$$P_{ij}(n+m) = \sum_k P_{ik}(m)\, P_{kj}(n) \Rightarrow P(n) = (P(1))^n.$$

Qualitative behavior. (Global) $\boxed{P_{ij}(n) \longrightarrow ?}$ $(n \to \infty)$

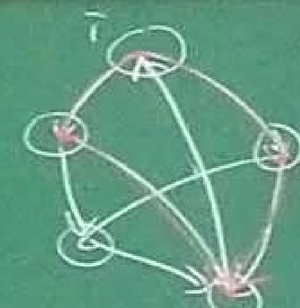
Asymptotic. $\longrightarrow$ Steady State. $\longrightarrow$ Classification of States

① Reachable. $i, j \in \Omega$. $i \to j \Leftrightarrow \exists n,\ P_{ij}(n) > 0$.

② Commutative. $i, j \in \Omega$. $i \leftrightarrow j \Leftrightarrow i \to j,\ j \to i$

$i \leftrightarrow j,\ j \leftrightarrow k \Rightarrow i \leftrightarrow k$.

Markov $\boxed{\text{Chains}}$ discrete



$$P(X_{n+1} = x_{n+1} \mid X_n = x_n,\ X_{n-1} = x_{n-1},\ \cdots,\ X_0 = x_0).$$

$$= P(X_{n+1} = x_{n+1} \mid X_n = x_n).$$

directed

$P(CA \mid B) = P(C \mid B)\, P(A \mid B)$

$$P_{ij}(n, m) = P(X_n = j \mid X_m = i)$$

$$\parallel\ P_{ij}(n-m)$$

Chapman-kolmogorov

③ Closed Set. $C \subseteq \Omega$. C is closed $\Leftrightarrow i \in C,\ j \bar{\in} C.$
$\Rightarrow i \not\to j.$

$$P \underset{I}{=} \begin{pmatrix} A & B \\ O & \boxed{D} \end{pmatrix}.$$

Reduction.

(O)

④ Irreducible (Chains)

$\boxed{C \subseteq \Omega.\ C \text{ is closed} \Rightarrow C = \Omega}$

Ω is irreducible $\Leftrightarrow \forall i, j \in \Omega,\ i \leftrightarrow j$

"$\Leftarrow$": Trivial. "$\Rightarrow$": $\forall i.\ A_i = \{ j \in \Omega \mid i \to j \} \Rightarrow A_i = \Omega$

$j \in A_i \quad k \bar{\in} A_i \Rightarrow j \not\to k.$

$j \to k.\quad j \in A_i,\ i \to j \Rightarrow i \to k.$ Contradiction!

⑤ Recurrent ($\boxed{\text{常}}$返). i. is recurrent $\Leftrightarrow \sum_{n=1}^{\infty} f_{ii}(n) = 1.$

First Passage Probability $f_{ij}(n) = P(X_n = j,\ X_k \neq j,\ k < n \mid X_0 = i)$

$\sum_{n=1}^{\infty} f_{ij}(n) \leq 1.$

$$\begin{cases} \text{Temporal} & P_{ij}(n) = \sum_{k=1}^{n} f_{ij}(k)\, P_{jj}(n-k). \\ \text{Spatial} & P_{ij}(n) = \sum_{l} P_{il}(k)\, P_{lj}(n-k)\ \forall k \end{cases}$$

$$P_{ij}(n) = P(X_n = j \mid X_0 = i) \qquad T_i = \min\{k \mid X_k = j, X_0 = i\}$$

$$k_1 \neq k_2$$

$$= \sum_{k=1}^{n} P(X_n = j, T_i = k \mid X_0 = i) \qquad \{T_i = k_1\} \cap \{T_i = k_2\} = \phi$$

$$= \sum_{k=1}^{n} P(X_n = j \mid T_i = k, X_0 = i)\, P(T_i = k \mid X_0 = i)$$

$$= \sum_{k=1}^{n} P(X_n = j \mid X_k = j)\, P(X_k = j, X_m \neq j, m < k \mid X_0 = i)$$

$$= \sum_{k=1}^{n} P_{jj}(n-k)\, f_{ij}(k)$$

$$P_{ij}(z) = \sum_{n=0}^{\infty} P_{ij}(n) z^n. \qquad F_{ij}(z) = \sum_{n=1}^{\infty} F_{ij}(n) z^n$$

