~ Mathematics 2 ~

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Dated :
Page :

WEEK 2

Limits & Continuity

CONTINUITY OF A FUNCTION

The function f is said to be continuous if it is continuous at all points in its domain i.e. for all points 'a' for which f(a) is defined.

$$\lim_{x \to a} f(x) = f(a) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Algebraically this means : if for a sequence of real numbers $\{a_n\}$ the limit $\lim a_n$ exists, then so does the limit $\lim f(a_n)$ and $\lim f(a_n) = f(\lim a_n)$

We can think of continuity of 'f' as being able to draw the graph of 'f' without lifting our pencils or equivalently that there are no jumps or breaks in the graph of the function.

EXAMPLES : Polynomials, rational functions with non-zero denominators, e^x, $\log(x)$; $x > 0$, $\sin(x)$, $\cos(x)$, etc.

ALGEBRA OF CONTINUOUS FN.

Theorem 1 : Suppose f and g be two real functions continuous at a real no. c. Then,

1. $f+g$ is continuous at $x=c$
2. $f-g$ is continuous at $x=c$
3. $f \cdot g$ is continuous at $x=c$
3. f/g is continuous at $x=c$ $[g(c) \neq 0]$

Theorem 2 : Suppose f and g are real valued functi-ons such that $(f o g)$ is defined at c. If 'g' is continuous at c and if 'f' is continuous at $g(c)$, then $(f o g)$ is continuous at c.

~ QUESTIONS ~

1. Examine whether the function f given by $f(x) = x^2$ is continuous at $x=0$.

Solution : for a f^n to be continuous it must satisfy the following :

$$\lim_{x \to a} f(x) = f(a)$$

$\therefore$ LHS $= \lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0^2 = 0$

RHS $= f(0) = (0)^2 = 0$

Dated: ____
Page: ____

Thus, LHS = RHS
Hence, f is continuous at 0.

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2. Discuss the continuity of the function f given by $f(x) = |x|$ at $x = 0$.

Solution: By definition $f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

Clearly the function is defined at 0 and $f(0) = 0$.

Left Hand Limit of f at 0 is
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0$$

Right Hand Limit of f at 0 is
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$$

Thus, the LHL, RHL and the value at fn coincide at $x = 0$. Hence, f is continuous at 0.

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3. Discuss the continuity of the function f given by $f(x) = x^3 + x^2 - 1$.

Solution: Clearly f is defined at every real number c and its value at c is $c^3 + c^2 - 1$.

We also know that,

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x^3 + x^2 - 1) = c^3 + c^2 - 1$$

Thus, $f(c) = \lim_{x \to c} f(x)$ and hence f is continuous at every real number. This means f is a continuous fn.

Date September 15, 2021

Dated :
Page :

Differentiability

Derivatives → Slope of tangent / Rate of Change

DEFINITION 1: suppose 'f' is a real valued function and 'a' is a point in its domain of definition. The derivative of f at a is defined by

$$\boxed{\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

provided this limit exists. Derivative of $f(x)$ at a is denoted by $f'(a)$

★ Observe that $f'(a)$ quatifies the change in $f(x)$ at a with respect to x.

DEFINITION 2: First Principle Of Derivative

Suppose f is a real valued function, the function defined by $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ wherever the limit exists is defined to be derivative of f at x and is denoted by $f'(x)$. It is also known as first principle of derivative.

The process of finding derivative of a function is called 'differentiation'.

Dated :
Page :

Thus, $$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

→ Domain of definition of $f'(x)$ is wherever the above limit exists.

→ Different notations used : $f'(x)$, $\frac{d}{dx}\left(f(x)\right)$,

if $f(x) = y \Rightarrow \frac{dy}{dx}$, $D\left[f(x)\right]$.

→ Derivative of f at $x = a$ is denoted by :

$$\left.\frac{d}{dx} f(x)\right|_a \quad ; \quad \left.\frac{df}{dx}\right|_a \quad ; \quad \left(\frac{df}{dx}\right)_{x=a}$$

ALGEBRA OF DERIVATIVE OF F^N

(1) $$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

(2) $$\frac{d}{dx}\left[f(x) - g(x)\right] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

Product Rule (3) $$\frac{d}{dx}\left[f(x) \cdot g(x)\right] = \frac{d}{dx} f(x) \cdot g(x) + f(x) \cdot \frac{d}{dx} g(x)$$

Quotient Rule (4) $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{d/dx\, f(x) \cdot g(x) - d/dx\, g(x) \cdot f(x)}{\left[g(x)\right]^2}$$

Geometrically, $f(x)$ is said to be derivable at $x=a$ if unique tangent of finite slope will exist at $x=a$.

Dated:

Page:

Differentiability implies Continuity

→ If a function f is differentiable at a point c, then it is also continuous at that point.

