Notes on Systems & Control

30.101 Systems & Control, Term 5 2020

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Contents

1 W1: Linear Time-Invariant Systems

1.1 Signals

- Signal: function changing in time and space

	Continuous signal	**Discrete signal**
Independent variable	Continuous	Discrete
Expression	$x(t), -\infty < t < \infty$	$x[k], k = 1, 2, \cdots$

	Deterministic signal	**Random/stochastic signal**
Value	Known	Unknown
Prediction accuracy	✓	×
Example	$x(t) = \cos(\omega t)$	$x(t) = \cos(\omega t + \phi),\ \phi = \{0, \frac{\pi}{2}, \pi\}$

	Periodic signal	**Non-periodic signal**
Satisfies $x(t) = x(t+T),\ T > 0$	✓	×
Example	$x(t) = \sin(t)$	$x(t) = \begin{cases} \cos t, & t < 0 \\ \sin t, & t \geq 0 \end{cases}$

	Bounded signal	**Unbounded signal**
$x(t) \to \infty$ **as** $t \to \infty$	×	✓

1.1.1 Basic signals

a. Unit impulse function

- Also known as delta function or Dirac distribution

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \qquad \int_{0^-}^{0^+} \delta(t)\, dt = 1$$

b. Unit step function

- Also known as Heaviside step function

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

c. Rectangular function

$$\mathrm{rect}\left(\frac{t}{T}\right) = \begin{cases} 1, & -\frac{T}{2} < t < \frac{T}{2} \\ 0, & \text{elsewhere} \end{cases}$$

- Linear combination of 2 step functions:

$$\mathrm{rect}\left(\frac{t}{T}\right) = u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)$$

d. Exponential growth/decay function

$$x(t) = Ce^{at}$$

- Exponential growth: $C > 0$
- Exponential decay: $C < 0$

1.2 Systems

- Converts input x to output y: $\quad y = S\{x\}$
- Transformation that map functions to other functions
  - Continuous time signal: $y(t) = S\{x(t)\}$
  - Discrete time signal: $y[n] = S\{x[n]\}$

	Dynamic system	**Static system**
Output depends on input from	Past	Present

1.2.1 Properties of systems

a. Causality

	Causal system	**Non-causal system**
Output depends on input at	Past and present	Past, present and future
Future affects past	$\times$	$\checkmark$

b. Linearity

- Has properties of superposition, i.e. additivity and scaling

c. Time Invariance

- Time shift in output = Time shift in input

$\Rightarrow$ Most physical systems can be modelled as Linear Time-Invariant (LTI) Systems.

1.3 Review of complex numbers

- $j = \sqrt{-1}$
- Rectangular form: $z = x + jy$
  - where $x = \mathrm{Re}(z); \quad y = \mathrm{Im}(z)$
- Polar form: $z = |z|e^{j\theta} = |z|\underline{/\theta}$

- $|z|$: magnitude of z; θ: phase angle; $\underline{/\theta}$: shorthand for $e^{j\theta}$

- Complex conjugate: $z^* = \overline{z} = x - jy = Re^{-j\theta} = R\angle - \theta$
  - $\overline{z} \cdot z = z \cdot \overline{z} = x^2 + y^2 = |z|^2$
- Addition & subtraction: easily performed in rectangular coordinates
- Multiplication & division: easily performed in polar coordinates

1.4 Complex variable and functions

- Complex variable: $s = \sigma + j\omega$
- Complex function: $G(s) = \mathrm{Re}[G(s)] + j\,\mathrm{Im}[G(s)] = G_x + jG_y$
  - $G(s)$: single-valued, one-one function
    - For every s in s-plane, there is only 1 value of $G(s)$ in $G(s)$-plane.
- General form of G(s):

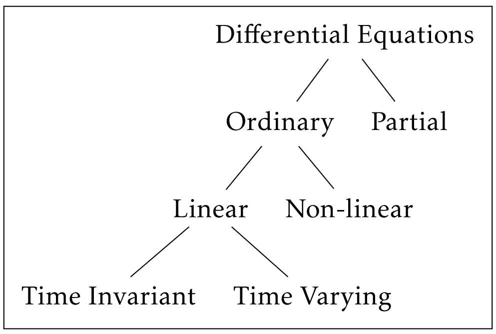
$$G(s) = \frac{K(s+z+1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = \frac{\mathrm{N(s)}}{\mathrm{D(s)}},$$

where N(s) is a polynomial of degree m and D(s) is a polynomial of degree n, and m < n.

  - Zeros (○): points where N(s) = 0 e.g. $s = -z_1, -z_2, \cdots, -z_m$
  - Poles/roots (×): points where D(s) = 0 e.g. $s = -p_1, -p_2, \cdots, -p_n$

1.5 Differential equations

- Model wide range of systems
- Involves derivatives of dependence variable with respect to independent variable



1.5.1 Ordinary Differential Equations (ODEs)

- General form:

$$g\left(\frac{d^n x}{dt^n}, \frac{d^{n-1}x}{dt^{n-1}}, \cdots, x, t\right) = f(t)$$

    - where x is the dependent variable;
    - t is the independent variable;
    - f, g are functions.

- Order of ODE = order of highest derivative of dependent variable

1.5.2 Linear ODEs

- General form:

$$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + x = f(t)$$

- $f(t)$ is the forcing function
- Dependent variable and its derivatives appear as a linear combination
    - Only pure functions of t in front of x and its derivatives
    - x and its derivatives are to power 1

1.5.3 Non-linear ODEs

$$\text{e.g. } (x^2-1)\frac{d^2x}{dt^2} + \frac{dx}{dt} + 3x = 0; \quad \frac{d^2x}{dt^2} + x^2 = \sin t$$

- Contains power or products of dependent variable and its derivatives

1.5.4 Time Invariant ODEs

$$\text{e.g. } \frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 10x = 0$$

- Coefficients are constants, independent of t

1.5.5 Time Varying ODEs

$$\text{e.g. } \frac{d^2x}{dt^2} + (\cos 2t)\frac{dx}{dt} + 10x = 0$$

- ≥ 1 coefficient(s) are functions of t

1.6 Solving LTI ODEs with Laplace Transform

- Initial conditions are taken care of
- Particular and complementary solutions are obtained simultaneously

1.7 Laplace Transform (LT)

- For a time function such that $f(t) = 0$ for $t < 0$,

$$F(s) = \mathscr{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}\,dt,\ t \geq 0$$

- where $s = \sigma + j\omega$
- $\int_0^{\infty}$ is an improper integral, thus Laplace Transform may not exist
  - Laplace Transform exists within Region of Convergence (ROC)

1.7.1 Properties of Laplace Transform

a. Linearity

$$\mathscr{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$$

b. Translation

$$\begin{aligned}
\mathscr{L}[g(t-T)] &= \int_0^{+\infty} g(t-T)e^{-st}\,dt \\
&= \int_0^{+\infty} g(\tau)e^{-s(\tau+T)}\,d\tau \text{ where } \tau = t - T \\
&= e^{-sT}\int_0^{+\infty} g(\tau)e^{-s\tau}\,d\tau \\
&= e^{-sT}F(s)
\end{aligned}$$

c. Differentiation

$$\mathscr{L}\left(\frac{d^n f(t)}{dt^n}\right) = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots$$

d. Integration

$$\mathscr{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s}$$

1.7.2 Laplace Transform Pairs

a. Exponential function

$$\begin{aligned}
f(t) &= \begin{cases} 0, & \text{for } t < 0 \\ Ae^{-\alpha t}, & \text{for } t \geq 0 \end{cases} \\
F(s) &= \int_0^{\infty} Ae^{-\alpha t}e^{-st}\,dt = \int_0^{\infty} Ae^{-(\alpha+s)t}\,dt \\
&= \frac{A}{s+\alpha},\ s > -\alpha
\end{aligned}$$

- ROC: F(s) exists when $\sigma > -\alpha$.
- Zeros: none
- Poles: $s = -\alpha$

b. Step function, $u(t)$

$$f(t) = \begin{cases} 0, & \text{for } t < 0 \\ 1, & \text{for } t \geq 0 \end{cases}$$

$$F(s) = \int_0^\infty 1 \cdot e^{-st}\, dt$$

$$= \frac{1}{s},\ s \geq 0$$

- ROC: $F(s)$ exists when $\sigma \geq 0$
- Zeros: none
- Poles: s = 0

c. Pulse function

- Considered as superposition of two step functions

$$f(t) = \begin{cases} 0, & t < 0,\ t > t_0 \\ \frac{A}{t_0}, & 0 < t < t_0 \end{cases}$$

$$F(s) = \mathscr{L}[f(t)] = \mathscr{L}\left[\frac{A}{t_0}u(t)\right] - \mathscr{L}\left[\frac{A}{t_0}u(t-t_0)\right]$$

$$= \frac{A}{t_0 s}(1 - e^{-st_0}),\ s \geq 0$$

- ROC: $F(s)$ exists when $\sigma \geq 0$
- Zeros: none
- Poles: s = 0

d. Impulse function, $\delta(t)$

- Special case of pulse function, when $t_0 \to 0$

$$f(t) = \begin{cases} 0, & t < 0,\ t > t_0 \\ \frac{1}{t_0}, & 0 < t < t_0 \end{cases}$$

- As $t_0 \to 0, f(t) \to \delta(t - t_0)$.

$$\text{When } t_0 \to 0,\ \mathscr{L}[\delta(t)] = \lim_{t_0 \to 0}\left[\frac{1}{t_0 s}(1 - e^{-st_0})\right] = \lim_{t_0 \to 0} \frac{\frac{d}{dt_0}(1 - e^{-st_0})}{\frac{d}{dt_0}(t_0 s)}$$

$$= 1$$

- ROC: $F(s)$ exists when $\sigma \geq 0$
- Zeros: none
- Poles: s = 0

e. Ramp function

$$f(t) = \begin{cases} 0, & t = 0 \\ At, & t \geq 0 \end{cases}$$

$$\begin{aligned} F(s) &= \int_0^\infty Ate^{-st}\,dt = A\left\{\left[-\frac{t}{s}e^{-st}\right]\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\,dt\right\} \\ &= \frac{A}{s^2},\ s > 0 \end{aligned}$$

- ROC: $F(s)$ exists when $\sigma \geq 0$
- Zeros: none
- Poles: s = 0

f. Sinusoidal function

$$f(t) = \begin{cases} 0, & t < 0 \\ A\sin\omega t, & t \geq 0 \end{cases}$$

$$\begin{aligned} F(s) &= \frac{A}{2j}\left[\mathscr{L}\left(e^{j\omega t}\right) - \mathscr{L}\left(e^{-j\omega t}\right)\right] = \frac{A}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) \\ &= \frac{A\omega}{s^2+\omega^2} \end{aligned}$$

- ROC: $F(s)$ exists when $-j\omega < \sigma < j\omega$
- Zeros: none
- Poles: $s = j\omega,\ s = -j\omega$

For more Laplace Transform pairs, refer to the table of Laplace Transforms in the textbook.

