

Elementary Row Operations

REF, RREF, System of Equations

Linear Algebra

Karthik Thiagarajan

Elementary Row Operations

- Swap two rows
- Scale a row by a **non-zero** constant
- Add a constant times one row to **another** row

Swap rows | Elementary Matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

Swap rows | Elementary Matrices

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$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

$$E_1 A = R$$

Swap rows | Elementary Matrices

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$$E_2 R = A$$

Swap rows | Elementary Matrices

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$$E_1 A = R$$

Reversible operation \implies Invertible Matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1 E_2 = I$$

$$E_2 R = A$$

$$\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

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Scale a row | Elementary Matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{R_1 \rightarrow 2R_1} \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Scale a row | Elementary Matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{R_1 \rightarrow 2R_1} \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

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$$E_1 A = R$$

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{R_1 \rightarrow \frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Scale a row | Elementary Matrices

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$$\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

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Scale a row | Elementary Matrices

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Reversible operation \implies Invertible Matrix

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Add multiple of row to another | Elementary Matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{bmatrix}$$

Add multiple of row to another | Elementary Matrices

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$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{bmatrix}$$

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$$E_1 A = R$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 + 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Add multiple of row to another | Elementary Matrices

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Reversible operation \implies Invertible Matrix

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$$E_1 E_2 = I$$

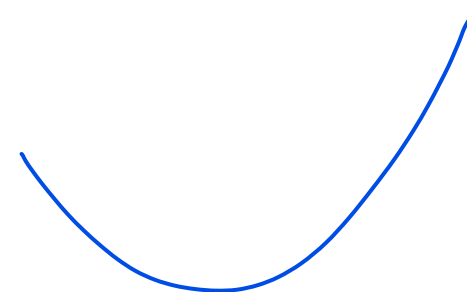
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Row Echelon Form (REF)

Leading entry (OR) **Pivot**: First non-zero entry in a row



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Leading entry (OR) **Pivot**: First non-zero entry in a row

Conditions

- All zero-rows are at the bottom
- The pivot in every non-zero row is 1.
- The pivot in a non-zero row is to the right of the pivot in the previous row.

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- The pivot in every non-zero row is 1.
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$$\begin{bmatrix} \mathbf{1} & 0 & 1 & 2 \\ 0 & \mathbf{1} & 2 & 3 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{1} & -3 \\ 0 & \mathbf{1} \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{1} & 0 & 1 & 2 \\ 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix}$$

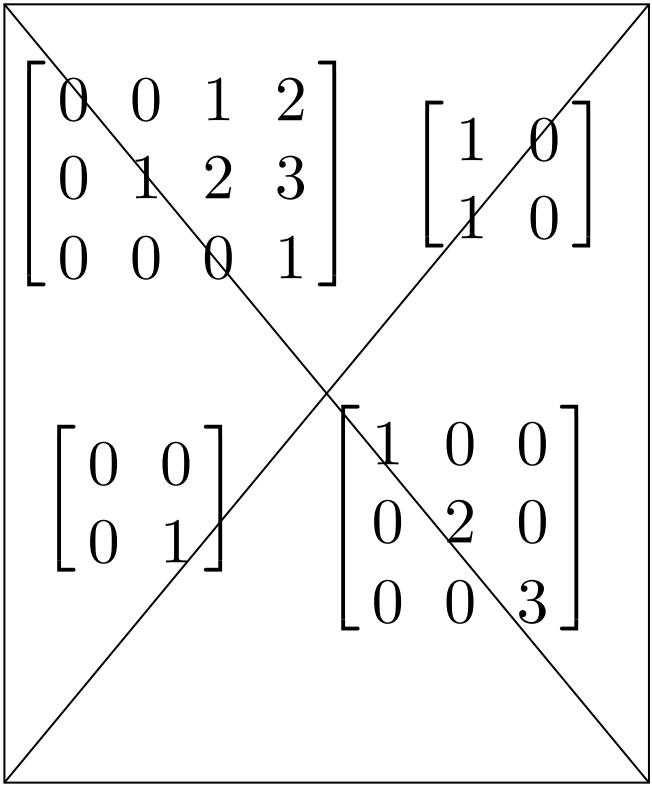
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$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & \mathbf{1} & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \mathbf{1} & 0 \\ 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{bmatrix} \quad \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Reduced Row Echelon Form (RREF)

Pivot column: A column that contains a pivot

Reduced Row Echelon Form (RREF)

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Conditions

- Matrix is in row echelon form.
- In every pivot column,
the pivot is the only non-zero entry.

