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~ PART 2 ~

Experiment : Process or phenomenon that we wish to study statistically

Outcome : Result of the experiment

Sample space : A sample space is a set that contains all outcomes of an experiment

→ Typically denoted by 'S'
→ Examples : 1. Tossing a coin
$S = \{Head, Tail\}$
2. Throw a die
$S = \{1, 2, 3, 4, 5, 6\}$

Events : An event is a subset of the sample space. (There is a technical restriction on what subsets can be events)

EXAMPLES :

(1) Toss a coin : $S = \{Head, Tail\}$
→ Events : empty set, $\{Head\}$, $\{Tail\}$, $\{Head, Tail\}$
→ 4 events

(2) Throw a die : $S = \{1, 2, 3, 4, 5, 6\}$
→ Events : ϕ, $\{1\}, \{2\}, \dots \{6\}, \{1,2\}, \{1,3\} \dots\dots \{1,2,3,4,5,6\}$
→ 64 events
→ Some can be described in words :
- getting an even no.
- getting a multiple of 3, etc.

★ Events are central objects in probability theory

→ An event is said to have 'occurred' if the actual outcome of the experiment belongs to the event

→ Events are sets
- All set theory notions apply to events

→ One event can be contained in another, i.e, $A \subseteq B$
- Throw a die : $A = \{2,6\}$, B = even no.
- If A occurred, B has also occurred
- If B occurred, A may or may not have occurred.

→ complement of an event A, denoted A^c = {outcomes of S not in A} = $\{S \setminus A\}$
- Throw a die : $A = \{2,4,6\}$ (even)

  $A^c = \{1,3,5\}$ (odd)
- If A occurred, A^c did not occur
- If A^c occurred, A did not occur

★ Combining Events To Create New Events

→ Since events are subsets, one can do complements, unions, intersections.

→ Union ('or' in English) denoted by '$\cup$'

→ Intersection ('and' in English) denoted by '$\cap$'

★ Disjoint Events

→ Two events with an empty intersection are said to be disjoint events

→ Suppose A & B are disjoint :
- If A occurred, B did not occur.
- If B occurred, A did not occur

→ Event & Its complement
- A & A^c are disjoint
- $A \cap A^c = \phi$
- $A \cup A^c = S$

DE - MORGAN'S LAW

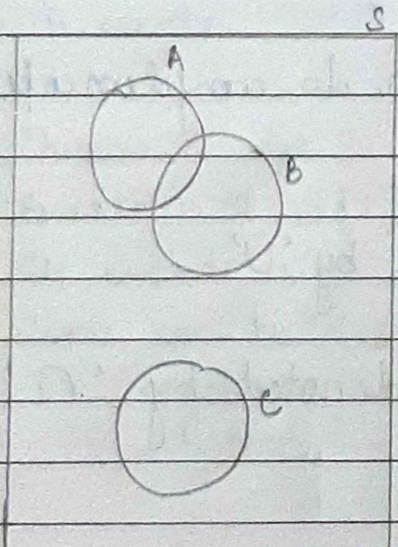
$(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

→ Multiple Events : $E_1, E_2, E_3 \ldots$ are disjoint if, for any $i \neq j$, $E_i \cap E_j = \phi$

→ Large Sample Space : Partition into disjoint events for study.

★ Events In Venn Diagrams



→ Union → all covered regions
→ Intersection → common region
→ Disjoint Events → no overlap in regions (A & C)

→ $A \cap B^c$ → region of A outside B

PROBABILITY

PROBABILITY

It is a function P that assigns to each event a real number between 0 and 1. The entire probability space (sample space, events & probability fn) should satisfy two axioms, which will be specified later.

→ Value assigned by the probability function is supposed to represent the 'chance' of the event occurring.

- Higher value means higher chance
- 0 means event can't occur
- 1 means event always occur
- Often, probability is mentioned as percentage

→ Ingredients of the Theory

- sample space, events, probability – these together are called 'probability space'.

PROBABILITY SPACE AXIOMS

(1) $P(S) = 1$ (probability of entire sample space equals 1)

(2) If $E_1, E_2, E_3, \ldots$ are disjoint events (how many events? could be infinitely many)

$$P(E_1 \cup E_2 \cup E_3 \cup \ldots) = P(E_1) + P(E_2) + P(E_3) + \ldots$$

→ Axioms are intuitive and restrict the functions that can be probability functions

→ They result in several intuitive deductions on probabilities of combination of events, probabilities of subsets of events, etc.

BASIC PROPERTIES OF PROBABILITY

(1) PROPERTY 1 : Empty Set

Probability of an empty set (denoted ϕ) equals 0.

→ $\boxed{P(\phi) = 0}$

(2) PROPERTY 2 : Complement

Let E^c be the complement of Event E. Then,

$$\boxed{P(E^c) = 1 - P(E)}$$

(3) PROPERTY 3 : Subset

If Event E is a subset of Event F, i.e, $E \subseteq F$, then

$$\boxed{P(F) = P(E) + P(F \setminus E)}$$

which implies that $P(E) \leq P(F)$.

