STATISTICS 2

NOTES BY GAGNEET KAUR

Date September 27, 2021

WEEK 4

Continuous Random Var.

CUMMULATIVE DISTRIBUTION F^N

Definition : The CDF of a random variable X, denoted $F_X(x)$, is a function from $\mathbb{R}$ to $[0,1]$, defined as

$$F_X(x) = P(X \leq x)$$

Properties :
- $F_X(b) - F_X(a) = P(a < X \leq b)$
- F_X : non-decreasing function taking non-negative values
- As $x \to -\infty$, F_X goes to 0
- As $x \to \infty$, F_X goes to 1
- F is continuous from the right

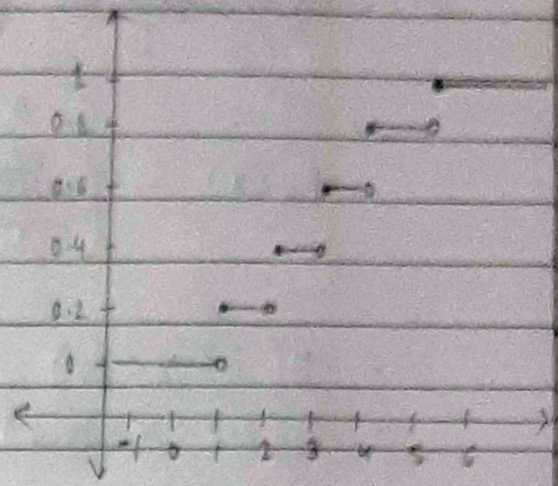
EXAMPLE : Bernoulli Random Variables

$$X \sim \{\overset{1-p}{0}, \overset{p}{1}\} \qquad F_X(x) = \begin{cases} 0 & , x < 0 \\ 1-p & , 0 \leq x < 1 \\ 1 & , 1 \leq x \end{cases}$$

EXAMPLE: Throw a die

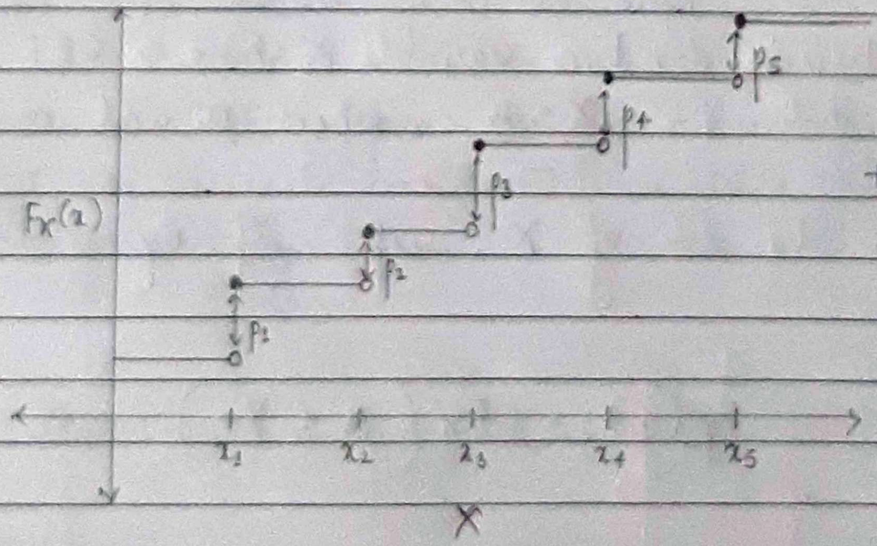
$$X \sim \left\{ \begin{matrix} 1, & 2, & 3, & 4, & 5, & 6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{matrix} \right\}$$

$$F_X(x) = \begin{cases} 0 & , \quad x < 1 \\ 1/6 & , \quad 1 \leq x < 2 \\ 2/6 & , \quad 2 \leq x < 3 \\ 3/6 & , \quad 3 \leq x < 4 \\ 4/6 & , \quad 4 \leq x < 5 \\ 5/6 & , \quad 5 \leq x < 6 \\ 1 & , \quad x \geq 6 \end{cases}$$