1.7.3 Initial Value Theorem

- If $f(t)$ and $\dfrac{df(t)}{dt}$ are both Laplace Transformable,
- and $\lim\limits_{s\to\infty} sF(s)$ exists,

$$f(0^+) = \lim_{s\to\infty} sF(s)$$

1.7.4 Final Value Theorem

- If $f(t)$ and $\frac{df(t)}{dt}$ are Laplace Transformable,
- $\lim\limits_{t\to\infty} f(t)$ exists,
- and $sF(s)$ has all its poles with **strictly negative real part**,

$$f(\infty) = \lim_{s\to 0} sF(s)$$

1.8 Inverse Laplace Transform (ILT)

$$\mathscr{L}^{-1}F(s) = f(t)$$

- For rational functions of F(s), ILT can be computed using partial fractions decomposition.

1.8.1 ILT Procedure

1. Express $F(s)$ as a proper rational fraction: $F(s) = \dfrac{N(s)}{D(s)}$, where degree of $N(s) < D(s)$

2. Check roots of D(s):

   Ⓐ Roots are Real and Distinct

   $$F(s) = \frac{N(s)}{D(s)} = \frac{a}{s+p_1} + \frac{a}{s+p_2} + \cdots + \frac{a}{s+p_n},$$
   $$\text{where } a_i = (s+p_i)F(s)\big|_{s=-p_i}$$

   Ⓑ Roots are Real and Repetitive

   $$F(s) = \frac{b_1}{s+p} + \frac{b_2}{(s+p)^2} + \cdots + \frac{b_n}{(s+p)^n}$$
   $$\text{where } b_i = \frac{1}{(n-1)!}\left[\frac{d^{n-i}}{ds^{n-i}}(s+p)^n F(s)\right]\Bigg|_{s=-p}$$

   Ⓒ Roots are Complex Conjugates

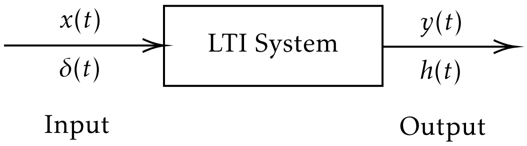
   $$F(s) = \frac{N(s)}{s^2+cs+d} = C_1\frac{\omega}{(s+a)^2+\omega^2} + C_2\frac{s+a}{(s+a)^2+\omega^2}$$
   $$\text{where poles, } s = -\frac{c}{2} \pm \frac{\sqrt{c^2-4d}}{2}$$

   Ⓓ Combination of Cases Ⓐ, Ⓑ, Ⓒ

   - Rewrite numerator in terms of denominator to simplify

3. Use Laplace Transform table pairs to infer $f(t)$ from $F(s)$.

2 W2: Convolution



2.1 Basic signals

1. Convolution: Delayed impulses
2. Fourier Analysis: Sinusoidal signals
3. Laplace Analysis: Complex exponentials

2.2 Properties of impulse function

1. $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$
2. $\displaystyle\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0)$

2.3 Convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \Longleftrightarrow \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau$$

- It can be written as $y(t) = x(t) * h(t)$

2.4 Causal systems

- For causal systems, $h(t) = 0,\ t < 0. \quad \therefore h(t-\tau) = 0,\ t < \tau$
- Only past and present values of $x(\tau)$ contribute to $y(t)$.
- If $x(t) = 0,\ t < 0$, then the convolution integral of a causal system is

$$y(t) = \int_0^t x(\tau)h(t-\tau)\,d\tau$$

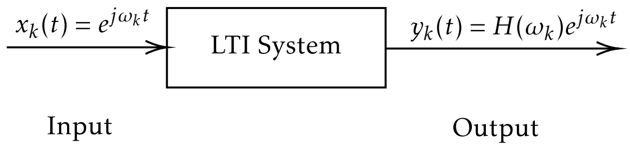
2.5 Graphical Method

1. Flip: $h(\tau) \to h(-\tau)$
2. Shift by t: $h(-\tau) \to h(t-\tau)$
3. Multiply by x: $x(\tau)h(t-\tau)$
4. Integrate over τ: $y(t) = \displaystyle\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$

2.6 Properties of Convolution

- Commutative: $x(t) * h(t) = h(t) * x(t)$
- Associative: $[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$
- Distributive: $x(t) * h_1(t) + x(t) * h_2(t) = x(t) * [h_1(t) + h_2(t)]$

3 W3: Fourier Analysis



3.1 Fourier Series

- A real periodic signal $x(t) = x(t + T_0)$ with period T_0 can be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

  - where a_k is the Fourier coefficients of the Fourier series,
  - ω_0 is the fundamental frequency of the Fourier series and can be found using the formula:

$$\omega_0 = \frac{2\pi}{T_0}$$

Synthesis: $\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$

Analysis: $\boxed{a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t}\, dt}$

- By using the Fourier Analysis formula, we can find the magnitude $|a_k|$ and phase θ_k of each Fourier coefficient, which can be plotted in the magnitude and phase spectrums respectively.

3.2 Forms of Fourier Series

1. $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$
2. $x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$
3. $x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t]$

3.3 Fourier Representation of Aperiodic Signals

- $\tilde{x}(t)$ is T_0 periodic, which is made by repeating the aperiodic signal $x(t)$
- $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},\ \omega_0 = \frac{2\pi}{T_0}$
- As $T_0 \to \infty,\ \omega_0 \to 0$
- Converges to Fourier Transform

3.4 Power of a signal

- Sum of squares of all the Fourier coefficients

$$\text{Power} = \sum_{k=-\infty}^{\infty} |a_k|^2 = \frac{1}{T} \int_T |x(t)|^2\, dt$$

3.5 Periodic x(t)

- Fourier Transform of x(t) is an impulse train

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

3.6 LTI System Response to Exponential Input

For stable systems:

$$y_{ss}(t) = \int_0^\infty x(\tau)h(t-\tau)\,d\tau = \int_0^\infty x(t-\tau)h(\tau)\,d\tau$$

Given $x(t) = e^{j\omega t}$:

$$\begin{aligned}
y_{ss}(t) &= \int_0^\infty e^{j\omega(t-\tau)}h(\tau)\,d\tau \\
&= e^{j\omega t}\int_0^\infty h(\tau)e^{-j\omega\tau}\,d\tau \\
&= e^{j\omega t}H(\omega) \\
&= |H(\omega)|e^{j\cdot\arg H(\omega)}e^{j\omega t} \\
&= |H\omega)|e^{j(\omega t+\arg H(\omega))}
\end{aligned}$$

3.7 Fourier Transform vs Laplace Transform

$$\text{Fourier Transform: } \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \qquad \text{Laplace Transform: } \int_0^\infty x(t)e^{-st}\,dt$$

- Limits of Integration: $-\infty$ to ∞ (FT), 0 to ∞ (LT)
- Location of complex variable: $j\omega$ lies on the imaginary axis (FT), s can be any complex number in the region of convergence (LT)
- Existence of FT and LT: If the imaginary axis is not in region of convergence of LT, FT does not exist while LT exists.
- Equivalence of FT and LT: If $x(t) = 0, t < 0$ and imaginary axis is in region of convergence of LT, FT is LT evaluated on the imaginary axis.
- Non-equivalence of FT and LT: If $x(t) \neq 0$ for $t < 0$, then FT $\neq$ LT.

4 W4: Modelling Physical Systems

4.1 Methodology

1. Draw schematic diagram of system and components, and define variables.
2. Use physical laws and write equations for each component, combining them together.
3. Identify unknown system parameters. Thereafter, verify the model with experiments.

4.2 Translational Mechanical Systems

	Mass	**Spring**	**Damper**
Force	$f = m\ddot{x}$	$f_k = k(x_2 - x_1)$	$f_b = b(\dot{x}_2 - \dot{x}_1)$
Conservative energies	$\text{KE} = \frac{1}{2}m\dot{x}^2$ $\text{PE} = mgh$	$\text{PE} = \frac{1}{2}kx^2$	NOT CONSERVATIVE
Other laws	Power $P = f\dot{x}$	N2L: $\sum f = ma = m\ddot{x}$	N3L

4.3 Rotational Mechanical Systems

	Mass	**Spring**	**Damper**
Torque	$\tau = J\ddot{\theta}$	$\tau_k = k(\theta_2 - \theta_1)$	$\tau_b = b(\dot{\theta}_2 - \dot{\theta}_1)$
Conservative energies	$\text{KE} = \frac{1}{2}J\dot{\theta}^2$	$\text{PE} = \frac{1}{2}k\theta^2$	NOT CONSERVATIVE
Other laws	Power $P = \tau\dot{\theta}$	N2L: $\sum \tau = J\alpha = J\ddot{\theta}$	N3L

4.4 Energy Method for Mechanical Systems

- Conservative systems only
- Do not dissipate energy due to friction

$$\Delta(\text{KE} + \text{PE}) = 0$$

$$\frac{d}{dt}(\text{KE} + \text{PE}) = 0$$

4.5 Electrical Systems

	Inductor	**Capacitor**	**Resistor**
Current or Voltage	$V_a - V_b = L\frac{di_L}{dt}$	$i_C = C\frac{d}{dt}(V_a - V_b)$	$V_a - V_b = i_R R$
Conservative energies	$E_L = \frac{1}{2}Li^2 = \frac{1}{2}L\dot{q}^2$	$E_C = \frac{1}{2}CV_{ab}^2 = \frac{q^2}{2C}$	NOT CONSERVATIVE
Other laws	Power $P = VI$	KVL, KCL	Ohm's Law

4.6 Kirchhoff's Laws

- **Kirchhoff's Voltage Law** (KVL): The algebraic sum of voltages in a loop is zero.

$$\sum_{i=1}^{n} V_n = 0$$

- **Kirchhoff's Current Law** (KCL): The algebraic sum of currents entering and leaving the node is zero.

$$\sum_{i=1}^{n} I_n = 0$$

4.7 Complex Impedance Method for Electrical Systems

- Ohm's Law: $E(s) = Z(s)I(s)$
- Impedances in series: $Z = Z_1 + Z_2 + Z_3 + \cdots$
- Impedances in parallel: $Z = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots$
- Impedances of electrical components:

	Inductor	**Capacitor**	**Resistor**
Current or Voltage	$v(t) = L\frac{di(t)}{dt}$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = Ri(t)$
Derivation of Z(s)	$V(s) = LsI(s)$ $Z_L(s) = \frac{V(s)}{I(s)} = Ls$	$I(s) = CsV(s)$ $Z_C(s) = \frac{V(s)}{I(s)} = \frac{1}{Cs}$	$V(s) = RI(s)$ $Z_R(s) = \frac{V(s)}{I(s)} = R$

4.8 Op-Amps

Ideal Op-Amp: $e_o = K(e_2 - e_1)$

- Differential gain of real op-amps: K $\approx 10^5$ to 10^6
- Infinite input impedance
- Zero output impedance
- Voltage at e_1 = Voltage at e_2
- Current at each input lead is zero

4.8.1 Examples of Op-Amps

a. Inverting amplifier

$$G(s) = \frac{E_o(s)}{E_i(s)} = -\frac{Z_f}{Z_i}$$

b. Non-inverting amplifier

$$G(s) = \frac{E_o(s)}{E_i(s)} = \frac{Z_1 + Z_2}{Z_1}$$

c. Summing amplifier

$$G(s) = \frac{E_o(s)}{E_i(s)} = -\left(\frac{Z_4}{Z_1}E_1(s) + \frac{Z_4}{Z_2}E_2(s) + \frac{Z_4}{Z_3}E_3(s)\right)$$

