$g(x,y)$. Nonlinear Function. $E(g(X,Y))$.

$$\frac{\partial E(g(X,Y))}{\partial \rho} = (\sigma_1\sigma_2)\, E\left(\frac{\partial^2 g(X,Y)}{\partial X\,\partial Y}\right).$$

$$E(g(X,Y)) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x}{\sigma_1}\right)^2+\left(\frac{y}{\sigma_2}\right)^2-2\rho\left(\frac{x}{\sigma_1}\right)\left(\frac{y}{\sigma_2}\right)\right)\right)dx\,dy$$

$E(\operatorname{sgn}(X)\operatorname{sgn}(Y))$ ① $g(x,y)=\operatorname{sgn}(x)\operatorname{sgn}(y)$

② $$\frac{\partial^2 g(x,y)}{\partial x\,\partial y} = 4\,\delta(x)\,\delta(y).$$

Nonlinear Operation.

$\operatorname{sgn}(x) = u(x) - u(-x)$.

$\frac{d}{dx}u(x) = \delta(x)$.

$g(x)=\operatorname{sgn}(x)$. $X(t) \longrightarrow \boxed{g} \longrightarrow Y(t)$. $R_X(\tau)$. W.S.S.

$$R_Y(t,s) = \frac{2}{\pi}\arcsin\left(\frac{R_X(t-s)}{R_X(0)}\right)$$

$g(x)=|x|$. $(X,Y)\sim N(0,0,\sigma^2,\sigma^2,\rho) \Rightarrow E|XY|$.

Price Theorem. $(X,Y)\sim N(0,0,\sigma_1^2,\sigma_2^2,\rho)$.

$$= \frac{2}{\pi}(\sigma_1\sigma_2)\arcsin(\rho). \qquad \underline{\sigma_1=\sigma_2=1.}$$

$$E|XY| = \frac{2}{\pi}\int \arcsin(\rho)\,d\rho + C = \frac{2}{\pi}\left(\rho\arcsin\rho+\sqrt{1-\rho^2}\right) + \overset{\overset{0}{\shortparallel}}{C}.$$

$$\int \arcsin(\rho)\,d\rho = \rho\arcsin(\rho) - \int \frac{\rho}{\sqrt{1-\rho^2}}\,d\rho = \rho\arcsin\rho+\sqrt{1-\rho^2}$$

$$\rho=0 \Rightarrow E|XY| = E|X|\cdot E|Y| = (E|X|)^2 = \frac{2}{\pi}\sigma^2$$

$$E|XY| = \frac{2}{\pi}\left(\rho\arcsin\rho+\sqrt{1-\rho^2}\right)$$

③. $$(\sigma_1\sigma_2)\, E\left(\frac{\partial^2 g(X,Y)}{\partial X\,\partial Y}\right) = \frac{4}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(x)\delta(y)\, f_{X,Y}(x,y)\,dx\,dy$$

$$= \frac{2}{\pi}\cdot\frac{1}{\sqrt{1-\rho^2}} = \frac{\partial E(g(X,Y))}{\partial\rho}. \Rightarrow E(g(X,Y)) = \frac{2}{\pi}\arcsin(\rho) + C$$

$\rho=0 \Rightarrow X, Y$ independent $\Rightarrow E(\operatorname{sgn}(X)\operatorname{sgn}(Y)) = E(\operatorname{sgn}(X))\overset{\overset{0}{\shortparallel}}{E}(\operatorname{sgn}(Y))$

$E|XY|$ ① $g(x,y)=|xy|$ ② $\dfrac{\partial g(x,y)}{\partial x\,\partial y} = \operatorname{sgn}(x)\operatorname{sgn}(y)$

③. $\dfrac{\partial E|XY|}{\partial\rho} = (\sigma_1\sigma_2)\,E(\operatorname{sgn}(X)\operatorname{sgn}(Y)).$

$$E(g(X,Y)) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

$$f_{X,Y}(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \phi_{X,Y}(\omega_1,\omega_2) \exp(-j(\omega_1 X + \omega_2 Y))\, d\omega_1 d\omega_2$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underline{\phi_{X,Y}(\omega_1,\omega_2)} \exp(-j(\omega_1 x + \omega_2 y))\, d\omega_1 d\omega_2\right) dx\, dy$$

$$\phi_{X,Y}(\omega_1,\omega_2) = \exp\left(-\frac{1}{2}(\omega_1^2 \cdot \sigma_1^2 + \omega_2^2\sigma_2^2 + 2\rho\omega_1\omega_2\sigma_1\sigma_2)\right)$$

$$\frac{\partial\phi_{X,Y}(\omega_1,\omega_2)}{\partial\rho} = -(\sigma_1\sigma_2)(\omega_1\omega_2)\,\phi_{X,Y}(\omega_1,\omega_2)$$