CDF of a Discrete Random Variable

$$X \sim \left\{ \begin{matrix} p_1 & p_2 & p_3 & p_4 & p_5 \\ x_1, & x_2, & x_3, & x_4, & x_5 \end{matrix} \right\}$$



the jump → PMF

Computing Probability of Intervals using CDF

$X \sim \text{Uniform}\{1, 2, \ldots, 100\}$

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ k/100, & k \le x < k+1, \quad k = 1, 2, \ldots, 99 \\ 1, & x \geq 100 \end{cases}$$

- $P(3 < X \le 10) = F_X(10) - F_X(3) = 10/100 + 3/100 = 7/100$
- $P(3.2 < X \le 10.6) = F_X(10.6) - F_X(3.2) = 7/100$

- $P(X \le 17) = F_X(17) = 17/100$
- $P(X \le 17.3) = F_X(17.3) = 17/100$

- $P(X > 87) = 1 - P(X \le 87) = 1 - 87/100 = 13/100$
- $P(X > 87.4) = 1 - P(X \le 87.4) = 13/100$

\# Let a discrete random variable X have a CDF F_X. Assume that $Y = \dfrac{X - \mu}{\sigma}$, where μ and σ are the mean & std. dev of X, resp. If F_Y is the CDF of Y, then

$$F_Y(y) = F_X(\mu + Y\sigma)$$

Date
sept 28, 2021

APPROXIMATING DISCRETE CDF WITH A CONTINUOUS FUNCTION

Cummulative Distribution Function

A function $F : \mathbb{R} \to [0,1]$ is said to be a CDF if

(i) F is a non-decreasing f^n taking values b/w 0 & 1
(ii) As $x \to -\infty$, F goes to 0
(iii) As $x \to \infty$, F goes to 1
(iv) Technical : F is continuous from the right

- A general CDF need not be like a CDF of a discrete random variable
  - → No need for step-like structure
  - → Can be smooth and continuous

- Continuous CDFs appear to be close approximations to CDFs of discrete random variables particularly when alphabet grows.

Example: Binomial using Continuous CDF

$X \sim$ Binomial $(100, 0.6)$

$$F_X(k) = \sum_{j=0}^{k} {}^{100}C_j \, (0.6)^j \, (0.4)^{n-j}$$

$$F(x) = \frac{1}{1 + \exp\left(\frac{-1.65451\,(x-60)}{\sqrt{24}}\right)}$$

- $P(40 < X \le 50) = 0.0271$, $F(50) - F(40) = 0.0318$
- $P(50 < X \le 60) = 0.5108$, $F(60) - F(50) = 0.4670$
- $P(60 < X \le 70) = 0.4473$, $F(70) - F(60) = 0.4670$
- $P(70 < X \le 80) = 0.0148$, $F(80) - F(70) = 0.0318$

CDF's & RANDOM VARIABLES

Theorem (Random Variable with CDF $F(x)$)
Given a valid CDF $F(x)$, there exists a random variable X taking values in $\mathbb{R}$ such that

$$P(X \le x) = F(x)$$

PROPERTIES

- $P(a < X \le b) = F(b) - F(a)$
- If $F(x)$ rises from F_1 to F_2 at x_1, $P(X = x_1) = F_2 - F_1$
- If $F(x)$ is continuous at x_0, $P(X = x_0) = 0$

CONTINUOUS RANDOM VARIABLE

Definition: A random variable X with CDF $F_X(x)$ is said to be a continuous random variable if $F_X(x)$ is continuous at every x.

- CDFs has no jumps or steps
- So, $P(X = x) = 0$ for all x

- Probability of X falling in an interval will be non-zero.