$P(A \setminus B) = P(A \cup B) - P(B)$

$P(A \setminus B) = P(A) - P(A \cap B)$

~~If A & B are disjoint, $P(A \setminus B) = P(A)$~~

(4) PROPERTY 4 : Difference & Intersection

If E & F are events, then

$$P(E) = P(E \cap F) + P(E \setminus F)$$
$$P(F) = P(E \cap F) + P(F \setminus E)$$

(5) PROPERTY 5 : Union & Intersection

If E & F are events, then

$$\boxed{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$$

WORKING WITH PROBABILITY SPACES

(1) Tossing A Coin

(2) Restaurant Hiring

→ A waiter and a cashier are to be hired. There are 4 applicants - David & Megha from Delhi, and Rajesh & Veronica from Mumbai. The restaurant hires one person at a random as waiter, and another from the remaining as cashier.

(a) Write out sample space.

$S = \{ (D,M), (D,R), (D,V), (M,D), (M,R), (M,V),$
$(R,D), (R,M), (R,V), (V,D), (V,M), (V,R) \}$

(b) Event A : Cashier is from Delhi

$A = \{(D,M), (R,M), (V,M), (M,D), (R,D), (V,D)\}$

(c) Event B : Exactly one position is filled by Delhite

$B = \{(D,R), (D,V), (M,R), (M,V), (R,D), (V,D), (R,M), (V,M)\}$

(d) Event C : Neither position is filled by a Delhite.

$C = \{(V,R), (R,V)\}$

(3) <u>Fishing Town</u>

→ Town with a few fishing boats going out to catch fish everyday.

→ Over the years, folks have observed the following:
- Chance of catching more than 400 kg of fish in a day is 35%.
- Chance of catching more than 500 kg of fish in a day is 10%.

→ What is the chance of catching between 400 kg & 500 kg of fish in a day?

$P(A) = P(>400\ kg) = 0.35$; $P(B) = P(>500\ kg) = 0.10$

$\therefore P(\geq 400 \text{ and } \leq 500) = P(A \setminus B) = P(A) - P(B)$
$= 0.35 - 0.10$
$= 0.25$

$\boxed{\text{Here, } B \subseteq A}$

(4) Weather Forecast

suppose you hear the following forecast for rain & temperature :

→ Chance of rain tomorrow = 60%
→ Chance of max. temp above 30 deg = 70%
→ Chance of rain & max temp above 30 deg = 40%

What is the chance of no rain and max temperature below 30 deg ?

A : rain ; B = temp > 30°

$P(A) = 60\%$ $P(B) = 70\%$ $P(A \cap B) = 40\%$

$\therefore$ $P(A^c) = 40\%$

$P(B^c) = 30\%$

We want to know, $P(A^c) \cap P(B^c)$, i.e. $P(A^c \cap B^c)$

De Morgan's Law : $P(A^c \cap B^c) = P(A \cup B)^c$

$$\therefore P(A \cup B)^c = 1 - P(A \cup B)$$
$$= 1 - (0.6 + 0.7 - 0.4)$$
$$= 0.1$$

DISTRIBUTIONS

→ Assign probabilities to individual outcomes

→ When is this possibles?
- When outcomes can be enumerated as first, second, third and so on (called countable sample space)
- Finite Sample space

→ EXAMPLE : uniform Distribution on a finite sample space.

- $S = \{$ finite no. of outcomes $\}$
- Equally likely outcomes - assign same probability to each outcome
- $P(\text{one outcome}) = \dfrac{1}{|S|}$, i.e, $\dfrac{1}{\text{no. of outcomes in } S}$

- $P(\text{event}) = \dfrac{\text{No. of outcomes in event}}{\text{No. of outcomes in } S}$

EXAMPLES

(1) 5 red and 8 blue marbles in an urn. Pick a marble from the urn at random.

→ $S = \{R_1, R_2, R_3, R_4, R_5, B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8\}$

$\underset{\uparrow \text{ event}}{P(\text{red})} = \dfrac{5 \rightarrow \text{no. of fav. outcomes}}{13 \rightarrow \text{total no. of outcomes}}$; $P(\text{blue}) = \dfrac{8}{13}$

2. Throw 2 dice. What is the probability that the sum of the two no.s is 8 ?

→ $S = \{(1,1), (1,2) \ldots (1,6), (2,1) \ldots (2,6) \ldots (6,1) \ldots (6,6)\}$

Event, $E = \{(2,6), (6,2), (3,5), (5,8), (4,4)\}$

$P(E) = \frac{5}{36}$

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3. The hats of 3 persons are identical and get mixed up. Each person picks a hat at random. What is the probability that none of the persons get their own hat ?

→ $S = \{(H_1, H_2, H_3), (H_1, H_3, H_2), (H_2, H_1, H_3), (H_2, H_3, H_1), (H_3, H_1, H_2), (H_3, H_2, H_1)\}$

$|S| = 6$, i.e $3!$

Event : No. person gets its ~~to~~ own hat

$P(E) = \frac{2}{6} = \frac{1}{3}$ Ans.

What if, there were 30 people ?

→ It will be $\frac{1}{e}$, i.e, 37%.

Derangement (With reference to Questions)

Derangements are arrangements of some number of objects into positions such that no object goes to its specified position.