$$E|X| = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} |x| \exp\left(-\frac{x^2}{2\sigma^2}\right) dx.$$

$$= \frac{2}{\sqrt{2\pi}\sigma}\int_0^{\infty} x \exp\left(-\frac{x^2}{2\sigma^2}\right) dx.$$

$$= \frac{2}{\sqrt{2\pi}\sigma}\left(-\sigma^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)\Big|_0^{\infty} = \sqrt{\frac{2}{\pi}}\,\sigma$$

$$\frac{1}{(2\pi)^{\frac{n}{2}}(\det\Sigma)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(X-\mu)^T \Sigma^{-1}(X-\mu)\right) \Rightarrow \exp\left(j\mu^T\omega - \frac{1}{2}\omega^T\Sigma\omega\right)$$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \qquad (\omega_1, \omega_2)\,\Sigma \begin{pmatrix}\omega_1 \\ \omega_2\end{pmatrix}$$

$$
\begin{aligned}
\frac{\partial g(x,y)}{\partial \rho} &= \int\int g(x,y) \int\int \sigma_1\sigma_2(-\omega_1\omega_2)\phi_{x,y}(\omega_1,\omega_2)exp(-j(\omega_1 x+\omega_2 y))d\omega_1 d\omega_2 dxdy \\
&= \sigma_1\sigma_2 \int\int g(x,y)\frac{\partial^2}{\partial x\partial y}\int\int \phi_{x,y}(\omega_1,\omega_2)exp(-j(\omega_1 x+\omega_2 y))d\omega_1 d\omega_2 dxdy \\
&= \sigma_1\sigma_2 \int\int g(x,y)\frac{\partial^2}{\partial x\partial y} f_{x,y}(x,y)dxdy \\
&= \sigma_1\sigma_2 \int\int \frac{\partial^2 g(x,y)}{\partial x\partial y} f_{x,y}(x,y)dxdy
\end{aligned}
$$

② $E(X h(Y)) = E_Y(E_X(X h(Y)|Y)) = E(h(Y) E(X|Y))$

$$= \frac{E(XY)}{EY^2} \cdot E(Y h(Y)) = C \cdot E(XY). \qquad E(X|Y) = \frac{E(XY)}{EY^2} Y.$$

$$E(h'(Y)) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} h'(y) \exp\left(-\frac{y^2}{2\sigma^2}\right) dy = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) dh(y)$$

$$= h(y) \exp\left(-\frac{y^2}{2}\right)\Big|_{-\infty}^{+\infty} + \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \frac{1}{\sigma^2} y h(y) \exp\left(-\frac{y^2}{2\sigma^2}\right) dy$$

$$= \frac{1}{\sigma^2} E(Y h(Y)).$$

Bussgang Property. $(X, Y) \sim N(0, 0, \sigma_1^2, \sigma_2^2, \rho)$.

$h(x)$. $E(X h(Y)) = C E(XY)$.

① Price. (1) $g(x, y) = x h(y)$. (2) $\frac{\partial^2 g(x,y)}{\partial x \partial y} = h'(y)$.

(3) $\frac{\partial E(x h(y))}{\partial \rho} = (\sigma_1 \sigma_2) \cdot E(h'(Y)) = C \Rightarrow E(X h(Y)) = C \cdot \rho$

$$\rho = \frac{E(XY)}{\sigma_1 \sigma_2} \Rightarrow E(X h(Y)) = E(h'(Y)) E(XY) = C E(XY)$$

$$\Phi_{X,Y}(\omega_1,\omega_2) = \exp\left(-\frac{1}{2}(\omega_1^2+\omega_2^2+2\rho\omega_1\omega_2)\right)$$

$$\Phi_{X,Y}(1,1) = \exp\left(-\frac{1}{2}(1+1+2\rho)\right) = \exp(-(1+\rho))$$

$$\Phi_{X,Y}(-1,-1) = \exp(-(1+\rho))$$

$$\Phi_{X,Y}(1,-1) = \Phi_{X,Y}(-1,1) = \exp(-(1-\rho))$$

$$E(\cos(X)\cos(Y)) = \frac{1}{2}\left(\exp(-(1+\rho)) + \exp(-(1-\rho))\right)$$

$X(t) \longrightarrow \boxed{\cos} \longrightarrow Y(t)$. $\quad (X,Y) \sim N(0,0,1,1,\rho)$.

$E(\cos(X)\cos(Y))$. $\quad g(x,y) = \cos(x)\cos(y)$.

