y}^{\mathbf{T}}\mathbf{a_4}}, \frac{\mathbf{c_5} - \mathbf{y}^{\mathbf{T}}\mathbf{a_5}}{\Delta\mathbf{y}^{\mathbf{T}}\mathbf{a_5}}\} = \min\{5,2\} = 2$
- Update $\mathbf{y}$: $\mathbf{y}^{\mathbf{T}} = \mathbf{y}^{\mathbf{T}} + \theta\Delta\mathbf{y}^{\mathbf{T}} = (8,3,2)$.

Iteration 3 III



- From the point of view of Dijkstra's algorithm:
  - Optimal solution to DRP is $\Delta \mathbf{y}^{\mathbf{T}} = (1, 1, 1)$: the explored vertex set $S = \{s, u, v\}$ in Dijkstra's algorithm. In fact, DRP is solved via identifying the nodes reachable from s.

- Step length $\theta = \min\{\frac{\mathbf{c_4 - y^T a_4}}{\mathbf{\Delta y^T a_4}}, \frac{\mathbf{c_5 - y^T a_5}}{\mathbf{\Delta y^T a_5}}\} = \min\{5, 2\} = 2$: finding the closest vertex to the nodes in S via comparing all edges going out from S.

Iteration 4 I

- Dual feasible solution: $\mathbf{y^T} = (\mathbf{8}, \mathbf{3}, \mathbf{2})$. Let's check the constraints in D:

$$
\begin{array}{ccccccccc}
y_s & & - & y_u & & & = & 5 & \\
y_s & & & & - & y_v & < & 8 & \Rightarrow x_2 = 0 \\
& & & y_u & - & y_v & = & 1 & \\
& & & y_u & & & < & 6 & \Rightarrow x_4 = 0 \\
& & & & & y_v & = & 2 &
\end{array}
$$

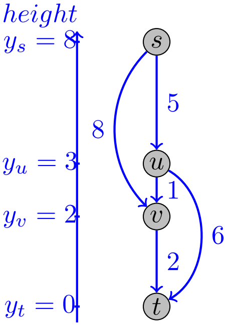
- Identifying tight constraints in D: $J = \{1, 3, 5\}$, implying that $x_2, x_4 = 0$.
- RP:

$$
\begin{array}{llllllllll}
\min & s_1 & +s_2 & +s_3 & & & & & & \\
s.t. & s_1 & & & +x_1 & +x_2 & & & & = 1 \quad \text{node } s \\
& & s_2 & & -x_1 & & +x_3 & +x_4 & & = 0 \quad \text{node } u \\
& & & s_3 & & -x_2 & -x_3 & & +x_5 & = 0 \quad \text{node } v \\
& s_1, & s_2, & s_3, & & & & & & \geq 0 \\
& & & & & x_2, & & x_4 & & = 0
\end{array}
$$

- DRP:

$$\begin{array}{rccccccl} \max & y_s & & & & & & \\ s.t. & y_s & - & y_u & & & \leq 0 \\ & & & y_u & - & y_v & \leq 0 \\ & & & & & y_v & \leq 0 \\ & y_s & , & y_u & , & y_v & \leq 1 \end{array}$$

- Solve DRP using combinatorial technique: optimal solution $\Delta\mathbf{y}^{\mathbf{T}} = (0, 0, 0)$. Done!



- From the point of view of Dijkstra's algorithm:
  - Optimal solution to DRP is $\Delta \mathbf{y}^{\mathbf{T}} = (0, 0, 0)$: there is a path from s to t, forcing $y_s = 0$ (note y_t is fixed to be 0). This corresponds to the explored node set $S = \{s, u, v, t\}$ in Dijkstra's algorithm.
- Another intuitive explanation: the **tightest** rope when picking up s.