~ Mathematics 2 ~

Notes By

Gagneet Kaur

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WEEK 4

VECTORS

Vectors in two dimensions

- Vectors in $\mathbb{R}^2$ are represented by ordered pairs (a, b).
- The first element in the ordered pair denotes the X-coordinate and the second one denotes the Y-coordinate in the Cartesian plane.
- The addition of the two vectors $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2) \in \mathbb{R}^2$, is given by $v_1 + v_2 = (x_1 + x_2, y_1 + y_2)$
- The scalar multiplication of a vector $v = (x, y) \in \mathbb{R}^2$ with a scalar $c \in \mathbb{R}$, is given by $cv = (cx + cy)$

Vectors in three dimensions

- The addition of two vectors $V_1 = (a_1, b_1, c_1)$ and $V_2 = (a_2, b_2, c_2)$ are done coordinate wise as follows

$$V_1 + V_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$$

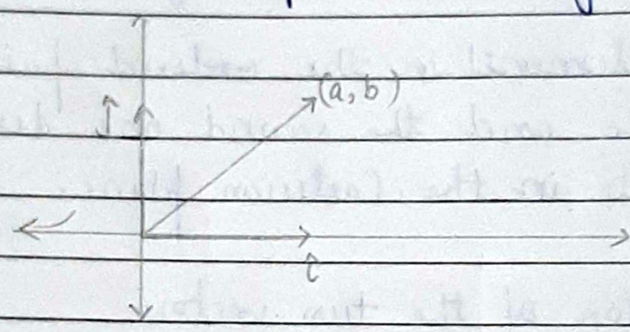
- The scalar multiplication of a vector $V = (a, b, c)$ by a scalar α is given by

$$\alpha(a, b, c) = (\alpha a, \alpha b, \alpha c)$$

Visualization of vectors in $\mathbb{R}^2$

point $(a, b) \equiv$ Vector $(a, b) = a\hat{i} + b\hat{j}$

Visualization : arrow from the origin to (a, b)



$\hat{i}$ → represents x-direction
$\hat{j}$ → represents y-direction

Vector in $\mathbb{R}^n$

Vectors in $\mathbb{R}^n$ are lists (or rows or columns) with n real entries.

Vectors with n entries $\equiv$ Vectors in $\mathbb{R}^n$ = points in $\mathbb{R}^n$

Vectors in Physical context

A vector has magnitude (size) and direction. The length of the line shows its magnitude & arrowhead - direction.

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MATRICES

What is a matrix?

→ A matrix is a rectangular array of numbers, arranged in rows and columns.

Example: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{2\times3}$

This is a 2X3 matrix (2 rows, 3 columns)

★ An 'm×n' matrix has 'm' rows and 'n' columns.

★ $(i,j)^{th}$ entry of a matrix is the entry occurring in the i^{th} row and j^{th} column.

SQUARE MATRIX

→ A square matrix is a matrix in which no. of rows is the same as the no. of columns.

Example: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3}$ This is a 3X3 matrix.

→ The i^{th} diagonal entry of a square matrix is the $(i,i)^{th}$ entry.

→ The diagonal of a square matrix is the set of diagonal entries.

DIAGONAL MATRIX

→ A square matrix in which all entries except the diagonal are 0 is called a diagonal matrix.

Example:
$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

SCALAR MATRIX

→ A diagonal matrix in which all the entries in the diagonal are equal is called a scalar matrix.

Example:
$$S = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

IDENTITY MATRIX

→ The scalar matrix with all diagonal entries 1 is called the identity matrix and is denoted by 'I'

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

LINEAR EQUATIONS & MATRICES

A set of linear equations can be represented in term of matrices.

Example : $3x + 4y = 5$
$4x + 6y = 10$

can be represented by matrix $\begin{bmatrix} 3 & 4 & 5 \\ 4 & 6 & 10 \end{bmatrix}$

ADDITION OF MATRICES

- Addition of Matrices $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and

$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$ is given by $A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$

- Addition of two matrices A and B is defined if both A and B have the same no. of rows and the same no. of columns. If both the matrices A and B have m rows and n columns, then the matrix A+B also has 'm' rows and 'n' columns.

- If $R = [a \quad b \quad c]$ and $C = [d \quad e \quad f]$, then
$R + C = [a+d \quad b+e \quad c+f]$.

SCALAR MULTIPLICATION

- Example : $3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$

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→ The product of a matrix A with a number c is denoted by cA and the $(i,j)^{th}$ entry of cA is product of $(i,j)^{th}$ entry of A with the number c.

$$c(A_{ij}) = (cA)_{ij}$$

MATRIX MULTIPLICATION

• If $R = [a \ b \ c]$ and $C = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$, then the product RC is given by,

$$RC = [ad + be + cf]$$

* Multiplication of matrices A and B is defined only when the no. of columns of A is the same as the number of rows of B.

