CS1200: Intro. to Algorithms and their Limitations Anshu & Vadhan

Lecture 19: NP and NP-completeness

Harvard SEAS - Fall 2024 *Nov. 7, 2024*

1 Announcements

- Salil OH 11-12pm; Anurag zoom OH Fri 1:30-2:30 pm
- Next SRE moved to *Thursday* 11/14.

Recommended Reading:

- MacCormick §12.0–12.3, Ch. 13

2 Polynomial-Time Reductions

Definition 2.1. For computational problems Π and Γ, we write $\Pi \leq_p \Gamma$ if there is a *polynomial-time* reduction R from Π to Γ. That is, there is a constant $c \geq 0$ such that R runs in time at most $O(N^c)$ on inputs of length N, counting oracle calls as one time step. Equivalently, there is a constant d such that $\Pi \leq_{O(N^d), O(N^d) \times O(N^d)} \Gamma$.

Some examples of polynomial-time reduction that we've seen include:

- 3-Coloring $\leq_p$ SAT (Lecture 15)
- LongPath $\leq_p$ SAT (SRE 5)
- IntervalScheduling-Decision $\leq_p$ Sorting (Lecture 4). In this case a simpler polynomial time reduction is to

Using polynomial-time reductions to compare problems fits nicely with the study of the classes $\mathsf{P}_{\mathsf{search}}$ and P, since they are "closed" under such reductions:

Lemma 2.2. *Let Π and Γ be computational problems such that $\Pi \leq_p \Gamma$. Then:*

1.

2.

Proof. 1.

2. Contrapositive of Item 1

□

This lemma means that we can use polynomial-time reductions both positively—to show that problems are in $\mathsf{P}_{\mathsf{search}}$— and negatively—to give evidence that problems are not in $\mathsf{P}_{\mathsf{search}}$. For example, under the *assumption* that 3-Coloring is not in $\mathsf{P}_{\mathsf{search}}$, it follows that SAT is not in $\mathsf{P}_{\mathsf{search}}$, by the above lemma and the fact that 3-Coloring $\leq_p$ SAT (SRE5). As always, *the direction of the reduction is crucial!*

Another very useful feature of polynomial-time reductions is that they compose with each other:

Lemma 2.3. *If* $\Pi \leq_p \Gamma$ *and* $\Gamma \leq_p \Theta$ *then* $\Pi \leq_p \Theta$.

This follows from Problem 2 in Problem Set 2, and then using the definition of polynomial time reduction.

3 NP

Figure 1: Can you find a cat?

Roughly speaking, NP consists of the computational problems where valid outputs can be *verified* in polynomial time. This is a very natural requirement; what's the point in searching for something if we can't recognize when we've found it?

Definition 3.1. A computational problem $\Pi = (\mathcal{I}, \mathcal{O}, f)$ is in $\mathsf{NP}_{\mathsf{search}}$ if the following conditions hold:

1. All valid outputs are of polynomial length:

2. All valid outputs are verifiable in polynomial time:

(Remark on terminology: $\mathsf{NP}_{\mathsf{search}}$ is often called FNP in the literature, and is closely related to, but slightly more restricted than, the class $\mathsf{PolyCheck}$ defined in the MacCormick text.)

Examples:

1. Satisfiability:

2. GraphColoring:

3. IndependentSet-ThresholdSearch:

Potential non-example:

1. IndependentSet-OptimizationSearch:

Even though this problem does not appear to be in $\mathsf{NP}_{\mathsf{search}}$ (its still an open question in the theory of computing!), it reduces in polynomial time to IndependentSet-ThresholdSearch, which is in $\mathsf{NP}_{\mathsf{search}}$ (to be discussed next week in the course).

