Gaussian Processes.

① $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$. $X \sim N(\mu, \sigma^2)$.

② $f_X(x) = \frac{1}{(2\pi)^{\frac{n}{2}}(\det \Sigma)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)$. $X \sim N(\mu, \Sigma)$

$\Phi_X(\omega) = \exp\left(j\underline{\omega^T\mu} - \underline{\frac{1}{2}\omega^T\Sigma\omega}\right)$

(1) Linearity. $X \in \mathbb{R}^n$. $X \sim N(\mu, \Sigma)$. $A \in \mathbb{R}^{m\times n}$. $Y = AX \in \mathbb{R}^m$

$Y \sim N(A\mu, A\Sigma A^T)$.

$\Phi_Y(\omega) = E(\exp(j\omega^T Y)) = E(\exp(j\boxed{\omega^T A}\boxed{X}))$.

$= E(\exp(j(A^T\omega)^T \cdot X)) = \Phi_X(\tilde{\omega})\Big|_{\tilde{\omega}=A^T\omega}$. Linear Invariance.

$= \exp\left(j\tilde{\omega}^T\mu - \frac{1}{2}\tilde{\omega}^T\Sigma\tilde{\omega}\right)\Big|_{\tilde{\omega}=A^T\omega} = \exp\left(j\omega^T\boxed{A\mu} - \frac{1}{2}\omega^T\boxed{A\Sigma A^T}\omega\right)$.

②. <u>Marginal Distribution</u> $X \in \mathbb{R}^n$. $X \in N(\mu, \Sigma)$. $X = (X_1, \cdots, X_n)^T$

$\hat{X} = (X_{n_1}, X_{n_2}, \cdots, X_{n_k})$, $\{n_1, \cdots, n_k\} \subseteq \{1, \cdots, n\}$

$$= \frac{1}{2}\left(P(X_1 \le y) + P(X_1 \ge -y)\right) = \frac{1}{2}\left(F_{X_1}(y) + 1 - F_{X_1}(-y)\right).$$

$$f_{Y_2}(y) = \frac{d}{dy} F_{Y_2}(y) = \frac{1}{2}\left(f_{X_1}(y) + f_{X_1}(-y)\right) = f_{X_1}(y).$$

$$F_{Y_2}(y) = \frac{1}{2}\left(P(|X_2| \le y) + P(|X_2| \ge -y)\right)$$

$$= \begin{cases} \frac{1}{2}\left(\underline{P(-y \le X_2 \le y)} + 1\right), & y \ge 0. \quad \frac{1}{2}\left(F_{X_2}(y) - F_{X_2}(-y) + 1\right) \\ \frac{1}{2}\left(0 + P(\underline{X_2 \ge -y} \text{ or } \underline{X_2 \le y})\right), & y < 0 \quad \frac{1}{2}\left(F_{X_2}(y) + 1 - F_{X_2}(-y)\right) \end{cases}$$

$$\tilde{X} = \begin{pmatrix} X_{n_1} \\ \vdots \\ X_{n_k} \end{pmatrix} = A\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \sim N(A\mu, A\Sigma A^T).$$

$$A = \begin{pmatrix} 0, \cdots, 0, \overset{n_1}{1}, 0, \cdots, 0 \\ 0, \cdots, 0, \cdots, 0, \overset{n_2}{1}, 0 \\ \cdots \\ 0, \cdots, 0, \overset{n_k}{1}, 0, \cdots, 0 \end{pmatrix}$$

$$Y_1 + Y_2 = X_1(1 + \operatorname{sgn}(X_2))$$

Marginal Gaussian $\Rightarrow$ Joint Gaussian.

$\underline{Y_2 = |X_2| \operatorname{sgn}(X_1)}$

①. X_1, X_2 independent. $X_1, X_2 \sim N(0,1)$. $\underline{Y_1 = X_1}$. $Y_2 = X_1 \operatorname{sgn}(X_2)$

$$F_{Y_2}(y) = P(Y_2 \le y) = P(X_1 \operatorname{sgn}(X_2) \le y) = P(X_1 \le y \mid X_2 \ge 0)P(X_2 \ge 0)$$
$$+ P(-X_1 \le y \mid X_2 \le 0)P(X_2 \le 0) = \frac{1}{2}\left(P(X_1 \le y) + P(-X_1 \le y)\right).$$

$$\int_{-\infty}^{+\infty} \exp(j\omega x)\,dx = 2\pi\,\delta(\omega).$$

$$= \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right)(1+\sin x\sin y)$$

