

Derivatives and Differentials

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April 5, 2013

Part I Theory

1 Optimization

We will be concerned with minimizing a non-linear least squares objective of the form

$$x^* = \arg \min_x \|h(x) - z\|_\Sigma^2 \quad (1.1)$$

where $x \in \mathcal{M}$ is a point on an n -dimensional manifold (which could be \mathbb{R}^n , an n -dimensional Lie group G , or a general manifold \mathcal{M}), $z \in \mathbb{R}^m$ is an observed measurement, $h : \mathcal{M} \rightarrow \mathbb{R}^m$ is a measurement function that predicts z from x , and $\|e\|_\Sigma^2 \triangleq e^T \Sigma^{-1} e$ is the squared Mahalanobis distance with covariance Σ .

To minimize (1.1) we need a notion of how the non-linear measurement function $h(x)$ behaves in the neighborhood of a linearization point a . Loosely speaking, we would like to define an $m \times n$ Jacobian matrix H_a such that

$$h(a \oplus \xi) \approx h(a) + H_a \xi \quad (1.2)$$

with $\xi \in \mathbb{R}^n$, and the operation \oplus “increments” $a \in \mathcal{M}$. Below we more formally develop this notion, first for functions from $\mathbb{R}^n \rightarrow \mathbb{R}^m$, then for Lie groups, and finally for manifolds.

Once equipped with the approximation (1.2), we can minimize the objective function (1.1) with respect to δx instead:

$$\xi^* = \arg \min_\xi \|h(a) + H_a \xi - z\|_\Sigma^2 \quad (1.3)$$

This can be done by setting the derivative of (1.3) to zero, yielding the **normal equations**,

$$H_a^T H_a \xi = H_a^T (z - h(a))$$

which can be solved using Cholesky factorization. Of course, we might have to iterate this multiple times, and use a trust-region method to bound ξ when the approximation (1.2) is not good.

2 Multivariate Differentiation

2.1 Derivatives

For a vector space \mathbb{R}^n , the notion of an increment is just done by vector addition

$$a \oplus \xi \triangleq a + \xi$$

and for the approximation 1.2 we will use a Taylor expansion using multivariate differentiation. However, loosely following [4], we use a perhaps unfamiliar way to define derivatives:

Definition 1. We define a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ to be **differentiable** at a if there exists a matrix $f'(a) \in \mathbb{R}^{m \times n}$ such that

$$\lim_{\delta x \rightarrow 0} \frac{|f(a) + f'(a)\xi - f(a + \xi)|}{|\xi|} = 0$$

where $|e| \triangleq \sqrt{e^T e}$ is the usual norm. If f is differentiable, then the matrix $f'(a)$ is called the **Jacobian matrix** of f at a , and the linear map $Df_a : \xi \mapsto f'(a)\xi$ is called the **derivative** of f at a . When no confusion is likely, we use the notation $F_a \triangleq f'(a)$ to stress that $f'(a)$ is a matrix.

The benefit of using this definition is that it generalizes the notion of a scalar derivative $f'(a) : \mathbb{R} \rightarrow \mathbb{R}$ to multivariate functions from $\mathbb{R}^n \rightarrow \mathbb{R}^m$. In particular, the derivative Df_a maps vector increments ξ on a to increments $f'(a)\xi$ on $f(a)$, such that this linear map locally approximates f :

$$f(a + \xi) \approx f(a) + f'(a)\xi$$

Example 1. The function $\pi : (x, y, z) \mapsto (x/z, y/z)$ projects a 3D point (x, y, z) to the image plane, and has the Jacobian matrix

$$\pi'(x, y, z) = \frac{1}{z} \begin{bmatrix} 1 & 0 & -x/z \\ 0 & 1 & -y/z \end{bmatrix}$$

2.2 Properties of Derivatives

This notion of a multivariate derivative obeys the usual rules:

Theorem 1. (Chain rule) If $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is differentiable at a and $g : \mathbb{R}^p \rightarrow \mathbb{R}^m$ is differentiable at $f(a)$, then the Jacobian matrix H_a of $h = g \circ f$ at a is the $m \times n$ matrix product

$$H_a = G_{f(a)} F_a$$

Proof. See [4] □

Example 2. If we follow the projection π by a calibration step $\gamma : (x, y) \mapsto (u_0 + fx, u_0 + fy)$, with

$$\gamma'(x, y) = \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix}$$

then the combined function $\gamma \circ \pi$ has the Jacobian matrix

$$(\gamma \circ \pi)'(x, y) = \frac{f}{z} \begin{bmatrix} 1 & 0 & -x/z \\ 0 & 1 & -y/z \end{bmatrix}$$

Theorem 2. (Inverse) If $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is differentiable and has a differentiable inverse $g \triangleq f^{-1}$, then its Jacobian matrix G_a at a is just the inverse of that of f , evaluated at $g(a)$:

$$G_a = [F_{g(a)}]^{-1}$$

Proof. See [4] □

Example 3. The function $f : (x, y) \mapsto (x^2, xy)$ has the Jacobian matrix

$$F_{(x,y)} = \begin{bmatrix} 2x & 0 \\ y & x \end{bmatrix}$$

and, for $x \geq 0$, its inverse is the function $g : (x, y) \mapsto (x^{1/2}, x^{-1/2}y)$ with the Jacobian matrix

$$G_{(x,y)} = \frac{1}{2} \begin{bmatrix} x^{-1/2} & 0 \\ -x^{-3/2}y & 2x^{-1/2} \end{bmatrix}$$

It is easily verified that

$$g'(a, b) f'(a^{1/2}, a^{-1/2}b) = \frac{1}{2} \begin{bmatrix} a^{-1/2} & 0 \\ -a^{-3/2}b & 2a^{-1/2} \end{bmatrix} \begin{bmatrix} 2a^{1/2} & 0 \\ a^{-1/2}b & a^{1/2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 1. Verify the above for $(a, b) = (4, 6)$. Sketch the situation graphically to get insight.

2.3 Computing Multivariate Derivatives

Computing derivatives is made easy by defining the concept of a partial derivative:

Definition 2. For $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the **partial derivative** of f at a ,

$$D_j f(a) \triangleq \lim_{h \rightarrow 0} \frac{f(a^1, \dots, a^j + h, \dots, a^n) - f(a^1, \dots, a^n)}{h}$$

which is the ordinary derivative of the scalar function $g(x) \triangleq f(a^1, \dots, x, \dots, a^n)$.

Using this definition, one can show that the Jacobian matrix F_a of a differentiable *multivariate* function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ consists simply of the $m \times n$ partial derivatives $D_j f^i(a)$, evaluated at $a \in \mathbb{R}^n$:

$$F_a = \begin{bmatrix} D_1 f^1(a) & \cdots & D_n f^1(a) \\ \vdots & \ddots & \vdots \\ D_1 f^m(a) & \cdots & D_n f^m(a) \end{bmatrix}$$

Problem 2. Verify the derivatives in Examples 1 to 3.