$$P(a < X \le b) = F(b) - F(a)$$

- Since $P(X = a) = 0$ & $P(X = b) = 0$, we have

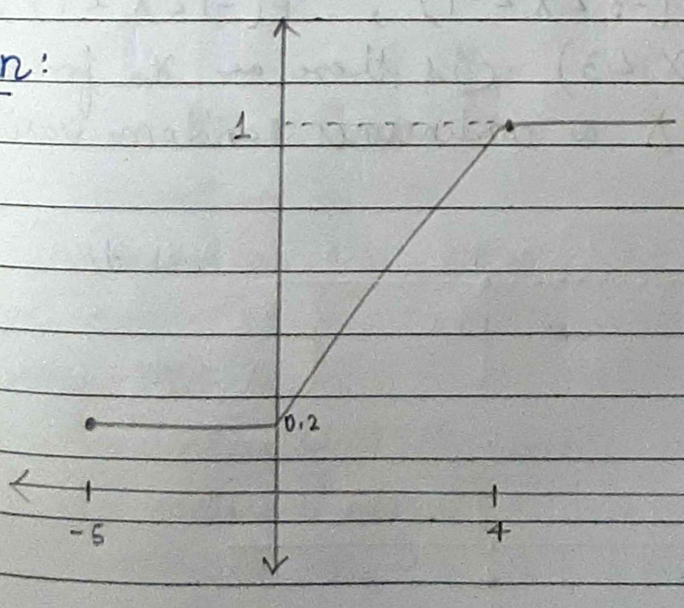
$$P(a \le X \le b) = P(a \le X < b) = P(a < X \le b) = P(a < X < b)$$

Problem : Consider a random variable X with CDF

$$F_X(x) = \begin{cases} 0 & , x < -5 \\ 0.2 & , x < 0 \ \& \ x \geq -5 \\ 0.2 + 0.2x & , 0 \le x < 4 \\ 1 & , x \geq 4 \end{cases}$$

Find $P(X < -3)$, $P(-3 < X < -1)$, $P(-1 < X < 1)$, $P(X \le -3)$, $P(X > -2)$, $P(X \geq 3)$, $P(0 \le X < 3)$. Is there an X_0 for which $P(X = X_0) > 0$? Is X a continuous random variable?

Solution:



$\Rightarrow P(X \le -3) = F(-3) = 0.2$

$\Rightarrow P(X = -3) = 0$

$\Rightarrow P(X < -3) = 0.2$

$\Rightarrow P(-3 < X < -1) = F(-1) - F(-3)$
$= 0.2 - 0.2 = 0$

$\Rightarrow P(-1 < X < 1) = F(1) - F(-1)$
$= 0.4 - 0.2$
$= 0.2$

$P(X > -2) = 1 - P(X \leq -2) = 1 - 0.2 = 0.8$

$P(X \geq 3) = P(X > 3) = 1 - P(X \leq 3) = 1 - 0.2 = 0.8$

$P(0 \leq X < 3) = F(3) - F(0) = 0.8 - 0.2 = 0.6$

Yes, the is an x_0, i.e, -5 for which $P(X = -5) > 0$, i.e,

$$P(X = -5) = 0.2$$

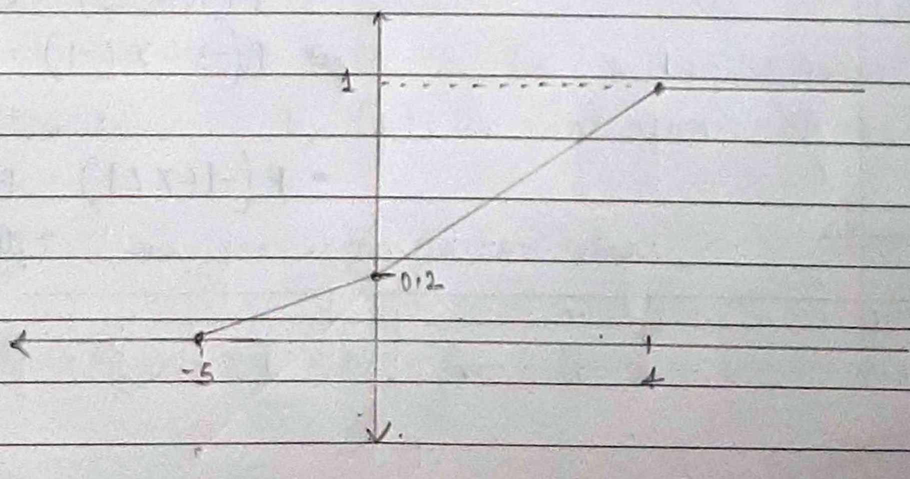
Thus, X is not a continuous random variable.