Reduced Row Echelon Form (RREF)

Pivot column: A column that contains a pivot

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$$\begin{bmatrix} \mathbf{1} & 0 & -1 & 0 \\ 0 & \mathbf{1} & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{1} & 0 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form (RREF)

Pivot column: A column that contains a pivot

Conditions

- Matrix is in row echelon form.
- In every pivot column, the pivot is the only non-zero entry.

$$\begin{bmatrix} \mathbf{1} & 0 & -1 & 0 \\ 0 & \mathbf{1} & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{1} & 0 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{1} & -3 \\ 0 & \mathbf{1} \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{1} & 2 \\ 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form (RREF)

Pivot column: A column that contains a pivot

Conditions

- Matrix is in row echelon form.
- In every pivot column, the pivot is the only non-zero entry.

- $\text{REF}(A)$ is not unique
- $\text{RREF}(A)$ is unique

$$\begin{bmatrix} \mathbf{1} & 0 & -1 & 0 \\ 0 & \mathbf{1} & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{1} & 0 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{1} & -3 \\ 0 & \mathbf{1} \end{bmatrix}$$
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$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A \rightarrow R$: Example

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 1 & 2 & -6 \\ 4 & 3 & 1 & -3 \end{bmatrix}$$

$A \rightarrow R$: Example

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 1 & 2 & -6 \\ 4 & 3 & 1 & -3 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - 3R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -15 \\ 4 & 3 & 1 & -3 \end{bmatrix}$$

$A \rightarrow R$: Example

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 1 & 2 & -6 \\ 4 & 3 & 1 & -3 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - 3R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -15 \\ 4 & 3 & 1 & -3 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 4R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -15 \\ 0 & -5 & 5 & -15 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -15 \\ 0 & -5 & 5 & -15 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_2} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A \rightarrow R$: Example

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 1 & 2 & -6 \\ 4 & 3 & 1 & -3 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - 3R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -15 \\ 4 & 3 & 1 & -3 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 4R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -15 \\ 0 & -5 & 5 & -15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -15 \\ 0 & -5 & 5 & -15 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_2} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -15 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \rightarrow \frac{-1}{5}R_2} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A \rightarrow R$: Example

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 1 & 2 & -6 \\ 4 & 3 & 1 & -3 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - 3R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -15 \\ 4 & 3 & 1 & -3 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 4R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -15 \\ 0 & -5 & 5 & -15 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

REF

RREF

$$A \rightarrow R$$

- m row operations
- m elementary matrices
- $E_1 \rightarrow \cdots \rightarrow E_m$
- $E = E_m \cdots E_1$
- Since E_i is invertible,
 E is invertible

$$A \rightarrow R$$

- m row operations
- m elementary matrices
- $E_1 \rightarrow \cdots \rightarrow E_m$
- $E = E_m \cdots E_1$
- Since E_i is invertible,
 E is invertible

$$R = E_m \cdots E_1 A$$

$$R = EA$$

$$A \rightarrow R$$

- m row operations
- m elementary matrices
- $E_1 \rightarrow \cdots \rightarrow E_m$
- $E = E_m \cdots E_1$
- Since E_i is invertible,
 E is invertible

$$R = E_m \cdots E_1 A$$

$$R = EA$$

Elementary row operations \rightarrow left-multiplication by an invertible matrix

REF, RREF and Invertibility

$$A \rightarrow n \times n$$

- A is invertible if and only if $REF(A)$ has n pivots
- A is invertible if and only if $RREF(A) = I$

$$R = EA \rightarrow I = EA$$

$$A^{-1} = E$$

System of Linear Equations

$$2x - 3y + 4z = 5$$

$$x + y - z = 2$$

$$3x - y + 5z = -1$$



System of Linear Equations

$$2x - 3y + 4z = 5$$

$$x + y - z = 2$$

$$3x - y + 5z = -1$$

$$\begin{bmatrix} 2 & -3 & 4 \\ 1 & 1 & -1 \\ 3 & -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

System of Linear Equations

$$2x - 3y + 4z = 5$$

$$x + y - z = 2$$

$$3x - y + 5z = -1$$

$$\boxed{Ax = b}$$

$$\begin{bmatrix} 2 & -3 & 4 \\ 1 & 1 & -1 \\ 3 & -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

