$N(t)$. $\underline{P(N(t)=k)} = \frac{(\lambda t)^k}{k!}\exp(-\lambda t)$. Waiting.

continuous.

Jump Processes 

$\{T_k\}$. r.v.

$\frac{1}{\lambda}$ $F_{T_1}(t) = P(T_1 \le t) = 1 - P(T_1 > t) = 1 - P(N(t)=0)$

$\frac{1}{\lambda^2}$ $= 1-\exp(-\lambda t)$. $f_{T_1}(t) = \frac{d}{dt}F_{T_1}(t) = \lambda\exp(-\lambda t)$. $t \ge 0$

T_1: Exponential Distribution. T_k. i.i.d!

$E(N(t)) = \lambda t. \Rightarrow \lambda = \frac{E(N(t))}{t}$. Intensity.

$S_1 = T_1$, $S_2 = T_1 + T_2$, $S_3 = T_1 + T_2 + T_3$, $\dots$

$$\Phi_{S_k}(\omega) = E(\exp(j\omega S_k)) = E\left(\exp\left(j\omega \sum_{i=1}^{k} T_i\right)\right) = \prod_{i=1}^{k} E(\exp(j\omega T_i))$$

$$= \prod_{i=1}^{k} \Phi_{T_i}(\omega) = (\Phi_{T_1}(\omega))^k = \left(\frac{\lambda}{\lambda - j\omega}\right)^k \Rightarrow \dots$$

$$\Phi_{T_1}(\omega) = \int_0^\infty \lambda\exp(-\lambda x)\exp(j\omega x)\,dx = \lambda\int_0^\infty \exp(-(\lambda - j\omega)x)\,dx = \frac{\lambda}{\lambda - j\omega}$$

$$F_{S_k}(t) = P(S_k \le t) = \sum_{i=k}^{\infty} \frac{(\lambda t)^i}{i!}\exp(-\lambda t).$$

$$f_{S_k}(t) = \frac{d}{dt}F_{S_k}(t) = \sum_{i=k}^{\infty} \frac{\lambda(\lambda t)^{i-1}}{(i-1)!}\exp(-\lambda t) - \lambda\frac{(\lambda t)^i}{i!}\exp(-\lambda t)$$

$$= \lambda\exp(-\lambda t)\sum_{i=k}^{\infty}\left(\frac{(\lambda t)^{i-1}}{(i-1)!} - \frac{(\lambda t)^i}{i!}\right) = \lim_{m\to\infty}\left(\frac{(\lambda t)^{k-1}}{(k-1)!} - \frac{(\lambda t)^m}{m!}\right) = \frac{(\lambda t)^{k-1}}{(k-1)!}$$

$$f_{S_k}(t) = \frac{\lambda(\lambda t)^{k-1}}{(k-1)!}\exp(-\lambda t),\ (t \ge 0)$$ Gamma Distribution

$$\Gamma(n+1) = n\,\Gamma(n) \Longrightarrow \Gamma(n) = (n-1)! \qquad \Gamma(1) = 1$$

Gamma Function. $\Gamma(P) = \int_0^\infty x^{P-1}\exp(-x)dx.$

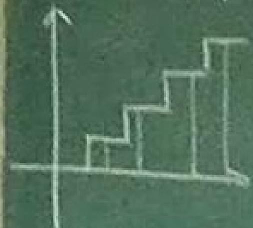
$$\Gamma(P+1) = \int_0^\infty x^P\exp(-x)dx = -\int_0^\infty x^P d\exp(-x) = -x^P\exp(-x)\Big|_0^\infty$$

$$+\int_0^\infty \exp(-x)dx^P = P\int_0^\infty x^{P-1}\exp(-x)dx = P\,\Gamma(P)$$

$$f_{S_k}(t) = \frac{\lambda(\lambda t)^{k-1}}{\Gamma(k)}\exp(-\lambda t).\quad S_k \sim \Gamma(k,\lambda)\quad \frac{\alpha^P}{\Gamma(P)}\int_0^\infty t^{P-1}\exp(-\alpha t)dt = 1$$

$$f_\xi(t) = \frac{\alpha^P t^{P-1}}{\Gamma(P)}\exp(-\alpha t),\ t \ge 0,\ \xi \sim \Gamma(P,\alpha)$$

$$F_{Y^2}(x) = P(Y^2 \le x) = P(-\sqrt{x} \le y \le \sqrt{x}) = 2\int_0^{\sqrt{x}} f_Y(y)\,dy.$$



$$= 2\int_0^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy. \qquad f_{Y^2}(x) = \frac{d}{dx} F_{Y^2}(x).$$