4.9 Analogous Systems

- Physically different systems but sharing the same differential equations and transfer functions
- More than 1 mechanical-electrical system analogy
  - Spring-Mass ↔ Series-RLC: Force-Voltage Analogy

Mechanical System	**Electrical System**
Force F	Voltage V
Mass m	Inductance L
Damping coefficient b	Resistance R
Spring constant k	Reciprocal of capacitance (Elastance) $\frac{1}{C}$
Displacement x	Charge q
Velocity v	Current i

  - Spring-Mass ↔ Parallel-RLC: Mass-Capacitance Analogy

Mechanical System	**Electrical System**
Force F	Current i
Mass m	Capacitance C
Damping coefficient b	Reciprocal of resistance (Conductance) $\frac{1}{R}$
Spring constant k	Reciprocal of inductance $\frac{1}{L}$
Displacement x	Magnetic flux linkage ψ
Velocity v	Voltage V

4.10 Transfer Function (TF)

$$G(s) = \left.\frac{\mathscr{L}(\text{output})}{\mathscr{L}(\text{input})}\right|_{\text{zero initial conditions}}$$

$$\text{e.g.} \quad a_0\frac{d^ny}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_{n-1}\frac{dy}{dt} + a_ny = b_0\frac{d^mx}{dt^m} + b_1\frac{d^{m-1}x}{dt^{m=1}} + \ldots + b_{m-1}\frac{dx}{dt} + b_mx$$

$$\begin{aligned}\therefore \text{Transfer function, } G(s) &= \left.\frac{\mathscr{L}(y(t))}{\mathscr{L}(x(t))}\right|_{\text{zero initial conditions}} \\ &= \frac{Y(s)}{X(s)} \\ &= \frac{b_0s^m + b_1s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n}\end{aligned}$$

- Order of system = highest power of s in denominator
- Mathematical model of system
- Property of system, unrelated to input
- If TF is known, output can be analyzed using inputs to understand systme
- If TF is unknown, output can be found by introducing known inputs and studying outputs.

4.11 Impulse-Response Function

- If input $x(t)$ is unit impulse function $\delta(t)$, $X(s) = 1$.

$$Y(s) = G(s)X(s) = G(s)$$

- Transfer function $G(s)$ is the LT of the unit impulse-response function of system $g(t)$:

$$G(s) = \mathscr{L}(g(t))$$

4.12 Characteristic Equation (CE)

$$\text{Denominator of TF} = 0$$

- Polynomial order $\leftrightarrow$ degree/order of system
- Solutions to CE are poles of system

5 W5: First Order Systems

5.1 LTI System Response

- Find system response
  - Input $\xleftrightarrow{\text{TF}}$ output
- Methods: time domain, frequency domain
- Standard input signals:
  - Unit impulse
  - Unit step
  - Unit ramp
  - Sine wave

5.2 Parts of System Response

- Transient Response: Immediate response after application of input response
- Steady-state Response: Long-time response after application of input response

5.3 Mathematical Model of First Order Systems

$$\begin{aligned} \text{DE:} \quad & T\frac{dy}{dx} + y = Ax \\ \text{TF:} \quad & \frac{Y(s)}{X(s)} = \frac{A}{Ts+1} \end{aligned}$$

- Time constant/characteristic time: T
- DC gain: A

5.4 Unit Step Response

- Input: $x(t) = u(t) \quad \Rightarrow X(s) = \frac{1}{s}$
- Output: $Y(s) = \frac{A}{s(Ts+1)} = A\left(\frac{1}{s} - \frac{1}{s+\frac{1}{T}}\right)$
  By ILT: $y(t) = A\left[1 - e^{-\frac{t}{T}}\right],\ t \geq 0$

1. Time constant: $y(T) \approx 0.63A$
2. Initial speed = $\left.\frac{dy}{dt}\right|_{t=0} = \frac{A}{T}$
3. 2% settling speed: When $y(t_{ss}) = 0.98A,\ t_{ss} = 4T$.
4. Steady state error, $e_{ss} = \lim_{t\to\infty} [u(t) - y(t)] = 1 - A$

5.5 Unit Impulse Response

- Input: $x(t) = \delta(t) \quad \Rightarrow X(s) = 1$
- Output: $Y(s) = \dfrac{A}{Ts+1} = \dfrac{A}{T}\left(\dfrac{1}{s+\frac{1}{T}}\right)$
  By ILT: $y(t) = \dfrac{A}{T}e^{-\frac{t}{T}},\ t \geq 0$

1. Time constant: $y(t) \approx 0.37A$
2. Initial speed $= \left.\dfrac{dy}{dt}\right|_{t=0} = -A$
3. Steady state error, $e_{ss} = \lim\limits_{t\to\infty}[\delta(t) - y(t)] = \lim\limits_{t\to\infty}\left[-\dfrac{A}{T}e^{-\frac{t}{T}}\right] = 0$

5.6 Unit Ramp Response

- Input: $x(t) = t \quad \Rightarrow X(s) = \dfrac{1}{s^2}$
- Output: $Y(s) = \dfrac{A}{s^2(Ts+1)} = \dfrac{A}{s^2} - \dfrac{AT}{s} + \dfrac{AT^2}{Ts+1}$
  By ILT: $y(t) = At - AT + ATe^{-\frac{t}{T}},\ t \geq 0$

1. Initial speed $= \left.\dfrac{dy}{dt}\right|_{t=0} = A - Ae^{-\frac{t}{T}}$
2. Steady state error, $e_{ss} = \lim\limits_{t\to\infty}[r(t) - y(t)] = \lim\limits_{t\to\infty}\left[t - AT\left(\dfrac{t}{T} - 1 + e^{-\frac{t}{T}}\right)\right] = AT + \lim\limits_{t\to\infty}[t(1-A)]$

5.7 Responses of First Order Systems

- Unit ramp function, $r(t)$: $y_r(t) = AT\left(\frac{t}{T} - 1 + e^{-\frac{t}{T}}\right),\ t \geq 0$
- Unit step function, $u(t)$: $y_u(t) = A(1 - e^{-\frac{t}{T}}),\ t \geq 0$
- Unit impulse function, $\delta(t)$: $y_\delta(t) = \dfrac{A}{T}e^{-\frac{t}{T}},\ t \geq 0$
- Properties:
  - $\dfrac{d}{dt}y_r(t) = y_u(t)$
  - $\dfrac{d}{dt}y_u(t) = y_\delta(t)$
  - Applies to higher order systems as well

5.8 Responses of First-Order Systems in Frequency and Time Domains

$T\dfrac{dy(t)}{dt} + y(t) = Ax(t)$		**Response to**		
		Unit step	**Unit impulse**	**Unit ramp**
Frequency domain: Transfer function $H(s) = \dfrac{A}{Ts+1}$		$X(s) = \dfrac{1}{s}$ $Y(s) = H(s)X(s)$ $= \dfrac{A}{s} - \dfrac{AT}{Ts+1}$	$X(s) = 1$ $Y(s) = H(s)X(s)$ $= \dfrac{A}{Ts+1}$	$X(s) = \dfrac{1}{s^2}$ $Y(s) = H(s)X(s)$ $= \dfrac{A}{s^2} - \dfrac{AT}{s} + \dfrac{AT^2}{Ts+1}$
Time domain: Impulse response $h(t) = \dfrac{A}{T}e^{-\frac{t}{T}},\ t \geq 0$		$x(\tau) = u(\tau)$ $y(t) = h(t) * x(t)$ $= A\left(1 - e^{-\frac{t}{T}}\right),\ t \geq 0$	$x(\tau) = \delta(\tau)$ $y(t) = h(t) * x(t)$ $= ATe^{-\frac{t}{T}},\ t \geq 0$	$x(\tau) = \tau$ $y(t) = h(t) * x(t)$ $= A\left(t - T + Te^{-\frac{t}{T}}\right),\ t \geq 0$
Poles	$s = -\dfrac{1}{T}$	$s = 0,\ s = -\dfrac{1}{T}$	$s = -\dfrac{1}{T}$	$s = 0,\ s = 0,\ s = -\dfrac{1}{T}$

6 W5 & W6: Second Order Systems

6.1 General form of Second Order Systems

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

- where ζ is the damping ratio and ω_n is the natural frequency of the system.
- Characteristic equation: $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$
- Poles: $s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$, $s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$

6.2 Parameters of Second Order Systems

- The poles of second order systems can be rewritten as:

$$\begin{aligned} s_{1,2} &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} \\ &= -\sigma \pm j\omega_d \end{aligned}$$

  - where σ is the attenuation of the system, $\boxed{\sigma = \zeta\omega_n}$
  - and ω_d is the damped natural frequency of the system. $\boxed{\omega_d = \omega_n\sqrt{1 - \zeta^2}}$

- For physical systems, the natural frequency ω_n is always positive.

6.3 Effects of Damping Ratio on Natural Response of Second Order Systems

- Case 1: Unstable ($\sigma < 0$, $\zeta < 0$)
- Case 2: Marginally stable, undamped conjugate complex poles ($\sigma = 0$, $\zeta = 0$)
- Case 3a: Stable, underdamped, conjugate complex poles ($0 < \zeta < 1$)
- Case 3b: Stable, critically damped, repeated real poles ($\zeta = 1$)
- Case 3c: Stable, overdamped, distinct real poles ($\zeta > 1$)

6.4 Natural Response of Second Order Systems

We have a generic second order equation of motion.

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

Taking the Laplace Transform of it:

$$\begin{aligned} s^2X(s) - sx(0) - \dot{x}(0) + 2\zeta\omega_n[sX(s) - x(0)] + \omega_n^2 X(s) = 0 \\ \therefore X(s) = \frac{sx(0) + 2\zeta\omega_n x(0) + \dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \end{aligned}$$

For the same set of initial conditions $x(0)$, $\dfrac{dx(0)}{dt}$ and ω_n, the free response of the system can be different based on the value of the damping ratio ζ.

6.4.1 Natural Response of Marginally Stable, Underdamped System (Case 2)

$$\begin{aligned} X(s) &= \frac{sx(0)+\dot{x}(0)}{s^2+\omega_n^2} \\ &= x(0)\frac{s}{s^2+\omega_n^2} + \frac{\dot{x}(0)}{\omega_n}\frac{\omega_n}{s^2+\omega_n^2} \end{aligned}$$

Taking ILT:

$$x(t) = x(0)\cos\omega_n t + \left(\frac{\dot{x}(0)}{\omega_n}\right)\sin\omega_n t,\ t \geq 0$$

6.4.2 Natural Response of Stable, Underdamped System (Case 3a)

$$\begin{aligned} X(s) &= \frac{sx(0)+2\zeta\omega_n x(0)+\dot{x}(0)}{s^2+2\zeta\omega_n s+\omega_n^2} \\ &= \frac{sx(0)+2\zeta\omega_n x(0)+\dot{x}(0)}{(s+\zeta\omega_n)^2+(\omega_n\sqrt{1-\zeta^2})^2} \\ &= \left(\frac{\zeta\omega_n x(0)+\dot{x}(0)}{\omega_n\sqrt{1-\zeta^2}}\right)\frac{\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+(\omega_n\sqrt{1-\zeta^2})^2} + x(0)\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+(\omega_n\sqrt{1-\zeta^2})^2} \end{aligned}$$

Taking ILT:

$$\begin{aligned} x(t) &= \left(\frac{\zeta\omega_n x(0)+\dot{x}(0)}{\omega_n\sqrt{1-\zeta^2}}\right)e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t) + x(0)e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}t),\ t>0 \\ &= e^{-\zeta\omega_n t}\left\{\left[\frac{\zeta}{\sqrt{1-\zeta^2}}x(0)+\frac{1}{\omega_d}\dot{x}(0)\right]\sin\omega_d t + x(0)\cos\omega_d t\right\},\ t>0 \end{aligned}$$

The system oscillates at the damping frequency ω_d.

