



IIT Madras
BSc Degree

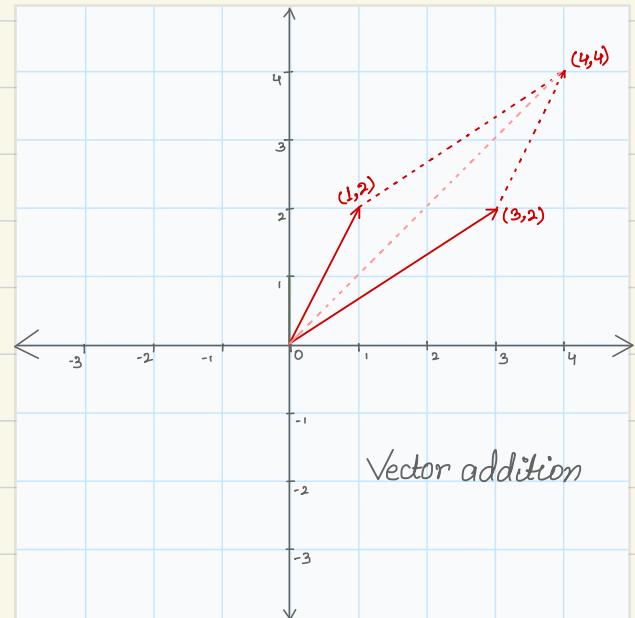
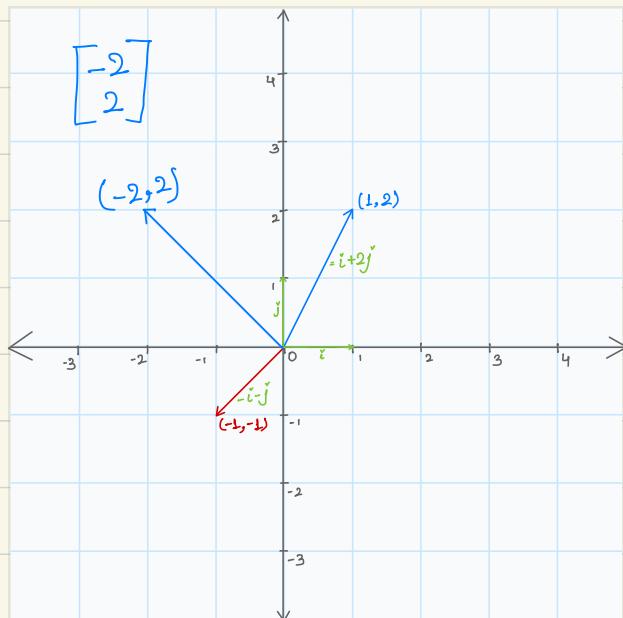
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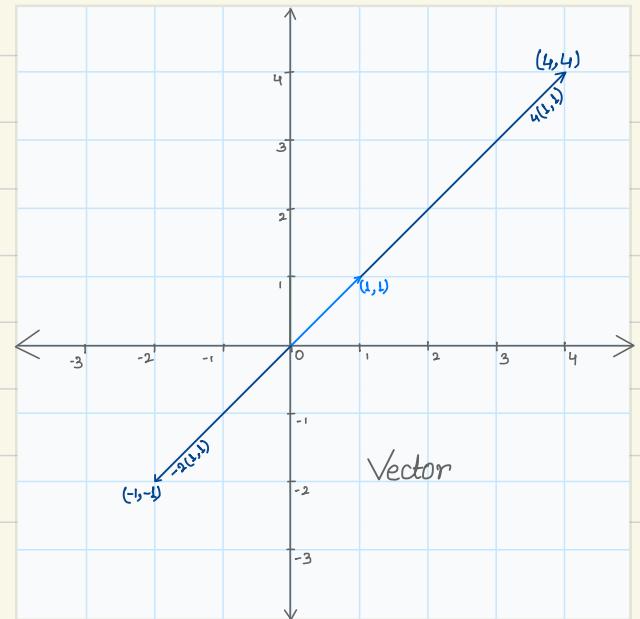
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◦ Vectors :- Vectors can be thought of list. It could be of rows or columns. It can be denote point in n-dimension and combining more than 1 vector form line. It can be used to perform arithmetic operation such as (sum / minus) and multiplication.



◦ Vector addition : To add 2 vector graphically place the graphical line on top point of other line in same direction or parallel to line which is been added.

◦ Vector multiplication : It is multiplying scalar (a number) with vector which stretches vector in same direction in case of +ve scalar & in opposite direction in -ve scalar.



□ Matrix: It is a rectangular/array of numbers, arranged in rows and columns.

Eg $\rightarrow \begin{bmatrix} 1, 2, 3 \\ 3, 9, 2 \end{bmatrix}$ This is 2×3 matrix.
(2 row, 3 columns)

- An $m \times n$ matrix has m rows and n columns.
- (i, j) -th entry of matrix is entry in i^{th} row & j^{th} column.
- Square matrix is matrix with same no. of rows & columns.
- Diagonal matrix has all no. 0 except diagonal.
- Scalar matrix is diagonal matrix with all same value.
- Identity matrix is scalar matrix with all value 1.

• Linear equation can be written in matrix:

Eg:
$$\begin{array}{l} 3x + 4y = 5 \\ 4x + 6y = 10 \end{array} = \begin{bmatrix} 3 & 4 & | & 5 \\ 4 & 6 & | & 10 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 6 & 10 \end{bmatrix}$$

- Matrix addition: Here we add corresponding entries of matrix. (Both matrices must have same size.)

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} + \begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} a+u & b+v & c+w \\ d+x & e+y & f+z \end{bmatrix}$$

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

- Scalar multiplication: Here we multiply scalar with each entry of matrix.

$$\text{Eg: } 3 \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 3 \\ 6 & 9 \end{bmatrix}$$

$$\circ (cA)_{ij} = c(A)_{ij}$$

- Matrix multiplication: It is multiplying row of matrix 1 & column of matrix 2 in consecutive place. Eg: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2 \times 2} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}_{2 \times 3} = \begin{bmatrix} 7 & 10 & 13 \\ 15 & 22 & 29 \end{bmatrix}_{2 \times 3}$

- Here, no. of rows in 1st matrix should be equal to no. of columns in 2nd matrix

$$A_{m \times n} B_{n \times b} = (AB)_{m \times p}$$

- The $(i,j)^{\text{th}}$ entry of AB is defined as:

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

- Multiplication by scalar matrices.

$$\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 2 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} c & 5c \\ 2c & 3c \\ 3c & 2c \end{bmatrix} = c \begin{bmatrix} 1 & 5 \\ 2 & 3 \\ 3 & 2 \end{bmatrix}$$

- If any matrix multiply by identity matrix. It gives the same matrix. It act as 1.

Properties of matrix addition & multiplication:

$$(A+B)+C = A+(B+C)$$

$$(AB)C = A(BC)$$

$$A+B = B+A$$

$$AB \neq BA$$

$$\lambda(A+B) = \lambda A + \lambda B, \lambda(AB) = (\lambda A)B = (\lambda B)A$$

$$A(B+C) = AB+AC, (A+B)C = AC+BC$$

° Linear equation in form of matrix:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

...

...

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

So, linear equation can be represented in form of $Ax = b$

Example:

$$\begin{array}{l} 3x + 2y + z = 6 \\ x + \frac{1}{2}y + \frac{2}{3}z = \frac{7}{6} \\ 4x + 6y - 10z = 0 \end{array} \quad A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & \frac{1}{2} & \frac{2}{3} \\ 4 & 6 & -10 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ \frac{7}{6} \\ 0 \end{bmatrix}$$

There are 3 possibility for solution of linear system of eq:

- The system has infinitely many solution.
- The system has single unique solution.
- The system has no solution.

○ Determinant : Every square matrix has a number associated with it, which is called determinant & can be denoted by $\det(A)$ or $|A|$. It can be used in:

- Solving system of linear equations.

- Finding inverse of matrix & calculus, more ...

- Determinant of 1×1 matrix : $A = [a]$, $a = 1 \times 1$ then $\det(A) = a$

- Determinant of 2×2 matrix : $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det(A) = ad - bc$

$$\text{Eg: } A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = 4 - 6 = -2$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Determinant of 3×3 matrix : $A =$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11} \times \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\text{Eg, } A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & a \end{bmatrix}$$

$$\det(A) = 2(72 - 42) - 4(27 - 35) - 1(18 - 40) = 70$$

□ Determinant of product of matrices :

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$AB = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$\begin{aligned} \det(AB) &= (ae+bg)(cf+dh) - (af+bh)(ce+dg) \\ &= \cancel{aecf} + \cancel{bgcf} + \cancel{aedh} + \cancel{bgdh} - \cancel{afe} - \cancel{fdg} - \cancel{bhc} - \cancel{hdg} \\ &= bgcf + aedh - bhce - afdg \\ &= bcfg + adeh - bceh - adfg \\ &= adeh - adfg - bceh + bcfg \\ &= ad(eh-fg) - bc(eh-fg) \\ &= (ad-bc)(eh-fg) \end{aligned}$$

$$\text{So, } \det(AB) = \det(A)\det(B)$$

○ Determinant of inverse of a matrix :

$$A A^{-1} = I = A^{-1} A$$

$$\Rightarrow \det(AA^{-1}) = \det(I)$$

$$\text{we saw, } \Rightarrow \det(A)\det(A^{-1}) = 1$$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

• Properties of determinant:

- Switching 2 rows/columns of matrix 'A' gives -ve of determinants of matrix 'A':

$$\text{Eg: } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tilde{A} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\det(\tilde{A}) = cb - ad = -(ad - bc) = -\det(A)$$

- Adding multiple of a row to another row in matrix 'A' gives same determinant as $\det(A)$

$$\text{Eg: } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tilde{A} = \begin{bmatrix} a+tc & b+td \\ c & d \end{bmatrix}$$

$$\begin{aligned}\det(\tilde{A}) &= (a+tc)d - (b+td)c = ad - \cancel{td} - bc - \cancel{tc} \\ &= ad - bc = \det(A)\end{aligned}$$

- Scalar multiplication of row/column with constant 't' of a matrix 'A' gives $t \cdot \det(A)$.

$$\text{Eg: } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tilde{A} = \begin{bmatrix} a & tb \\ c & td \end{bmatrix}$$

$$\begin{aligned}\det(\tilde{A}) &= tad - tbc = t(ad - bc) \\ &= t \cdot \det(A)\end{aligned}$$

○ Upper triangular matrix: These are matrix which has zeros below the diagonal.

- For such matrix the determinant is product of diagonal. Eg:

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 8 & 7 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\det(A) = 2(72 - 0) - 4(0 - 0) + 3(0 - 0) \\ = 2 \times 72 - 4 \times 0 + 3 \times 0 = 144 = 2 \times 8 \times 9$$

○ The transpose of a matrix:

The transpose of matrix ' $A_{m \times n}$ ' is ' $n \times m$ ' matrix with $(i, j)^{\text{th}}$ entry is A_{ji} .

Notation: A^T Definition $\Rightarrow (A^T)_{ij} = A_{ji}$

$$\text{let, } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then, } A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \det(A) &= a_{11} \times \det \begin{bmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{bmatrix} - a_{21} \times \det \begin{bmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{bmatrix} + a_{31} \times \det \begin{bmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) - a_{31}(a_{12}a_{23} - a_{22}a_{13}) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} - a_{31}a_{12}a_{23} + a_{31}a_{22}a_{13} \\ &= \det(A) \end{aligned}$$

\therefore The determinant of transpose of a matrix is the determinant of original matrix.

○ Minors and cofactors:

Minor as we seen in calculating determinant of 3×3 matrix.

So, as we have $n \times n$ square matrix. [$n \leq 4$]
 Then the minor of entry of i^{th} row & j^{th} column
 is determinant of submatrix formed by
 deleting i^{th} row & j^{th} column.

$(i, j)^{\text{th}}$ minor

$$\text{Eg, } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{11} = \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

The $(i, j)^{\text{th}}$ cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

$$\text{For above example: } C_{11} = (-1)^{1+1} M_{11} = M_{11}$$

$$\&, C_{23} = (-1)^{2+3} M_{23} = -M_{23}$$

○ Determinants in term of minors & cofactors:
 For above example ($A_{3 \times 3}$) determinant will be:

$$\begin{aligned} \text{det}(A) &= a_{11} \times \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11} \times M_{11} - a_{12} \times M_{12} + a_{13} \times M_{13} \\ &\hookrightarrow \text{Here we will replace with minors.} \end{aligned}$$

$$= a_{11} \times C_{11} + a_{12} \times C_{12} + a_{13} \times C_{13}$$

\hookrightarrow To make all sign +ve we replace M_{ij} with cofactors.

for $A_{2 \times 2}$ also, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$$\det(A) = A_{11}A_{22} - A_{12}A_{21} \\ = A_{11}M_{11} - A_{12}M_{12} = A_{11}C_{11} + A_{12}C_{12}$$

It is generalised for $A_{4 \times 4}$ also:

$$\det(A) = \sum_{j=1}^4 (-1)^{1+j} a_{1j} M_{1j} = \sum_{j=1}^4 a_{1j} C_{1j}$$

□ Inductive definition of determinant:

So, for $A_{n \times n}$ matrix. The determinant will be similar to above like $A_{4 \times 4}$.

By directly replacing cofactor and minor. We get:

$$\det(A) = \sum_{j=1}^n (-1)^{1+j} a_{1j} M_{1j} = \sum_{j=1}^n a_{1j} C_{1j}$$

□ Expansion along any row or column:

It means determinant can not only calculated by 1st row only. It can be calculated by selecting any fixed row or a column.

$$\text{Det}(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} M_{ij} \text{ for a fixed } i$$

$$= \sum_{i=1}^n (-1)^{i+j} a_{ij} M_{ij} \text{ for a fixed } j$$

Eg, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$+ \quad - \quad + \quad - \quad \dots$	$\swarrow (-1)^{i+j}$
$- \quad + \quad - \quad + \quad \dots$	
$+ \quad - \quad + \quad - \quad \dots$	
$\dots \quad \dots \quad \dots \quad \dots \quad \dots$	

Determinant by 2nd row of A:

$$= -a_{21} \times \det \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} + a_{22} \times \det \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} - a_{23} \det \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

Determinant by 2nd column of A:

$$= -a_{12} \times \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{22} \times \det \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} - a_{32} \det \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

□ Important properties and identities:

Property 1: Determinant of a product is product of a determinant.

$$\det(AB) = \det(A)\det(B)$$

$$\det(A^n) = \det(A)^n$$

$$\det(A^{-1}) = \frac{1}{\det(A)} = \det(A)^{-1}$$

$$\det(P^TAP) = \det(A)$$

$$\det(AB) = \det(BA)$$

$$\det(A^TA) = \det(A)^2$$

$$\det(A^T) = \det(A)$$

Property 2: Switching 2 rows or columns changes sign:

$$\det(\tilde{A}) = -\det(A)$$

Property 3: Adding multiples of a row to another row leaves determinant unchanged and vice-versa for columns.

$$\det(\tilde{A}) = \det(A)$$

Property 4: Scalar multiplication of a row by constant 't' multiply determinant by t^n and vice-versa for column.

$$\det(tA) = t \det(A)$$

(for 1 row or column)

For all row it will be:

$$\det(tA_{n \times n}) = t^n \det(A)$$

○ Useful tips :

- The determinant of a matrix with a row or column of 0 is 0.
- The determinant of a matrix with a row or column is linear combination of other column is zero.
- Scalar multiplication of a row/column by 't' multiplies determinant by 't'.
- While calculating determinant, you can choose to compute it using expansion along any suitable row/column.

□ Cramer's Rule :

Consider a solution of linear equation $Ax = b$ where A is a $n \times n$ invertible matrix and b is a column vector with ' n ' entries.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

...

...

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$$

Then A_{xi} to be matrix obtained by replacing the i th column of A by the column vector b . Cramer's rule states the unique solution is:

$$x_i = \frac{\det(A_{xi})}{\det(A)}$$

Example:

Consider eq:

$$4x_1 - 3x_2 = 11$$

$$6x_1 + 5x_2 = 7$$

$$\text{Sol, } x_1 = 2, x_2 = -1$$

Matrix representation $Ax = b$:

$$A = \begin{bmatrix} 4 & -3 \\ 6 & 5 \end{bmatrix}, b = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$$

$$\det(A) = 38$$

$$Ax_1 = \begin{bmatrix} 11 & -3 \\ 7 & 5 \end{bmatrix}, \det(Ax_1) = 76$$

$$Ax_2 = \begin{bmatrix} 4 & 11 \\ 6 & 7 \end{bmatrix}, \det(Ax_2) = -38$$

Replace 1st row of A with b than 2nd row and on.

$$\text{Calculate } \frac{\det(Ax_1)}{\det(A)} = \frac{76}{38} = 2 \quad \frac{\det(Ax_2)}{\det(A)} = \frac{-38}{38} = -1$$

Similarly we can solve any invertible matrix on 'n' dimension by constantly replacing ith column with b and finding determinant and dividing by determinant of A.

○ Converse of matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det(A) = ab - bc \neq 0$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Here, $\left. \begin{array}{l} ae + bg = 1 \\ ce + dg = 0 \\ af + bh = 0 \\ cf + dh = 1 \end{array} \right\}$

$$\left. \begin{array}{l} ade + bdg = d \\ bce + bdg = 0 \\ e(ad - bc) = d \\ e = \frac{d}{ad - bc} \end{array} \right\}$$

○ Adjugate of square matrix:-

The adjugate matrix is transpose of

cofactor matrix which is obtain by minors.

$$\text{adj}(A) = C^T$$

Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 5 & 6 & 0 \end{bmatrix}$

$$\det(A) = 1(2 \times 0 - 8 \times 6) - 2(0 \times 0 - 8 \times 5) + 3(0 \times 6 - 2 \times 5)$$

$$= -48 + 80 - 30 = 2$$

Minors \Rightarrow
 Remove i, j row &
 col. & find det of rest

$$\begin{bmatrix} -48 & -40 & -10 \\ -18 & -15 & -4 \\ 10 & 8 & 2 \end{bmatrix}$$

$$(\text{minors})(-1)^{ij}$$

$$\text{Cofactors} = \begin{bmatrix} -48 & 40 & -10 \\ 18 & -15 & 4 \\ 10 & -8 & 2 \end{bmatrix}$$

$$\text{Adj}(A) = C^T = \begin{bmatrix} -48 & 18 & 10 \\ 40 & -15 & -8 \\ -10 & 4 & 2 \end{bmatrix}$$

Compute $A \frac{1}{\det(A)} \text{adj}(A) \times \frac{1}{\det(A)} \text{adj}(A) A$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 5 & 6 & 0 \end{bmatrix} \begin{bmatrix} -24 & 9 & 5 \\ 20 & -\frac{15}{2} & -4 \\ -5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -24 & 9 & 5 \\ 20 & \frac{15}{2} & -4 \\ -5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 5 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$

○ Adjugate and inverse:

If A is a $n \times n$ matrix and $\det(A) \neq 0$, then A^{-1} .

Then A^{-1} exists and equal:

$$\text{Hence, } A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

○ Solution of system of linear eq. with invertible coeff. matrix:

Consider a system of linear equation $Ax = b$
where coeff. matrix A is invertable.

Multiplying both side by A^{-1} :

$$\begin{aligned} Ax &= b \\ A^{-1}Ax &= A^{-1}b \\ Ix &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

→ Complete example till now:

Calculate inverse of linear equation:

$$\begin{aligned} 8x_1 + 8x_2 + 4x_3 &= 1960 \\ 12x_1 + 5x_2 + 7x_3 &= 2215 \\ 3x_1 + 2x_2 + 5x_3 &= 1135 \end{aligned}$$

$$A = \begin{bmatrix} 8 & 8 & 4 \\ 12 & 5 & 7 \\ 3 & 2 & 5 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 1960 \\ 2215 \\ 1135 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 8(25-14) - 8(60-21) + 4(24-15) \\ &= 88 - 312 + 36 = -188 \end{aligned}$$

$$\text{Minors} = \begin{bmatrix} 11 & 39 & 9 \\ 32 & 28 & -8 \\ 36 & 8 & -56 \end{bmatrix} \quad \text{Coff. matrix} = \begin{bmatrix} 11 & -39 & 9 \\ -32 & 28 & 8 \\ 36 & -8 & -56 \end{bmatrix}$$

$$\text{Adjuicate} = \begin{bmatrix} 11 & -32 & 36 \\ -39 & 28 & 8 \\ 9 & 8 & -56 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{-188} \begin{bmatrix} 11 & -32 & 36 \\ -32 & 28 & -8 \\ 36 & 8 & -56 \end{bmatrix}$$

$$x = A^{-1}b = \frac{1}{-188} \begin{bmatrix} 11 & -32 & 36 \\ -32 & 28 & -8 \\ 36 & 8 & -56 \end{bmatrix} \begin{bmatrix} 1960 \\ 2215 \\ 1135 \end{bmatrix} = \frac{1}{-188} \begin{bmatrix} 1960 \\ 2215 \\ 1135 \end{bmatrix} = \begin{bmatrix} 45 \\ 125 \\ 150 \end{bmatrix}$$

means $x_1 = 45, x_2 = 125, x_3 = 150$

○ Homogenous system of linear equation :

- It is in form $Ax = 0$. This solution is called 'trivial solution'.

