Math 2 Week 2 : Solving linear equations

Lecture 2.1 - Determinants (Part-3)

Recall :

- For a 1×1 matrix $[a]$, the determinant is defined by $det([a]) = a$.
- For an $n \times n$ matrix, the determinant is defined inductively via the minors M_{1j} or cofactors C_{1j} corresponding to the first row.
- The (i, j)-th minor is the determinant of the submatrix formed by deleting the i-th row and j-th column.
- The (i, j)-th cofactor $C_{ij} = (-1)^{i+j} M_{ij}$.

Definition

$det(A) = \Sigma_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j} = \Sigma_{i=1}^{n} a_{1j} C_{1j}$

Expansion along any row or column

$$det(A) = \Sigma_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \quad \text{for a fixed } i$$

$$= \Sigma_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \quad \text{for a fixed } j$$

Expansion along any row or column

$$det(A) = \Sigma_{\underline{\underline{j=1}}}^{n}(-1)^{i+j}a_{ij}M_{ij} \quad \text{for a fixed } i$$

expansion along the i^{th} row

$$= \Sigma_{\underline{\underline{i=1}}}^{n}(-1)^{i+j}a_{ij}M_{ij} \quad \text{for a fixed } j$$

expansion along the j^{th} column

$$det(A_{3\times3}) = (-1)^{2+1}a_{21} \times M_{21} + (-1)^{2+2}a_{22} \times M_{22} + (-1)^{2+3}a_{23} \times M_{23}$$

$$= -a_{21} \times det\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{22} \times det\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} - a_{23}\ det\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= (-1)^{1+2}a_{12} \times det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + (-1)^{2+2}a_{22} \times det\begin{bmatrix} & \\ & \end{bmatrix} + (-1)^{2+3}a_{32} \times det\begin{bmatrix} & \\ & \end{bmatrix}$$

Important properties and identities

Property 1 : Determinant of a product is product of the determinants. Related identity : $det(AB) = det(A)det(B)$

$$\det(A^n) = \det(A)^n.$$

$$\det(A^{-1}) = \frac{1}{\det(A)} = \det(A)^{-1}.$$

$$\det(P^{-1}AP) = \det(A).$$

$$\det(AB) = \det(BA)$$

$$\det(A^TA) = \det(A)^2.$$

$$\det(A^T) = \det(A)$$

expand along 1st column

+ induction

Property 2 : Switching two rows or columns changes the sign.

$$A = \begin{array}{c} \\ i\text{th} \rightarrow \\ j\text{th} \rightarrow \\ \\ \end{array} \begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ & & & \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} a_{j1} & a_{j2} & \cdots & a_{jn} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix} \begin{array}{c} \leftarrow i\text{th row} \\ \leftarrow j\text{th row} \\ \\ \end{array}$$

$$\det(\tilde{A}) = -\det(A)$$

Expand along ith row & use induction

Property 3 : Adding multiples of a row to another row leaves the determinant unchanged.

Property 3' : Adding multiples of a column to another column leaves the determinant unchanged.

Property 4 : Scalar multiplication of a row by a constant t multiplies the determinant by t.

Property 4' : Scalar multiplication of a column by a constant t multiplies the determinant by t.

Useful computational tips

1) The determinant of a matrix with a row or column of zeros is 0.
2) The determinant of a matrix in which one row (or column) is a linear combination of other rows (resp. columns) is 0.
3) Scalar multiplication of a row by a constant t multiplies the determinant by t.
4) While computing the determinant, you can choose to compute it using expansion along a suitable row or column.

QN : 2,3,4,8,9

Lecture 2.2 - Cramer's Rule

An example of using Cramer's rule

Consider the following system of linear equations

$$4x_1 - 3x_2 = 11$$
$$6x_1 + 5x_2 = 7$$

$12x_1 - 9x_2 = 33$
$12x_1 + 10x_2 = 14$
$19x_2 = 14 - 33$
$= -19.$
$\Rightarrow x_2 = -1.$
$\Rightarrow x_1 = 2.$

Matrix representation : $Ax = b$ where
the matrix A is given by $A = \begin{bmatrix} 4 & -3 \\ 6 & 5 \end{bmatrix}$, $\quad b = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$.

Unique solution : $x_1 = 2, x_2 = -1$

Example (Contd.) : Steps to apply Cramer's rule

1.

- Coefficient matrix $A = \begin{bmatrix} 4 & -3 \\ 6 & 5 \end{bmatrix}$
- Calculate $det(A)$. $= 4 \times 5 - (-3) \times 6 = 20 + 18 = 38.$

2.

- Coefficient matrix $A = \begin{bmatrix} 4 & -3 \\ 6 & 5 \end{bmatrix}$
- Calculate $det(A)$. $= 38.$
- Replace the first column of A by the column vector b and call it A_{x_1} . $A_{x_1} = \begin{bmatrix} 11 & -3 \\ 7 & 5 \end{bmatrix}$
- Replace the second column of A by the column vector b and call it A_{x_2}. $A_{x_2} = \begin{bmatrix} 4 & 11 \\ 6 & 7 \end{bmatrix}$
- Calculate $det(A_{x_1}) = 76$.
- Calculate $det(A_{x_2}) = -38$.

3.

Calculate $\dfrac{det(A_{x_1})}{det(A)}$ $\dfrac{det(A_{x_2})}{det(A)}$

The solutions are :

$$x_1 = \frac{det(A_{x_1})}{det(A)} = \frac{76}{38} = 2. \qquad x_2 = \frac{det(A_{x_2})}{det(A)} = \frac{-38}{38} = -1.$$

Cramer's rule for invertible 2 × 2 matrices

Consider the following system of linear equations of two variables.

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Matrix representation : $Ax = b \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Define $A_{x_1} = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$ and $A_{x_2} = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$.

The solution of the system of equations in 2 variables is:

$$x_1 = \frac{det(A_{x_1})}{det(A)} = \frac{b_1 a_{22} - a_{12} b_2}{a_{11} a_{22} - a_{12} a_{21}}$$

$$x_2 = \frac{det(A_{x_2})}{det(A)} = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}$$

Cramer's rule for invertible 3 × 3 matrices

Consider the following system of linear equations in 3 variables :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

Matrix representation : $Ax = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Define

$$A_{x_1} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

$$A_{x_2} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix} \qquad A_{x_3} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

Define
$$A_{x_1} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

$$A_{x_2} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix} \qquad A_{x_3} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

The solution of the system of equations of 3 variables is:

$$x_1 = \frac{det(A_{x_1})}{det(A)} \qquad x_2 = \frac{det(A_{x_2})}{det(A)} \qquad x_3 = \frac{det(A_{x_3})}{det(A)}$$

Example of Cramer's rule for a 3 × 3 invertible matrix

Consider the system of linear equations $Ax = b$ where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 4 & 3 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