6.4.3 Natural Response of Stable, Critically Damped System (Case 3b)

$$\begin{aligned} X(s) &= \frac{sx(0)+2\omega_n x(0)+\dot{x}(0)}{s^2+2\omega_n s+\omega_n^2} \\ &= \frac{x(0)}{s+\omega_n} + \frac{\omega_n x(0)+\dot{x}(0)}{(s+\omega_n)^2} \end{aligned}$$

Taking ILT:

$$x(t) = x(0)e^{-\omega_n t} + [\omega_n x(0)+\dot{x}(0)]\,te^{-\omega_n t},\ t>0$$

6.4.4 Natural Response of Stable, Overdamped System (Case 3c)

$$\begin{aligned} X(s) &= \frac{sx(0)+2\omega_n x(0)+\dot{x}(0)}{s^2+2\zeta\omega_n s+\omega_n^2} \\ &= \frac{(s+2\zeta\omega_n)x(0)+\dot{x}(0)}{(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})} \\ &= \frac{\hat{a}}{(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})} + \frac{\hat{b}}{(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})} \end{aligned}$$

Taking ILT:

$$x(t) = \hat{a}e^{-(\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})} + \hat{b}e^{-(\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})},\ t>0$$

$$\text{where } \hat{a} = \frac{x(0)(-\zeta+\sqrt{\zeta^2-1})}{2\sqrt{\zeta^2-1}} - \frac{\dot{x}(0)}{2\omega_n\sqrt{\zeta^2-1}}, \qquad \hat{b} = \frac{x(0)(\zeta+\sqrt{\zeta^2-1})}{2\sqrt{\zeta^2-1}} - \frac{\dot{x}(0)}{2\omega_n\sqrt{\zeta^2-1}}$$

6.5 Comparison of Natural Response of Stable Systems (Cases 3a, 3b, 3c)

- If the poles of X(s) are stable, steady state value of $x(t)$ can be computed:

$$\begin{aligned} x(t \to \infty) &= \lim_{s \to 0} sX(s) \\ &= \lim_{s \to 0} \frac{s^2 x(0) + 2\zeta\omega_n x(0) + \dot{x}(0)s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= 0 \end{aligned}$$

- For Cases 3a, 3b and 3c, their responses return to equilibrium position as $t \to \infty$.

6.6 Logarithmic Decrement Method

- Used to find the damping ratio ζ.
- Procedure:
  1. Find the amplitude of the first peak x_1 and that of the nth peak x_n.
  2. The logarithmic decrement δ can be found using the formula:
  $$\delta = \frac{1}{n-1} \ln \frac{x_1}{x_n}$$
  3. The damping ratio ζ can be found by applying this formula:
  $$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$$

6.7 Unit Step Response of Second Order Systems

The transfer function $G(s)$ of a second order system is as follows:

$$G(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

If $R(s)$ is a unit step input, $R(s) = \dfrac{1}{s}$.

$$\begin{aligned} C(s) &= R(s)G(s) \\ &= \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \\ &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \end{aligned}$$

The unit step response of the system $c(t)$ depends on the poles of $C(s)$.

6.7.1 Unit Step Response of Marginally Stable, Underdamped System (Case 2)

$$C(s) = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

Taking ILT,

$$c(t) = u(t) - \cos\omega_n t, \ t > 0$$

6.7.2 Unit Step Response of Stable, Underdamped System (Case 3a)

$$
\begin{aligned}
C(s) &= \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2} \\
&= \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}
\end{aligned}
$$

Taking ILT,

$$
c(t) = u(t) - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right],\ t>0
$$

6.7.3 Unit Step Response of Stable, Critically Damped System (Case 3b)

$$
\begin{aligned}
C(s) &= \frac{\omega_n^2}{s(s^2+2\omega_n s+\omega_n)} \\
&= \frac{\omega_n^2}{s(s+\omega_n)^2} \\
&= \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}
\end{aligned}
$$

Taking ILT,

$$
\begin{aligned}
c(t) &= u(t) - e^{-\omega_n t} - \omega_n t e^{-\omega_n t} \\
&= u(t) - e^{-\omega_n t}(1+\omega_n t),\ t>0
\end{aligned}
$$

6.7.4 Unit Step Response of Stable, Overdamped System (Case 3c)

$$
\begin{aligned}
C(s) &= \frac{\omega_n^2}{s(s^2+2\omega_n s+\omega_n)} \\
&= \frac{\omega_n^2}{s(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})} \\
&= \frac{1}{s} + \frac{\bar{a}}{s+\omega_n(\zeta+\sqrt{\zeta^2-1})} + \frac{\bar{b}}{s+\omega_n(\zeta-\sqrt{\zeta^2-1})}
\end{aligned}
$$

Taking ILT,

$$
c(t) = u(t) + \bar{a}e^{-\omega_n t(\zeta+\sqrt{\zeta^2-1})} + \bar{b}e^{-\omega_n t(\zeta-\sqrt{\zeta^2-1})},\ t>0
$$

$$
\text{where } \bar{a} = \frac{1}{2\sqrt{\zeta^2-1}(\zeta+\sqrt{\zeta^2-1})}, \qquad \bar{b} = -\frac{1}{2\sqrt{\zeta^2-1}(\zeta-\sqrt{\zeta^2-1})}
$$

6.8 Comparison of Unit Step Response for Stable Systems (Cases 3a, 3b, 3c)

- If the poles of C(s) are stable, steady state value of $c(t)$ can be computed:

$$
\begin{aligned}
c(t\to\infty) &= \lim_{s\to 0} sC(s) \\
&= \lim_{s\to 0} \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \\
&= 1
\end{aligned}
$$

- For Cases 3a, 3b and 3c, their responses tend to 1 as $t\to\infty$.
- Among underdamped systems (Case 3a), systems with $0.5<\zeta<0.8$ converges the quickest.
- Critically damped systems (Case 3b) exhibits the fastest response for all values of ζ.
- Overdamped systmes (Case 3c) converges to the steady-state value without oscillating.

6.9 Transient Parameters of Second Order Systems

- Standard initial conditions of zero and unit-step input $r(t) = u(t)$ are used as common practice.

1. **Delay time**, t_d: Time required for response to reach 50% of final value the first time.

2. **Rise time**, t_r: Time required for response to rise from 0% to 100% of the final value.

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi - \beta}{\omega_d}$$

3. **Peak time**, t_p: Time required for response to reach first peak of overshoot

$$t_p = \frac{\pi}{\omega_d}$$

4. **Maximum (Percent) Overshoot**, M_p: Maximum peak value of response measured from steady state value

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \qquad M_p(\%) = \frac{c(t_p) - c(\infty)}{c(\infty)} = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

5. **Settling time**, t_s: Time required for response to reach and stay within 2% or 5% of the final value

$$t_s = 4T = \frac{4}{\zeta\omega_n} \quad (2\%)$$
$$t_s = 3T = \frac{3}{\zeta\omega_n} \quad (5\%)$$

6.10 Higher Order Systems

- Unit step response of higher order systems will be a linear sum of first order and second order systems

6.11 Dominant Poles

- Slowest poles of systems are responsible for longest lasting terms in transient response.

- Let $-p$ be the first order pole and $-\zeta\omega_n$ be the second order pole.

6.11.1 Dominant First Order Behaviour

- If $\zeta\omega_n \geq 10p$, then $G(s) \approx \frac{\frac{K}{\omega_n^2}}{s+p}$.

6.11.2 Dominant Second Order Behaviour

- If $p \geq 10\zeta\omega_n$, then $G(s) \approx \frac{\frac{K}{p}}{s^2 + 2\zeta\omega_n s + \omega_n^2}$.

7 W8: Fluid & Thermal Systems

7.1 Introduction to Fluid Systems

- 2 main types of fluid systems:
  - Hydraulic systems (containing liquids)
  - Pneumatic systems (containing gases)
- 2 types of fluid flow:
  - Laminar flow (viscous forces dominate)
  - Turbulent flow (inertial forces dominate)

7.2 Parameters of Fluid Flow

- **Fluid Resistance**, R: measure of change in pressure required to change unit change in flow rate

$$R = \frac{\Delta \text{ Pressure}}{\Delta \text{ Flow rate}} = \frac{dP}{dQ}$$

- **Fluid Capacitance**, C: measure of change in stored fluid required to cause unit change in pressure

$$C = \frac{\Delta \text{ Capacity}}{\Delta \text{ Pressure}} = \frac{Q\,dt}{dP}$$

  - With a constant cross-sectional area A:

$$C = \frac{\Delta \text{ Capacity}}{\Delta \text{ Pressure Head}} = \frac{Q\,dt}{dh}$$

- **Fluid Inductance**, I: measure of change in pressure to make unit rate of change in the flow rate

$$I = \frac{\Delta \text{ Pressure}}{\text{Rate of change of flow rate}} = \frac{dP\,dt}{dQ}$$

  - Negligible in liquid systems

7.3 Modelling Liquid Level Systems

- Fluid inductance neglected.
- Assign variables to flow rates and pressure/pressure head.
- Using continuity of flow, conservation of mass and Newton's 2nd Law, write equations for each parameter.
- Useful equations:
  - Continuity of flow: Fluid accumulated = Net inflow of fluid

$$C\frac{dh}{dt} = q_i - q_o$$

  - Fluid resistance: $R = \dfrac{dh}{dQ}$
  - Power = $P \times Q$

7.4 Modelling Hydraulic Systems

- Assume hydraulic fluid is incompressible.
- Fluid inductance neglected.
- Assign variables to mass flow rates, pressure, displacement and velocities.
- Using continuity of flow, conservation of mass and Newton's 2nd Law, write equations for each parameter,
- Useful equations:
  - Continuity of flow: Fluid passing throgugh piston = Movement of piston

$$q = \rho A(\dot{y} - \dot{z})$$

  - Fluid resistance: $R = \dfrac{dP}{dQ}$
  - Power = $P \times Q$

7.5 Modelling Pneumatic Systems

- Assume subsonic fluid flow.
- Fluid inductance neglected.
- Assign variables to mass flow rates and pressure.
- Using continuity of flow, conservation of mass and Newton's 2nd Law, write equations for each parameter.
- Useful equations:
  - Continuity of flow: Fluid accumulated = Net inflow of fluid

$$C\frac{dp_o}{dt} = q$$

  - Fluid resistance: $R = \dfrac{dP}{dQ} = \dfrac{p_i - p_o}{q}$
  - Power = $P \times Q$

7.6 Introduction to Thermal Systems

- 3 modes of heat transfer:
  - Conduction
  - Convection
  - Radiation (negligible unless very high temperatures)
- Rate of heat flow for conductive and convective heat transfer, $q = K\Delta\theta$
  - where K is the heat transfer coefficient and $\Delta\theta$ is the temperature difference between the two mediums.