- For this system there only 2 possibility :

- 0 is unique solution

- There is infinitely many solution other than 0.

- In a homogenous system of equation, if there are more variable than eq, then it guaranteed have non-trivial solution.

○ Echelon form:

□ Row echelon form:

The matrix is in row echelon form if:

- The 1st entry of each row (leading entry) is 1.
- Each leading entry in column to right of the leading entry of previous row.
- Row with all 0 element are below non-zero row.
- For row with all 0 leading entry will be 0.

$$\text{Eg: } A_{\text{ref}} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{Row echelon form}$$

□ Reduced Row echelon form: In this

the matrix is in row echelon form and the column having leading coefficient 1 should have all other element 0.

$$\text{Eg: } A_{\text{ref}} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{Reduced Row echelon form}$$

Solving this, $A_{\text{ref}}x = b$

$$x_1 + 2x_2 = b_1 \quad x_1 = b_1 - 2x_2$$

$$x_3 = b_2 \quad x_2 = C \text{ (constant)}$$

$$x_4 = b_3 \quad x_1 = b_1 - 2C$$

$$x = \begin{bmatrix} b_1 - 2C \\ C \\ b_2 \\ b_3 \end{bmatrix}$$

- let $Ax = b$ a system of linear equation and suppose A is in reduced row echelon form.
- Assume for every zero row of A , the corresponding entry of b is 0. (if the i^{th} row of $A = 0$ so be $b_i = 0$)
 - If i^{th} column has leading entry of some row, we call x_i a dependent var.
 - The i^{th} column don't have leading entry we call x_i an independent variable.

Example:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

$x_1, x_2 \rightarrow \text{dep. var}$

$x_3 \rightarrow \text{Ind. var}$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_1, x_3, x_4 \rightarrow \text{dep. var.}$

$x_2 \rightarrow \text{Ind. var.}$

Reduced row echelon form gives all sol of $Ax = b$

○ Elementary row operations:

Type 1: $\det(A) = -\det(B)$

Interchanging 2 rows

$$\begin{array}{c} \begin{bmatrix} 3 & 2 & 1 & 1 \end{bmatrix} R_1 \longleftrightarrow R_2 \rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \\ \sim \sim \sim \sim \rightarrow \begin{bmatrix} 3 & 2 & 1 & 1 \\ 0 & 7 & 1 & 1 \end{bmatrix} \end{array}$$

Type 2: $\det(A) = c\det(B)$

Scalar mult. of a row by constant

$$\begin{array}{c} \begin{bmatrix} 3 & 2 & 1 & 1 \end{bmatrix} R_1/3 \rightarrow \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \end{bmatrix} \\ \sim \sim \sim \sim \rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \end{array}$$

Type 3: $\det(A) = \det(B)$

Adding multiple of a row to another row

$$\begin{array}{c} \begin{bmatrix} 3 & 2 & 1 & 1 \end{bmatrix} R_1 - 3R_2 \rightarrow \begin{bmatrix} 0 & -1 & 1 & 1 \end{bmatrix} \\ \sim \sim \sim \sim \rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \end{array}$$

○ Row reduction (Row echelon form) :

Here we have to do row wise operations to convert a matrix to reduced row echelon form.

Action

1) Find leftmost non-zero column.

Example with notation

$$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$$

2) Use row operation to get 1 in top of column.

If we have 0 on top we can replace it with other row.

$$\begin{array}{c} \begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1/3} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \\ \sim \sim \sim \end{array}$$

3) Once we got 1 on top then we try to make '0' all element below it by sub. multiple of 1st row.

$$\begin{array}{c} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1/3 & -1/3 & -1/3 \\ 0 & 7 & 1 & 1 \end{bmatrix} \\ \sim \sim \sim \end{array}$$

4) If all element that row after 1 become 0. we come to next row.

$$\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1/3 & -1/3 & -1/3 \\ 0 & 7 & 1 & 1 \end{bmatrix}$$

5) Here we find next non-zero column by ignoring 1st row. By repeating same operation from 1-5.

- Removing 7 to 0

$$\begin{array}{c} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \times 3} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 7 & 1 & 1 \end{bmatrix} \\ \sim \sim \sim \end{array}$$

$$\begin{array}{c} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - 7R_2} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 8 & 8 \end{bmatrix} \end{array}$$

- Repeating (1-5)

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 8 & 8 \end{bmatrix} \xrightarrow{R_3/8} \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Now we get matrix in Row echelon form:

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

○ Reduced row echelon form:

We can reduce row echelon form matrix to reduced row echelon form.

Action

Take a column containing 1 in extreme right.

Example with notation

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Do row operation to make elements '0' of that column.

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2+R_3, R_1 - \frac{R_3}{3}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Remove zero from previous column and keep doing until all coll. left with 1.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The resulting matrix is in Reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

○ Calculating determinant by row reduction :

As we saw row echelon form of matrix produces an upper triangular matrix with diagonal entry all 1 if it is invertible (square) or some 1 and 0.

- Row reduced A into row echelon form.
- If diagonal of reduced matrix has 0, then its determinant is 0.
- We can find determinant by doing reverse mapping of operation by finding effect on determinant. (bcz in row echelon form of square matrix determinant is 1.)

Type 1: $\det(A) = -\det(B)$ Interchanging 2 rows.

Type 2: $\det(A) = c\det(B)$ Scalar mult. of a row by constant.

Type 3: $\det(A) = \det(B)$ Adding multiple of a row to another row.

○ Augmented matrix :

Here we have a system of linear equation where A is ' $m \times n$ ' matrix and b is ' $m \times 1$ ' column vector.

An augmented matrix is a system defined as matrix of size ' $m \times (n+1)$ ' whose first ' n ' columns are column of A and last column is column of b.

We denote it as $[A|b]$

Augmented matrix: $[A|b] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{bmatrix}$

Example :

$$\begin{aligned} 3x_1 + 2x_2 + x_3 + x_4 &= 6 \\ x_1 + x_2 &= 2 \\ 7x_2 + x_3 + x_4 &= 8 \end{aligned} \quad A = \begin{bmatrix} 3 & 2 & 1 & 1 & 6 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 2 \\ 8 \end{bmatrix}$$

$$[A|b] = \begin{bmatrix} 3 & 2 & 1 & 1 & 6 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{bmatrix}$$

○ Gaussian elimination method :

Consider system of linear equation $Ax = b$

- Form augmented matrix of system $[A|b]$
- Perform reduced row echelon form on $[A|b]$
- After step 1 & 2 , R & C be the new matrix obtained $[R|C]$

Hence, solution of $Ax = b$ is precisely solution of $Rx = c$.

- From corresponding system of eq, $Ax=b$ find all sol. of $Rx=c$ & hence $Ax=b$.

Example :

$$\begin{array}{c|ccccc} 3 & 2 & 1 & 1 & 6 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array} \xrightarrow{R_1/3} \begin{array}{c|ccccc} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array} \xrightarrow{R_2-R_1} \begin{array}{c|ccccc} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1/3 & -1/3 & -1/3 & 0 \\ 0 & 7 & 1 & 1 & 8 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 2 \\ 0 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 7 & 1 & 1 & 8 \end{array} \right] \xrightarrow{R_2 \times 3} \left[\begin{array}{cccc|c} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 7 & 1 & 1 & 8 \end{array} \right] \xrightarrow{R_3 - 7R_2} \left[\begin{array}{cccc|c} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 8 & 8 & 8 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 8 & 8 & 8 \end{array} \right] \xrightarrow{R_3/8} \left[\begin{array}{cccc|c} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array} \right] \text{ Row echelon form}$$

$$\left[\begin{array}{cccc|c} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array} \right] \xrightarrow{R_2 + R_3} \left[\begin{array}{cccc|c} 1 & \frac{2}{3} & 0 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array} \right] \xrightarrow{R_1 - \frac{2R_2}{3}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array} \right]$$

$R \quad c$

$$R_x = c$$

x_1, x_2, x_3 = dependent variable.

x_4 = independent variable

$$x_4 = c$$

$$\text{Here, } x_1 = 1, x_2 = 1,$$

$$x_3 + x_4 = 1, x_3 = 1 - c$$

Reduced row echelon form

Set of sol of $R_x = c$ & $A_x = b$

is $\{x_1 = 1, x_2 = 1, x_3 = 1 - c, x_4 = c \mid c \in \mathbb{R}\}$

$$\left\{ \left[\begin{array}{c} 1 \\ 1 \\ 1-c \\ c \end{array} \right] \mid c \in \mathbb{R} \right\}$$

○ If we found any 0 row then it doesn't have sol.

○ Vector space :

A vector space is a set with two operations (called addition & scalar multiplication with below (i)-(viii) properties).

► Properties of addition and scalar multiplication:

Let v, w, v' be vectors in \mathbb{R}^n & $a, b \in \mathbb{R}$

- 1) $v + w = w + v$
- 2) $(v + w) + v' = v + (w + v)$
- 3) $v + 0 = 0 + v = v$
- 4) $v + (-v) = 0$
- 5) $1v = v$
- 6) $(ab)v = a(bv)$
- 7) $a(v + w) = av + aw$
- 8) $(a+b)v = av + bv$

Any set satisfy these conditions is a vector space.

A vector space V over \mathbb{R} is a set along 2 fn:

$$+: V \times V \rightarrow V \quad \text{and} \quad \cdot: \mathbb{R} \times V \rightarrow V$$

(i.e. for each pair of element v_1 and v_2 in V , there is unique element $v_1 + v_2$ in V , and for each $c \in \mathbb{R}$ and $v \in V$ there is a unique element $c \cdot v$ in V)

Non-example: Let's define scalar mult. in \mathbb{R}^2 as follows:

$$\Rightarrow (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 - y_2)$$

$$\Rightarrow c(x_1, x_2) = (cx_1, cx_2)$$

That's why,

$$(x_1, x_2) + (y_1, y_2) \neq (y_1, y_2) + (x_1, x_2)$$

○ Cancellation law of vector addition:

If $v_1, v_2, v_3 \in V$ such that, $v_1 + v_3 = v_2 + v_3$, then $v_1 = v_2$.

$$\text{Eg: } (x_1, x_2, x_3) + (y_1, y_2, y_3) = (z_1, z_2, z_3) + (y_1, y_2, y_3)$$

$$\Rightarrow x_i + y_i = z_i + y_i, \quad i=1,2,3$$

Subtracting y_i from both side we get $x_i^{\circ} = z_i^{\circ}$.

- $0_V = 0$ for each $v \in V$

- $(-c)v = -cv = c(-v)$ for each $c \in \mathbb{R}$ & each $v \in V$.

- $c0 = 0$ for each $c \in \mathbb{R}$

○ Linear Combination of vectors:-

Let V be vector space & $v_1, v_2, \dots, v_n = V$.

The linear comb. of v_1, v_2, \dots, v_n with coefficient $a_1, a_2, \dots, a_n \in \mathbb{R}$
A vector $v \in V$ is linear comb. of v_1, v_2, \dots, v_n if
there exist $a_1, a_2, \dots, a_n \in \mathbb{R}$.

$$\text{So that, } v = \sum_{i=1}^n a_i v_i$$

Example in \mathbb{R}^2

$$: 2(1,2) + (2,1) = (4,5)$$

Each vector is expression of
linear comb. of other 2 vectors:

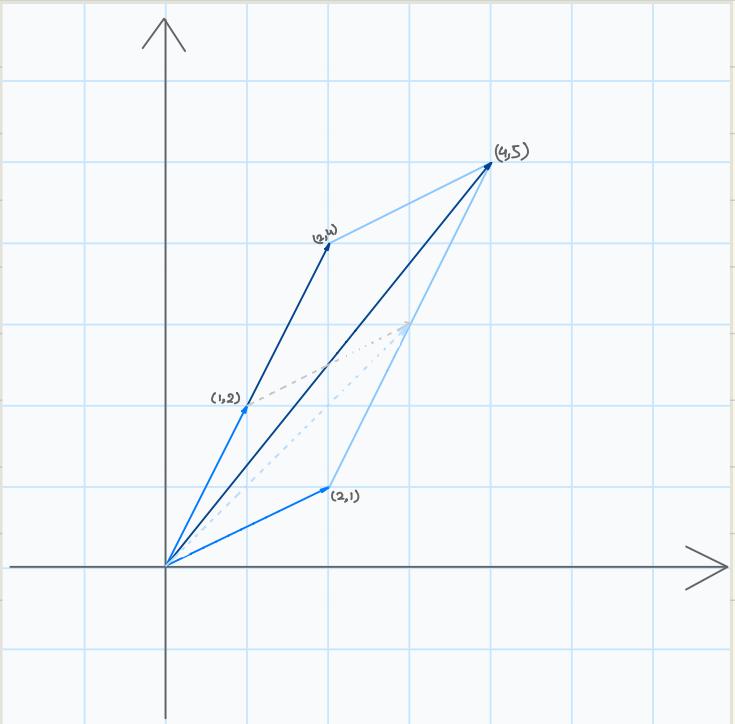
$$\Rightarrow \frac{1}{2}(4,5) - \frac{1}{2}(2,1) = (1,2)$$

$$\Rightarrow (4,5) - 2(1,2) = (2,1)$$

We can write expression as:

$$2(1,2) + (2,1) - (4,5) = (0,0)$$

Observe, zero vector is linear
comb. of $(1,2), (2,1), (4,5)$ with
non zero coeff. $(2,1, -1)$.



Example in \mathbb{R}^3 :

$$2(0,2,1) + \frac{3}{2}(2,2,0)$$

$$(0,4,2) + (3,3,0) = (3,7,2)$$

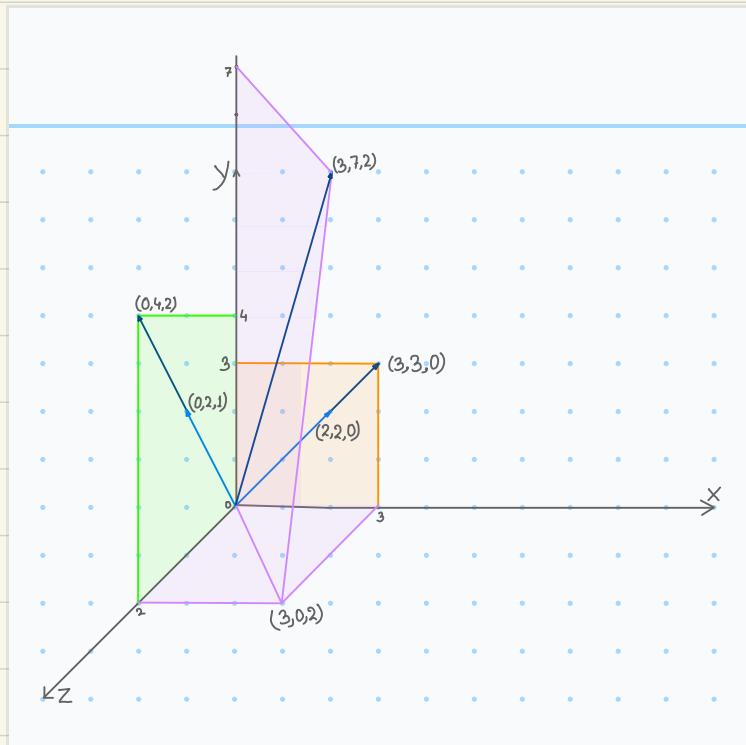
Moreover, each of vectors in expression is linear comb. of other 2 vectors:

- $\frac{1}{2}(3,7,2) - \frac{3}{4}(2,2,0) = (0,2,1)$
- $\frac{2}{3}(3,7,2) - \frac{4}{3}(0,2,1) = (2,2,0)$

Further we can rewrite expression:

$$2(0,2,1) + \frac{3}{2}(2,2,0) - (3,7,2) = (0,0,0)$$

with non-zero coefficient.



The plain of 2 vectors $(0,2,1)$ & $(2,2,0)$ can expressed by eq: $2x - 2y + 4z = 0$

let's choose vector $(1,2,0)$ which is not on plane. Then we can claim that, $(1,2,0)$ can be liner comb of $(0,2,1)$ & $(2,2,0)$. The only way 0 vectors is linear comb. if coeff. = 0.

Linear dependence :

A set of vector v_1, v_2, \dots, v_n from vector space V is linearly dependent if there exist scalars a_1, a_2, \dots, a_n , not all 0. such that:

$$a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$$

So, 0 vector is linear comb. of v_1, v_2, \dots, v_n with non-zero coeff.
Ex: \mathbb{R}^3 : $(2,3,7)$ & $(\frac{5}{3}, \frac{5}{2}, \frac{35}{6})$ can easily be,
 $5(2,3,7) - 6(\frac{5}{3}, \frac{5}{2}, \frac{35}{6}) = (0,0,0)$ Hence, it is linearly dependent.

- ▷ In \mathbb{R}^2 if 2 vectors are linearly dependent, they must be on same line.
- ▷ In \mathbb{R}^3 if 3 vectors are linearly dependent, they must be on same plane.

Linear independence :

A set of vector v_1, v_2, \dots, v_n from a vector space V is said to be linearly independent if v_1, v_2, \dots, v_n are not linearly dependent. 😊

if eq: $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$

can only be satisfied when $a_i=0$ for all $i=1,2,\dots,n$.

A set of vector v_1, v_2, \dots, v_n from vector space V is linearly independent if the only linear comb. of v_1, v_2, \dots, v_n which equals 0 is linear combination with all coefficients 0.

Ex in \mathbb{R}^2 , 2 vector $(-1, 3)$ and $(2, 0)$

Consider eq: $a(-1, 3) + b(2, 0) = (0, 0)$

Hence eq: $-a+2b=0$ and $3a=0$

$\Rightarrow 0, a=0$ and $b=0$

Zero vector :

Let v_1, v_2, \dots, v_n set of vector containing the 0 vector. Suppose, $v_i=0$. Then we can choose $a_i=1$ & $a_j=0$ for $j \neq i$ than linear comb. $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$ but not all coefficient is 0.

Hence, a set of vectors v_1, v_2, \dots, v_n containing 0 vectors is always a linearly dependent set.

- Two non-zero vectors are linearly independent when they are not multiples of each other.
- If 3 vectors are linearly independent, then none of these vectors is linear combination of other two.

□ Linear independence in \mathbb{R}^n :

Vectors are said to be linearly independent if we equate these vectors to 0 it is only possible if all the coefficients are zero.

