Elementary Row Operations
REF, RREF, System of Equations

Linear Algebra

Karthik Thiagarajan

Elementary Row Operations

- Swap two rows
- Scale a row by a **non-zero** constant
- Add a constant times one row to **another** row

Swap rows | Elementary Matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

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$$E_1 A = R$$

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Reversible operation $\implies$ Invertible Matrix

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$$E_2 R = A$$

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Scale a row | Elementary Matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{R_1 \to 2R_1} \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

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Add multiple of row to another | Elementary Matrices

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Leading entry (OR) **Pivot**: First non-zero entry in a row

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$$\begin{bmatrix} \mathbf{1} & 0 & 1 & 2 \\ 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix}$$

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Pivot column: A column that contains a pivot

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$$\begin{bmatrix} \mathbf{1} & 0 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix}$$

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- REF(A) is not unique
- RREF(A) is unique

$A \to R$: Example

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 1 & 2 & -6 \\ 4 & 3 & 1 & -3 \end{bmatrix}$$

$A \to R$: Example

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 1 & 2 & -6 \\ 4 & 3 & 1 & -3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -15 \\ 4 & 3 & 1 & -3 \end{bmatrix}$$

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$A \to R$: Example

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REF RREF

$A \to R$

- m row operations
- m elementary matrices
- $E_1 \to \cdots \to E_m$
- $E = E_m \cdots E_1$
- Since E_i is invertible,
  E is invertible

$A \to R$

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$$R = E_m \cdots E_1 A$$

$$R = EA$$

$A \to R$

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Elementary row operations $\to$ left-multiplication by an invertible matrix

REF, RREF and Invertibility

$A \rightarrow n \times n$

- A is invertible if and only if $REF(A)$ has n pivots
- A is invertible if and only if $RREF(A) = I$

$$R = EA \;\rightarrow\; I = EA$$

$$A^{-1} = E$$

System of Linear Equations

$$2x - 3y + 4z = 5$$

$$x + y - z = 2$$

$$3x - y + 5z = -1$$

System of Linear Equations

$$2x - 3y + 4z = 5$$

$$x + y - z = 2$$

$$3x - y + 5z = -1$$

$$\begin{bmatrix} 2 & -3 & 4 \\ 1 & 1 & -1 \\ 3 & -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

System of Linear Equations

$$2x - 3y + 4z = 5$$

$$x + y - z = 2$$

$$3x - y + 5z = -1$$

$$\boxed{Ax = b}$$

$$\begin{bmatrix} 2 & -3 & 4 \\ 1 & 1 & -1 \\ 3 & -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

System of Linear Equations

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$b = 0 \Longrightarrow$ homogeneous system

Gaussian Elimination

$$
\begin{array}{c}
Ax = b \\
\Updownarrow \\
EAx = Eb \\
\Updownarrow \\
Rx = c
\end{array}
$$

Gaussian Elimination

$$
\begin{tikzcd}
Ax = b \arrow[d, leftrightarrow] \arrow[dd, bend left=60, "\begin{array}{c}\textbf{Forward Elimination} \\ \text{(using row reduction)}\end{array}"] \\
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(using row reduction)

Augmented matrix $\quad [A|b] \rightarrow [R|c]$

Gaussian Elimination

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Augmented matrix $[A|b] \to [R|c]$

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$$\boxed{Rx = c}$$

Pivot columns $\rightarrow$ dependent variables

Non-pivot columns $\rightarrow$ independent variables

$$\begin{tikzcd} Ax = b \arrow[d, leftrightarrow] \arrow[dd, bend left, "\substack{\textbf{Forward Elimination}\\ \text{(using row reduction)}}"] \\ EAx = Eb \arrow[d, leftrightarrow] \\ Rx = c \end{tikzcd}$$

Augmented matrix $[A|b] \rightarrow [R|c]$

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Non-pivot columns $\rightarrow$ independent variables

Forward Elimination

(using row reduction)

Backward Substitution

- Set aribitrary values for independent variables.

Augmented matrix $[A|b] \rightarrow [R|c]$

Gaussian Elimination

$$\boxed{Rx = c}$$

$$\begin{tikzcd} Ax = b \arrow[d, leftrightarrow] \arrow[dd, bend left] \\ EAx = Eb \arrow[d, leftrightarrow] \\ Rx = c \end{tikzcd}$$

Pivot columns $\rightarrow$ dependent variables
Non-pivot columns $\rightarrow$ independent variables

Forward Elimination
(using row reduction)

Backward Substitution

- Set aribitrary values for independent variables.
- Solve for dependent variables.

Augmented matrix $\quad [A|b] \rightarrow [R|c]$

Gaussian Elimination

$$\begin{bmatrix} \mathbf{1} & -1 & 0 & -1 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

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$y, w \rightarrow$ independent

$x, z \rightarrow$ dependent

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$y, w \rightarrow$ independent

$x, z \rightarrow$ dependent

$$y = t_1$$
$$w = t_2$$

Gaussian Elimination

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$y, w \rightarrow$ independent

$x, z \rightarrow$ dependent

$$\begin{aligned} y &= t_1 \\ w &= t_2 \end{aligned} \longrightarrow \begin{aligned} x &= 1 + t_1 + t_2 \\ z &= -1 - 2t_2 \end{aligned}$$

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$x, z \to$ dependent

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$$S = \{(1 + t_1 + t_2, t_1, \, -1 - 2t_2, t_2) : \; t_1, t_2 \in \mathbb{R}\}$$

Gaussian Elimination

$Ax = b,\ A \ \rightarrow \ m \times n, \quad [A|b] \rightarrow [R|c]$

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Gaussian Elimination

$Ax = b, \; A \rightarrow \; m \times n, \quad [A|b] \rightarrow [R|c]$

- # pivots = # dependent variables
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Gaussian Elimination

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Gaussian Elimination

$Ax = b,\ A \ \to\ m \times n, \quad [A|b] \to [R|c]$

- # pivots = # dependent variables
- # dependent variables + # independent variables = n
- If $[R|c]$ has a pivot in the last column, $Ax = b$ has no solution
- If not, it has at least one solution:
  - If there are no independent variables, $Ax = b$ has a unique solution
  - If there is at least one independent variable, $Ax = b$ has infinitely many solutions.