→ Suppose $A = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ and $B = [C_1 \ C_2 \ C_3]$, where R_i's denote the rows of matrix A and C_i's denote the columns of Matrix B. Moreover, assume that the no. of columns of A and the no. of rows of B are the same. The product AB is

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{bmatrix}$$

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- If A is a '$m \times n$' matrix & B is a '$n \times p$' matrix, then AB is a well defined and is an $m \times p$ matrix.

MULTIPLICATION WITH SPECIAL MATRICES

→ Scalar multiplication by c = multiplication by scalar matrix cI.

$$\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} c & 2c \\ 3c & 4c \\ 5c & 6c \end{bmatrix} = c \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

→ $I \times A_{3\times3} = A_{3\times3} = A_{3\times3} \times I$

→ $I \times A_{3\times n} = A_{3\times n}$

→ $A_{m\times3} \, I = A_{m\times3}$

PROPERTIES OF MATRIX ADDITION & MULTIPLICATION

(1) $(A+B)+C = A+(B+C)$
(2) $(AB)C = A(BC)$
{ Associativity property }

(3) $A+B = B+A$ (Commutativity)

(4) $AB \neq BA$ (commutativity doesn't apply of multiplication)

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(5) $\lambda(A+B) = \lambda A + \lambda B$

(6) $\lambda(AB) = (\lambda A)B = A(\lambda B)$

(7) $A(B+C) = AB + AC$

(8) $(A+B)C = AC + BC$

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System of Linear Eq^{n}s

LINEAR EQUATIONS

A linear eq^n is an equation of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where $x_1, x_2, \dots, x_n$ are the variables (or unknowns) and $a_1, a_2, a_3, \dots, a_n$ are the coefficients, which are real numbers.

EXAMPLE: $2x + 3y + 5z = -9$, where x, y, z are variables & $2, 3, 5$ are the coeffs.

SYSTEM OF LINEAR EQUATIONS

A system of linear eq^n is a collection of one or more linear equations involving the same set of variables. For example,

$$\begin{aligned} 3x + 2y + z &= 6 \\ x - \tfrac{1}{2}y + \tfrac{2}{3}z &= \tfrac{7}{6} \\ 4x + 6y + 10z &= 0 \end{aligned}$$

is a system of 3 eq^{n}s in the three variables x, y, z. A solution to a linear system is an

assignment of values to the variables such that all the equations are simultaneously satisfied.
A solution to the above system is given by

$$x = 1, \quad y = 1, \quad z = 1$$

GENERAL FORM OF SYSTEM OF LINEAR EQUATIONS

A general system of m linear equations with n unknowns can be written as.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

MATRIX REPRESENTATION

The system of linear equations is equivalent to a matrix equation of the form

$$Ax = b$$

where A is an $m \times n$ matrix, x is a column vector with n entries and b is a column vector with m entries.

coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

EXAMPLE: The following system of linear equations ~~can be~~

$$3x + 2y + z = 6$$
$$x - \tfrac{1}{2}y + \tfrac{2}{3}z = \tfrac{7}{6}$$
$$4x + 6y + -10z = 0$$

can be represented as

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -1/2 & 2/3 \\ 4 & 6 & -10 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 7/6 \\ 0 \end{bmatrix}$$

SOLUTION TO A LINEAR SYSTEM

There are 3 possibilities for the solutions to a linear system of equations:

1. The system has infinitely many solutions

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

2. The system has only 1 solution

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

3. The system has no solution.

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

Determinants

Every square matrix A has an associated number, called its determinants and denoted by det(A) or |A|. It is used in

- → solving a system of linear eqn
- → finding the inverse of a matrix
- → calculus and more

→ Determinant of a 1X1 matrix :
If A = [a], a 1X1 matrix, then det(A) = a

Determinant of 2X2 matrix :

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad det(A) = ad - bc$$

Example : $\begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix}$ $\therefore$ det $= (2\times10) - (3\times6) = 20 - 18 = 2$

$B = \begin{bmatrix} 5 & 2/3 \\ 6 & 3/7 \end{bmatrix}$ $\therefore$ det(B) $= \left(5\times\frac{3}{7}\right) - \left(\frac{2}{3}\times6\right) = \frac{15}{7} - 4 = -\frac{13}{7}$

Determinant of a 3x3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We will obtain the determinant by expanding with respect to the 1st row :

$$\det(A) = a_{11} \times \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$\therefore \det(A) = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Example :

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 2(8\times 9 - 7\times 6) - 4(3\times 9 - 5\times 7) + 1(3\times 6 - 8\times 5) \\ &= 2(72-42) - 4(27-35) + 1(18-40) \\ &= 2(30) - 4(-8) + (-22) \\ &= 60 + 32 - 22 \\ \Rightarrow \det(A) &= 70 \end{aligned}$$

Determinant of the Identity Matrix

$\det(I) = 1$ (always) $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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Determinant of product of matrices

det $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$\therefore$ $\det(AB) = \det(A) \cdot \det(B)$

This equality holds for 3×3 matrix too.

