Power Spectral Density.

$$R_X(\tau) = R_X(t-s)$$

$X(t)$. w.s.s. $E(X(t)\overline{X(s)}) = R_X(t,s) = R_X(t-s)$. $\tau = t-s$.

$$\begin{cases} \int_{-\infty}^{+\infty} R_X(\tau)\exp(-j\omega t)\,dt = S_X(\omega) \\ \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_X(\omega)\exp(j\omega t)\,d\omega = R_X(\tau) \end{cases}$$

$$S_X(\omega) = \lim_{T\to\infty} \frac{1}{T} E\left| \int_{-\frac{T}{2}}^{\frac{T}{2}} X(t)\exp(-j\omega t)\,dt \right|^2.$$ (Wiener-Khinchine)

$S_X(\omega) \geq 0$. $\quad S_{X+Y}(\omega) \overset{?}{=} S_X(\omega) + S_Y(\omega)$.

Linear System. $\xrightarrow{X(t)} \boxed{H} \xrightarrow{Y(t)}$. H: Linear (LTI) Time-Invariant Shift

$$Y(t) = \int_{-\infty}^{+\infty} h(t-s)X(s)\,ds = (h \otimes X)(t).$$

$$E(Y(t)) = \int_{-\infty}^{+\infty} h(t-s)E(X(s))\,ds = \int_{-\infty}^{+\infty} h(t-s)m_X\,ds = m_X\int_{-\infty}^{+\infty} h(t)\,dt$$

$C\,m_X$

$$R_Y(t,s) = E(Y(t)\overline{Y(s)}) = E\left(\int_{-\infty}^{+\infty} X(\tau)h(t-\tau)\,d\tau\,\overline{\int_{-\infty}^{+\infty} X(r)h(t-r)\,dr}\right)$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E\left(X(\tau)\overline{X(r)}\right) h(t-\tau)\,\overline{h(s-r)}\,d\tau\, dr.$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_X(\tau-r)\, h(t-\tau)\,\overline{h(s-r)}\,d\tau\, dr. \quad \tilde{h}(t) = h(-t).$$

$$(\tau-r)+(t-\tau)+(s-r)$$

$$\int_{-\infty}^{+\infty} X(\tau)\, h(t-\tau)\,d\tau = (X \circledast h)(t)$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_X(\tau-r)\, h(t-\tau)\,\overline{\tilde{h}(r-s)}\,d\tau\, dr$$

$Y(t)$ is W.S.S.

$$= (R_X \circledast h \circledast \overline{\tilde{h}})(t-s), \quad R_Y = R_X \circledast h \circledast \overline{\tilde{h}}$$

$$S_Y(\omega) = \int_{-\infty}^{+\infty} R_Y(\tau)\exp(-j\omega\tau)\,d\tau.$$

$$= \int_{-\infty}^{+\infty} (R_X \circledast h \circledast \overline{\tilde{h}})(\tau)\exp(-j\omega\tau)\,d\tau.$$

$$= S_X(\omega)\cdot H(\omega)\cdot \overline{H(\omega)} = S_X(\omega)\,|H(\omega)|^2$$

$$\int_{-\infty}^{+\infty} \overline{\tilde{h}(t)}\exp(-j\omega t)\,dt = \int_{-\infty}^{+\infty} \overline{h(-t)}\exp(-j\omega t)\,dt = \overline{\int_{-\infty}^{+\infty} h(-t)\exp(j\omega t)\,dt}$$

$$= \overline{\int_{-\infty}^{+\infty} h(t')\exp(-j\omega t')\,dt'} = \overline{H(\omega)}.$$

Spectral Representation. $\int_{-\infty}^{+\infty} \xi(t) \exp(-j\omega t)\,dt$.

$\boxed{\xi(t) = \int_{-\infty}^{+\infty} \exp(j\omega t)\, d\boxed{F_\xi(\omega)}}$ Stieltjes Integral.

$\int f(\omega)\,d\omega \Longrightarrow \int f(\omega)\,dg(\omega) = \lim_{k\to\infty} \sum_n f(\omega_n)(g(\omega_n) - g(\omega_{n+1}))$

$= \int f(\omega)\, g'(\omega)\, d\omega$ Notation! $\rightarrow$ Logic

$F(\omega)$ Integral Spectral Density Orthogonal Increment.

$\forall\ \omega_1 < \omega_2 \le \omega_3 < \omega_4.\quad F_\xi(\omega_4) - F_\xi(\omega_3) \perp F_\xi(\omega_2) - F_\xi(\omega_1)$.