And if By creating matrix from eq. & vectors find determinant. If determinant of that matrix is non-zero. Means that matrix is invertible. Then those vectors are linearly independent.

$$\therefore \det(A) \neq 0 \quad (\text{Linearly independent})$$

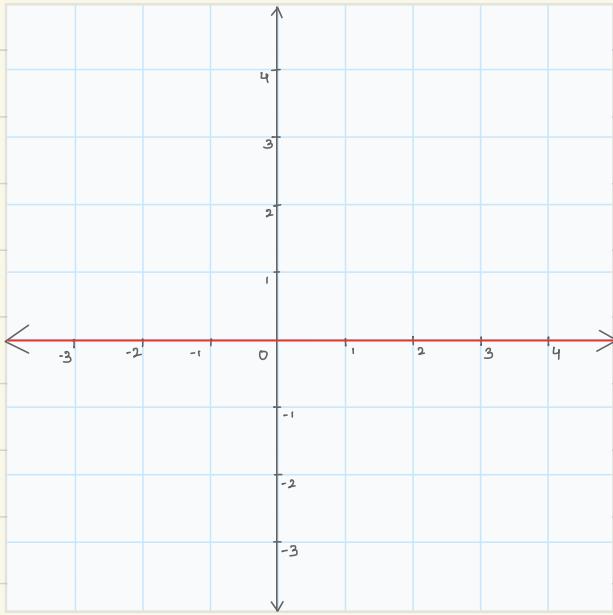
○ Span of set of vectors:

It is set of all finite linear combination of (vector) elements. denoted by $\text{Span}(S)$

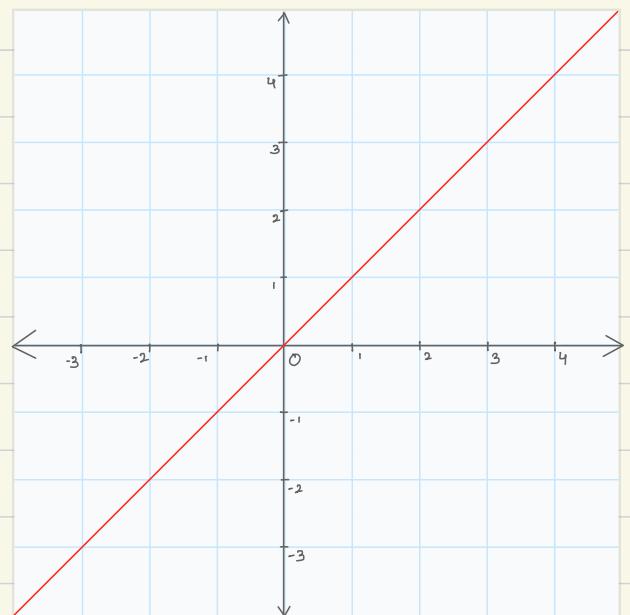
i.e.,

$$\text{Span}(S) = \left\{ \sum_{i=1}^n a_i v_i \mid v_i \in V, a_1, a_2, \dots, a_n \in \mathbb{R} \right\}$$

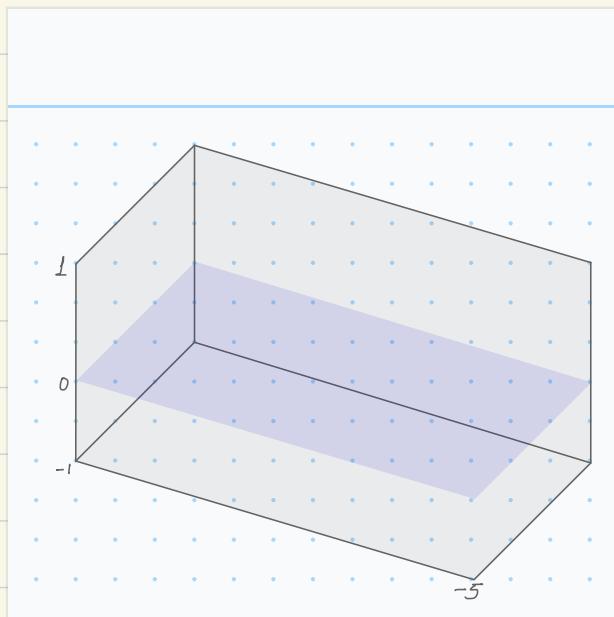
Eg: Span of $\{(1, 0)\} \subset \mathbb{R}^2$
 $= \{a(1, 0) | a \in \mathbb{R}\} = (a, 0)$



Span of $\{(1, 1)\} \subset \mathbb{R}^2$
 $= \{a(1, 1) | a \in \mathbb{R}\} = (a, a)$



Span of $\{(1, 0, 0), (0, 1, 0)\} \subset \mathbb{R}^3$
 then, $\{a(1, 0, 0) + b(0, 1, 0) | a, b \in \mathbb{R}\} = \{(a, b, 0) | a, b \in \mathbb{R}\}$



○ Spanning set of a vector space :

It is a set $S \subseteq V$ is spanning set V if $\text{Span}(S) = V$
 It is combination of those vectors whose comb.

can produce any vector in V of vector space V .

Eg: $S = \{(1, 0), (0, 1)\}$ than $\text{Span}(S) = \mathbb{R}^2$

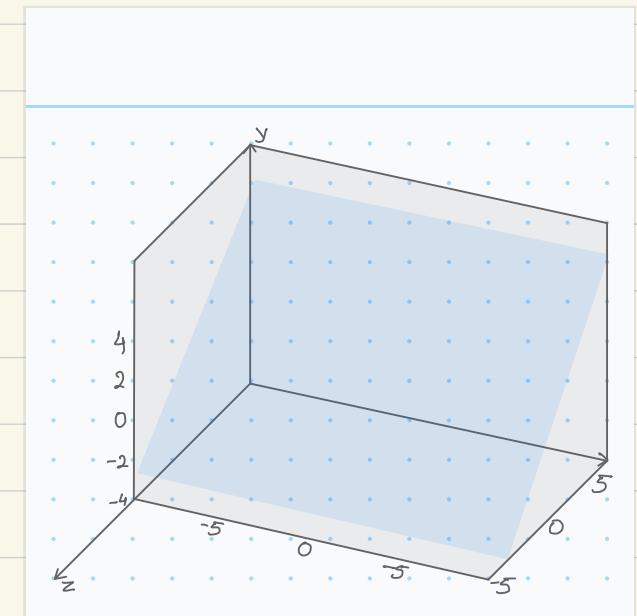
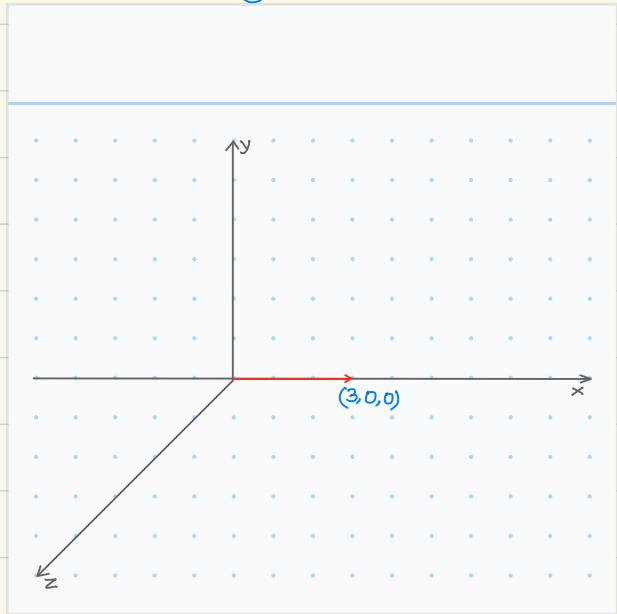
$S = \{(1, 0), (0, 1), (1, 3)\}$ than $\text{Span}(S) = \mathbb{R}^2$

$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ than $\text{Span}(S) = \mathbb{R}^3$

For creating spanning set start with empty set and keep adding new vector which is not in the span of vector we have so far and in this way we create spanning set of n dimension (\mathbb{R}^n).

Eg:- Start with empty set \emptyset as before.

- Thus, $S_0 = \emptyset$ and hence $\text{Span}(S_0) = \text{Span}(\emptyset) = \{(0, 0, 0)\}$
- Append any vector not in $\text{Span}(S_0)$ e.g. $(3, 0, 0)$ & call new S_1 .



- Choose vector outside S_1 , eg $(2, 2, 1)$ and append it to S_1 & call the new set S_2 .
Then $S_2 = S_1 \cup \{(2, 2, 1)\}$ & $\text{Span}(S_2)$ is the plain above.

- Again choose a vector outside $\text{Span}(S_2)$, eg $(1, 3, 3)$
Append it to S_2 & call new set S_3 .

$$\text{Then } S_3 = S_2 \cup \{(1, 3, 3)\}$$

An arbitrary vector $(x, y, z) \in \mathbb{R}^3$ can be written as:

$$(x, y, z) = \underbrace{3x - 5y + 4z}_9 (3, 0, 0) + (y - z)(2, 2, 1) + \underbrace{2z - y}_3 (1, 3, 3)$$

$$\text{Hence, } \text{Span}(S_3) = \mathbb{R}^3$$

○ Basis: Basis B of vector space V is linearly independent subset of V that spans V .

The basis of a vector space is a set of linearly independent vectors that span the full space.

- Equivalent condition for B to be basis:

The following conditions are equivalent to the subset $B \subseteq V$ being a basis:

- B is linearly independent & $\text{span}(B) = V$
- B is maximal linearly independent set. (adding vectors make linearly dependent)
- B is minimal spanning set. (Not spanning by removing 1 vector)

○ How to find basic vectors:-

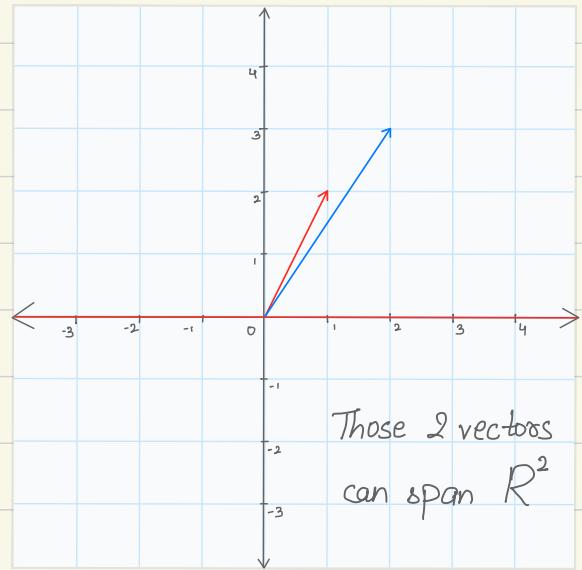
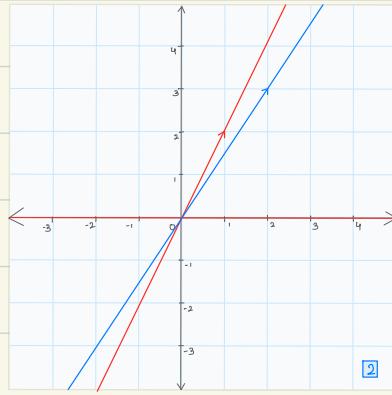
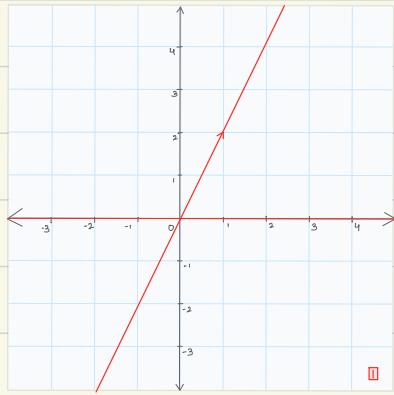
For creating spanning set • start with empty set and keep adding new vector which is not in the span of set obtained. Until we obtain a spanning set.

- Take a spanning set and keep deleting vectors which are linear combination of other vectors until you

remain with vectors that satisfy that they are not linear combination of other one.

Eg. in \mathbb{R}^2 : (Method 1)

- let start with empty set $\{\emptyset\}$ and append non-zero vector $(1,2)$ ①



- Now choose other vector not span of prev. vector eg. $(2,3)$
 $\text{span } \{(1,2), (2,3)\} = \mathbb{R}^2$

Eg in \mathbb{R}^3 (Method 2):

Start with set:

$$S = \{(1,0,0), (1,2,0), (1,0,3), (0,2,3), (0,4,2)\}$$

- Check span of \mathbb{R}^3

$$\text{Now, observe, } (0,4,2) = 2(1,2,0) + \frac{2}{3}(1,0,3) - \frac{8}{3}(1,0,0)$$

$$\text{So, removing } (0,4,2) \quad \{(1,0,0), (1,2,0), (1,0,3), (0,2,3)\}$$

$$- 8/4(0,2,3) = (1,2,0) + (1,0,3) - 2(1,0,0)$$

by deleting $(0,2,3)$ we left with $\{(1,0,0), (1,1,0), (1,0,1)\}$

Final vector is not combination of other vectors.

Hence S_2 forms a basis of \mathbb{R}^3 .

Rank/dimensions of vector space:

It is size or cardinality of basis of vector space.

Eg: If B is basis of V , then the rank is number of elements in B .

Dimension of $\mathbb{R}^n = n$.

Subspace W of \mathbb{R}^3 spanned by $\{(1,0,0), (0,1,0), (3,5,0)\}$

(Dependent) Observe that, $3(1,0,0) + 5(0,1,0) = (3,5,0)$

By removing $\{(1,0,0), (0,1,0)\}$ form subspace W .

Dimension of W is 2.

In term of matrix: Eg: Write a vector which spans W as row of matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix} \xrightarrow{\text{Applying row-reduction}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 0 \end{bmatrix} \xrightarrow{R_3 - 5R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, no' of non-zero rows is dimension of $\text{span}(W) = 2$

Rank of matrix:

Let A be $m \times n$ matrix.

- Column space of A is subspace of \mathbb{R}^m spanned by col. vectors of A .
- Row space of A is subspace of \mathbb{R}^m spanned by row vectors of A .
- Dimension of col. space is def. by col. rank of A .
- Dimension of row space is def. by row rank of A .

Fact: Column rank = row rank which is also called rank of A .

Eg: $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix}$ In reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Non-zero rank & $\text{Rank}(A) = 2$.

□ Rank & dimension using Gaussian elimination:

- Finding dimension and basis with given spanning set:

Considered a vector space W spanned by set S .

Eg: let consider a vector space W spanned by set X :

$$\{(1, 0, 1), (-2, -3, 1), (3, 3, 0)\}$$

→ Forming matrix of vector in the spanning set as rows and reduce to row echelon form:

$$\begin{array}{ccc|c} 1 & 0 & 1 & R_3 - 3R_1 \\ -2 & -3 & 1 & \sim \\ 3 & 3 & 0 & R_2 + 2R_1 \end{array} \sim \begin{array}{ccc|c} 1 & 0 & 1 & R_2/3 \\ 0 & -3 & 3 & \sim \\ 0 & 3 & -3 & R_3 - 3R_2 \end{array} \sim \begin{array}{ccc|c} 1 & 0 & 1 & \\ 0 & 1 & -1 & \\ 0 & 0 & 0 & \end{array}$$

- The dimension of vector space $W = 2$ (non-zero rows)
- The basis of vector space $W = 2$ (non-zero rows)

- Column method:

let W be subspace of R^3 spanned by set S .
 $S = \{(1, 0, 1), (-2, -3, 1), (3, 3, 0)\}$

Form matrix with vectors in S as columns:

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

Row reduced from

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Here first two vectors form basis of W .

○ Solution space of homogeneous system of linear equation or Null space:

→ Let A be an $m \times n$ matrix.

The subspace $W = \{x \in \mathbb{R}^n \mid Ax = 0\}$ of \mathbb{R}^n is called solution space of homogenous system of linear equation $Ax = 0$ or null space of A .

Nullspace is subspace in \mathbb{R}^n . The dimension of null space is called nullity of A .

Null space is the collection of vector x which is multiplied by the given matrix gives zero vector.

□ Finding nullity and basis for nullspace:

We can find nullity by row reduction and gaussian elimination. For homogenous system of linear eq. $Ax = 0$

Eg:- $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

Row reduction - Gaussian elimination

The augment matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array} \right]$$

$$R_3 - 3R_1$$

$$R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Independent variable = x_2, x_3 (nullity = 2)

dependent variable = x_1

Putt $x_2 = t_1$ and $x_3 = t_2$ than equation will be:

$$x_1 = -x_2 - x_3 \quad [\text{from above metrix}]$$

$$x_1 = -t_1 - t_2$$

Hence, nullspace of A (the solution space of $Ax=0$) is $\{(-t_1, -t_2, t_1, t_2) | t_1, t_2 \in \mathbb{R}\}$

Example of solution, $(-5, 2, 3), (-20, 10, 10)$

We get basic vector by putting value of t_1 & t_2 0 & 1 interchangably.

$t_1 = 1, t_2 = 0$ yields the basis vector $(-1, 1, 0)$

$t_1 = 0, t_2 = 0$ yields the basis vector $(-1, 0, 1)$

Hence, basic of nullspace is $(-1, 1, 0), (-1, 0, 1)$

Example 3×4 :

Homogenous system matrix $Ax=0$

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{Aug. mat.} \div \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 2 & 3 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} \text{Row} \\ \text{Reduced} \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$$

Ind. variable = x_4 , dep. var. = x_1, x_2, x_3 . Nullity = 1.

We put $x_4 = t$

Eq. will be:-

$$x_1 - 3x_4 = 0 \quad x_1 = 3t$$

$$x_2 + 3x_4 = 0 \quad x_2 = -3t$$

$$x_3 + 2x_4 = 0 \quad x_3 = -2t$$

Hence nullspace of A (sol. space of $Ax=0$) is $\{(3t, -3t, -2t, t) | t \in \mathbb{R}\}$

Put any thing in place of t and get basic vector.

○ Rank-nullity theorem:

Let A be a $M \times N$ matrix.

Rank (A) calculated as no. of non-zero row (dependent var.) of a matrix in row-echelon form.

Hence,

$$\text{rank}(A) = (\text{no. of row} \neq 0 \text{ of } R) = \text{no. of dep. vars. in } Rx=0$$

$$\text{nullity}(A) = \text{no. of independent variable of } Rx=0$$

So, we have rank-nullity theorem:

For an $m \times n$ matrix A, $\text{rank}(A) + \text{nullity}(A) = n$.

Q How to check if set of n vectors is basis of \mathbb{R}^n :

- We are given n vectors of \mathbb{R}^n
- Then we write them as columns of matrix & get $n \times n$ matrix.
- If determinant of that matrix is 0, then given set of vectors doesn't form a basis, otherwise it form a basis.

Eg: vectors for standard basis are $(1, 0), (0, 1)$ give matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ identity matrix with determinant = 1. so it form basis.

2) We take vectors $(1, -2), (5, -10)$ yields to $\begin{bmatrix} 1 & 5 \\ -2 & -10 \end{bmatrix}$ with 0 determinant. doesn't form basis of \mathbb{R}^2 .

Because they are linear multiple of each other.

3) Is a set $\{(1, 2, 3), (0, 1, 2), (1, 3, 0)\}$ basic of \mathbb{R}^3 ?

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 2 & 0 \end{bmatrix} \quad \det(A) = -5$$

$\det \neq 0$ so it is basis of \mathbb{R}^3 .

° Expression linear combination :

The term $45x_1 + 125x_2$ is an expression.

We can think of it as f_{nA} from \mathbb{R}^3 to \mathbb{R} . Since every value of $x_1, x_2 \& x_3$, we obtain a real no.

Combination can be expressed as:

$$\text{Eg: } C_A(x_1, x_2, x_3) = 45x_1, 125x_2, 150x_3 = [45 \ 125 \ 150]$$

let x_1 = rice, x_2 = daal, x_3 = oil.

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

day 1 office 1 order 30 tiffin with

20kg rice, 10kg daal and 4lts oil.

day 2 office 2 order 40 tiffin with

30kg rice, 12kg daal and 2lts oil.

day 3 office 1 & office 2 with 15 & 50 tiffin.

from above 2 we can get for 15 tiffin office 1: $\frac{1}{2}$

10kg rice, 5kg daal, 2lts oil.

office 2 with 50 tiffin $\frac{5}{4}$

37.5kg rice, 15kg daal and 2.5lts oil.

total req = 47.5kg rice, 20kg daal & 4.5lts oil.

From above coff.

$$45 \times 47.5 + 125 \times 20 + 150 \times 4.5 = 5312.5 \text{ rupees.}$$

We can also calculate it by:

Cost of caterer office 1 day 1 = 2750 rs

Cost of caterer office 2 day 2 = 3150 rs

$$\text{day 3 off. 1\&2} = \frac{1}{2} \times 2750 + \frac{5}{4} \times 3150 = 5312.5 \text{ rs.}$$

Extended example: $\begin{array}{c|ccc} & \text{Rice/kg} & \text{Daal/kg} & \text{Oil/l} \\ \hline \text{Shop A} & 45 & 125 & 150 \\ \text{Shop B} & 40 & 120 & 170 \\ \text{Shop C} & 50 & 130 & 160 \end{array}$

$$C_A(x_1, x_2, x_3) = 45x_1, 125x_2, 150x_3 = \begin{bmatrix} 45 & 125 & 150 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$C_B(x_1, x_2, x_3) = 40x_1, 120x_2, 170x_3 = \begin{bmatrix} 40 & 120 & 170 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$C_C(x_1, x_2, x_3) = 50x_1, 130x_2, 160x_3 = \begin{bmatrix} 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We can compare by creating vectors:

$$\begin{aligned} C(x_1, x_2, x_3) &= (C_A(x_1, x_2, x_3) \quad C_B(x_1, x_2, x_3) \quad C_C(x_1, x_2, x_3)) \\ &= (45x_1 + 125x_2 + 150x_3, 40x_1 + 120x_2 + 170x_3, 50x_1 + 130x_2 + 160x_3) \end{aligned}$$

Now when we are given x_1, x_2 and x_3 we put in this cost fn & see result to find lowest cost.
Apart from this we can also use matrix:

$$C(x_1, x_2, x_3) = \begin{bmatrix} 45 & 125 & 150 \\ 40 & 120 & 170 \\ 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

If we have to buy 2 kg rice, 1 kg Dal, 2 kg oil.

$$\begin{bmatrix} 45 & 125 & 150 \\ 40 & 120 & 170 \\ 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 515 \\ 540 \\ 550 \end{bmatrix}$$

This will be the cost to compare.

