$\mu_1 = E(X_1)$, $\mu_2 = E(X_2)$ $\sigma_1^2 = Var(X_1)$, $\sigma_2^2 = Var(X_2)$.

$$\rho = \frac{E(X_1-\mu_1)(X_2-\mu_2)}{(E(X_1-\mu_1)^2)^{\frac{1}{2}}(E(X_2-\mu_2)^2)^{\frac{1}{2}}}$$

n : $f_X(x) = \frac{1}{(2\pi)^{\frac{n}{2}}(\det\Sigma)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)$. $X, \mu \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n\times n}$

$\Sigma = E(X-\mu)(X-\mu)^T$, $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}^{-1}$

Characteristic Function. $X \in \mathbb{R}$ $\Phi_X(\omega) = E(\exp(j\omega X))$

Gaussian Processes.

$X(t)$. $\forall n$. $\forall t_1 \le t_2 \le \cdots \le t_n$. $X = (X(t_1), \cdots, X(t_n))$.

$\Leftrightarrow$ $X \sim N(\mu, \Sigma)$. n-dimensional Joint Gaussian Distribution

$n=1$: $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

$n=2$: $f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right)\right)$

$(X_1, X_2) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$.

$$\Phi_X(\omega) = \int_{-\infty}^{+\infty} \exp(j\omega x) f_X(x)\,dx. \qquad E(g(X,Y)) = E_Y(E_X(g(X,Y)|Y))$$

$$P(A) = E(I_A).$$

Sum of Random Variables. X, Y, independent. $f_X(x), f_Y(y)$

$$Z = X + Y. \quad f_Z(z) = (f_X \circledast f_Y)(z) = \int_{-\infty}^{+\infty} f_X(z-t) f_Y(t)\,dt.$$

$$F_Z(t) = P(Z \le t) = \underline{P(X+Y \le t)} = \int_{-\infty}^{+\infty} P(X+y \le t \mid Y=y) f_Y(y)\,dy$$

$$\Omega : A \subseteq \Omega \quad P(A) = \sum_{x \in A} P(x) = \sum_{\omega \in \Omega} I_A(\omega) P(\omega) = E(I_A)$$

$$F_Z(t) = \int_{-\infty}^{+\infty} P(X \le t-y) f_Y(y)\,dy = \int_{-\infty}^{+\infty} F_X(t-y) f_Y(y)\,dy.$$

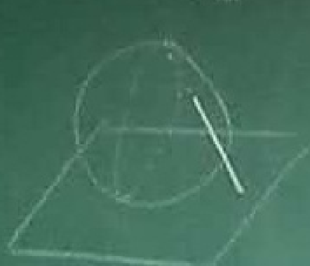
$$f_Z(t) = \frac{d}{dt} F_Z(t) = \frac{d}{dt} \int_{-\infty}^{+\infty} F_X(t-y) f_Y(y)\,dy.$$

$$= \int_{-\infty}^{+\infty} f_X(t-y) f_Y(y)\,dy = \boxed{(f_X \circledast f_Y)}(t).$$

$$\Phi_Z(\omega) = E(\exp(j\omega Z)) = E(\exp(j\omega(X+Y))) = E(\exp(j\omega X)\exp(j\omega Y))$$

$$= E(\exp(j\omega X)) E(\exp(j\omega Y)) = \Phi_X(\omega)\Phi_Y(\omega)$$

$$\Phi_X(\omega) = \int_{-\infty}^{+\infty} f_X(x)\exp(j\omega x)\,dx.$$



$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2} + j\omega x\right)dx.$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2}(x^2-2\mu x+\mu^2)+j\omega x\right)dx$$

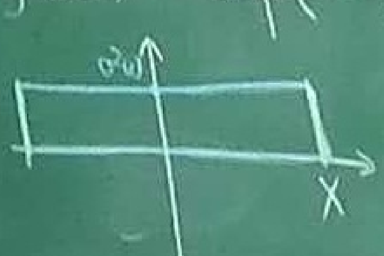
$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2}\left(x^2-2\mu x-2j\sigma^2\omega x+(\mu+j\sigma^2\omega)^2\right) + \frac{1}{2\sigma^2}(\mu+j\sigma^2\omega)^2 - \frac{1}{2\sigma^2}\mu^2\right)dx$$

$$= \underline{\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} \exp\left(-\frac{(x-\mu-\boxed{j\sigma^2\omega})^2}{2\sigma^2}\right)dx}\,\exp\left(j\mu\omega - \frac{1}{2}\sigma^2\omega^2\right).$$