As described in the procedure, calculate $det(A) = -37$. Since it is non-zero, we can apply Cramer's rule. Follow the next steps in the procedure :

$$A_{x_1} = \begin{bmatrix} 0 & 0 & 3 \\ 2 & 2 & 5 \\ 1 & 3 & 1 \end{bmatrix} \qquad A_{x_2} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 4 & 1 & 1 \end{bmatrix} \qquad A_{x_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

$$det(A_{x_1}) = 12 \qquad det(A_{x_2}) = -27 \qquad det(A_{x_3}) = 4.$$

Applying Cramer's rule, the solution of the system of equations is :

$$x_1 = \frac{det(A_{x_1})}{det(A)} = -\frac{12}{37}$$
$$x_2 = \frac{det(A_{x_2})}{det(A)} = \frac{27}{37}$$
$$x_3 = \frac{det(A_{x_3})}{det(A)} = \frac{4}{37}$$

Cramer's rule for invertible n × n matrices

Consider the system of linear equations $Ax = b$ where A is an $n \times n$ invertible matrix and b is a column vector with n entries.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Define A_{x_i} to be the matrix obtained by replacing the $i-th$ column of A by the column vector b. Cramer's rule states that the (unique) solution is :

$$x_i = \frac{det(A_{x_i})}{det(A)}.$$

QN : 1,4
4th - get 2 eqn and find x3 values and take them equal

Lecture 2.3 : Solutions to a system of linear equations with an invertible coefficient matrix

The solution of a system of linear equations with an invertible coefficient matrix

Square Matrix (Recall)

A square matrix is a matrix in which the number of rows is the same as the number of columns.

Example

$$\begin{bmatrix} 3 & 5 & -7 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix}_{3\times 3}, \begin{bmatrix} 2.5 & 1 \\ 0 & 2 \end{bmatrix}_{2\times 2}$$

The inverse of a Square Matrix (recall)

Let A be an $n \times n$ matrix. The inverse of A is another $n \times n$ matrix B such that $AB = BA = \mathrm{I}_{n\times n}$ and is denoted by A^{-1}.

Let A be an $n \times n$ matrix. The inverse of A is another $n \times n$ matrix B such that $\boxed{AB = BA = \mathrm{I}_{n\times n}}$ and is denoted by A^{-1}.

Example

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

A B

$A = B^{-1}$ $B = A^{-1}$.

Uniqueness of Inverse:

$AB = BA = I$, $AC = CA = I$.

$C(AB) = (CA)B$

$= C I_n = C$; $(CA)B = I_n B = B$

$\Rightarrow B = C$.

The determinant of an invertible matrix (recall)

Recall that $det(A)det(A^{-1}) = 1$ and hence $det(A^{-1}) = \frac{1}{det(A)}$.

Conclusion : inverse of A exists $\quad \Rightarrow \quad det(A)$ has to be non-zero.

Recall that $det(A)det(A^{-1}) = 1$ and hence $det(A^{-1}) = \frac{1}{det(A)}$.
Conclusion : inverse of A exists $\Rightarrow$ $det(A)$ has to be non-zero.

What about the converse i.e. does $det(A) \neq 0 \Rightarrow A$ is invertible?

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad det(A) = \boxed{ad - bc \neq 0}$

$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$ae + bg = 1$
$af + bh = 0$
$ce + dg = 0$
$cf + dh = 1$

$ade + bdg = d.$
$bce + bdg = 0.$
$(ad-bc)e = d.$
$\Rightarrow e = \frac{d}{ad-bc}$

The adjugate of a square matrix

Recall that the (i,j)-th minor is the determinant of the submatrix formed by deleting the i-th row and j-th column. Notation : M_{ij}.

The (i,j)-th cofactor is defined as : $C_{ij} := (-1)^{i+j}M_{ij}$.

The cofactor matrix C is the matrix whose (i,j)-th entry is C_{ij}.

Definition

The adjugate matrix of A is defined as : $adj(A) := C^T$.

A 3 × 3 example of adjugate and inverse

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 5 & 6 & 0 \end{bmatrix}$$

$$\begin{aligned} det(A) &= 1(2 \times 0 - 8 \times 6) - 2(0 \times 0 - 8 \times 5) + 3(0 \times 6 - 2 \times 5) \\ &= -48 + 80 - 30 = 2 \end{aligned}$$

$$\begin{array}{lll} M_{11} = -48, & M_{12} = -40, & M_{13} = -10 \\ M_{21} = -18, & M_{22} = -15, & M_{23} = -4 \\ M_{31} = 10, & M_{32} = 8, & M_{33} = 2 \end{array}$$

$$\text{The cofactor matrix} \quad C = \begin{bmatrix} -48 & 40 & -10 \\ 18 & -15 & 4 \\ 10 & -8 & 2 \end{bmatrix}$$

The adjugate matrix $\quad adj(A) = \begin{bmatrix} -48 & 18 & 10 \\ 40 & -15 & -8 \\ -10 & 4 & 2 \end{bmatrix}.$

Let us compute $A \frac{1}{det(A)} adj(A)$[redacted] and $\frac{1}{det(A)} adj(A) A$.

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} \end{aligned}$$

Hence $A^{-1} = \frac{1}{det(A)} adj(A)$.

Adjugate and Inverse

If A is an $n \times n$ matrix and $det(A) \neq 0$, then A^{-1} exists and equals

$$A^{-1} = \frac{1}{det(A)} adj(A).$$

$$\sum_{j=1}^{n} a_{ij} C_{ij} = det(A)$$

$$\sum_{j=1}^{n} a_{ij} \boxed{\frac{1}{det(A)} C_{ij}} = 1.$$

$$\sum_{j=1}^{n} a_{ij} \left(adj(A)_{ji} \frac{1}{det(A)} \right) = 1.$$

$$\sum_{j=1}^{n} a_{ij} \left(\frac{1}{det(A)} adj(A) \right)_{ji} = 1.$$

$$\sum_{j=1}^{n} a_{ij} \left(\frac{1}{det(A)} adj(A) \right)_{ii} = 1.$$

$$\Rightarrow \left(A \frac{1}{det(A)} \right.$$

$$i \neq k$$

$$\frac{1}{det(A)} \boxed{\sum_{j=1}^{n} a_{ij} C_{kj}} = 0$$

The solution of a system of linear equations with an invertible coefficient matrix

Consider the system of linear equations $Ax = b$ where the coefficient matrix A is an invertible matrix.

