Midterms Revision Guide

30.101 Systems & Control, Term 5 2020

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Contents

1 W1: Linear Time-Invariant Systems

1.1 Signals

- Signal: function changing in time and space

Continuous signal e.g. $x(t)$, $-\infty < t < \infty$

Discrete signal e.g. $x[k]$, $k = 1, 2, \dots$

Determinstic signal e.g. $x(t) = \cos \omega t$ exact value of x(t) at any t is known

Random/stochastic signal e.g $x(t) = \cos(\omega t + \phi)$, $\phi = \{0, \frac{\pi}{2}, \pi\}$ esact value of x(t) at any t is unknown

Periodic signal e.g. $x(t) = \sin t$, where $x(t) = x(t + T)$

Non-periodic signal e.g. $x(t) = \begin{cases} \cos t, & t < 0 \\ \sin t, & t \geq 0 \end{cases}$, where $x(t) \neq x(t + T)$

Bounded signal: $x(t)$ does not $\to \infty$ as $t \to \infty$

Unbounded signal: $x(t) \to \infty$ as $t \to \infty$

1.1.1 Basic signals

a. Unit impulse function $\delta(t)$

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \qquad \int_{0^-}^{0^+} \delta(t)\, dt = 1$$

b. Unit step function $u(t)$

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

c. Rectangular function $\mathrm{rect}\left(\frac{t}{T}\right)$

$$\begin{aligned} \mathrm{rect}\left(\frac{t}{T}\right) &= \begin{cases} 1, & -\frac{T}{2} < t < \frac{T}{2} \\ 0, & \text{elsewhere} \end{cases} \\ &= u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right) \end{aligned}$$

d. Exponential growth/decay function

$$x(t) = Ce^{at}$$

- Exponential growth: $C > 0$
- Exponential decay: $C < 0$

1.2 System Properties

1. Causal: output depends on input at present, past
2. Linearity: has property of superposition
3. Time Invariance: time shift in output = time shift in inuput

1.3 Complex exponential sinusoidal signals

- $\sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$
- $\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$
- $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

1.4 Zeros and Poles

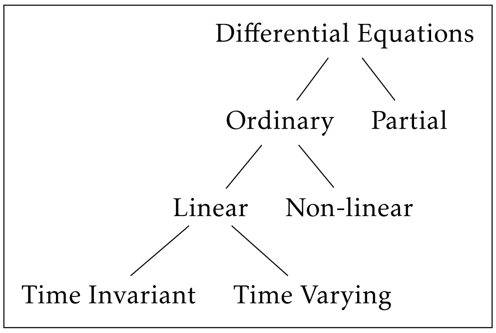
- General form of G(s):

$$G(s) = \frac{K(s+z+1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = \frac{\mathrm{N(s)}}{\mathrm{D(s)}},$$

  where N(s) is a polynomial of degree m and D(s) is a polynomial of degree n, and m < n.

  - Zeros (○): points where N(s) = 0 e.g. $s = -z_1, -z_2, \cdots, -z_m$
  - Poles/roots (×): points where D(s) = 0 e.g. $s = -p_1, -p_2, \cdots, -p_n$

1.5 Differential equations



1.5.1 Ordinary Differential Equations (ODEs)

- General form:

$$g\left(\frac{d^n x}{dt^n}, \frac{d^{n-1}x}{dt^{n-1}}, \cdots, x, t\right) = f(t)$$

  - where x is the dependent variable;
  - t is the independent variable.

- Linear ODE: output and its derivatives are pure functions of input, and are to power 1
- Non-linear ODE: output and its derivatives are not pure functions of input
- Time Invariant ODE: coefficients are independent of t
- Time Varying ODE: coefficients are functions of t

1.6 Laplace Transform (LT)

If $f(t) = 0$ for $t < 0$:

$$F(s) = \mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt,\ t \geq 0$$

- where $s = \sigma + j\omega$
- $\displaystyle\int_0^\infty$ is an improper integral, thus Laplace Transform may not exist
  - Laplace Transform exists within Region of Convergence (ROC)