3 Multivariate Functions on Lie Groups

3.1 Lie Groups

Lie groups are not as easy to treat as the vector space \mathbb{R}^n but nevertheless have a lot of structure. To generalize the concept of the total derivative above we just need to replace $a \oplus \xi$ in (1.3) with a suitable operation in the Lie group G . In particular, the notion of an exponential map allows us to define an incremental transformation as tracing out a geodesic curve on the group manifold along a certain **tangent vector** ξ ,

$$a \oplus \xi \triangleq a \exp(\hat{\xi})$$

with $\xi \in \mathbb{R}^n$ for an n -dimensional Lie group, $\hat{\xi} \in \mathfrak{g}$ the Lie algebra element corresponding to the vector ξ , and $\exp \hat{\xi}$ the exponential map. Note that if G is equal to \mathbb{R}^n then composing with the exponential map $ae^{\hat{\xi}}$ is just vector addition $a + \xi$.

Example 4. For the Lie group $SO(3)$ of 3D rotations the vector ξ is denoted as ω and represents an angular displacement. The Lie algebra element $\hat{\xi}$ is a skew symmetric matrix denoted as $[\omega]_{\times} \in \mathfrak{so}(3)$, and is given by

$$[\omega]_{\times} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Finally, the increment $a \oplus \xi = ae^{\hat{\xi}}$ corresponds to an incremental rotation $R \oplus \omega = Re^{[\omega]_{\times}}$.

3.2 Derivatives

We can generalize Definition 1 to map exponential coordinates ξ to increments $f'(a)\xi$ on $f(a)$, such that the linear map Df_a locally approximates a function f from G to \mathbb{R}^m :

$$f(ae^{\hat{\xi}}) \approx f(a) + f'(a)\xi$$

Definition 3. We define a function $f : G \rightarrow \mathbb{R}^m$ to be **differentiable** at $a \in G$ if there exists a matrix $f'(a) \in \mathbb{R}^{m \times n}$ such that

$$\lim_{\xi \rightarrow 0} \frac{|f(a) + f'(a)\xi - f(ae^{\hat{\xi}})|}{|\xi|} = 0$$

If f is differentiable, then the matrix $f'(a)$ is called the **Jacobian matrix** of f at a , and the linear map $Df_a : \xi \mapsto f'(a)\xi$ is called the **derivative** of f at a .

Note that the vectors ξ can be viewed as lying in the tangent space to G at a , but defining this rigorously would take us on a longer tour of differential geometry. Informally, ξ is simply the direction, in a local coordinate frame, that is locally tangent at a to a geodesic curve $\gamma : t \mapsto ae^{t\hat{\xi}}$ traced out by the exponential map, with $\gamma(0) = a$.

3.3 Derivative of an Action

The (usual) action of an n -dimensional matrix group G is matrix-vector multiplication on \mathbb{R}^n , i.e., $f : G \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ with

$$f(T, p) = Tp$$

Since this is a function defined on the product $G \times \mathbb{R}^n$ the derivative is a linear transformation $Df : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ with

$$Df_{(T,p)}(\xi, \delta p) = D_1 f_{(T,p)}(\xi) + D_2 f_{(T,p)}(\delta p)$$

Theorem 3. *The Jacobian matrix of the group action $f(T, P) = Tp$ at (T, p) is given by*

$$F_{(T,p)} = \begin{bmatrix} TH(p) & T \end{bmatrix} = T \begin{bmatrix} H(p) & I_n \end{bmatrix}$$

with $H : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ a linear mapping that depends on p , and I_n the $n \times n$ identity matrix.

Proof. First, the derivative $D_2 f$ with respect to p is easy, as its matrix is simply T :

$$f(T, p + \delta p) = T(p + \delta p) = Tp + T\delta p = f(T, p) + D_2 f(\delta p)$$

For the derivative $D_1 f$ with respect to a change in the first argument T , we want

$$f(Te^{\hat{\xi}}, p) = Te^{\hat{\xi}} p \approx Tp + D_1 f(\xi)$$

Since the matrix exponential is given by the series $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$ we have, to first order

$$Te^{\hat{\xi}} p \approx T(I + \hat{\xi})p = Tp + T\hat{\xi}p$$

Hence, we need to show that

$$\hat{\xi}p = H(p)\xi \tag{3.1}$$

with $H(p)$ an $n \times n$ matrix that depends on p . Expressing the map $\xi \rightarrow \hat{\xi}$ in terms of the Lie algebra generators G^i , using tensors and Einstein summation, we have $\hat{\xi}_j^i = G_{jk}^i \xi^k$ allowing us to calculate $\hat{\xi}p$ as

$$\left(\hat{\xi}p\right)^i = \hat{\xi}_j^i p^j = G_{jk}^i \xi^k p^j = \left(G_{jk}^i p^j\right) \xi^k = H_k^i(p) \xi^k$$

□

Example 5. For 3D rotations $R \in SO(3)$, we have $\hat{\omega} = [\omega]_{\times}$ and

$$G_{k=1} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad G_{k=2} : \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad G_{k=3} : \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrices $(G_k^i)_j$ are obtained by assembling the j^{th} columns of the generators above, yielding $H(p)$ equal to:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} p^1 + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} p^2 + \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} p^3 = \begin{pmatrix} 0 & p^3 & -p^2 \\ -p^3 & 0 & p^1 \\ p^2 & -p^1 & 0 \end{pmatrix} = [-p]_{\times}$$

Hence, the Jacobian matrix of $f(R, p) = Rp$ is given by

$$F_{(R,p)} = R \begin{pmatrix} [-p]_{\times} & I_3 \end{pmatrix}$$

3.4 Derivative of an Inverse Action

Applying the action by the inverse of $T \in G$ yields a function $g : G \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$g(T, p) = T^{-1}p$$

Theorem 4. *The Jacobian matrix of the inverse group action $g(T, p) = T^{-1}p$ is given by*

$$G_{(T,p)} = \begin{bmatrix} -H(T^{-1}p) & T^{-1} \end{bmatrix}$$

where $H : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ is the same mapping as before.