Multiplying both sides by A^{-1} we obtain :

$$A x = b$$

$$A^{-1} A x = A^{-1} b$$

$$I_n x = A^{-1} b$$

$$x = A^{-1} b.$$

Example

$$
\begin{aligned}
8x_1 + 8x_2 + 4x_3 &= 1960\\
12x_1 + 5x_2 + 7x_3 &= 2215\\
3x_1 + 2x_2 + 5x_3 &= 1135
\end{aligned}
$$

$$
A = \begin{bmatrix} 8 & 8 & 4\\ 12 & 5 & 7\\ 3 & 2 & 5 \end{bmatrix} \qquad x = \begin{bmatrix} x_1\\ x_2\\ x_3 \end{bmatrix} \qquad b = \begin{bmatrix} 1960\\ 2215\\ 1135 \end{bmatrix}
$$

$det(A) = 8(25 - 14) - 8(60 - 21) + 4(24 - 15) = 88 - 312 + 36 = -188.$

So A is invertible and we compute the inverse as follows :

$$
\begin{aligned}
M_{11} &= 11, & M_{12} &= 39, & M_{13} &= 9\\
M_{21} &= 32, & M_{22} &= 28, & M_{23} &= -8\\
M_{31} &= 36, & M_{32} &= 8, & M_{33} &= -56
\end{aligned}
$$

The cofactor matrix: $C = \begin{bmatrix} 11 & -39 & 9\\ -32 & 28 & 8\\ 36 & -8 & -56 \end{bmatrix}$

The adjugate matrix $adj(A) = \begin{bmatrix} 11 & -32 & 36 \\ -39 & 28 & -8 \\ 9 & 8 & -56 \end{bmatrix}$.

$A^{-1} = \frac{1}{det(A)} adj(A) = \frac{1}{-188} \begin{bmatrix} 11 & -32 & 36 \\ -39 & 28 & -8 \\ 9 & 8 & -56 \end{bmatrix}$

$$x = A^{-1}b = \frac{1}{-188} \begin{bmatrix} 11 & -32 & 36 \\ -39 & 28 & -8 \\ 9 & 8 & -56 \end{bmatrix} \begin{bmatrix} 1960 \\ 2215 \\ 1135 \end{bmatrix}$$

$$= -\frac{1}{188} \begin{bmatrix} -8460 \\ -23500 \\ -28200 \end{bmatrix} = \begin{bmatrix} 45 \\ 125 \\ 150 \end{bmatrix}$$

Hence the solution is $x_1 = 45, x_2 = 125, x_3 = 150$.

Homogeneous System of Linear Equations

A system of linear equations is homogeneous if all of the constant terms are 0 i.e. $b = 0$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= \underline{0} \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= \underline{0} \\ &\ldots \\ &\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= \underline{0} \end{aligned}$$

Solutions of a homogeneous system

The matrix form of a homogeneous system is $Ax = 0$.

If A is an invertible matrix then multiplying both sides by A^{-1}, we obtain $x = A^{-1}0 = 0$.

A homogeneous system of linear equations with n equations in n unknowns :

- has a unique solution 0 if its coefficient matrix is invertible, i.e. its determinant is non-zero.
- has an infinite number of solutions if its coefficient matrix is not invertible i.e. its determinant is 0.

QN : 6,7,4,9,10

Lecture 2.4 The echelon form

System of linear equations

A general system of m linear equations with n unknowns can be written as

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\ldots \\
&\ldots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

Matrix Representation

The matrix representation of this system of linear equations is $Ax = b$ where :

$$
A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

A solution is an assignment of values for x so that the equations are satisfied (i.e. hold true).

Example

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

$Ax = 0$ $\Rightarrow$ $x_1 = -2x_3$

$x_1 + 2x_3 = 0$ $\quad x_2 = -3x_3$

$x_2 + 3x_3 = 0$ $\quad x = \begin{bmatrix} -10 \\ -15 \\ 5 \end{bmatrix}$

$x_3 = 5.$ $x_1 = -10, x_2 = -15$ $\quad x = \begin{bmatrix} -2c \\ -3c \\ c \end{bmatrix}$

$x_3 = c.$ $x_1 = -2c$, $x_2 = -3c$

$Ax = b$ $\quad x_1 + 2x_3 = b_1$

$x_2 + 3x_3 = b_2.$

(Reduced) Row echelon form

A matrix is in row echelon form if :

- The first non-zero element in each row, called the leading entry, is 1.
- Each leading entry is in a column to the right of the leading entry in the previous row.
- Rows with all zero elements, if any, are below rows having a non-zero element.
- For a non-zero row, the leading entry in the row is the only non-zero entry in its column.

Examples

$$A_{ref} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_{rref} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.6.2 The row echelon form and reduced row echelon form

A matrix is in row echelon form if :

- The first non-zero element (the leading entry) in a row is 1.
- The column containing the leading 1 of a row is to the right of the column containing the leading 1 of the row above it. In different words, all subsequent non-zero rows which also have their leading entries (i.e. first non-zero entries) as 1 and they should appear to the right of the leading entry in the previous row.
- Any non-zero rows are always above rows with all zeros.

A matrix is in reduced row echelon form if :

- The first non-zero element in the first row (the leading entry) is the number 1.
- The column containing the leading 1 of a row is to the right of the column containing the leading 1 of the row above it. In different words, all subsequent non-zero rows which also have their leading entries (i.e. first non-zero entries) as 1 and they should appear to the right of the leading entry in the previous row.
- The leading entry in each row must be the only non-zero number in its column.
- Any non-zero rows are always above rows with all zeros.

Note: Any matrix which is in reduced row echelon form is also in row echelon form.

Solutions of Ax = b when A is in reduced row echelon form

Let $Ax = b$ be a system of linear equations and suppose A is in reduced row echelon form.

Suppose for some i, the i^{th} row of A is a zero row but $b_i \neq 0$. Then this system has no solution.

Reason : This means if we write the corresponding system of linear equations, the i^{th} equation reads

$$0x_1 + 0x_2 + \dots 0x_n = b_i.$$

Since $b_i \neq 0$ this equation cannot be satisfied.

2.

Let $Ax = b$ be a system of linear equations and suppose A is in reduced row echelon form.

Assume that for every zero row of A, the corresponding entry of b is also 0 (i.e. if the i^{th} row of A is zero, then so is b_i).

- If the i-th column has the leading entry of some row, we call x_i a dependent variable.
- If the i-th column does not have the leading entry of some row, we call x_i an independent variable.

Solutions of $Ax = b$ when A is in reduced row echelon form

- Assign arbitrary values to independent variables.
- For a dependent variable, there is a unique equation in which it occurs. All other variables in that equation are independent variables and thus have values assigned. Hence, we can compute the value of the dependent variable from this equation substituting the assigned values for the other independent variables in the equation.
- The obtained values for x_i give a solution to the system.
- In fact every solution is obtained in this way.

Conclusion : If A is in reduced row echelon form, this easy procedure provides us with ALL the solutions of $Ax = b$.