It has linearity:

$$\text{means: } C(\alpha(x_1, x_2, x_3) + (y_1, y_2, y_3)) = \alpha C(x_1, x_2, x_3) + C(y_1, y_2, y_3)$$

matrix

$$\begin{bmatrix} 45 & 125 & 150 \\ 40 & 120 & 170 \\ 50 & 130 & 160 \end{bmatrix} \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right\} = \alpha \begin{bmatrix} 45 & 125 & 150 \\ 40 & 120 & 170 \\ 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

□ linear mapping :

A linear mapping f from R^n to R^m can be defined as:

$$f(x_1, x_2, \dots, x_n) = \left(\sum_{j=1}^n a_{1j} x_j, \sum_{j=1}^n a_{2j} x_j, \dots, \sum_{j=1}^n a_{mj} x_j \right)$$

Where coefficients a_{ij} s are real no (scalar). A linear mapping can be thought of collection of linear comb.

We can write it in matrix form.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

Linearity of linear mapping :

Linear mapping satisfy linearity i.e for $c \in R$

$$f(x_1 + cy_1, x_2 + cy_2, \dots, x_n + cy_n) = f(x_1, x_2, \dots, x_n) + cf(y_1, y_2, \dots, y_n)$$

$$f(x_1 + cy_1, x_2 + cy_2, \dots, x_n + cy_n) = A$$

$$\begin{bmatrix} x_1 + cy_1 \\ x_2 + cy_2 \\ \dots \\ x_n + cy_n \end{bmatrix}$$

$$A = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} + c \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix} \right\} = A \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} + cA \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix} = f(x_1, x_2, \dots, x_n) + cf(y_1, y_2, \dots, y_n)$$

○ Linear transformation :

A fn $f: V \rightarrow W$ b/w two vector space V & W is said to be linear transformation if for any 2 vectors v_1 and v_2 in vector space V & for any scalar.

Finally, A linear transformation transformation from $T: R^n \rightarrow R^m$ which satisfy:

- $f(v_1 + v_2) = f(v_1) + f(v_2)$
- $f(cv_1) = cf(v_1)$

Linear mapping are basically linear transformation.

→ For linear transformation, being 1-1 is equivalent to $f(v) = 0$ implies $v = 0$. (one to one)

→ A linear transformation $f: V \rightarrow W$ b/w 2 vector space V and W said to be an isomorphism if it is bijective (onto)

○ Basic determine linear transformation :

Let V be vector space with basic $\{v_1, v_2, \dots, v_n\}$

let, $f: V \rightarrow W$ be linear transformation. Then offered vector $f(v_1), f(v_2), \dots, f(v_n)$ uniquely determine f .

let $v \in V$, $v = \sum_{i=1}^n c_i v_i$

$$f(v) = f\left(\sum_{i=1}^n c_i v_i\right) = \sum_{i=1}^n c_i f(v_i)$$

Eg: linear transformation of matrix.

Consider linear transformation:

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2; f(x, y) = (2x, y)$$

We can represent it in $f(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

By considering standard basis $(1, 0)$ and $(0, 1)$ for \mathbb{R}^2

$$f(1, 0) = (2, 0) = 2(1, 0) + 0(0, 1)$$

$$f(0, 1) = (0, 1) = 0(1, 0) + 1(0, 1)$$

○ Recovering linear transformation:

Let $B = v_1, v_2, \dots, v_n$ & $T = w_1, w_2, \dots, w_m$ ordered basis of V, W .

Suppose A is a $m \times n$ matrix. What is corresponding lin. tran?

Let $v \in V$. Express $v = \sum_{j=1}^n c_j v_j$. Define

$$f(v) = \sum_{j=1}^n c_j \sum_{i=1}^m A_{ij} w_i$$

Check that f is linear transformation.

Letting $c_k = 1$ & $c_j = 0$ for $j \neq k$, we get

$$f(v_k) = A_{1k} w_1 + A_{2k} w_2 + \dots + A_{mk} w_m$$

Hence the matrix corresponding to f is indeed A .

○ Kernel: Kernel is the set of v for which gives output '0' after transformation.

Kernel is denoted as $\text{Ker}(f)$.

$$\text{Ker}(f) = \{v \in V \mid f(v) = 0\}$$

° Image : Image is function's output after transformation. It is range of function.
Image of function denoted as $\text{Im}(f)$

$$\text{Im}(f) = \{w \in W \mid \exists v \in V \text{ for which } f(v) = w\}$$

Eg: Consider $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ $f(x, y) = (2x, y)$

Then $\text{Ker}(f) = (0, 0)$, $\text{Im}(f) = \mathbb{R}^2$	$f(u, y) = (2u, y)$
	$\text{Ker}(f) = (0, y)$
	$\text{Im}(f) = (u, 0) = \mathbb{R}$

Example: Consider $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$, defined by:

$$T(x_1, x_2, x_3, x_4) = (2x_1 + 4x_2 + 6x_3 + 8x_4, x_1 + 3x_2 + 5x_4, x_1 + x_2 + 6x_3 + 3x_4)$$

Choose β & γ to standard (ordered) basis of \mathbb{R}^4 & \mathbb{R}^3

The corresponding matrix is:

$$\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix}$$

Row reduced form

$$\begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Non-pivot or independent variable are x_3 & x_4
so putting $x_3 = s$ & $x_4 = t$ we get, $x_1 = -9s - 2t$ & $x_2 = 3s - t$

By, substituting $s=1, t=0$ and $s=0, t=1$ gives basic vectors $(-9, 3, 1, 0), (-2, -1, 0, 1)$ for null space.

Since we have chosen β to be standard basis, the basis for $\text{ker}(T)$ is also same,

$$-9e_1 + 3e_2 + 1e_3 + 0e_4 = (-9, 3, 1, 0)$$

$$-2e_1 - 1e_2 + 0e_3 + 1e_4 = (-2, -1, 0, 1)$$

Moreover, the pivot columns are 1st & 2nd columns.
Hence column space is first 2 columns of the matrix A, i.e., $(2, 1, 1), (4, 3, 1)$

Since we choose T to standard basis, the basis for $\text{im}(T)$ is also same: ie,

$$2e_1 + e_2 + e_3 = (2, 1, 1), \text{ and}$$

$$4e_1 + 3e_2 + e_3 = (4, 3, 1)$$

Ex-2 (Where basis we use are not standard basis)

$$V = \mathbb{R}^2, W = \{(x, y, z) | x + y + z = 0\}$$

$$\text{let respective ordered basis be } \beta = (1, 1), (1, -1)$$

$$\gamma = (-1, 1, 0), (-1, 0, 1)$$

$T(x, y) = (0, x+2y, -x-2y)$ linear transform from V to W.

$$(1, 1) = 0, 1+2, -1-2 = (0, 3, -3)$$

$$(1, -1) = 0, 1-2, -1+2 = (0, -1, 1)$$

$$\text{So, } (0, 3, -3) \text{ is 3 times } (-1, 1, 0) \text{ & } -3 \text{ times } (-1, 0, 1)$$

$$\xrightarrow{-3} (-3, 3, 0) \xrightarrow{3} (3, 0, -3)$$

$$\text{Similarly } (0, -1, 1) \text{ is } -1 \text{ times } (-1, 1, 0) \text{ & } 1 \text{ times } (-1, 0, 1)$$

$$\xrightarrow{1} (1, -1, 0) \xrightarrow{-1} (-1, 0, 1)$$

$$\text{Corresponding matrix } \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix} \text{ Reduced row echelon form } \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$$

Pivot column = x_1 (dependent)

non pivot col = x_2 (independent), So, $x_2 = t$

$$\text{Then } x_1 = t/3$$

The basis for null space is singleton set consist by substituting $t = 1$ i.e. $\{(1/3, 1)\}$

Therefore $\text{Ker}(t) = \frac{1}{3}(1, 1) + 1(1, 1) = (4/3, -2/3)$

Since pivot col is 1st col of orig. matrix $= (3, -3)$

Therefore $\text{Ker}(t)$ in singleton set consist of $3(-1, 1, 0) + (-3)(-1, 0, 1) = (0, 3, -3)$ (Basis of img of lin trans T)

Rank nullity theorem of linear transformation:

Let $T: V \rightarrow W$ be a lin. Trans.

The rank of T (denoted $\text{rank}(T)$) is dim. of $\text{Im}(T)$

The nullity of T (denotes $\text{nullity}(T)$) is dim. of $\text{ker}(T)$

Reinterpreting rank-nullity theorem of matrices, we obtain: $\text{rank}(t) + \text{nullity}(t) = \dim(V)$

• Equivalence of matrices (denoted by \sim)

Let A & B are $m \times n$ matrices. We say A is equivalent to B if $B = QAP$ for invertible matrices ($P_{n \times n}$ & $Q_{m \times m}$).

• Characters: $\rightarrow A$ can be transformed into B by combining element by row and column operations.
 $\rightarrow \text{rank}(A) = \text{rank}(B)$

Equivalence of matrices is an equivalence relation i.e.:
 $\triangleright A$ is equivalent to itself. ($A = I_{m \times m} A I_{n \times n}$) $A \sim A$

- A is equiv. to B means B is equiv to A.
 $(B = QAP \text{ and } A = Q^{-1}BP^{-1}) \quad | \quad A \sim B \Rightarrow B \sim A$
- A is equiv. to B & B is equiv. to C means C is equiv to A.
 $(B = QAP, C = Q^{-1}BP^{-1}, C = Q'Q A P P^{-1}) \quad | \quad A \sim B, B \sim C \Rightarrow A \sim C$

• Linear transformation and equivalence of matrices:

Consider linear trans. $T: V \rightarrow W$, two ordered basis β_1 & β_2 for V , and two ordered basis γ_1 & γ_2 for W .

Let A be a matrix corresponding to T with respect to basis β_1 & γ_1 and B be matrix corresponding to T with respect to β_2 and γ_2 .

$P \rightarrow$ express the ordered basis β_2 in term of β_1 .

$Q \rightarrow$ Express ordered basis γ_1 in term of γ_2 .

Then $B = QAP$

Eg:- Consider linear transformation $f: R^3 \rightarrow R^2$, defined as:
 $f(x, y, z) = (x+y, y+z)$

Considered 2 ordered basis for R^3

$\beta_1 = (1, 0, 0), (0, 1, 0), (0, 0, 1)$, & $\beta_2 = (1, 1, 0), (0, 1, 1), (0, 0, 1)$

Similarly consider 2 ordered bases for R^2
 $\gamma_1 = (1, 0), (0, 1)$ & $\gamma_2 = (1, 1), (1, 0)$

From Transformation $f(x, y, z) = (x+y, y+z)$ we get:

$$f(1, 0, 0) = (1, 0)$$

$$f(0, 1, 0) = (1, 1) = 1(1, 0) + 1(0, 1)$$

$$f(0, 0, 1) = (0, 1)$$

Hence matrix corresponds to f with respect to basis β_1 and γ_1 is $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Similarly 2nd basis: in $(x+y, y+z)$

$$\begin{aligned} f(1,1,0) &= (2,1) &= 1(1,0) + 1(1,1) \\ f(0,1,1) &= (1,2) &= -1(1,0) + 2(1,1) \\ f(0,0,1) &= (0,1) &= 1(1,0) + 1(1,1) \end{aligned}$$

Hence matrix corresponds to f with respect to basis β_2 and γ_2 is $B = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

choose $P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Then
 (Produced out of air randomly)

$$QAP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix} = B$$

Hence A & B are equivalent to each other.

○ Similar matrices:

An $n \times n$ matrix A is similar to an $n \times n$ matrix B if there exist an $n \times n$ matrix P such that, $B = P^{-1}AP$
 Similarity of matrices is an equivalence relation i.e:
 ▷ A is similar to itself. ($P = I : A = I^{-1}AI$)

► A is similar to B means B is similar to A.
 $(B = P^{-1}AP \Rightarrow PBP^{-1} = A \Rightarrow A = (P^{-1})^{-1}B(P^{-1}))$

► A is similar to B & B is similar to C means C is similar to A.
 $(B = P^{-1}AP, C = Q^{-1}BQ, C = Q^{-1}(P^{-1}AP)Q = Q^{-1}P^{-1}A(PQ) = (PQ)^{-1}A(PQ))$

Important properties (Similar matrices):

→ A and B are equivalent.

→ A and B have same rank.

→ $\det(B) = \det(A)$ (determinants are same)

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P)$$

$$= \frac{1}{\det(P)} \det(A) \det(P) = \det(A)$$

→ Several other invariants of A and B are the same such as characteristic polynomial, minimal polynomial & eigen values with multiplicity.

Eg: Consider a linear transf. $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ where

$$f(x, y, z) = (-x+y+z, x-y+z, x+y-z)$$

Let $\beta = \gamma$ both be the standard ordered bases of \mathbb{R}^3 .

We get-

$$f(1, 0, 0) = (-1, 1, 1) = -1(1, 0, 0) + 1(0, 1, 0) + 1(0, 0, 1)$$

$$f(0, 1, 0) = (1, -1, 1) = 1(1, 0, 0) - 1(0, 1, 0) + 1(0, 0, 1)$$

$$f(0, 0, 1) = (1, 1, -1) = 1(1, 0, 0) + 1(0, 1, 0) - 1(0, 0, 1)$$

Hence the matrix of f corresponding standard basis

$$\text{is : } \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Consider other basis $B' =$

Then we have:

$$f(1,1,1) = (1,1,1) = 1(1,1,1) + 0(-1,1,0) + 0(-1,0,1)$$

$$f(-1,1,0) = (2, -2, 0) = 0(1,1,1) - 2(-1,1,0) + 0(-1,0,1)$$

$$f(-1,0,1) = (2, 0, -2) = 0(1,1,1) + 0(-1,1,0) - 2(-1,0,1)$$

Hence the matrix of f corresponding ordered basis B'

$$\text{is : } \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\text{let } P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ then } P^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

$$\text{Then } P^{-1} A P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & -4/3 & 2/3 \\ 2/3 & 2/3 & -4/3 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Hence A and B are similar matrix.

• Linear transformation and similarity of matrices:

Consider linear trans. $T: V \rightarrow V$, two ordered basis β & γ for V .

Let A be a matrix corresponding to T with respect to basis β and B be matrix corresponding to T with respect to γ .

$P \rightarrow$ express the ordered basis γ in term of B .
 $Q \rightarrow$ Express ordered basis β in term of γ .
 Then, $B = P^{-1}AP$.

-Affine subspace: It is moving subspace from base to top of vector.

○ Dot product of 2 vectors in R^2 :

By generalizing in (x_1, y_1) and (x_2, y_2) in R^2 . Dot product we get $(x_1, y_1) \cdot (x_2, y_2) = x_1x_2 + y_1y_2$

Consider 2 vector $(3, 4)$ and $(2, 7)$ in R^2 . Dot product of these two vectors gives us scalar as follows:

$$\square (3, 4) \cdot (2, 7) = (3 \times 2) + (4 \times 7) = 6 + 28 = 34$$

○ The length of vector in R^2 :

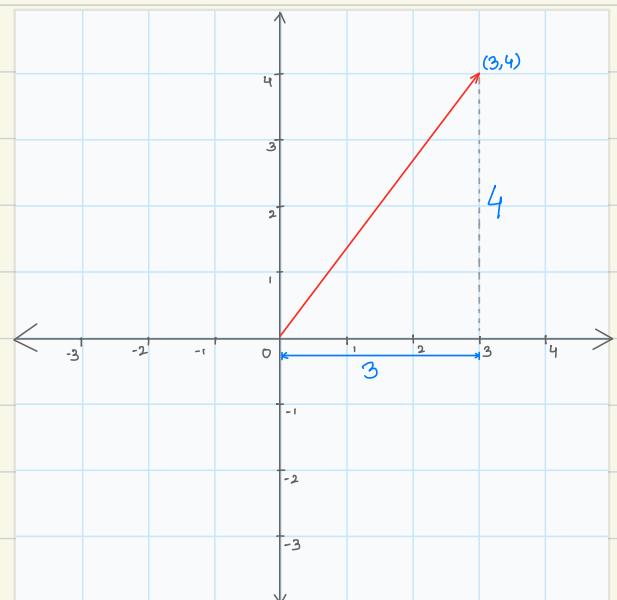
For previous example length of vector $(3, 4)$ in R^2 is:

Using pythagoras' theorem, the length of vector $(3, 4)$ is

$$\sqrt{3^2 + 4^2} = 5$$

To generalise it length of vector (x, y)

$$= \sqrt{x^2 + y^2}$$



- The relation between length and dot product in \mathbb{R}^2 :

Observe that dot product of $(3, 4) \cdot (3, 4)$ is $3^2 + 4^2$.

Then the length of $(3, 4)$ is square root of the dot product of the vector with itself.

$$\text{Length of vector } (3, 4) = \sqrt{(3, 4) \cdot (3, 4)} = \sqrt{3^2 + 4^2} = 5$$

More generally, the length of vector $(x, y) \in \mathbb{R}^2$ is

$$\sqrt{x^2 + y^2} = \sqrt{(x, y) \cdot (x, y)}$$

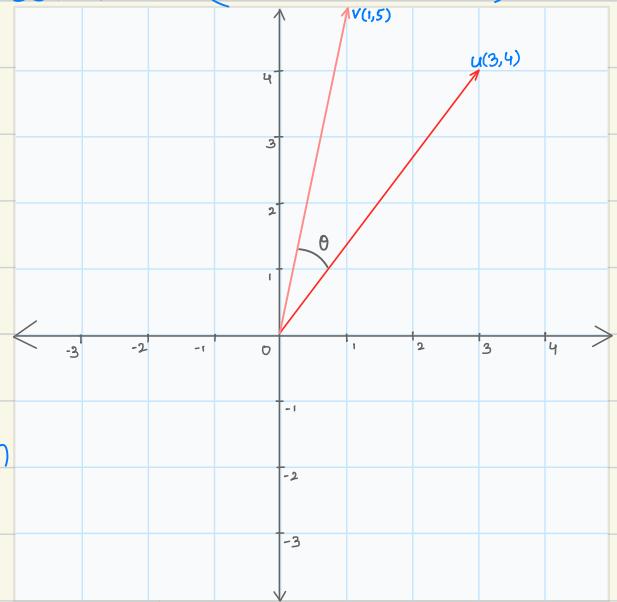
- The angle b/w 2 vectors in \mathbb{R}^2 :

The angle between vector 'u' and 'v' and measure how far the direction of v from u (or viceversa).

Eg. θ be angle b/w $u(3, 4)$ & $v(1, 5)$.

- It can be measured in degree (0-360) or in radian (0- 2π)

- Angle is often described by computing trigonometric fn eg. sin, cos, tan



- The dot product of 2 vectors in \mathbb{R}^3 :

Similar to \mathbb{R}^2 here also after multiplying at same level we add it.

$$\text{Eg. } (1, 2, 3) \cdot (2, 0, 1) = (1 \times 2) + (2 \times 0) + (3 \times 1) = 2 + 0 + 3 = 5$$

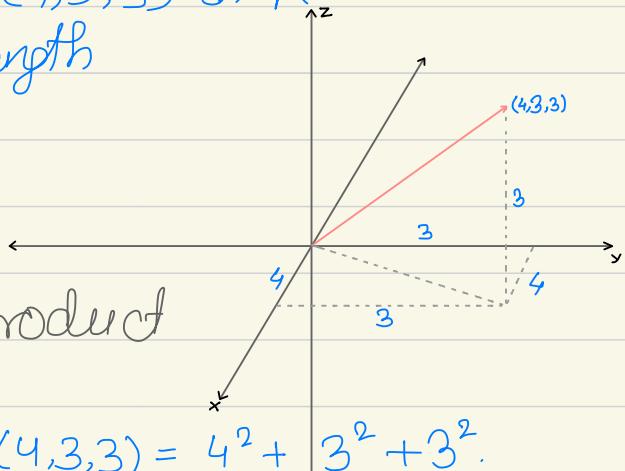
$$\text{Generalizing: } (x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = x_1 x_2 + y_1 y_2 + z_1 z_2$$

○ Length of vector in \mathbb{R}^3 :

lets find length of vector $(4,3,3)$ in \mathbb{R}^3 .

Here also by pythagoras we find length of $(4,3,3)$ vector is:

$$\sqrt{4^2 + 3^2 + 3^2} = \sqrt{34} \text{ unit}$$



○ Similarly the length and dot product in \mathbb{R}^3

Observe that dot product of $(4,3,3) \cdot (4,3,3) = 4^2 + 3^2 + 3^2$.