Proof. Again, the derivative D_2g with respect to in p is easy, the matrix of which is simply T^{-1} :

$$g(T, p + \delta p) = T^{-1}(p + \delta p) = T^{-1}p + T^{-1}\delta p = g(T, p) + D_2g(\delta p)$$

Conversely, a change in T yields

$$g(Te^{\hat{\xi}}, p) = \left(Te^{\hat{\xi}}\right)^{-1} p = e^{-\hat{\xi}} T^{-1}p$$

Similar to before, if we expand the matrix exponential we get

$$e^{-A} = I - A + \frac{A^2}{2!} - \frac{A^3}{3!} + \dots$$

so

$$e^{-\hat{\xi}} T^{-1}p \approx (I - \hat{\xi}) T^{-1}p = g(T, p) - \hat{\xi} (T^{-1}p)$$

□

Example 6. For 3D rotations $R \in SO(3)$ we have $R^{-1} = R^T$, $H(p) = -[p]_{\times}$, and hence the Jacobian matrix of $g(R, p) = R^T p$ is given by

$$G_{(R,p)} = \begin{pmatrix} [R^T p]_{\times} & R^T \end{pmatrix}$$

4 Instantaneous Velocity

For matrix Lie groups, if we have a matrix $T_b^n(t)$ that depends on a parameter t , i.e., $T_b^n(t)$ follows a curve on the manifold, then it would be of interest to find the velocity of a point $q^n(t) = T_b^n(t)p^b$ acted upon by $T_b^n(t)$. We can express the velocity of $q(t)$ in both the n-frame and b-frame:

$$\dot{q}^n = \dot{T}_b^n p^b = \dot{T}_b^n (T_b^n)^{-1} p^n \quad \text{and} \quad \dot{q}^b = (T_b^n)^{-1} \dot{q}^n = (T_b^n)^{-1} \dot{T}_b^n p^b$$

Both the matrices $\hat{\xi}_{nb}^n \triangleq \dot{T}_b^n (T_b^n)^{-1}$ and $\hat{\xi}_{nb}^b \triangleq (T_b^n)^{-1} \dot{T}_b^n$ are skew-symmetric Lie algebra elements that describe the **instantaneous velocity** [3, page 51 for rotations, page 419 for SE(3)]. We will revisit this for both rotations and rigid 3D transformations.

5 Differentials: Smooth Mapping between Lie Groups

5.1 Motivation and Definition

The above shows how to compute the derivative of a function $f : G \rightarrow \mathbb{R}^m$. However, what if the argument to f is itself the result of a mapping between Lie groups? In other words, $f = g \circ \varphi$, with $g : G \rightarrow \mathbb{R}^m$ and where $\varphi : H \rightarrow G$ is a smooth mapping from the n -dimensional Lie group H to the p -dimensional Lie group G . In this case, one would expect that we can arrive at Df_a by composing linear maps, as follows:

$$f'(a) = (g \circ \varphi)'(a) = G_{\varphi(a)} \varphi'(a)$$

where $\varphi'(a)$ is an $n \times p$ matrix that is the best linear approximation to the map $\varphi : H \rightarrow G$. The corresponding linear map $D\varphi_a$ is called the **differential** or **pushforward** of the mapping φ at a .

Because a rigorous definition will lead us too far astray, here we only informally define the pushforward of φ at a as the linear map $D\varphi_a : \mathbb{R}^n \rightarrow \mathbb{R}^p$ such that $D\varphi_a(\xi) \stackrel{\Delta}{=} \varphi'(a)\xi$ and

$$\varphi\left(ae^{\widehat{\xi}}\right) \approx \varphi(a) \exp\left(\widehat{\varphi'(a)\xi}\right) \quad (5.1)$$

with equality for $\xi \rightarrow 0$. We call $\varphi'(a)$ the **Jacobian matrix** of the map φ at a . Below we show that even with this informal definition we can deduce the pushforward in a number of useful cases.

5.2 Left Multiplication with a Constant

Theorem 5. Suppose G is an n -dimensional Lie group, and $\varphi : G \rightarrow G$ is defined as $\varphi(g) = hg$, with $h \in G$ a constant. Then $D\varphi_a$ is the identity mapping and

$$\varphi'(a) = I_n$$

Proof. Defining $y = D\varphi_a x$ as in (5.1), we have

$$\begin{aligned} \varphi(a)e^{\widehat{y}} &= \varphi(ae^{\widehat{x}}) \\ hae^{\widehat{y}} &= hae^{\widehat{x}} \\ y &= x \end{aligned}$$

□

5.3 Pushforward of the Inverse Mapping

A well known property of Lie groups is the fact that applying an incremental change $\widehat{\xi}$ in a different frame g can be applied in a single step by applying the change $Ad_g \widehat{\xi}$ in the original frame,

$$ge^{\widehat{\xi}}g^{-1} = \exp\left(Ad_g \widehat{\xi}\right) \quad (5.2)$$

where $Ad_g : \mathfrak{g} \rightarrow \mathfrak{g}$ is the **adjoint representation**. This comes in handy in the following:

Theorem 6. Suppose that $\varphi : G \rightarrow G$ is defined as the mapping from an element g to its *inverse* g^{-1} , i.e., $\varphi(g) = g^{-1}$, then the pushforward $D\varphi_a$ satisfies

$$(D\varphi_a x)^\wedge = -Ad_a \hat{x} \quad (5.3)$$

In other words, and this is intuitive in hindsight, approximating the inverse is accomplished by negation of \hat{x} , along with an adjoint to make sure it is applied in the right frame. Note, however, that (5.3) does not immediately yield a useful expression for the Jacobian matrix $\varphi'(a)$, but in many important cases this will turn out to be easy.

Proof. Defining $y = D\varphi_a x$ as in (5.1), we have

$$\begin{aligned} \varphi(a)e^{\hat{y}} &= \varphi(ae^{\hat{x}}) \\ a^{-1}e^{\hat{y}} &= (ae^{\hat{x}})^{-1} \\ e^{\hat{y}} &= -ae^{\hat{x}}a^{-1} \\ \hat{y} &= -Ad_a \hat{x} \end{aligned}$$

□

Example 7. For 3D rotations $R \in SO(3)$ we have

$$Ad_g(\hat{\omega}) = R\hat{\omega}R^T = [R\omega]_\times$$

and hence the pushforward for the inverse mapping $\varphi(R) = R^T$ has the matrix $\varphi'(R) = -R$.

5.4 Right Multiplication with a Constant

Theorem 7. Suppose $\varphi : G \rightarrow G$ is defined as $\varphi(g) = gh$, with $h \in G$ a constant. Then $D\varphi_a$ satisfies

$$(D\varphi_a x)^\wedge = Ad_{h^{-1}} \hat{x}$$

Proof. Defining $y = D\varphi_a x$ as in (5.1), we have

$$\begin{aligned} \varphi(a)e^{\hat{y}} &= \varphi(ae^{\hat{x}}) \\ ahe &= ae^{\hat{x}}h \\ e^{\hat{y}} &= h^{-1}e^{\hat{x}}h = \exp(Ad_{h^{-1}}\hat{x}) \\ \hat{y} &= Ad_{h^{-1}}\hat{x} \end{aligned}$$

□

Example 8. In the case of 3D rotations, right multiplication with a constant rotation R is done through the mapping $\varphi(A) = AR$, and satisfies

$$[D\varphi_A x]_\times = Ad_{R^T} [x]_\times$$

For 3D rotations $R \in SO(3)$ we have

$$Ad_{R^T}(\hat{\omega}) = R^T \hat{\omega} R = [R^T \omega]_\times$$

and hence the Jacobian matrix of φ at A is $\varphi'(A) = R^T$.