$E\left((F_\xi(\omega_4) - F_\xi(\omega_3))\overline{(F_\xi(\omega_2) - F_\xi(\omega_1))}\right) = 0.$

$\underset{\Vert}{\xi(t)}\quad \xi(t+T) = \xi(t) \Rightarrow R_\xi(\tau + T) = R_\xi(\tau)$

$\sum_k \overset{\nearrow F(\omega)}{\alpha_k} \exp(j \overset{\nearrow \omega}{\frac{2k\pi}{T}} t),\ [0, \overset{\nearrow \infty}{T}].\quad E(\alpha_k \overline{\alpha_m}) = 0,\ (k \neq m)$

$\rightarrow \int dF_\xi(\omega) \exp(j\omega t)\quad E\left(dF_\xi(\omega_1)\, \overline{dF_\xi(\omega_2)}\right) = 0$

$$S_Y(\omega) = \int_{-\infty}^{+\infty} R_Y(\tau)\exp(-j\omega\tau)\,d\tau.$$

$$S_X(\omega) = |A_X(\omega)\exp(j\Phi_X(\omega))|$$

$$A_X(\omega) \geq 0. \quad \Phi_X(\omega)?$$

$$S_X(\omega) \to \Phi_X(\omega).$$

$S_X(\omega)$

Phase Loss

$$= \int_{-\infty}^{+\infty} (R_X \otimes h \otimes \bar{h})(\tau)\exp$$

$$= S_X(\omega)\cdot H(\omega)\cdot\overline{H(\omega)} = S_X(\omega$$

(Spectral Factorization)

$$\int_{-\infty}^{+\infty} \bar{h}(t)\exp(-j\omega t)\,dt = \int_{-\infty}^{+\infty} \overline{h(-t)}\exp(-j\omega t)\,dt = \overline{\int_{-\infty}^{+\infty} h(-t)\exp(j\omega t)\,dt}$$

$$= \overline{\int_{-\infty}^{+\infty} h(t')\exp(-j\omega t')\,dt'} = \overline{H(\omega)}.$$

$$X(t) \to \int_{-\infty}^{+\infty} X(t)\exp(-j\omega t)\,dt.$$

$\tilde{h}(t) = h(-t)$

$\bar{h}(t) = h(t-$

$$E\left(\int_{-\infty}^{+\infty} X(t)\exp(-j\omega_1 t)\,dt\ \overline{\int_{-\infty}^{+\infty} X(s)\exp(-j\omega_2 s)\,ds}\right).$$

$E|X(t)|^2 < \infty$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_X(t-s)\exp(-(j\omega_1 t - j\omega_2 s))\,dt\,ds$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_X(\overset{t'}{t-s})\exp(-j\omega_1(\overset{t'}{t-s}))\,d\overset{t'}{t}\ \exp(-j(\omega_1-\omega_2)s)\,ds$$

$$= S_X(\omega_1)\cdot\int_{-\infty}^{+\infty}\exp(-j(\omega_1-\omega_2)s)\,ds = S_X(\omega_1)\,\delta(\omega_1-\omega_2)$$

$$\left\{\begin{array}{l}\text{Make } \boxed{\text{Difference}} \\ \text{Acknowledge.}\end{array}\right.$$

$$\underline{\xi(t)} = \int_{-\infty}^{+\infty} \boxed{\exp(j\omega t)}\, \underline{dF_\xi(\omega)}.$$

t. S.

$$\xi(t) \longleftrightarrow \exp(j\omega t). \qquad \underline{\text{Isometry}}.$$

$$H_1: \underline{\xi(t)}, \underline{\xi(s)}. \qquad \underline{\exp(j\omega t)}, \underline{\exp(j\omega s)} = H_2$$

$$\|\xi(s)-\xi(t)\|^2 \qquad \|\exp(j\omega t)-\exp(j\omega s)\|^2$$

$$\|$$

$$E|\xi(t)-\xi(s)|^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\exp(j\omega t)-\exp(j\omega s)|^2 \boxed{S_\xi(\omega)}\, d\omega.$$

$$\forall\, \omega_1 < \omega_2 \le \omega_3 < \omega_4. \quad F_\xi(\omega_4)-F_\xi(\omega_3) \perp F_\xi(\omega_2)-F_\xi(\omega_1).$$

$$E\left((F_\xi(\omega_4)-F_\xi(\omega_3))\overline{(F_\xi(\omega_2)-F_\xi(\omega_1))}\right) = 0.$$

$$\xi(t) \qquad \xi(t+T) = \xi(t) \Rightarrow R_\xi(\tau+T) = R_\xi(\tau)$$

$$\|$$

$$\sum_k \overset{F(\omega)}{\alpha_k} \underline{\exp\left(j\frac{\overset{\omega}{2k\pi}}{T}t\right)}, [0, \overset{\infty}{T}] \qquad E(\alpha_k\overline{\alpha_m}) = 0, (k \neq m)$$

$$\to \int dF_\xi(\omega) \exp(j\omega t) \qquad E\left(dF_\xi(\omega_1)\overline{dF_\xi(\omega_2)}\right) = 0$$

Continuous: $f_{\xi}(x) = \frac{d}{dx} F_{\xi}(x)$. $\quad E(\xi) = \int_{\Omega} \xi(\omega)\, P(d\omega)$.