In the language of permutations, a derangement is a permutation σ of n elements with no fixed point, i.e,

$$\sigma(i) \neq i \text{ for all } i \in \{1, 2, \dots, n\}$$

Let $D(n)$ be the no. of derangements for n different objects, then

$$\boxed{D_n = n! \sum_{n=0}^{n} \frac{(-1)^n}{n!}}$$

This tends to 'e' for larger 'n'

$\therefore$

$D(1) = 0$	$D(5) = 44$
$D(2) = 1$	$D(6) = 265$
$D(3) = 2$	$D(7) = 1848$
$D(4) = 9$	$D(8) = 14792$

The Derangement formula can be recursively written as:

$$\boxed{D(n) = n \times D(n-1) + (-1)^n}$$

\# **PROBABILITY** : $P(n) = \dfrac{D(n)}{n!} = \dfrac{n!}{e} \times \dfrac{1}{n!} = \dfrac{1}{e}$

CONDITIONAL PROBABILITY

Conditional Probability Space

→ Consider a probability space : Sample space S , collection of Events , Probability function P.
Let B be an event with $P(B) > 0$

→ <u>DEFINITION</u> (Conditional Probability space Given B)

<u>Sample space</u> : B
<u>Events</u> : $A \cap B$ for every event A in original space
<u>Probability Function</u> : $P(A \cap B) / P(B)$

denoted by $P(A|B)$ and called conditional probability of A given B.

→ For any event A in original space,

$$P(A \cap B) = P(B) \cdot P(A|B)$$

EXAMPLES

(1) Throw a die

$S = \{1, 2, 3, 4, 5, 6\}$, uniform distribution

$$P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\}) = \frac{1}{6}$$

Event $E = \{2, 4, 6\}$, $P(E) = \frac{1}{2}$

Conditional probability space given E,

- $P(\{2\}|E) = \frac{P(\{2\} \cap E)}{P(E)} = \frac{P(\{2\})}{P(E)} = \frac{1/6}{1/2} = \frac{1}{3}$

- $P(\{4\}|E) = \frac{P(\{4\} \cap E)}{P(E)} = \frac{P(\{4\})}{P(E)} = \frac{1/6}{1/2} = \frac{1}{3}$

- $P(\{1\}|E) = \frac{P(\{1\} \cap E)}{P(E)} = \frac{P(\phi)}{P(E)} = \frac{0}{1/2} = 0$

- $P(\{3\}|E) = P(\{5\}|E) = 0$

- $P(\{2,5\}|E) = \frac{P(\{2,5\} \cap E)}{P(E)} = \frac{P(\{2\})}{P(E)} = \frac{1/6}{1/2} = \frac{1}{3}$

- $P(\{2,3,4\}|E) = \frac{P(\{2,3,4\} \cap E)}{P(E)} = \frac{P(\{2,4\})}{P(E)} = \frac{2/6}{1/2} = \frac{2}{3}$

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(2) Consider a class with 15 students - 4 from state 1, 8 from state 2 and 3 from state 3. Three different students are chosen at random one after another. What is the probability that the selected 3 are from State 1, state 3 and state 1 again in that order?

Solution: Total Students = 15

State 1 → 4 students

State 2 → 8 students

State 3 → 3 students

classmate

Date

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Basically we need to find, $P(E_1 \cap E_2 \cap E_3)$ where

E_1 → selecting student from State 1

E_2 → selecting student from State 3

E_3 → selecting student from State 1

$\therefore$ $P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P(E_2 \cap E_3 \mid E_1)$

$= P(E_1) \cdot P(E_2 \mid E_1) \cdot P(E_3 \mid E_1 \cap E_2)$

$= \frac{\cancel{4}^{2}}{\cancel{15}_{5}} \times \frac{\cancel{3}}{\cancel{14}_{7}} \times \frac{3}{13} = \frac{6}{455}$ Ans.

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(3) A family has two children. What is the probability that both are girls, given that at least one is girl?

Sample space $= \{BB, BG, GG, GB\}$

Event, E_1 → $\overset{\text{at least}}{\text{one}}$ is a girl $= \{BG, GG, GB\}$

~~$P(\{GG\} \cap E_1)$~~ = We need to find,

$P(\{GG\} \mid E_1) = \frac{P(\{GG\} \cap E_1)}{P(E_1)} = \frac{1/4}{3/4} = \frac{1}{3}$

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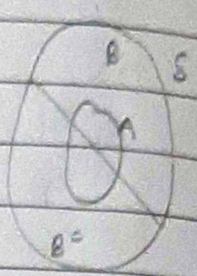
LAW OF TOTAL PROBABILITY

Law of Total Probability



$$P(A) = P(A \cap B) + P(A \cap B^c)$$
$$= P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c)$$

Proof: A: disjoint union of $A \cap B$ and $A \cap B^c$

→ By Axiom 2, $P(A) = P(A \cap B) + P(A \cap B^c)$

→ Using conditional probability on each term above, we get the result.

Example

(1) An economic model predicts that if interest rates rise, then there is a 60% chance that unemployment will increase, but that if interest rates do not rise, then there is only a 30% chance that unemployment will increase. If the economist believes there is a 40% chance that interest rates will rise, what should she calculate is the probability that unemployment will increase?

Solution: Let B be the event that unemployment will increase and A be the event that interest rates rises.