Then the length of $(4,3,3)$ is square root of the dot product of the vector with itself.

$$\begin{aligned} \text{Length of vector } (4,3,3) &= \sqrt{(4,3,3) \cdot (4,3,3)} \\ &= \sqrt{4^2 + 3^2 + 3^2} = \sqrt{34} \end{aligned}$$

More generally, the length of vector $(x, y, z) \in \mathbb{R}^3$ is

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{(x, y, z) \cdot (x, y, z)}$$

○ The angle b/w 2 vectors in \mathbb{R}^3 and the dot product:

The angle between vector 'u' and 'v' in \mathbb{R}^3 is angle b/w them computed by passing a plane through them.

It measures how far the direction of v from u (viceversa) let u & v be 2 vectors in \mathbb{R}^3 . Then we can compute angle θ b/w the vectors u & v using dot product as:

$$\cos \theta = \frac{u \cdot v}{\sqrt{(v \cdot v) \cdot (u \cdot u)}}$$

$$\text{i.e. } \theta = \cos^{-1} \left(\frac{u \cdot v}{\sqrt{(v \cdot v) \cdot (u \cdot u)}} \right)$$

Example:

Compute angle in \mathbb{R}^3

lets compute angle θ b/w $(1,0,0)$ & $(1,0,1)$

$$(1,0,0) \cdot (1,0,1) = 1(1,0,1) \cdot (1,0,1) = 2(1,0,0) \cdot (1,0,0) = 1$$

$$\text{Hence, } \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \text{ radians or } 45^\circ$$

° Dot product in \mathbb{R}^n : length and angle:

- Let $u = (u_1, u_2, \dots, u_n)$ & $v = (v_1, v_2, \dots, v_n)$ be vectors in \mathbb{R}^n
- The dot product of 2 vectors u and v is defined as $u \cdot v = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$.
- The length of vector u is denoted by $\|u\|$ and defined by $\|u\| = \sqrt{u \cdot u}$
- The angle θ b/w 2 vector u and v is measured on the 2-dimensional plane spanned by u and v and can be computed as:

$$\cos \theta = \frac{u \cdot v}{\|u\| \times \|v\|}$$

$$\text{i.e. } \theta = \cos^{-1}\left(\frac{u \cdot v}{\sqrt{(v \cdot v) \cdot (u \cdot u)}}\right)$$

° Inner product of vector space:

An inner product of a vector space V in a fn $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ satisfies following:
 $\rightarrow \langle v, v \rangle > 0$ for all $v \in V \setminus \{0\}$; $\langle v, v \rangle = 0$ only if $v = 0$.

$$\rightarrow \langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$$

$$\rightarrow \langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$$

$$\rightarrow \langle cv_1, v_2 \rangle = c \langle v_1, v_2 \rangle = \langle v_1, cv_2 \rangle$$

- A vector space V together with an inner product $\langle \cdot, \cdot \rangle$ is called an inner product space.
- The dot product is an example of an inner product.

Example of inner product in R^2 :

$$\langle \cdot, \cdot \rangle : R^2 \times R^2 \rightarrow R$$

$$\langle u, v \rangle = x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2 x_2 y_2$$

where $u = (x_1, x_2)$ and $v = (y_1, y_2)$ be in R^2 .

Can be written as:

$$[x_1 \ x_2] \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Eg

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$u^T v = [u_1 \ u_2 \ u_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

If $u^T v = 0$ then u and v are orthogonal \perp .

○ Norm on vector space:

A norm of a vector space V is a function

$$\| \cdot \| : V \rightarrow \mathbb{R}$$

$$x \mapsto \| x \|$$

Satisfying foll. cond:

- $\|x+y\| \leq \|x\| + \|y\|$, for all $x, y \in V$
- $\|cx\| = |c|\|x\|$ for all $c \in R$ and for all $x \in V$
- $\|x\| \geq 0$ for all $x \in V$; $\|x\|=0$ if and only if $x=0$

length as an example of norm:

Recall length of a vector $u = (x_1, x_2, \dots, x_n) \in R^n$ is
 $\|u\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

The length fn $R^n \rightarrow R$ is norm on R^n

Example of norm on R^n :

Define $\|u\| = |x_1| + |x_2| + \dots + |x_n|$ for $u = (x_1, x_2, \dots, x_n) \in R^n$

○ The inner product includes a norm:

let V is an inner product space with inner product $\langle \cdot, \cdot \rangle$

Then the fn $\| \cdot \| : V \rightarrow R$ defined by $\|v\| = \sqrt{\langle v, v \rangle}$ is a norm of V .

○ Geometric intuition of orthogonal vectors:

If angle θ b/w 2 vectors u and v in R^n is right angle 90° then:

$$\cos(\theta) = 0 = \frac{u \cdot v}{\|u\| \|v\|}$$

then $u \cdot v = 0$

e.g. $(1, 2, 3)$ and $(2, 2, -1)$ $= 1 \cdot 2 + 2 \cdot 2 - 6 = 2 + 4 - 6 = 0$.

Orthogonal vector:

Two vectors u and v of an inner product space V are said to be orthogonal if $\langle u, v \rangle = 0$

Eg: Consider \mathbb{R}^2 with an inner product

$$\langle u, v \rangle = x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2x_2 y_2$$

where $u = (x_1, x_2)$ and $v = (y_1, y_2)$ be in \mathbb{R}^2 .

Then vector $(1, 1)$ and $(1, 0)$ are orthogonal by:

$$\begin{aligned}\langle (1, 1), (1, 0) \rangle &= 1 \times 1 - (1 \times 0 + 1 \times 1) + 2 \times 1 \times 0 \\ &= 1 - 1 + 0 = 0\end{aligned}$$

Orthogonal set of vector:

It is set of vectors whose elements are mutually orthogonal.

Explicitly, if $S = \{v_1, v_2, \dots, v_k\} \subseteq V$, then S is an orthogonal set of vectors if:

$$\langle v_i, v_j \rangle = 0 \text{ if } i, j \in \{1, 2, \dots, k\} \text{ are } i \neq j.$$

Eg: In \mathbb{R}^3 with usual dot product. The set

$S = \{(4, 3, 2), (-3, 2, -3), (-5, 18, 17)\}$ is an orthogonal set of vectors.

like:

$$(4, 3, 2) \cdot (-3, 2, -3) = -12 + 6 + 6 = 0$$

$$(4, 3, 2) \cdot (-5, 18, 17) = -20 + 54 - 34 = 0$$

$$(-3, 2, 3) \cdot (-5, 18, 17) = 15 + 36 - 51 = 0$$

○ Orthogonality and linear independence:

Let $\{v_1, v_2, \dots, v_k\}$ be an orthogonal set of non-zero vectors in the inner product space V .

Then $\{v_1, v_2, \dots, v_k\}$ is linearly independent set of vectors.

◦ Orthogonal basis:

Let V be an inner product space. A basis consist of mutually orthogonal vectors is called orthogonal basis.

Since an orthogonal set of vectors is already linearly independent, an orthogonal set is a basis when it is maximal orthogonal set. (there is no orthogonal set strictly containing this one)

If $\dim(V) = n$, then

orthogonal basis \equiv orthogonal set of n vectors.

Example :

1. Standard basis.

2. $\{(4, 3, -2), (-3, 2, -3), (-5, 18, 17)\} \subseteq \mathbb{R}^3$

3. Consider \mathbb{R}^2 with inner product

◦ Orthonormal set:

It is an orthogonal set of vector such that the norm(length) of each vector of set is 1.

Simply it is all the vectors in B which have length 1 and all vectors are orthogonal 90° to each other.

Explicitly: $S = \{v_1, v_2, \dots, v_k\} \subseteq V$, then S is is orthonormal set of vectors if orthogonal $\langle v_i, v_j \rangle = 0$ for $i, j \in \{1, 2, \dots, k\}$ and $i \neq k$ and length $\|v_i\| = 1 \quad \forall i \in \{1, 2, \dots, k\}$

It is same as orthogonal set with norm(length) of each vector 1.

□ Orthonormal basis:

A orthonormal set which form a basis is called orthonormal basis.

Equivalently: An orthonormal basis is an orthogonal basis where the norm of each vector is 1.

- An orthonormal basis is a maximal orthonormal set.

Eg: The standard basis with respect to usual inner product forms an orthonormal basis.

$$\langle e_i, e_j \rangle = (0, 0, \dots, 0, 1, 0, \dots, 0) \cdot (0, 0, \dots, 1, 0, 0, \dots, 0)$$

$$= 0 \times 0 + 0 \times 0 + \dots + 0 \times 1 + 1 \times 0 + 0 \times 0 + \dots + 0 \times 0 = 1$$

$$\|e_i\| = \sqrt{\langle e_i, e_i \rangle} = \sqrt{0 \times 0 + \dots + 1 \times 1 + 0 \times 0 + \dots + 0 \times 0} = \sqrt{1} = 1$$

Eg: Consider \mathbb{R}^3 with usual inner product and the set $\beta = \left\{ \frac{1}{3}(1, 2, 2), \frac{1}{3}(-2, -1, 2), \frac{1}{3}(2, -2, 1) \right\}$, Then β form an orthogonal basis of \mathbb{R}^3 .

$$\text{In this, } \|v_1\|^2 = \|v_2\|^2 = \|v_3\|^2 = 1 \quad (\text{norm/length})$$

$$\text{and, } \langle v_1, v_2 \rangle = \langle v_2, v_3 \rangle = \langle v_1, v_3 \rangle = 0$$

cardianility $|\beta| = 3$ & β is linearly independent.

& $\text{Dim}(\mathbb{R}^3) = 3$ so, β is an orthonormal basis.

○ Obtaining orthonormal sets from orthogonal sets:

let V be an inner product space. if $\Gamma = \{v_1, v_2, \dots, v_k\}$ is an orthogonal set of vectors, then we can obtain orthonormal set of vectors β from Γ by:

$$\beta = \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \dots, \frac{v_k}{\|v_k\|} \right\} \quad \begin{array}{l} \langle v_i, v_j \rangle = 0 \\ \langle \frac{v_i}{\|v_i\|}, \frac{v_j}{\|v_j\|} \rangle = 0 \end{array} \quad \left\| \frac{v_i}{\|v_i\|} \right\| = \frac{1}{\|v_i\|} \|v_i\| = 1$$

Consider example in \mathbb{R}^2 with usual innerproduct and the orthogonal basis $\Gamma = \{(1,3), (-3,1)\}$

norm, $\|(1,3)\| = 10$, $\|(-3,1)\| = 10$

then $\beta = \left[\frac{1}{\sqrt{10}}(1,3), \frac{1}{\sqrt{10}}(-3,1) \right]$ is orthonormal basis of \mathbb{R}^2 .

Importance of orthonormal basis:

Suppose $\Gamma = \{v_1, v_2, \dots, v_n\}$ is an orthonormal basis of an inner product space V and let $v \in V$.

Then v can be written as:

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n.$$

Since Γ is orthonormal, we can use inner product and compute $c_i = \langle v, v_i \rangle$

$$\begin{aligned} \langle v, v_i \rangle &= \langle c_1 v_1 + c_2 v_2 + \dots + c_n v_n, v_i \rangle \\ &= c_1 \langle v_1, v_i \rangle + c_2 \langle v_2, v_i \rangle + \dots + c_i \langle v_i, v_i \rangle + \dots + c_n \langle v_n, v_i \rangle \\ &= c_i \langle v_i, v_i \rangle \text{ all other become 0.} \\ &= c_i \|v_i\|^2 = c_i \end{aligned}$$

Eg: $\left[\frac{1}{\sqrt{10}}(1,3), \frac{1}{\sqrt{10}}(-3,1) \right]$ is orthonormal basis of \mathbb{R}^2 . Write $(2,5)$ as lin. comb. in term of basis vector.

$$(2,5) = c_1 \frac{1}{\sqrt{10}}(1,3) + c_2 \frac{1}{\sqrt{10}}(-3,1)$$

$$c_1 = \left\langle (2,5), \frac{1}{\sqrt{10}}(1,3) \right\rangle = \frac{1}{\sqrt{10}} (2 \times 1 + 5 \times 3) = \frac{1}{\sqrt{10}} \times 17 = \frac{17}{\sqrt{10}}$$

$$c_2 = \left\langle (2,5), \frac{1}{\sqrt{10}}(-3,1) \right\rangle = \frac{1}{\sqrt{10}} (2 \times (-3) + 5 \times 1) = \frac{-1}{\sqrt{10}}$$

We got the coefficient

$$(2, 5) = \frac{17}{\sqrt{10}} v_1 + \frac{-1}{\sqrt{10}} v_2 = \frac{17}{\sqrt{10}} \times \frac{1}{\sqrt{10}} (1, 3) - \frac{1}{\sqrt{10}} \times \frac{1}{\sqrt{10}} (-3, 1)$$

$$= \frac{17}{10} (1, 3) - \frac{1}{10} (-3, 1)$$

Gram-Schmidt process overview:

In an inner product space:

Any basis
 x_1, x_2, \dots, x_n

Gram-Schmidt
 process

Orthonormal basis
 v_1, v_2, \dots, v_n

- Example and intuition:

Consider the basis $\beta = \{(1, 2, 2), (-1, 0, 2), (0, 0, 1)\}$ for \mathbb{R}^3 .

Orthonormal basis for \mathbb{R}^3 will be?

Let $v_1 = (1, 2, 2)$. Vector orthogonal to v_1 , i.e. a vector in $\langle v_1 \rangle$, so we use projection P_{v_1} to v_1

Define $v_2 = (-1, 0, 2) - P_{v_1}((-1, 0, 2))$

$$= (-1, 0, 2) - \frac{\langle (-1, 0, 2), (1, 2, 2) \rangle}{\langle (1, 2, 2), (1, 2, 2) \rangle} (1, 2, 2)$$

$$= \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3} \right)$$

We want a vector which is orthogonal to both v_1 & v_2 , i.e. a vector in $\text{span}(\{v_1, v_2\})^\perp$, so we use Projection $P_{\text{span}(\{v_1, v_2\})}^{\perp}$ to $\text{span}(\{v_1, v_2\})$.

Define $v_3 = (0, 0, 1) - P_{v_1}((0, 0, 1)) - P_{v_2}((0, 0, 1))$

$$= (0, 0, 1) - \frac{\langle (0, 0, 1), (-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}) \rangle}{\langle (-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}), (-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}) \rangle} (-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3})$$

$$= \left(\frac{2}{9}, -\frac{2}{9}, \frac{1}{9} \right)$$

Thus v_1, v_2, v_3 is an orthogonal basis and dividing each vector by its norm yields an orthonormal basis:

$$= \left\{ \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right), \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right), \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \right\}$$

You can check by $\langle v_1, v_2 \rangle = \langle v_2, v_3 \rangle = \langle v_1, v_3 \rangle = 0$

o The Gram-Schmidt process :

Let V be inner product space with a basis $\{x_1, x_2, \dots, x_n\}$. Define the orthogonal basis $\{v_1, v_2, \dots, v_n\}$ and the corresponding orthonormal basis $\{w_1, w_2, \dots, w_n\}$ as follows:

$$v_1 = x_1; w_1 = \frac{v_1}{\|v_1\|}$$

$$v_2 = x_2 - \langle x_2, w_1 \rangle w_1; w_2 = \frac{v_2}{\|v_2\|}$$

... ...

$$v_i = x_i - \langle x_i, w_1 \rangle w_1 - \langle x_i, w_2 \rangle w_2 - \dots - \langle x_i, w_{i-1} \rangle w_{i-1}; w_i = \frac{v_i}{\|v_i\|}$$

... ...

$$v_n = x_n - \langle x_n, w_1 \rangle w_1 - \langle x_n, w_2 \rangle w_2 - \dots - \langle x_n, w_{n-1} \rangle w_{n-1}; w_n = \frac{v_n}{\|v_n\|}$$

- Any finite-dimensional vector space with an inner product has an orthogonal basis.
- Any basis can be changed to an orthonormal basis using Gram-Schmidt Process.

Orthogonal transformation :-

Let V be an inner product space and T be a linear transformation from V to V . T is said to be orthogonal transformation if

$$\langle Tv, Tw \rangle = \langle v, w \rangle \quad \forall v, w \in V.$$

When $V = \mathbb{R}^n$ with usual inner product, a linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is orthogonal if it preserves angle and length.

If length is preserved then angle automatically got preserved.

Example:



Finding rotation matrix in \mathbb{R}^2 :

Consider the standard basis $\{(1,0), (0,1)\}$ of \mathbb{R}^2 .

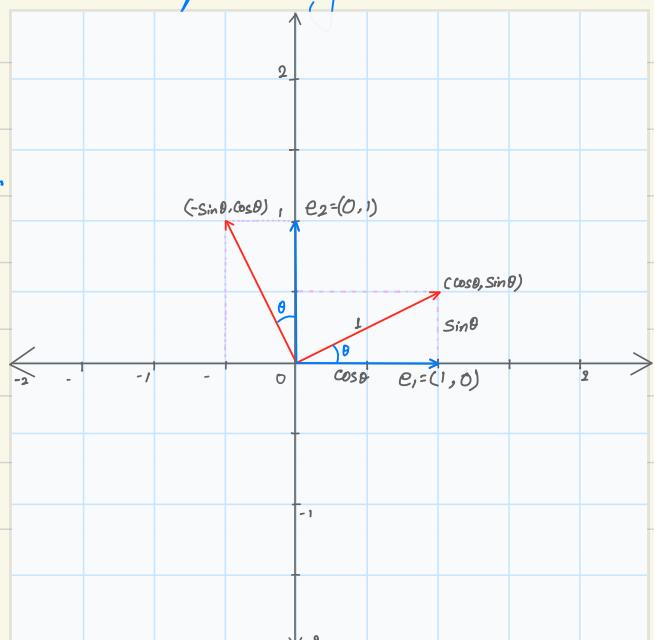
Rotate the plane by angle θ . The vector obtained after rotation tells us matrix corresponding to linear transformation.

Blue one is basic vectors and reds are transformed vector.

Let T be corresponding linear transformation. Then:

$$T(1,0) = (\cos(\theta), \sin(\theta))$$

$$\text{and, } T(0,1) = (-\sin(\theta), \cos(\theta)).$$



Thus matrix corresponding to linear transformation
is: $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$

Note: $R_\theta^T = R_{-\theta}$ & $R_\theta^T R_\theta = R_\theta R_\theta^T = I$

So, length and angle are preserved & std. basis are orthonormal then rotated vector are also orthonormal & yields an orthonormal basis of \mathbb{R}^2 .

° Orthogonal matrix:

As $\{v_1, v_2, v_3\}$ is orthonormal set, the linear transformation T is an orthogonal transformation.

Observe that: $A A^T = A^T A = I_3$

A square matrix A is called an orthogonal matrix.

if: $A A^T = A^T A = I$

□ Multivariable functions : Visualization.

○ Scalar - valued multivariable function :

A scalar - valued multivariable function is a function $f: D \rightarrow \mathbb{R}$ where D is domain in \mathbb{R}^n ($n \geq 1$)
Examples:

- Linear transformation.
- Polynomial transformation.
- (Arithmetic) Combinations or compositions.

○ Vector - valued multivariable function :

A vector - valued multivariable function is a function $f: D \rightarrow \mathbb{R}^m$ where D is domain in \mathbb{R}^n ($m, n \geq 1$)
It can be thought of as a vector of scalar-valued multivariable function.

Example of linear transformation :

$$f(x, y, z) = (x^2 + y^2, y^2 + z^2, z^2 + x^2)$$
$$\mathbb{R}^3 \rightarrow \mathbb{R}^3$$

○ Multivariable function (Function of several variable):

It is either a scalar - valued multivariable function or a vector - valued multivariable function .

When considering multivariable fn we write $f: D \rightarrow \mathbb{R}^m$ where D is domain in \mathbb{R}^n where $n \geq 1$ and with no restriction on m (i.e m can be 1).

To refer element in D without bothering about coordinate we use $x \in D$.

Examples:

$$f(x, y) = 2.5x - 3.4y$$

$$f(x, y) = 3x^3 - 3y^2 + 4.8x^2y - 9.9xy + \pi$$

$$f(x, y) = \sin(x^2 + y^2)$$

$$f(x, y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

$$f(x, y) = 10e^{-2x-5y}$$

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

$$f(x, y) = u^2 + y^2 + z^2$$

$$f(x, y) = (2x + 2y + 2z)$$

$$f(u, y) = (\sin(u)\cos(y), \tan(y+z), \ln(u^2 + y^2 + z^2), e^{xyz}) | R^3 \rightarrow R^4$$

o Arithmetic operations on multivariable function:

let $D \subset R^n$ and $f: D \rightarrow R^m$, $g: D \rightarrow R^m$ be multivariable function on D .