$$P_{ij}(0) z^0 + \sum_{n=1}^{\infty} \qquad \delta_{ij} = P_{ij}(0). \qquad f_{ij}(0) = 0.$$

$$P_{ij}(z) = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} f_{ij}(k) P_{jj}(n-k)\right) z^n = \sum_{n=1}^{\infty} \sum_{k=1}^{n} f_{ij}(k) z^k P_{jj}(n-k) z^{n-k}$$

$$= \sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty} P_{jj}(n-k) z^{n-k}\right) f_{ij}(k) z^k \qquad + \boxed{P_{ij}(0) \cdot z^0}$$

$$= \left(\sum_{n=0}^{\infty} P_{jj}(n) z^n\right)\left(\sum_{k=1}^{\infty} f_{ij}(k) z^k\right) + \delta_{ij}$$

$$P_{ij}(z) = \delta_{ij} + F_{ij}(z)P_{jj}(z).$$

i is Non-Recurrent $\Downarrow$ $\boxed{P_{ii}(n) \to 0.}$

$$i = j: \quad P_{ii}(z) = 1 + F_{ii}(z)P_{ii}(z).$$

$$P_{ii}(z) = \frac{1}{1 - F_{ii}(z)} \xRightarrow{z \to 1} \sum_{n=0}^{\infty} P_{ii}(n) = \frac{1}{1 - \boxed{\sum_{k=1}^{\infty} f_{ii}}}$$

$$\sum_{n=0}^{\infty} P_{ii}(n) = \begin{cases} \infty, & \text{Recurrent} \\ < \infty, & \text{Non-Recurrent} \end{cases}$$

j is Non-Recurrent $\quad P_{ij}(n) \to 0, \ \forall i, \ n \to \infty.$

$$i \neq j. \quad P_{ij}(z) = F_{ij}(z)P_{jj}(z) \xRightarrow{z \to 1} \underbrace{\sum_{n=0}^{\infty} P_{ij}(n)}_{<\infty} = \underbrace{\sum_{n=1}^{\infty} f_{ij}(n)}_{\leq 1} \underbrace{\sum_{n=0}^{\infty} P_{jj}(n)}_{<\infty}$$

$i \Leftrightarrow j. \quad \exists n_1, \ P_{ij}(n_1) > 0. \quad \exists n_2, \ P_{ji}(n_2) > 0.$

$$P_{ii}(n_1 + n_2 + n) \geq P_{ij}(n_1) P_{ji}(n_2) P_{jj}(n)$$

$$\sum_{n=0}^{\infty} P_{ii}(n_1 + n_2 + n) \geq P_{ij}(n_1) P_{ji}(n_2) \boxed{\sum_{n=0}^{\infty} P_{jj}(n)} = \infty$$

Finite States $\Rightarrow$ Existence of Recurrent State.

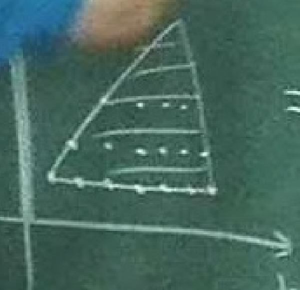
$$\sum_j P_{ij}(n) = 1,\ \forall n \Rightarrow \lim_{n\to\infty} \sum_j P_{ij}(n) = 1.$$

$$\Rightarrow \underline{\sum_j \lim_{n\to\infty} P_{ij}(n) = 1} . \Rightarrow 1 = \sum_j 0 = 0.\ \text{Contradiction!}$$