$\therefore \quad P(A) = 0.4 \quad \Rightarrow \quad P(A^c) = 0.6$

$P(B|A) = 0.6 \quad$ and $\quad P(B|A^c) = 0.3$

Thus, $P(B) = P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c)$

$= (0.6 \times 0.4) + (0.3 \times 0.6)$

$= 0.24 + 0.18$

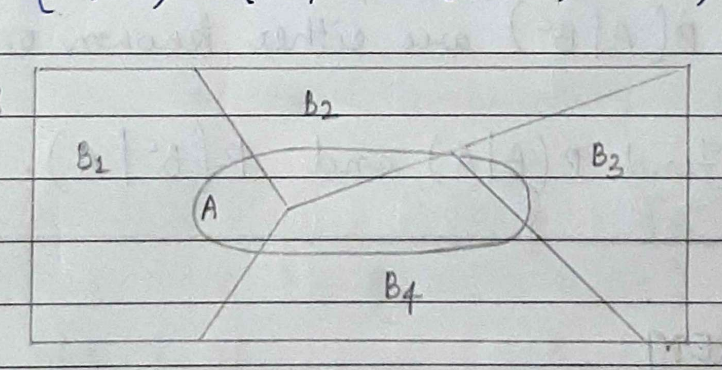
$\Rightarrow \quad P(B) = 0.42$

There is 42% chance that unemployment will increase.

Generally, $B_1, B_2, B_3 \ldots$: Partition of S

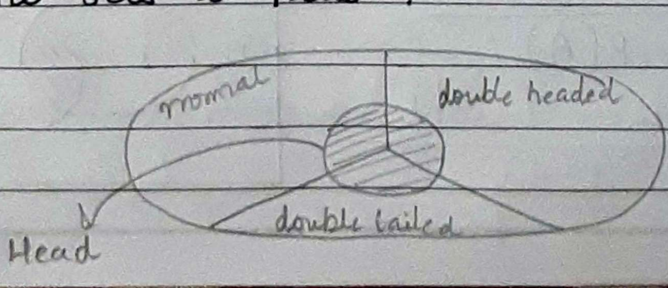
$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) \ldots + P(A \cap B_n)$

$= P(A|B_1) \cdot P(B_1) + P(A|B_2) \cdot P(B_2) + \ldots + P(A|B_n) \cdot P(B_n)$



Example 2 : A man has 5 coins - 2 are double-headed, 1 is double-tailed, 2 are normal. He picks a coin at random and tosses it. What is the probability that he sees a head?

Solution :



$$P(\text{head}) = P(h|dh) \cdot P(dh) + P(h|dt) \cdot P(dt) + P(h|n) \cdot P(n)$$
$$= \left(1 \times \frac{2}{5}\right) + \left(0 \times \frac{1}{5}\right) + \left(\frac{1}{2} \times \frac{2}{5}\right)$$

$$\Rightarrow P(\text{head}) = \frac{2}{5} + \frac{1}{5} = \frac{3}{5} \text{ Ans.}$$

BAYES' THEOREM

Problem

Two events : A and B

$P(A)$ and $P(B)$ are known.

$P(A|B)$ and $P(A|B^c)$ are either known or easy to find.

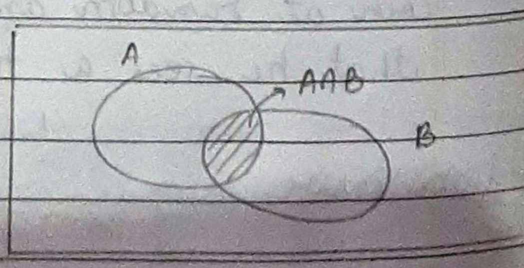
QUESTION : Find $P(B|A)$ and $P(B^c|A)$.

BAYES' THEOREM

A, B : events with $P(A) > 0$, $P(B) > 0$

$$P(A \cap B) = P(B) \cdot P(A|B) = P(A) \cdot P(B|A)$$

$$\Rightarrow P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$



EXAMPLES

(1) 1% of people in a city have swine Flu. Flu Test : 95% of people with Swine Flu test positive, 2% of people without the disease will test positive. A person is randomly chosen from city and tests positive. What is the probability that the person actually has swine Flu?

Solution : Let the events be as follows :

A : person tests positive

B : person has swine flu

$P(B) = 0.01$

$P(B^c) = 0.99$

$P(A|B) = .95$; $P(A|B^c) = 0.02$

∴ From the law of total probability, we have,

$$P(A) = P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c)$$
$$= (0.95 \times 0.01) + (0.02 \times 0.99)$$
$$= 0.0293$$

Applying Bayes' Theorem,

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)} = \frac{0.01 \times 0.95}{0.0293} = 0.3242$$

∴ 32.42% of is the probality percent that person actually has swine flu.

(2) A student attempting an MCQ with 4 choices (of which one is correct) knows the correct answer with probability 3/4. If she does not know, she guesses a random choice. Given that a question was answered correctly, what is conditional probability that she knew the answer?

Sol. Let the events be as follows:

A → answer is correct
B → knows the answer
B^c → doesn't know the answer

Given that,

$$P(A|B) = 1 \quad ; \quad P(A|B^c) = \frac{1}{4} \quad ; \quad P(B) = \frac{3}{4}$$

$$P(B^c) = \frac{1}{4}$$

Using Law of Total Probability,

$$P(A) = P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c)$$
$$= (1 \times 3/4) + (1/4 \times 1/4)$$
$$= 3/4 + 1/16$$
$$\Rightarrow P(A) = \cancel{\frac{7}{8}}\ \frac{13}{16}$$

Applying Bayes' Theorem,

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)} = \frac{3/4 \times 1}{13/16} = \frac{12}{13}$$ Ans.

(3) You first roll a fair die, then toss as many fair coins as the no. that showed on the die. Given that 5 heads are obtained, what is the probability that the die showed 5 ?