- The sum function $f+g$ is defined on D by:
$$(f+g)(x) \rightarrow f(x) + g(x), x \in D$$

- Let $c \in R$. The function cf defined on D by:
$$(cf)(x) \equiv c \times f(x), x \in D$$

- If $m=1$, the product function fg is defined on D by:
$$fg(x) = f(x) \times g(x), x \in D$$

- If $m=1$, & $g(x) \neq 0$, $x \in D$, the quotient f/g is defined on D by:
$$(f/g)(x) = f(x)/g(x), x \in D$$

○ Function obtained by composition:

let $D \in \mathbb{R}^n$ & $f: D \rightarrow \mathbb{R}^m$ be a multivariable function.

let $g: E \rightarrow \mathbb{R}^p$ be a function on E where $\text{Range}(f) \subseteq E \subseteq \mathbb{R}^m$

Then for each $x \in D$, $f(x) \in E$ & therefore $g(f(x))$ yields a well-defined element in \mathbb{R}^p .

We can obtain multivariable fn: $gof: D \rightarrow \mathbb{R}^p$ called composition of f and g defined as:

$$gof(x) = g(f(x)), x \in D.$$

Example: $f(u, v) = u^2 + v^2$ is fn from \mathbb{R}^2 to \mathbb{R} .

$g(x) = \sqrt{x}$ is fn from $E = \{x \in \mathbb{R} \mid x \geq 0\}$ to \mathbb{R}

$$\text{Then } gof(x) = \sqrt{u^2 + v^2}$$

○ Curves in \mathbb{R}^m :

A curve in \mathbb{R}^m refers to the range of a function $f: D \rightarrow \mathbb{R}^m$ where D is a domain in \mathbb{R} .

Example: - line in \mathbb{R}^m

- $\Gamma(f)$ where f is function of one variable.

- Conicks in \mathbb{R}^2

- Helix in \mathbb{R}^3 : $s(t) = (\cos(t), \sin(t), t)$

- The subset $\{(u, v) \mid v^2 = u^3\}$ of \mathbb{R}^2 .

□ Things to remember for further course:

- From now on function means scalar-valued multivariable function

- If a is point in \mathbb{R}^n , then open ball of radius r around a is defined:

$$\{x \in \mathbb{R}^n \mid \|x - a\| < r\}$$

- e_1, e_2, \dots, e_n is the standard ordered basis of \mathbb{R}^n .

◦ Rate of change w.r.t a particular variable at a point:
 let $f(u_1, u_2, \dots, u_n)$ be a fn defined on D in \mathbb{R}^n containing a point \underline{a} and an open ball around it (radius).

Then rate of change of f at \underline{a} w.r.t the variable x_i is

$$\lim_{h \rightarrow 0} \frac{f(\underline{a} + h e_i) - f(\underline{a})}{h} \quad \rightarrow \text{i}^{\text{th}} \text{ place}$$

here, $\underline{a} = (a_1, a_2, \dots, a_n) \mid e_i = (0, 0, 0, \dots, 0, 1, 0, 0, \dots, 0, 0)$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f((a_1, a_2, \dots, a_n) + h(0, 0, \dots, 1, 0, 0, \dots, 0)) - f(a_1, a_2, \dots, a_n)}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(a_1, a_2, \dots, a_{i-1}, a_i + h, a_{i+1}, \dots, a_n) - f(a_1, a_2, \dots, a_n)}{h}$$

Example:-

◦ Rate of change $f(u, y) = ux + y$ at $(0, 0)$ w.r.t. x

$$\rightarrow \lim_{x \rightarrow 0} \frac{f((0, 0) + h(1, 0)) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{h-0}{h} \Rightarrow \lim_{x \rightarrow 0} \frac{h}{h} = 1$$

◦ Rate of change of $f(u, y, z) = uy + yz + zx$ at $(1, 2, 3)$ w.r.t. y

$$\Rightarrow \lim_{x \rightarrow 0} \frac{f((1, 2, 3) + h(0, 1, 0)) - f(1, 2, 3)}{h} = \lim_{x \rightarrow 0} \frac{f(1, 2+h, 3) - f(1, 2, 3)}{h}$$

$$\lim_{x \rightarrow 0} \frac{1(2+h)+2(1+h)3+1 \cancel{x}3 - 1 \cancel{x}2 - 2 \cancel{x}3 - 1 \cancel{x}3}{h} = \lim_{x \rightarrow 0} \frac{h+3h}{h} = 4$$

○ Rate of change of $f(u, y) = \sin(u, y)$ at $(1, 0)$ w.r.t x .

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(1, 0) + h(1, 0) - f(1, 0)}{h} = \lim_{h \rightarrow 0} \frac{f(1+h, 0) - f(1, 0)}{h} = \lim_{h \rightarrow 0} \frac{0-0}{h} = 0.$$

w.r.t (with respect to) y :

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(1, 0) + h(0, 1) - f(1, 0)}{h} = \lim_{h \rightarrow 0} \frac{\sin(h) - 0}{h} = \lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1.$$

○ Partial derivatives:

Let $f(x_1, x_2, \dots, x_n)$ be function defined on domain D in \mathbb{R}^n . The partial derivative of f w.r.t x_i is the fn denoted by :

$\partial f / \partial x_i$ or $\frac{\partial f}{\partial x_i}(x)$ as defined as:

$\partial = \text{dell}$

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h \rightarrow 0} \frac{f(x + hei) - f(x)}{h}$$

Its domain consists of those points of D at which the limits exists.

Example :

$$f(x, y) = x + y$$

Bcz in respect to x .

$$\begin{aligned} \frac{\partial f}{\partial x}(u, y) &= \lim_{h \rightarrow 0} \frac{f((u, y) + h(1, 0)) - f(u, y)}{h} = \lim_{h \rightarrow 0} \frac{f(u+h, y) - f(u, y)}{h} \\ &= \lim_{h \rightarrow 0} \frac{u+h+y - (u+y)}{h} = 1 \end{aligned}$$

Similary it can be calculated with respect to y .

○ Calculating partial derivatives:

To calculate partial derivative w.r.t x_i , think of f only as a function of x_i while treating all other variables as constants. Then calculate it as derivative of one variable.

Example:

$$\rightarrow f(u, y, z) = uy + yz + zu$$

$$\frac{\partial f}{\partial u}(u, y, z) = y + 0 + z \quad (\text{treated } y \text{ as constant } x^3 = u^2, 3u = 3)$$

$$\frac{\partial f}{\partial u}(u, y, z) = u + z \quad \frac{\partial f}{\partial u}(u, y, z) = u + y$$

$$\rightarrow f(u, y, z) = \sin(uy)$$

$$\frac{\partial f}{\partial u}(u, y, z) = \cos(uy) \cdot y = y \cos(uy)$$

$$\frac{\partial f}{\partial u}(u, y) = u \cos(uy)$$

$$\rightarrow f(u, y) = \begin{cases} \frac{uy}{u^2+y^2} & \text{if } (u, y) \neq (0, 0) \\ 0 & \text{if } (u, y) = (0, 0) \end{cases}$$

$$\frac{\partial f}{\partial u}(u, y) = \frac{(u^2+y^2)y - uy(2u)}{(u^2+y^2)^2} = \frac{y^3+u^2y}{(u^2+y^2)^2}$$

$$\frac{\partial f}{\partial u} = \frac{u^3 - y^2u}{(u^2+y^2)^2}$$

$$(u, y) = (0, 0)$$

$$\frac{\partial f}{\partial u}(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$$

$$\frac{\partial f}{\partial u}(0, 0) = 0$$

◦ Rate of change in a particular point:

Let $f(u_1, u_2, \dots, u_n)$ be a fn defined on a domain D in R^n containing a pt. \underline{a} and an open ball around it.

Instead of direction of axes, we are interested in rate of change of fn f at \underline{a} in some other direction.

We can use same idea as for partial derivatives & choose unit vector $u \in R^n$ in direction we want to compute.

$$\lim_{h \rightarrow 0} \frac{f(\underline{a} + hu) - f(\underline{a})}{h}$$

if, $\underline{a} = (a_1, a_2, \dots, a_n)$, $u = (u_1, u_2, \dots, u_n)$

$$\lim_{h \rightarrow 0} \frac{f(a_1 + hu_1, a_2 + hu_2, \dots, a_n + hu_n) - f(a_1, \dots, a_n)}{h}$$

This is fn of one variable:

we can think as $g(h) = f(a_1 + hu_1, \dots, a_n + hu_n)$

it means, $\lim_{x \rightarrow 0} \frac{g(h) - g(0)}{h}$

Example :

◦ Rate of change of $f(x, y) = x+y$ at $(0,0)$ in direction of the $y=x$ line.

$\Rightarrow y=x$ means 45° line so $u = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$$\lim_{h \rightarrow 0} \frac{f(0 + h\frac{1}{\sqrt{2}}, 0 + h\frac{1}{\sqrt{2}}) - f(0,0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(h\frac{1}{\sqrt{2}} + h\frac{1}{\sqrt{2}}) - (0+0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{2}h}{h} = \sqrt{2}$$

◦ Rate of change of $f(x, y, z) = xy + yz + zx$ at $(1, 2, 3)$ in direction of vector $(4, 3, 0) = v$.

Here first we have to convert vectors in unit vectors.
by dividing by its norm.

$$\|v\| = \sqrt{4^2 + 3^2 + 0^2} = \sqrt{16+9} = 5 \quad u = \frac{1}{5}(4, 3, 0)$$

$$\lim_{h \rightarrow 0} \frac{f(1+h^{4/5}, 2+h^{3/5}, 3+h^{0/5}) - f(1, 2, 3)}{h}$$

$$\lim_{h \rightarrow 0} \frac{h^2/25 + 2h^{4/5} + h^{3/5} + 3h^{3/5} + 3h^{4/5}}{h} = \frac{8}{5} + \frac{3}{5} + \frac{9}{5} + \frac{12}{5} = \frac{32}{5}$$

○ The rate of change of $f(x, y) = \sin(x, y)$ at $(1, 0)$ is the direction 60° (from x axis) $u = (\cos 60^\circ, \sin 60^\circ) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$

$$= \lim_{h \rightarrow 0} \frac{f(1+h^{1/2}, 0+h^{3/2}) - f(1, 0)}{h} = \lim_{h \rightarrow 0} \frac{\sin((1+\frac{1}{2})\frac{\sqrt{3}}{2}h) - \sin(1, 0)}{h}$$

by defining $g(h) = \sin((1+\frac{1}{2})\frac{\sqrt{3}}{2}h)$

$$\Rightarrow g'(h) = \cos(\theta) \times \left\{ \frac{\sqrt{3}}{2}h \times \frac{1}{2} + (1+h^{1/2})\frac{\sqrt{3}}{2} \right\}$$

$$\Rightarrow g'(0) = \cos(0) \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

○ Directional derivatives:

let $f(x_1, x_2, \dots, x_n)$ be fn in domain D in \mathbb{R}^n . The directional derivative of f in direction of unit vector u is the fn denoted by $f_u(\underline{x})$ as defined as:

$$f_u(\underline{x}) = \lim_{h \rightarrow 0} \frac{f(\underline{x} + hu) - f(\underline{x})}{h}$$

Its domain consists of those points of D at which limit exists.

Properties : Linearity and product

Linearity :- let $c \in \mathbb{R}$. If the directional derivative at point \underline{a} in direction of unit vector u exists for both fn $f(\underline{u})$ and $g(\underline{u})$, then it also exists for $(cf+g)(\underline{u})$ and

$$(cf+g)_u(\underline{a}) = cf_u(\underline{a}) + gu(\underline{a})$$

Product rule :- If directional derivative at pt \underline{a} in the direction of the unit vector u exists for both fn $f(\underline{u})$ and $g(\underline{u})$, then it also exists for $(fg)(\underline{u})$ and

$$(fg)_u(\underline{a}) = f_u(\underline{a})g(\underline{a}) + f(\underline{a})g_u(\underline{a})$$

Quotient rule :- If directional derivative at pt \underline{a} in the direction of the unit vector u exists for both fn $f(\underline{u})$ and $g(\underline{u})$, and $g(\underline{a}) \neq 0$, then it also exists for $f/g(\underline{u})$ and

$$(f/g)_u(\underline{a}) = \frac{f_u(\underline{a})g(\underline{a}) - f(\underline{a})g_u(\underline{a})}{g(\underline{a})^2}$$

Example:-

$$\square f(u, y) = u + y, u = (u_1, u_2)$$

$$f_u(u, y) = \lim_{h \rightarrow 0} \frac{f(u+hu_1, y+hu_2) - f(u, y)}{h}$$

$$\lim_{h \rightarrow 0} \frac{x+hu_1 + y + hu_2 - (u+y)}{h} = \lim_{h \rightarrow 0} \frac{hu_1 + hu_2}{h} = u_1 + u_2 = u_1x_1 + u_2x_2$$

$$\square f(u, y, z) = uy + yz + zu, u = (u_1, u_2, u_3)$$

$$f_u(u, y, z) = \lim_{h \rightarrow 0} \frac{(u+h u_1)(y+h u_2) + (y+h u_1)(z+h u_3) + (z+h u_2)(u+h u_1) - (uy + yz + zu)}{h}$$

$$\lim_{h \rightarrow 0} \frac{h^2(u_1 u_2 + u_2 u_3 + u_3 u_1) + h(u_1 y + u_2 y + u_3 y + u_2 z + u_1 z + u_3 z)}{h}$$

$$= u_1 y + u_2 y + u_3 y + u_2 z + u_1 z + u_3 z$$

$$= u_1(y+z) + u_2(z+u) + u_3(u+y)$$

○ Limits of sequence in R^P :

Let $\{q_n\}$ be sequence in R^P . Denote the coordinates of $q_n = \{a_{n1}, a_{n2}, \dots, a_{nP}\}$

We say $\{q_n\}$ has limit $q = \{a_1, a_2, \dots, a_p\} \in R^P$ if as n increases, the sequence in the i^{th} coordinate has limit a_i . i.e. $\{q_{ni}\} \rightarrow a_i$ for each i .

- A sequence $\{q_n\}$ in R^P is called convergent if it converges to some limit.
- A sequence $\{q_n\}$ in R^P is called divergent if it is not convergent.

$$\lim 1/n \rightarrow 0$$

■ Example: $\left\{\left(\frac{1}{n}, n \sin\left(\frac{1}{n}\right)\right)\right\} = \lim n \sin\left(\frac{1}{n}\right) \rightarrow 1 \quad \text{so } q = (0, 1)$

$\left\{\left((-1)^n, n \sin\left(\frac{1}{n}\right)\right)\right\}$ it don't converge.

○ Limit of a scalar-valued multivariable fn at a point:

Let f be scalar-valued multivariable fn defined on a domain D in R^K and q be a point that exists a sequence in D which converges to q .

If there exists a real no. L such that $f(a_n) \rightarrow L$ for all seq. a_n such that $a_n \rightarrow a$, then we say limit of f at a exists and equals L . We denote this by:

$$\lim_{x \rightarrow a} f(x) = L.$$

It is equivalent to: as x comes closer and closer to a , $f(x)$ comes closer & closer to L . If there is no L we say limit don't exists.

○ Rules about scalar-valued multivariable functions:

- If $\lim_{x \rightarrow a} f(x) = F$, $\lim_{x \rightarrow a} g(x) = G$ and $c \in R$, then

$$\lim_{x \rightarrow a} (cf + g)(x) = cF + G$$

- If $\lim_{x \rightarrow a} f(x) = F$, $\lim_{x \rightarrow a} g(x) = G$ and $c \in R$, then

$$\lim_{x \rightarrow a} (fg)(x) = FG.$$

- If $\lim_{x \rightarrow a} f(x) = F$, $\lim_{x \rightarrow a} g(x) = G \neq 0$, then $\frac{f}{g}$ is

defined in at least a small interval around a and

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{F}{G}.$$

Example: $h(x, y, z) = x^2y^3 + y^3z^2 + xyz$

$$\lim_{(x,y,z) \rightarrow (1,2,3)} h(u, y, z) = \lim_{(x,y,z) \rightarrow (1,2,3)} x^2y^3 + \lim_{(x,y,z) \rightarrow (1,2,3)} y^3z^2 + \lim_{(x,y,z) \rightarrow (1,2,3)} xyz$$
$$= 1^2 \times 2^3 + 2^3 \times 3^2 + 1 \times 2 \times 3$$
$$= 8 + 72 + 6 = 86.$$

Composition: Suppose f is a scalar-valued multivariable fn and g is a fn of one variable such that the composition $g \circ f$ is well-defined. If

- If $\lim_{x \rightarrow a} f(x) = F$, $\lim_{x \rightarrow a} g(x) = L$, then $\lim_{x \rightarrow a} (g \circ f)(x) = L$.

Example:

$$h(x, y, z) = e^{xyz}$$

We can write it in two ways other is:

$$f(u, y, z) = xyz, g(x) = e^x$$
$$\Rightarrow (g \circ f)(xyz) = g(f(u, y, z)) = e^{xyz}$$

$$\lim_{(u, y, z) \rightarrow (1, 2, 3)} f = (1 \times 2 \times 3) = 6 \quad \lim_{x \rightarrow 6} e^x$$

Sandwich principle :

- If $\lim_{x \rightarrow a} f(x) = L$, $\lim_{x \rightarrow a} g(x) = L$, and $h(x)$ is fn such

that $f(x) \leq h(x) \leq g(x)$, then $\lim_{x \rightarrow a} h(x) = L$

◦ Finding limit by substitution:

Suppose we want to find the limit of $f(x)$ at pt a i.e. $\lim_{x \rightarrow a} f(x)$. Often we can substitute value of a in the expression for $f(x)$ & obtain limit.

But in complicated fn it don't work or when a don't belong to domain of $f(x)$

Example:

$$\lim_{x \rightarrow (0,0)} \frac{x^3 - y^2 x}{(x^2 + y^2)^2}$$

$$a_n = (\frac{1}{n}, 0)$$

$$f(a_n) = \frac{(\frac{1}{n})^3 - 0^2 \times \frac{1}{n}}{(\frac{1}{n})^2 + 0^2} = \frac{\frac{1}{n^3}}{\frac{1}{n^2}} = n \text{ divergent.}$$

$$b_n = (0; \frac{1}{n})$$

$$f(b_n) = \frac{0^3 - \frac{1}{n^2} \times 0}{(0^2 + \frac{1}{n^2})^2} = 0 \text{ convergent}$$

Both giving diff. result so limit don't exist.

◦ Limit of a vector-valued function at a point:

Let $f: D \rightarrow \mathbb{R}^m$ be a vector-valued multivariable function defined on a domain D in \mathbb{R}^k and a be a point such that there exists a sequence in D which converges to a .

If f_i is i^{th} component function of f , then f_i is a scalar-valued function from D to \mathbb{R} . Suppose each i -the limit $\lim_{x \rightarrow a} f_i(x)$ exists and equation L_i

Define: $\underline{L} = (L_1, L_2, \dots, L_n)$. Then $\lim_{\underline{x} \rightarrow \underline{a}} f(\underline{x}) = \underline{L}$

- This is equivalent to as \underline{x} comes closer & closer to \underline{a} , $f(\underline{x})$ eventually comes closer & closer to \underline{L} .
- If for some i , the limit f_i at \underline{a} don't exist, then the limit of f at \underline{a} don't exist.

Example: $\lim_{\underline{x} \rightarrow (1, 2)} (x^2y + y^3, e^{xy}, \frac{x^2-1}{y^3-2})$

If limit for each vector exist than only \lim exist.

$$\lim_{\underline{x} \rightarrow (1, 2)} x^2y + y^3 = 1^2 \cdot 2 + 2^3 = 10, \lim_{\underline{x} \rightarrow (1, 2)} e^{xy} = e^{1 \cdot 2} = e^2, \lim_{\underline{x} \rightarrow (1, 2)} \frac{x^2-1}{y^3-2} = \frac{1^2-1}{2^3-2} = 0$$

Each limit exist so it will be $(10, e^2, 0)$

- $\lim_{\underline{x} \rightarrow (0, 0)} \left(\frac{\sin(xy)}{xy}, \frac{x^3 - y^2x}{(x^2 + y^2)^2} \right)$

$$\lim_{\underline{x} \rightarrow (0, 0)} \frac{\sin(xy)}{xy} = \frac{\sin(0)}{0} = 1$$

$$\lim_{\underline{x} \rightarrow (0, 0)} \frac{x^3 - y^2x}{(x^2 + y^2)^2} = \text{Does not exist}$$

Limit don't exist for this vector value fn don't have limit.

○ limit of a fn at a point along a curve:

Let f be a scalar valued multivariable fn in domain D in \mathbb{R}^k and \underline{a} pt that sequence \underline{a} exist in D which converges to \underline{a} . Let C is a curve passing through pt \underline{a} belong to domain D .

