$$\frac{1-P(N(t+\Delta t)-N(t)=0)}{\Delta t} \xrightarrow{\Delta t \to 0} \lambda(t).$$

$\lambda t.$

$\int_0^t \lambda(s)ds.$

$$\frac{d}{dt} G_{N(t)}(z) = G_{N(t)}(z)(\lambda(t)(z-1)). \quad G_{N(0)}(z) = 1.$$

$$G_{N(t)}(z) = \exp\left(\left(\int_0^t \lambda(s)ds\right)(z-1)\right).$$

Non-Stationary

Homogeneous

$$P(N(t)=k) = \frac{\left(\int_0^t \lambda(s)ds\right)^k}{k!} \exp\left(-\int_0^t \lambda(s)ds\right)$$

Poisson Processes

① Independent Increment

② Stationary Increment

③ Sparsity ← Burst. Impulsive

$$E(N(t)) = \int_0^t \lambda(s)ds / t.$$

$$\frac{E(N(t))}{t} \Rightarrow \lim_{t \to 0} \frac{E(N(t))}{t}.$$

$$\left(E\left(z^{N(t+\Delta t)-N(t)}\right)-1\right)$$

$$G_{N(t)}(z) = E(z^{N(t)}). \quad G_{N(t+\Delta t)}(z) - G_{N(t)}(z) = G_{N(t)}(z)$$

$$P(N(t+\Delta t)-N(t)=0) - 1 + zP(N(t+\Delta t)-N(t)=1) + \sum_{k\geq 2} z^k P(N(t+\Delta t)-N(t)=k)$$

$$\sum_{k\ge 2} z^k P(N(t+\Delta t)-N(t)=k). \Longrightarrow \sum_{k\ge 2} z^k \frac{P(N(t+\Delta t)-N(t)=k)}{P(N(t+\Delta t)-N(t)=1)}.$$

$$\boxed{\frac{P(N(t+\Delta t)-N(t)\geqq 2)}{P(N(t+\Delta t)-N(t)=1)} \xrightarrow{\Delta t\to 0} 0} \Rightarrow \boxed{\frac{P(N(t+\Delta t)-N(t)=k)}{P(N(t+\Delta t)-N(t)\geqq 1)} \xrightarrow{\Delta t\to 0} P_k}.$$

$(k=1,2,\cdots,n,\cdots)$

$$\underbrace{\frac{P(N(t+\Delta t)-N(t)=0)-1}{\Delta t}}_{-\lambda(t)} + \underbrace{\frac{P(N(t+\Delta t)-N(t)\geqq 1)}{\Delta t}}_{\lambda(t)}\left(\sum_{k\ge 1} \boxed{\underbrace{\frac{P(N(t+\Delta t)-N(t)=k)}{P(N(t+\Delta t)-N(t)\geqq 1)}}_{\to P_k}} z^k\right)$$

"rare"

$$\frac{d}{dt}G_{N(t)}(z) = G_{N(t)}(z)\left(\lambda(t)\left(\sum_{k\ge 1} P_k z^k - 1\right)\right).$$

$$G_{N(0)}(z)=1. \quad G_{N(t)}(z) = \exp\left(\int_0^t \lambda(s)ds \overbrace{\left(\sum_{k\ge 1} P_k z^k - 1\right)}^{z-1}\right).$$

$$\frac{P(N(t+\Delta t)-N(t)=k)}{P(N(t+\Delta t)-N(t)\ge 1)} = P(N(t+\Delta t)-N(t)=k \mid N(t+\Delta t)-N(t)\ge 1).$$

Generalized Poisson $E(z^{N(t)}) = \sum_k z^k P(N(t)=k)$

$\underline{N(t)}$ ① Independent Increment $E(N(t)) = \int_0^t \lambda(s)ds / t$.

+1 ② Stationary Increment $\frac{E(N(t))}{t} \Rightarrow \lim_{t\to 0} \frac{E(N(t))}{t}$

(1') ③ Sparsity ← Burst. Impulsive $(E(z^{N(t+\Delta t)-N(t)})-1)$

$G_{N(t)}(z) = E(z^{N(t)})$. $G_{N(t+\Delta t)}(z) - G_{N(t)}(z) = G_{N(t)}(z)$

$P(N(t+\Delta t)-N(t)=0)-1+$ $\sum_{k\geq 1} z^k P(N(t+\Delta t)-N(t)=k)$

$X_1, X_2, X_3, X_4, \ldots$ Insurance. Compound. Poisson Processes.