Solution : For a fair die,

$$P(1) = P(2) = P(3) = P(5) = P(6) = 1/6$$

$$P(5 \text{ heads} \mid \text{die showed } i) = 0 \quad , \quad i = 1, 2, 3, 4$$

Now, $P(5 \text{ heads} \mid \text{die showed } 5) = \frac{1}{32}$

$$P(5 \text{ heads} \mid \text{die showed } 6) = \frac{6}{64} = \frac{3}{32}$$

$$\therefore P(5 \text{ heads}) = \left(\frac{1}{32} \times \frac{1}{6}\right) + \left(\frac{1}{\cancel{6}_2} \times \frac{\cancel{3}}{32}\right) = \frac{1}{48}$$

Applying Bayes'. Theorem,

$$P(\text{die showed } 5 \mid 5 \text{ heads}) = \frac{P(\text{die showed } 5) \cdot P(5h \mid \text{die show } 5)}{P(5 \text{ heads})}$$

$$= \frac{1/6 \times 1/32}{1/48} = \frac{1}{4}$$

INDEPENDENCE OF EVENTS

Independence of 2 events

Two events A and B are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

→ If $P(B) > 0$, $P(A|B) = P(A)$ [if A & B are independent]
→ Probability of A is unaffected by occurrence of B, i.e. A and B are independent

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Example : Throw A Die

(1) A : even , B : odd
$A = \{2, 4, 6\}$; $B = \{1, 3, 5\}$

$A \cap B = \phi$
$P(A \cap B) = 0 \neq P(A) \cdot P(B)$

* POINTS TO REMEMBER

- Disjoint events are never independent

- Why? If A and B are disjoint and B occurs, A definitely did not occur
  i.e. Occurrence of B definitely impacts conditional probability of A.

- For events to be independent, they should have a non empty intersection.

(2) A: even no. $A = \{2, 4, 6\}$
B: multiple of 3 $B = \{3, 6\}$

$A \cap B = \{6\}$
$P(A \cap B) = \underset{P(A)}{3/6} \times \underset{P(B)}{2/6} = 1/6$

→ A and B are independent.

* TYPICAL POINTS OF CONFUSION

- A and B have an intersection. How can they be independent ?

- There is only 1 throw of die. How can two events be independent when there is only one throw ?

(3) CARD FROM A PACK

A: Card is a spade   B: Card is a king
$A \cap B$ = Card is king of spades.

$$P(A \cap B) = \frac{1}{52} = \frac{13}{52} \times \frac{4}{52} = P(A) \cdot P(B)$$

→ A and B are independent.

TYPICAL POINTS OF CONFUSION :

- Isn't there a spade-king card? How can card being a spade be independent of card being a king when there is a spade king card?

- There is only one card drawn. How can two events be defined and be independent when there is only one card?

MUTUAL INDEPENDENCE OF 3 EVENTS

Events A, B and C are independent mutually if

$P(A \cap B) = P(A) \cdot P(B)$
$P(A \cap C) = P(A) \cdot P(C)$
$P(B \cap C) = P(B) \cdot P(C)$
$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

* Two Constraints

→ A and B are independent, A and C are independent, B and C are independent.

→ Additional constraint on $A \cap B \cap C$ - this is important

MUTUAL INDEPENDENCE OF MULTIPLE EVENTS

Events $A_1, A_2, \ldots, A_n$ are mutually independent if, for all $i1, i2, \ldots, ik$,

$$P(A_{i1} \cap A_{i2} \cap \ldots \cap A_{ik}) = P(A_{i1})\, P(A_{i2}) \ldots P(A_{ik})$$

★ Almost 2^n constraints

→ P(intersection of any subset of events) = Prod. of Prob(events)

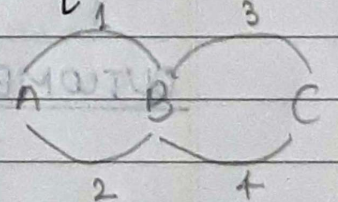
★ Interesting result : A & B are independent, it means, A & B^c are independent

$$\rightarrow P(A \cap B^c) = P(A \setminus B) = P(A) - P(A \cap B)$$
$$= P(A)(1 - P(B)) = P(A) \cdot P(B^c)$$

★ Also, A & B are independent ⇒ A^c & B^c are independent

★ n events are mutually independent ⇒ any subset with or without complementing are independent as well.

Example : Two roads each connect A and B and B and C. Each of the four roads gets blocked with probability p independent of all other roads. What is the probability that there is an open route from A to B given that there is no open route from A to C?



Sol : $P\left(\text{no open route from A to B}\right) = P(1 \text{ is blocked } \& \text{ 2 is blocked})$
$$= p \cdot p = p^2$$

$P\left(\text{open route from A to B}\right) = P(1 \text{ is open or } 2 \text{ is open})$
$$= 1 - p^2$$

$$\therefore P\big(\underbrace{\text{no open route from A to C}}_{E}\big) = P\left(\text{no open route from A to C} \,\middle|\, \underbrace{\text{open route from A to B}}_{F}\right) \cdot P\left(\text{open route from A to B}\right) + P(E|F^c) \cdot P(F^c)$$
$$= p^2(1-p^2) + 1(p^2) = 2p^2 - p^4$$

$$\therefore P(F|E) = \frac{P(E|F) \cdot P(F)}{P(E)} = p^2(1-p^2) / 2p^2 - p^4 = \frac{1-p^2}{2-p^2} \text{ Ans.}$$

BERNOULLI TRIALS

A single Bernoulli Trial

SETTING : Occurence of Event A in a sample space is considered "success". Non-occurrence of A is considered "failure". Let $p = P(A)$

BERNOULLI TRIAL : Sample space is {success, failure} with P{success} = p or,

$\{0, 1\}$ with $\boxed{P(1) = p}$, $\boxed{P(0) = 1 - p}$

This distribution is denoted Bernoulli (p).

