~ Mathematics 2 ~

Notes By
Gagneet Kaur

October 5, 2

WEEK 5

Determinants

CRAMER'S RULE

This rule is applied to any system of linear equations with 'n' equations and 'n' variables.

- Consider a system of linear equations as follows:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

- Let the matrix representation of the above system be $Ax = b$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- Let A_{x_i} be the matrix obtained by replacing the i-th column of A by b, for $i = 1, 2, 3$.

Dated :
Page :

$$A_{x_1} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}, \quad A_{x_2} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}, \quad A_{x_3} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

- If $\det(A) \neq 0$, then the solutions to the above system are

$$x_i = \frac{\det A x_i}{\det A}, \quad i = 1, 2, 3$$

i.e., $x_1 = \dfrac{\det A x_1}{\det A}$, $x_2 = \dfrac{\det A x_2}{\det A}$, $x_3 = \dfrac{\det A x_3}{\det A}$

INVERSE OF A MATRIX A

→ Find out the cofactor matrix C, whose ij^{th} element is C_{ij} : the ij^{th} cofactor of A.

→ Adjoint of A is transpose of C.

→ Calculate the determinant of A and check whether it is non-zero or not.

→ If determinant is non-zero, then the inverse of A exists and is given by

$$\boxed{A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)}$$

Dated :
Page :

Also, let A be $n \times n$ matrix. The inverse of A is another $n \times n$ matrix B such that

$$\boxed{AB = BA = I_{n \times n}}$$

and is denoted by A^{-1}.

Example: $\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$

and $\boxed{\det(A^{-1}) = \dfrac{1}{\det(A)}}$.

\# For the inverse of A to exist → det(A) has to be non-zero

ADJUGATE OF MATRIX

$\text{adj}(A) = C^T$ (Transpose of cofactor matrix of A)

SOLUTION OF A SYSTEM OF LINEAR EQUATIONS WITH AN INVERTIBLE COEFF. MATRIX

Consider the system of linear equations $Ax = b$ where the coefficients matrix A is an invertible matrix.

Dated :
Page :

Multiplying both sides by A^{-1} we get,

$$Ax = b$$

$$\Rightarrow A^{-1}Ax = A^{-1}b$$

$$\Rightarrow Ix = A^{-1}b$$

$$\Rightarrow \boxed{x = A^{-1}b}$$

HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

A system of linear equations is homogeneous if all of the constant terms are 0, i.e., $b = 0$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

SOLUTIONS OF A HOMOGENEOUS SYSTEM

The matrix form of a homogeneous system is $Ax = 0$

If A is invertible matrix then multiplying both sides by A^{-1}, we obtain $x = A^{-1}0 = 0$

A homogeneous system of linear equations with 'n' equations in 'n' unknowns:

→ has a unique solution 0 if its coeff matrix is invertible, i.e, its determinant is non-zero.

→ has an infinite no. of solutions if its coeff matrix is not invertible, i.e, determinant is 0.

The echelon form

A matrix is in row echelon form if:

→ The first non-zero element (the leading entry) in a row is 1.

→ The column containing the leading 1 of a row is to the right of the column containing the leading 1 of the row above it.

→ Any non-zero are always above rows with all zeros.

A matrix is in reduced row echelon form if:

→ The first non-zero element in the first row (the leading entry) is the number 1.

Dated:

Page:

- All subsequent non-zero rows must also have their leading entries (i.e first non-zero entries) as 1 and they should appear to the right of the leading entry in the previous row.

- The leading entry in each row must be the only non-zero number in its column.

- Any non-zero rows are always above rows with all zeros.

NOTE: Any matrix which is in reduced row echelon form is also in row echelon form.

SOLUTIONS OF $Ax = b$ when A is in REDUCED ROW ECHELON FORM

Let $Ax = b$ be a system of linear equations and suppose A is in reduced row echelon form.

(1) Suppose for some i, the i^{th} row of A is zero row but $b_i \neq 0$. Then this system has <u>no solution</u>.

REASON: This means if we write the corresponding system of linear equations, the i^{th} equation reads

$$0x_1 + 0x_2 + \dots + 0x_n = b_i$$

Since, $b_i \neq 0$, this equation cannot be satisfied.

(2) Assume that for every zero row of A, the corresponding entry of b is also 0 (i.e. if the i^{th} row of A is zero, then so is b_i)

→ If the i^{th} column has the leading entry of some row, we call x_i a dependent variable

→ If the i^{th} column does not have the leading entry of some row, we call x_i an independent variable.

→ Assign arbitrary values to independent variables.

→ For a dependent variable, there is a unique equation in which it occurs. All other variables in that equation are independent variables and thus have values assigned. Hence, we can compute the value of the dependent variable from this eq^n substituting the assigned values for the other independent variables in the eq^n.

→ The obtained values for x_i give a solution to the system.

→ In fact every solution is obtained in this way.

CONCLUSION: If A is in reduced row echelon form, this easy procedure provides us with ALL the solutions of $Ax = b$.

Dated :
Page :

The Gaussian Elimination

THE AUGMENTED MATRIX

Let $Ax = b$ be a system of linear equations where A is an $m \times n$ matrix and b is an $m \times 1$ column vector.

The augmented matrix of this system is defined as the matrix of size ~~(m×n)~~ $m \times (n+1)$ whose first ~~m~~ n columns are the columns of A and the last column is b.

We denote the augmented matrix by $[A|b]$ and put a vertical line between the first n columns ~~of~~ and the last column b while writing it.

Example :

$$\begin{aligned} 3x_1 + 2x_2 + x_3 + x_4 &= 6 \\ x_1 + x_2 &= 2 \\ 7x_2 + x_3 + x_4 &= 8 \end{aligned}$$

Here, $A = \begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 2 \\ 8 \end{bmatrix}$

The augmented matrix is $[A|b] = \left[\begin{array}{cccc|c} 3 & 2 & 1 & 1 & 6 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right]$

GAUSSIAN ELIMINATION METHOD

Consider the system of linear equations $Ax = b$

1. Form the augmented matrix of the system $[A|b]$

2. Perform the same operations on $[A|b]$ that were used to bring A into reduced row echelon form.

3. Let R be the submatrix of the obtained matrix of the first n columns and c be the submatrix of the obtained matrix consisting of the last col.

We write the obtained matrix as $[R|c]$. Notice that R is the reduced row echelon matrix obtained by row reducing A.

→ The solutions of $Ax = b$ are precisely the solutions of $Rx = c$.

4. Form the corresponding system of linear equations $Rx = c$.

5. Find all the solutions of $Rx = c$ and hence of $Ax = b$.

Since, R is in the reduced row echelon form, we can find ALL its solutions.