QN : 2,3,6,7,8,10

Degrees of Freedom

In general, for a linear system with

- n variables, and
- m independent equations,

the number of independent (free) variables is:

Independent variables= n−m

In **row echelon form (REF)**, the rule is:

- The **leading entry** (first nonzero entry from the left) of each nonzero row **must be to the right of the leading entry of the row above it**.
- **All entries below a leading entry must be 0**.
- Entries **above the leading entry** can be anything — there is no restriction in REF.
- Rows consisting of all zeros, if any, are at the **bottom**.

So, **entries below a leading entry must be 0** in REF.

For **reduced row echelon form (RREF)**, the rules are stricter than REF:

1. **Leading 1s:** Each nonzero row has a **leading entry of 1** (called a pivot).
2. **Rightward progression:** The leading 1 of each row is **to the right of the leading 1 in the row above**.
3. **Zeros below and above: All entries in the column containing a leading 1 must be 0**, except for the leading 1 itself.
4. **Zero rows at bottom:** Any rows consisting entirely of zeros are at the **bottom**.

 # Key difference from REF:

- In REF, entries **above** the pivot can be anything.
- In RREF, entries **above and below** the pivot must be **0**.

Lecture 2.5 Row reduction

Contents

- What are elementary row operations?
- Reducing any matrix to (reduced) row echelon form using elementary row operations.
- Computing the determinant using row reduction.

Elementary Row operations

Type	Action	Example and notation
1	Interchange two rows	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \overset{R_1 \longleftrightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 \\ 0 & 7 & 1 & 1 \end{bmatrix}$
2	Scalar multiplication of a row by a constant t.	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \overset{R_1/3}{\rightsquigarrow} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$
3	Adding multiples of a row to another row.	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \overset{R_1 - 3R_2}{\rightsquigarrow} \begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$

Row reduction : Row echelon form

Action	Example and notation
Find the left most non-zero column	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$
Use elementary row operations to get 1 in the top position of that column	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \overset{R_1/3}{\rightsquigarrow} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$
Use type 3 elementary row operations to make the entries below the 1 into 0.	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \overset{R_2-R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1/3 & -1/3 & -1/3 \\ 0 & 7 & 1 & 1 \end{bmatrix}$

Action	Example and notation
If there are no non-zero rows below the current row, the matrix is in row echelon row. Else find the next non-zero row and by switching rows (type 1 elementary operation), move all the zero rows between the current row and that row to below the non-zero row. Repeat the above steps for the submatrix below the current row.	$\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1/3 & -1/3 & -1/3 \\ 0 & 7 & 1 & 1 \end{bmatrix}$

(type 1 elementary operation), move all the zero rows between the current row and that row to below the non-zero row. Repeat the above steps for the submatrix below the current row.

$$\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & \boxed{1/3} & -1/3 & -1/3 \\ 0 & 7 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{3R_2} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 7 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3-7R_2} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 8 & 8 \end{bmatrix}$$

$$\xrightarrow{R_3/8} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Row reduction : Reduced row echelon form

Assume the matrix is in row echelon form $\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Action	Example and notation
Take the columns containing a 1 in the leading position of some row. Use type 3 elementary row operations to make all the entries in those columns 0.	$\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $\xrightarrow{R_2 - R_3, R_1 - \frac{1}{3}R_3} \begin{bmatrix} 1 & 2/3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Action	Example and notation
Take the columns containing a 1 in the leading position of some row. Use type 3 elementary row operations to make all the entries in those columns 0.	$\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $\xrightarrow{R_2 + R_3, R_1 - \frac{1}{3}R_3} \begin{bmatrix} 1 & 2/3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$$\xrightarrow{R_1 - \frac{2}{3}R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix}$$

Example

$$A = \begin{bmatrix}2&4&1\\3&8&7\\5&6&9\end{bmatrix} \xrightarrow{R_1/2} \begin{bmatrix}1&2&1/2\\3&8&7\\5&6&9\end{bmatrix} \xrightarrow{R_2-3R_1,R_3-5R_1}$$

$$\begin{bmatrix}1&2&1/2\\0&2&11/2\\0&-4&13/2\end{bmatrix} \xrightarrow{R_2/2} \begin{bmatrix}1&2&1/2\\0&1&11/4\\0&-4&13/2\end{bmatrix} \xrightarrow{R_3+4R_2} \begin{bmatrix}1&2&1/2\\0&1&11/4\\0&0&35/2\end{bmatrix} \xrightarrow{\frac{2}{35}R_3}$$

$$\begin{bmatrix}1&2&1/2\\0&1&11/4\\0&0&1\end{bmatrix} \xrightarrow[R_2-\frac{11}{4}R_3]{R_1-\frac{1}{2}R_3} \begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Example

$$A = \begin{bmatrix}2&4&1\\3&8&7\\5&6&9\end{bmatrix}$$

Recall from determinants

$$
\begin{aligned}
det(A) &= 2 \times det \begin{bmatrix} 8 & 7 \\ 6 & 9 \end{bmatrix} - 4 \times det \begin{bmatrix} 3 & 7 \\ 5 & 9 \end{bmatrix} + 1 \times det \begin{bmatrix} 3 & 8 \\ 5 & 6 \end{bmatrix} \\
&= 2(72 - 42) - 4(27 - 35) + 1(18 - 40) \\
&= 2(30) - 4(-8) + 1(-22) \\
&= 60 + 32 - 22 \\
&= 70
\end{aligned}
$$

Type	Notation	Effect on determinant
1	$A \overset{R_i \leftrightarrow R_j}{\rightsquigarrow} B$	$det(A) = -det(B)$
2	$A \overset{R_i / c}{\rightsquigarrow} B$	$det(A) = c det(B)$
3	$A \overset{R_i + cR_j}{\rightsquigarrow} B$	$det(A) = det(B)$

Computing the determinant via row reduction

For a square matrix A :

Observe : Row reducing A into row echelon form produces an upper triangular matrix with diagonal entries either all 1 (if it is invertible) or some 1s and some 0s.

1. Row reduce A into row echelon form.
2. If the diagonal entries of the reduced matrix contain a 0, then its determinant is 0 and tracing the determinant back along the row reduction procedure shows that the determinant of A must be 0.
3. If the diagonal entries of the reduced matrix are all 1s its determinant is 1. Tracing back along the procedure used to row reduce using the table of how the determinant changes according to elementary row operations, we can compute the determinant of A.