Repeated Bernoulli Trials

SETTING : Repeat a Bernoulli Trial multiple times independently.

Perform n independent Bernoulli (p) trials.

OUTCOME : 0 or 1 (Trial 1), 0 or 1 (Trial 2), 0 or 1 (T3) ..... 0 or 1 (Trial n)

Sample Space : 2^n outcomes

Eg : $n = 3 \rightarrow S = \{000, 001, 010, 011, 100, 101, 110, 111\}$

no. of outcomes $= 2^3 = 8$

Probabilities : Use independence

Let $n = 3$.

$$P(000) = P(\text{trial 1 is 0 \& trial 2 is 0 \& trial 3 is 0})$$
$$= (1-p) \times (1-p) \times (1-p) = (1-p)^3$$

$$P(101) = P(\text{T1 is 1 \& T2 is 0 \& T3 is 1})$$
$$= p \times (1-p) \times p = p^2(1-p)$$

EXAMPLE

(1) Toss a fair coin 5 times

Toss a fair coin 5 times.
H is 0 and T is 1
sample space has $2^5 = 32$ equally likely outcomes.

For example,

$$P(HHHHH) = \frac{1}{32}, \quad P(HTHTH) = \frac{1}{32}, \text{ etc.}$$

$$P(0 \text{ tails}) = P(HHHHH) = 1/32 = \frac{5C_0}{32}$$

$$P(1 \text{ tail}) = P(\{THHHH, HTHHH, HHTHH, HHHTH, HHHHT\})$$
$$= 1/32 + 1/32 + 1/32 + 1/32 + 1/32$$
$$= \frac{5}{32} = \frac{5C_1}{32}$$

$P(2 \text{ tails}) = P(\{TTHHH, THTHH, THHTH, THHHT, HTTHH, HTHTH, HTHHT, HHTTH, HHTHT, HHHTT\})$

$\Rightarrow P(2 \text{ tails}) = \frac{10}{32} = \frac{^5C_2}{32}$

$P(3 \text{ tails}) = P(\{HHTTT, HTHTT, HTTHT, HTTTH, THHTT, THTHT, THTTH, TTHHT, TTHTH, TTTHH\})$

$\Rightarrow P(3 \text{ tails}) = \frac{10}{32} = \frac{^5C_3}{32}$

$P(4 \text{ tails}) = P(\{HTTTT, THTTT, TTHTT, TTTHT, TTTTH\})$
$= \frac{5}{32} = \frac{^5C_4}{32}$

$P(5 \text{ tails}) = P(\{TTTTT\}) = \frac{1}{32} = \frac{^5C_5}{32}$

Now,

$P(\text{at least 4 tails}) = P(4 \text{ tails or } 5 \text{ tails})$
$= \frac{5}{32} + \frac{1}{32} = \frac{6}{32}$

$P(\text{at most 3 heads}) = P(0 \text{ or } 1 \text{ or } 2 \text{ or } 3 \text{ heads})$
$= \frac{1}{32} + \frac{5}{32} + \frac{10}{32} + \frac{10}{32} = \frac{26}{32}$

<u>Computing Probabilities for Bernoulli Trials</u>

Perform n independent Bernoulli (p) trials.

Example: $n = 10$

$$p(0110001011) = (1-p)\cdot p\cdot p\cdot(1-p)(1-p)(1-p)\cdot p\cdot(1-p)\cdot p\cdot p$$
$$= p^5(1-p)^5$$

In general,

★ $\boxed{p(b_1 b_2 \ldots b_n) = p^w(1-p)^{n-w}}$, where w = no. of 1's in $b_1 b_2 \ldots b_n$

- The actual sequence of 0s and 1s does not matter. Only the no. of successes matters in the probability computation.

Ex. $p(b_1 b_2 \ldots b_{200} \text{ with } 36 \text{ 1s}) = p^{36}(1-p)^{200-36} = p^{36}(1-p)^{164}$

EXAMPLE: Toss a Biased coin 5 times

H is 0 and T is 1.

$P(H) = 1/3$, $P(T) = 2/3$

$P(HHHHH) = (1/3)^5$, $P(TTTTT) = (2/3)^5$

$$P(1 \text{ tail}) = P(\{THHHH, HTHHH, HHTHH, HHHTH, HHHHT\})$$
$$= 5 \times \left(\frac{2}{3}\right) \times \left(\frac{1}{3}\right)^4$$

$$P(2 \text{ tails}) = P(\{TTHHH, THTHH, THHTH, THHHT, HTTHH, HTHTH, HTHHT, HHTTH, HHTHT, HHHTT\})$$

$\Rightarrow P(2 \text{ tails}) = 10 \times \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right)^3$

$P(3 \text{ tails}) = 10 \times \left(\frac{2}{3}\right)^3 \times \left(\frac{1}{3}\right)^2$

$P(4 \text{ tails}) = 5 \times \left(\frac{2}{3}\right)^4 \times \left(\frac{1}{3}\right)$

BINOMIAL DISTRIBUTION

Binomial (n, p) : Binomial Distribution

Perform n independent Bernoulli (p) trials.