7.7 Parameters of Heat Flow

- **Thermal Resistance**, R:
  measure of the change in temperature difference required to cause a unit change in heat flow rate

$$R = \frac{\Delta \text{ Temperature difference}}{\Delta \text{ Heat flow rate}} = \frac{d(\Delta\theta)}{dq} = \frac{1}{K}$$

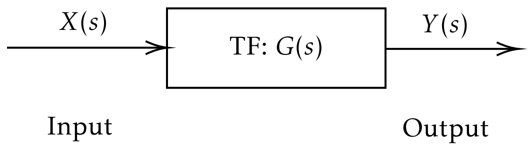
- **Thermal Capacitance**, C: measure of the change in quantity of heat energy stored required to cause a unit change in temperature

$$C = \frac{\Delta \text{ Heat energy capacity}}{\Delta \text{ Temperature}} = \frac{q\ dt}{d\theta}$$

7.8 Modelling Thermal Systems

- Assign variables to heat flow rates and temperature.
- Using continuity of heat flow and conservation of energy, write equations for each parameter.
- Useful equations:
  - Continuity of heat flow: $q = C\dfrac{d\theta}{dt}$
  - Thermal resistance: $R = \dfrac{d(\Delta\theta)}{dq} = \dfrac{\theta_a - \theta_b}{q}$
  - Power $= \theta \times q$

8 W8: Block Diagrams

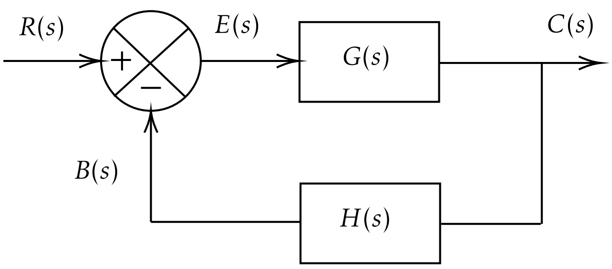


- Pictorial representation of system
- Variables linked to each other via TF blocks
- Signals travel in direction of arrow
- Systems can share same block diagram

8.1 Elements of A Block Diagram

- **Block**: represents the transfer function of a system component, with a single input and a single output
- **Summing point**: represents point of a system component, with more than 1 input and a single output
- **Branch point**: represents the point where the signal from one block goes to other blocks or summing points at the same time

8.2 Feedback Loop Transfer Functions



This is a **NEGATIVE** feedback loop.

- Parameters:
  - $R(s)$: input
  - $E(s)$: error
  - $C(s)$: output
  - $B(s)$: feedback
- **Open Loop Transfer Function** (OLTF): ratio of feedback signal to error signal

$$\mathrm{OLTF} = \frac{B(s)}{E(s)} = G(s)H(s)$$

- **Feedforward Transfer Function** (FTF): ratio of output signal to error signal

$$\mathrm{FTF} = \frac{C(s)}{E(s)} = G(s)$$

- **Negative Closed Loop Transfer Function** (Negative CLTF):

  ratio of output signal to input signal in a negative feedback loop

$$\text{Negative CLTF} = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

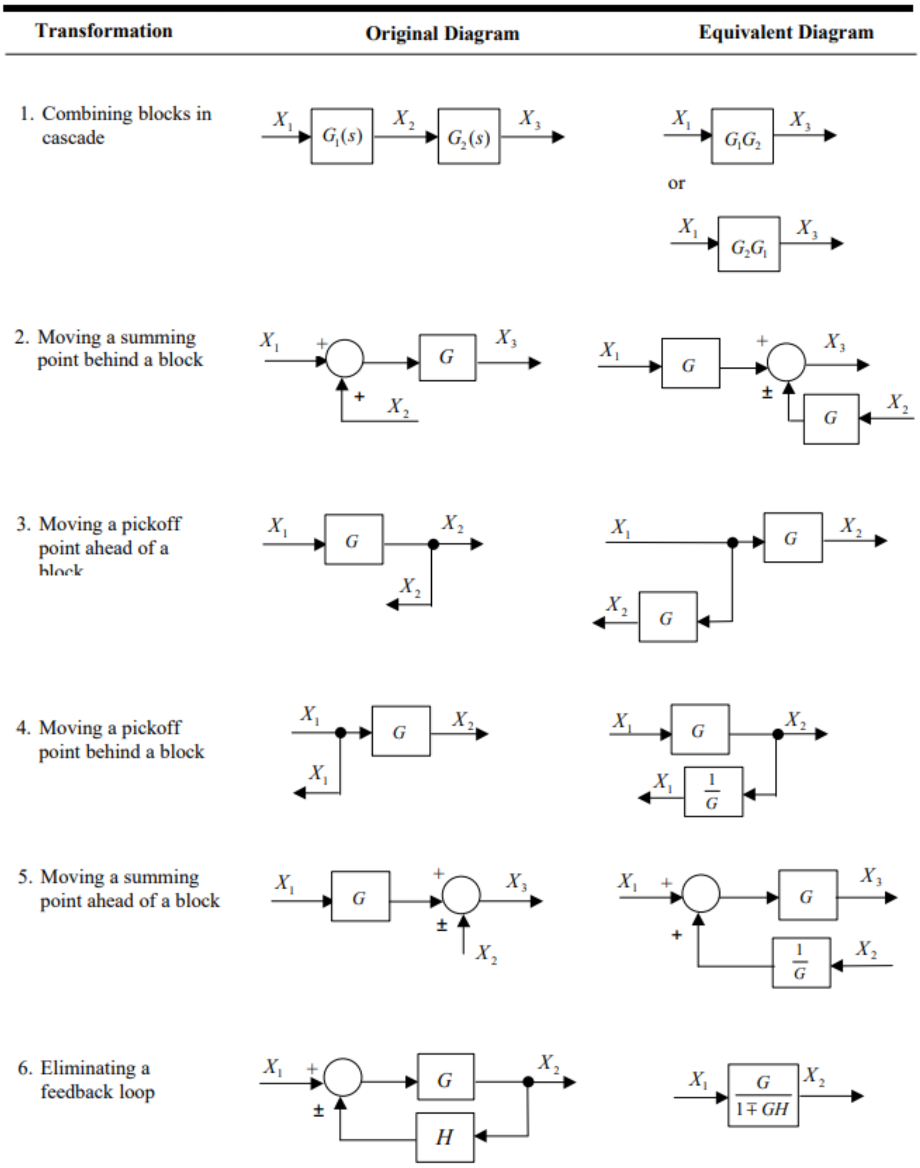
- **Positive Closed Loop Transfer Function** (Positive CLTF):

  ratio of output signal to input signal in a positive feedback loop

$$\text{Negative CLTF} = \frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)}$$

8.3 Block Diagram Reduction

- Image taken from *Modern Control Systems* by Dorf and Bishop



9 W8: PID Controllers

9.1 Automatic Controllers

- Compares actual value of plant/system output with desired value
- Controller determines deviation and produces control signal to reduce deviation (control action)
- Examples of automatic controllers:
  - On-off controllers
  - PID controllers
  - Lead-Lag compensators
  - State Space controllers (30.114)
- **Purpose**:
  - Improve transient response
  - Enhance steady state performance
  - Augment or introduce stability into system

9.2 On-off Controllers

- Output depends on value of error between desired and actual values, e
- Without differential control:

$$U(t) = \begin{cases} U_1, & e > 0 \\ U_2, & e < 0 \end{cases}$$

Switch-over point: $e = 0$

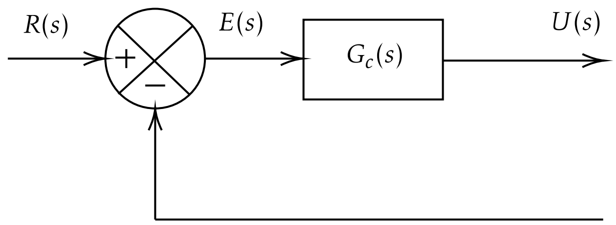
- With differential control:

$$U(t) = \begin{cases} U_1, & e > e^+ \\ U_2, & e < e^- \end{cases}$$

Differential gap: $e^- \leq e \leq e^+$

  - Differential gap allows controller value to maintain value until error has moved significantly from 0
  - Tradeoff of accuracy (more switchings) vs operating life

9.3 PID Controllers



- Uses error between desired and actual values to minimize errors by changing controller output
- Parameters:
  - Proportional: depends on present error $\quad K_p e(t)$
  - Integral: accumulation of past error $\quad K_i \int_{-\infty}^{t} e(t)\, dt$
  - Derivative: prediction of future error $\quad K_d \frac{de(t)}{dt}$
- Output in Time Domain:

$$
\begin{aligned}
u(t) &= K_p e(t) + K_i \int_{-\infty}^{t} e(t) + K_d \frac{de(t)}{dt} \\
&= K_p \left[e(t) + \frac{1}{T_i} \int_{-\infty}^{t} e(t)\, dt + T_d \frac{de(t)}{dt} \right]
\end{aligned}
$$

  - where K_p, K_i and K_d are the proportional, integral and derivative gains respectively,
  - and T_i, T_d are the integral and derivative times respectively.
- Output in Laplace Domain:

$$
\begin{aligned}
U(s) &= K_p E(s) + \frac{K_i}{s} E(s) + K_d s E(s) \\
G_c(s) &= \frac{U(s)}{E(s)} \\
&= K_p + \frac{K_i}{s} + K_d s \\
&= K_p \left(1 + \frac{1}{T_i s} + T_d s \right)
\end{aligned}
$$

9.4 Types of PID Controllers

- P Controller:

$$G_c(s) = K_p$$

- I Controller:

$$
\begin{aligned}
G_c(s) &= \frac{K_i}{s} \\
&= \frac{K_p}{T_i s}
\end{aligned}
$$

- PI Controller:

$$
\begin{aligned}
G_c(s) &= K_p + \frac{K_i}{s} \\
&= K_p \left(1 + \frac{1}{T_i s} \right)
\end{aligned}
$$

- PD Controller:

$$\begin{aligned} G_c(s) &= K_p + K_d s \\ &= K_p(1 + T_d s) \end{aligned}$$

- PID Controller:

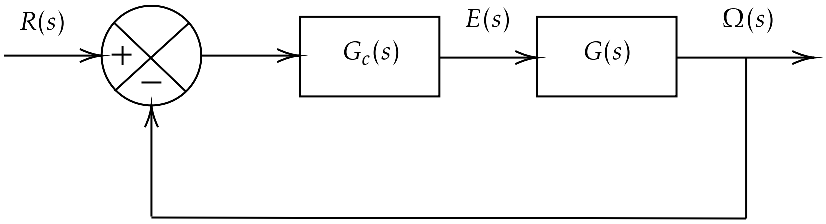
$$\begin{aligned} G_c(s) &= K_p + \frac{K_i}{s} + K_d s \\ &= K_p\left(1 + \frac{1}{T_i s} + T_d s\right) \end{aligned}$$