5.5 Pushforward of Compose

Theorem 8. *If we define the mapping $\varphi : G \times G \rightarrow G$ as the product of two group elements $g, h \in G$, i.e., $\varphi(g, h) = gh$, then the pushforward will satisfy*

$$D\varphi_{(a,b)}(x, y) = D_1\varphi_{(a,b)}x + D_2\varphi_{(a,b)}y$$

with

$$(D_1\varphi_{(a,b)}x)^\wedge = Ad_{b^{-1}}\hat{x} \text{ and } D_2\varphi_{(a,b)}y = y$$

Proof. Looking at the first argument, the proof is very similar to right multiplication with a constant b . Indeed, defining $y = D\varphi_a x$ as in (5.1), we have

$$\begin{aligned} \varphi(a, b)e^{\hat{y}} &= \varphi(ae^{\hat{x}}, b) \\ abe^{\hat{y}} &= ae^{\hat{x}}b \\ e^{\hat{y}} &= b^{-1}e^{\hat{x}}b = \exp(Ad_{b^{-1}}\hat{x}) \\ \hat{y} &= Ad_{b^{-1}}\hat{x} \end{aligned} \tag{5.4}$$

In other words, to apply an incremental change \hat{x} to a we first need to undo b , then apply \hat{x} , and then apply b again. Using (5.2) this can be done in one step by simply applying $Ad_{b^{-1}}\hat{x}$.

The second argument is quite a bit easier and simply yields the identity mapping:

$$\begin{aligned} \varphi(a, b)e^{\hat{y}} &= \varphi(a, be^{\hat{x}}) \\ abe^{\hat{y}} &= abe^{\hat{x}} \\ y &= x \end{aligned} \tag{5.5}$$

□

Example 9. For 3D rotations $A, B \in SO(3)$ we have $\varphi(A, B) = AB$, and $Ad_{B^T}[\omega]_\times = [B^T \omega]_\times$, hence the Jacobian matrix $\varphi'(A, B)$ of composing two rotations is given by

$$\varphi'(A, B) = \begin{bmatrix} B^T & I_3 \end{bmatrix}$$

5.6 Pushforward of Between

Finally, let us find the pushforward of **between**, defined as $\varphi(g, h) = g^{-1}h$. For the first argument we reason as:

$$\begin{aligned} \varphi(g, h)e^{\hat{y}} &= \varphi(ge^{\hat{x}}, h) \\ g^{-1}he^{\hat{y}} &= (ge^{\hat{x}})^{-1}h = -e^{\hat{x}}g^{-1}h \\ e^{\hat{y}} &= -(h^{-1}g)e^{\hat{x}}(h^{-1}g)^{-1} = -\exp Ad_{(h^{-1}g)}\hat{x} \\ \hat{y} &= -Ad_{(h^{-1}g)}\hat{x} = -Ad_{\varphi(h, g)}\hat{x} \end{aligned} \tag{5.6}$$

The second argument yields the identity mapping.

Example 10. For 3D rotations $A, B \in SO(3)$ we have $\varphi(A, B) = A^T B$, and $Ad_{B^T A}[-\omega]_\times = [-B^T A \omega]_\times$, hence the Jacobian matrix $\varphi'(A, B)$ of between is given by

$$\varphi'(A, B) = \begin{bmatrix} (-B^T A) & I_3 \end{bmatrix}$$

5.7 Numerical PushForward

Let's examine

$$f(g) e^{\hat{y}} = f(g e^{\hat{x}})$$

and multiply with $f(g)^{-1}$ on both sides:

$$e^{\hat{y}} = f(g)^{-1} f(g e^{\hat{x}})$$

We then take the log (which in our case returns y , not \hat{y}):

$$y(x) = \log \left[f(g)^{-1} f(g e^{\hat{x}}) \right]$$

Let us look at $x = 0$, and perturb in direction i , $e_i = [0, 0, 1, 0, 0]$. Then take derivative,

$$\frac{\partial y(d)}{\partial d} \triangleq \lim_{d \rightarrow 0} \frac{y(d) - y(0)}{d} = \lim_{d \rightarrow 0} \frac{1}{d} \log \left[f(g)^{-1} f(g e^{\widehat{de}_i}) \right]$$

which is the basis for a numerical derivative scheme.

5.8 Derivative of the Exponential and Logarithm Map

Theorem 9. *The derivative of the function $f : \mathbb{R}^n \rightarrow G$ that applies the wedge operator followed by the exponential map, i.e., $f(\xi) = \exp \hat{\xi}$, is the identity map for $\xi = 0$.*

Proof. For $\xi = 0$, we have

$$\begin{aligned} f(\xi) e^{\hat{y}} &= f(\xi + x) \\ f(0) e^{\hat{y}} &= f(0 + x) \\ e^{\hat{y}} &= e^{\hat{x}} \end{aligned}$$

□

Corollary 1. *The derivative of the inverse f^{-1} is the identity as well, i.e., for $T = e$, the identity element in G .*

For $\xi \neq 0$, things are not simple, see . <http://deltaepsilons.wordpress.com/2009/11/06/helgasons-formula-for-the-differential-of-the-exponential/>.

6 General Manifolds

6.1 Retractions

General manifolds that are not Lie groups do not have an exponential map, but can still be handled by defining a **retraction** $\mathcal{R} : \mathcal{M} \times \mathbb{R}^n \rightarrow \mathcal{M}$, such that

$$a \oplus \xi \triangleq \mathcal{R}_a(\xi)$$

A retraction [1] is required to be tangent to geodesics on the manifold \mathcal{M} at a . We can define many retractions for a manifold \mathcal{M} , even for those with more structure. For the vector space \mathbb{R}^n the retraction is just vector addition, and for Lie groups the obvious retraction is simply the exponential map, i.e., $\mathcal{R}_a(\xi) = a \cdot \exp \hat{\xi}$. However, one can choose other, possibly computationally attractive retractions, as long as around a they agree with the geodesic induced by the exponential map, i.e.,

$$\lim_{\xi \rightarrow 0} \frac{|a \cdot \exp \hat{\xi} - \mathcal{R}_a(\xi)|}{|\xi|} = 0$$

Example 11. For $SE(3)$, instead of using the true exponential map it is computationally more efficient to define the retraction, which uses a first order approximation of the translation update

$$\mathcal{R}_T \left(\begin{bmatrix} \omega \\ v \end{bmatrix} \right) = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{[\omega] \times} & v \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} Re^{[\omega] \times} & t + Rv \\ 0 & 1 \end{bmatrix}$$

6.2 Derivatives

Equipped with a retraction, then, we can generalize the notion of a derivative for functions f from general a manifold \mathcal{M} to \mathbb{R}^m :

Definition 4. We define a function $f : \mathcal{M} \rightarrow \mathbb{R}^m$ to be **differentiable** at $a \in \mathcal{M}$ if there exists a matrix $f'(a)$ such that

$$\lim_{\xi \rightarrow 0} \frac{|f(a) + f'(a)\xi - f(\mathcal{R}_a(\xi))|}{|\xi|} = 0$$

with $\xi \in \mathbb{R}^n$ for an n -dimensional manifold, and $\mathcal{R}_a : \mathbb{R}^n \rightarrow \mathcal{M}$ a retraction \mathcal{R} at a . If f is differentiable, then $f'(a)$ is called the **Jacobian matrix** of f at a , and the linear transformation $Df_a : \xi \mapsto f'(a)\xi$ is called the **derivative** of f at a .