Finite States + Irreducible $\Rightarrow$ All are recurrent

$$P_{ij}(z) - \sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} f_{ij}(k)\, P_{jj}(n-k)\right) z^n = \sum_{n=1}^{\infty}\sum_{k=1}^{n} f_{ij}(k) z^k\, P_{jj}(n-k) z^{n-k}$$

$$= \sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty} P_{jj}(n-k) z^{n-k}\right) f_{ij}(k) z^k \quad + \boxed{P_{ij}(0)\cdot z^0}$$



$$= \left(\sum_{n=0}^{\infty} P_{jj}(n) z^n\right)\left(\sum_{k=1}^{\infty} f_{ij}(k) z^k\right) + \delta_{ij}$$

$$n! = \prod_{k=1}^{n} k = \exp\left(\ln \prod_{k=1}^{n} k\right) = \exp\left(\sum_{k=1}^{n} \ln k\right)$$

$$\sum_{k=1}^{n} \ln k \approx \int_1^n \ln x \, dx = x(\ln x - 1)\Big|_1^n = \ln\left(\frac{n}{e}\right)^n + 1$$

$$\exp\left(\sum_{k=1}^{n} \ln k\right) \approx \left(\frac{n}{e}\right)^n \cdot c.$$

$$4pq \le (p+q)^2 = 1.$$

$$\begin{cases} p = q = \frac{1}{2}. & \Rightarrow \text{Recurrent} \\ p \ne q. & \Rightarrow \text{Non-Recurrent} \end{cases}$$

$$\sum_{k=0}^{\infty} \binom{2k}{k} p^k q^k.$$

$$= \frac{\left(\frac{2k}{e}\right)^{2k} \sqrt{2\pi \cdot 2k}}{\left(\frac{k}{e}\right)^{2k} 2\pi k} \cdot (4pq)^k$$

$$\sim \frac{(4pq)^k}{\sqrt{k}} \qquad \frac{1}{\sqrt{k}}$$

$$n! \sim \left(\frac{n}{e}\right)^n \boxed{\sqrt{2\pi n}}$$

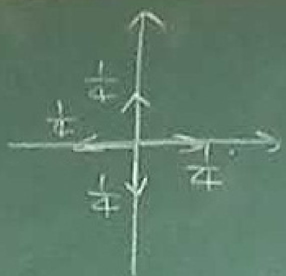
Order

$$\frac{1}{k^{\alpha}}$$

$$\frac{1}{k^{1+s}}$$

$$\sum_{k=1}^{n}\frac{1}{k}\sim \ln n.$$

$$\sum_{k=1}^{n}\frac{1}{k\ln k}$$



$$(1+x)^{2k}$$

$$(1+x)^{k}(1+x)^{k}$$

$$P_{00}(2k)=\left(\frac{1}{4}\right)^{2k}\sum_{m=0}^{k}\frac{(2k)!}{m!\,m!\,(k-m)!\,(k-m)!}$$

$$=\left(\frac{1}{4}\right)^{2k}\frac{(2k)!}{k!\,k!}\sum_{m=0}^{k}\frac{k!\,k!}{m!\,(k-m)!\,m!\,(k-m)!}$$

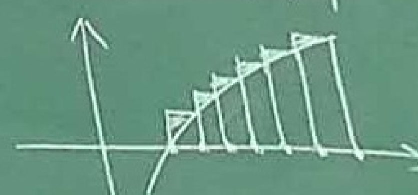
$$=\left(\frac{1}{4}\right)^{2k}\binom{2k}{k}\sum_{m=0}^{k}\binom{k}{m}\binom{k}{k-m}$$

$$=\left(\frac{1}{4}\right)^{2k}\binom{2k}{k}\binom{2k}{k}=\left(\frac{1}{4}\right)^{2k}\binom{2k}{k}^{2}\sim\frac{1}{k}$$

$$n!=\prod_{k=1}^{n}k=\exp\left(\ln\prod_{k=1}^{n}k\right)=\exp($$

$$\sum_{k=1}^{n}\ln k\approx\int_{1}^{n}\ln x\,dx=x(\ln x-1)\Big|$$

$$\exp\left(\sum_{k=1}^{n}\ln k\right)\approx\left(\frac{n}{e}\right)^{n}\cdot c.$$



$$\binom{2k}{k}\sim\frac{4^{k}}{\sqrt{k}}$$

$$\binom{2k}{k}^{2}\sim\frac{4^{2k}}{k}$$

$$\frac{1}{k^{3/2}}$$

$$4pq\le(p+q)^{2}=1.$$

$$\begin{cases} p=q=\frac{1}{2} \Rightarrow \text{Recurrent} \\ p\neq q \Rightarrow \text{Non-Recurrent}\end{cases}$$