Correlation. $X, Y \in \mathbb{R}. \ E(XY)$.

$\in \mathbb{C} \ E(X\bar{Y})$.

Correlation Function

$X(t), \ X(s)$. $R_X(t,s) = E(X(t)X(s))$. Positive Definite.

$X(t)$, $X(t+\tau)$, $\tau > 0$.

Wide-Sense Stationary (Invariance)

$$E(X(t)X(s)) = \underset{\substack{\| \\ R_X(t-s)}}{R_X(t,s)} = R_X(t+\tau, s+\tau) = E(X(t+\tau)X(s+\tau))$$

Let $\tau = t-s$ $\quad R_X(\tau)$

$X(t) = \cos(\omega t + \Phi)$. ω: deterministic, $\Phi \sim U(0, 2\pi)$.

$$R_X(t,s) = E(X(t)X(s)) = E(\cos(\omega t+\Phi)\cos(\omega s+\Phi))$$

$$= \frac{1}{2} E(\cos(\omega(t+s)+2\Phi) + \cos(\omega(t-s))) \qquad X \sim f_X(x)$$

$$= \frac{1}{2} E(\cos(\omega(t+s)+2\Phi)) + \frac{1}{2} E(\cos(\omega(t-s))) \qquad E(g(X))$$

$$= \frac{1}{2}\int_0^{2\pi} \cos(\omega(t+s)+2\phi)\frac{1}{2\pi}\,d\phi + \frac{1}{2}\cos(\omega(t-s)) \qquad = \int_{\mathbb{R}} g(x) f_X(x)\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} \cos(\omega(t+s)+2\phi)\,d\phi + \frac{1}{2}\cos(\omega(t-s)).$$

$$= \frac{1}{4\pi}\int_0^{2\pi} \cos(2\phi)\,d\phi + \frac{1}{2}\cos(\omega(t-s))$$

$\gamma(t)$ is w.s.s.

$$= \frac{1}{2}\cos(\omega(t-s)) = \frac{1}{2}\cos(\omega\tau), \quad \tau = t-s.$$

$\gamma(t,\Omega) = \cos(\omega t + \phi(\Omega))$ $\quad$ $\gamma(t) = A\cos(\omega t+\phi)$, $\phi \sim U(0,2\pi)$, A r.v. independent

$$R_\gamma(t,s) = E(A\cos(\omega t+\phi)\,A\cos(\omega s+\phi)) = \underline{E(A^2)}\,\underline{E(\cos(\omega t+\phi)\cos(\omega s+\phi))}$$

$\gamma(t) = \cos(\omega t+\phi)$, $\phi \sim U(0,2\pi)$ $\quad$ ω r.v. $f(\omega)$.

independent

$$R_\gamma(t,s) = E(\cos(\omega t+\phi)\cos(\omega s+\phi)).$$

$$= \frac{1}{2}E(\cos(\omega(t+s)+2\phi)) + \frac{1}{2}E(\cos(\omega(t-s))).$$

$f(\omega,\phi) = f(\omega)f(\phi)$.

$$E_{\omega,\phi}(\cos(\omega(t+s)+2\phi)) = \int_{\mathbb{R}}\int_{\mathbb{R}} \cos(\omega(t+s)+2\phi)\,f(\omega,\phi)\,d\omega\,d\phi$$

Joint Density

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \cos(\omega(t+s)+2\phi)\,f(\phi)f(\omega)\,d\phi\,d\omega$$

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \cos(\omega(t+s)+2\phi) f(\phi)\, d\phi \right) f(\omega)\, d\omega .$$

$$= \int_{\mathbb{R}} \left(\int_0^{2\pi} \cos(\omega(t+s)+2\phi) \frac{d\phi}{2\pi} \right) f(\omega)\, d\omega = 0.$$

$$= \int_{\mathbb{R}} \int_0^{2\pi} \cos(\omega(t+s)+2\phi) \frac{d\phi}{2\pi}$$

Conditional Expectation. $E(g(X,Y)) = E_Y(E_X(g(X,Y)\,|\,Y))$

$$E(g(X,Y)) = \int_{\mathbb{R}}\int_{\mathbb{R}} g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(x,y) f_{X|Y}(x|y)\, dx \right) . f_Y(y)\, dy .$$

$$= \int_{\mathbb{R}} E_X(g(X,y)|y)\, f_Y(y)\, dy . = E_Y(E_X(g(X,Y)\,|\,Y)).$$

$X_1, \cdots, X_n$ i.i.d. $E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n) = nE(X_1)$