For manifolds that are also Lie groups, the derivative of any function $f : G \rightarrow \mathbb{R}^m$ will agree no matter what retraction \mathcal{R} is used.

Part II

Practice

Below we apply the results derived in the theory part to the geometric objects we use in GTSAM. Above we preferred the modern notation $D_1 f$ for the partial derivative. Below (because this was written earlier) we use the more classical notation

$$\frac{\partial f(x,y)}{\partial x}$$

In addition, for Lie groups we will abuse the notation and take

$$\left. \frac{\partial \varphi(g)}{\partial \xi} \right|_a$$

to be the Jacobian matrix $\varphi'(a)$ of the mapping φ at $a \in G$, associated with the pushforward $D\varphi_a$.

7 SLAM Example

Let us examine a visual SLAM example. We have 2D measurements z_{ij} , where each measurement is predicted by

$$z_{ij} = h(T_i, p_j) = \pi(T_i^{-1} p_j)$$

where T_i is the 3D pose of the i^{th} camera, p_j is the location of the j^{th} point, and $\pi : (x, y, z) \mapsto (x/z, y/z)$ is the camera projection function from Example 1.

8 BetweenFactor

BetweenFactor is often used to summarize

Theorem 9 about the derivative of the exponential map $f : \xi \mapsto \exp \xi$ being identity only at $\xi = 0$ has implications for GTSAM. Given two elements T_1 and T_2 , BetweenFactor evaluates

$$g(T_1, T_2; Z) = f^{-1}(\text{between}(Z, \text{between}(T_1, T_2))) = f^{-1}(Z^{-1}(T_1^{-1}T_2))$$

but because it is assumed that $Z \approx T_1^{-1}T_2$, and hence we have $Z^{-1}T_1^{-1}T_2 \approx e$ and the derivative should be good there. Note that the derivative of *between* is identity in its second argument.

9 Point3

A cross product $a \times b$ can be written as a matrix multiplication

$$a \times b = [a]_{\times} b$$

where $[a]_{\times}$ is a skew-symmetric matrix defined as

$$[x, y, z]_{\times} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

We also have

$$a^T [b]_{\times} = -([b]_{\times} a)^T = -(a \times b)^T$$

The derivative of a cross product

$$\frac{\partial(a \times b)}{\partial a} = [-b]_{\times} \quad (9.1)$$

$$\frac{\partial(a \times b)}{\partial b} = [a]_{\times} \quad (9.2)$$

10 2D Rotations

10.1 Rot2 in GTSAM

A rotation is stored as $(\cos \theta, \sin \theta)$. An incremental rotation is applied using the trigonometric sum rule:

$$\begin{aligned}\cos \theta' &= \cos \theta \cos \delta - \sin \theta \sin \delta \\ \sin \theta' &= \sin \theta \cos \delta + \cos \theta \sin \delta\end{aligned}$$

where δ is an incremental rotation angle.

10.2 Derivatives of Actions

In the case of $SO(2)$ the vector space is \mathbb{R}^2 , and the group action $f(R, p)$ corresponds to rotating the 2D point p

$$f(R, p) = Rp$$

According to Theorem 3, the Jacobian matrix of f is given by

$$f'(R, p) = \begin{bmatrix} RH(p) & R \end{bmatrix}$$

with $H : \mathbb{R}^2 \rightarrow \mathbb{R}^{2 \times 2}$ a linear mapping that depends on p . In the case of $SO(2)$, we can find $H(p)$ by equating (as in Equation 3.1):

$$[w]_{+p} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} \omega = H(p)\omega$$

Note that

$$H(p) = \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = R_{\pi/2} p$$

and since 2D rotations commute, we also have, with $q = Rp$:

$$f'(R, p) = \begin{bmatrix} R(R_{\pi/2} p) & R \end{bmatrix} = \begin{bmatrix} R_{\pi/2} q & R \end{bmatrix}$$

10.3 Pushforwards of Mappings

Since $Ad_R[\omega]_{+} = [\omega]_{+}$, we have the derivative of **inverse**,

$$\frac{\partial R^T}{\partial \omega} = -Ad_R = -1$$

compose,

$$\frac{\partial (R_1 R_2)}{\partial \omega_1} = Ad_{R_2^T} = 1 \text{ and } \frac{\partial (R_1 R_2)}{\partial \omega_2} = 1$$

and **between**:

$$\frac{\partial (R_1^T R_2)}{\partial \omega_1} = -Ad_{R_2^T R_1} = -1 \text{ and } \frac{\partial (R_1^T R_2)}{\partial \omega_2} = 1$$

11 2D Rigid Transformations

11.1 The derivatives of Actions

The action of $SE(2)$ on 2D points is done by embedding the points in \mathbb{R}^3 by using homogeneous coordinates

$$f(T, p) = \hat{q} = \begin{bmatrix} q \\ 1 \end{bmatrix} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = T \hat{p}$$

To find the derivative, we write the quantity $\hat{\xi} \hat{p}$ as the product of the 3×3 matrix $H(p)$ with ξ :

$$\hat{\xi} \hat{p} = \begin{bmatrix} [\omega]_+ & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} [\omega]_+ p + v \\ 0 \end{bmatrix} = \begin{bmatrix} I_2 & R_{\pi/2} p \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} = H(p) \xi \quad (11.1)$$

Hence, by Theorem 3 we have

$$\frac{\partial (T \hat{p})}{\partial \xi} = TH(p) = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_2 & R_{\pi/2} p \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & RR_{\pi/2} p \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & R_{\pi/2} q \\ 0 & 0 \end{bmatrix} \quad (11.2)$$

Note that, looking only at the top rows of (11.1) and (11.2), we can recognize the quantity $[\omega]_+ p + v = v + \omega (R_{\pi/2} p)$ as the velocity of p in \mathbb{R}^2 , and $\begin{bmatrix} R & R_{\pi/2} q \end{bmatrix}$ is the derivative of the action on \mathbb{R}^2 .