Outcome : $\boxed{\text{no. of successes}}$, which will denote $B(n,p)$ or B in short

Sample Space : $\{0, 1, 2, \ldots, n\}$

Example : $n = 3$

$P(B=0) = P(\text{trials result in } 000) = (1-p)^3$

$P(B=1) = P(\text{trials result in } 001 \text{ or } 010 \text{ or } 100) = 3p(1-p)^2$

$P(B=2) = P(\text{trials result in } 110 \text{ or } 101 \text{ or } 011) = 3p^2(1-p)$

$P(B=3) = P(\text{trials result in } 111) = p^3$

Binomial $(5, p)$

$n = 5$, B = no. of successes in n Bernoulli (p) trials.

$P(B=0) = P(\text{trials result in } 00000) = (1-p)^5$

$P(B=1) = 5p(1-p)^4$

$P(B=2) = 10p^2(1-p)^3$

$P(B=3) = 10p^3(1-p)^2 = P(\text{trials result in 3 1s})$ $\quad {}^5C_3\, p^3(1-p)^2$

$P(B=4) = 5p^4(1-p) = P(\text{trials result in 4 1s})$ $\quad {}^5C_4\, p^4(1-p)$

$P(B=5) = p^5 = P(\text{trials result in 5 1s})$

Binomial (n, p) : small n

PROBABILITY OF NO. OF SUCCESSES

	$B=0$	$B=1$	$B=2$	$B=3$	$B=4$	$B=5$
$n=1$	$1-p$	p				
$n=2$	$(1-p)^2$	$2p(1-p)$	p^2			
$n=3$	$(1-p)^3$	$3p(1-p)^2$	$3p^2(1-p)$	p^3		
$n=4$	$(1-p)^4$	$4p(1-p)^3$	$6p^2(1-p)^2$	$4p^3(1-p)$	p^4	
$n=5$	$(1-p)^5$	$5p(1-p)^4$	$10p^2(1-p)^3$	$10p^3(1-p)^2$	$5p^4(1-p)$	p^5

Binomial (n, p) : Expression for probability

In general, what is $P(B(n,p) = k)$? $\{k = 0, 1, 2 \dots n\}$

$$P(B(n,p) = k) = P(\text{trial results in } b_1 b_2 \dots b_n \text{ with exactly } k \text{ 1s})$$

no. of successes

$$= (\text{no. of } b_1 b_2 \dots b_n \text{ with exactly } k \text{ 1s})\, p^k (1-p)^{n-k}$$

* No. of $b_1 b_2 \dots b_n$ with exactly k 1s $= {}^nC_k = \dfrac{n!}{k!\,(n-k)!}$

$$\therefore \boxed{P[B(n,p) = k] = {}^nC_k\, p^k (1-p)^{n-k}}$$

Graph of Binomial (n,p) : Some Observations

- Starts at $(1-p)^n$ → increases and reaches a peak → falls to p^n
- Where is the peak ? Near np
  → Exactly, floor $(p(n+1))$, i.e, largest integer $\leq p(n+1)$
- $P(B=0 \text{ or } B=1 \text{ or } B=2 \text{ or} \dots B=n) = 1$

$$\Rightarrow P(B=0) + P(B=1) + P(B=2) + \dots + P(B=n) = 1$$

$$\Rightarrow \boxed{(1-p)^n + {}^nC_1\, p (1-p)^{n-1} + {}^nC_2\, p^2 (1-p)^{n-2} + \dots + p^n = 1}$$

EXAMPLES

(1) Each person has a disease with probability 0.1 independently. Out of 100 random persons tested for the disease, what is the probability that 20 persons test positive?

Solution: $p = 0.1$
$n = 100$

$$P\left(B(100, 0.1) = 20\right) = {}^{100}C_{20}\,(0.1)^{20}\,(0.9)^{80}$$

$$=$$

---

(2) Suppose a fair coin is tossed 10 times.

(a) What is the probability that the no. of heads is a multiple of 3?

(b) What is a probability that the no. of heads is even?

Solution: $n = 10$ $p = 0.5$

(a) S_H {: No. of heads is multiple of 3 $= \{0, 3, 6, 9\}$

$$\therefore P\left(B(10, 0.5) \in \{0,3,6,9\}\right) = P(B=0) + P(B=3) + P(B=6) + P(B=9)$$

$$\Rightarrow {}^{10}C_0(0.5)^0(0.5)^{10} + {}^{10}C_3(0.5)^3(0.5)^7 + {}^{10}C_6(0.5)^6(0.5)^4 + {}^{10}C_9(0.5)^9(0.5)^1$$

$$= \left(\frac{1}{2}\right)^{10} + {}^{10}C_3\left(\frac{1}{2}\right)^{10} + {}^{10}C_6\left(\frac{1}{2}\right)^{10} + {}^{10}C_9\left(\frac{1}{2}\right)^{10}$$ Ans.

classmate

Date

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(b) No. of heads is even = $\{0,2,4,6,8,10\}$

$P(\text{no. of heads even}) = P(B=0) + P(B=2) + P(B=4) + P(B=6) + P(B=8) + P(B=10)$

$= {}^{10}C_0 (1/2)^{10} + {}^{10}C_2 (1/2)^{10} + {}^{10}C_4 (1/2)^{10} + {}^{10}C_6 (1/2)^{10} + {}^{10}C_8 (1/2)^{10} + {}^{10}C_{10} (1/2)^{10}$

$= \dfrac{1 + {}^{10}C_2 + {}^{10}C_4 + {}^{10}C_6 + {}^{10}C_8 + 1}{2^{10}}$ Ans.