$E(X_1 + \cdots + X_N)$ N. r.v. $E(X_1) + \cdots + E(X_N) = NE(X)$

$$= E_N(E_X(X_1 + \cdots + X_N\,|\,N)) = E(E(X_1) + \cdots + E(X_N)).$$

$$= E(NE(X_1)) = E(X_1)E(N)$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \cos(\omega(t+s)+2\phi)\, f(\omega,\phi)\,d\omega\,d\phi$$ Joint

$$= \int_{\phi}\int_{\mathbb{R}} \cos(\omega(t+s)+2\phi)\, f(\phi)\, f(\omega)\,d\phi\,d\omega$$ Density

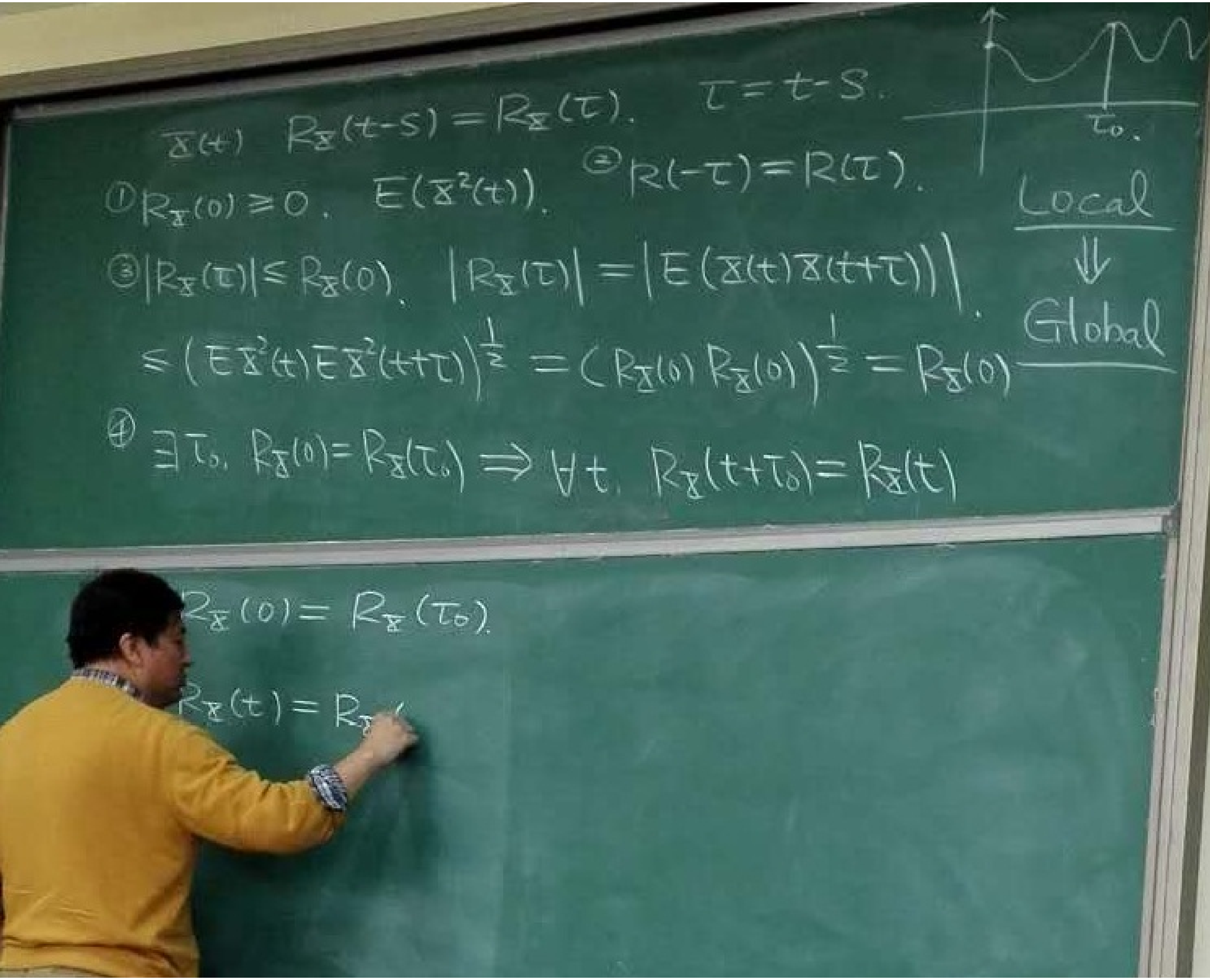
$$E(\cos(\omega(t+s)+2\phi)) = E_{\omega}(E_{\phi}(\cos(\omega(t+s)+2\phi)\,|\,\omega))$$