Problem: Consider a random variable X with CDF

$$F_X(x) = \begin{cases} 0 & , \quad x < -5 \\ 0.04x + 0.2 & , \quad -5 \leq x < 0 \\ 0.2 + 0.2x & , \quad 0 \leq x < 4 \\ 1 & , \quad x \geq 4 \end{cases}$$

Find $P(X < -3)$, $P(-3 < X < -1)$, $P(-1 < X < 1)$, $P(X \leq -3)$, $P(0 \leq X < 3)$. Is there an x_0 for which $P(X = x_0) > 0$? Is X a continuous random variable?

Solution:



$p(X < -3) = 0.08$

$p(-3 < X < -1) = F(-1) - F(-3) = 0.16 - 0.08 = 0.08$

$p(-1 < X < 1) = F(1) - F(-1) = 0.4 - 0.16 = 0.24$

$p(X \leq -3) = 0.08$

$p(0 \leq X < 3) = 0.8 - 0.2 = 0.6$

- No, there is no x_0 for which $P(X = x_0) > 0$.
  Thus, X is a continuous R.V.

PROBABILITY DENSITY FUNCTION

Definition: A continuous random variable X with CDF $F_X(x)$ is said to have a PDF $f_X(x)$ if, for all x_0,

$$F_X(x_0) = \int_{-\infty}^{x_0} f_X(x)\, dx$$

- CDF is the integral of PDF
  → Derivative of the CDF (wherever it exists) is usually taken as the PDF

- Why PDF?
  ↳ Value of PDF around $f_X(x_0)$ is related to X taking value around x_0
  → Higher the PDF, higher the chance that X lies there
  → Contrast with value of CDF at x_0, $F_X(x_0)$

Area under the curve of f_X over the entire range of X is always 1.
$\int_a^b f_X(x)\,dx = P(a \le X \le b)$

→ PDF is much easier in probability computations.

Properties of PDF

A function $f: \mathbb{R} \to \mathbb{R}$ is said to be a density f^n if
(i) $f(x) \geq 0$
(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$
(iii) $f(x)$ is piecewise continuous

→ Given a density function f, there is a continuous random variable X with PDF as f

→ Support of the random variable X with PDF f_X is
$\text{supp}(X) = \{x : f_X(x) > 0\}$

→ supp (X) contains intervals in which X can fall with positive probability
→ Remember: $P(X = x) = 0$ for a continuous random var.

→ For a random variable X with PDF f_X, an event A is a subset of the real line and its probability is computed as $P(A) = \int_A f(x)\,dx$.

Problem: consider the function $f(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

Show that f is a PDF. Consider a random variable X with density f. Find $P(X = 1/5)$, $P(X = 2/5)$, $P(X \in [1/5 - \epsilon, 1/5 + \epsilon])$, $P[X \in [2/5 - \epsilon, 2/5 + \epsilon]]$.

Solution: $P(X = 1/5) = 0$; $P(X = 2/5) = 0$

$$P(X \in [1/5 - \epsilon, 1/5 + \epsilon]) = \int_{1/5-\epsilon}^{1/5+\epsilon} 3x^2 \cdot dx = [x^3]_{1/5-\epsilon}^{1/5+\epsilon}$$

$$= \left(\frac{1}{5} + \epsilon\right)^3 - \left(\frac{1}{5} - \epsilon\right)^3 = \frac{6}{25}\epsilon + 2\epsilon^3$$

$$P(X \in [2/5 - \epsilon, 2/5 + \epsilon]) = \int_{2/5-\epsilon}^{2/5+\epsilon} 3x^2\, dx = [x^3]_{2/5-\epsilon}^{2/5+\epsilon} = \left(\frac{2}{5} + \epsilon\right)^3 - \left(\frac{2}{5} - \epsilon\right)^3$$

$$= \frac{24}{25}\epsilon + 2\epsilon^3$$

Problem: Consider a R.V. X with density $f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

Find $P(X \in [0.1, 0.3])$, $P(X \in (0.1, 0.03])$, $P(X \in [0.1, 0.03))$, $P(X \in (0.1, 0.03))$.