1.7 Initial Value Theorem

If $f(t)$ and $\dfrac{df(t)}{dt}$ are both Laplace Transformable and $\lim\limits_{s\to\infty} sF(s)$ exists,

$$f(0^+) = \lim_{s\to\infty} sF(s)$$

1.8 Final Value Theorem

If $f(t)$ and $\frac{df(t)}{dt}$ are Laplace Transformable and $\lim\limits_{t\to\infty} f(t)$ exists,
and $sF(s)$ has all its poles with **strictly negative real part**,

$$f(\infty) = \lim_{s\to 0} sF(s)$$

1.9 Inverse Laplace Transform (ILT)

1. Express $F(s)$ as a proper rational fraction: $F(s) = \dfrac{N(s)}{D(s)}$, where degree of $N(s) < D(s)$
2. Check roots of D(s):

   Ⓐ Roots are Real and Distinct

   $$F(s) = \frac{N(s)}{D(s)} = \frac{a}{s+p_1} + \frac{a}{s+p_2} + \cdots + \frac{a}{s+p_n},$$

   $$\text{where } a_i = (s+p_i)F(s)\big|_{s=-p_i}$$

   Ⓑ Roots are Real and Repetitive

   $$F(s) = \frac{b_1}{s+p} + \frac{b_2}{(s+p)^2} + \cdots + \frac{b_n}{(s+p)^n}$$

   $$\text{where } b_i = \frac{1}{(n-1)!}\left[\frac{d^{n-i}}{ds^{n-i}}(s+p)^n F(s)\right]\Bigg|_{s=-p}$$

   Ⓒ Roots are Complex Conjugates

   $$F(s) = \frac{N(s)}{s^2+cs+d} = C_1\frac{\omega}{(s+a)^2+\omega^2} + C_2\frac{s+a}{(s+a)^2+\omega^2}$$

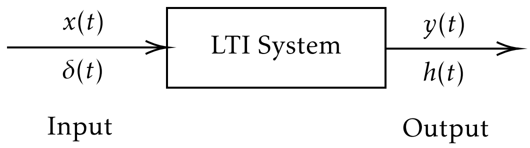
   $$\text{where poles, } s = -\frac{c}{2} \pm \frac{\sqrt{c^2-4d}}{2}$$

   Ⓓ Combination of Cases Ⓐ, Ⓑ, Ⓒ

   - Rewrite numerator in terms of denominator to simplify

3. Use Laplace Transform table pairs to infer $f(t)$ from $F(s)$.

2 W2: Convolution



2.1 Properties of impulse function

1. $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$
2. $\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\, dt = x(t_0)$

2.2 Convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau \Longleftrightarrow \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\, d\tau$$

- It can be written as $y(t) = x(t) * h(t)$

2.3 Graphical Method

1. Flip: $h(\tau) \to h(-\tau)$
2. Shift by t: $h(-\tau) \to h(t-\tau)$
3. Multiply by x: $x(\tau)h(t-\tau)$
4. Integrate over τ: $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$

2.4 Properties of Convolution

- Commutative: $x(t) * h(t) = h(t) * x(t)$
- Associative: $[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$
- Distributive: $x(t) * h_1(t) + x(t) * h_2(t) = x(t) * [h_1(t) + h_2(t)]$

3 W3: Fourier Analysis

3.1 Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},\ \omega_0 = \frac{2\pi}{T_0}$$

Synthesis: $\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$

Analysis: $\boxed{a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t}\, dt}$

3.2 Convergence Conditions (Dirichlet Conditions)

1. Signal is integral over any period. $\int_{T_0} |x(t)| < \infty$
2. Signal must be bounded. $x(t)$ cannot be $\pm\infty$.
3. Finite number of discontinuities in interval T.

3.3 Forms of Fourier Series

1. $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$
2. $x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$
3. $x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t]$

3.4 Fourier Representation of Aperiodic Signals

- $\tilde{x}(t)$ is T_0 periodic, which is made by repeating the aperiodic signal $x(t)$
- $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},\ \omega_0 = \frac{2\pi}{T_0}$
- As $T_0 \to \infty,\ \omega_0 \to 0$
- Converges to Fourier Transform

3.5 Power of a signal

- Sum of squares of all the Fourier coefficients

$$\text{Power} = \sum_{k=-\infty}^{\infty} |a_k|^2$$

3.6 Fourier Transform Pairs

$$\text{Fourier Transform} \quad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$\text{Inverse Fourier Transform} \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

3.7 Periodic x(t)

- Fourier Transform of x(t) is an impulse train

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

3.8 Fourier Transform vs Laplace Transform

$$\text{Fourier Transform: } \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \qquad \text{Laplace Transform: } \int_{0}^{\infty} x(t)e^{-st}\,dt$$

- Limits of Integration: $-\infty$ to ∞ (FT), 0 to ∞ (LT)
- Location of complex variable: $j\omega$ lies on the imaginary axis (FT),
  s can be any complex number in the region of convergence (LT)
- Existence of FT and LT: If the imaginary axis is not in region of convergence of LT, FT does not exist while LT exists.
- Equivalence of FT and LT: If $x(t) = 0, t < 0$ and imaginary axis is in region of convergence of LT, FT is LT evaluated on the imaginary axis.
- Non-equivalence of FT and LT: If $x(t) \neq 0$ for $t < 0$, then FT $\neq$ LT.