The derivative of the inverse action $g(T, p) = T^{-1} \hat{p}$ is given by Theorem 4 specialized to $SE(2)$:

$$\frac{\partial (T^{-1} \hat{p})}{\partial \xi} = -H(T^{-1} p) = \begin{bmatrix} -I_2 & -R_{\pi/2} (T^{-1} p) \\ 0 & 0 \end{bmatrix}$$

11.2 Pushforwards of Mappings

We can just define all derivatives in terms of the adjoint map, which in the case of $SE(2)$, in twist coordinates, is the linear mapping

$$Ad_T \xi = \begin{bmatrix} R & -R_{\pi/2} t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

and we have

$$\frac{\partial T^{-1}}{\partial \xi} = -Ad_T$$

$$\frac{\partial (T_1 T_2)}{\partial \xi_1} = Ad_{T_2^{-1}} \text{ and } \frac{\partial (T_1 T_2)}{\partial \xi_2} = I_3$$

$$\frac{\partial (T_1^{-1} T_2)}{\partial \xi_1} = -Ad_{T_2^{-1} T_1} = -Ad_{\text{between}(T_2, T_1)} \text{ and } \frac{\partial (T_1^{-1} T_2)}{\partial \xi_2} = I_3$$

12 3D Rotations

12.1 Derivatives of Actions

In the case of $SO(3)$ the vector space is \mathbb{R}^3 , and the group action $f(R, p)$ corresponds to rotating a point

$$q = f(R, p) = Rp$$

To calculate $H(p)$ for use in Theorem (3) we make use of

$$[\omega]_{\times} p = \omega \times p = -p \times \omega = [-p]_{\times} \omega$$

so $H(p) \triangleq [-p]_{\times}$. Hence, the final derivative of an action in its first argument is

$$\frac{\partial (Rp)}{\partial \omega} = RH(p) = -R[p]_{\times}$$

Likewise, according to Theorem 4, the derivative of the inverse action is given by

$$\frac{\partial (R^T p)}{\partial \omega} = -H(R^T p) = [R^T p]_{\times}$$

12.2 Instantaneous Velocity

For 3D rotations R_b^n from a body frame b to a navigation frame n we have the spatial angular velocity ω_{nb}^n measured in the navigation frame,

$$[\omega_{nb}^n]_{\times} \triangleq \dot{R}_b^n (R_b^n)^T = \dot{R}_b^n R_n^b$$

and the body angular velocity ω_{nb}^b measured in the body frame:

$$[\omega_{nb}^b]_{\times} \triangleq (R_b^n)^T \dot{R}_b^n = R_n^b \dot{R}_b^n$$

These quantities can be used to derive the velocity of a point p , and we choose between spatial or body angular velocity depending on the frame in which we choose to represent p :

$$v^n = [\omega_{nb}^n]_{\times} p^n = \omega_{nb}^n \times p^n$$

$$v^b = [\omega_{nb}^b]_{\times} p^b = \omega_{nb}^b \times p^b$$

We can transform these skew-symmetric matrices from navigation to body frame by conjugating,

$$[\omega_{nb}^b]_{\times} = R_n^b [\omega_{nb}^n]_{\times} R_b^n$$

but because the adjoint representation satisfies

$$Ad_R[\omega]_{\times} \triangleq R[\omega]_{\times} R^T = [R\omega]_{\times}$$

we can even more easily transform between spatial and body angular velocities as 3-vectors:

$$\omega_{nb}^b = R_n^b \omega_{nb}^n$$

12.3 Pushforwards of Mappings

For $SO(3)$ we have $Ad_R[\omega]_{\times} = [R\omega]_{\times}$ and, in terms of angular velocities: $Ad_R\omega = R\omega$. Hence, the Jacobian matrix of the **inverse** mapping is (see Equation 5.3)

$$\frac{\partial R^T}{\partial \omega} = -Ad_R = -R$$

for **compose** we have (Equations 5.4 and 5.5):

$$\frac{\partial (R_1 R_2)}{\partial \omega_1} = R_2^T \text{ and } \frac{\partial (R_1 R_2)}{\partial \omega_2} = I_3$$

and **between** (Equation 5.6):

$$\frac{\partial (R_1^T R_2)}{\partial \omega_1} = -R_2^T R_1 = -\text{between}(R_2, R_1) \text{ and } \frac{\partial (R_1 R_2)}{\partial \omega_2} = I_3$$

12.4 Retractions

Absil [1, page 58] discusses two possible retractions for $SO(3)$ based on the QR decomposition or the polar decomposition of the matrix $R[\omega]_{\times}$, but they are expensive. Another retraction is based on the Cayley transform $\mathcal{C} : \mathfrak{so}(3) \rightarrow SO(3)$, a mapping from the skew-symmetric matrices to rotation matrices:

$$Q = \mathcal{C}(\Omega) = (I - \Omega)(I + \Omega)^{-1}$$

Interestingly, the inverse Cayley transform $\mathcal{C}^{-1} : SO(3) \rightarrow \mathfrak{so}(3)$ has the same form:

$$\Omega = \mathcal{C}^{-1}(Q) = (I - Q)(I + Q)^{-1}$$

The retraction needs a factor $-\frac{1}{2}$ however, to make it locally align with a geodesic:

$$R' = \mathcal{R}_R(\omega) = R\mathcal{C}\left(-\frac{1}{2}[\omega]_{\times}\right)$$

Note that given $\omega = (x, y, z)$ this has the closed-form expression below

$$\begin{aligned} & \frac{1}{4 + x^2 + y^2 + z^2} \begin{bmatrix} 4 + x^2 - y^2 - z^2 & 2xy - 4z & 2xz + 4y \\ 2xy + 4z & 4 - x^2 + y^2 - z^2 & 2yz - 4x \\ 2xz - 4y & 2yz + 4x & 4 - x^2 - y^2 + z^2 \end{bmatrix} \\ &= \frac{1}{4 + x^2 + y^2 + z^2} \left\{ 4(I + [\omega]_{\times}) + \begin{bmatrix} x^2 - y^2 - z^2 & 2xy & 2xz \\ 2xy & -x^2 + y^2 - z^2 & 2yz \\ 2xz & 2yz & -x^2 - y^2 + z^2 \end{bmatrix} \right\} \end{aligned}$$

so it can be seen to be a second-order correction on $(I + [\omega]_{\times})$. The corresponding approximation to the logarithmic map is:

$$[\omega]_{\times} = \mathcal{R}_R^{-1}(R') = -2\mathcal{C}^{-1}(R^T R')$$

13 3D Rigid Transformations

13.1 The derivatives of Actions

The action of $SE(3)$ on 3D points is done by embedding the points in \mathbb{R}^4 by using homogeneous coordinates

$$\hat{q} = \begin{bmatrix} q \\ 1 \end{bmatrix} = f(T, p) = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = T \hat{p}$$

The quantity $\hat{\xi} \hat{p}$ corresponds to a velocity in \mathbb{R}^4 (in the local T frame), and equating it to $H(p)\xi$ as in Equation 3.1 yields the 4×6 matrix $H(p)$ ¹:

$$\hat{\xi} \hat{p} = \begin{bmatrix} [\omega]_{\times} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} \omega \times p + v \\ 0 \end{bmatrix} = \begin{bmatrix} [-p]_{\times} & I_3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix} = H(p)\xi$$

Note how velocities are analogous to points at infinity in projective geometry: they correspond to free vectors indicating a direction and magnitude of change. According to Theorem 3, the derivative of the group action is then

$$\frac{\partial (T \hat{p})}{\partial \xi} = TH(p) = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} [-p]_{\times} & I_3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R[-p]_{\times} & R \\ 0 & 0 \end{bmatrix}$$

$$\frac{\partial (T \hat{p})}{\partial \hat{p}} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

in homogenous coordinates. In \mathbb{R}^3 this becomes $R \begin{bmatrix} -[p]_{\times} & I_3 \end{bmatrix}$.