PROOF: Since f is differentiable at c, we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

But for $x \neq c$, we have

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c} \cdot (x - c)$$

$$\therefore \lim_{x \to c} \left[f(x) - f(c) \right] = \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \cdot (x - c) \right]$$

$$\Rightarrow \lim_{x \to c} \left[f(x) \right] - \lim_{x \to c} \left[f(c) \right] = \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \right] \cdot \lim_{x \to c} \left[(x - c) \right]$$

$$= f'(c) \times 0 = 0$$

$$\Rightarrow \lim_{x \to c} f(x) = \lim_{x \to c} f(c)$$

$$\Rightarrow \lim_{x \to c} f(x) = f(c)$$

Hence, f is continuous at $x = c$.

Dated :
Page :

Some Useful Rules

(1) Product Rule : $(u \cdot v)' = u'v + v'u$

(2) Quotient Rule : $\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$

(3) Chain Rule : Let f be a real valued function that is a composition of two of functions, i.e, u and v. So, $f = v \circ u$

$$\therefore \frac{df}{dx} = \frac{du}{dx} \cdot \frac{dv}{du}$$

first derivative of complete function

or $f(g)'(x) = f'(g(x)) \cdot g'(x)$ then derivation of inner f^n.

Some Standard Derivatives

1. $(x^n)' = nx^{n-1}$
2. $(\sin x)' = \cos x$
3. $(\cos x)' = -\sin x$
4. $(\tan x)' = \sec^2 x$
5. ~~ta~~ $\sin^2 x = \sin 2x$
6. $(e^x)' = e^x$
7. $[\ln(x)]' = \frac{1}{x}$

Indeterminate limits

Let $f(x)$ and $g(x)$ be functions and suppose $f(x)$ and $g(x)$ are defined on an interval around the point c.

- Further, suppose $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ or both limits diverge to ∞ or both limits diverge to $-\infty$.

- Suppose we are interested in computing $\lim_{x \to a} \frac{f(x)}{g(x)}$.

Then we cannot use the quotient rule to compute it since the quotient rule yields an inderminate form $\left(\text{i.e.}, \frac{0}{0}, \frac{\infty}{\infty}, -\frac{\infty}{\infty} \right)$

In the situation, we can try and use L' Hôpital's rule.

~ L' Hospital's Rule ~

★ In this situation of the inderminate form, suppose the following conditions hold:

(1) $f'(x)$ and $g'(x)$ exist on this interval.
(2) $g'(x) \neq 0$ on this interval
(3) $L = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Then, $$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = L = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

For Example,

$$\lim_{x \to 0} \frac{\log_e(1+x)}{x} \quad \left[\text{It's of the form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{1/1+x}{1} = \lim_{x \to 0} \frac{1}{1+x} = \frac{1}{1+0} = 1$$

More Examples

(1) $$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x-2}$$

Sol. Its of the form $\frac{0}{0}$.

We will apply L'Hospital's Rule.

$$\therefore \lim_{x \to 2} \frac{2x - 5}{1-0} = \frac{2(2) - 5}{1} = 4-5 = -1$$

$\overset{0}{(b)'}e^x + (e^x)'b$

(2) $$\lim_{x \to 0} \frac{\sin(x)}{x}$$

$$= \lim_{x \to 0} \frac{\cos x}{1}$$

$$= 1$$

(3) $$\lim_{x \to \infty} \frac{a + be^x}{c + de^x}$$

$$= \lim_{x \to \infty} \frac{be^x}{de^x}$$

$$= \lim_{x \to \infty} \frac{b}{d}$$

$$= \frac{b}{d}$$

Date September 6, 2021

Dated :
Page :

Derivatives & Tangents

\# Let f be a function defined on an open interval around a. Then f is **differentiable** at a if $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists.

$$f'(x) = \frac{df}{dx}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

\# A **tangent** to $f(x)$ at a is a line which represents the instantaneous direction in which the graph $\Gamma(f)$ moves at $(a, f(a))$.

Traditionally, the tangent to $f(x)$ at a is thought of as a line which just touches $\Gamma(f)$ at $(a, f(a))$

Tangents as limits of secants

Recall the notion of a tangent to the function f at a point a (i.e. a tangent to the graph $\Gamma(f)$ at $(a, f(a))$).

If it exists, we can think of a tangent as a 'limit' of secants joining $(a, f(a))$ and nearby points $(a+h, f(a+h))$!

<u>$\mathcal{E}q^n$ of a Secant</u> : $y - f(a) = \dfrac{f(a+h) - f(a)}{a+h-a}(x-a)$

What happens to the limit of this eq^n ?