χ^2

$$= \frac{1}{\sqrt{2\pi}} \cdot x^{-\frac{1}{2}} \exp\left(-\frac{x}{2}\right). \quad x \ge 0.$$ Chi-Square Distribution

$N(t)$. $R_N(t,s) = E(N(t)N(s)) = E((N(t)-N(s)+N(s))N(s))$

$$= \underbrace{E(N(t)-N(s))}_{\lambda(t-s)}\underbrace{E(N(s))}_{\lambda s} + \underbrace{E(N^2(s))}_{\lambda^2 s^2+\lambda s} = \lambda(t-s)\cdot\lambda s + (\lambda s)^2 + \lambda s.$$

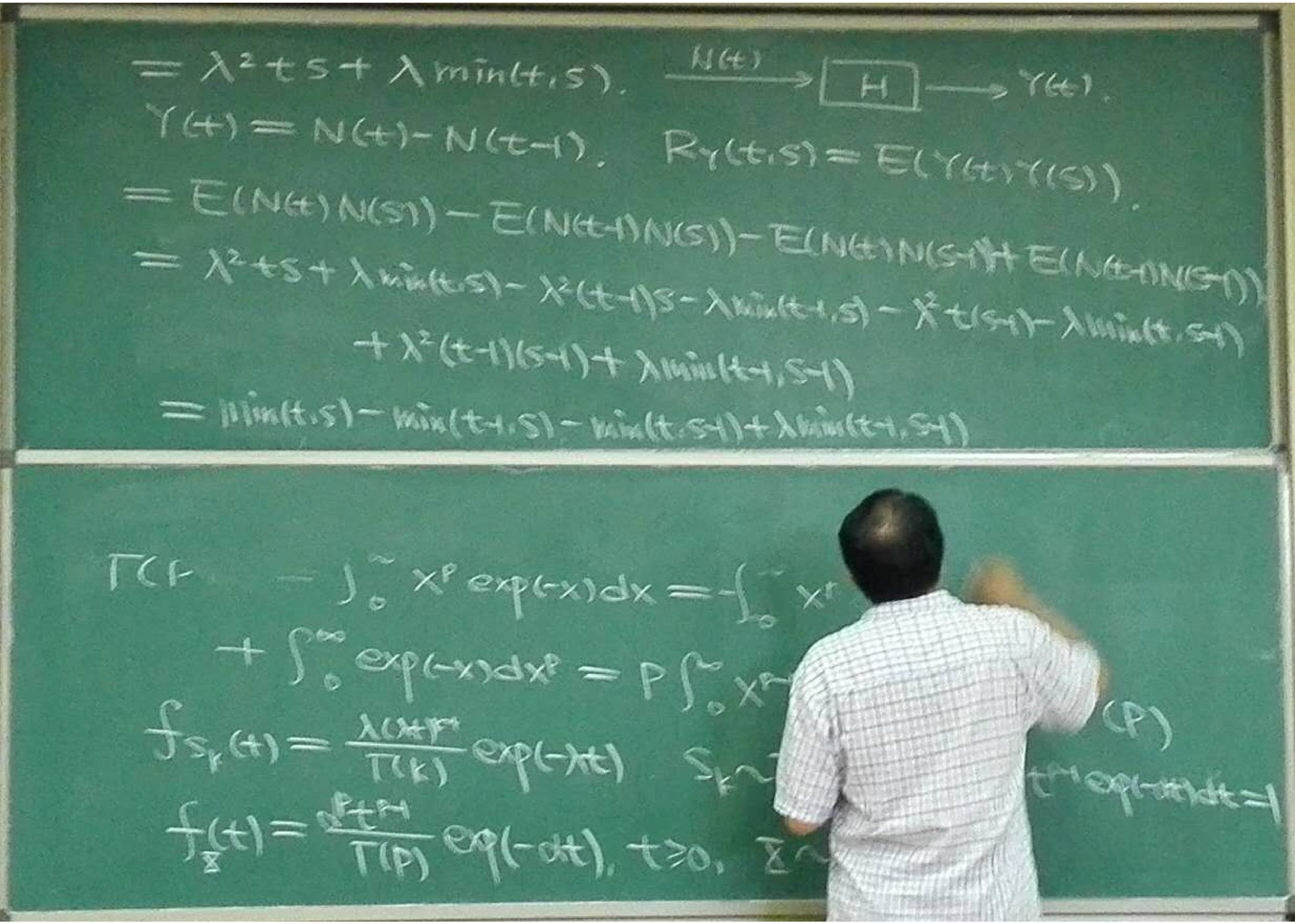
$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty x^{-\frac{1}{2}} \exp(-x)\,dx = 2\int_0^\infty \exp(-x)\,dx^{\frac{1}{2}}$$