$Y(t) = \sum_{k=1}^{N(t)} X_k$. $N(t)$: Poisson. X_k: i.i.d. X_k independent

$G_{Y(t)}(z) = E(z^{Y(t)}) = E(z^{\sum_{k=1}^{N(t)} X_k}) = E(E(z^{\sum_{k=1}^{n} X_k} | N(t)=n))$

$= E(\prod_{k=1}^{N(t)} E(z^{X_k})) = E(\prod_{k=1}^{N(t)} G_{X_k}(z)) = E((\overset{z'}{\boxed{G_X(z)}})^{N(t)}) = G_{N(t)}(G_X(z))$

$$= \exp(\lambda_1 t(z-1)) \cdot \exp(\lambda_2 t(z-1))$$

$$= \exp((\lambda_1+\lambda_2)t(z-1)). \quad N(t). \text{ Poisson. } \lambda_1+\lambda_2$$

$$N_1(t) - N_2(t) = N(t). \quad P(N(t)<0)>0.$$

$$G_{N(t)}(z) = E(z^{N_1(t)-N_2(t)}) = E(z^{N_1(t)})E(z^{-N_2(t)})$$

$$= \exp(\lambda_1 t(z-1)+\lambda_2 t(z^{-1}-1)) = \exp\left((\lambda_1+\lambda_2)t\left(\frac{\lambda_1}{\lambda_1+\lambda_2}z+\frac{\lambda_2}{\lambda_1+\lambda_2}z^{-1}-1\right)\right)$$

$$G_Y(z) = G_{N(t)}(G_\xi(z)) = \exp(\lambda t(z'-1))\Big|_{z'=G_\xi(z)}$$

$$= \exp(\lambda t(\boxed{G_\xi(z)}-1)). \quad P_k = P(\xi=k).$$

$$\exp\left(\lambda t\left(\boxed{\sum_{k\geq 1} P_k z^k}-1\right)\right)$$

$N_1(t)$, λ_1. $N_2(t)$, λ_2. $N(t)=N_1(t)+N_2(t)$ $N_1(t)$, $N_2(t)$, independent

$$G_{N(t)}(z) = E(z^{N(t)}) = E(z^{N_1(t)+N_2(t)}) = E(z^{N_1(t)})E(z^{N_2(t)})$$

$X_1, X_2, X_3, X_4, \dots$ Insurance. Compound. Poisson Processes

$$Y(t) = \sum_{k=1}^{N(t)} X_k(t). \quad N(t): \text{Poisson}. \quad X_k: \text{i.i.d.} \quad X_k \text{ independent}$$

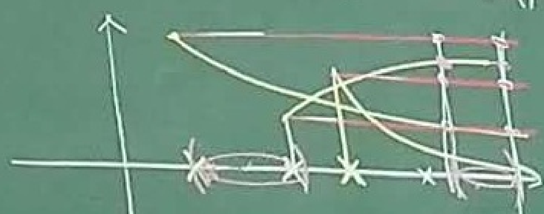
$$G_{Y(t)}(z) = E(z^{Y(t)}) = E\left(z^{\sum_{k=1}^{N(t)} X_k}\right) = E\left(E\left(z^{\sum_{k=1}^{n} X_k} \,\middle|\, N(t)=n\right)\right)$$

$$= E\left(\prod_{k=1}^{N(t)} E(z^{X_k})\right) = E\left(\prod_{k=1}^{N(t)} G_{X_k}(z)\right) = E\left(\big(\overset{z'}{\boxed{G_X(z)}}\big)^{N(t)}\right) = G_{N(t)}(G_X(z))$$

$$G_X(z) = \frac{\lambda_1}{\lambda_1+\lambda_2} z + \frac{\lambda_2}{\lambda_1+\lambda_2} z^{-1}.$$

$$P(X=1) = \frac{\lambda_1}{\lambda_1+\lambda_2}, \quad P(X=-1) = \frac{\lambda_2}{\lambda_1+\lambda_2}$$

$$G_{N(t+\Delta t)} - G_{N(t)} = E\left(z^{N(t)}\left(z^{N(t+\Delta t)-N(t)} - 1\right)\right)$$



$$Y(t) = \sum_{k=1}^{N(t)} X_k(t, \tau_k). = \sum_{k=1}^{N(t)} X(t, \tau_k, A_k)$$

$\boxed{\text{Filtered}}$ Poisson.