System of Linear Equations

$$2x - 3y + 4z = 5$$

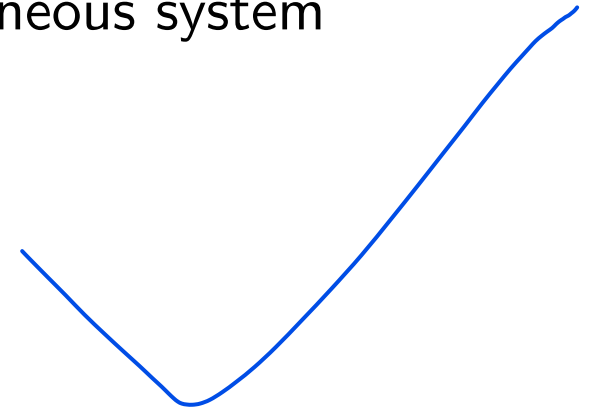
$$x + y - z = 2$$

$$3x - y + 5z = -1$$

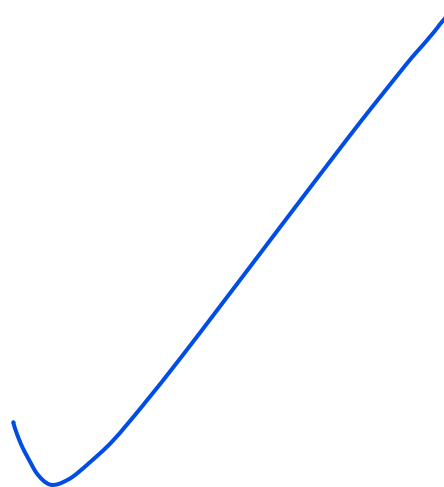
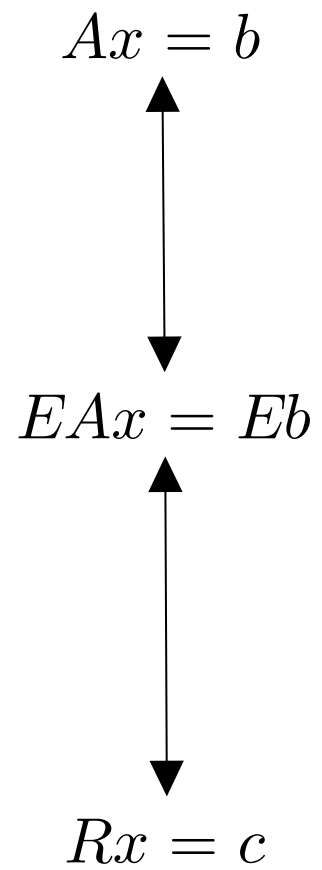
$$\boxed{Ax = b}$$

$$\begin{bmatrix} 2 & -3 & 4 \\ 1 & 1 & -1 \\ 3 & -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