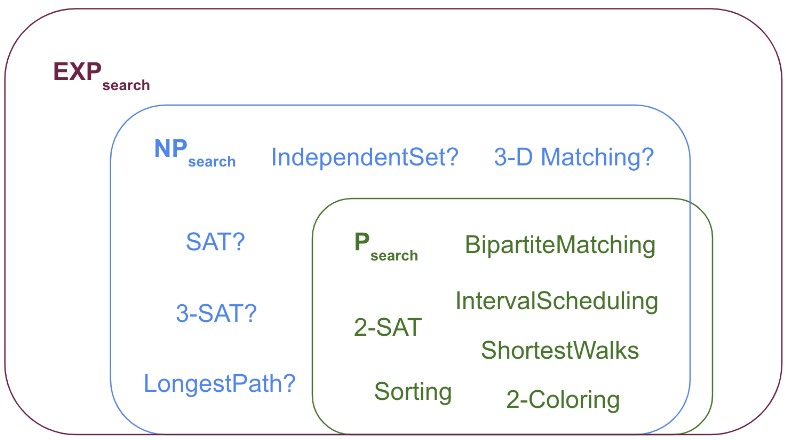
The following proposition shows that every problem in $\mathsf{NP}_{\mathsf{search}}$ can be solved in exponential time.

Proposition 3.2. $\mathsf{NP}_{\mathsf{search}} \subseteq \mathsf{EXP}_{\mathsf{search}}$.

Proof.

□

So now our diagram of complexity classes looks like this:



Remarks:

- **$\mathsf{P}_{\mathsf{search}}$ vs $\mathsf{NP}_{\mathsf{search}}$:** Somewhat counterintuitively, $\mathsf{P}_{\mathsf{search}} \not\subseteq \mathsf{NP}_{\mathsf{search}}$. This due to artificial examples that you may see later in the course, but most of the natural problems in $\mathsf{P}_{\mathsf{search}}$ are also in $\mathsf{NP}_{\mathsf{search}}$ (like all of the green problems in the above diagram).

- **Class NP:** Every problem in $\mathsf{NP}_{\mathsf{search}}$ has a corresponding decision problem (deciding whether or not there is a solution). The class of such decision problems is called NP. We will discuss the class NP more next week.

We still have question marks next to all of the blue problems; we don't know whether they (and thousands of other important problems in $\mathsf{NP}_{\mathsf{search}}$) are in $\mathsf{P}_{\mathsf{search}}$ or not. We will now try to get a handle on these questions.

4 $\mathsf{NP}_{\mathsf{search}}$-Completeness

Unfortunately, although it is widely conjectured, we do not know how to prove that $\mathsf{NP}_{\mathsf{search}} \not\subseteq \mathsf{P}_{\mathsf{search}}$. As we will see next week, this is an equivalent formulation of the famous P vs. NP problem, considered one of the most important open problems in computer science and mathematics. However, even without resolving the P vs. NP conjecture, we can give strong evidence that problems are not solvable in polynomial time by showing that they are $\mathsf{NP}_{\mathsf{search}}$-*complete*:

Definition 4.1 (NP-completeness, search version)**.** A problem Π is $\mathsf{NP}_{\mathsf{search}}$-*complete* if:

1.

2.

We can think of the NP-complete problems as the "hardest" problems in NP. Indeed:

Proposition 4.2. *Suppose* Π *is* $\mathsf{NP}_{\mathsf{search}}$*-complete. Then* $\Pi \in \mathsf{P}_{\mathsf{search}}$ *iff* $\mathsf{NP}_{\mathsf{search}} \subseteq \mathsf{P}_{\mathsf{search}}$.

In other words, if any $\mathsf{NP}_{\mathsf{search}}$−complete problem is in $\mathsf{P}_{\mathsf{search}}$, then all problems in $\mathsf{NP}_{\mathsf{search}}$ are in $\mathsf{P}_{\mathsf{search}}$. Remarkably, there are natural NP-complete problems. The first one is CNF-Satisfiability:

Theorem 4.3 (Cook–Levin Theorem)**.** *SAT is* $\mathsf{NP}_{\mathsf{search}}$*-complete.*

This can be interpreted as strong evidence that SAT is not solvable in polynomial time. If it were, then *every* problem in $\mathsf{NP}_{\mathsf{search}}$ would be solvable in polynomial time. We will return to a proof of the Cook–Levin Theorem later in the course.

5 More $\mathsf{NP}_{\mathsf{search}}$-complete Problems

Once we have one $\mathsf{NP}_{\mathsf{search}}$-complete problem, we can get others via reductions from it. Consider the computational problem 3-SAT, which is obtained when we restrict the number of literals in each clause of SAT.