$$\sum_{k=1}^{N(t)} \delta(t-\tau_k) \longrightarrow \boxed{X(t)} \longrightarrow Y(t)$$

$$\Phi_{Y(t)}(\omega) = E(\exp(j\omega Y(t))) = E\left(\exp\left(j\omega \sum_{k=1}^{N(t)} g(t,\tau_k,A_k)\right)\right)$$

$$= E_{N,\tau}\left(E_A\left(\exp\left(j\omega \sum_{k=1}^{n} g(t,\tau_k,A_k)\right) \middle| N(t)=n,\tau_1,\cdots,\tau_n\right)\right)$$

Let $B(t,\tau) = E_A(\exp(j\omega\, g(t,\tau,A)))$.

$$= E_{N,\tau}\left(\prod_{k=1}^{N(t)} B(t,\tau_k)\right) = E_N\left(E_\tau\left(\prod_{k=1}^{n} B(t,\tau_k) \middle| N(t)=n\right)\right)$$

$$= E_N\left(\frac{n!}{t^n}\int\cdots\int_{0\le\tau_1\le\cdots\le\tau_n\le t} \boxed{\prod_{k=1}^{n} B(t,\tau_k)}\, d\tau_1\cdots d\tau_n\right)$$

$$\prod_{k=1}^{n} B(t,\tau_k) = B(t,\tau_1)B(t,\tau_2)\cdots B(t,\tau_n).$$

$$= E_N\left(\frac{1}{t^n}\int_0^t\cdots\int_0^t \prod_{k=1}^{n} B(t,\tau_k)\,d\tau_1\cdots d\tau_n\right)$$

$$= E_N\left(\frac{1}{t^n}\prod_{k=1}^{n}\int_0^t B(t,\tau_k)\,d\tau_k\right) = E\left(\left(\frac{1}{t}\int_0^t B(t,\tau)\,d\tau\right)^{N(t)}\right)$$

$$= \exp\left(\lambda t\left(\frac{1}{t}\int_0^t B(t,\tau)\,d\tau - 1\right)\right) = \exp\left(\lambda\int_0^t (B(t,\tau)-1)\,d\tau\right)$$

$$= \exp\left(\lambda \int_0^t \left(E_A(\exp(j\omega \gamma(t,\tau,A))) - 1\right) d\tau\right) \cdot \Phi_{Y(t)}(\omega)$$

$$E(Y(t)) = \frac{1}{j}\frac{d}{d\omega}\Phi_{Y(t)}(\omega)\Big|_{\omega=0} = \lambda\int_0^t E_A(\gamma(t,\tau,A))\,d\tau$$

$$E\sum_{k=1}^{N(t_1)}(t_1-\tau_k) \to \gamma(t_1,\tau_k,A_k) = t_1 - \tau_k$$

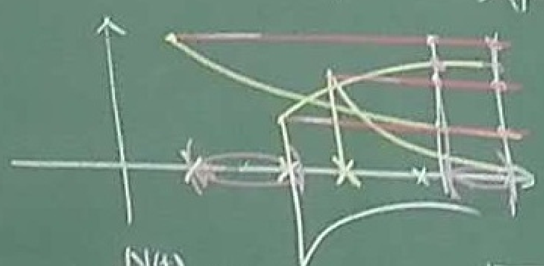
$$\min \sum_{k=1}^{n} \frac{\lambda}{2} t_k^2,\quad \text{s.t.} \sum_{k=1}^{n} t_k = t,\ t_k \ge 0 \qquad \lambda\int_0^{t_1}(t_1-\tau)\,d\tau = \frac{\lambda t_1^2}{2}$$

$$\Rightarrow t_1 = \cdots = t_n = \frac{t}{n}. \qquad \frac{\lambda t^2}{2n}$$

$$G_\gamma(z) = \frac{\lambda_1}{\lambda_1+\lambda_2}z + \frac{\lambda_2}{\lambda_1+\lambda_2}z^{-1}.$$

$$P(\gamma=1) = \frac{\lambda_1}{\lambda_1+\lambda_2}, \qquad P(\gamma=-1) = \frac{\lambda_2}{\lambda_1+\lambda_2}$$

$$G_{N(t+\Delta t)} - G_{N(t)} = E\left(z^{N(t)}\left(z^{N(t+\Delta t)-N(t)}-1\right)\right)$$



$$Y(t) = \sum_{k=1}^{N(t)} \gamma_k(t,\tau_k) = \sum_{k=1}^{N(t)} \gamma(t,\tau_k,A_k)$$

$\boxed{\text{Filtered}}$ Poisson.

$$\sum_{k=1}^{N(t)} \delta(t-\tau_k) \to \boxed{\gamma(t)} \to Y(t)$$