Example

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix} \xrightarrow{R_1/2} \begin{bmatrix} 1 & 2 & 1/2 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix} \xrightarrow{R_2-3R_1, R_3-5R_1} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 2 & 11/2 \\ 0 & -4 & 13/2 \end{bmatrix}$$

$$\xrightarrow{R_2/2} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & -4 & 13/2 \end{bmatrix} \xrightarrow{R_3+4R_2} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & 0 & 35/2 \end{bmatrix} \xrightarrow{2R_3/35} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

$\det(A) = 70 = 35\times 2$ $\qquad\qquad 35 \qquad\qquad 2\times 35/2 = 35$

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix} \xrightarrow{R_1/2} \begin{bmatrix} 1 & 2 & 1/2 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix} \xrightarrow{R_2-3R_1,\, R_3-5R_1} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 2 & 11/2 \\ 0 & -4 & 13/2 \end{bmatrix}$$

$$\Big\downarrow R_2/2$$

$$\begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & 0 & 1 \end{bmatrix} \xleftarrow{2R_3/35} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & 0 & 35/2 \end{bmatrix} \xleftarrow{R_3+4R_2} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & -4 & 13/2 \end{bmatrix}$$

$1 \qquad\qquad 35/2 \qquad\qquad 35/2$

QN : 1,2,3,5,6,8,9,10

Lecture 2.6 The Gaussian Elimination Method

The Gaussian Elimination Method

Recall

We have seen the following methods to find the solutions to a system of linear equations $Ax = b$:

- If A is invertible, then the solution is unique and is given by $A^{-1}b$. The solution can be found by using :
  1. Cramer's rule.
  2. the adjugate matrix to calculate A^{-1}.
- If A is in (reduced) row echelon form, we can find all the solutions as follows :
  1. Find the dependent variables (corr. to columns with leading entries) and independent variables (corr. to other columns).
  2. Assign a value to each independent variable.Calculate the values of each dependent variable using the unique equation in which it occurs.

Contents

- The augmented matrix for a system of linear equations.
- The Gaussian elimination method to determine all solutions of a system of linear equations.
- Computing the inverse using Gaussian elimination.

The augmented matrix

Let $Ax = b$ be a system of linear equations where A is an $m \times n$ matrix and b is a $m \times 1$ column vector.

The augmented matrix of this system is defined as the matrix of size $m \times n + 1$ whose first n columns are the columns of A and the last column is b.

We denote the augmented matrix by $[A|b]$ and put a vertical line between the first n columns and the last column b while writing it.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

$$A$$

Example

$$
\begin{aligned}
3x_1 + 2x_2 + x_3 + x_4 &= 6 \\
x_1 + x_2 &= 2 \\
7x_2 + x_3 + x_4 &= 8
\end{aligned}
$$

where $A = \begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 2 \\ 8 \end{bmatrix}$.

The augmented matrix is $[A|b] = \left[\begin{array}{cccc|c} 3 & 2 & 1 & 1 & 6 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right]$.

The Gaussian elimination method

Consider the system of linear equations $Ax = b$.

1. Form the augmented matrix of the system $[A|b]$.

2. Perform the same operations on $[A|b]$ that were used to bring A into reduced row echelon form.

3. Let R be the submatrix of the obtained matrix of the first n columns and c be the submatrix of the obtained matrix consisting of the last column.

We write the obtained matrix as $[R|c]$. Notice that R is the reduced row echelon matrix obtained by row reducing A.

The solutions of $Ax = b$ are precisely the solutions of $Rx = c$.

4. Form the corresponding system of linear equations $Rx = c$.

5. Find ALL the solutions of $Rx = c$ and hence of $Ax = b$.

Since R is in reduced row echelon form, we can find ALL its solutions (as described earlier).

Example

$$\left[\begin{array}{cccc|c} 3 & 2 & 1 & 1 & 6 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right] \xrightarrow{R_1/3} \left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1/3 & -1/3 & -1/3 & 0 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right] \xrightarrow{3R_2} \left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right]$$

$$\xrightarrow{R_3 - 7R_2} \left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 8 & 8 & 8 \end{array}\right] \xrightarrow{R_3/8} \left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$
\left[\begin{array}{cccc|c}
1 & 2/3 & 1/3 & 1/3 & 2 \\
0 & 1 & -1 & -1 & 0 \\
0 & 0 & 1 & 1 & 1
\end{array}\right]
\xrightarrow[R_1 - \frac{1}{3}R_3]{R_2 + R_3}
\left[\begin{array}{cccc|c}
1 & 2/3 & 0 & 0 & 5/3 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1
\end{array}\right]
$$

$$
\xrightarrow{R_1 - \frac{2}{3}R_2}
\underbrace{\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{array}\right]}_{R}
\left|
\underbrace{\begin{array}{c} 1 \\ 1 \\ 1 \end{array}}_{c}
\right]
$$

$Rx = c$

$x_1, x_2, x_3 \to$ dependent

$x_4 \to$ independent

$x_4 = c.$

$x_1 = 1$

$x_2 = 1$

$x_3 + x_4 = 1 \implies x_3 = 1 - c$

Set of solns. of $Rx = c$ & hence $Ax = b$ is $\left\{ \begin{array}{l} x_1 = 1, x_2 = 1, \\ x_3 = 1-c, x_4 = c \end{array} \;\middle|\; c \in \mathbb{R} \right\}$

$= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1-c \\ c \end{bmatrix} \;\middle|\; c \in \mathbb{R} \right\}.$

Another example

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_2 - 3x_3 &= 1 \\ 2x_1 + x_2 + 5x_3 &= 0 \end{aligned}$$

The matrix representation of this system of linear equations is:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 5 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

The augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 1 \\ 2 & 1 & 5 & 0 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 1 \\ 2 & 1 & 5 & 0 \end{array}\right] \xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & 3 & -4 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & \cancel{1} & 1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 1 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{-3} \end{array}\right]$$

$R \qquad c$

$Rx = c$.

This system does not have solutions.

$\therefore$ $Ax = b$ does not have solutions

Homogeneous system of linear equations

0 is always a solution of a homogeneous system of linear equations $Ax = 0$. This solution is called the *trivial solution*.

For a homogeneous system, there are only two different possibilities :

- 0 is the unique solution.
- there are infinitely many solutions other than 0.

$Ax = 0$
$x_1 = w_1, x_2 = w_2, \ldots, x_n = w_n$ is a soln.
then so is
$x_1 = tw_1, x_2 = tw_2, \ldots, x_n = tw_n$.

0 is always a solution of a homogeneous system of linear equations $Ax = 0$. This solution is called the *trivial solution*.

For a homogeneous system, there are only two different possibilities :

- 0 is the unique solution.
- there are infinitely many solutions other than 0.

In a homogeneous system of equations, if there are more variables than equations, then it is guaranteed to have nontrivial solutions.

$\left[\quad\right]_{m \times n}$

Computing the inverse

Computing the inverse of an invertible matrix A is equivalent to :

Finding solutions of $Ax = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $Ay = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $Az = \begin{bmatrix}0\\0\\1\end{bmatrix}$.