Solution: $P(0.1 \le X \le 0.3) = \int_{0.1}^{0.3} 2x\, dx = [x^2]_{0.1}^{0.3} = 0.09 - 0.01 = 0.08$

$P(0.8 \le X \le 1) = (1)^2 - (0.8)^2 = 0.36$

Problem: Consider the f^n. Find k such that $f(x)$ is valid PDF.

$$f(x) = \begin{cases} k, & 0 \le x < 1/4 \\ 2k, & 1/4 \le x < 3/4 \\ 3k, & 3/4 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_0^1 f(x) \cdot dx = 1$$

$$\Rightarrow \int_0^{1/4} k \cdot dx + \int_{1/4}^{3/4} 2k\, dx + \int_{3/4}^{1} 3k \cdot dx = 1$$

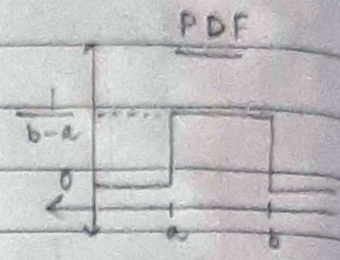
$$\Rightarrow \frac{1}{4}k + 2k \cdot \frac{1}{2} + 3k \cdot \frac{1}{4} = 1 \Rightarrow \boxed{k = \frac{1}{2}}$$

Date September 29, 2021

COMMON DISTRIBUTIONS

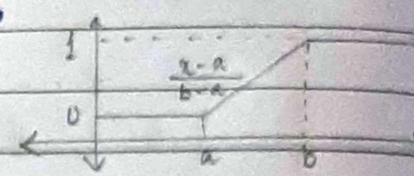
1. Uniform Distribution

$X \sim$ Uniform $[a, b]$



- PDF $\quad f_X(x) = \begin{cases} 1/b-a & , a < x < b \\ 0 & , \text{otherwise} \end{cases}$

- CDF $\quad F_X(x) = \begin{cases} 0 & , x \le a \\ \frac{x-a}{b-a} & , a < x < b \\ 1 & , x \ge b \end{cases}$



Example: Suppose $X \sim$ Uniform $[-10, 10]$. Find $P(-3 \le X \le 2)$, $P(5 < |X| < 7)$, $P(1-\epsilon < X < 1+\epsilon)$, $P(9-\epsilon < X < 9+\epsilon)$, $P(X>7 \mid X>3)$.

Solution: $f(x) = \begin{cases} 1/20 & , -10 < x < 10 \\ 0 & , \text{otherwise} \end{cases}$

$$P(-3 \le X \le 2) = \int_{-3}^{2} \frac{1}{20} \cdot dx = \left[\frac{x}{20}\right]_{-3}^{2} = \left[\frac{2}{20}\right] - \left[\frac{-3}{20}\right] = \frac{5}{20}$$

$$P(5 < |X| < 7) = P(5 < X < 7) + P(-7 < X < -5)$$
$$= \frac{2}{20} + \frac{2}{20} = \frac{4}{20} = \frac{1}{5}$$

$$P(X>7 \mid X>3) = \frac{P(X>7 \text{ and } X>3)}{P(X>3)} = \frac{P(X>7)}{P(X>3)} = \frac{3/20}{7/20} = \frac{3}{20}$$
$$= \frac{3}{7}.$$

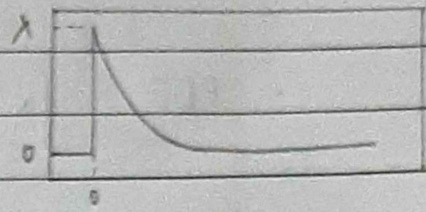
$P(X > s+t \mid X > s) = e^{-t}$

independent of s

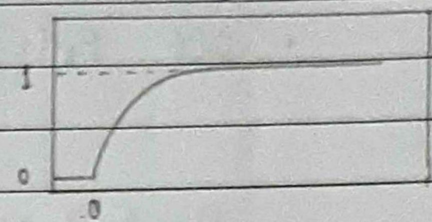
2. Exponential Distribution

$X \sim Exp(\lambda)$

PDF



CDF



- PDF $\quad f_X(x) = \begin{cases} \lambda \exp(-\lambda x), & x > 0 \\ 0, & \text{otherwise} \end{cases}$

- CDF $\quad F_X(x) = \begin{cases} 0, & x \leq 0 \\ 1 - \exp(-\lambda x), & x > 0 \end{cases}$

Example: Suppose $X \sim Exp(2)$. Find $P(5 < X < 7)$, $P(1-\epsilon < X < 1+\epsilon)$, $P(9-\epsilon < X < 9+\epsilon)$, $P(X > 4)$, $P(X > 7 \mid X > 3)$.

