Conditional Distribution.

X, Y. $X \in \mathbb{R}^n$, $Y \in \mathbb{R}^m$. $(X, Y) \sim N(\mu, \Sigma)$.

$\mu = (\mu_1, \mu_2)^T$. $\mu_1 = E(X)$, $\mu_2 = E(Y)$. $\mu \in \mathbb{R}^{m+n}$.

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}$$

$\Sigma_{11} \in \mathbb{R}^{n\times n} = E(X-\mu_1)(X-\mu_1)^T$

$\Sigma_{22} \in \mathbb{R}^{m\times m} = E(Y-\mu_2)(Y-\mu_2)^T$

$$P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)}.$$

$$P_{X,Y}(x,y) = C\exp\left(-\frac{1}{2}(x-\mu_1, y-\mu_2)^T \cdot \Sigma^{-1} \begin{pmatrix} x-\mu_1 \\ y-\mu_2 \end{pmatrix}\right).$$

$$P_X(x) = C_1 \exp\left(-\frac{1}{2}(x-\mu_1)^T \Sigma_{11}^{-1} (x-\mu_1)\right).$$

$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}^{-1}$ Matrix Inversion Formula. "Hole"

$AH+B=0 \Rightarrow H=-A^{-1}B$. $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I & H \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & \underline{AH+B} \\ C & CH+D \end{pmatrix}$

$$\begin{pmatrix} I & 0 \\ H' & I \end{pmatrix}\begin{pmatrix} A & 0 \\ C & D-CA^{-1}B \end{pmatrix} = \begin{pmatrix} A & 0 \\ H'A+C & D-CA^{-1}B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{pmatrix}$$

$$H'A+C=0 \Rightarrow H' = -CA^{-1}$$

$$\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{pmatrix}$$

$$(H_1 H_2 H_3)^{-1} = H_3^{-1} H_2^{-1} H_1^{-1}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & (D-CA^{-1}B)^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix}$$

$$= \begin{pmatrix} A^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ 0 & (D-CA^{-1}B)^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix}$$

$$= \begin{pmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$$

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}^{-1} = \begin{pmatrix} I & -\Sigma_{11}^{-1}\Sigma_{12} \\ 0 & I \end{pmatrix}\begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & (\Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I \end{pmatrix}$$

$$(x-\mu_1, y-\mu_2)^T \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}^{-1} \begin{pmatrix} x-\mu_1 \\ y-\mu_2 \end{pmatrix}$$

$$= ((x-\mu_1)^T, (y-\mu_2)^T) \begin{pmatrix} I & -\Sigma_{11}^{-1}\Sigma_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & (\Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}) \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ \Sigma_{21}\Sigma_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} x-\mu_1 \\ y-\mu_2 \end{pmatrix}$$

$$= (x-\mu_1)^T \Sigma_{11}^{-1} (x-\mu_1) + \underline{(y-\mu_2-\Sigma_{21}\Sigma_{11}^{-1}(x-\mu_1))^T (\Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})^{-1}}$$

$$(y-\mu_{Y|X})^T \Sigma_{Y|X}^{-1} (y-\mu_{Y|X}) = \underline{(y-\mu_2-\Sigma_{21}\Sigma_{11}^{-1}(x-\mu_1))}$$

$$\mu_{Y|X} = \mu_2 + \boxed{\frac{\sigma_{21}}{\sigma_{11}}}(x-\mu_1).$$

$$X \sim N(\mu, \sigma_{11}^2).$$

$$\frac{X-\mu}{\sigma_{11}} \sim N(0,1).$$

$E(X)=E(Y)=0$ $\quad \alpha X \to Y$. Linear Representation

$$\min_\alpha E(Y-\alpha X)^2 \Rightarrow \alpha = \underline{E(XY)/E(X^2)}.$$ Projection!

$$\mu_{Y|X} = E(Y|X) = \arg\min_g E(Y-g(X))^2$$

$\forall g.$

$$\underbrace{E(Y-g(X))^2}_{\vee\!/} = E(\underline{Y-E(Y|X)} + \underline{E(Y|X)-g(X)})^2$$

$$= \underline{E(Y-E(Y|X))^2} + \underline{E(E(Y|X)-g(X))^2}$$

$$+ E((Y-E(Y|X))(E(Y|X)-g(X)))$$

$$G(X,Y) = (Y-E(Y|X))\underline{(E(Y|X)-g(X))}.$$

$$E(G(X,Y)).$$

$$= E_X(E_Y(G(X,Y)|X))$$

$$E(h(X)Y|X).$$

$$= h(X)E(Y|X)$$

$$\mu_{Y|X} = \boxed{\mu_2} + \boxed{\Sigma_{21}\Sigma_{11}^{-1}(X-\mu_1)}.$$

$$\Sigma_{Y|X} = \underline{\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}}$$

$$\mu_{Y|X} = \mu_2 + \boxed{\frac{\sigma_{21}}{\sigma_{11}}}(X-\mu_1).$$

$$E(X) = E(Y) = 0 \qquad \alpha X \to Y. \text{ Linear Representation}$$

$$\min_\alpha E(Y-\alpha X)^2 \Rightarrow \alpha_{opt} = \underline{E(XY)/E(X^2)}. \text{ Projection!}$$