Input	**:** A CNF formula φ on n variables $z_0, \ldots z_{n-1}$ in which each clause has width at most 3 (i.e. contains at most 3 literals)
Output	**:** An $\alpha \in \{0,1\}^n$ such that $\varphi(\alpha) = 1$, or $\perp$ if no satisfying assignment exists

Computational Problem 3-SAT

Theorem 5.1. *3-SAT is* $\mathsf{NP}_{\mathsf{search}}$*-complete.*

Proof. The proof follows in two steps.

1. 3SAT is in $\mathsf{NP}_{\mathsf{search}}$:

2. 3SAT is $\mathsf{NP}_{\mathsf{search}}$-hard: Since every problem in $\mathsf{NP}_{\mathsf{search}}$ reduces to SAT (Theorem 4.3), all we need to show is SAT $\leq_p$ 3SAT (since reductions compose - Lemma 2.3).

The reduction algorithm from SAT to 3SAT has the following components (Figure 2). First, we give an algorithm R which takes a SAT instance φ to a 3SAT instance φ'.

$$\text{SAT instance } \varphi \xrightarrow{\text{polytime R}} \text{3SAT instance } \varphi'$$

Then we feed the instance φ' to our 3SAT oracle and obtain a satisfying assignment β to φ' or $\perp$ if none exists. If we get $\perp$ from the oracle, we return $\perp$, else we transform β into a satisfying assignment to φ using another algorithm S.

$$\text{SAT assignment } \alpha \xleftarrow{\text{polytime S}} \text{3SAT assignment } \beta$$

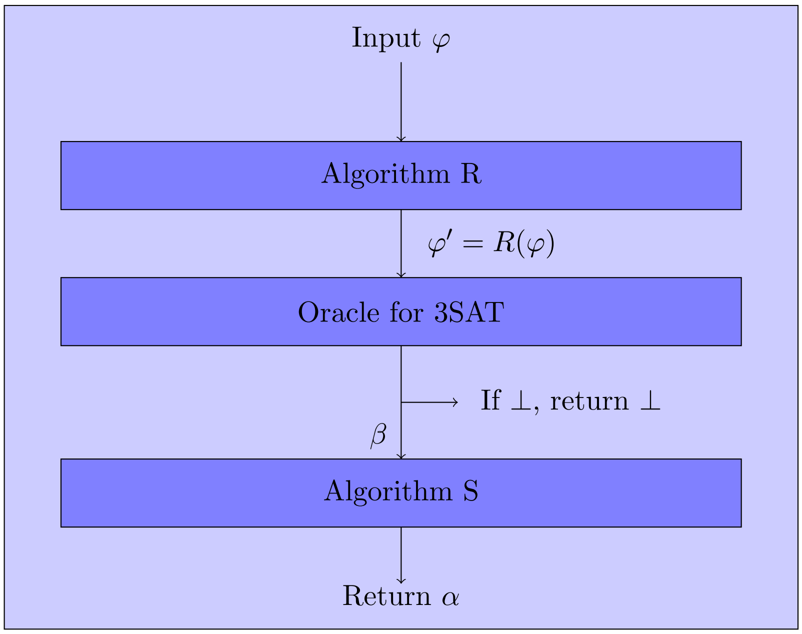


Figure 2: Reduction algorithm from SAT to 3SAT.

Algorithm R: The intuition behind this algorithm is that when we have a clause $(\ell_0 \vee \ell_1 \vee ... \vee \ell_{k-1})$ in the SAT instance ϕ (with large width $k > 3$), we want to break it into multiple clauses of width 3. But simply breaking it up doesn't preserve information about φ being satisfiable. Instead, we introduce the clauses $(y \vee \ell_0 \vee \ell_1)$ and $(\neg y \vee \ell_2 \ldots \ell_{k-1})$, where y is a new boolean variable. We repeat this process until all the clauses are of width 3. Note that this way of reducing the width of the clauses is the reverse of the resolution procedure. Formally, the algorithm is as follows:

$R(\varphi)$:
Input : A CNF formula φ
Output : A CNF formula φ' with each clause of width 3
$\varphi' = \varphi$
$i = 0$
while *φ' has a clause $C = (\ell_0 \vee \ldots \vee \ell_{k-1})$ of width $k > 3$* **do**
    Remove C
    Add clauses
return φ'

Note that φ' is **not** an equivalent formula to φ. While φ is on variables $z_0, \dots z_{n-1}$, the formula φ' is on variables $z_0, \dots z_{n-1}, y_0, \dots y_{t-1}$, where t is the number of iterations of the while loop.