||

Cauchy Integral. $f(z)$ Analytic $\oint_C f(z)\,dz = 0.$

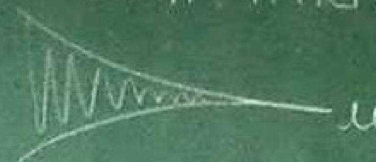
$f(z) = \exp\left(-\frac{z^2}{2\sigma^2}\right).$ $\int_0^{\sigma^2\omega} \exp\left(-\frac{(x+jy)^2}{2\sigma^2}\right)dy = \int_0^{\sigma^2\omega}\exp\left(-\frac{x^2}{2\sigma^2}\right)\exp\left(-\frac{y^2}{2\sigma^2}\right)$



$$= \boxed{\exp\left(-\frac{x^2}{2\sigma^2}\right)}\int_0^{\sigma^2\omega} \exp\left(-\frac{y^2}{2\sigma^2}\right)\exp\left(-j\frac{xy}{\sigma^2}\right)dy \xrightarrow{x\to\infty} 0 \qquad \exp\left(-j\frac{xy}{\sigma^2}\right)dy$$

Law of Large Number: $X_1, \cdots, X_n$ i.i.d. $E(X_1)=\mu$.

$$\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{n\to\infty} \mu.$$



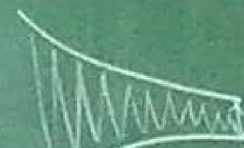
$$(1+\frac{1}{n})^n \to e.$$

$$(1+\frac{a}{n})^n \to \exp(a)$$

Central Limit Theorem: $E(X_k)=0$, $Var(X_k)=1$

$$\frac{X_1+\cdots+X_n}{\sqrt{n}} \xrightarrow{n\to\infty} N(0,1)$$

$$(1+\frac{a}{n}+o(\frac{1}{n}))^n \to e^a$$



$$\Phi_{X_1}(\frac{\omega}{n}) = E(\exp(jX_1\frac{\omega}{n})) = E(1+\frac{j\omega X_1}{n}+o(\frac{1}{n})) = 1+\frac{j\omega\mu}{n}+o(\frac{1}{n}).$$

$$\Phi_{\frac{X_1+\cdots+X_n}{n}}(\omega) = E(\exp(j\omega(\frac{X_1+\cdots+X_n}{n})))$$

$$= \prod_{k=1}^{n} E(\exp(j\omega\frac{X_k}{n})) = \prod_{k=1}^{n}\Phi_{X_k}(\frac{\omega}{n}) = (\Phi_{X_1}(\frac{\omega}{n}))^n.$$

$$= (1+\frac{j\omega\mu}{n}+o(\frac{1}{n}))^n \xrightarrow{n\to\infty} \exp(j\mu\omega) = \Phi_\mu(\omega)$$

$$\Phi_{\frac{X_1+\cdots+X_n}{\sqrt{n}}}(\omega) = (\Phi_{X_1}(\frac{\omega}{\sqrt{n}}))^n = (1+\frac{j\omega E(X_1)}{\sqrt{n}}+\frac{1}{2}E(\frac{j\omega X_1}{\sqrt{n}})^2+o(\frac{1}{n}))^n$$

$$= \left(1 - \frac{E(\mathbb{X}_1)^2 \omega^2}{2n} + o\left(\frac{1}{n}\right)\right)^n \doteq \left(1 - \frac{\omega^2}{2n} + o\left(\frac{1}{n}\right)\right)^n$$

$\mathbb{X} \sim \boxed{F}$

$a\mathbb{X} \sim \boxed{F}$

$$\xrightarrow{n\to\infty} \exp\left(-\frac{\omega^2}{2}\right) = \Phi_{\mathbb{X}}(\omega). \quad \mathbb{X} \sim N(0,1).$$

$Var(a\mathbb{X}) = a^2 Var(\mathbb{X})$.

Maximum Entropy $\Omega \to (\#\Omega) \to \log(\#\Omega)$ Boltzmann.

$$E(f) = \boxed{-\int_{-\infty}^{+\infty} f_{\mathbb{X}}(x)\log f_{\mathbb{X}}(x)\,dx} = E(\log f(\mathbb{X})). \quad \text{Shannon Randomness}$$

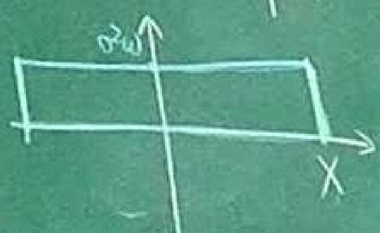
$$\max_f E(f). \quad \text{s.t.} \int_{-\infty}^{+\infty} x f(x)\,dx = \mu. \quad \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \sigma^2$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} \exp\left(-\frac{(x-\mu-\boxed{j\sigma^2\omega})^2}{2\sigma^2}\right)dx \quad \boxed{\exp\left(j\mu\omega - \frac{1}{2}\sigma^2\underline{\underline{\omega^2}}\right)}.$$

$\Phi_{\mathbb{X}}(\underline{\underline{\omega}})$.