Solution: $f_X(x) = \begin{cases} 2\exp(-2x), & x > 0 \\ 0, & \text{otherwise} \end{cases}$

$$P(5 < x < 7) = \int_5^7 2e^{-2x}\,dx = \left[-e^{-2x}\right]_5^7 = -e^{-14} + e^{-10}$$
$$= e^{-10} - e^{-14}$$

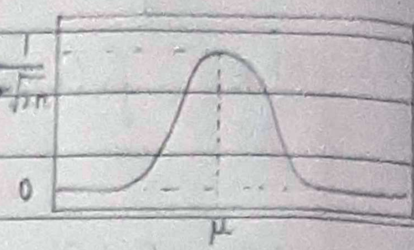
$$P(X > 4) = 1 - P(X \leq 4) = 1 - (1 - e^{-8}) = e^{-8}$$

$$P(X > 7 \mid X > 3) = \frac{P(X > 7)}{P(X > 3)} = \frac{e^{-14}}{e^{-6}} = e^{-8}$$

3. Normal Distribution

$X \sim \text{Normal}(\mu, \sigma^2)$

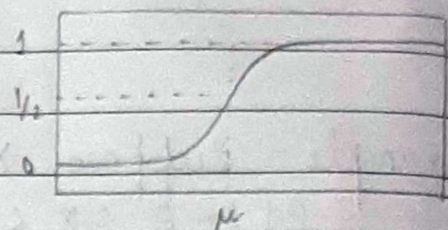
PDF



- PDF $\quad f_X(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right)$

- CDF $\quad F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$

CDF



- Standard Normal : Normal (0, 1)

Probability Computations with Normal Distribution

★ CDF of $X \sim \text{Normal}(\mu, \sigma^2)$ does not have a closed form expression

★ Standardization : If $X \sim \text{Normal}(\mu, \sigma^2)$, then $\dfrac{(X-\mu)}{\sigma} \sim \text{Normal}(0, 1)$

→ $Z \sim \text{Normal}(0, 1)$, PDF : $f_Z(z) = \dfrac{1}{\sqrt{2\pi}} \exp(-z^2/2)$

CDF : $F_Z(z) = \int_{-\infty}^{z} \dfrac{1}{\sqrt{2\pi}} \exp(-u^2/2)\, du$

★ How to compute probabilities for a normal distribution?
→ convert probability computation to that of a standard normal
→ use normal tables or computing systems

Example: Suppose $X \sim$ Normal $(2, 5)$. Find $P(X<5)$, $P(X<10)$, $P(X<-5)$, $P(X<-10)$, $P(X>5)$, $P(X>10)$.

Solution: $Z = \frac{X-2}{\sqrt{5}} \sim$ Normal $(0,1)$ $\quad \{X = \sqrt{5}Z + 2\}$

Assume $F_Z(z)$ is known.

$X<5 \leftrightarrow \sqrt{5}Z+2 < 5 \leftrightarrow Z < 3/\sqrt{5}$

$\therefore P(X<5) = P(Z < 3/\sqrt{5}) = F_Z(3/\sqrt{5})$

Similarly for other probabilities.

Date: September 30, 2021

Page No. | Date | YOUVA

Functions of Continuous Random Variable

GENERAL CASE : CDF of $g(X)$

- Suppose X is a continuous random variable ~~random~~ with CDF F_X and PDF f_X

- Suppose $g : \mathbb{R} \to \mathbb{R}$ is a function

Then, $Y = g(X)$ is a random variable with CDF F_Y determined as follows :

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P\left(X \in \{x : g(x) \leq y\}\right)$$

- How to evaluate the above probability?

→ Convert the subset $A_y = \{x : g(x) \leq y\}$ into intervals in real line

→ Find the probability that X falls in those intervals

→ $F_Y(y) = P(X \in A_y) = \int_{A_y} f_X(x)\, dx$

- If F_Y has no jumps, you may be able to differentiate and find a PDF.

