$X(t)$ is cyclostationary. T Period

$\Rightarrow Y(t) = X(t+\theta)$. θ independent of $X(t)$. $\boxed{\theta \sim U(0,T)}$

Phase. Random Phase Modulation.

$Y(t)$ is W.S.S. $R_Y(t,s) = E(Y(t)Y(s)) = E(X(t+\theta)X(s+\theta))$

$$= E_\theta(E_X(X(t+\theta)X(s+\theta)|\theta)) = E_\theta(R_X(t+\theta, s+\theta))$$

$$= \frac{1}{T}\int_0^T R_X(\underbrace{t+\theta}_{t-s+\theta'}, \underbrace{s+\theta}_{\theta'})\,d\theta. \quad \theta' = s+\theta.\ d\theta' = d\theta.$$

Non-Stationary Stochastic Processes.

① Cyclostationary Processes

② Orthogonal Increment Processes

$X(t)$. $R_X(t,s) = E(X(t)X(s)) = R_X(t+T, s+T)$. $\forall T \in \mathbb{R}$

$\exists T. \Rightarrow R_X(t,s) = R_X(t+T, s+T) \Rightarrow R_X(t,s) = R_X(t+nT, s+nT)$

$$= \frac{1}{T}\int_{S}^{S+T} R_X(\underline{t-S}+\theta', \theta')\,d\theta' = \frac{1}{T}\int_{0}^{T} R_X(\boxed{t-S}+\theta', \theta')\,d\theta'$$

$R_Y(t, S)$

Communication Signal. PAM (Pulse Amplitude Modulation)

BPSK, QPSK, QAM, ...

$X(t) = \sum_{k=-\infty}^{+\infty} \boxed{\alpha_k}\,\Phi(t-kT)$; α_k: information Symbol. 0, 1   00 01 10 11

$E(\alpha_k)=0$, W.S.S. $\underline{\Phi(t)}$: Baseband Waveform $4\pi A_{eff}/\lambda^2$

Spectrum Shaping

Cognitive Radio

$$R_X(t, S) = E\left(\sum_{k=-\infty}^{+\infty}\alpha_k \Phi(t-kT)\sum_{n=-\infty}^{+\infty}\alpha_n \Phi(S-nT)\right).$$

$$= \sum_{k=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} E(\alpha_k\alpha_n)\,\Phi(t-kT)\,\Phi(S-nT).$$

$k'=k-n$ $k' \to k$

$$= \sum_{k=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} R_\alpha(\underline{k-n})\,\Phi(t-kT)\,\Phi(S-nT)$$

$$= \sum_{k'=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} R_\alpha(k')\,\Phi(t-(k'+n)T)\,\Phi(S-nT)$$

$$= \sum_{k'=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} R_\alpha(k')\,\Phi(\overset{+T}{\underline{t}}-nT-k'T)\,\Phi(\overset{+T}{\underline{S}}-nT)$$

$T'=T$

$$E(\bar{X}(t)\bar{X}(s)) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} R_\alpha(k)\int_{-\infty}^{+\infty}\phi(\theta''+\boxed{t-s}\boxed{-kT})\,\phi(\theta'')\,d\theta''$$

$$\int_{-\infty}^{+\infty} x(t+\tau)\,x(t)\,dt$$

Let $R_\phi(\tau) = \int_{-\infty}^{+\infty}\phi(\theta+\boxed{\tau})\,\phi(\theta)\,d\theta.$

$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty} R_\alpha(k)\,R_\phi(\tau-kT). \qquad \tau = t-s$$

$$\tau' = \tau - kT$$

$$S_Y(\omega) = \int_{-\infty}^{+\infty} R_Y(\tau)\exp(-j\omega\tau)\,d\tau = \frac{1}{T}\sum_{k=-\infty}^{+\infty} R_\alpha(k)\int_{-\infty}^{+\infty} R_\phi(\tau-kT)\exp(-j\omega\tau)\,d\tau$$