$$\overset{y=x^{\frac{1}{2}}}{=} 2\int_0^\infty \exp(-y^2)\,dy = 2\int_0^\infty \exp\left(-\frac{y^2}{2\cdot\frac{1}{2}}\right) dy.$$

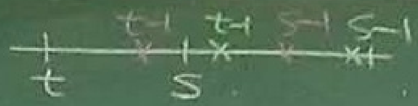
$$= 2\cdot\sqrt{2\pi}\cdot\sqrt{\tfrac{1}{2}}\cdot\tfrac{1}{2} = \sqrt{\pi}$$

$Z \sim \Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$ $\quad f_Z(x) = \dfrac{\left(\frac{1}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \cdot x^{-\frac{1}{2}} \exp\left(-\frac{x}{2}\right). \quad x \ge 0.$

$Y \sim N(0,1) \Rightarrow Z = Y^2$ $\quad = \dfrac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}} \exp\left(-\frac{x}{2}\right). \quad x \ge 0$

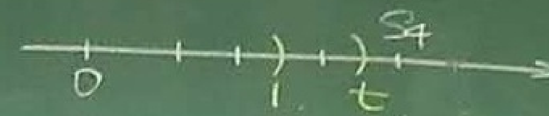


$= \lambda^2 t s + \lambda \min(t, s)$.
$\xrightarrow{N(t)} \boxed{H} \longrightarrow Y(t)$.
$Y(t) = N(t) - N(t-1)$. $R_Y(t,s) = E(Y(t)Y(s))$.
$= E(N(t)N(s)) - E(N(t-1)N(s)) - E(N(t)N(s-1)) + E(N(t-1)N(s-1))$
$= \lambda^2 t s + \lambda \min(t,s) - \lambda^2 (t-1)s - \lambda \min(t-1,s) - \lambda^2 t(s-1) - \lambda \min(t,s-1)$
$+ \lambda^2 (t-1)(s-1) + \lambda \min(t-1,s-1)$
$= \min(t,s) - \min(t-1,s) - \min(t,s-1) + \lambda \min(t-1,s-1)$
$\Gamma(p) = \int_0^\infty x^p \exp(-x)dx = -\int_0^\infty x^p$
$+ \int_0^\infty \exp(-x)dx^p = p\int_0^\infty x^{p-1}$
(p)
$f_{S_k}(t) = \frac{\lambda(\lambda t)^{k-1}}{\Gamma(k)}\exp(-\lambda t)$ $S_k \sim$
$t^{p-1}\exp(-\alpha t)dt = 1$
$f_X(t) = \frac{\alpha^p t^{p-1}}{\Gamma(p)}\exp(-\alpha t)$, $t \geq 0$, $X \sim$



$S - S - (S-1)$

$S-(t-1)-(S-1$



$S_k = T_1 + \cdots + T_k$. $E(S_k) = E(T_1 + \cdots + T_k) = \sum_{i=1}^{k} E(T_i) = \frac{k}{\lambda}$

$E(S_4) = \frac{4}{\lambda}$. $E(S_4 \mid N(1)=2)$ $\quad F_{S_4}(t \mid N(1)=2) = P(S_4 \le t \mid N(1)=2)$

