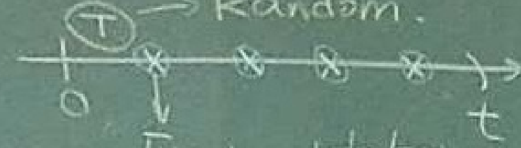
Poisson Processes.

Noise.
↑

$X(t)$ $\quad R_X(t,s)=E(X(t)X(s))$ Moment. Gaussian

Discrete States 

→ Random. Counting Processes (Point).

Events $\lfloor t/T \rfloor$

$X\sim P(\lambda)$. $X\in\{0,1,2,\cdots\}$ $\quad P(X=k)=\frac{\lambda^k}{k!}\exp(-\lambda)$ $\boxed{\text{Waiting}}$

Binomial: $X\sim B(n,p)$ $\quad P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$.

$n\to\infty$. $p\to 0$. $n\cdot p=\lambda$. $E(X)=np$. "Rare".

$$P(X=k)=\frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}.$$

$$\xrightarrow{n\to\infty}\frac{\lambda^k}{k!}\exp(-\lambda).$$ (Binomial Approximation)

$N(t)$ $\quad [0,t]$. Numbers of Events. $\boxed{P(N(t)=k)}=?$

① Independent Increments $\forall\, t_1<t_2\le t_3<t_4$

$X(t_4)-X(t_3)$ $\quad X(t_2)-X(t_1)$ independent.

② **Stationary** Increment. $N(t)-N(s) \sim P(t-s)$.

③ **Sparsity**. $P(N(t+\Delta t)-N(t)\geq 2) / P(N(t+\Delta t)-N(t)=1) \xrightarrow{\Delta t\to 0} 0$.

Moment Generating Functions. $G_X(z)=E(z^X)$ $\quad \Phi_X(\omega) = G_X(z)\big|_{z=\exp(j\omega)}$

$\Phi_X(\omega) = \int_{-\infty}^{+\infty} f_X(x)\exp(j\omega x)\,dx$. $\quad G_X(z) = \sum_k P(X=k)z^k$ (Z-Transform)

$G_{N(t)}(z) = E(z^{N(t)}) = \sum_{k=0}^{+\infty} P(N(t)=k)z^k$ $\qquad$ Rate of Time.

$f(t) \Leftarrow \frac{d}{dt}f(t) = g(f(t))$ $\quad$ Differential Equations.

$$G_{N(t+\Delta t)}(z) - G_{N(t)}(z) = E(z^{N(t+\Delta t)}) - E(z^{N(t)})$$

$$= E(z^{N(t+\Delta t)} - z^{N(t)}) = E(z^{N(t)}(z^{N(t+\Delta t)-N(t)}-1))$$

$$= E(z^{N(t)})E(z^{N(t+\Delta t)-N(t)}-1) = G_{N(t)}(z)E(z^{N(t+\Delta t)-N(t)}-1)$$

$$= G_{N(t)}(z)E(z^{N(\Delta t)}-1)$$

$$\left|\sum_{k\ge 2}\frac{P(N(\Delta t)=k)}{P(N(\Delta t)=1)}z^k\right| \le \sum_{k\ge 2}\frac{P(N(\Delta t)=k)}{P(N(\Delta t)=1)}|z|^k \le \frac{P(N(\Delta t)\ge 2)}{P(N(\Delta t)=1)} \xrightarrow{\Delta t\to 0} 0$$

$$P(N(t)=0) = P(N(s)=0,\ N(t)-N(s)=0)\quad (0<s<t)$$

$$= P(N(s)=0)\,P(N(t)-N(s)=0) = P(N(s)=0)\,P(N(t-s)=0)$$

Let $F(t)=P(N(t)=0) \Rightarrow F(t)=F(s)F(t-s).\ \forall s\in(0,t)$

$\forall t,s.\ F(t+s)=F(t)F(s) \Rightarrow F(t)=\exp(-\lambda t).\ (\lambda\ge 0)$

$$\frac{d}{dt}G_{N(t)}(z) \leftarrow \frac{1}{\Delta t}\left(G_{N(t+\Delta t)}(z)-G_{N(t)}(z)\right) = G_{N(t)}(z)\,\frac{E(z^{N(\Delta t)}-1)}{\Delta t}.$$

$$E(z^{N(\Delta t)}-1) = \sum_{k=0}^{\infty}P(N(\Delta t)=k)z^k - 1.$$

$$= \underline{P(N(\Delta t)=0)-1} + \underline{zP(N(\Delta t)=1)+\sum_{k\ge 2}P(N(\Delta t)=k)z^k}$$

$$zP(N(\Delta t)=1)+\sum_{k\ge 2}P(N(\Delta t)=k)z^k$$

$$= P(N(\Delta t)=1)\left(z+\boxed{\sum_{k\ge 2}\frac{P(N(\Delta t)=k)}{P(N(\Delta t)=1)}z^k}\right)$$

Domain of Converg

$|z|\le 1$

① $x\in\mathbb{N}$. $x=n$. $H(n)=H(n-1)+H(1)=H(n-2)+2H(1)$
$=\cdots=n\cdot H(1)$. Let $H(1)=\lambda$. $H(n)=\lambda\cdot n$.