Determinant of Inverse of ~~an~~ Matrix

$AA^{-1} = I = A^{-1}A$

$\therefore$ $\det(AA^{-1}) = \det(I)$

$\Rightarrow$ $\det(A) \cdot \det(A^{-1}) = 1$

$\Rightarrow$ $\det(A^{-1}) = \dfrac{1}{\det(A)}$

Property : Switching 2 rows

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define $\tilde{A} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$\det(\tilde{A}) = cb - ad = -(ad - cb) = -\det(A)$

$\rightarrow$ Same applies for switching 2 columns.

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Property : Adding multiples of a row to another row

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define $\tilde{A} = \begin{bmatrix} a+tc & b+td \\ c & d \end{bmatrix}$

$$\det(\tilde{A}) = (a+tc)d - (b+td)c$$
$$= ad + \cancel{tcd} - bc - \cancel{tcd}$$
$$= ad - bc$$
$$= \det(A)$$

Property : Scalar Multiplication of a row by a constant t

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define $\tilde{A} = \begin{bmatrix} a & tb \\ c & dt \end{bmatrix}$

$$\det(\tilde{A}) = adt - ctb$$
$$= t(ad - bc)$$
$$= t \cdot \det(A)$$

TRANSPOSE OF A MATRIX & ITS DETERMINANT

The transpose of $A_{m \times n}$ is the $n \times m$ matrix with $(i,j)^{th}$ entry A_{ji}

Notation : A^T , Definition : $(A^T)_{ij} = A_{ji}$

$\det A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

* $\det(A^T) = \det(A)$

Minors & Cofactors

If A is an $n \times n$ square matrix with $n \leq 4$. Then the minor of the entry in the i^{th} row and j^{th} column is the determinant of the submatrix formed by deleting the i^{th} row and j^{th} column.

NAME : the $(i,j)^{th}$ minor ; Notation : M_{ij}

Example : $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $M_{11} = \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

The $(i,j)^{th}$ cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

Example: $C_{11} = (-1)^{1+1} M_{11} = M_{11}$

$C_{23} = (-1)^{2+3} M_{23} = -M_{23}$

Determinants in terms of Minors & Cofactors

Observe that : For $A_{3\times3}$

$$\det(A) = a_{11} \times \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$\Rightarrow \det(A) = (a_{11} \times M_{11}) - (a_{12} \times M_{12}) + (a_{13} \times M_{13})$$

$$\Rightarrow \det(A) = a_{11} \times C_{11} + a_{12} \times C_{12} + a_{13} \times C_{13}$$

This formula holds for $A_{2\times2}$. We use it to generalize the determinant beyond $n = 3$. Generalization to $A_{4\times4}$:

$$\det(A) = \sum_{j=1}^{4} (-1)^{1+j} a_{1j} M_{1j} = \sum_{i=1}^{4} a_{1j} C_{1j}$$

Inductive Definition of the Determinant

Suppose $A_{n\times n}$ is given and we know how to define determinants for $(n-1)\times(n-1)$ matrices. Define minors and cofactors as before :

The $(i,j)^{th}$ minor is the determinant of the submatrix formed by deleting the i^{th} row & j^{th} column.

The $(i,j)^{th}$ cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

$$\therefore \det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j} = \sum_{i=1}^{n} a_{1j} C_{1j}$$

Expansion along any row or column

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \quad \text{for a fixed } i$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \quad \text{for a fixed } j$$

Important Properties and Identities

Property 1: Determinant of a product is a product of the determinants.

$$\det(AB) = \det(A) \cdot \det(B)$$

Property 2: Switching 2 rows or columns changes the sign.

Property 3: Adding multiples of a row to another row leaves the determinant unchanged (same in case of columns)

Property 4: Scalar multiplication of a row by a constant t multiplies the determinant by t. (same in case of columns)

Useful Computation Tips

1. The determinant of a matrix with a row or column

of zeroes is 0.

(2) The determinant of a matrix in which one row (or column) is a linear combination of other rows (resp. columns) is 0.

(3) Scalar multiplication of a row by a constant t multiplies the determinant by t.

(4) While computing the determinant you can choose to compute it using expansion along a suitable row or column.