Prior

$$Y|X \sim N(\mu_{Y|X}, \Sigma_{Y|X}).$$

$$X \sim N(\mu, \sigma_{11}^2).$$

$$\frac{X-\mu}{\sigma_{11}} \sim N(0,1).$$

$$\begin{pmatrix} I & O \\ H' & I \end{pmatrix}\begin{pmatrix} A & O \\ C & D-CA^{-1}B \end{pmatrix} = \begin{pmatrix} A & O \\ H'A+C & D-CA^{-1}B \end{pmatrix} = \begin{pmatrix} A & O \\ O & D-CA^{-1}B \end{pmatrix}$$

$$H'A+C=0 \Rightarrow H' = -CA^{-1}$$

$$\begin{pmatrix} I & O \\ -CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{pmatrix}$$

$$(H_1 H_2 H_3)^{-1} = H_3^{-1} H_2^{-1} H_1^{-1}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & (D-CA^{-1}B)^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix}$$

$$E((Y-E(Y|X))(E(Y|X)-g(X)))$$

$$= E_X\Big(\underbrace{E((Y-E(Y|X))(E(Y|X)-g(X))\,|\,X)}_{=}\Big)$$

$$(E(Y|X)-g(X))\underbrace{E(Y-E(Y|X)\,|\,X)}_{=}$$

$$E(Y|X) - E(Y|X)E(1|X) = 0$$

$X \sim f_X(x) \quad E(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx. \quad Y = G(X).$

"Magic"

$$E(Y) = E(G(X)) = \int_{-\infty}^{+\infty} y f_Y(y)\,dy = \int_{-\infty}^{+\infty} G(x) f_X(x)\,dx.$$

$$F_Y(y) = P(Y \le y) = P(G(X) \le y) = P(X \lesseqqgtr G^{-1}(y))\,?$$

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_{-\infty}^{0} F_X(x)\,dx + \int_{0}^{\infty} (1 - F_X(x))\,dx$$

$$= \int_\Omega X(\omega)\, \underline{\underline{P(d\omega)}} = \sum_k X(\omega_k) P(\{\omega_k\}).$$

$$\mu_{Y|X} = \boxed{\mu_2} + \boxed{\Sigma_{21}\Sigma_{11}^{-1}(x - \mu_1)}.$$

$$\Sigma_{Y|X} = \underline{\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}} \ge 0.$$

$$\mu_{Y|X} = \mu_2 + \boxed{\frac{\sigma_{21}}{\sigma_{11}}}(x - \mu_1).$$

$$E(G(X)) = \boxed{\int_\Omega G(X(\omega))\, P(d\omega)} = \int_\Omega Y(\omega)\, P(d\omega).$$

$$\frac{X - \mu}{\sigma_{11}} \sim N(0,1).$$

$E(X) = E(Y) = 0 \quad \alpha X \to Y.$ Linear Representation

$\min_\alpha E(Y - \alpha X)^2 \Rightarrow \alpha_{opt} = \underline{E(XY)/E(X^2)}.$ Projection!

$$\Sigma_{Y|X} = \sigma_{22} - \frac{\sigma_{12}\sigma_{21}}{\sigma_{11}} = \sigma_{22} - \frac{(\sigma_{12})^2}{\sigma_{11}} \geq 0. \qquad (E(XY))^2 \leq EX^2 EY^2$$

$$\sigma_{12}^2 \leq \sigma_{11}\cdot\sigma_{22} \quad \left(E((X-\mu_1)(Y-\mu_2))\right)^2 \leq E(X-\mu_1)^2 E(Y-\mu_2)^2$$

$$E(Y) = E(E(Y|X)). \quad Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$$

$$Var(Y|X) = E((Y - E(Y|X))^2 | X)$$

$$Var(Y) = E(Y-EY)^2 = E(Y - E(Y|X))^2 + E(E(Y|X) - EY)^2$$

$$= E(\overset{Var(Y|X)}{\underline{E(Y - E(Y|X))^2 | X}}) + E(E(Y|X) - E(E(Y|X)))^2.$$

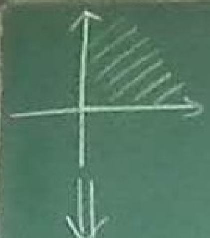
Nonlinear. Operation. Case by Case.