$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty} R_\alpha(k)\exp(-j\omega kT)\int_{-\infty}^{+\infty} R_\phi(\tau')\exp(-j\omega\tau')\,d\tau'$$

$Y(t) = X(t+\theta)$. $\quad R_Y(t,s)$. $\quad d\theta' = d\theta$.

$$= \frac{1}{T}\int_0^T \sum_{k=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} R_\alpha(k)\,\phi(\boxed{t+\theta-nT-kT})\,\phi(\boxed{s+\theta-nT})\,d\theta.$$

$$\theta' = \theta - nT.$$

$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty}\boxed{\sum_{n=-\infty}^{+\infty}} R_\alpha(k)\int_0^T \phi(\theta-nT+t-kT)\,\phi(\theta-nT+s)\,d\theta.$$

$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty}\boxed{\sum_{n=-\infty}^{+\infty}} R_\alpha(k)\int_{-nT}^{-(n-1)T}\phi(\theta'+t-kT)\,\phi(\theta'+s)\,d\theta'$$

$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty} R_\alpha(k)\int_{-\infty}^{+\infty}\phi(\theta'+t-kT)\,\phi(\theta'+s)\,d\theta' \qquad \theta'' = \theta'+s.$$

$$R_X(t,s) = E\left(\sum_{k=-\infty}^{+\infty} \alpha_k \phi(t-kT) \boxed{\sum_{n=-\infty}^{+\infty} \boxed{\alpha_n} \boxed{\phi}(s-nT)}\right).$$

$$= \sum_{k=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} E(\alpha_k \alpha_n)\, \phi(t-kT)\, \phi(s-nT).$$

"Convolution"

$$k' = k-n \qquad k' \to k$$

$$= \sum_{k=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} R_\alpha(k-n)\, \phi(t-kT)\, \phi(s-nT)$$

$$= \sum_{k'=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} R_\alpha(k')\, \phi(t-(k'+n)T)\, \phi(s-nT)$$

$$= \sum_{k'=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} R_\alpha(k')\, \phi(\overset{+T}{t}-nT-k'T)\, \phi(\overset{+T}{s}-nT) \qquad T' = T$$

$$S_Y(\omega) = \frac{1}{T}\left(\sum_{k=-\infty}^{+\infty} R_\alpha(k) \exp(-j\omega kT)\right) S_\phi(\omega).$$

$$= \frac{1}{T} \boxed{S_\alpha(\omega) \cdot S_\phi(\omega)}.$$

$$S_\alpha(\omega) = \sum_k R_\alpha(k) \exp(-j\omega kT).$$

$$S_\phi(\omega) = \int_{-\infty}^{+\infty} R_\phi(\tau) \exp(-j\omega\tau)\, d\tau$$

$= E(\xi^2(s)) = E(\xi^2(\min(s,t))) = \boxed{g(\min(s,t))}$

$E((\xi(t_4)-\xi(t_3))(\xi(t_2)-\xi(t_1)))$

$= R_\xi(t_4,t_2)+R_\xi(t_3,t_1)-R_\xi(t_4,t_1)-R_\xi(t_3,t_2)$

$= g(t_2)+g(t_1)-g(t_1)-g(t_2)=0$

$B(t)-B(s)$

Brown Motion: ① $B(0)=0$. ② Orthogonal Increment. ③ $\sim N(0,\sigma^2(t-s))$

Irregular Botany 1820's $\boxed{\text{Bachelier}}$. 1900.

Orthogonal Increment. $\xi(t)$. $t_1 \le t_2 \le t_3 < t_4$.

$\xi(t_4)-\xi(t_3) \perp \xi(t_2)-\xi(t_1)$. $\xi(0)=0$. Poincare Chaos

$E((\xi(t_4)-\xi(t_3))(\xi(t_2)-\xi(t_1)))=0$. $E(\xi(t))=0$.