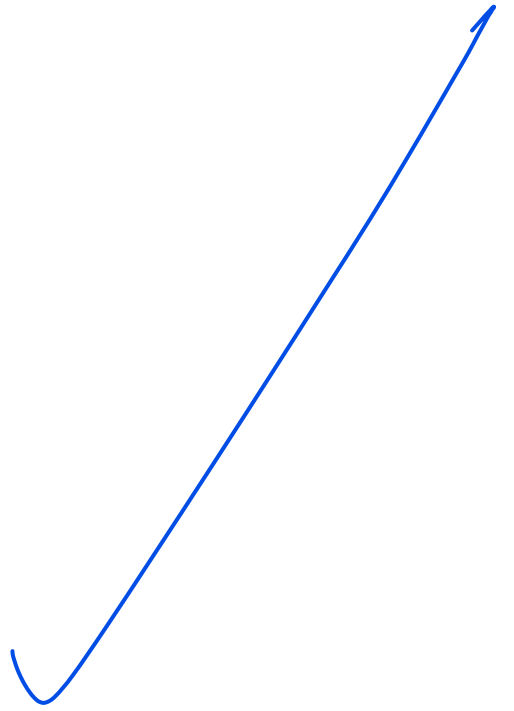
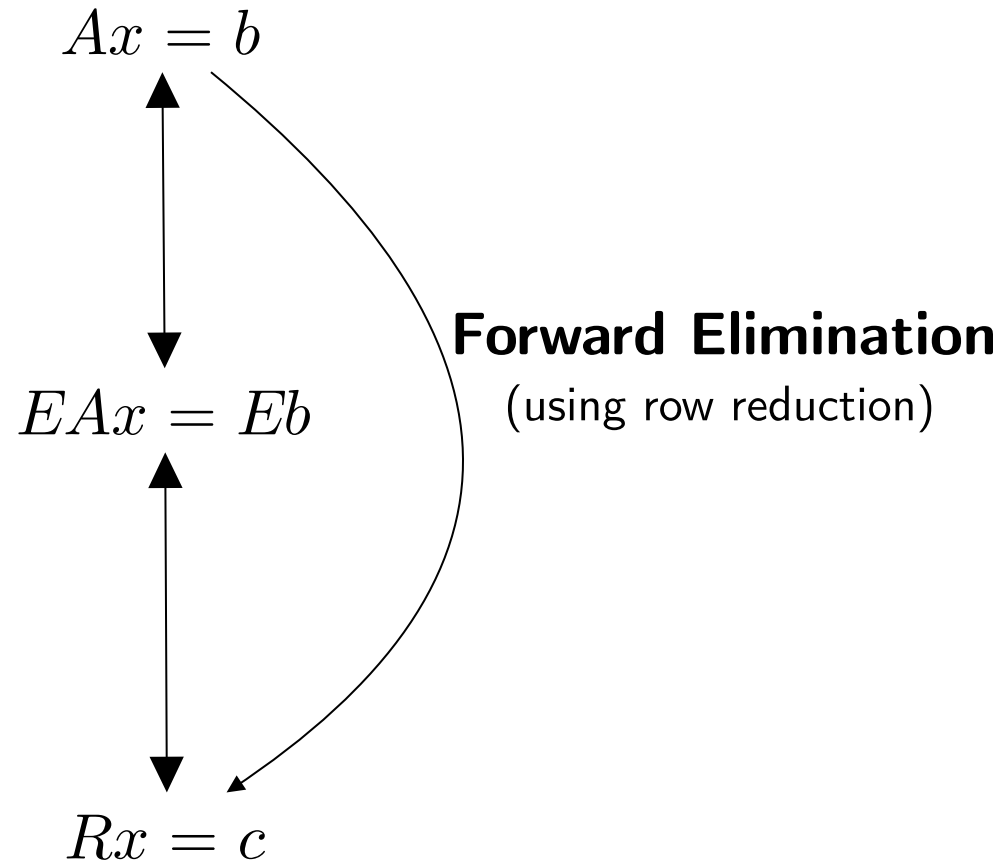
$b = 0 \implies$ homogeneous system