Tangents and Derivatives

Let f be a function differentiable at point a. Then the tangent to f at 'a' exists and is given by

$$\boxed{y = f'(a)(x-a) + f(a)}$$

Converse → Suppose the tangent to f at 'a' exists and is not vertical (i.e. is not a line $x = a$). Then, f is differentiable at a and the eq^n of the tangent is

$$y = f'(a)(x-a) + f(a)$$

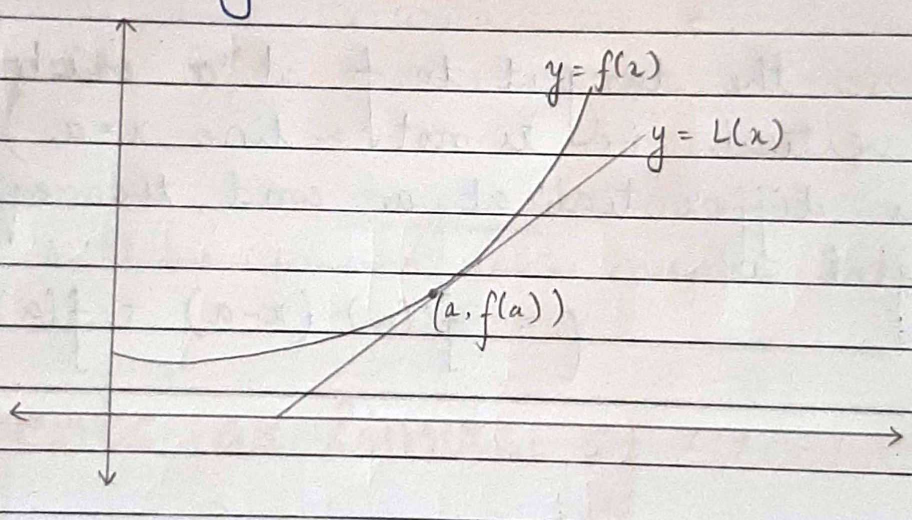
Linear Approximation

Application of Tangent Line to a Function

Given a function, $f(x)$, we can find its tangent at $x = a$. The equation of the tangent line is,

$$L(x) = f(a) + f'(a)(x - a)$$

Take a look at the following graph of a function and its tangent line.



From this graph we can see that near $x = a$ the tangent line and the function have nearly the same graph. On occasion we will use the tangent line, $L(x)$, as an approximation to the fn, $f(x)$, near ~~x~~ $x = a$.

In these cases we call the tangent line the <u>linear approximation</u> to the fn $x = a$.

Why would we do this ?

- Linear approximations do a very good job of approximating values of $f(x)$ as long as we stay 'near' $x = a$. However, the farther away from $x = a$ we get, the worse the approximation is liable to be.

QUESTIONS

(1) Using the linear appromation of $f(x) = \sqrt{x}$ at $x = 4$, find the approximate value of $\sqrt{4.4}$.

Sol. We know, $L(x) = f(a) + f'(a)(x-a)$

Here, $f(a) = \sqrt{a}$ $\quad f'(a) = \dfrac{1}{2\sqrt{a}}$ $\quad x = 4.4$, $a = 4$

$\therefore L(x) = f(4) + f'(4)(x-4)$

$\Rightarrow L(x) = \sqrt{4} + \dfrac{1}{2\sqrt{4}}(x-4)$

$\Rightarrow L(x) = 2 + \dfrac{1}{4}(x-4) \Rightarrow L(x) = 2 + \dfrac{x}{4} - 1$

$\Rightarrow L(x) = 1 + \dfrac{x}{4}$

$\therefore L(4.4) = 1 + \dfrac{4.4}{4} = \dfrac{4+4.4}{4} = \dfrac{8.4}{4} = 2.1$

$\Rightarrow \sqrt{4.4} = 2.1$ Ans

Dated:
Page:

(2) If $L_p(t) = e^4(At+B) + Ce^5$ denotes the best linear approximation of the function $q(t)$ at the point $t=3$, then find the value of $A+B+C$.

Solution: $q(t) = \dfrac{e^{t+2} - e^4}{t-2}$, if $t>2$

$$\Rightarrow q(3) = \frac{e^5 - e^4}{1} = e^5 - e^4$$

Now, $q'(t) = \dfrac{(t-2)e^{t+2} - (e^{t+2} - e^4)}{(t-2)^2}$

$$q'(3) = e^4$$

$\therefore$ The best linear approximation $L_q(t)$ of the function $q(t)$ at the point $t=3$ is

$$\begin{aligned} L_q(t) &= q(3) + q'(3)(t-3) \\ &= e^5 - e^4 + e^4(t-3) \\ &= e^5 - e^4 + te^4 - 3e^4 \\ &= e^5 + te^4 - 4e^4 \end{aligned}$$

$$L_q(t) = e^5 + e^4(t-4) \quad \text{— (1)}$$

$\therefore$ Comparing (1) with $e^4(At+B) + Ce^5$

$A=1$, $B=-4$, $C=1$

$\Rightarrow A+B+C = 1+1-4 = -2$ Ans.