$\frac{\partial^2 g(x,y)}{\partial x \partial y} = \sin(x)\sin(y) \longrightarrow \cos(x)\cos(y)$. $\quad \frac{\partial^2 h}{\partial \rho^2} = h$. $\left(\begin{array}{l} h(0) \\ h'(0) \end{array}\right.$

$$\frac{1}{2}E(\cos(X+Y)+\cos(X-Y)) = \frac{1}{2}\left(\frac{1}{2}E(\exp(j(X+Y))+\exp(-j(X+Y)))\right.$$

$$\left. + \frac{1}{2}E(\exp(j(X-Y))+\exp(-j(X-Y)))\right.$$

$$E(\exp(j(X+Y))) = \Phi_{X,Y}(1,1).$$

$$f_r(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) I_{(0,\infty)}(r).$$ Rayleigh Distribution.

$$f_\Phi(\Phi) = \frac{1}{2\pi} I_{[0,2\pi]}(\Phi).$$ Uniform $U(0,2\pi)$.

$$f_{r,\Phi}(r,\Phi) = f_r(r) f_\Phi(\Phi)$$ independent

$X \sim (\mu_x, \sigma^2)$. $Y \sim N(\mu_Y, \sigma^2)$ $(X^2+Y^2)^{\frac{1}{2}} = r$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2}\left((x-\mu_x)^2+(y-\mu_Y)^2\right)\right)$$

①. X, Y. independent $X \sim N(0,\sigma^2)$. $Y \sim N(0,\sigma^2)$.

(X, Y)

$\Downarrow$

(r, Φ).

$$Z = \underset{\sim}{X} \cos(2\pi f_c t) + Y \sin(2\pi f_c t).$$ Phase

$$r = |Z| = (X^2+Y^2)^{\frac{1}{2}}, \quad \Phi = \arctan\left(\frac{Y}{X}\right).$$

$$\sqrt{X^2+Y^2} \cos(2\pi f_c t + \Phi)$$

envelope

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right). \underline{I_{[-\infty,\infty]}(x) I_{[-\infty,\infty]}(y)}$$

$\Downarrow$

$$f_{r,\Phi}(r,\Phi) = \underline{\frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right)} \underline{I_{(0,\infty)}(r)} \underline{I_{[0,2\pi]}(\Phi)}$$

$$= \sqrt{\mu_x^2+\mu_Y^2}\ \sin(\theta+\phi) = \sqrt{\mu_x^2+\mu_Y^2}\ \cos(\theta+\phi+\tfrac{\pi}{2})$$

Let $R = \sqrt{\mu_x^2+\mu_Y^2}$ $\qquad I_0(x) = \int_0^{2\pi} \exp(x\cos\phi)\,d\phi$ Zero-Order Modified Bessel Function of first Kind

$$f_r(r) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{R^2+r^2}{2\sigma^2}\right)\cdot \boxed{\int_0^{2\pi}} \exp\left(\frac{r}{\sigma^2}\cos(\phi+\theta+\tfrac{\pi}{2})\right)d\phi$$

$$= \frac{1}{2\pi\sigma^2}\exp\left(-\frac{R^2+r^2}{2\sigma^2}\right)\int_0^{2\pi}\exp\left(\frac{r}{\sigma^2}\cos\phi\right)d\phi$$

Rician Distribution $\quad = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{R^2+r^2}{2\sigma^2}\right) I_0\left(\frac{r}{\sigma^2}\right) I_{(0,\infty)}(r)$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{\mu_x^2+\mu_Y^2}{2\sigma^2}\right)\exp\left(-\frac{1}{2\sigma^2}(x^2+y^2-2\mu_x x-2\mu_Y y)\right)$$

$$(x,y)\to(r,\phi).$$

$$I_{(0,\infty)}(r)\ I_{[0,2\pi]}(\phi)$$

$$f_{r,\phi}(r,\phi) = \frac{r}{2\pi\sigma^2}\exp\left(-\frac{\mu_x^2+\mu_Y^2}{2\sigma^2}\right)\exp\left(-\frac{1}{2\sigma^2}(r^2-2r(\mu_x\cos\phi+\mu_Y\sin\phi))\right)$$

$$f_r(r) = \int_{-\infty}^{+\infty} f_{r,\phi}(r,\phi)\,d\phi = \frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right)\exp\left(-\frac{\mu_x^2+\mu_Y^2}{2\sigma^2}\right).$$

$\mu_x\cos\phi+\mu_Y\sin\phi.\ \tan\theta = \frac{\mu_Y}{\mu_x}$ $\qquad \int_0^{2\pi}\exp\left(\frac{r}{\sigma^2}(\mu_x\cos\phi+\mu_Y\sin\phi)\right)d\phi$

$$= \sqrt{\mu_x^2+\mu_Y^2}\left(\sin\theta\cos\phi+\cos\theta\sin\phi\right)$$