$$R_X(t,s) = \frac{1}{2}E(\cos(\omega(t-s))) \qquad = E_{\omega}(E_{\phi}(\cos(2\phi)\,|\,\omega)) = 0.$$

$$\tau = t-s. \quad = \frac{1}{2}E(\cos(\omega\tau)) = \frac{1}{2}\int_{\mathbb{R}} \cos(\omega\tau)\, f(\omega)\,d\omega = \frac{1}{2}\mathrm{Re}(\Phi_{\omega}(\tau))$$

Characteristic Function. X, r.v. $\Phi_X(\omega) = E(\exp(j\omega X))$. $j = \sqrt{-1}$.

$$\Phi_X(\omega) = \int_{\mathbb{R}} \exp(j\omega x)\, f_X(x)\,dx \qquad \Phi_X(\omega) \longleftrightarrow f_X(x)$$


$X(t)$ $R_X(t-s)=R_X(\tau)$. $\tau=t-s$.
τ_0.
① $R_X(0)\geq 0$. $E(X^2(t))$.
② $R(-\tau)=R(\tau)$.
③ $|R_X(\tau)|\leq R_X(0)$. $|R_X(\tau)|=|E(X(t)X(t+\tau))|$.
$\leq (EX^2(t)EX^2(t+\tau))^{\frac{1}{2}}=(R_X(0)R_X(0))^{\frac{1}{2}}=R_X(0)$
④ $\exists\tau_0$. $R_X(0)=R_X(\tau_0)\Rightarrow\forall t$. $R_X(t+\tau_0)=R_X(t)$
Local
⇓
Global
$R_X(0)=R_X(\tau_0)$.
$R_X(t)=R_X($

W.S.S. $\begin{cases} \text{Second Order} & R_X(t,s) = R_X(t-s) \\ \text{First Order} & E(X(t)) = m(t) = m \end{cases}$

Covariance

Correlation

$E(XY)$ $\quad Cov(X,Y) = E((X - EX)(Y - EY))$

$= E(XY) - \underline{EXEY}$ Centering

$$|R_X(t) - R_X(t+\tau_0)| = |E(X(0)X(t)) - E(X(0)X(t+\tau_0))|$$

$$= |E(\underline{X(0)}(\underline{X(t) - X(t+\tau_0)}))| \le (\underline{E(X^2(t))}\, E(X(t) - X(t+\tau_0))^2)^{\frac{1}{2}} = 0$$

mean square periodicity

$\underline{R_X(0) = R_X(\tau_0)}$ $\overset{①}{\Longrightarrow}$ $E|X(t) - X(t+\tau_0)|^2 = 0$

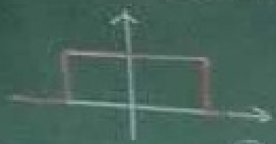
$\Updownarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad \Updownarrow$

$R_X(t) = R_X(t+\tau_0)$ $\overset{②}{\Longleftarrow}$ $\qquad X(t) = X(t+\tau_0)$

①
$$E|X(t) - X(t+\tau_0)|^2 = E(X^2(t)) + E(X^2(t+\tau_0)) - 2E(X(t)X(t+\tau_0))$$

$$= R_X(0) + R_X(0) - 2R_X(\tau_0)$$

$$= 2R_X(0) - 2R_X(\tau_0) = 0$$



Continuity. $R_X(\tau)$ is continuous at 0 $\overset{①}{\Longrightarrow}$ $\lim_{\delta \to 0} E|\boxed{X(t+\delta)} - \boxed{X(t)}| = 0$

$\Updownarrow$

$R_X(\tau)$ is continuous at every^{②} t.