The limit of f at \underline{a} along the curve C exists & equals L if for every sequence \underline{a}_n contained in C which converges to \underline{a} , the seq $f(\underline{a}_n)$ converges to L .

Example:

$$g(x,y) = \frac{x^3 - y^2 x}{(x^2 + y^2)^2}$$

$$\lim_{(x,y) \rightarrow (0,0)} g(x,y) = \lim_{x \rightarrow 0} \frac{1}{x} \text{ (DNE)} \quad \left(g(n,0) = \frac{n^3}{n^4} = \frac{1}{n} \right)$$

along x axis (Along x axis means yaxis=0 so it will be $g(n,0)$)

$$\lim_{(x,y) \rightarrow (0,0)} g(x,y) = \lim_{y \rightarrow 0} 0 \quad \left(g(0,y) = \frac{0}{y^2} \right)$$

along y axis (Along axis means xaxis=0 so it will be $g(0,y)$)

○ Limit of a fn along curve and limit of fn:

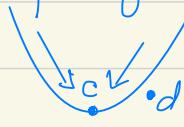
let f be a scalar-valued multivariable fn defined on a domain D in \mathbb{R}^k and a be a point such that there exists a sequence in D which converges to a .

Theorem:

The limit of f at a exists and equal L precisely when for every curve C in the domain D passes through a

the limit of f at a along C exist and equals L .

→ If curve from both side reach point c then limit exist

 but if it don't reach 'd' then it don't exist.

Example:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{u^2 - y^2}{u^2 + y^2}$$

Along u axis $(x,0) \rightarrow \textcircled{1}$

Along y axis $(0,y) \rightarrow \textcircled{11}$

$$\textcircled{1} \lim_{x \rightarrow 0} \frac{x^2}{x^2} = 1$$

$$\textcircled{11} \lim_{y \rightarrow 0} \frac{-y^2}{y^2} = -1$$

they don't match so limit don't exist.

Another example: along x axis $\lim_{x \rightarrow 0} \frac{0}{u^2+y^2} = 0$

$$\lim_{(u,y) \rightarrow (0,0)} \frac{uy}{u^2+y^2}$$

along y axis $\lim_{y \rightarrow 0} \frac{0}{u^2+y^2} = 0$

but it match and by substituting $u=y$ it don't match
then it is not defined $\lim_{u \rightarrow 0} \frac{u^2}{u^2+y^2} = \frac{1}{2} \neq 0$ limit DNE.

3rd example

$$\lim_{(u,y) \rightarrow (0,0)} \frac{uy^2}{u^2+y^2} = \text{the line } y=mx = \lim_{x \rightarrow 0} \frac{x(mx)^2}{x^2+(mx)^4} = \frac{mx^3}{x^2(1+m^2x^2)}$$

along line $x=0$ $\lim_{y \rightarrow 0} \frac{0}{\sqrt{y}} = 0$

○ Continuity of function:

Suppose f : multivar. fn in domain D in \mathbb{R}^k & $a \in D$ be a pt exist a seq. in D which converges to a .

f is continuous at a if the limit of f at a exists and $f(a) = \lim f(x)$

$$f(a_n) \rightarrow f(a) \text{ whenever } a_n \rightarrow a.$$

The fn f is continuous if it is continuous at all pt. in domain. For all pt a for which $f(a)$ is defined,

$$\lim_{x \rightarrow a} f(x) = f(a)$$

□ Computing directional derivative vs partial derivative:
Directional derivative:

$$\text{Eg: } f(x, y, z) = uy + yz + zu$$

$$u = (u_1, u_2, u_3)$$

$$f_u(x) = \lim_{h \rightarrow 0} \frac{f(x+hu_1, y+hu_2, z+hu_3) - f(x, y, z)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(u_1+hu_1)(y+hu_2) + (y+hu_2)(z+hu_3) + (z+hu_3)(u_1+hu_1) - (xy + yz + zx)}{h}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{h^2(u_1u_2 + u_2u_3 + u_3u_1) + h(xu_2 + yu_1 + yu_3 + zu_1 + zu_2)}{h} \\
 &= \lim_{h \rightarrow 0} h(u_1u_2 + u_2u_3 + u_3u_1) + u_1(y+z) + u_2(u+z) + u_3(x+y) \\
 &\quad \text{It become 0 bcz of constant} \\
 &= u_1(y+z) + u_2(x+z) + u_3(x+y) \\
 &\quad \text{It can be written as } \nabla f(\underline{u}) \cdot \underline{u}
 \end{aligned}$$

Partial derivative

$$f_u = y+z, \quad f_y = x+z, \quad f_z = y+x.$$

○ Gradient vector/function :

let $f(u_1, u_2, \dots, u_n)$ be fn in domain D in \mathbb{R}^n containing open ball around point \underline{a} .

Suppose all partial derivative of f at \underline{a} exist. Then gradient vector of f at \underline{a} is vector $(f_{x_1}(\underline{a}), f_{x_2}(\underline{a}), \dots, f_{x_n}(\underline{a}))$ in \mathbb{R}^n . It is denoted by $\nabla f(\underline{a})$.

The gradient fn of f is a fn taking value in \mathbb{R}^n obtained by associating every pt \underline{a} to its gradient vector $\nabla f(\underline{a})$.
 - The domain of $\nabla f(\underline{a})$ is set of points in D where all partial derivative exist.

Example:

$$- f(u, y) = \sin(uy) \quad \frac{\partial f}{\partial u} = y \cos(uy), \quad \frac{\partial f}{\partial y} = u \cos(uy)$$

$$\nabla f(u, y) = (y \cos(uy), u \cos(uy))$$

$$\nabla f(0, 0) = (0, 0) \quad \nabla f(\pi, 1) = (1 \cos(\pi), \pi \cos(\pi)) = (-1, -\pi)$$

$$- f(u, y, z) = u^2 + y^2 + z^2 \rightarrow \text{Partial der: } \frac{\partial f}{\partial u} = 2u, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = 2z$$

$$\nabla f(u, y, z) = (2u, 2y, 2z)$$

$$\nabla f(1, 2, 3) = (2, 4, 6)$$

Prev. calc.
Partial deri.

$$- f(x, y) = \frac{xy}{x^2+y^2}$$

$\frac{\partial f}{\partial x}$	$\frac{x^3-y^2x}{(x^2+y^2)^2} \because (x, y) \neq (0, 0)$
$\frac{\partial f}{\partial y}$	$\frac{y^3-x^2y}{(x^2+y^2)^2} \because (x, y) \neq (0, 0)$
0	$\because (x, y) = (0, 0)$
0	$\because (x, y) = (0, 0)$

$$\nabla f(x, y) = \begin{cases} \frac{1}{(x^2+y^2)^2} (y^2-y^2x, x^3-x^2y) & \because (x, y \neq 0, 0) \\ (0, 0) & \because (x, y = 0, 0) \end{cases}$$

Properties of gradients:

Linearity: Let $c \in \mathbb{R}$

$$\nabla(cf+g)(x) = c\nabla f(x) + \nabla g(x)$$

Product rule:

$$\nabla(fg)(x) = g(x)\nabla f(x) + f(x)\nabla g(x)$$

Quotient rule:

$$\nabla(f/g)(x) = \frac{1}{g(x)^2} (g(x)\nabla f(x) - f(x)\nabla g(x))$$

Directional derivatives and gradients:

(Let $f(x_1, \dots, x_m)$ be fn in D in \mathbb{R}^m containing open ball around a)

a. Suppose ∇f exists and is continuous on some open ball around point a . Then for every unit vector u , the directional derivative $f_u(a)$ exists and equals $\nabla f(a) \cdot u$

Example

$$- f(x, y) = x+y$$

$$f_u(x, y) = (1, 1) \cdot u = u_1 + u_2$$

$$\frac{\partial f}{\partial x} = 1, \frac{\partial f}{\partial y} = 1, \nabla f = (1, 1)$$

$$- f(u, y, z) = uy + yz + zu$$

$$\frac{\partial f}{\partial u} = y + z, \quad \frac{\partial f}{\partial y} = u + z, \quad \frac{\partial f}{\partial z} = y + u$$

$$\nabla f = (y+z, u+z, y+u)$$

$$f_u(u, y, z) = \nabla f \cdot u$$

$$= (y+z, u+z, y+u) \cdot (u_1, u_2, u_3)$$

$$= u_1(y+z) + u_2(u+z) + u_3(y+u)$$

$$- f(u, y) = \sin(uy)$$

$$\nabla f = (y \cos(uy), u \cos(uy))$$

$$f_u(u, y) = \nabla f \cdot u$$

$$= (y \cos(uy), u \cos(uy)) \cdot u$$

$$= u_1 y \cos(uy) + u_2 u \cos(uy)$$

○ Water flow in 3D

○ In what direction is directional derivative minimized?

let $f(u_1, \dots, u_n)$ be f_n in D in \mathbb{R}^n . With open ball around a . Suppose ∇f exists and is continuous on some open ball around point a .

$$f_u = \nabla f(a) \cdot u \quad \text{when } \theta \text{ is } \angle b/w \\ = \|\nabla f(a)\| \cdot \|u\| \cdot \cos(\theta) \quad \nabla f(a) \& u$$

$$u = \text{unit vector} = 1 \quad = \|\nabla f(a)\| \cdot \cos(\theta)$$

$\cos(\theta)$ is minimized when $\theta = \pi$, i.e. u is pointing in direction opp to $\nabla f(a)$

\therefore The min. value of f_u is attained when $u = -\nabla f(a)/\|\nabla f(a)\|$ & equal to $-\|\nabla f(a)\|$

- Direction in which the directional derivative is maximized or remain unchanged:

Assume same hypothesis the gradient exist and is continuous on open ball.

$$f_u = \|\nabla f(a)\| \cdot \|u\| \cos(\theta) = \|\nabla f(a)\| \cos(\theta)$$

It is maximized when $\theta = 0$, i.e. 'u' in same direction $\nabla f(a)$ i.e. $u = \frac{\nabla f(a)}{\|\nabla f(a)\|}$

It remain 0 when (rate of change)/ $f_u = 0$ i.e when cos in is 0 when $\theta = \pi/2$ i.e 'u' is orthogonal or perpendicular to $\nabla f(a)$

- Derivative:

- It is minimized when it is opp. direction of gradient vector.
- It is maximised when it is in same direction to gradient vector.
- It is constant when it is in perpendicular to gradient vector.

Directions: steepest ascent, steepest descent, no change.

Property

In term of directional derivative

Direction

Steepest

ascent

f_u is +ve and maximum

$$u = \nabla f / \|\nabla f\|$$

Steepest
descent

f_u is -ve and minimum

$$u = -\nabla f / \|\nabla f\|$$

No
Change

$$f_u = 0$$

u is orthogonal
to ∇f

Example:

1) $f(x,y) = \sin(xy)$ Compute gradient

$$\nabla f(x,y) = (y \cos(xy), x \cos(xy))$$

at $(\pi, 1)$ what is direction of steepest descent.

$$\nabla f(\pi, 1) (\cos(\pi), \pi \cos(\pi)) = (-1, -\pi)$$

$$u = \frac{-\nabla f(\pi, 1)}{\|\nabla f(\pi, 1)\|} = \frac{(-1, -\pi)}{\sqrt{1+\pi^2}}$$

2) $f(x,y,z) = x^2 + y^2 + z^2$

$$\nabla f(x,y,z) = (2x, 2y, 2z)$$

At $(1, 1, 1)$ what is direction in which f increases fastest.

$$\nabla f(1, 1, 1) = (2, 2, 2)$$

In direction of $u = \frac{(2, 2, 2)}{\sqrt{2^2+2^2+2^2}} = \frac{1}{\sqrt{3}}(1, 1, 1)$

In which direction f remains constant?

e.g., $\frac{1}{\sqrt{2}}(1, -1, 0), \frac{1}{\sqrt{2}}(1, 0, -1)$

○ Tangents for $f(x,y) :$

Let $f(x,y)$ be a fn in domain D of \mathbb{R}^2 with open ball around point \underline{a} .

Consider a line L in D passing through \underline{a} & restrict f to L .

If line L is in the direction of unit vector ' u ', then tangent will be line with slope $f_u(\underline{a})$ & passing through point $(\underline{a}, f(\underline{a}))$

Example:

— $f(u, y) = u + y$; tangent at $(1, 1)$ in direction of $(1, 0)$

- First we have to find directional derivative.

$$f_u(1, 1) = \frac{\partial f}{\partial u}(u, y) = 1$$

$(a, b, f(a, b))$ $u_1, u_2, f_u(a, b)$

$$(u(t), y(t), z(t)) = (1, 1, 2) + t(1, 0, 1)$$

$$u(t) = 1+t, y(t) = 1, z(t) = 2+t$$

— $f(u, y) = uy$; tangent at $(1, 1)$ in direction of $(3, 4)$

$$u = \left(\frac{3}{5}, \frac{4}{5}\right) \quad f_u(1, 1) = 1 \times \frac{3}{5} + 1 \times \frac{4}{5} = \frac{7}{5} \quad \nabla f(u, y) = (y, u) \\ \nabla f(1, 1) = (1, 1)$$

$$(u(t), y(t), z(t)) = (1, 1, 1) + t\left(\frac{3}{5}, \frac{4}{5}, \frac{7}{5}\right)$$

$$= \left(1 + \frac{3t}{5}, 1 + \frac{4t}{5}, 1 + \frac{7t}{5}\right)$$

— $f(u, y) = \sin(uy)$; tangent at $(\pi, 1)$ in direction of $(1, 2)$

$$u = \frac{1}{\sqrt{5}}(1, 2) \quad \nabla f(u, y) = (y \cos(uy), u \cos(uy)) \quad \nabla f(\pi, 1) \\ f_u(\pi, 1) = -1 \times \frac{1}{\sqrt{5}} + (-\pi) \times \frac{2}{\sqrt{5}} = \frac{-2\pi - 1}{\sqrt{5}} = (-1, -\pi)$$

$$(x(t), y(t), z(t)) = (\pi, 1, 0) + t\left(\frac{1}{\sqrt{5}}, 2/\sqrt{5}, \frac{(-2\pi - 1)}{\sqrt{5}}\right)$$

○ Tangents for scalar-valued multivariable functions:

All same and if line L is in the direction of unit vector u , then the tangent will be the line with slope $f_u(a)$ and passing through $f(a, f(a))$

○ Parametric equation with example:

Similar to 2 variable we can derive equation of

tangent line as:

$\underline{x} = (x_1, x_2, \dots, x_n)$, z is variable in which we are measuring the fn.

$\underline{a} = (a_1, \dots, a_n)$, $u = (u_1, \dots, u_n)$

Line through \underline{a} in direction of u is

$$x_i(t) = a_i + t u_i, z = 0$$

$$(\underline{x}(t), z(t)) = (\underline{a}, 0) + t(u, 0)$$

\therefore The tangent line to f at \underline{a} above L is:

$$(\underline{x}(t), z(t)) = (\underline{a}, f(\underline{a})) + t(u, f_u(\underline{a}))$$

$$x_i(t) = a_i + t u_i, z(t) = f(\underline{a}) + t f_u(\underline{a}) \quad \boxed{\text{Tangent line}}$$

Example: $f(u, y) = uy + yz + zy$; tangent at $(1, 1, 1)$ in direction of $(-1, -2, 2)$

$$u = 1/3(-1, -2, 2)$$

$$\nabla f(u, y, z) = (y+z, z+u, u+y)$$

$$\nabla f(1, 1, 1) = (2, 2, 2)$$

$$f_u(1, 1, 1) = -2/3$$

$$(u(t), y(t), z(t), v(t)) = (1, 1, 1, 3) + t(-1/3, -2/3, 2/3, -2/3)$$

$$u(t) = 1 - t/3, y(t) = 1 - 2t/3, z(t) = 1 + 2t/3, v(t) = 3 - 2t/3$$

○ When do all tangent exist:

This is equivalent to asking when all directional derivative exists.

Let $f(u_1, u_2, \dots, u_n)$ be fn defined in domain $D \subset \mathbb{R}^n$ containing open ball around point \underline{a} .

- Suppose ∇f exist and is continuous on some open ball around \underline{a} . Then for every unit vector u , the directional derivative $f_u(\underline{a})$ exists and equals $\nabla f(\underline{a}) \cdot u$.

○ The collection of all tangents:

let $f(u, y)$ a fn in domain D in \mathbb{R}^2 with open ball around point (a, b) .

Suppose ∇f exists and is continuous on some open ball around point (a, b) . Then all tangent around point (a, b) exist and eq. of tangent in unit vector u direction is:

$$\begin{aligned} f_u(a, b) &= \nabla f(a, b) \cdot u \\ &= \frac{\partial f}{\partial x}(a, b) \cdot u_1 + \frac{\partial f}{\partial y}(a, b) u_2 \end{aligned}$$

$$\begin{aligned} u(t) &= a + u_1 t, \quad y(t) = b + u_2 t, \quad z(t) = f(a, b) + f_u(a, b)t \\ &= f(a, b) + \left(\frac{\partial f}{\partial x}(a, b) u_1 + \frac{\partial f}{\partial y}(a, b) u_2 \right) t \end{aligned}$$

○ The equation of tangent plane:

let $f(u, y)$ be a fn defined on domain D in \mathbb{R}^2 with open ball around point (a, b)

Suppose gradient ∇f exist and continuous. Then eq. of tangent plane to f at (a, b) is given by:

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(u-a) + \frac{\partial f}{\partial y}(a, b)(y-b)$$

$$\frac{\partial f}{\partial x}(a, b)u + \frac{\partial f}{\partial y}(a, b)y - z = \frac{\partial f}{\partial x}(a, b)a + \frac{\partial f}{\partial y}(a, b)b.$$

Example:

$$\begin{aligned} f(u, y) &= u + y; \text{ tangent at } (1, 1) \\ \Rightarrow \nabla f(1, 1) &= (1, 1), \quad z = 2 + 1(u-1) + 1(y-1) \end{aligned}$$

$$\begin{aligned} z &= 2 + u - 1 + y - 1 = u + y \\ z &= u + y \end{aligned}$$

$$f(u, y) = uy; \text{ tangent at } (1, 1)$$

$$\Rightarrow \nabla f(1, 1) = (1, 1), z = 1 + 1(u-1) + 1(y-1)$$

$$z = u + y - 1$$

$$f(u, y) = \sin(uy); \text{ tangent at } (1, 0)$$

$$\Rightarrow \nabla f(1, 0) = (0, 1), z = 0 + 0(u-1) + 1(y-0)$$

$$z = y$$

○ The tangent hyperplane:

let $f(\underline{u})$ be a fn defined on domain D in \mathbb{R}^2
with open ball around point \underline{a} .

Suppose ∇ exists and continuous around point \underline{a} .
Then eq. of tangent hyperplane to f at (\underline{a}, b) is
given by:

$$z = f(\underline{a}) + \sum \frac{\partial f}{\partial u_i}(\underline{a})(u_i - a_i)$$

$$= f(\underline{a}) + \nabla f(\underline{a})$$

Example:

$$f(u, y, z) = xy + yz + zx; \text{ tangent at } (1, 1, 1)$$

$$\nabla f(u, y, z) = (y+z, x+z, x+y)$$

$$\nabla(1, 1, 1) = (2, 2, 2)$$

Tangent hyperplane is:

$$z = f(1, 1, 1) + \nabla f(1, 1, 1) \cdot (u-1, y-1, z-1)$$

$$= 3 + (2, 2, 2) \cdot (u-1, y-1, z-1)$$

$$\text{Eq, is } u = 3 + 2(u-1) + 2(y-1) + 2(z-1)$$

$$f(u, y, z) = u^2 + y^2 + z^2; \text{ tangent at } (2, 3, -1)$$

$$\nabla f = (2u, 2y, 2z)$$

$$\nabla f(2, 3, -1) = (4, 6, -2)$$

$$L = f(2, 3, -1) + \nabla f(2, 3, -1) \cdot (x-2, y-3, z+1)$$

$$= 14 + (4, 6, -2) \cdot (x-2, y-3, z+1)$$

$$L_{eq} \text{ is } L = 14 + 4(x-2) + 6(y-3) - 2(z+1)$$

Linear approximation:

Let $f(u, y)$ be a fn defined on domain D in \mathbb{R}^2 with open ball around point a .

Suppose ∇f exists and continuous around point a .

Then the fn $L(u, y) = f(a) + \nabla f(a) \cdot (u-a)$ is best linear approximation for fn f close to a .