② $x\in\mathbb{Z}$. $H(0)=H(0)+H(0)\Rightarrow H(0)=0$.
$H(-n)=H(0)-H(n)=-H(n)=-\lambda n=\lambda(-n)$

③ $x\in\mathbb{Q}$. $H(1)=H(\frac{1}{n}+\cdots+\frac{1}{n})=nH(\frac{1}{n})\Rightarrow H(\frac{1}{n})=H(1)\cdot\frac{1}{n}=\lambda(\frac{1}{n})$
$H(\frac{m}{n})=m\cdot H(\frac{1}{n})=\lambda\cdot(\frac{m}{n})$

$F(t)$ is continuous (Bounded). Cauchy Equation $H(t)=-\lambda t$.

$H(t)=\log F(t)$. $F(t+s)=F(t)F(s)\Rightarrow H(t+s)=H(t)+H(s)$

$F(t)=F(\frac{t}{2})F(\frac{t}{2})=(F(\frac{t}{2}))^2\geq 0$. Assume $\exists x$. $F(x)=0$.

$F(x)=0\Rightarrow F(\frac{x}{2})=0\Rightarrow F(\frac{x}{4})=0\Rightarrow\cdots\Rightarrow F(\frac{x}{2^n})=0\Rightarrow F(0)$
$=\lim_{n\to\infty}F(\frac{x}{2^n})=0$

$F(x)=F(0)\cdot F(x)=0\quad\forall x$

④ $x \in \mathbb{R}$. $\exists \{q_k\}$. s.t $x = \lim_{n\to\infty} q_n$. $\quad f(x) \quad x_0$. $\lim_{y\to x_0} f(y) = f(x_0)$.

$0.a_1a_2a_3 \cdots$. $q_1 = 0.a_1$, $q_2 = 0.a_1a_2$, $q_3 = 0.a_1a_2a_3$, $\cdots$

$$H(x) = H(\lim_{n\to\infty} q_n) = \lim_{n\to\infty} H(q_n) = \lim_{n\to\infty}(\lambda q_n) = \lambda \lim_{n\to\infty} q_n = \lambda x$$

$\forall x \in \mathbb{R}$. $H(x) = \lambda x \Rightarrow \forall x \in \mathbb{R}$ $F(x) = \exp(\lambda x)$

$P(N(\Delta t)=0) = \exp(-\lambda \Delta t)$. $\lambda \geq 0$ $\quad \dfrac{\exp(-\lambda\Delta t) - 1}{\Delta t} \xrightarrow{\Delta t \to 0} -\lambda$.

$$\left| \sum_{k\geq 2} \frac{P(N(\Delta t)=k)}{P(N(\Delta t)=1)} z^k \right| \leq \sum_{k\geq 2} \frac{P(N(\Delta t)=k)}{P(N(\Delta t)=1)} |z|^k \leq \frac{P(N(\Delta t)\geq 2)}{P(N(\Delta t)=1)} \xrightarrow{\Delta t \to 0} 0$$

$$\frac{1}{\Delta t}\left(E(z^{N(\Delta t)}) - 1\right) = \underbrace{\boxed{\frac{P(N(\Delta t)=0)-1}{\Delta t}}}_{\downarrow\, -\lambda} + \underbrace{\boxed{\frac{P(N(\Delta t)=1)}{\Delta t}}}_{\downarrow\, \lambda}\left(z + \boxed{\sum_{k\geq 2} \frac{P(N(\Delta t)=k)}{P(N(\Delta t)=1)} z^k}\right)$$

$$P(N(\Delta t)=0) + P(N(\Delta t)=1) + P(N(\Delta t)\geq 2) = 1.$$

$$\frac{P(N(\Delta t)=1)}{\Delta t}\left(1 + \boxed{\frac{P(N(\Delta t)\geq 2)}{P(N(\Delta t)=1)}}\right) = \boxed{\frac{1-P(N(\Delta t)=0)}{\Delta t}} \qquad \frac{P(N(\Delta t)=1)}{\Delta t} \xrightarrow{\Delta t \to 0} \lambda$$

$$\frac{d}{dt} G_{N(t)}(z) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(G_{N(t+\Delta t)}(z) - G_{N(t)}(z) \right).$$

$$= \underline{\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(E(z^{N(\Delta t)} - 1) \right)} G_{N(t)}(z)$$

$$\frac{d}{dt} G_{N(t)}(z) = \boxed{(-\lambda + \lambda z)} G_{N(t)}(z) \Rightarrow G_{N(0)}(z) \exp((-\lambda + \lambda z)t) = G_{N(t)}(z)$$

$$G_{N(0)}(z) = E(z^{N(0)}) = E(z^0) = 1$$

$$G_{N(t)}(z) = \exp((-\lambda + \lambda z)t).$$

$$\begin{array}{cc} \| & \| \\ \sum_{k=0}^{\infty} \boxed{P(N(t)=k)} \cdot z^k & \exp(-\lambda t) \exp((\lambda t) z) = \exp(-\lambda t) \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} z^k \end{array}$$

$$= \sum_{k=0}^{\infty} \boxed{\left(\frac{(\lambda t)^k}{k!} \exp(-\lambda t) \right)} z^k$$

$$P(N(t)=k) = \frac{(\lambda t)^k}{k!} \exp(-\lambda t)$$