$$(x,y)\,dx = \int_{-\infty}^{+\infty} g(x,y)\,dy = 0.$$

$$\boxed{\int_{-\infty}^{+\infty} \sin x\,dx} \neq 0$$

$$\int_{-\infty}^{+\infty} h(x)\,dx = \lim_{\boxed{\substack{T_1\to\infty \\ T_2\to-\infty}}} \int_{T_2}^{T_1} h(x)\,dx$$

$$\int_{-\infty}^{+\infty} h(x)\,dx = \lim_{T\to\infty}\int_{-T}^{T} h(x)\,dx$$

Cauchy P.V

(Principal Value)

$$\int_{-\infty}^{+\infty} \sin x\,dx = \frac{1}{2j}\int_{-\infty}^{+\infty}\left(\exp(jx)-\exp(-jx)\right)dx$$

$$= \frac{\pi}{j}\left(\delta(1)+\delta(-1)\right) = 0.$$

$X_1 \sim N,\ X_2 \sim N,\ \cdots,\ X_n \sim N.$ Independent

$$\Downarrow$$

$\Sigma = \text{diag}!$

$$X_k \sim N(\mu_k, \sigma_k^2).$$

$$f_{X_1,\cdots,X_n}(x_1,\cdots,x_n) = \prod_{k=1}^{n} f_{X_k}(x_k) = \prod_{k=1}^{n}\left(\frac{1}{\sqrt{2\pi}\,\sigma_k}\exp\left(-\frac{(x_k-\mu_k)^2}{2\sigma_k^2}\right)\right)$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}\prod_{k=1}^{n}\sigma_k}\exp\left(-\frac{1}{2}\sum_{k=1}^{n}\frac{(x_k-\mu_k)^2}{\sigma_k^2}\right) = \frac{1}{(2\pi)^{\frac{n}{2}}(\det\Sigma)^{\frac{1}{2}}}\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

$$\mu = (\mu_1,\cdots,\mu_n)^T. \quad \Sigma = \text{diag}(\sigma_1^2,\cdots,\sigma_n^2).$$

$$= \frac{1}{2}\left(P(X_1 \le y) + P(X_1 \ge -y)\right) = \frac{1}{2}\left(F_{X_1}(y) + 1 - F_{X_1}(-y)\right).$$

$$f_{Y_2}(y) = \frac{d}{dy} F_{Y_2}(y) = \frac{1}{2}\left(f_{X_1}(y) + f_{X_1}(-y)\right) = f_{X_1}(y).$$

$$F_{Y_2}(y) = \frac{1}{2}\left(P(|X_2| \le y) + P(|X_2| \ge -y)\right)$$

$$= \begin{cases} \frac{1}{2}\left(\underline{P(-y \le X_2 \le y)} + 1\right), & y \ge 0. \quad \frac{1}{2}\left(F_{X_2}(y) - F_{X_2}(-y) + 1\right) \\ \frac{1}{2}\left(0 + P(\underline{X_2 \ge -y} \text{ or } \underline{X_2 \le y})\right), & y < 0 \quad \frac{1}{2}\left(F_{X_2}(y) + 1 - F_{X_2}(-y)\right) \end{cases}$$

$X \in \mathbb{R}^n$, $\quad X \sim N \Leftrightarrow \forall \alpha \in \mathbb{R}^n$. $\underline{\alpha^T X \sim N}$.