Algorithm S: Given an assignment β to the variables $z_0, \dots z_{n-1}, y_0, \dots y_{t-1}$, the algorithm simply takes part of the assignment to the variables $z_0, \dots z_{n-1}$.

Next we consider the runtime and correctness of the overall reduction algorithm.

Runtime of the reduction algorithm: We first consider the runtime of the algorithm R:

Then, we consider the runtime of the algorithm S, which is simply $O(n)$. Overall, the runtime of the reduction algorithm is $O(nm)$.

Proof of correctness: We will show that if φ is satisfiable, then the reduction algorithm produces a satisfying assignment and if φ is unsatisfiable, the reduction algorithm will output $\perp$. This is based on the following two claims.

Claim 5.2. *If φ is satisfiable then $\varphi' = R(\varphi)$ is satisfiable.*

Proof of claim. Assume that φ is satisfiable. Let $\varphi = \varphi_0, \varphi_1, \dots, \varphi_t = R(\varphi)$ be the formula as it evolves through the t loop iterations. We will prove by induction on i that φ_i is satisfiable for $i = 0, \dots, t$. constructed through the t loop iterations.

Base case ($i = 0$):

Induction step: By the induction hypothesis, we can assume that φ_{i-1} is satisfiable, and now we need to show that φ_i is satisfiable:

□

Claim 5.3. *If β satisfies $R(\varphi)$, then $\alpha = S(\beta)$ also satisfies φ.*

Proof of claim. We prove by "backwards induction" that β satisfies φ_i for $i = t, \dots, 0$. We can then drop the extra t variables that don't appear in φ without changing the satisfiability. (We call this "backwards induction" since our base cases is $i = t$.)

The base case ($i = t$) follows because β satisfies $R(\varphi) = \varphi_t$ by assumption.

For the induction step:

□

To finish the correctness proof, suppose φ is satisfiable. Then from Claim 5.2, φ' is also satisfiable. The 3SAT oracle returns a satisfying assignment β, which is turned into a satisfying assignment for φ via the algorithm S (Claim 5.3). If φ is unsatisfiable, then by Claim 5.3, φ' is also unsatisfiable. In this case, the 3SAT oracle returns $\perp$ - as a result the reduction algorithm also returns $\perp$.

This completes the proof that 3-SAT is $\mathsf{NP}_{\mathsf{search}}$-complete. □

6 Mapping Reductions

The usual strategy for proving that a problem Γ in $\mathsf{NP}_{\mathsf{search}}$ is also $\mathsf{NP}_{\mathsf{search}}$-hard (and hence $\mathsf{NP}_{\mathsf{search}}$-complete) follows a structure similar to the proof of Theorem 5.1.

1. Pick a known $\mathsf{NP}_{\mathsf{search}}$-complete problem Π to try to reduce to Γ.

2. Come up with an algorithm R mapping

3. Show that R runs in polynomial time.

4. Show that if x has an answer,

5. Conversely, show that if $R(x)$ has an answer, then so does x. Moreover,

Reductions with the structure outlined above are called *mapping reductions*, and they are what are typically used throughout the theory of NP-completeness. A formal definition follows (but we won't expect you to use this formalism, you can stick with the general definition of polynomial-time reductions):

Definition 6.1. Let $\Pi = (\mathcal{I}, \mathcal{O}, f)$ and $\Gamma = (\mathcal{J}, \mathcal{P}, g)$ be search problems. A *polynomial-time mapping reduction* from Π to Γ consists of two polynomial-time algorithms R and S such that for every $x \in \mathcal{I}$:

1.

2.

3.

Note that the above outline only proves $\mathsf{NP}_{\mathsf{search}}$-hardness; a proof that Γ is $\mathsf{NP}_{\mathsf{search}}$-complete should also check that it's in $\mathsf{NP}_{\mathsf{search}}$.