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(3) A bit (0, 1) sent by alice to Bob gets flipped with $p = 0.1$.

(a) If 5 bits are sent by Alice independently, what is the probability that at most 2 bits gets flipped?

(b) If 10 bits are sent by Alice independently, what is the probability that at most 2 bits gets flipped?

Solution: (a) $P\left(B(5, 0.1) \in \{0,1,2\}\right) = P(B=0) + P(B=1) + P(B=2)$

$= {}^5C_0 (0.9)^5 + {}^5C_1 (0.1)^1 (0.9)^4 + {}^5C_2 (0.1)^2 (0.9)^3$

$= (0.9)^5 + 5(0.1)(0.9)^4 + 10(0.1)^2(0.9)^3$

$=$

(b) $P\left(B(10, 0.1) \in \{0,1,2\}\right) = P(B=0) + P(B=1) + P(B=2)$

$= {}^{10}C_0 (0.9)^{10} + {}^{10}C_1 (0.1)(0.9)^9 + {}^{10}C_2 (0.1)^2 (0.9)^8$

GEOMETRIC DISTRIBUTION

Geometric (p) : Geometric Distribution

Perform independent Bernoulli (p) trials indefinitely.

OUTCOME : No. of trials needed for first success, which we denote $G(p)$ or G.

sample space : {1, 2, 3, 4, ..., (goes on and on)}

$P(G=1)$ = P(first trial is success) = p

$P(G=2)$ = P(first trial is failure & second trial is success)
= $(1-p)\,p$

$P(G=3)$ = P(trial result 001) = $(1-p)^2\,p$

....

$P(G=k)$ = P(first k-1 trials will be failure & k^{th} trial success)
= $(1-p)^{k-1}\,p$

Graph of Geometric Distribution : Observations

- Starts at p and keep falling
- Keeps on decreasing but, if $p<1$, never goes all the way to zero.

What if? $P(G>k) = 1 - P(G \le k)$
$= (1-p)^k$

- $P(G \le k) = P(G=1 \text{ or } G=2 \text{ or} \ldots \text{ or } G=k)$

$\Rightarrow P(G \le k) = P(G=1) + P(G=2) + \ldots + P(G=k)$

$\Rightarrow P(G \le k) = p + (1-p)p + (1-p)^2 p + \ldots + (1-p)^{k-1} p$

$$\boxed{P(G \le k) = 1 - (1-p)^k}$$ Important

EXAMPLE

(1) In Ludo, a player needs to repeatedly throw a die till she gets a 1. What is the probability that she needs lesser than 6 throws? What is the probability that she needs lesser than 11 throws? What is the probability that she needs lesser than 21 throws?

Sol. ~~P(lesser than 6 throws) = P(G=1 or ... or G=5)~~
$\cancel{(1/6 + 5/6) \times (1/6) + (5/6)^2 (1/6)} \times$

$P(\text{lesser than 6 throws}) = P(G=1 \text{ or } \ldots \text{ or } G=5)$
$= 1/6 + (5/6 \times 1/6) + (5/6)^2 (1/6) + (5/6)^3 (1/6) + (5/6)^4 (1/6)$
$= 1 - (5/6)^6 = 0.5981$

$P(\text{lesser than 11 throws}) = 1/6 + (5/6 \times 1/6) + \ldots + (5/6)^{10} (1/6)$
$= 1 - \left(\frac{5}{6}\right)^{\cancel{10}\,11} = 0.8654$

$P(\text{lesser than 21 throws}) = 1 - \left(\frac{5}{6}\right)^{21} = 0.9783$

(2) Player 1 is 40% free throw shooter, while Player 2 is a 70% shooter. Each throw is independent of all previous throws. The 2 players alternate shooting with Player 1 starting till the first basket is scored.

(a) What is the probability that player 1 wins before the 3rd round?

(b) What is the probability that Player 1 wins?

Solution : $p(P1) = 0.4$ $\qquad$ $p(P2) = 0.7$

(a) Favourable outcomes : 1_{P1} , $0_{P1}\ 0_{P2}\ 1_{P1}$

$\therefore\ P(G \leq 3) = (0.4) + (0.6 \times 0.3 \times 0.4) = 0.472$

(b) Favourable Outcomes : 1_{P1} , $0_{P1}\ 0_{P2}\ 1_{P1}$, $0_{P1}\ 0_{P2}\ 0_{P1}\ 0_{P2}\ 1_{P1}$, ...

$$\therefore\ P(\text{player 1 wins}) = (0.4) + (0.6 \times 0.3 \times 0.4) + (0.6 \times 0.3 \times 0.6 \times 0.3 \times 0.4) + (0.6 \times 0.3 \times 0.6 \times 0.3 \times 0.6 \times 0.3 \times 0.4) + \ldots$$

$$= (0.4) + (0.18 \times 0.4) + (0.18)^2 \times 0.4 + (0.18)^3 \times 0.4 + \ldots$$

Using GP formula, we have,

$$P(P1 \text{ wins}) = \frac{0.4}{1 - 0.18} = \frac{0.4}{0.82} = 0.487$$