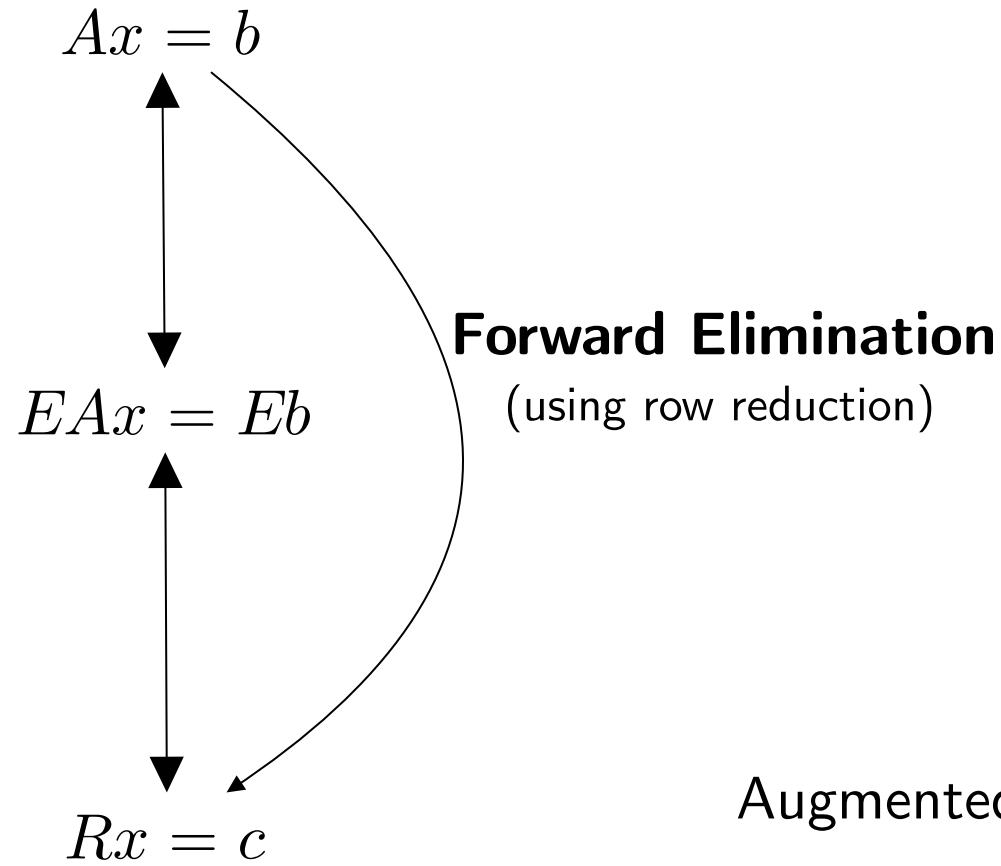
Gaussian Elimination



Gaussian Elimination



Gaussian Elimination

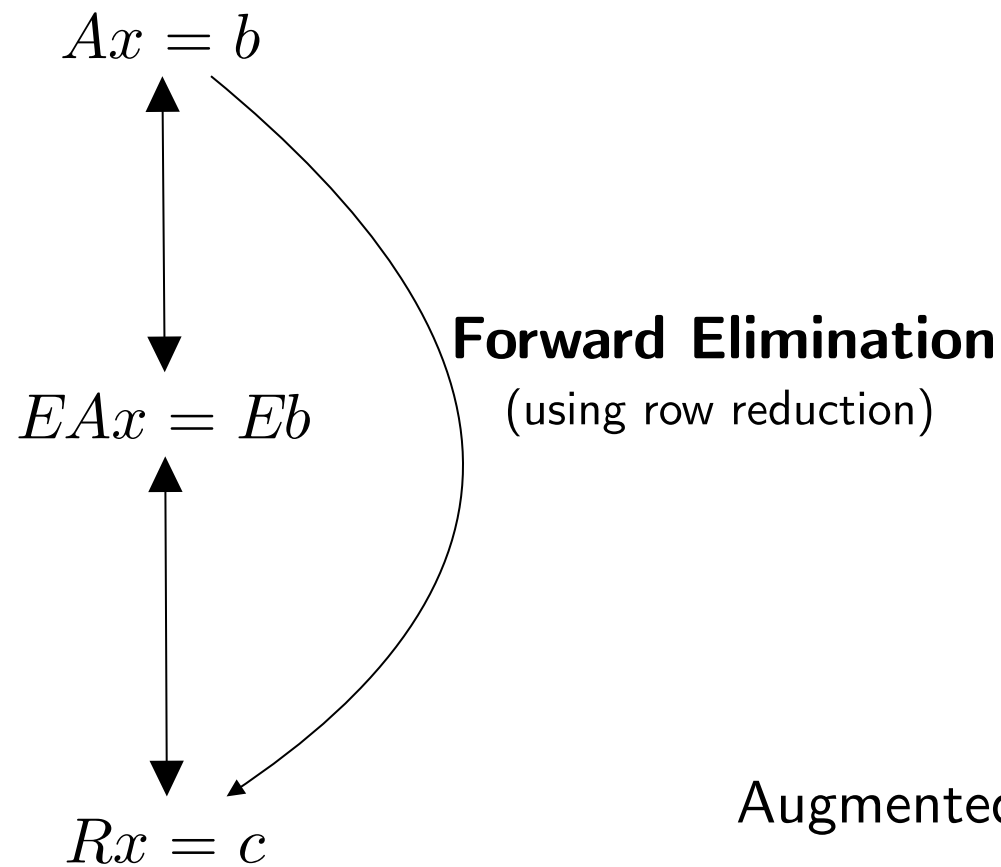


Augmented matrix

$$[A|b] \rightarrow [R|c]$$

Gaussian Elimination

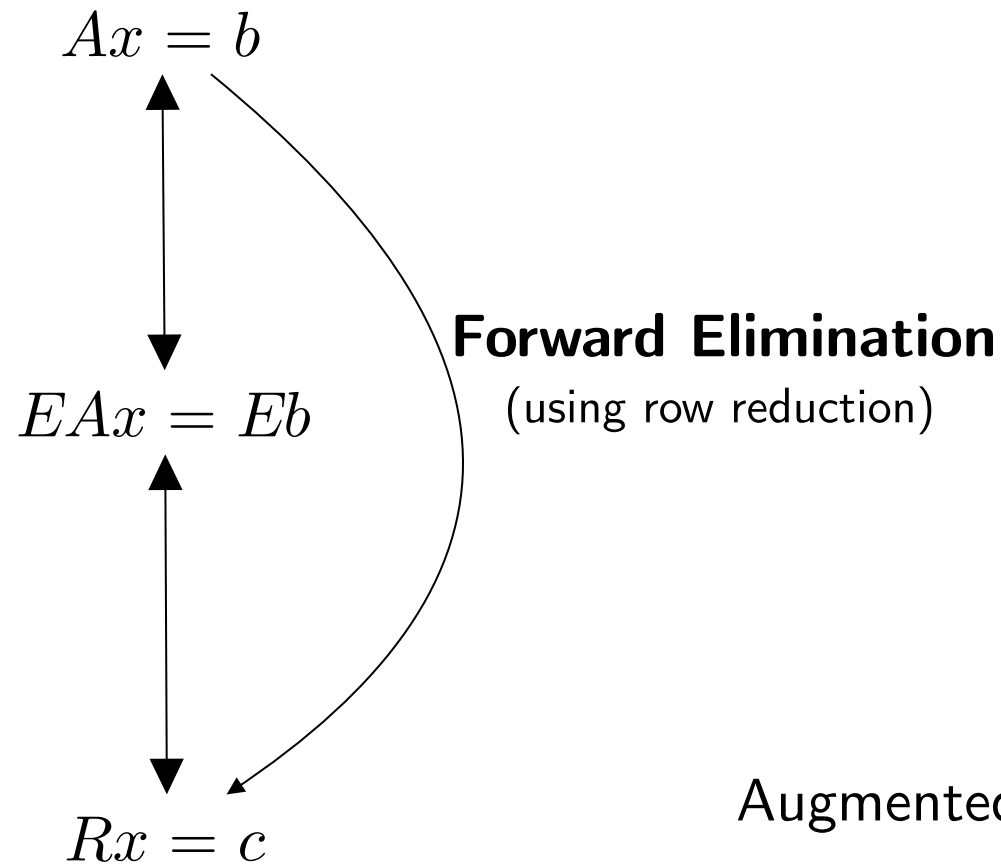
$$\boxed{Rx = c}$$



$$[A|b] \rightarrow [R|c]$$

Gaussian Elimination

$$\boxed{Rx = c}$$