MONOTONIC , Differentiable Functions

Theorem : Suppose X is a continuous random variable with PDF f_X. Let $g(x)$ be monotonic for $x \in supp(X)$ with derivative $g'(x) = \frac{d\,g(x)}{dx}$.
Then, the PDF of $Y = g(X)$ is

$$f_Y(y) = \frac{1}{|g'(g^{-1}(y))|} f_X(g^{-1}(y))$$

- Translation:

$Y = X + a$

$f_Y(y) = f_X(y - a)$

- Scaling:

$Y = aX$

$f_Y(y) = \frac{1}{|a|} f_X(y/a)$

- Affine :

$Y = aX + b$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{(y-b)}{a}\right)$$

Affine transformation of Normal Distribution

(1) $X \sim$ Normal $(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

Now, $Y = \sigma X + \mu$

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

$\therefore Y \sim \text{Normal}(\mu, \sigma^2)$

(2) $X \sim \text{Normal}(\mu, \sigma^2)$

$\therefore Y = \dfrac{X-\mu}{\sigma}$

$\Rightarrow Y \sim \text{Normal}(0, 1)$

Thus, Affine transfomation of a normal R.V. is normal.

PROBLEMS

(1) Let $X \sim \text{Exp}(\lambda)$. Find PDF of X^2

Sol: We know, for Exp. Dist., $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$

$\rightarrow \text{supp}(X) = \{x : x > 0\}$

$g(x) = x^2 \quad , \quad g'(x) = 2x \quad , \quad g^{-1}(y) = \sqrt{Y}$

$\Rightarrow f_Y(y) = \dfrac{1}{2\sqrt{y}} \lambda e^{-\lambda\sqrt{Y}} \quad , \quad Y > 0 \qquad \left[f_Y(y) = \dfrac{1}{g'(g^{-1}(y))} f_X(g^{-1}(y)) \right]$

---

(2) Let $X \sim \text{Unif}[-3, 1]$. Find the PDF of X^2.

Page No.:
Date:

Solution: $supp(X) = [-3, 1]$

$g(x) = x^2$ is not monotonic in $supp(X)$

$Y = X^2 \in [0, 9]$

$F_Y(y) = P(Y \le y) = P(X^2 \le y)$

- $y \in [0,1]$: $(x^2 \le y) \Rightarrow -\sqrt{y} < X < \sqrt{y} \Rightarrow F_Y(y) = \frac{2\sqrt{Y}}{4}$
- $y \in [1, 9]$: $(x^2 \le y) \Rightarrow -\sqrt{y} < X < 1 \Rightarrow F_Y(y) = \frac{1+\sqrt{Y}}{4}$

$$f_Y(y) = \begin{cases} 1/2 \cdot 1/2\sqrt{Y} & , \quad 0 < y < 1 \\ 1/4 \cdot 1/2\sqrt{Y} & , \quad 1 < y < 9 \end{cases}$$

---

(3) Let $X \sim$ Uniform $[-3, 1]$. Find the PDF of $\max(X, 0)$

$$g(x) = \max(X, 0) = \begin{cases} 0 & , \quad -3 < x < 0 \\ x & , \quad 0 < x < 1 \end{cases}$$

$x \in supp(X) = [-3, 1]$

- $y < 0$: $F_Y(y) = P(y \le \text{negative values}) = 0$
- $y = 0$: $F_Y(y) = P(Y \le 0) = P(Y = 0) = P(g(x) = 0)$
  $= P(-3 < X < 0) = 3/4$
- $0 < y < 1$: $F_Y(y) = 3/4 + Y/4$
- $y > 1$: $F_Y(y) = 1$

Expected Value : C.R.V.

continuous random variable

Expected Value : Function of a continuous R.V.

Let X be a continuous random variable with density $f_X(x)$. Let $g: \mathbb{R} \to \mathbb{R}$ be a function. The expected value of $g(X)$, denoted $E[g(X)]$, is given by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx$$

whenever the above integral exists.

- If X is discrete with range T_X and PMF p_X,

$$E[g(X)] = \sum_{x \in T_X} g(x)\, p_X(x)$$

→ Summation in discrete case is replaced by integration in continuous case.

- The integral may diverge to $\pm\infty$ or may not exist in some cases.