9.5 Recap: DC Motor

- First order system

$$G(s) = \frac{A}{Ts+1}, \text{ where } A = \frac{K}{R_a b + KK_b} \text{ and } T = \frac{R_a J}{R_a b + KK_b}$$

9.6 Proportional Control of First Order System



- P Controller: $G_c(s) = K_p$
- DC Motor: $G(s) = \dfrac{A}{Ts+1}$
- Closed Loop Transfer Function:

$$\begin{aligned} \frac{\Omega(s)}{R(s)} &= \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{K_p A}{Ts + 1 + K_p A} \\ &= A_p\left(\frac{1}{T_p s + 1}\right), \text{ where } A_p = \frac{K_p A}{1 + K_p A} \text{ and } T_p = \frac{T}{1 + K_p A} \end{aligned}$$

  - A_p and T_p are the DC gain and time constants of the closed loop system
  - A_p and T_p can be changed by adjusting K_p

- With a unit step input $R(s)$:

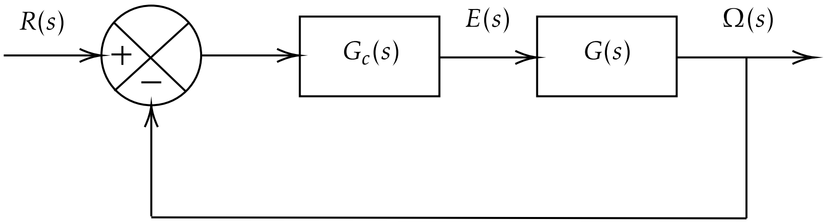
$$\begin{aligned} \Omega(s) &= R(s)\frac{A_p}{T_p s + 1} \\ &= \frac{A_p}{s(T_p s + 1)} \\ \Rightarrow \omega(t) &= A_p\left(1 - e^{-\frac{t}{T_p}}\right) \\ &= \frac{K_p A}{1 + K_p A}\left(1 - e^{-\frac{t}{T_p}}\right) \end{aligned}$$

- Steady state error, e_{ss}:

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty}[r(t)-\omega(t)] \\ &= 1-\frac{K_pA}{1+K_pA} \\ &= \frac{1}{1+K_pA} \end{aligned}$$

  - As $K_p \to \infty$, $e_{ss} \to \infty$.

9.7 Integral Control of First Order System



- I Controller: $G_c(s) = \dfrac{K_i}{s}$
- DC Motor: $G(s) = \dfrac{A}{Ts+1}$
- Closed Loop Transfer Function:

$$\begin{aligned} \frac{\Omega(s)}{R(s)} &= \frac{G_c(s)G(s)}{1+G_c(s)G(s)} = \frac{K_iA}{Ts^2+s+K_iA} \\ &= \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}, \text{ where } \omega_n^2 = \frac{K_iA}{T} \text{ and } 2\zeta\omega_n = \frac{1}{T} \end{aligned}$$

  - ω_n can be adjusted by changing K_i.
  - ζ is indirectly affected by ω_n and T.

- Error function $E(s)$:

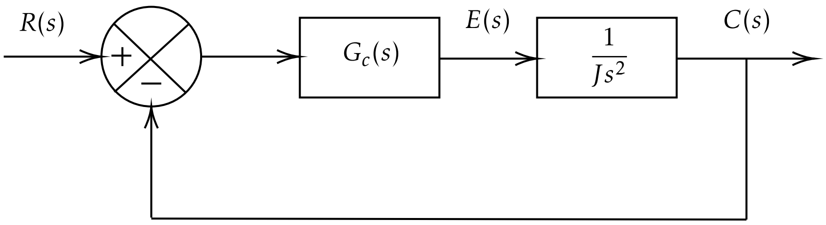
$$\begin{aligned} \frac{E(s)}{R(s)} &= 1-\frac{\Omega(s)}{R(s)} = 1-\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \\ &= \frac{s^2+2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2} \\ \Rightarrow E(s) &= R(s)\frac{E(s)}{R(s)} \\ &= \frac{s^2+2\zeta\omega_n s}{s(s^2+2\zeta\omega_n s+\omega_n^2)} \end{aligned}$$

- Steady state error, e_{ss}:

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0}\frac{s^2+2\zeta\omega_n s}{s^2+2\zeta\omega_n s+\omega_n^2} \\ &= 0 \end{aligned}$$

  - $\therefore$ I controller can remove the steady state error in a first order system.

9.8 Proportional Control of Second Order System

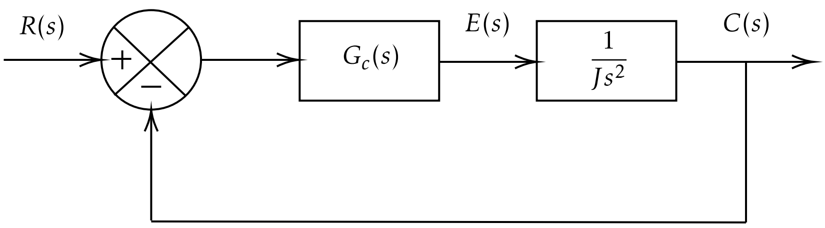


- P Controller: $G_c(s) = K_p$
- Second Order System: $G(s) = \dfrac{1}{Js^2}$
- Closed Loop Transfer Function:

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{G_c(s)G(s)}{1+G_c(s)G(s)} \\ &= \frac{K_p}{Js^2+K_p}\end{aligned}$$

- Characteristic equation: $Js^2 + K_p = 0$
- Poles: $s = \pm j\sqrt{\dfrac{K_p}{J}}$
  - System oscillates indefinitely
  - $\therefore$ P controller cannot remove disturbances in a second order system.

9.9 PD Control of Second Order System



- PD Controller: $G_c(s) = K_p(1+T_d s)$
- Second Order System: $G(s) = \frac{1}{Js^2}$
- Closed Loop Transfer Function:

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{G_c(s)G(s)}{1+G_c(s)G(s)} \\ &= \frac{K_p(1+T_d s)}{Js^2+K_pT_d s+K_p}\end{aligned}$$

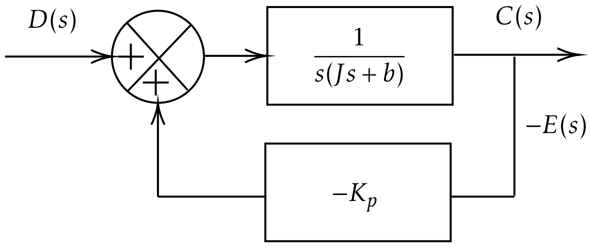
- Characteristic equation:

$$\begin{aligned} & Js^2 + K_pT_d s + K_p = 0 \\ \Rightarrow\quad & Js^2 + K_d s + K_p = 0\end{aligned}$$

  - Derivative control action introduces damping effect which stabilizes system

9.10 Disturbance Rejection Without Integrator

- Assume a unit step disturbance $D(s)$ is inserted between $G_c(s)$ and $G(s)$.
- Let input $R(s) = 0$ and $D(s) = \frac{1}{s}$.
- Simplifying the block diagram:



- Closed Loop Transfer Function:

$$\frac{C(s)}{D(s)} = \frac{\frac{1}{s(Js+b)}}{1 + \frac{K_p}{s(Js+b)}}$$
$$= \frac{1}{Js^2 + bs + K_p}$$

- Error function $E(s)$:

$$\frac{E(s)}{D(s)} = -\frac{C(s)}{D(s)}$$
$$= -\frac{1}{Js^2 + bs + K_p}$$
$$\Rightarrow E(s) = -D(s)\frac{C(s)}{D(s)}$$
$$= -\frac{1}{s(Js^2 + bs + K_p)}$$

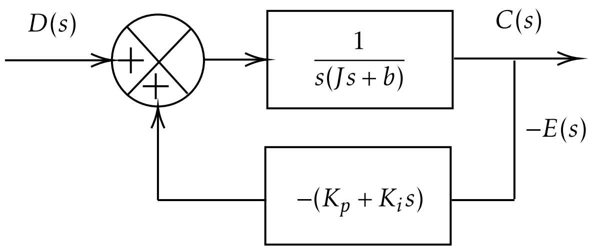
- Steady state error, e_{ss}:

$$e_{ss} = \lim_{s\to 0} sE(s)$$
$$= -\lim_{s\to 0} \frac{1}{Js^2 + bs + K_p}$$
$$= -\frac{1}{K_p}$$

  - $\therefore$ P controller cannot remove the steady state error in a first order system.

9.11 Disturbance Rejection With Integrator

- Assume a unit step disturbance $D(s)$ is inserted between $G_c(s)$ and $G(s)$.
- Let input $R(s) = 0$ and $D(s) = \frac{1}{s}$.
- Simplifying the block diagram:



- Closed Loop Transfer Function:

$$\frac{C(s)}{D(s)} = \frac{\frac{1}{s(Js+b)}}{1+\left(K_p+\frac{K_p}{T_i}\right)\frac{1}{s(Js+b)}}$$
$$= \frac{s}{Js^3+bs^2+K_ps+\frac{K_p}{T_i}}$$

- Error function, $E(s)$:

$$\frac{E(s)}{D(s)} = -\frac{C(s)}{D(s)}$$
$$= -\frac{s}{Js^3+bs^2+K_ps+\frac{K_p}{T_i}}$$
$$\Rightarrow E(s) = -D(s)\frac{C(s)}{D(s)}$$
$$= -\frac{s}{s\left(Js^3+bs^2+K_ps+\frac{K_p}{T_i}\right)}$$

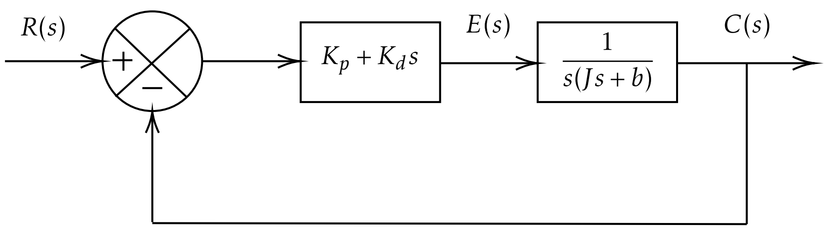
- Steady state error, e_{ss}:

$$e_{ss} = \lim_{s\to 0} sE(s)$$
$$= -\lim_{s\to 0}\frac{s}{Js^3+bs^2+K_ps+\frac{K_p}{T_i}}$$
$$= 0$$

  - $\therefore$ PI controller can remove the steady state error in a second order system.