The derivative of the inverse action $T^{-1}p$ is given by Theorem 4:

$$\frac{\partial (T^{-1} \hat{p})}{\partial \xi} = -H(T^{-1} \hat{p}) = \begin{bmatrix} [T^{-1} \hat{p}]_{\times} & -I_3 \end{bmatrix}$$

$$\frac{\partial (T^{-1} \hat{p})}{\partial \hat{p}} = \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}$$

Example 12. Let us examine a visual SLAM example. We have 2D measurements z_{ij} , where each measurement is predicted by

$$z_{ij} = h(T_i, p_j) = \pi(T_i^{-1} p_j) = \pi(q)$$

where T_i is the 3D pose of the i^{th} camera, p_j is the location of the j^{th} point, $q = (x', y', z') = T_i^{-1} p_j$ is the point in camera coordinates, and $\pi : (x, y, z) \mapsto (x/z, y/z)$ is the camera projection function from Example 1. By the chain rule, we then have

$$\frac{\partial h(T, p)}{\partial \xi} = \frac{\partial \pi(q)}{\partial q} \frac{\partial (T^{-1} p)}{\partial \xi} = \frac{1}{z'} \begin{bmatrix} 1 & 0 & -x'/z' \\ 0 & 1 & -y'/z' \end{bmatrix} \begin{bmatrix} [q]_{\times} & -I_3 \end{bmatrix} = \begin{bmatrix} \pi'(q)[q]_{\times} & -\pi'(q) \end{bmatrix}$$

$$\frac{\partial h(T, p)}{\partial p} = \pi'(q) R^T$$

¹ $H(p)$ can also be obtained by taking the j^{th} column of each of the 6 generators to multiply with components of \hat{p}

13.2 Instantaneous Velocity

For rigid 3D transformations T_b^n from a body frame b to a navigation frame n we have the instantaneous spatial twist ξ_{nb}^n measured in the navigation frame,

$$\hat{\xi}_{nb}^n \triangleq \dot{T}_b^n (T_b^n)^{-1}$$

and the instantaneous body twist ξ_{nb}^b measured in the body frame:

$$\hat{\xi}_{nb}^b \triangleq (T_b^n)^T \dot{T}_b^n$$

13.3 Pushforwards of Mappings

As we can express the Adjoint representation in terms of twist coordinates, we have

$$\begin{bmatrix} \omega' \\ v' \end{bmatrix} = \begin{bmatrix} R & 0 \\ [t]_{\times} R & R \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix}$$

Hence, as with $SO(3)$, we are now in a position to simply posit the derivative of **inverse**,

$$\frac{\partial T^{-1}}{\partial \xi} = -Ad_T = - \begin{bmatrix} R & 0 \\ [t]_{\times} R & R \end{bmatrix}$$

compose in its first argument,

$$\frac{\partial (T_1 T_2)}{\partial \xi_1} = Ad_{T_2^{-1}}$$

in its second argument,

$$\frac{\partial (T_1 T_2)}{\partial \xi_2} = I_6$$

between in its first argument,

$$\frac{\partial (T_1^{-1} T_2)}{\partial \xi_1} = -Ad_{T_2^{-1} T_1}$$

and in its second argument,

$$\frac{\partial (T_1^{-1} T_2)}{\partial \xi_2} = I_6$$

13.4 Retractions

For $SE(3)$, instead of using the true exponential map it is computationally more efficient to design other retractions. A first-order approximation to the exponential map does not quite cut it, as it

yields a 4×4 matrix which is not in $SE(3)$:

$$\begin{aligned}
T \exp \hat{\xi} &\approx T(I + \hat{\xi}) \\
&= T \left(I_4 + \begin{bmatrix} [\omega]_{\times} & v \\ 0 & 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_3 + [\omega]_{\times} & v \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R(I_3 + [\omega]_{\times}) & t + Rv \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

However, we can make it into a retraction by using any retraction defined for $SO(3)$, including, as below, using the exponential map $Re^{[\omega]_{\times}}$:

$$\mathcal{R}_T \left(\begin{bmatrix} \omega \\ v \end{bmatrix} \right) = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{[\omega]_{\times}} & v \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} Re^{[\omega]_{\times}} & t + Rv \\ 0 & 1 \end{bmatrix}$$

Similarly, for a second order approximation we have

$$\begin{aligned}
T \exp \hat{\xi} &\approx T \left(I + \hat{\xi} + \frac{\hat{\xi}^2}{2} \right) \\
&= T \left(I_4 + \begin{bmatrix} [\omega]_{\times} & v \\ 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} [\omega]_{\times} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} [\omega]_{\times} & v \\ 0 & 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} I_3 + [\omega]_{\times} + \frac{1}{2} [\omega]_{\times}^2 & v + \frac{1}{2} [\omega]_{\times} v \\ 0 & 1 \end{bmatrix} \right) \\
&= \begin{bmatrix} R(I_3 + [\omega]_{\times} + \frac{1}{2} [\omega]_{\times}^2) & t + R[v + (\omega \times v)/2] \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

inspiring the retraction

$$\mathcal{R}_T \left(\begin{bmatrix} \omega \\ v \end{bmatrix} \right) = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{[\omega]_{\times}} & v + (\omega \times v)/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} Re^{[\omega]_{\times}} & t + R[v + (\omega \times v)/2] \\ 0 & 1 \end{bmatrix}$$

14 2D Line Segments (Ocaml)

The error between an infinite line (a, b, c) and a 2D line segment $((x1, y1), (x2, y2))$ is defined in Line3.ml.

15 Line3vd (Ocaml)

One representation of a line is through 2 vectors (v, d) , where v is the direction and the vector d points from the origin to the closest point on the line.

In this representation, transforming a 3D line from a world coordinate frame to a camera at (R_w^c, t^w) is done by

$$\begin{aligned} v^c &= R_w^c v^w \\ d^c &= R_w^c (d^w + (t^w v^w) v^w - t^w) \end{aligned}$$

16 Line3 (Ocaml)

For 3D lines, we use a parameterization due to C.J. Taylor, using a rotation matrix R and 2 scalars a and b . The line direction v is simply the Z-axis of the rotated frame, i.e., $v = R_3$, while the vector d is given by $d = aR_1 + bR_2$.