$x' \qquad y' \qquad z'$

$$\left[A \;\middle|\; \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right] = [A \mid I] \xrightarrow[\text{row echelon form}]{\text{reduction to reduced}} [I \mid \underbrace{Q}_{A^{-1}}]$$

$$A\begin{bmatrix}x' & y' & z'\end{bmatrix} = \begin{bmatrix}Ax' & Ay' & Az'\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I \qquad 3\times 3$$

$$\left[\begin{matrix}1&1&1\\2&4&8\\3&9&27\end{matrix}\;\middle|\;\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{matrix}1&1&1\\0&2&6\\0&6&24\end{matrix}\;\middle|\;\begin{matrix}1&0&0\\-2&1&0\\-3&0&1\end{matrix}\right] \xrightarrow{R_2/2} \left[\begin{matrix}1&1&1\\0&1&3\\0&6&24\end{matrix}\;\middle|\;\begin{matrix}1&0&0\\-1&1/2&0\\-3&0&1\end{matrix}\right] \xrightarrow{R_3-6R_2} \left[\begin{matrix}1&1&1\\0&1&3\\0&0&6\end{matrix}\;\middle|\;\begin{matrix}1&0&0\\-1&1/2&0\\3&-3&1\end{matrix}\right] \xrightarrow{R_3/6} \left[\begin{matrix}1&1&1\\0&1&3\\0&0&1\end{matrix}\;\middle|\;\begin{matrix}1&0&0\\-1&1/2&0\\1/2&-1/2&1\end{matrix}\right] \leftarrow$$

QN : 5,10

Week 2 :

Tutorial 1 :

Maths 2 Week 2 Tutorial 1

Note Title 28-04-2021

$$f(x) = x^3$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x+h) = (x+h)^3$

$= x^3 + h^3 + 3x^2h + 3xh^2$

$f(x) = x^3$

$$= \lim_{h\to 0} \frac{h^3 + 3x^2h + 3xh^2}{h}$$

$\lim_{x\to a}(f+g) = \lim_{x\to a} f + \lim_{x\to a} g$

$$= \lim_{h\to 0}\left(h^2 + 3x^2\left(\frac{h}{h}\right) + 3x\left(\frac{h^2}{h}\right)\right)$$

$f(x+h) - f(x)$

$= \cancel{x^3} + h^3 + 3x^2h + 3xh^2 - \cancel{x^3}$

$= h^3 + 3x^2h + 3xh^2$

$$= \lim_{h\to 0} h^2 + \boxed{\lim_{h\to 0}(3x^2)} + \lim_{h\to 0} 3xh$$

$$= 0 + 3x^2 + 0 = \underline{\underline{3x^2}}$$

$f'(x) = 3x^2.$

Tutorial 2 :

Solving System of linear equations:

$$\left.\begin{aligned} -x_1 + x_2 - x_3 &= 0 \\ 2x_1 + 2x_2 - 2x_3 &= 2 \\ x_2 + x_3 &= -1 \end{aligned}\right\}$$

$$A = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 2 & -2 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \text{coefficient matrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1}$$

$$\underset{3\times3}{A}\ \underset{3\times1}{x} = b \ \ \text{(3×1 matrix).}$$

$$b = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

$$\boxed{Ax = b}$$

$[A|b] \longrightarrow$ Augmented matrix

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 2 & 2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right].$$

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 2 & 2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 4 & -4 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4} R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 1 & 1 & -1 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 2 & -\frac{3}{2} \end{array}\right] \xrightarrow{\frac{1}{2} R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{4} \end{array}\right]$$

Row echelon form.

$$R' x = b'$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{3}{4} \end{bmatrix}$$

$$x_1 - x_2 + x_3 = 0$$
$$x_2 - x_3 = \frac{1}{2}$$
$$\boxed{x_3 = -\frac{3}{4}}$$

$$x_2 + \frac{3}{4} = \frac{1}{2}$$
$$\Rightarrow \boxed{x_2 = \frac{1}{2} - \frac{3}{4} = -\frac{1}{4}}$$

$$x_1 + \frac{1}{4} - \frac{3}{4} = 0$$
$$\Rightarrow x_1 + \left(-\frac{1}{2}\right) = 0$$
$$\Rightarrow \boxed{x_1 = \frac{1}{2}}$$

$$\left.\begin{array}{l} x_1 = \frac{1}{2} \\ x_2 = -\frac{1}{4} \\ x_3 = -\frac{3}{4} \end{array}\right\} \text{unique solution}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{4} \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{4} \end{array}\right]$$

Row echelon form

$$\xrightarrow{R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} - \frac{3}{4} \\ 0 & 0 & 1 & -\frac{3}{4} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{3}{4} \end{array}\right]$$

Reduced Row echelon form.

$$R'' x = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ -\frac{3}{4} \end{bmatrix}$$

$$\left.\begin{array}{l} x_1 = \frac{1}{2} \\ x_2 = -\frac{1}{4} \\ x_3 = -\frac{3}{4} \end{array}\right\} \text{unique Soln.}$$

Tutorial 3 :

Solving System of Linear Equations:

$$\left.\begin{aligned} x_1 - x_3 &= 0 \\ -x_1 + x_2 + x_3 &= -1 \\ x_1 - x_2 - x_3 &= 0 \end{aligned}\right\}$$

$$Ax = b$$

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

Reduced row echelon form

$$Rx = b'$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad x_1 - x_3 = 0$$
$$x_2 = -1$$
$$\boxed{0 = -1} \longrightarrow \text{absurd.}$$

No Solution

Tutorial 4 :

Solving system of linear equations:

$$\left.\begin{aligned} x_2 - x_3 &= 1 \\ x_1 \quad + 2x_3 &= -1 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}\right\}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced Row echelon form.

$$Rx = b'$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 + 2x_3 &= -1 \\ x_2 - x_3 &= 1 \end{aligned}\right\}$$
$$\boxed{0 = 0}$$

Infinitely many Solutions.

$$x_1 = -2x_3 - 1$$
$$x_2 = x_3 + 1$$
$$x = \begin{pmatrix} -2x_3 - 1 \\ x_3 + 1 \\ x_3 \end{pmatrix}$$

Tutorial 5 :

Computing the inverse of an invertible matrix A is equivalent to :

Finding solutions of $Ax = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $Ay = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $Az = \begin{bmatrix}0\\0\\1\end{bmatrix}$.

x' y' z'

$$\left[A \,\middle|\, \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right] = [A \mid I] \underbrace{\xrightarrow{\text{reduction to reduced}}}_{\text{row echelon form}} [I \mid \underset{A^{-1}}{?}]$$

$$A\,[x' \; y' \; z'] = [Ax' \; Ay' \; Az'] = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I \qquad 3\times 3$$

$$\left[\begin{matrix}1&1&1\\2&4&8\\3&9&27\end{matrix}\,\middle|\,\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{matrix}1&1&1\\0&2&6\\0&6&24\end{matrix}\,\middle|\,\begin{matrix}1&0&0\\-2&1&0\\-3&0&1\end{matrix}\right] \xrightarrow{R_2/2} \left[\begin{matrix}1&1&1\\0&1&3\\0&6&24\end{matrix}\,\middle|\,\begin{matrix}1&0&0\\-1&1/2&0\\-3&0&1\end{matrix}\right]$$

$$\xrightarrow{R_3-6R_2} \left[\begin{matrix}1&1&1\\0&1&3\\0&0&6\end{matrix}\,\middle|\,\begin{matrix}1&0&0\\-1&1/2&0\\3&-3&1\end{matrix}\right] \xrightarrow{R_3/6} \left[\begin{matrix}1&1&1\\0&1&3\\0&0&1\end{matrix}\,\middle|\,\begin{matrix}1&0&0\\-1&1/2&0\\1/2&-1/2&1/6\end{matrix}\right] \longleftarrow$$