$R_\xi(t,s)=E(\xi(t)\xi(s)) = E((\xi(t)-\xi(s)+\xi(s))\xi(s))$

$= E(\underbrace{(\xi(t)-\xi(s))}_{s\ t.}\underbrace{(\xi(s)-\xi(0))}_{0.\ s.}) + E(\xi^2(s))$

Einstein $\boxed{1905}$

$\frac{d}{dx}\mathrm{sgn}(x) = 2\delta(x)$. $\mathrm{sgn}(x) = u(x) - u(-x)$.

$u(x) = \begin{cases} 1 & x \geq 0. \\ 0 & x < 0. \end{cases}$ Heaviside. Function.

L. Schwartz 1950's.

$\frac{d}{dx}u(x) = \boxed{\delta(x)}$. $\delta(0) = \infty$. $\delta(x) = 0,\ x \neq 0$. $\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$

Functional → Linear Operator. $(\delta(x))(f) = \langle \delta(x), f \rangle$

$$= 2\int \delta(x) f(x)\,dx = \int \left(\frac{d}{dx}\mathrm{sgn}(x)\right) f(x)\,dx.$$

$$= \mathrm{sgn}(x) f(x) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \mathrm{sgn}(x) \frac{d}{dx} f(x)\,dx.$$

$f(x) \in C_0^\infty(\mathbb{R})$. $f(x) = 0.\ |x| \geq C$.

$$= -\int_{-\infty}^{+\infty} \mathrm{sgn}(x) \frac{d}{dx} f(x)\,dx = \int_{-\infty}^{0} \frac{d}{dx} f(x)\,dx - \int_{0}^{\infty} \frac{d}{dx} f(x)\,dx$$

$$= \underline{2f(0)} \qquad \underline{\int_{-\infty}^{+\infty} \delta(x) f(x)\,dx = f(0)}$$

$Y(t) = \frac{d}{dt} B(t)$. $\int \Rightarrow S \to$ Sum.

$$R_Y(t,s) = E(Y(t)Y(s)) = E\left(\frac{d}{dt}B(t)\frac{d}{ds}B(s)\right).$$

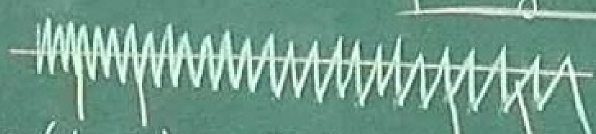
$$= \frac{\partial^2}{\partial t \partial s} E(B(t)B(s)) = \frac{\partial^2}{\partial t \partial s} R_B(t,s)$$

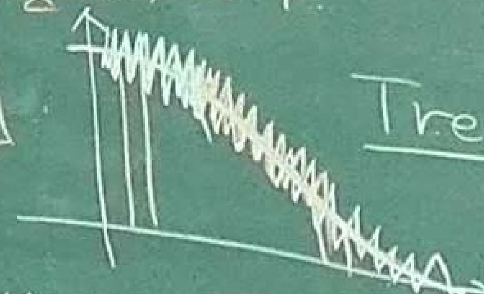
$$= \sigma^2 \frac{\partial^2}{\partial t \partial s} \min(s,t) = \sigma^2 \frac{\partial^2}{\partial t \partial s}\left(\frac{1}{2}(s+t-|s-t|)\right)$$

$$= \frac{\partial}{\partial t}\left(-\frac{1}{2}\,\mathrm{sgn}(s-t)\right) = \boxed{\delta(t-s)} \quad \boxed{\text{White Noise}}$$

$$R_\gamma(\tau) = \delta(\tau) \Longleftrightarrow S_\gamma(\omega) = 1.$$

Filter $\boxed{\text{High-Pass}}$





Trend

t

$$R_\gamma(t,s) = E(\gamma(t)\gamma(s)) = E((\gamma(t)-\gamma(s)+\gamma(s))\gamma(s))$$

$$= E((\underbrace{\gamma(t)-\gamma(s)}_{s,\,t})(\underbrace{\gamma(s)-\gamma(0)}_{0,\,s})) + E(\gamma^2(s)) \quad \text{Einstein } \boxed{1905}$$

t

$\sqrt{-15}$