Cauchy Integral. $f(z)$ Analytic $\oint_C f(z)\,dz = 0$.

$$f(z) = \exp\left(-\frac{z^2}{2\sigma^2}\right). \quad \int_0^{\sigma^2\omega} \exp\left(-\frac{(x+jy)^2}{2\sigma^2}\right)dy = \int_0^{\sigma^2\omega}\exp\left(-\frac{x^2}{2\sigma^2}\right)\exp\left(-\frac{y^2}{2\sigma^2}\right)$$



$$= \boxed{\exp\left(-\frac{x^2}{2\sigma^2}\right)}\int_0^{\sigma^2\omega}\exp\left(-\frac{y^2}{2\sigma^2}\right)\exp\left(-j\frac{xy}{\sigma^2}\right)dy \xrightarrow{x\to\infty} 0$$

$$\exp\left(-j\frac{xy}{\sigma^2}\right)dy$$

$$\tilde{G}(t) = \int_{-\infty}^{+\infty} (f_0 + tg)\log(f_0 + tg)\,dx + \lambda_1\left(\int_{-\infty}^{+\infty} x(f_0 + tg)\,dx - \mu\right) + \lambda_2\int_{-\infty}^{+\infty} x^2(f_0 + tg)\,dx - \sigma^2).$$

$$\frac{d}{dt}\tilde{G}(t) = \int_{-\infty}^{+\infty} g(x)\left(\log(f_0 + tg) + \lambda_1 x + \lambda_2 x^2 + 1\right)dx.$$

$$\frac{d}{dt}\tilde{G}(t)\Big|_{t=0} = \int_{-\infty}^{+\infty} g(x)\left(\log(f_0) + \lambda_1 x + \lambda_2 x^2 + 1\right)dx = 0$$

$$f_0(x) = \exp(-\lambda_2 x^2 - \lambda_1 x - 1)$$

Variational Technique.

$E(f)$. Functional. f_0 optimal Function. $E(f) \le E(f_0)$ $\forall f$.

$G(t) = E(f_0 + tg)$. $\forall g$. $G(t) \le G(0)$. $\forall t$.

$$\frac{d}{dt}G(t) = \frac{d}{dt}\int_{-\infty}^{+\infty} (f_0 + tg)\log(f_0 + tg)\,dx$$

$$= \int_{-\infty}^{+\infty} g\log(f_0 + tg)\,dx + \int_{-\infty}^{+\infty} g\,dx$$

$\mathbb{X} \in \mathbb{R}^n$. $\mathbb{X} = (\mathbb{X}_1, \cdots, \mathbb{X}_n)$. $\Phi_{\mathbb{X}}(\omega) = E(\exp(j\omega^T \mathbb{X}))$

$\exp(j\mu\omega - \frac{1}{2}\sigma^2\omega^2) \rightarrow \exp(j\mu^T\omega - \frac{1}{2}\omega^T \Sigma \omega)$. $\omega \in \mathbb{R}^n$.

$$f_{\mathbb{X}}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}(\det\Sigma)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right) \quad x, \mu \in \mathbb{R}^n$$

$$\int_{\mathbb{R}} f_{\mathbb{X}}(x)dx = \frac{1}{(2\pi)^{\frac{n}{2}}(\det\Sigma)^{\frac{1}{2}}} \int_{\mathbb{R}} \exp\left(-\frac{1}{2}(\underbrace{x-\mu}_{x'})^T \Sigma^{-1}(x-\mu)\right)dx$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}(\det\Sigma)^{\frac{1}{2}}} \int_{\mathbb{R}} \exp\left(-\frac{1}{2}(x')^T \Sigma^{-1}(x')\right)dx'$$

$$= \left(1 - \frac{E(\mathbb{X}_1)^2 \cdot \omega^2}{2n} + o\left(\frac{1}{n}\right)\right)^n = \left(1 - \frac{\omega^2}{2n} + o\left(\frac{1}{n}\right)\right)^n$$

$$\xrightarrow{n\to\infty} \exp\left(-\frac{\omega^2}{2}\right) = \Phi_{\mathbb{X}}(\omega). \quad \mathbb{X} \sim N(0,1).$$

$\mathbb{X} \sim \boxed{F}$

$a\mathbb{X} \sim \boxed{F}$

$Var(a\mathbb{X}) = a^2 Var(\mathbb{X})$.