Discrete: $F_{\xi}(x) = \sum_k P_k U(x - x_k)$.

$$f_{\xi}(x) = \sum_k P_k \delta(x - x_k), \quad E(\xi) = \int_{-\infty}^{+\infty} x\, dF_{\xi}(x) = \int_{-\infty}^{+\infty} x f_{\xi}(x)\, dx$$

$$= \sum_k \int_{-\infty}^{+\infty} P_k x \delta(x - x_k)\, dx = \sum_k x_k P_k = \sum_k P_k P(\xi = x_k)$$

$$\|a\|^2 = \langle a, a \rangle. \Rightarrow \|a-b\|^2 = \langle a-b, a-b \rangle.$$

H_1: $\langle \xi(t), \xi(s) \rangle = E(\boxed{\xi(t)}\, \boxed{\overline{\xi(s)}}) = R_{\xi}(t-s)$.

H_2: $\langle \exp(j\omega t), \exp(j\omega s) \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(j\omega t) \overline{\exp(j\omega s)}\, S_{\xi}(\omega)\, d\omega$

$\underline{\underline{\xi(\omega, t)}} \longleftrightarrow \underline{\underline{\exp(j\omega t)}}$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{\xi}(\omega) \exp(j\omega(t-s))\, d\omega$$

$$E(\xi) = \int_{\mathbb{R}} x f_{\xi}(x)\, dx = \int_{\mathbb{R}} x\, dF_{\xi}(x)$$

$$= \frac{1}{2\pi}\int_{-\Omega}^{\Omega} S_X(\omega)\left| \exp(j\omega t) - \sum_{k=-\infty}^{+\infty} \exp(j\omega kT)\,\mathrm{sinc}(\Omega(t-kT))\right|^2 d\omega.$$

$$\exp(j\omega t) = \sum_{k=-\infty}^{+\infty} \beta_k \exp(jkT\omega). \quad \beta_k = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} \exp(j\omega t)\exp(-j\omega kT)\,d\omega$$

$[-\Omega, \Omega]$

$$\beta_k = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} \exp(j\omega(t-kT))\,d\omega$$

$$= \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} \cos(\omega(t-kT))\,d\omega$$

$$= \frac{1}{2\Omega}\left.\frac{\sin(\omega(t-kT))}{(t-kT)}\right|_{-\Omega}^{\Omega} = \frac{\sin(\Omega(t-kT))}{\Omega(t-kT)} = \mathrm{sinc}(\Omega(t-kT))$$

Sampling Theorem. $\underline{X(t)} = \sum_{k=-\infty}^{+\infty} \boxed{X(kT)}\,\boxed{\mathrm{sinc}}(\Omega(t-kT))$

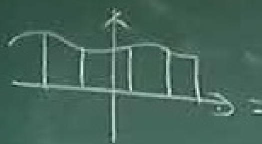
Sampling $\Updownarrow$ Expansion

$\underline{X(t)} = \sum_k$ $\quad X(t) = \sum_k \alpha_k \boxed{\phi_k(t)}$

$$E\left| X(t) - \sum_{k=-\infty}^{+\infty} X(kT)\,\mathrm{sinc}(\Omega(t-kT))\right|^2 = 0$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_X(\omega)\left| \exp(j\omega t) - \sum_{k=-\infty}^{+\infty} \exp(j\omega kT)\,\mathrm{sinc}(\Omega(t-kT))\right|^2 d\omega$$

$\hookrightarrow [-\Omega, \Omega]$.



$$= \frac{1}{2\pi}\int_{-B}^{B} S_\xi(\omega)\,(\omega\tau)^2\,d\omega$$

$$\le \frac{1}{\pi} B^2\tau^2 \int_{-B}^{B} S_\xi(\omega)\,d\omega$$

$$\sum_k \delta(t-kT) \Leftrightarrow \quad = 2B^2\tau^2 R_\xi(0)$$

$$= CB^2$$

$$E(\xi)=\int_\Omega \xi(\omega)\,P(d\omega)$$

$$E\xi(x)=\int_{-\infty}^{+\infty} x f_\xi(x)\,dx$$

$$-\sum_k P_k P(\xi=x_k)$$

$$S_\xi(\omega)=0,\quad |\omega|>\boxed{B}.$$

Fluctuation

$$E|\xi(t+\tau)-\xi(t)|^2 \le CB^2.$$

$$= \frac{1}{2\pi}\int_{-B}^{B} S_\xi(\omega)\,|\exp(j\omega(t+\tau))-\exp(j\omega t)|^2\,d\omega$$

$|\sin x|<|x|$ $\quad 1-\cos(x)=2\sin^2\frac{x}{2}$

$$= \frac{1}{2\pi}\int_{-B}^{B} S_\xi(\omega)\,(2-2\cos(\omega\tau))\,d\omega$$

$$\boxed{\underline{X}(\underline{\omega},t) \longleftrightarrow \exp(j\omega t)}$$

$$= \frac{1}{2\pi}\int_{-B}^{B} S_\xi(\omega)\,4\sin^2\frac{\omega\tau}{2}\,d\omega$$

$$\le \frac{2}{\pi}\int_{-B}^{B} S_\xi(\omega)\cdot\left(\frac{\omega\tau}{2}\right)^2 d\omega$$