① $E|X(t+\delta)-X(t)|^2 = 2(R_X(\delta)-R_X(0)) \xrightarrow{\delta \to 0} 0.$ Mean square Continuous.

② $|R_X(\tau+\delta)-R_X(\tau)| \le \left(R_X(0)\, \underline{E(X(\tau+\delta)-X(\tau))^2}\right)^{\frac{1}{2}} \xrightarrow{\delta \to 0} 0$

Positive Definite. $\forall n$. $\forall t_1, \cdots, t_n$ $(g(t_i - t_j))_{ij}$ is pd

Bochner-Khinchine.

"Insight" $= \int_{\mathbb{R}} f(\omega) \left(\sum_{i=1}^{n} \exp(j\omega t_i) z_i\right) \overline{\left(\sum_{j=1}^{n} \exp(j\omega t_j) z_j\right)} \, d\omega$.

$$= \int_{\mathbb{R}} f(\omega) \left| \sum_{i=1}^{n} \exp(j\omega t_i) z_i \right|^2 d\omega \geq 0.$$

"$\Rightarrow$" $R_X(\tau)$ is. p.d. $\Rightarrow \exists X(t)$. w.s.s. s.t $R_X(\tau) = E(\overline{X(t)} X(t+\tau))$

$$0 \leq \frac{1}{T} E \left| \int_{-\frac{T}{2}}^{\frac{T}{2}} X(t) \exp(-j\omega t) dt \right|^2 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} E(\overline{X(t)} X(s)) \exp(-j\omega(t-s)) dt ds$$

Bochner–Khinchine. $\underline{R_X(\tau)} \Leftrightarrow \underline{\int_{\mathbb{R}} R_X(\tau) \exp(-j\omega\tau) d\tau \geq 0}$.

"$\Leftarrow$" $\exists f(\omega) \geq 0$. s.t $R_X(\tau) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\omega) \exp(j\omega\tau) d\omega$.

$\forall n$. $\forall t_1, t_2, \cdots, t_n$. $\forall z_1, z_2, \cdots, z_n \in \mathbb{C}$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} R_X(t_i - t_j) z_i \bar{z}_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{R}} f(\omega) \exp(j\omega(t_i - t_j)) d\omega \, z_i \bar{z}_j$$

$$= \int_{\mathbb{R}} f(\omega) \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \exp(j\omega(t_i - t_j)) z_i \bar{z}_j \right) d\omega$$

$$\cdots \exp(-j\omega(t-s))\,\underline{dtds}.$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} R_X(t-s)\exp(-j\omega(t-s))$$

① $u = t-s$. $v = t+s$. ② $dtds \to dudv$.

$$\frac{1}{T}\iint_{\Diamond} R_X(u)\exp(-j\omega u)\frac{1}{2}dudv \qquad \left|\det\left(\frac{\partial(t,s)}{\partial(u,v)}\right)\right| = \frac{1}{2}$$

$$\frac{1}{2}$$

$$= \frac{1}{T}\left(\int_{-T}^{0}\int_{-u-T}^{u+T} + \int_{0}^{T}\int_{u-T}^{-u+T}\right) R_X(u)\exp(-j\omega u)\,dvdu.$$

$$\| \quad \frac{1}{2}$$

$$= \frac{1}{T}\int_{-T}^{T}\int_{|u|-T}^{-|u|+T} R_X(u)\exp(-j\omega u)\,dvdu.$$

$$= \frac{1}{T}\int_{-T}^{T}\underline{\frac{2T-2|u|}{2}}\, R_X(u)\exp(-j\omega u)\,du.$$

$$= \int_{-T}^{T}\left(1-\frac{|u|}{T}\right) R_X(u)\exp(-j\omega u)\,du \geq 0.$$

$$\Downarrow T\to\infty$$

$$\int_{-\infty}^{+\infty} R_X(u)\exp(-j\omega u)\,du \geq 0$$

$$\boxed{\text{Lebesgue Dominated Converge}}$$