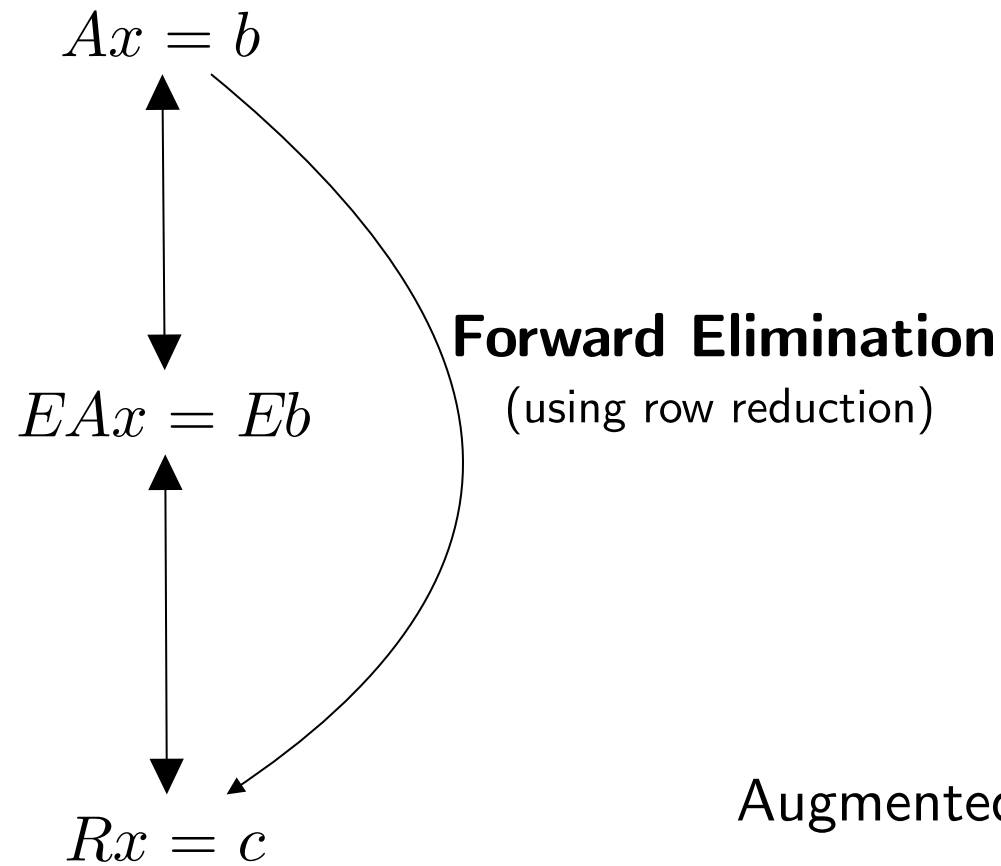


Pivot columns \rightarrow dependent variables
Non-pivot columns \rightarrow independent variables

$$[A|b] \rightarrow [R|c]$$

Gaussian Elimination

$$\boxed{Rx = c}$$



Pivot columns \rightarrow dependent variables
Non-pivot columns \rightarrow independent variables

Backward Substitution

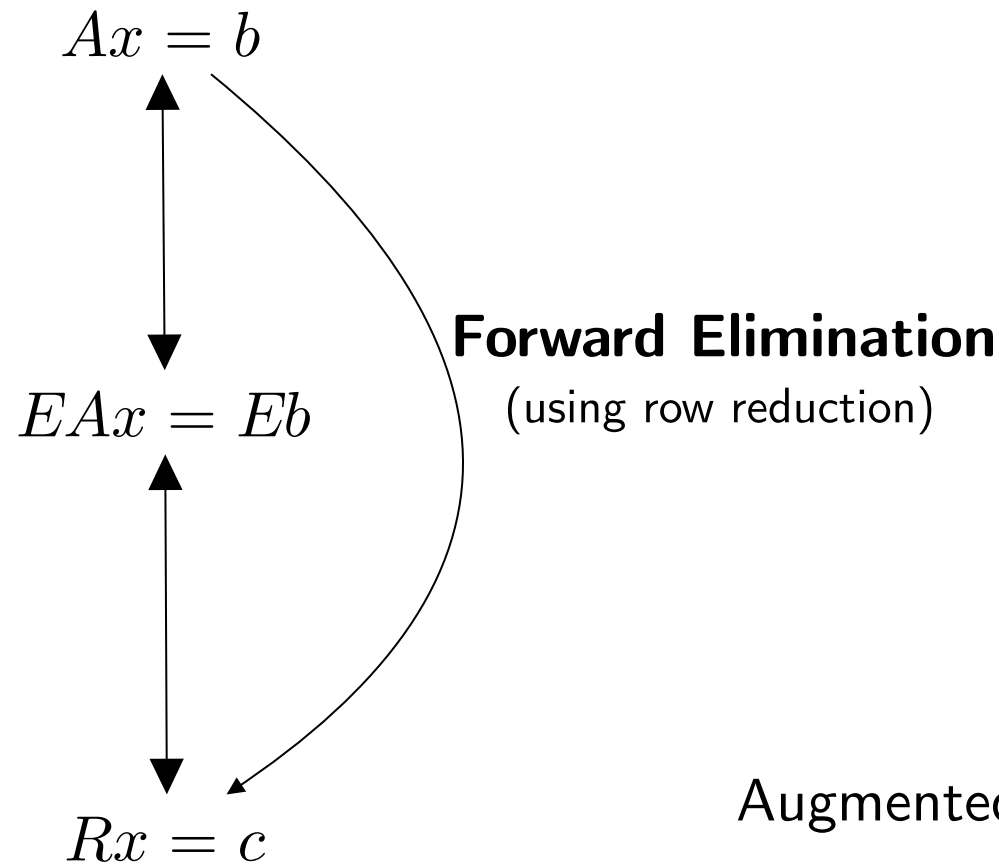
- Set arbitrary values for independent variables.

Augmented matrix

$$[A|b] \rightarrow [R|c]$$

Gaussian Elimination

$$\boxed{Rx = c}$$



Pivot columns \rightarrow dependent variables
Non-pivot columns \rightarrow independent variables

Backward Substitution

- Set arbitrary values for independent variables.
- Solve for dependent variables.