$= P(N(t) \ge 4 \mid N(1)=2) = 1 - P(N(t) \le 3 \mid N(1)=2)$

$= 1 - P(N(t)=2 \mid N(1)=2) - P(N(t)=3 \mid N(1)=2)$

$$P(N(t)=2 / N(1)=2) = \frac{P(N(1)=2,\ N(t)-N(1)=0)}{P(N(1)=2)} = \frac{P(N(1)=2)\, P(N(t)-N(1)=0)}{P(N(1)=2)}$$

$$= P(N(t)-N(1)=0) = \exp(-\lambda(t-1))$$

$$P(N(t)=3 \mid N(1)=2) = P(N(t)-N(1)=1) = \lambda(t-1)\exp(-\lambda(t-1))$$

$$F_{S_4}(t \mid N(1)=2) = 1 - \exp(-\lambda(t-1)) - \lambda(t-1)\exp(-\lambda(t-1)), \quad t \ge 1$$

$$f_{S_4}(t \mid N(1)=2) = \frac{d}{dt} F_{S_4}(t \mid N(1)=2) = \lambda \exp(-\lambda(t-1)) - \lambda \exp(-\lambda(t-1)) + \lambda^2 (t-1)\exp(-\lambda(t-1))$$

$$f_{S_4}(t \mid N(1)=2) = \lambda^2 (t-1)\exp(-\lambda(t-1)),\quad t \ge 1.$$

$E(X)=\frac{1}{\lambda}$

$Var(X)=\frac{1}{\lambda^2}$

$E(X^2)=\frac{2}{\lambda^2}$

$$\int_1^\infty \lambda^2 (t-1)\exp(-\lambda(t-1))\,dt = \lambda^2\int_0^\infty t\exp(-\lambda t)\,dt = 1.$$

$$\lambda^2\int_1^\infty t(t-1)\exp(-\lambda(t-1))\,dt = \lambda^2\int_0^\infty (t+1)t\exp(-\lambda t)\,dt$$

$$= \lambda^2\int_0^\infty t^2\exp(-\lambda t)\,dt + \lambda^2\int t\exp(-\lambda t)\,dt$$

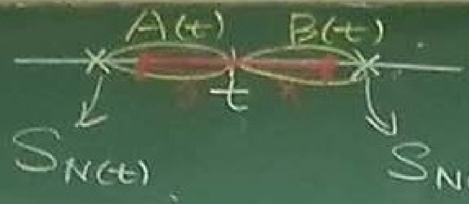
$$E(S_4 \mid N(1)=2) = \lambda E(X^2) + \lambda E(X) = 1 + \frac{2}{\lambda} \qquad \frac{4}{\lambda}$$

$$X\sim Exp(\lambda). \quad F_X(x)=P(X\le x)=1-\exp(-\lambda x),\ x\ge 0.$$

$$F_X(x \mid X>a) = P(X\le x \mid X>a) = \frac{P(a<X\le x)}{P(X>a)}$$

$$= \frac{\exp(-\lambda a)-\exp(-\lambda x)}{\exp(-\lambda a)} = 1-\exp(-\lambda(x-a)),\ (x\ge a)$$

$E(X) > E(Y)$ $\boxed{\text{Inspection}}$ Paradoxes



$$A(t) = t - S_{N(t)}.$$

$$B(t) = S_{N(t)+1} - t.$$

$$P(B(t) > x)$$

$$= P(A(t) > 0,\ B(t) > x)$$

$$P(A(t) > y,\ B(t) > x).$$

$$= P(N(x+y) = 0) = \exp(-\lambda(x+y)).$$

$$P(B(t) \le x) = F_{B(t)}(x).$$

$$P(B(t) > x) = \exp(-\lambda x) = 1 - F_{B(t)}(x)$$

$$E(S_{N(t)+1} - S_{N(t)}) \ne E(S_{k+1} - S_k),\ \forall k$$

$$E(S_{N(t)+1} - S_{N(t)}) = E_N(E_S(S_{k+1} - S_k \mid N(t) = k))$$

$$\ne E\left(\frac{1}{\lambda}\right) \cancel{= \frac{1}{\lambda}}$$