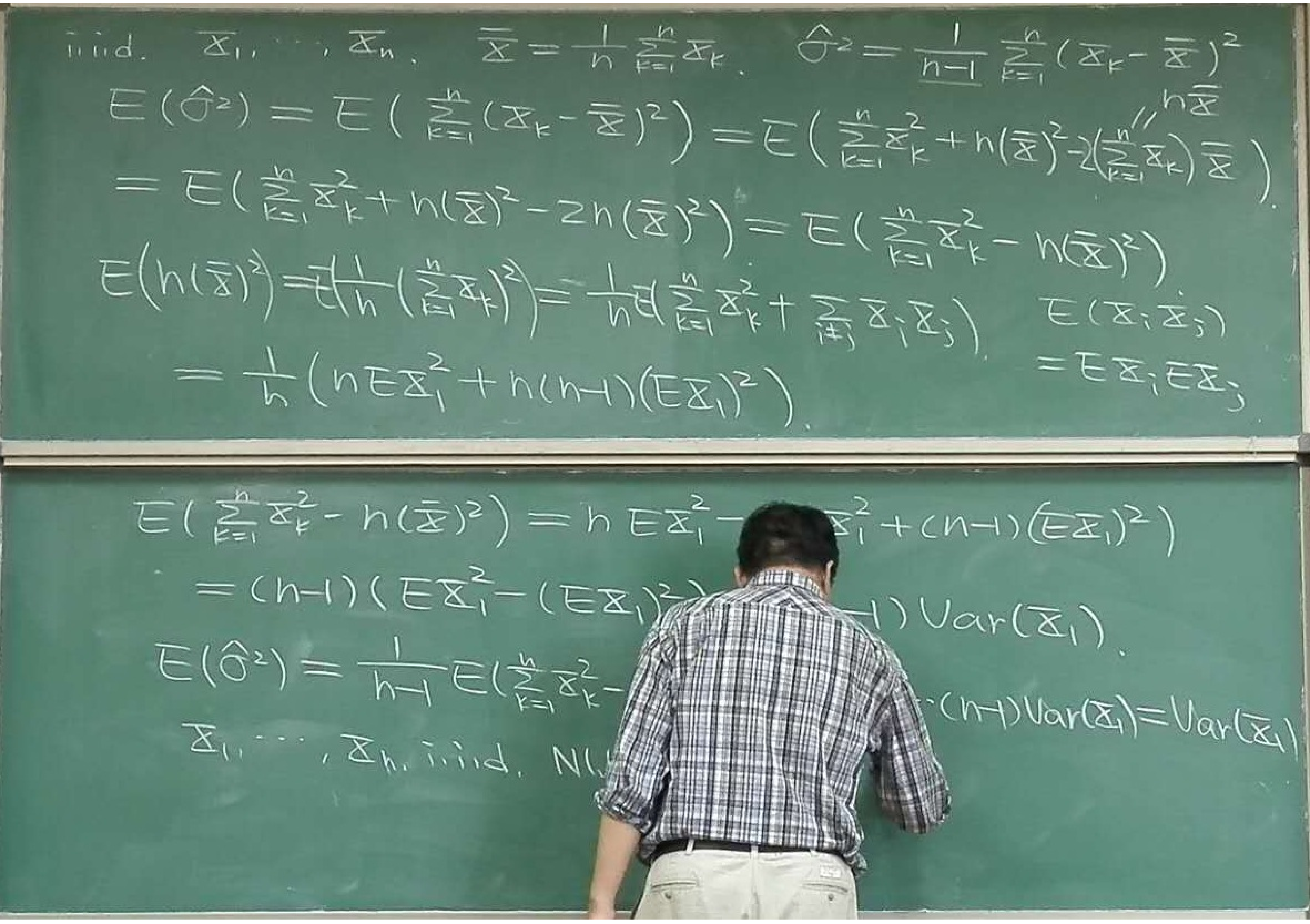
"$\Rightarrow$" Trivial! $E(\exp(j1Y)) = \Phi_Y(1)$ $\boxed{\exp(j\omega^T\mu - \frac{1}{2}\omega^T \Sigma_X \omega)}$

"$\Leftarrow$". $\Phi_X(\omega) = E(\exp(j\boxed{\omega^T X})) = \Phi_{\omega^T X}(1) = \exp(j\mu_{\omega^T X} - \frac{1}{2}\sigma^2_{\omega^T X})$.

$$\mu_{\omega^T X} = E(\omega^T X) = \omega^T E(X) = \omega^T \mu.$$

$$\sigma^2_{\omega^T X} = E(\omega^T X - \omega^T \mu)^2 = E(\omega^T(X - \mu))^2 = E(\omega^T (X-\mu)(X-\mu)^T \omega)$$