Now, we will *not* use the incremental rotation scheme we used for rotations: because the matrix R translates from the line coordinate frame to the world frame, we need to apply the incremental rotation on the right-side:

$$R' = R(I + \Omega)$$

Projecting a line to 2D can be done easily, as both v and d are also the 2D homogenous coordinates of two points on the projected line, and hence we have

$$\begin{aligned} l &= v \times d \\ &= R_3 \times (aR_1 + bR_2) \\ &= a(R_3 \times R_1) + b(R_3 \times R_2) \\ &= aR_2 - bR_1 \end{aligned}$$

This can be written as a rotation of a point,

$$l = R \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}$$

but because the incremental rotation is now done on the right, we need to figure out the derivatives again:

$$\frac{\partial (R(I + \Omega)x)}{\partial \omega} = \frac{\partial (R\Omega x)}{\partial \omega} = R \frac{\partial (\Omega x)}{\partial \omega} = R[-x]_{\times} \quad (16.1)$$

and hence the derivative of the projection l with respect to the rotation matrix R of the 3D line is

$$\frac{\partial(l)}{\partial \omega} = R \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}_{\times} = \begin{bmatrix} aR_3 & bR_3 & -(aR_1 + bR_2) \end{bmatrix} \quad (16.2)$$

or the a, b scalars:

$$\begin{aligned} \frac{\partial(l)}{\partial a} &= R_2 \\ \frac{\partial(l)}{\partial b} &= -R_1 \end{aligned}$$

Transforming a 3D line $(R, (a, b))$ from a world coordinate frame to a camera frame (R_w^c, t^w) is done by

$$\begin{aligned} R' &= R_w^c R \\ a' &= a - R_1^T t^w \\ b' &= b - R_2^T t^w \end{aligned}$$

Again, we need to redo the derivatives, as R is incremented from the right. The first argument is incremented from the left, but the result is incremented on the right:

$$\begin{aligned} R'(I + \Omega') &= (AB)(I + \Omega') = (I + [S\omega]_{\times})AB \\ I + \Omega' &= (AB)^T (I + [S\omega]_{\times}) (AB) \\ \Omega' &= R'^T [S\omega]_{\times} R' \\ \Omega' &= [R'^T S\omega]_{\times} \\ \omega' &= R'^T S\omega \end{aligned}$$

For the second argument R we now simply have:

$$\begin{aligned} AB(I + \Omega') &= AB(I + \Omega) \\ \Omega' &= \Omega \\ \omega' &= \omega \end{aligned}$$

The scalar derivatives can be found by realizing that

$$\begin{pmatrix} a' \\ b' \\ \dots \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} - R^T t^w$$

where we don't care about the third row. Hence

$$\frac{\partial((R(I + \Omega_2))^T t^w)}{\partial \omega} = -\frac{\partial(\Omega_2 R^T t^w)}{\partial \omega} = -[R^T t^w]_{\times} = \begin{bmatrix} 0 & R_3^T t^w & -R_2^T t^w \\ -R_3^T t^w & 0 & R_1^T t^w \\ \dots & \dots & 0 \end{bmatrix}$$

17 Aligning 3D Scans

Below is the explanation underlying Pose3.align, i.e. aligning two point clouds using SVD. Inspired but modified from CVOnline...

Our model is

$$p^c = R(p^w - t)$$

i.e., R is from camera to world, and t is the camera location in world coordinates. The objective function is

$$\frac{1}{2} \sum (p^c - R(p^w - t))^2 = \frac{1}{2} \sum (p^c - Rp^w + Rt)^2 = \frac{1}{2} \sum (p^c - Rp^w - t')^2 \quad (17.1)$$

where $t' = -Rt$ is the location of the origin in the camera frame. Taking the derivative with respect to t' and setting to zero we have

$$\sum (p^c - Rp^w - t') = 0$$

or

$$t' = \frac{1}{n} \sum (p^c - Rp^w) = \bar{p}^c - R\bar{p}^w \quad (17.2)$$

here \bar{p}^c and \bar{p}^w are the point cloud centroids. Substituting back into (17.1), we get

$$\frac{1}{2} \sum (p^c - R(p^w - t))^2 = \frac{1}{2} \sum ((p^c - \bar{p}^c) - R(p^w - \bar{p}^w))^2 = \frac{1}{2} \sum (\hat{p}^c - R\hat{p}^w)^2$$

Now, to minimize the above it suffices to maximize (see CVOnline)

$$\text{trace}(R^T C)$$

where $C = \sum \hat{p}^c (\hat{p}^w)^T$ is the correlation matrix. Intuitively, the cloud of points is rotated to align with the principal axes. This can be achieved by SVD decomposition on C

$$C = USV^T$$

and setting

$$R = UV^T$$

Clearly, from (17.2) we then also recover the optimal t as

$$t = \bar{p}^w - R^T \bar{p}^c$$

Appendix

Differentiation Rules

Spivak [4] also notes some multivariate derivative rules defined component-wise, but they are not that useful in practice:

- Since $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined in terms of m component functions f^i , then f is differentiable at a iff each f^i is, and the Jacobian matrix F_a is the $m \times n$ matrix whose i^{th} row is $(f^i)'(a)$:

$$F_a \triangleq f'(a) = \begin{bmatrix} (f^1)'(a) \\ \vdots \\ (f^m)'(a) \end{bmatrix}$$

- Scalar differentiation rules: if $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ are differentiable at a , then

$$(f + g)'(a) = F_a + G_a$$

$$(f \cdot g)'(a) = g(a)F_a + f(a)G_a$$

$$(f/g)'(a) = \frac{1}{g(a)^2} [g(a)F_a - f(a)G_a]$$

Tangent Spaces and the Tangent Bundle

The following is adapted from Appendix A in [3].

The **tangent space** $T_p M$ of a manifold M at a point $p \in M$ is the vector space of **tangent vectors** at p . The **tangent bundle** TM is the set of all tangent vectors

$$TM \triangleq \bigcup_{p \in M} T_p M$$

A **vector field** $X : M \rightarrow TM$ assigns a single tangent vector $x \in T_p M$ to each point p .

If $F : M \rightarrow N$ is a smooth map from a manifold M to a manifold N , then we can define the **tangent map** of F at p as the linear map $F_{*p} : T_p M \rightarrow T_{F(p)} N$ that maps tangent vectors in $T_p M$ at p to tangent vectors in $T_{F(p)} N$ at the image $F(p)$.

Homomorphisms

The following *might be* relevant [2, page 45]: suppose that $\Phi : G \rightarrow H$ is a mapping (Lie group homomorphism). Then there exists a unique linear map $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$

$$\phi(\hat{x}) \triangleq \lim_{t \rightarrow 0} \frac{d}{dt} \Phi(e^{t\hat{x}})$$

such that

1. $\Phi(e^{\hat{x}}) = e^{\phi(\hat{x})}$
2. $\phi(T\hat{x}T^{-1}) = \Phi(T)\phi(\hat{x})\Phi(T^{-1})$
3. $\phi([\hat{x}, \hat{y}]) = [\phi(\hat{x}), \phi(\hat{y})]$

In other words, the map ϕ is the derivative of Φ at the identity. As an example, suppose $\Phi(g) = g^{-1}$, then the corresponding derivative *at the identity* is

$$\phi(\hat{x}) \triangleq \lim_{t \rightarrow 0} \frac{d}{dt} (e^{t\hat{x}})^{-1} = \lim_{t \rightarrow 0} \frac{d}{dt} e^{-t\hat{x}} = -\hat{x} \lim_{t \rightarrow 0} e^{-t\hat{x}} = -\hat{x}$$

In general it suffices to compute ϕ for a basis of \mathfrak{g} .

References

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- [3] R.M. Murray, Z. Li, and S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, 1994.
- [4] M. Spivak. *Calculus on manifolds*. Benjamin New York;, 1965.