9.12 PD Controller vs P Controller and Velocity Feedback

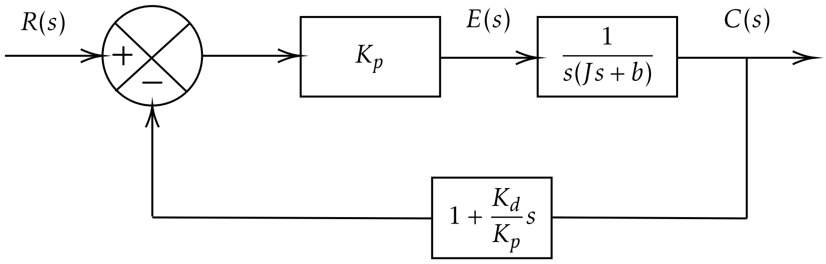
- PD Controller



  - Closed Loop Transfer Function:

$$\frac{C(s)}{R(s)} = \frac{\frac{K_p+K_ds}{s(Js+b)}}{1+\frac{K_p+K_ds}{s(Js+b)}}$$
$$= \frac{K_p+K_ds}{Js^2+(b+K_d)s+K_p}$$

- P Controller and Velocity Feedback



  - Closed Loop Transfer Function:

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{\frac{K_p}{s(Js+b)}}{1+\left(1+\frac{K_d}{K_p}s\right)\frac{K_p}{s(Js+b)}} \\ &= \frac{K_p}{Js^2+(b+K_d)s+K_p}\end{aligned}$$

- Both systems have the same poles but different poles.

9.13 Effect of Zeros on Transient Response

- Zero near a pole
  - A zero near a pole reduces that term in overall response

$$\begin{aligned}G_1(s) &= \frac{2}{(s+1)(s+2} = \frac{2}{s+1} - \frac{2}{s+2} \\ G_2(s) &= \frac{2(s+1.1)}{(s+1)(s+2)} = \frac{0.18}{s+1} + \frac{1.64}{s+2}\end{aligned}$$

- Left and right hand plane zeroes
  - Consider a second order system $H(s)$:

$$H(s) = \frac{\frac{\omega_n s}{\alpha\zeta} + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\frac{s}{\alpha\zeta\omega_n} + 1}{\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1}$$

  - Let $s = \dfrac{s}{\omega_n}$:

$$\begin{aligned}\widetilde{H}(s) &= \frac{\frac{s}{\alpha\zeta}+1}{s^2+2\zeta s+1} = \frac{1}{s^2+2\zeta s+1} - \frac{1}{\alpha\zeta}\frac{s}{s^2+2\zeta s+1} \\ &= H_o(s) + \frac{1}{\alpha\zeta}H_d(s),\end{aligned}$$

    - where H_o is the second order system with no zero,
    - and H_d is the time derivative of H_o.
  - The zero introduces the term $\dfrac{1}{\alpha\zeta}H_d(s)$.
  - The second order system has a zero at $s = -\alpha\zeta\omega_n$.
    - If zero is in LHP ($\alpha > 0$), the system is a minimum phase system.
    - If zero is in RHP ($\alpha < 0$), the system is a non-minimum phase system.
    - More on minimum and non-minimum phase systems in Week 11.

9.14 Summary of PID Control Action

Parameter	Rise time, t_r	Overshoot, M_p	Settling time, t_s	Steady state error, e_{ss}	System stability
K_p	$\downarrow$	$\uparrow$	Small Δ	$\downarrow$	$\downarrow$
K_i	$\downarrow$	$\uparrow$	$\uparrow$	Eliminates	$\downarrow$
K_d	Small Δ	$\downarrow$	$\downarrow$	No effect	$\downarrow$ if K_d is small

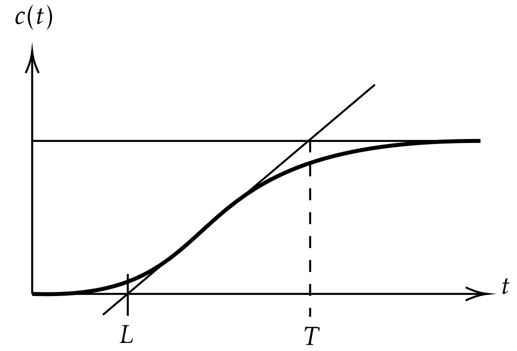
- Proportional Control Action
  - Increases speed of response and ω_n
  - Reduces system stability
- Integral Control Action
  - Improves steady state performance
  - Slows down system
  - Reduces system stability
  - Increases system order
  - Rejects disturbances
- Derivative Control Action
  - Adds damping to system
  - Improves stability of system

9.15 Tuning PID Controllers

- Heuristic method formulated in by Ziegler and Nichols in 1942
- Ziegler-Nicholas Tuning has 2 methods:
  - Open loop tuning method
  - Closed loop tuning method
- Gives educated guess for controller gains
- Provides good starting point for further fine tuning

9.16 Open Loop Ziegler-Nichols Tuning

- Draw a tangent line at the inflection point of the open loop unit step output:



- Comprises 2 experimental parameters:
  - Lag time, L: where the tangent line cuts the time axis
  - Approximated time constant, T: where the tangent line cuts the steady state response
- Recommended PID controller gains:

Type of Controller	**Proportional gain,** K_p	**Integral time,** T_i	**Differential time,** T_d
P	$\frac{T}{L}$	-	-
PI	$0.9\frac{T}{L}$	$\frac{L}{0.3}$	-
PID	$1.2\frac{T}{L}$	$2L$	$0.5L$

9.17 Closed Loop Ziegler-Nicholas Tuning

- With only the P controller, increase K_p until the output shows sustained oscillations.
- K_p at this point is known as the critical value K_{cr}.
- The corresponding period P_{cr} can also be found.
- Recommended PID controller gains:

Type of Controller	**Proportional gain,** K_p	**Integral time,** T_i	**Differential time,** T_d
P	$0.5K_{cr}$	-	-
PI	$0.45K_{cr}$	$\frac{P_{cr}}{1.2}$	-
PID	$0.6K_{cr}$	$0.5P_{cr}$	$0.125P_{cr}$

10 W10: Linearization

10.1 Purpose of Linearization

- Allows linear analysis methods to be used on non-linear systems
- e.g. small angle approximation $\sin\theta \approx \theta$

10.2 Linearization about a Point (x, z)

- Choose an operating point $(\bar{x},\ \bar{z})$.
- If $x-\bar{x}$ is small and $\bar{z} = f(\bar{x})$, ignoring the higher order terms:

$$
\begin{aligned}
z - \bar{z} &= \left.\frac{df(x)}{dx}\right|_{x=\bar{x}} (x-\bar{x}) \\
\Rightarrow \hat{z} &= \left.\frac{df(x)}{dx}\right|_{x=\bar{x}} \hat{x}
\end{aligned}
$$

  - where $\hat{x} = x - \bar{x}$ and $\hat{z} = z - \bar{z}$.

10.3 Linearization about a Point (x, y, z)

- Choose an operating point $(\bar{x},\ \bar{y},\ \bar{z})$.
- If $x-\bar{x}$ and $y-\bar{y}$ are small and $\bar{z} = f(\bar{x},\ \bar{y})$, ignoring the higher order terms:

$$
\begin{aligned}
z - \bar{z} &= \left.\frac{\partial f(x,\ y)}{\partial x}\right|_{\substack{x=\bar{x}\\ y=\bar{y}}} (x-\bar{x}) + \left.\frac{\partial f(x,\ y)}{\partial y}\right|_{\substack{x=\bar{x}\\ y=\bar{y}}} (y-\bar{y}) \\
\Rightarrow \hat{z} &= \left.\frac{\partial f(x,\ y)}{\partial x}\right|_{\substack{x=\bar{x}\\ y=\bar{y}}} \hat{x} + \left.\frac{\partial f(x,\ y)}{\partial y}\right|_{\substack{x=\bar{x}\\ y=\bar{y}}} \hat{y}
\end{aligned}
$$

  - where $\hat{x} = x - \bar{x}$, $\hat{y} = y - \bar{y}$ and $\hat{z} = z - \bar{z}$.

11 W10: System Stability

11.1 Stability Analysis

- Stability is a critical concern.
- Stability is a system property, does not depend on inputs.
- Types of stability:
  - Absolute/ internal stability: LHP poles
  - Neutral stability: Non-repeated $j\omega$ axis poles
  - Unstable: Repeated $j\omega$ axis poles, RHP poles

11.2 Routh-Hurwitz Stability Criterion

1. Write characteristic equation in descending powers of s.
2. If coefficients contain zero or negative values, there are imaginary poles or RHP poles.
   $\Rightarrow$ System is not stable.
3. If coefficients are positive, construct a Routh Array.
4. No. of RHP poles in CE = no. of changes in sign of coefficients in 1st column of Routh array

e.g. $s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$

$$\begin{array}{llll}
s^4 & a_1 = 1 & a_2 = 3 & a_3 = 5 \\
s^3 & b_1 = 2 & b_2 = 4 & \\
s^2 & c_1 & c_2 & \\
s^1 & d_1 & & \\
s^0 & e_1 & &
\end{array}$$

$$c_1 = \frac{b_1 a_2 - a_1 b_2}{b_1} = 1, \quad c_2 = \frac{b_2 a_3 - 0}{b_2} = 5$$

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1} = -6$$

$$e_1 = \frac{d_1 c_2 - 0}{d_1} = 5$$

There are 2 sign changes and thus 2 RHP poles, $\therefore$ the system is unstable.

11.3 Special Cases of Routh Array

11.3.1 Zero in First Column

e.g. $s^3 + 2s^2 + s + 2 = 0$

$$\begin{array}{lll}
s^3 & 1 & 1 \\
s^2 & 2 & 2 \\
s^1 & 0 \approx \epsilon & \\
s^0 & 2 &
\end{array}$$

- Replace zero with a small positive number, ϵ.
- Indicates presence of a pair of imaginary roots.

11.3.2 Zeros in Entire Derived Row

e.g. $s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$

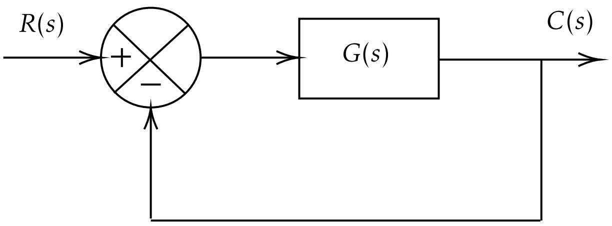
s^5	1	24	-25
s^4	2	48	-50
s^3	$\cancel{0}$ 8	$\cancel{0}$ 96	
s^2	24	-50	
s^1	112.7		
s^0	-50		

- Create an auxiliary polynomial $P(s)$ from row above.
- Replace the coefficients of the zero row with $\dfrac{dP(s)}{ds}$:

$$\boxed{\begin{array}{rcl} P(s) & = & 2s^4 + 48s^2 - 50 \\ \Rightarrow \frac{dP(s)}{ds} & = & 8s^3 + 96s \end{array}}$$

- Denotes presence of radially opposite poles in s-plane.

