Multi-Variate Analysis. (Statistics)

$X = (X_1, \cdots, X_N)^T$ Correlation (Binary) $\rightarrow$ Correlation Function

$E(X_i X_j) = E(X_j X_i)$. $\frac{n(n+1)}{2}$ $\frac{n(n-1)}{2}$ Global

$(E(X_i X_j))_{ij} = E(XX^T) = R_X$. Correlation Matrix

① Decorrelation. $X \in \mathbb{R}^N$ $Y = g(X) \in \mathbb{R}^N$ $E(Y_i Y_j) = \delta_{ij}$

Color Noise $\Rightarrow$ White Noise. "Whiten".

$R_X = R_X^T$ $g(X) = AX$. $A \in \mathbb{R}^{N\times N}$. $E(YY^T) = E(AX(AX)^T)$.

$R_X \geq 0$. $= E(AXX^TA^T) = AE(XX^T)A = AR_XA^T = \text{diag}$.

$R_X = \sum_{k=1}^{N} \lambda_k U_k U_k^T$. $R_X U_k = \lambda_k U_k$. $U_k^T U_m = \delta_{km}$.

$\lambda_k \geq 0$. $= U \,\text{diag}(\lambda_1, \cdots, \lambda_N) \cdot U$. $U = (U_1, \cdots, U_N)$

$A = U$.

$\max_{\alpha} \alpha^T R_X \alpha$ s.t. $\alpha^T \alpha = 1$. Lagrange Multiplier.

$g(\alpha) = \alpha^T R_X \alpha - \lambda(\alpha^T \alpha - 1)$. $\quad \nabla_\alpha (\alpha^T A \alpha) = (A + A^T)\alpha$.

$\nabla_\alpha g(\alpha) = 2R_X \cdot \alpha - 2\lambda\alpha = 0 \Rightarrow \boxed{R_X \cdot \alpha = \lambda \cdot \alpha}$

$\alpha^T R_X \cdot \alpha = \lambda \Rightarrow \max_{\alpha} \alpha^T R_X \alpha = \max_k \lambda_k$ The First PC

Orthogonal Complement of the first PC

② Principal Component Analysis (PCA).

(X_1, X_2) $\quad X = (X_1, \cdots, X_N)^T$. $\quad E(X) = 0$.

$\alpha \in \mathbb{R}^N$. $\quad \mathrm{Proj}_\alpha X = \frac{\alpha^T X}{\underline{\alpha^T \alpha}} \cdot \underline{\alpha}$. $\quad \alpha^T \alpha = 1$.

$\mathrm{Proj}_\alpha (\alpha^T X) \cdot \alpha \to \alpha^T X \to \mathrm{Var}(\alpha^T X) = E(\alpha^T X - E(\alpha^T X))^2$

$= E(\alpha^T X)^2 = E(\alpha^T X)(X^T \alpha) = \alpha^T E(X X^T)\alpha = \alpha^T R_X \alpha$

$$\max_{\alpha_2} \alpha_2^T R_x \alpha_2 \quad \text{s.t} \quad \alpha_2^T\alpha = 1, \quad \alpha_2^T\alpha_1 = 0.$$

$$g(\alpha_2) = \alpha_2^T R_x \alpha_2 + \lambda_1(\alpha_2^T\alpha_2 - 1) + \lambda_2\alpha_2^T\alpha_1$$

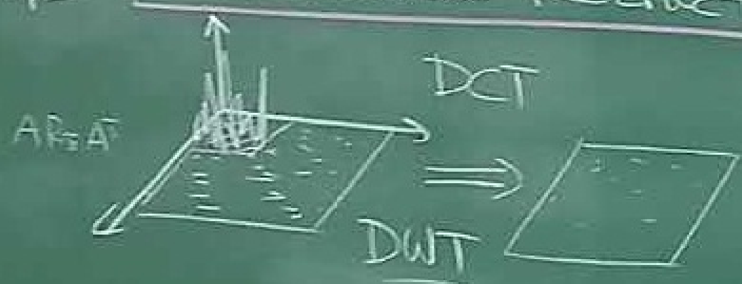
$$\nabla_{\alpha_2} g(\alpha_2) = 2R_x\alpha_2 + 2\lambda_1\alpha_2 + \lambda_2\alpha_1 = 0.$$

$$2\alpha_1^T R_x\alpha_2 + \underline{2\lambda_1\alpha_1^T\alpha_2} + \lambda_2\overset{1}{\underline{\alpha_1^T\alpha_1}} = 0$$

$$\lambda_2 = 2\alpha_1^T R_x \cdot \alpha_2 = 2\alpha_2^T R_x\alpha_1 = 2\lambda_1\alpha_2^T\alpha_1 = 0$$

降维 $\quad R_x\alpha_2 = \lambda\alpha_2. \quad \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_N$. (PCA)

$\boxed{\text{Dimensional Reduction}}$ $\left\{ \begin{array}{l} \text{Compression.} \\ \text{Classification. (AI)} \end{array} \right.$



Eigen-based Eigenface.