4 W4: Modelling Physical Systems

4.1 Translational Mechanical Systems

	Mass	**Spring**	**Damper**
Force	$f = m\ddot{x}$	$f_k = k(x_2 - x_1)$	$f_b = b(\dot{x}_2 - \dot{x}_1)$
Conservative energies	$\text{KE} = \frac{1}{2}m\dot{x}^2$ $\text{PE} = mgh$	$\text{PE} = \frac{1}{2}kx^2$	NOT CONSERVATIVE
Other laws	Power $P = f\dot{x}$	N2L: $\sum f = ma = m\ddot{x}$	N3L

4.2 Rotational Mechanical Systems

	Mass	**Spring**	**Damper**
Torque	$\tau = J\ddot{\theta}$	$\tau_k = k(\theta_2 - \theta_1)$	$\tau_b = b(\dot{\theta}_2 - \dot{\theta}_1)$
Conservative energies	$\text{KE} = \frac{1}{2}J\dot{\theta}^2$	$\text{PE} = \frac{1}{2}k\theta^2$	NOT CONSERVATIVE
Other laws	Power $P = \tau\dot{\theta}$	N2L: $\sum \tau = J\alpha = J\ddot{\theta}$	N3L

4.3 Energy Method for Mechanical Systems

- Conservative systems only
- Do not dissipate energy due to friction

$$\frac{d}{dt}(\text{KE} + \text{PE}) = 0$$

4.4 Electrical Systems

	Inductor	**Capacitor**	**Resistor**
Current or Voltage	$V_a - V_b = L\frac{di_L}{dt}$	$i_C = C\frac{d}{dt}(V_a - V_b)$	$V_a - V_b = i_R R$
Conservative energies	$E_L = \frac{1}{2}Li^2 = \frac{1}{2}L\dot{q}^2$	$E_C = \frac{1}{2}CV_{ab}^2 = \frac{q^2}{2C}$	NOT CONSERVATIVE
Other laws	Power $P = VI$	KVL, KCL	Ohm's Law

4.5 Energy Method for Electrical Systems

- Conservative systems only
- Do not dissipate energy due to heat loss (no resistors)

$$\frac{d}{dt}(E_L + E_C) = 0$$

4.6 Complex Impedance Method for Electrical Systems

- Ohm's Law: $E(s) = Z(s)I(s)$
- Impedances in series: $Z = Z_1 + Z_2 + Z_3 + \cdots$
- Impedances in parallel: $Z = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots$

4.7 Op-Amps

$$\text{Ideal Op-Amp: } e_o = K(e_2 - e_1)$$

- Differential gain of real op-amps: K $\approx 10^5$ to 10^6
- Infinite input impedance
- Zero output impedance
- Voltage at e_1 = Voltage at e_2
- Current at each input lead is zero

4.7.1 Examples of Op-Amps

a. Inverting amplifier

$$G(s) = \frac{E_o(s)}{E_i(s)} = -\frac{Z_f}{Z_i}$$

b. Non-inverting amplifier

$$G(s) = \frac{E_o(s)}{E_i(s)} = \frac{Z_1 + Z_2}{Z_1}$$

c. Summing amplifier

$$G(s) = \frac{E_o(s)}{E_i(s)} = -\left(\frac{Z_4}{Z_1}E_1(s) + \frac{Z_4}{Z_2}E_2(s) + \frac{Z_4}{Z_3}E_3(s)\right)$$

4.8 Analogous Systems

- Physically different systems but sharing the same differential equations and transfer functions
- More than 1 mechanical-electrical system analogy
  - Spring-Mass $\leftrightarrow$ Series-RLC: Force-Voltage Analogy
  - Spring-Mass $\leftrightarrow$ Parallel-RLC: Mass-Capacitance Analogy