Example.

$$L.A. \text{ to } f(u, y) = uy \text{ at } (1, 1)$$

$$\nabla f(u, y) = (y, u)$$

$$\nabla f(1, 1) = (1, 1)$$

$$L(u, y) = f(1, 1) + \nabla f(1, 1) \cdot (u-1, y-1)$$

$$= 1 + (1, 1) \cdot (u-1, y-1)$$

$$= 1 + u - 1 + y - 1 = u + y - 1 \text{ is best linear approx.}$$

$$L.A. \text{ of } f(u, y, z) = u^2 + y^2 + z^2 \text{ at } (2, 3, -1)$$

$$\nabla f(2, 3, -1) = (4, 6, -2)$$

$$L_f(u, y, z) = f(2, 3, -1) + \nabla f(2, 3, -1) \cdot (u-2, y-3, z+1)$$

$$= 3 + 4(u-2) + 6(y-3) - 2(z+1)$$

$$= 4u + 6y - 2z - 29 \text{ is best linear approx close to } (2, 3, -1)$$

- Point of local extrema for multivariable function:
 let $f(\underline{u})$ be a fn on domain $D \subset \mathbb{R}^n$ & suppose $\underline{a} \in D$.
 - The point \underline{a} is local maximum of f for some open ball B containing \underline{a} , $f(\underline{u}) \leq f(\underline{a})$ where $\underline{x} \in B \cap D$.
 - The point \underline{a} is local minimum of f for some open ball B containing \underline{a} , $f(\underline{a}) \leq f(\underline{u})$ where $\underline{x} \in B \cap D$.
 - A local extremum of f is either a local maximum or a local minimum of f .

- The gradient vector at point of local extrema:
 Let $f(\underline{u})$ be a fn defined on a domain D in \mathbb{R}^n containing some open ball around point \underline{a} of local extremum.

Restrict f to a line L passing through \underline{a} and view it as a vector of one variable on L .

Then \underline{a} is local extremum for restricted fn on L & hence the directional derivative of f in the direction of line L (if it exist) at \underline{a} is 0.

In particular, those partial derivative which exists at \underline{a} must be 0.

If $\nabla f(\underline{a})$ exist for a local extremum \underline{a} then $\nabla(\underline{a}) = 0$.

- Critical point: A point \underline{a} is critical pt of a fn $f(\underline{u})$ if either $\nabla f(\underline{a})$ doesn't exist or $\nabla f(\underline{a})$ exists & $\nabla f(\underline{a}) = 0$.

Example: critical pt of $f(u, y) = u^2 + 6uy + 4y^2 + 2x - 4y$.

$$\frac{\partial f}{\partial u} = 2u + 6y + 2$$

$$\nabla f(u, y) = (2u + 6y + 2, 6u + 8y - 4)$$

Set $\nabla f = 0$ i.e. $(2u + 6y + 2, 6u + 8y - 4) = (0, 0)$

$$2u + 6y + 2 = 0$$

$$6u + 8y - 4 = 0 \quad u = 2, y = -1$$

∴ critical point is $(2, -1)$

◦ Saddle point: Every local extremum is a critical point.
Unfortunately not all critical points are local extrema.

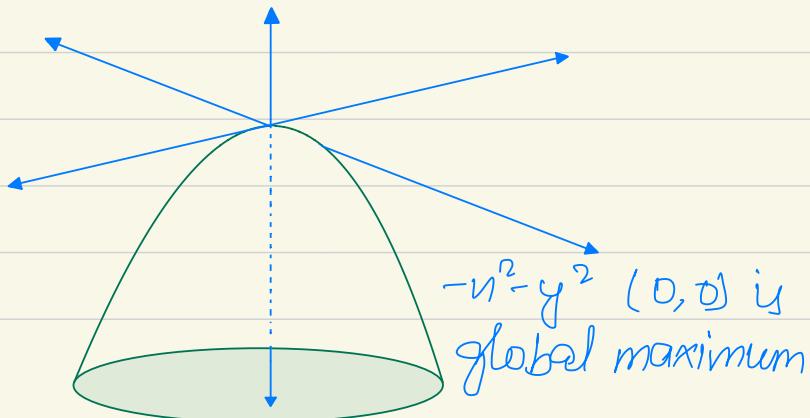
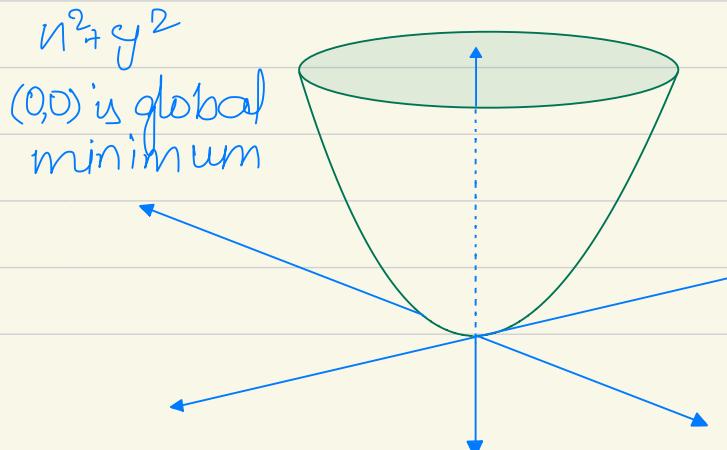
Example: $f(u, y) = u^3$

A saddle pt is a critical pt \underline{a} such that $\nabla f(\underline{a}) = 0$
Exists & $\nabla f(\underline{a}) = 0$ but \underline{a} is not a local extremum.

◦ Absolute (or global) extrema:

let $f(\underline{u})$ be a fn defined on a domain D in R^n &
suppose $\underline{a} \in D$.

- The point \underline{a} is an absolute maximum (global) of f if $f(\underline{u}) \leq f(\underline{a})$ for all $\underline{u} \in D$.
- The point \underline{a} is an absolute minimum (global) of f if $f(\underline{a}) \leq f(\underline{u})$ for all $\underline{u} \in D$.



◦ Existence of absolute maximum/minimum:

A domain D in \mathbb{R}^n is called closed if it contains all its boundary points. A domain in \mathbb{R}^n is called bounded if it is contained inside a ball around 0 with finite radius.

If the domain D is closed and bounded & f is continuous on D , then global maximum & minimum exist.

Global max & min are in particular local max & min unless they are boundary pt.

So, to find global max & min we find critical pt.

- Inside domain D

- On boundary of D .

- On the boundary of boundary of D .

& check value of f in all critical points.

Example:

Find absolute max & min of the fn

$f(x, y) = x^3 + y^3 - 3x - 3y^2 + 1$ over square with vertices

$(0,0), (2,0), (2,2), (0,2)$

$$\Rightarrow \nabla f = (3x^2 - 3, 3y^2 - 6y)$$

$$\text{Set to } 0. 3x^2 - 3 = 0, 3y^2 - 6y = 0$$

$$x^2 = 1, y(y-2) = 0$$

$$x = \pm 1, y = 0 \text{ or } 2$$

$(1,0), (1,2)$ (critical pt for square) $(0,0), (0,2), (2,2), (2,0)$

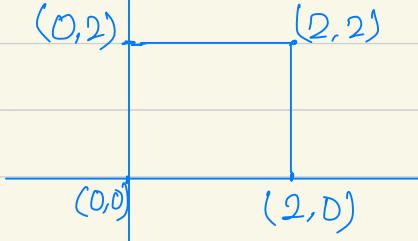
$f(x,y) = x^3 - 3x + 1 : 0 < x < 2$ (boundary with end points)

$$f'(x) = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow x = \pm 1 \Rightarrow (1,0)$$

$$f(1,2) = 1^3 + 8 - 3 \cdot 1 - 12 + 1 = 1^3 - 3 \cdot 1 - 3$$

$$f'(y) = 0 \Rightarrow y$$

$$f(0,y) = y^3 - 6y = 0$$



$$f'(y) = 0$$

$$3y^2 - 6y = 0$$

$$y(y-2) = 0 \quad (0,0), (2,0)$$

$$f(2)y = y^3 - 3y^2 + C$$

$$(1,0), (1,2), (0,0), (0,2), (2,2), (2,0)$$

-1 -5 1 -3 -1 3

so abs. max is (2,0) with value 3

" " min " (1,2) " " " -5

Second order partial derivative for $f(x,y)$:

Let $f(u,y)$ be fn defined on domain D in \mathbb{R}^2 .
Then 2nd order partial derivative of f are partial derivatives of partial derivative.

Notation:

$$f_{xx} \text{ or } \frac{\partial^2 f}{\partial u^2}$$

$$f_{yy} \text{ or } \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} \text{ or } \frac{\partial^2 f}{\partial y \partial u}$$

$$f_{yx} \text{ or } \frac{\partial^2 f}{\partial x \partial y}$$

Mixed partial derivative

Example:

$$f(u,y) = u+y \quad : \frac{\partial f}{\partial u} = 1, \quad \frac{\partial f}{\partial y} = 1$$

$$\frac{\partial^2 f}{\partial u^2} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 0, \quad \frac{\partial^2 f}{\partial u \partial y} = 0, \quad \frac{\partial^2 f}{\partial y \partial u} = 0$$

$$f(u, y) = \sin(uy) : \frac{\partial f}{\partial u} = y \cos(uy), \frac{\partial f}{\partial y} = u \cos(uy)$$

$$\frac{\partial^2 f}{\partial u^2} = y (-\sin(uy)) = -y^2 \sin(uy)$$

$$\frac{\partial^2 f}{\partial y^2} = -u^2 \sin(uy)$$

$$\frac{\partial^2 f}{\partial u \partial y} = 1(\cos(uy)) + u(-ysin(uy)) = \cos(uy) - uysin(uy)$$

$$\frac{\partial^2 f}{\partial y \partial u} = 1(\cos(uy)) + y(-usin(uy)) = \cos(uy) - uysin(uy)$$

○ Cliraut's theorem about mixed partial:

Let $f(u, y)$ be a fn defined on a domain D in \mathbb{R}^2 containing a point g and an open ball around it.

If 2nd order mixed partial derivative f_{xy} & f_{yx} are continuous on open ball around g then, $f_{uy}(g) = f_{yu}(g)$

○ Second order partial derivative :

Let $f(u_1, u_2, \dots, u_n)$ be a fn defined on domain D in \mathbb{R}^n . Then 2nd order partial derivative of f be defined analogously as the partial derivatives of partial derivative.

$$f_{x_i x_i} \text{ or } \frac{\partial^2 f}{\partial u_i^2}$$

$$f_{x_i x_j} = (f_{x_i})_{x_j} \text{ or } \frac{\partial^2 f}{\partial u_j \partial u_i} = \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial u_i} \right)$$

Example: $f(u, y, z) = uy + yz + zu$

$$\frac{\partial f}{\partial u} = y+z, \quad \frac{\partial f}{\partial y} = u+z, \quad \frac{\partial f}{\partial z} = u+y$$

$$\frac{\partial^2 f}{\partial u^2} = 0, \quad \frac{\partial^2 f}{\partial y \partial u} = 1, \quad \frac{\partial^2 f}{\partial z \partial u} = 1$$

$$1, \quad \frac{\partial^2 f}{\partial y^2} = 0, \quad \frac{\partial^2 f}{\partial z \partial y} = 1, \quad \frac{\partial^2 f}{\partial z^2} = 0$$

from Clairaut's theorem:

$$\frac{\partial^2 f}{\partial u \partial y} = 1, \quad \frac{\partial^2 f}{\partial u \partial z} = 1, \quad \frac{\partial^2 f}{\partial y \partial z} = 1$$

○ Higher order partial derivatives:

Let $f(u_1, u_2, \dots, u_n)$ be a fn in domain D in R^n .

Then higher order partial derivative of f are defined analogously by taking successive partial derivatives.

$$f_{x_1, x_2, \dots, x_n} = ((f_{u_i})_{u_1}) \dots)_{u_k}$$

$$\text{or } \frac{\partial^k f}{\partial u_{i_k} \dots \partial u_{i_2} \partial u_{i_1}} = \frac{\partial}{\partial u_{i_k}} \left(\frac{\partial}{\partial u_{i_{k-1}}} \left(\dots \left(\frac{\partial f}{\partial u_{i_1}} \right) \right) \right)$$

Its domain is where derivative exists.

○ Hessian Matrix:

Let $f(u_1, u_2, \dots, u_n)$ be fn defined on domain D in R^n .

Then Hessian matrix defined as:

$$Hf = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ \frac{\partial^2 f}{\partial x_i \partial x_1} & \frac{\partial^2 f}{\partial x_i \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_i \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_i \partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Example:

$$f(u, y) = u + y$$

$$Hf = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2}$$

$$f(u, y) = \sin(uy)$$

$$Hf = \begin{bmatrix} -y^2 \sin(uy) & \cos(uy) - uy \sin(uy) \\ \cos(uy) - uy \sin(uy) & -u^2 \sin(uy) \end{bmatrix}$$

$$f(uyz) = uy + y^2 z + z^2 u$$

$$Hf \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{3 \times 3}$$

◻ 2nd derivative test to check nature of critical point:

- 1) If a is a critical pt $f''(a) > 0$, then it is local minimum.
- 2) If a is a critical pt $f''(a) < 0$, then it is local maximum.
- 3) If a is a critical pt $f''(a) = 0$, then test is inconclusive.

◦ Critical point in multivariable fn:

A point \underline{a} is critical pt of a fn $f(u)$ if either $\nabla f(\underline{a})$ doesn't exist or $\nabla f(\underline{a})$ exists & $\nabla f(\underline{a}) = 0$.

A saddle pt is a critical pt \underline{a} such that $\nabla f(\underline{a})$ exists and $\nabla f(\underline{a}) = 0$ but \underline{a} not local extremum.

◻ The Hessian test: Classifying critical point of $f(u,y)$

Let $f(u,y)$ be a fn defined on domain $D \in \mathbb{R}^2$.
let \underline{a} is a critical pt of f such that the first & second order partial derivatives continuous in open ball around \underline{a} .

Hessian test can applied to check nature of critical pt \underline{a} .

- 1) If $\det(H_f(\underline{a})) > 0$ and $f_{xx}(\underline{a}) > 0$ than \underline{a} is local minimum.
- 2) If $\det(H_f(\underline{a})) > 0$ and $f_{xx}(\underline{a}) < 0$ than \underline{a} is local maximum.
- 3) If $\det(H_f(\underline{a})) < 0$ than \underline{a} is a saddle point.
- 4) If $\det(H_f(\underline{a})) = 0$ than test is inconclusive

Example:

$$f(u, y) = u^2 + y^2$$

$\nabla f = (2u, 2y)$. critical pt $(0,0)$

$$H_f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad H_f(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\det(H_f(0,0)) = 4 > 0$$

$$f_{xx}(0,0) = 2 > 0$$

$\therefore (0,0)$ a local minimum.

$$- f(u, y) = u^2 - y^2 \quad \nabla f = (-2u, -2y), \text{ Critical pt } (0,0)$$

$$Hf = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \quad Hf(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\det(Hf(0,0)) = 4 > 0$$

$$f_{xx} = -2 < 0$$

$\therefore (0,0)$ is local maximum.

$$- f(u, y) = u^2 - y^2 \quad \nabla f = (2u, -2y) \quad C.P = (0,0)$$

$$Hf = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \quad Hf(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\det(Hf(0,0)) = -4 < 0$$

$\therefore (0,0)$ is a saddle point.

$$- f(u, y) = u^4 + y^4 \quad \nabla f = (4u^3, 4y^3). \quad C.P = (0,0)$$

$$Hf = \begin{bmatrix} 12u^2 & 0 \\ 0 & 12y^2 \end{bmatrix} \quad Hf(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\det(Hf(0,0)) = 0$$

\therefore The test is inconclusive.

Example = $f(u, y) = u^2 + 6uy + 4y^2 + 2u - 4y$

$$\nabla f = (2u + 6y + 2, 6u + 8y - 4) \text{ equating to 0}$$

$$C.P = (2, 1)$$

$$Hf = \begin{bmatrix} 2 & 6 \\ 6 & 8 \end{bmatrix} = Hf(2, 1)$$

$$\det(Hf(2, 1)) = 16 - 36 = -20 < 0$$

$\therefore (2, 1)$ is a saddle point.

○ The Hessian test : Classifying critical point of $f(x, y, z)$ [3 variable]

Let $f(x, y, z)$ be a fn defined in D in \mathbb{R}^3 . Let \underline{x} be a critical pt. of f such that 1st, 2nd order partial derivatives are continuous in an open ball around \underline{x} .

Hessian test can be applied to check nature of critical point \underline{x} .

1.) If $f_{xx}(\underline{x}) > 0$, $(f_{xx}f_{yy} - f_{xy}^2)(\underline{x}) > 0$, $\det(Hf(\underline{x})) > 0$,

then \underline{x} is a local minimum.

2.) If $f_{xx}(\underline{x}) < 0$, $(f_{xx}f_{yy} - f_{xy}^2)(\underline{x}) > 0$, $\det(Hf(\underline{x})) < 0$,

then \underline{x} is a local maximum.

3.) If $\det(Hf(\underline{x})) \neq 0$ & cases 1 or 2 occur, then \underline{x} is saddle point.

4.) If $\det(Hf(\underline{x})) = 0$, then the test is inconclusive.

○ Understanding the term :

Terms are: f_{xx} , $(f_{xx}f_{yy} - f_{xy}^2)(\underline{x})$ & $\det(Hf(\underline{x}))$

$$Hf(\underline{x}) = \begin{bmatrix} f_{xx}(\underline{x}) & f_{xy}(\underline{x}) & f_{xz}(\underline{x}) \\ f_{yx}(\underline{x}) & f_{yy}(\underline{x}) & f_{yz}(\underline{x}) \\ f_{zx}(\underline{x}) & f_{zy}(\underline{x}) & f_{zz}(\underline{x}) \end{bmatrix}$$

1×1

2×2

3×3

Signs:

+

+

+

local minimum

approach

$\xleftarrow{-}$

other non-zero cases

$\xrightarrow{+}$

local maximum

$\det(Hf(\underline{x})) \neq 0$

Saddle point

Degenerate case

$\det(Hf(\underline{x})) = 0$

Inconclusive

Example:

$$f(u, y, z) = u^2 + y^2 + z^2$$

$$\nabla f = (2u, 2y, 2z), \text{ Critical pt : } (0, 0, 0)$$

$$Hf = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = Hf(0, 0, 0)$$

$$\det(Hf(0, 0, 0)) = 8 > 0 \therefore (0, 0, 0)$$

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = 4 > 0$$

$$f_{xx}(0, 0, 0) = 2 > 0 \quad \text{local minima.}$$

$$f(u, y, z) = -u^2 - y^2 - z^2$$

$$\nabla f = (-2u, -2y, -2z), \text{ Critical pt : } (0, 0, 0)$$

$$Hf = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = Hf(0, 0, 0)$$

$$\det(Hf(0, 0, 0)) = -8 < 0 \therefore (0, 0, 0)$$

$$\det\left(\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}\right) = 4 > 0$$

$$f_{xx}(0, 0, 0) = -2 < 0 \quad \text{local maxima.}$$

$$f(u, y, z) = u^2 - y^2 + z^2$$

$$Hf = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = Hf(0, 0, 0)$$

$$\det(Hf(0, 0, 0)) = -8 < 0 \therefore (0, 0, 0)$$

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\right) = -4 < 0$$

$$f_{xx}(0, 0, 0) = 2$$

Saddle point.

$$f(u, y, z) = u^4 + y^4 + z^4$$

$$\nabla f = (4u^3, 4y^3, 4z^3), \text{ critical pt. } (0, 0, 0)$$

$$Hf = \begin{bmatrix} 12u^2 & 0 & 0 \\ 0 & 12y^2 & 0 \\ 0 & 0 & 12z^2 \end{bmatrix} \quad Hf(0, 0, 0) = \underset{3 \times 3}{0} \quad \text{Inconclusive}$$

$$f(u, y, z) = uy + yz + zx$$

$$\nabla f = (y+z, z+u, u+y) \text{ equating to 0, we get}$$

$$\text{critical pt} = (0, 0, 0)$$

$$Hf = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = Hf(0, 0, 0)$$

$$\det(Hf(0, 0, 0)) = 0 \times \det \begin{bmatrix} \end{bmatrix} - 1 \times \det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 1 \times \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 1 - 1 = 0 < 0$$

$\therefore (0, 0, 0)$ is a saddle point.

○ Differentiability, tangents and best linear approximation:

f is differentiable at \underline{a}

Best linear approximation to f at \underline{a} exists

Tangent plane to f at \underline{a} exists

○ Differentiability for scalar-valued multivariable function:

Let f be a scalar-valued multivariable fn defined on a domain D in \mathbb{R}^n containing an open ball around a point a .

Then f is differentiable at a if:

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a) - h \cdot \nabla f(a)}{\|h\|} = 0.$$