① Matching feature

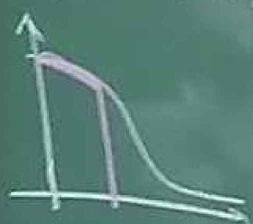
$(x_1^{(1)}, \dots, x_{10000}^{(1)}) = x^{(1)}$

$$= E(\alpha^T X)^2 = E(\alpha^T X)(X^T \alpha) = \alpha^T E(X X^T)\alpha = \alpha R_X \alpha^T$$

R_X ② $X^{(1)}, X^{(2)}, \cdots, X^{(M)}$. $R_X = \frac{1}{M}\sum_{k=1}^{M} X^{(k)}(X^{(k)})^T$

$\in \mathbb{R}^{N\times N}$. $\lambda_1 \geq \cdots \geq \lambda_M$. $\underline{U_1, \cdots, U_N}$. $\mathrm{Rank}(R_X) \leq M$

$N > M$. $\boxed{\cancel{U_1}}, \underline{U_2, \cdots, U_L}) \to F(i)$. $i = 1, 2, \cdots, P$



③ $Y \in \mathbb{R}^N$ $\max_P \|Y^T F(P)\|$ Variable-Selection

Feature

$X - EX$

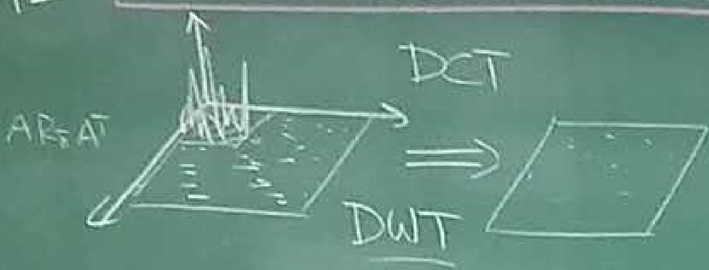
$\hat{X}^{(k)} = X^{(k)} - \sum_{i=1}^{M} X^{(i)}$

Make $\boxed{\text{Difference}}$!

降维

$R_X \alpha_2 = \lambda \alpha_2$. $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_N$. (PCA)

$\boxed{\text{Dimensional Reduction}}$ $\begin{cases} \text{Compression.} \\ \text{Classification. (AI)} \end{cases}$



Eigen-based $\boxed{\text{Eigenface}}$.

eig(A)

Matching feature

① $(X_1^{(1)}, \cdots, X_{10000}^{(1)}) = X^{(1)}$

③ $X = (X_1, \cdots, X_N)^T$. $R_X = U\,\mathrm{diag}(\lambda_1, \cdots, \lambda_n)\,U^T$.

$Y = U^T X = (Y_1, \cdots, Y_N)^T$. $U = (U_1, \cdots, U_N)$.

$\boxed{X} = UY = \sum_{k=1}^{N} Y_k U_k$. Karhunan-Loeve Expansion

$Y_k \in \mathbb{R}$. $U_k \in \mathbb{R}^N$ $E(Y_k Y_m) = \delta_{km}$ $U_k^T U_m = \delta_{km}$

Biorthogonal

本征 $X(t) = \sum_{k=1} \alpha_k \boxed{\phi_k(t)}$.

$\boxed{\int_I R_X(t,s)\,\phi_k(s)\,ds = \lambda_k \phi_k(t)}$.

$R_X \cdot \phi_k = \lambda_k \phi_k$

$\sum_{j=1}^{N} R_X(i,j)\,\overline{\phi_k(j)} = \lambda_k \phi_k(i)$

$X(t) = \sum_{k=1}^{\infty} \alpha_k \underline{\exp(j\omega t)}$

$E(\alpha_k \alpha_m) = \delta_{km}$.

$\langle \phi_k(t), \phi_m(t) \rangle = \delta_{km}$.

$\| \int_I \phi_k(t)\phi_m(t)\,dt$

$\phi_k(t) = \underline{\exp(j\omega_k t)}$

Fourier Series.

R_X ② $X^{(1)}, X^{(2)}, \cdots, X^{(M)}$. $R_X = \frac{1}{M}\sum_{k=1}^{M} X^{(k)}\left(X^{(k)}\right)^T$

$\in \mathbb{R}^{N\times N}$. $\lambda_1 \geq \cdots \geq \lambda_N$. $\underline{U_1, \cdots, U_N}$. $\mathrm{Rank}(R_X) \leq M$

$N > M$. $(\boxed{\cancel{U_1}}, \underline{U_2, \cdots, U_L}) \rightarrow F(i)$. $i = 1, 2, \cdots, P$

$X - EX$

③ $Y \in \mathbb{R}^N$ $\max_P \|Y^T F(P)\|$ Variable-Selection

Feature

$\hat{X}^{(k)} = X^{(k)} - \sum_{i=1}^{M} X^{(i)}$

Make $\boxed{\text{Difference}}$!

Eigen-based

$\boxed{\text{Eigenface}}$

Asymptotic eig(A)

① Matching feature

$(\bar{\gamma}^{(1)}_{1,1}, \dots, \bar{\gamma}^{(1)}_{19600}) = \bar{\gamma}^{(1)}$

$$\int_{-\infty}^{+\infty} R_x(t - s)\exp(j\omega_k s)\,ds. \qquad \tau = t - s. \qquad \phi_k(t+\tau) \neq \beta_k \phi_k(t)$$

$$= \int_{-\infty}^{+\infty} R_x(\tau)\exp(j\omega_k(t-\tau))\,d\tau \qquad \exp(j\omega_k(t+\tau))$$

$$= \exp(j\omega_k t)\boxed{\int_{-\infty}^{+\infty} R_x(\tau)\exp(-j\omega_k\tau)\,d\tau} \qquad = \exp(j\omega_k\tau)\exp(j\omega_k t)$$

$$= \underline{\lambda_k}\,\underline{\exp(j\omega_k t)} \qquad = \beta_k\exp(j\omega_k t)$$