4.9 Transfer Function (TF)

$$G(s) = \left.\frac{\mathscr{L}(\text{output})}{\mathscr{L}(\text{input})}\right|_{\text{zero initial conditions}}$$

- Order of system = highest power of s in denominator

4.10 Impulse-Response Function

- $g(t)$: unit impulse-response function of system

$$G(s) = \mathscr{L}(g(t))$$

4.11 Characteristic Equation (CE)

$$\text{Denominator of TF} = 0$$

- Polynomial order $\leftrightarrow$ degree/order of system
- Solutions to CE are poles of system

5 W5: First Order Systems

5.1 LTI System Response

- Find system response
  - Input $\xleftrightarrow{\text{TF}}$ output
- Methods: time domain, frequency domain
- Standard input signals:
  - Unit impulse
  - Unit step
  - Unit ramp
  - Sine wave

5.2 Parts of System Response

- Transient Response: Immediate response after application of input response
- Steady-state Response: Long-time response after application of input response

5.3 Mathematical Model of First Order Systems

$$\begin{aligned} \text{DE:} \quad & T\frac{dy}{dx} + y = Ax \\ \text{TF:} \quad & \frac{Y(s)}{X(s)} = \frac{A}{Ts+1} \end{aligned}$$

- Time constant/characteristic time: T
- DC gain: A

5.4 Unit Step Response

- Input: $x(t) = u(t) \quad \Rightarrow X(s) = \frac{1}{s}$
- Output: $Y(s) = \frac{A}{s(Ts+1)} = A\left(\frac{1}{s} - \frac{1}{s+\frac{1}{T}}\right)$
  By ILT: $y(t) = A\left[1 - e^{-\frac{t}{T}}\right],\ t \geq 0$

1. Time constant: $y(T) \approx 0.63A$
2. Initial speed = $\left.\frac{dy}{dt}\right|_{t=0} = \frac{A}{T}$
3. 2% settling speed: When $y(t_{ss}) = 0.98A,\ t_{ss} = 4T$.
4. Steady state error, $e_{ss} = \lim_{t\to\infty}[u(t) - y(t)] = 1 - A$

5.5 Unit Impulse Response

- Input: $x(t) = \delta(t) \quad \Rightarrow X(s) = 1$
- Output: $Y(s) = \frac{A}{Ts+1} = \frac{A}{T}\left(\frac{1}{s+\frac{1}{T}}\right)$
  By ILT: $y(t) = \frac{A}{T}e^{-\frac{t}{T}},\ t \geq 0$

1. Time constant: $y(t) \approx 0.37A$
2. Initial speed = $\left.\frac{dy}{dt}\right|_{t=0} = -A$
3. Steady state error, $e_{ss} = \lim_{t\to\infty}[\delta(t) - y(t)] = \lim_{t\to\infty}\left[-\frac{A}{T}e^{-\frac{t}{T}}\right] = 0$

5.6 Unit Ramp Response

- Input: $x(t) = t \quad \Rightarrow X(s) = \frac{1}{s^2}$
- Output: $Y(s) = \frac{A}{s^2(Ts+1)} = \frac{A}{s^2} - \frac{AT}{s} + \frac{AT^2}{Ts+1}$
  By ILT: $y(t) = At - AT + ATe^{-\frac{t}{T}},\ t \geq 0$

1. Initial speed = $\left.\frac{dy}{dt}\right|_{t=0} = A - Ae^{-\frac{t}{T}}$
2. Steady state error, $e_{ss} = \lim_{t\to\infty}[r(t) - y(t)] = \lim_{t\to\infty}\left[t - AT\left(\frac{t}{T} - 1 + e^{-\frac{t}{T}}\right)\right] = AT + \lim_{t\to\infty}[t(1-A)]$

5.7 Responses of First Order Systems

- Unit ramp function, $r(t)$: $y_r(t) = AT\left(\frac{t}{T} - 1 + e^{-\frac{t}{T}}\right),\ t \geq 0$
- Unit step function, $u(t)$: $y_u(t) = A(1 - e^{-\frac{t}{T}}),\ t \geq 0$
- Unit impulse function, $\delta(t)$: $y_\delta(t) = \dfrac{A}{T}e^{-\frac{t}{T}},\ t \geq 0$
- Properties:
    - $\dfrac{d}{dt}y_r(t) = y_u(t)$
    - $\dfrac{d}{dt}y_u(t) = y_\delta(t)$
    - Applies to higher order systems as well