$$= \omega^T E((X-\mu)(X-\mu)^T)\omega = \omega^T \Sigma_X \omega.$$


i.i.d. $X_1, \cdots, X_n$. $\bar{X} = \frac{1}{n}\sum_{k=1}^{n} X_k$. $\hat{\sigma}^2 = \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \bar{X})^2$
$E(\hat{\sigma}^2) = E(\sum_{k=1}^{n}(X_k - \bar{X})^2) = E(\sum_{k=1}^{n} X_k^2 + n(\bar{X})^2 - 2(\underbrace{\sum_{k=1}^{n} X_k}_{= n\bar{X}})\bar{X})$
$= E(\sum_{k=1}^{n} X_k^2 + n(\bar{X})^2 - 2n(\bar{X})^2) = E(\sum_{k=1}^{n} X_k^2 - n(\bar{X})^2)$
$E(n(\bar{X})^2) = E(\frac{1}{n}(\sum_{k=1}^{n} X_k)^2) = \frac{1}{n}E(\sum_{k=1}^{n} X_k^2 + \sum_{i \neq j} X_i X_j)$. $E(X_i X_j) = E X_i E X_j$
$= \frac{1}{n}(n E X_1^2 + n(n-1)(E X_1)^2)$.
$E(\sum_{k=1}^{n} X_k^2 - n(\bar{X})^2) = n E X_1^2 -$ $X_1^2 + (n-1)(E X_1)^2)$
$= (n-1)(E X_1^2 - (E X_1)^2$ $-1)\,Var(X_1)$.
$E(\hat{\sigma}^2) = \frac{1}{n-1}E(\sum_{k=1}^{n} X_k^2 -$ $\cdot(n-1)Var(X_1) = Var(X_1)$
$X_1, \cdots, X_n$ i.i.d. N(

i.i.d. $X_1, \cdots, X_n$. $\boxed{\overline{X} = \frac{1}{n} \sum_{k=1}^{n} X_k}$ $\boxed{\hat{\sigma}^2 = \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \overline{X})^2}$

① $Y_1^2 + \cdots + Y_n^2 = X_1^2 + \cdots + X_n^2 = \sum_{k=1}^{n} (X_k - \overline{X})^2 + n(\overline{X})^2$

$\boxed{\sum_{k=1}^{n} (X_k - \overline{X})^2 = \sum_{k=1}^{n} X_k^2 - \boxed{n(\overline{X})^2}}$ $= (n-1)\hat{\sigma}^2 + n(\overline{X})^2$

② $Y_1 = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k = \sqrt{n}\,\overline{X} \Rightarrow Y_1^2 = n(\overline{X})^2$ Chi-Square $\chi^2(n-1)$

$\Rightarrow Y_1^2 = n(\overline{X})^2$. $Y_2^2 + \cdots + Y_n^2 = (n-1)\cdot\hat{\sigma}^2$ $\chi^2(1)$.

$$E\left(\sum_{k=1}^{n} X_k^2 - n(\overline{X})^2\right) = nEX_1^2 - \left(EX_1^2 + (n-1)(EX_1)^2\right)$$

$$= (n-1)\left(EX_1^2 - (EX_1)^2\right) = (n-1)\,Var(X_1).$$

$$E(\hat{\sigma}^2) = \frac{1}{n-1} E\left(\sum_{k=1}^{n} X_k^2 - n(\overline{X})^2\right) = \frac{1}{n-1}\cdot(n-1)Var(X_1) = Var(X_1)$$

$X_1, \cdots, X_n$ i.i.d. $N(\mu, \sigma^2)$ $\overline{X}$ $\hat{\sigma}^2$ independent (Cochran)

$BB^T = I$. $\begin{pmatrix} \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}} \\ \ast \end{pmatrix} = B$. $Y = B\overline{X}$. $\overline{X} = (X_1, \cdots, X_n)^T$.

Moment. $X \sim N(0, \sigma^2)$.

$$E(X^n) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} x^n \exp\left(-\frac{x^2}{2\sigma^2}\right) dx := I_n.$$

$$= \frac{1}{\sqrt{2\pi}\sigma}(-\sigma^2)\int_{-\infty}^{+\infty} x^{n-1}\, d\exp\left(-\frac{x^2}{2\sigma^2}\right) = \frac{1}{\sqrt{2\pi}\sigma}(-\sigma^2)\left(-\int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx^{n-1}\right)$$

$(n-1)!!$

$$= (n-1)\sigma^2 \cdot \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} x^{n-2} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx = (n-1)\sigma^2 I_{n-2}$$

$$I_n = (n-1)\sigma^2 I_{n-2} = (n-1)(n-3)\sigma^4 I_{n-4} = \cdots = \begin{cases} 0 & n = 2k-1 \\ \boxed{(2k-1)!!\,\sigma}^{2k}, & n = 2k \end{cases}$$

$E(X_1 X_2 X_3 X_4)$. $(X_1, \cdots, X_4) \sim N$. $E(X_1) = \cdots = E(X_4) = 0$.

$$= E(X_1X_2)E(X_3X_4) + E(X_1X_3)E(X_2X_4) + E(X_1X_4)E(X_2X_3)$$

$E(X_1X_2X_3) = 0$.

$2n$. $(2n-1)!!$

$$E(X_1X_2X_3X_4X_5X_6) = \sum E(X_iX_j)E(X_kX_l)E(X_mX_n) \quad 15$$