Maximum Entropy. $\Omega \rightarrow (\#\Omega) \rightarrow \log(\#\Omega)$ Boltzmann.

$$E(f) = \boxed{-\int_{-\infty}^{+\infty} f_{\mathbb{X}}(x)\log f_{\mathbb{X}}(x)dx} = E(\log f(\mathbb{X})).$$ Shannon Randomness

$$\max_f E(f). \quad \text{s.t} \quad \int_{-\infty}^{+\infty} x f(x)dx = \mu. \quad \int_{-\infty}^{+\infty} x^2 f(x)dx = \sigma^2$$

$$\Sigma = E(\bar{X}-\mu)(\bar{X}-\mu)^T. \qquad \Sigma^T =$$

$$\Sigma^{-1} = U^T \Lambda U. \qquad U U^T = U^T$$

$$\Sigma^{-1} = U^T \Lambda^{\frac{1}{2}} \Lambda^{\frac{1}{2}} U = L^T L.$$

$$y = Lx' \qquad = \underline{U^T \Lambda^{\frac{1}{2}} U} \cdot \underline{U^T \Lambda^{\frac{1}{2}} U} = L^2$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} (\det \Sigma)^{\frac{1}{2}}} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} y^T y\right) \frac{1}{|\det L|} dy$$

$$y = Lx'$$

$$dy = \left|\det\left(\frac{dy}{dx'}\right)\right| \cdot dx'$$

$$= |\det L| \cdot dx'$$

$$dx' = \frac{1}{|\det L|} dy \qquad \Sigma^{-1} = L^T L$$

$$\det \Sigma^{-1} = (\det L)^2$$

$$(\det \Sigma)^{\frac{1}{2}} = \frac{1}{|\det L|}$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} y^T y\right) dy$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} \sum_{k=1}^{n} y_k^2\right) dy_1 \cdots dy_n$$

$$= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) dy\right)^n = 1$$

$$\cdot \left(\int_{-\infty}^{+\infty} x (f_0 + tg)\, dx - \mu\right)$$

$$x^2 (f_0 + tg)\, dx - \sigma^2).$$

$$\frac{dy}{dx'} = \frac{\partial(y_1, \cdots, y_n)}{\partial(x'_1, \cdots, x'_n)}$$

$$= \begin{pmatrix} \frac{\partial y_1}{\partial x'_1} & \cdots & \frac{\partial y_1}{\partial x'_n} \\ \frac{\partial y_n}{\partial x'_1} & \cdots & \frac{\partial y_n}{\partial x'_n} \end{pmatrix} = L$$

$$\frac{1}{(2\pi)^{\frac{n}{2}}(\det\Sigma)^{\frac{1}{2}}}\int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)+j\omega^T x\right)dx.$$

$(AB)^T = B^T A^T$

$L^{-1}(L^{-1})^T.$
$=(L^T L)^{-1}$
$=(\Sigma^{-1})^{-1}$
$=\Sigma.$

$\Sigma^{-1}=L^T L \qquad y = L(x-\mu) \Longrightarrow x = L^{-1}y+\mu.$

$$=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}\underline{y^T y}+\underline{j\omega^T(L^{-1}y+\mu)}\right)dy$$

$$=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}(y-j(L^{-1})^T\omega)^T(y-j(L^{-1})^T\omega)+j\omega^T\mu-\frac{1}{2}\omega^T(L^{-1})(L^{-1})^T\omega\right)$$

$$=\boxed{\exp\left(j\mu^T\omega-\frac{1}{2}\omega^T\Sigma\omega\right)}\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}(y-j(L^{-1})^T\omega)^T(y-j(L^{-1})^T\omega)\right)dy$$

$X\in\mathbb{R}^n.\quad X=(X_1,\dots,X_n).\quad \Phi_X(\omega)=E(\exp(j\omega^T X))$

$$\exp\left(j\mu\omega-\frac{1}{2}\sigma^2\omega^2\right)\to\exp\left(j\mu^T\omega-\frac{1}{2}\omega^T\Sigma\omega\right).\quad \omega\in\mathbb{R}^n.$$

$$f_X(x)=\frac{1}{(2\pi)^{\frac{n}{2}}(\det\Sigma)^{\frac{1}{2}}}\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)\quad x,\mu\in\mathbb{R}^n$$

$$\int_{\mathbb{R}}f_X(x)dx=\frac{1}{(2\pi)^{\frac{n}{2}}(\det\Sigma)^{\frac{1}{2}}}\int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}\overset{x'}{(x-\mu)}^T\Sigma^{-1}(x-\mu)\right)dx$$

$$=\frac{1}{(2\pi)^{\frac{n}{2}}(\det\Sigma)^{\frac{1}{2}}}\int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}(x')^T\Sigma^{-1}(x')\right)dx'=\frac{1}{(2\pi)^{\frac{n}{2}}(\det\Sigma)^{\frac{1}{2}}}\int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}(x')^T L^T L\,(x')\right)dx'$$