$(X, Y) \sim N(0, 0, \sigma_1^2, \sigma_2^2, \rho)$. $(X, Y) \to \boxed{G} \to (X', Y')$.

Hard Limiter. $G(x) = sgn(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0. \end{cases}$

$X' \sim \begin{pmatrix} -1 & 1 \\ 1-p & p \end{pmatrix}$ $\mu_1 = 0 \Rightarrow p = \frac{1}{2}$ $Y' \sim \begin{pmatrix} -1 & 1 \\ 1-p' & p' \end{pmatrix}$ $\mu_1 = \mu_2 = 0$

$E(X') = E(Y') = 0 \quad E(X'Y') = 1 \cdot P(XY \geq 0) + (-1) P(XY < 0)$

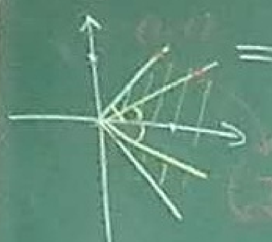


$$x' = u + v, \quad y' = u - v.$$

$$dx'dy' = \left|\det \frac{\partial(x',y')}{\partial(u,v)}\right| dudv, \quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$x'^2 + y'^2 - 2\rho x'y' = 2u^2 + 2v^2 - 2\rho(u^2 - v^2)$$

$$= 2((1-\rho)u^2 + (1+\rho)v^2).$$



$$= \frac{2}{\pi\sqrt{1-\rho^2}} \iint_{\Delta} \exp\left(-\frac{1}{(1-\rho^2)} \cdot \left((1-\rho)u^2 + (1+\rho)v^2\right)\right) dudv$$

$$u' = \frac{u}{\sqrt{1+\rho}}, \quad v' = \frac{v}{\sqrt{1-\rho}}$$

$$= \frac{2}{\pi\sqrt{1-\rho^2}} \iint_{\Delta'} \exp\left(-(u'^2 + v'^2)\right) \sqrt{1-\rho^2}\, du'dv'$$

$$E(X'Y') = P(XY \geq 0) - P(XY < 0) = 2P(XY \geq 0) - 1.$$

$$P(XY \geq 0) = 2\int_0^\infty \int_0^\infty f_{X,Y}(x,y)\, dxdy. \quad \mu_1 = \mu_2 = 0.$$

$$= \frac{2}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_0^\infty \int_0^\infty \exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right)\right) dxdy$$

$$x' = \frac{x-\mu_1}{\sigma_1}, \quad y' = \frac{y-\mu_2}{\sigma_2}$$

$$= \frac{2}{2\pi\sqrt{1-\rho^2}} \int_0^\infty \int_0^\infty \exp\left(-\frac{1}{2(1-\rho^2)}\left(x'^2 + y'^2 - 2\rho x'y'\right)\right) dx'dy'$$

$$\pi \iint_{\Delta'} \exp(-(u'^2+v'^2))\,du'dv', \quad u'=r\cos\theta,\ v'=r\sin\theta.$$

$$= \frac{2}{\pi}\iint \exp(-r^2)\,r\,dr\,d\theta. \qquad \phi = \arctan\sqrt{\frac{1+\rho}{1-\rho}}.$$

$$= \frac{2}{\pi}\left(\int_0^\infty \exp(-r^2)\,r\,dr\right)\cdot 2\phi = \frac{2}{\pi}\phi = \frac{2}{\pi}\arctan\sqrt{\frac{1+\rho}{1-\rho}}$$

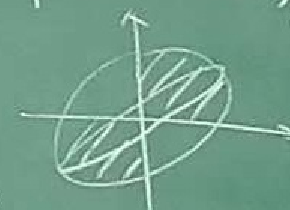
$\tan(\theta)=t$

$$= \frac{1}{\pi}\arccos\left(\frac{1-\frac{1+\rho}{1-\rho}}{1+\frac{1+\rho}{1-\rho}}\right) = \frac{1}{\pi}\arccos(-\rho)$$

$\Rightarrow \cos(2\theta) = \frac{1-t^2}{1+t^2}$

$\arcsin(-\rho) + \arccos(-\rho) = \frac{\pi}{2}$

$P(XY \geq 0)$.



$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \arcsin(-\rho)\right) = \frac{1}{\pi}\left(\frac{\pi}{2} + \arcsin(\rho)\right).$$

$$= \frac{1}{2} + \underline{\frac{1}{\pi}\arcsin(\rho)}. \quad \text{(Arcsin Law)}$$

$$E(X'Y') = 2P(XY \geq 0) - 1 = \frac{2}{\pi}\arcsin(\rho)$$