MEAN & VARIANCE

X : continuous random variable

- Mean, denoted $E[X]$ or μ_X or simply μ

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

→ Mean is the average or expected value of X

- Variance, denoted $Var(X)$ or σ_X^2 or simply σ^2

$$Var(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\, dx$$

→ Variance is a measure of spread of X about its mean
→ $Var(X) = E[X^2] - (E[X])^2$

EXAMPLES of Mean & Variance

(1) $X \sim$ Uniform $[a, b]$

$$f_X(x) = \frac{1}{b-a}, \quad a < x < b$$

$$E[X] = \frac{a+b}{2} \quad ; \quad Var(X) = \frac{(b-a)^2}{12}$$

(2) $X \sim Exp(\lambda)$

$$f_X(x) = \lambda \exp(-\lambda x), \quad x > 0$$

$$E[X] = 1/\lambda \quad ; \quad Var(X) = \frac{1}{\lambda^2}$$

(3) $X \sim$ Normal (μ, σ^2)

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-(x-\mu)^2/2\sigma^2\right)$$

$$E[X] = \mu \quad ; \quad Var(X) = \sigma^2$$

Bounds

MARKOV & CHEBYSHEV INEQUALITIES

Markov Inequality

→ X : continuous random variable with mean μ
→ supp(X): non-negative, i.e., $P(X<0) = 0$

$$P(X > c) \le \frac{\mu}{c}$$

Chebyshev Inequality

→ X : continuous random variable with mean μ and variance σ^2

$$P(|X-\mu| \geq k\sigma) \le \frac{1}{k^2}$$

Probability space & its axioms

• Discrete Case

→ Sample space : finite or countable set
→ Events : power set of sample space
→ Probability f^n : PMF

- <u>Continuous Case</u>

→ Sample space : interval of real line

→ Events : intervals in the sample space along with their complements and countable unions.
  * This avoids some 'bizzarre' subsets that defy our sense of measure.

→ Probability Function : function from intervals inside sample space to [0,1] satisfying the axioms.
  * Possibly only if $P(X=x) = 0$

PROBLEMS

(1) A continuous random variable X has PDF $f_X(x)$,

$$f_X(x) = \begin{cases} 1-|x| & , -1 \leq x \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

Find the CDF of X, $E[X]$ and $Var(X)$.

<u>Solution</u>:

$$F_X(x) = \begin{cases} 0 & , x < -1 \\ \frac{1}{2} + x + \frac{x^2}{2} & , -1 \leq x < 0 \\ \frac{1}{2} + x - \frac{x^2}{2} & , 0 \leq x < 1 \\ 1 & , x \geq 1 \end{cases}$$

$$\therefore \quad E[X] = \int_{-1}^{1} x \cdot f(x)\,dx = \int_{-1}^{0} x(1+x)\,dx + \int_{0}^{1} x(1-x)\,dx = 0$$

and $\quad Var(X) = E[X^2] = \frac{1}{6}$

(2) A continuous random variable X has PDF

$$f_X(x) = \begin{cases} \frac{1}{2}\cos x & , -\pi/2 \le x \le \pi/2 \\ 0 & , \text{otherwise} \end{cases}$$

Find the CDF of X, $E[X]$, $Var(X)$.

Solution:

$$F(x) = \begin{cases} 0 & , x < -\pi/2 \\ ? & -\pi/2 \le x \le \pi/2 \\ 1 & , x > \pi/2 \end{cases}$$

for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$,

$$F_X(x) = \int_{-\pi/2}^{x} f(x)\cdot dx = \int_{-\pi/2}^{x} f_x(u)\,du = \frac{1}{2}\Big[\sin x\Big]_{-\pi/2}^{x}$$

$$\Rightarrow F_X(x) = \frac{1}{2}\left[\sin x - \sin\left(\frac{-\pi}{2}\right)\right] = \frac{1+\sin x}{2} \quad , \frac{-\pi}{2} \le x \le \frac{\pi}{2}$$

Now, $E[X] = 0$

and $Var(X) = E[X^2] = \int_{-\pi/2}^{\pi/2} x^2 \cdot \frac{1}{2}\cos x\, dx = \frac{\pi^2}{4} - 2$