Calculating Inverse of a matrix using Row operations:

$$A = \begin{pmatrix}2&4&6\\-1&3&-3\\0&4&2\end{pmatrix} \qquad \text{Find } A^{-1}$$

$$\left(\begin{matrix}2&4&6\\-1&3&-3\\0&4&2\end{matrix}\,\middle|\,\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right) \xrightarrow{\frac{1}{2}R_1} \left(\begin{matrix}1&2&3\\-1&3&-3\\0&4&2\end{matrix}\,\middle|\,\begin{matrix}1/2&0&0\\0&1&0\\0&0&1\end{matrix}\right)$$

$$\xrightarrow{R_2+R_1} \left(\begin{matrix}1&2&3\\0&5&0\\0&4&2\end{matrix}\,\middle|\,\begin{matrix}1/2&0&0\\1/2&1&0\\0&0&1\end{matrix}\right) \xrightarrow{\frac{1}{5}R_2} \left(\begin{matrix}1&2&3\\0&1&0\\0&4&2\end{matrix}\,\middle|\,\begin{matrix}1/2&0&0\\1/10&1/5&0\\0&0&1\end{matrix}\right)$$

$$\xrightarrow{R_3-4R_2} \left(\begin{matrix}1&2&3\\0&1&0\\0&0&2\end{matrix}\,\middle|\,\begin{matrix}1/2&0&0\\1/10&1/5&0\\-4/10&-4/5&1\end{matrix}\right) \xrightarrow{R_1-2R_2} \left(\begin{matrix}1&0&3\\0&1&0\\0&0&2\end{matrix}\,\middle|\,\begin{matrix}3/10&-2/5&0\\1/10&1/5&0\\-4/10&-4/5&1\end{matrix}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 3/10 & -2/5 & 0 \\ 0 & 1 & 0 & 1/10 & 1/5 & 0 \\ 0 & 0 & 2 & -4/10 & -4/5 & 1 \end{array}\right)$$

$$\frac{12}{10} - \frac{2}{5} = \frac{12-4}{10} = \frac{8}{10}$$

$$\xrightarrow{\frac{1}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 3/10 & -2/5 & 0 \\ 0 & 1 & 0 & 1/10 & 1/5 & 0 \\ 0 & 0 & 1 & -2/10 & -4/10 & 1/2 \end{array}\right)$$

$$\xrightarrow{R_1 - 3R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 9/10 & 8/10 & -3/2 \\ 0 & 1 & 0 & 1/10 & 1/5 & 0 \\ 0 & 0 & 1 & -2/10 & -4/10 & 1/2 \end{array}\right)$$

$I_{3\times3}$

$$AA^{-1} = I = A^{-1}A$$

$$A^{-1} = \begin{pmatrix} 9/10 & 4/5 & -3/2 \\ 1/10 & 1/5 & 0 \\ -1/5 & -2/5 & 1/2 \end{pmatrix}$$

Tutorial 6 :

MATHEMATICS FOR DATA SCIENCE - 2 - WEEK 5

Note Title 05-09-2021

$$\begin{bmatrix} 2x + y - 2z = -1 \\ 3x - 3y - z = 5 \\ x - 2y + 3z = 6 \end{bmatrix}$$

$$\underbrace{\begin{pmatrix} 2 & 1 & -2 \\ 3 & -3 & -1 \\ 1 & -2 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} -1 \\ 5 \\ 6 \end{pmatrix}}_{b}$$

$\tilde{A}X = \tilde{b}$

row echelon

$[A|b]$

$$\left[\begin{array}{ccc|c} 2 & 1 & -2 & -1 \\ 3 & -3 & -1 & 5 \\ 1 & -2 & 3 & 6 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 3 & -3 & -1 & 5 \\ 2 & 1 & -2 & -1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 0 & 3 & -10 & -13 \\ 0 & 5 & -8 & -13 \end{array}\right]$$

$$\downarrow R_2 \to \frac{1}{3}R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 0 & 1 & -10/3 & -13/3 \\ 0 & 5 & -8 & -13 \end{array}\right] \xrightarrow{R_3 \to R_3 - 5R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 0 & 1 & -\frac{10}{3} & -\frac{13}{3} \\ 0 & 0 & \frac{26}{3} & \frac{26}{3} \end{array}\right] \xrightarrow{R_3 \to \frac{3}{26}R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 0 & 1 & -\frac{10}{3} & -\frac{13}{3} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\tilde{A}$ $\tilde{b}$

$x - 2y + 3z = 6$

$y - \frac{10}{3}z = -\frac{13}{3}$

$\boxed{z = 1}$

$y = -1 \Rightarrow x = 6 - 2 - 3 = 1$

$\tilde{A}X = \tilde{b}$ $(1, -1, 1)$

1/1

Tutorial 7 :

Mathematics for Data Science -2 - Week 5.

Note Title 20-08-2021

$$\begin{pmatrix} 2 & 1 & -3 \\ 4 & 2 & -6 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Homogeneous system $Ax = 0$.

$$\left(\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 4 & 2 & -6 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 4 & 2 & -6 & 0 \\ 2 & 1 & -3 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 4R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 6 & -10 & 0 \\ 0 & 3 & -5 & 0 \end{array}\right)$$

$$\downarrow R_2 \to \tfrac{1}{6}R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & -\frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xleftarrow{R_1 \to R_1 + R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & \frac{-10}{6} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xleftarrow{R_3 \to R_3 - 3R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -10/6 & 0 \\ 0 & 3 & -5 & 0 \end{array}\right)$$

$1 - \frac{10}{6}$

$z = c$

$y - \frac{5}{3}z = 0$

$y = \frac{5}{3}z = \frac{5}{3}c$

$x - \frac{2}{3}z = 0 \Rightarrow x = \frac{2}{3}z = \frac{2}{3}c$

$(x, y, z) = \left(\frac{2}{3}c, \frac{5}{3}c, c\right)$ //

1/1

Tutorial 8 :

MATHEMATICS FOR DATA SCIENCE - 2 - WEEK 5

Note Title 27-08-2021

$$\begin{bmatrix} x - 3y + z = 4 \\ -x + 2y - 5z = 3 \\ 5x - 13y + 13z = 8 \end{bmatrix}$$ no soln.