Augmented matrix

$$[A|b] \rightarrow [R|c]$$

Gaussian Elimination

$$\begin{bmatrix} \mathbf{1} & -1 & 0 & -1 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} \mathbf{1} & -1 & 0 & -1 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$y, w \rightarrow$ independent

$x, z \rightarrow$ dependent

Gaussian Elimination

$$\begin{bmatrix} \mathbf{1} & -1 & 0 & -1 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$y, w \rightarrow$ independent

$x, z \rightarrow$ dependent

$$y = t_1$$

$$w = t_2$$

Gaussian Elimination

$$\begin{bmatrix} \mathbf{1} & -1 & 0 & -1 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$y, w \rightarrow$ independent

$x, z \rightarrow$ dependent

$$\begin{array}{l} y = t_1 \\ w = t_2 \end{array} \longrightarrow \begin{array}{l} x = 1 + t_1 + t_2 \\ z = -1 - 2t_2 \end{array}$$

Gaussian Elimination

$$\begin{bmatrix} \mathbf{1} & -1 & 0 & -1 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$y, w \rightarrow$ independent

$x, z \rightarrow$ dependent

$$\begin{array}{ccc} y = t_1 & \xrightarrow{\hspace{2cm}} & x = 1 + t_1 + t_2 \\ w = t_2 & & z = -1 - 2t_2 \end{array}$$

$$S = \{(1 + t_1 + t_2, t_1, -1 - 2t_2, t_2) : t_1, t_2 \in \mathbb{R}\}$$

Gaussian Elimination

$$Ax = b, \quad A \rightarrow m \times n, \quad [A|b] \rightarrow [R|c]$$

Gaussian Elimination

$$Ax = b, \quad A \rightarrow m \times n, \quad [A|b] \rightarrow [R|c]$$

- # pivots = # dependent variables

Gaussian Elimination

$$Ax = b, \quad A \rightarrow m \times n, \quad [A|b] \rightarrow [R|c]$$

- # pivots = # dependent variables
- # dependent variables + # independent variables = n ✓

Gaussian Elimination

$$Ax = b, \quad A \rightarrow m \times n, \quad [A|b] \rightarrow [R|c]$$

- $\#$ pivots = $\#$ dependent variables
- $\#$ dependent variables + $\#$ independent variables = n
- If $[R|c]$ has a pivot in the last column, $Ax = b$ has no solution

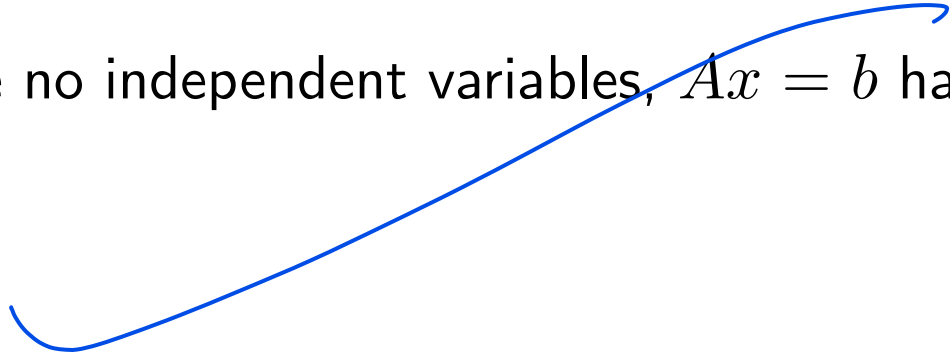
Gaussian Elimination

$$Ax = b, \quad A \rightarrow m \times n, \quad [A|b] \rightarrow [R|c]$$

- $\#$ pivots = $\#$ dependent variables
- $\#$ dependent variables + $\#$ independent variables = n
- If $[R|c]$ has a pivot in the last column, $Ax = b$ has no solution
- If not, it has at least one solution:

Gaussian Elimination

$$Ax = b, \quad A \rightarrow m \times n, \quad [A|b] \rightarrow [R|c]$$

- $\# \text{ pivots} = \# \text{ dependent variables}$
 - $\# \text{ dependent variables} + \# \text{ independent variables} = n$
 - If $[R|c]$ has a pivot in the last column, $Ax = b$ has no solution
 - If not, it has at least one solution:
 - If there are no independent variables, $Ax = b$ has a unique solution
- 

Gaussian Elimination

$$Ax = b, \quad A \rightarrow m \times n, \quad [A|b] \rightarrow [R|c]$$

- $\#$ pivots = $\#$ dependent variables
- $\#$ dependent variables + $\#$ independent variables = n
- If $[R|c]$ has a pivot in the last column, $Ax = b$ has no solution
- If not, it has at least one solution:
 - If there are no independent variables, $Ax = b$ has a unique solution
 - If there is at least one independent variable, $Ax = b$ has infinitely many solutions.