$Ax = b$

$$\begin{bmatrix} 1 & -3 & 1 \\ -1 & 2 & -5 \\ 5 & -13 & 13 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix}$$

$[A|b]$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ -1 & 2 & -5 & 3 \\ 5 & -13 & 13 & 8 \end{array}\right] \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{cccc} 1 & -3 & 1 & 4 \\ 0 & -1 & -4 & 7 \\ 0 & 2 & 8 & -12 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{2}R_3]{R_2 \to -R_2} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & 1 & 4 & -7 \\ 0 & 1 & 4 & -6 \end{array}\right]$$

$$\downarrow R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

no soln

$$\begin{bmatrix} x - 3y + z = 4 \\ y + 4z = -7 \\ 0 = 1 \end{bmatrix}$$ absurd.

Notes :

1. Key Points:

Calculating minors and cofactors of a matrix:
If A is a square matrix, then the minor of the entry in the i-th row and j-th column (denoted by M_{ij}) is the determinant of the submatrix formed by deleting the i-th row and j-th column.
The ij-th cofactor (denoted by C_{ij}) is defined to be $(-1)^{i+j} M_{ij}$.
Calculating the determinant:

$det(A) = \sum_{i=1}^{n} a_{ij}(-1)^{i+j} M_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}$
Let us consider a 3×3 matrix A as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Calculating M_{11} and C_{11}:

• Find out the sub-matrix by deleting the first row and the first column:
$$\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & a_{22} & a_{23} \\ \bullet & a_{32} & a_{33} \end{bmatrix}$$

• Calculating the determinant of the sub-matrix:
$$det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) = a_{22}a_{33} - a_{32}a_{23}$$
$M_{11} = a_{22}a_{33} - a_{32}a_{23}$
$C_{11} = (-1)^{1+1}(a_{22}a_{33} - a_{32}a_{23}) = a_{22}a_{33} - a_{32}a_{23}$
Calculating M_{23} and C_{23} :

Calculating M_{23} and C_{23} :

• Find out the sub-matrix by deleting the second row and the third column:
$$\begin{bmatrix} a_{11} & a_{12} & \bullet \\ \bullet & \bullet & \bullet \\ a_{31} & a_{32} & \bullet \end{bmatrix}$$
• Calculating the determinant of the sub-matrix:
$$det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}\right) = a_{11}a_{32} - a_{31}a_{12}$$
$M_{23} = a_{11}a_{32} - a_{31}a_{12}$
$C_{23} = (-1)^{2+3}(a_{11}a_{32} - a_{31}a_{12}) = -a_{11}a_{32} + a_{31}a_{12}$
Calculating the determinant of A:

Expanding with respect to the first row:

$$det(A) = a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} + a_{13}(-1)^{1+3}M_{13}$$

Expanding with respect to the second row:

$$det(A) = a_{21}(-1)^{2+1}M_{21} + a_{22}(-1)^{2+2}M_{22} + a_{23}(-1)^{2+3}M_{23}$$

Expanding with respect to the third row:

$$det(A) = a_{31}(-1)^{3+1}M_{31} + a_{32}(-1)^{3+2}M_{32} + a_{33}(-1)^{3+3}M_{33}$$

The determinant can also be calculated by expanding along columns.

Expanding with respect to the first column:
$det(A) = a_{11}(-1)^{1+1}M_{11} + a_{21}(-1)^{2+1}M_{21} + a_{31}(-1)^{3+1}M_{31}$

2. Key Points :

Cramers' Rule: (This rule is applied to any system of linear equations with n equations and n variables. Here we recall the method for a system of linear equations with 3 equations and 3 variables.)

Consider a system of linear equations as follows:
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

Let the matrix representation of the above system be $Ax = b$, where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Let A_{x_i} be the matrix obtained by replacing the i-th column of A $\left(i.e., \begin{bmatrix} a_{1i} \\ a_{2i} \\ a_{3i} \end{bmatrix} \right)$ by b, for $i = 1, 2, 3$.

$$A_{x_1} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$
$$A_{x_2} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}$$
$$A_{x_3} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

If $det(A) \neq 0$, then the solutions to the above system are $x_i = \frac{det\ A_{x_i}}{det\ A}$, for $i = 1, 2, 3$. i.e., $x_1 = \frac{det\ A_{x_1}}{det\ A}$, $x_2 = \frac{det\ A_{x_2}}{det\ A}$, $x_3 = \frac{det\ A_{x_3}}{det\ A}$

3. Key Points:

Calculating the inverse of a matrix A:

Find out the cofactor matrix C, whose ij-th element is C_{ij}: the ij-th cofactor of A.
Adjoint of A is transpose of C.
Calculate the determinant of A and check whether it is non-zero or not.
If determinant is non-zero then the inverse of A exists and is given by
$$A^{-1} = \frac{1}{det(A)} adj(A).$$

Solve With US 2.1 : 2,3,4,7,11

Key Points: 1

A matrix is in row echelon form if :

- The first non-zero element (the leading entry) in a row is 1.
- The column containing the leading 1 of a row is to the right of the column containing the leading 1 of the row above it.
- Any non-zero rows are always above rows with all zeros.

A matrix is in reduced row echelon form if :

- The first non-zero element in the first row (the leading entry) is the number 1.
- All subsequent non-zero rows must also have their leading entries (i.e. first non-zero entries) as 1 and they should appear to the right of the leading entry in the previous row.
- The leading entry in each row must be the only non-zero number in its column.
- Any non-zero rows are always above rows with all zeros.

Note: Any matrix which is in reduced row echelon form is also in row echelon form.

Key Points: 2

Let $Ax = b$ be a system of linear equations and suppose A is in reduced row echelon form. Assume that for every zero row of A, the corresponding entry of b is also 0 (i.e., if the $i-th$ row of A is 0, then so is b_i.)

- If the $i-th$ column has the leading entry of some row, we call x_i a dependent variable.
- If the $i-th$ column does not have the leading entry of any row, we call x_i an independent variable.

Key Points: 3

The three different types of elementary row operations that can be performed on a matrix are:

- **Type 1:** Interchanging two rows.
- **Type 2:** Multiplying a row with some constant.
- **Type 3:** Adding a scalar multiple of a row to another row.

Key Points: 4

Let $Ax = b$ denote the matrix representation of a system of linear equations.
The augmented matrix is denoted by $[A|b]$.
The matrix obtained by performing the operations that transform A to its reduced row echelon form R on the augmented matrix is denoted by $[R|c]$.
The solutions of $Rx = c$ are the same as the solutions of $Ax = b$.

QN:9,10,

PA : 1,2,3,4,6,
In 2nd question let a(0) as investment and then further
GA : 1,2,3,4,5,7,8,13