12 W10: System Types



- Consider under unity feedback $H(s) = 1$

$$G(s) = K\frac{(\tau_1 s+1)(\tau_2 s+1)\cdots(\tau_m s+1)}{s^N(T_1 s+1)(T_2 s+1)\cdots(T_n s+1)}$$

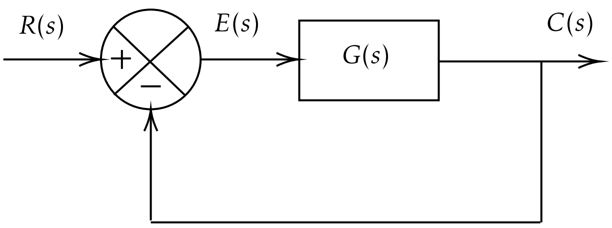
  - System Order: $N + n$
  - No. of poles: $N + n$

- System is classified by pole multiplicity N at the origin:

$$\begin{aligned} N &= 0 \Rightarrow \text{Type 0 system} \\ N &= 1 \Rightarrow \text{Type 1 system} \\ N &= 2 \Rightarrow \text{Type 2 system} \end{aligned}$$

  - System Order $\neq$ System Type

12.1 Static Position Error Constant



- R(s) is a unit step input, $\therefore R(s) = \frac{1}{s}$
- Closed Loop Transfer Function:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

- Error function, $E(s)$:

$$\begin{aligned} \frac{E(s)}{R(s)} &= 1 - \frac{C(s)}{R(s)} = \frac{1}{1+G(s)} \\ E(s) &= R(s)\frac{E(s)}{R(s)} = \frac{1}{s(1+G(s))} \end{aligned}$$

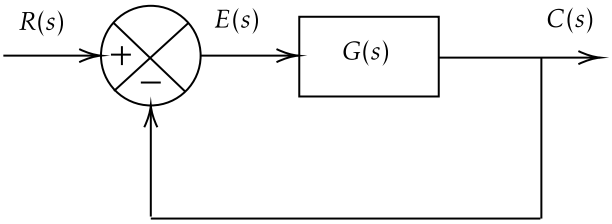
- Steady state error, e_{ss}:

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0}\frac{1}{1+G(s)} \\ &= \frac{1}{1+K_P} \end{aligned}$$

- Static Position Error Constant, K_P:

$$K_P = \lim_{s\to 0} G(s)$$

12.2 Static Velocity Error Constant



- R(s) is a unit ramp input, $\therefore R(s) = \frac{1}{s^2}$
- Closed Loop Transfer Function:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

- Error function, $E(s)$:

$$\frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = \frac{1}{1 + G(s)}$$
$$E(s) = R(s)\frac{E(s)}{R(s)} = \frac{1}{s^2(1 + G(s))}$$

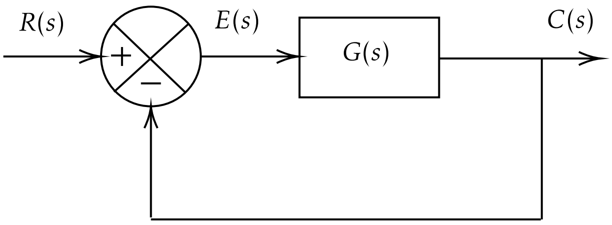
- Steady state error, e_{ss}:

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} sE(s) \\ &= \lim_{s \to 0} \frac{1}{s(1 + G(s))} \\ &= \frac{1}{K_V} \end{aligned}$$

- Static Velocity Error Constant, K_V:

$$K_V = \lim_{s \to 0} sG(s)$$

12.3 Static Acceleration Error Constant



- R(s) is a unit parabolic input, $\therefore R(s) = \frac{1}{s^3}$
- Closed Loop Transfer Function:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

- Error function, $E(s)$:

$$\frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = \frac{1}{1 + G(s)}$$
$$E(s) = R(s)\frac{E(s)}{R(s)} = \frac{1}{s^3(1 + G(s))}$$

- Steady state error, e_{ss}:

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} \frac{1}{s^2(1+G(s))} \\ &= \frac{1}{K_A} \end{aligned}$$

- Static Acceleration Error Constant, K_A:

$$K_A = \lim_{s\to 0} s^2 G(s)$$

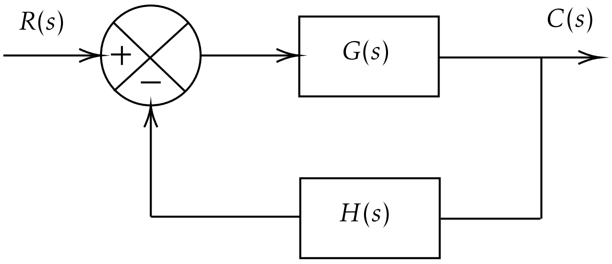
12.4 System Types and Steady State Errors

	Steady state error, e_{ss}		
	Step input $r(t) = 1$	Ramp input $r(t) = t$	Parabolic input $r(t) = 0.5t^2$
Type 0 System	$\frac{1}{1+K_P}$	-	-
Type 1 System	-	$\frac{1}{K_V}$	-
Type 2 System	-	-	$\frac{1}{K_A}$

- To reduce steady state error e_{ss}, increase error constants K_P, K_V or K_A.
- Another way is to add integrators $\frac{1}{s}$ in feed forward path.
  - However it reduces system stability.

13 W11: Root Locus

13.1 Root Locus Method



- Closed Loop Transfer Function: $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$
- Characteristic Equation: $1 + G(s)H(s) = 0 \qquad \Rightarrow G(s)H(s) = -1$
  - We rewrite $G(s)H(s)$ in polynomial form with amplitude K, zeros and poles.

$$K\frac{N(s)}{D(s)} = -1$$
$$\therefore K\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = -1$$

- Steps to construct Root Locus:
  1. Write characteristic equation in Root Locus form.
  2. Locate open loop poles and zeros.
  3. Find the number of loci and root loci on the real axis.
  4. Determine asymptotes of root locus.
  5. Locate break points.
  6. Derive the departure and arrival angles.
     - Complex poles and zeros only
  7. Determine where the root loci crosses the imaginary axis.
  8. Locate closed loop poles for a certain value of K.

13.2 Step 1: Characteristic Equation in Root Locus form

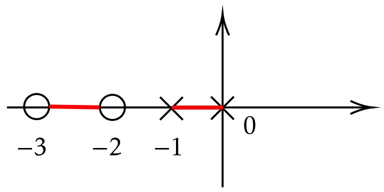
- CE not in RL form: e.g. $1 + G(s)H(s) = 0$
- General RL form: $1 + K\dfrac{N(s)}{D(s)} = 0$
- **CE in RL form:** e.g. $1 + K\dfrac{s}{(s+1)(s+1)} = 0$

13.3 Step 2: Open Loop Poles and Zeros

- e.g. CE: $1 + K\dfrac{1}{s(s+1)(s+2)}$
- $N(s) = 1$
- $D(s) = s(s+1)(s+2)$
- **Open Loop Zeros:** no zeros
- **Open Loop Poles:** $s = 0,\ s = -1,\ s = -2$

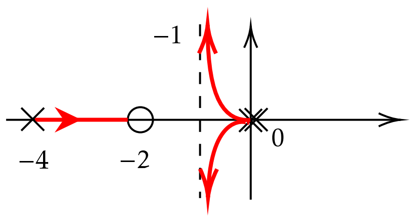
13.4 Step 3: No. of Loci & Real Axis Loci

- No. of loci = no. of open loop poles
- Real axis loci lies to the left of ODD no. of poles and zeros on real axis
- e.g.
  - Open loop zeros: $s = -2,\ s = -3$
  - Open loop poles: $s = 0,\ s = -1$
  - 2 poles $\Rightarrow$ 2 loci



13.5 Step 4: Asymptotes of Root Loci

- Find center of asymptotes on real axis σ_c and angle of asymptotes β.
- e.g.
  - Open loop zeros: $s = -2$
  - Open loop poles: $s = 0,\ s = 0,\ s = -4$
  - No. of open loop zeros, $m = 1$
  - No. of open loop poles, $n = 3$



$$\sigma_c = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} = \frac{(0+0-4)-(-2)}{3-1} = -1$$

$$\beta = \frac{(2\ell+1)180^\circ}{n-m} = \frac{(2\ell+1)180^\circ}{3-1} = 90^\circ,\ -90^\circ$$

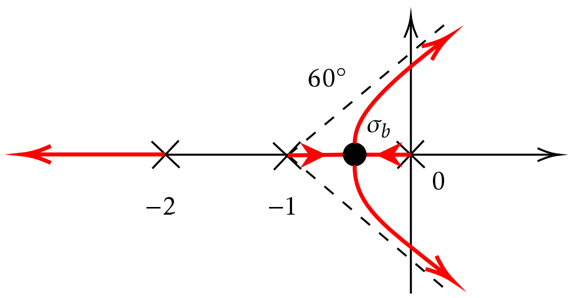
13.6 Step 5: Locate Break Points

- Breakaway point: root locus on real axis $\longrightarrow$ complex plane
- Breakin point: root locus on complex plane $\longrightarrow$ real axis
- Find corresponding value of s at breakaway/breakin point:
- e.g. $N(s) = 1,\ D(s) = s^3 + 3s^2 + 2s$

$$\Rightarrow \frac{dK}{ds} = -\frac{D'(s)N(s) - D(s)N'(s)}{N^2(s)} = 0$$

$$D'(s)N(s) - D(s)N'(s) = (3s^2 + 6s + 2)(1) = 0$$

$$\therefore s = -0.4226,\ s = -1.5774$$



- Check break points (K must be positive)

$$s = -0.4226 \quad \Rightarrow K = -\frac{D(s)}{N(s)} = 0.3849$$

$$s = -1.5774 \quad \Rightarrow K = -\frac{D(s)}{N(s)} = -0.3849 \text{ (rej. } \because K > 0)$$

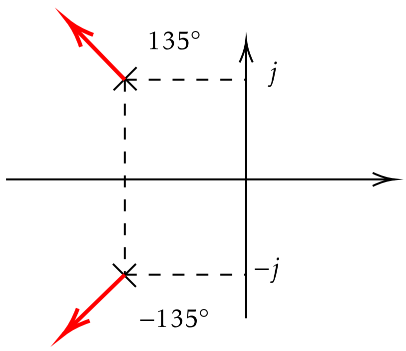
$\therefore$ Breakaway point, $\sigma_b = -0.4226$

13.7 Step 6: Departure & Arrival Angles

- Root locus 'departs' from complex poles
- Root locus 'arrives' from complex zeros
- Departure angle, $\theta_D = 180° + \angle\left[\dfrac{N(s)}{D(s)}\right]'$
- Arrival angle, $\theta_A = 180° - \angle\left[\dfrac{N(s)}{D(s)}\right]'$
- $\angle\left[\dfrac{N(s)}{D(s)}\right]'$: phase engle of $\dfrac{N(s)}{D(s)}$ at complex zero/pole, ignoring contribution of that pole
- e.g. $\text{CE} = 1 + K\dfrac{(s+2)}{(s+1+j)(s+1-j)} = 0$

  OL poles: $s = -1-j$, $s = -1+j$

$$
\begin{aligned}
\text{For } s = -1+j,\ \theta_D &= 180° + \angle\left[\frac{N(s)}{D(s)}\right]' \\
&= 180° + \angle\left.\frac{s+2}{s+1+j}\right|_{s=-1+j} \\
&= 180° + \angle(-1+j+2) - \angle(-1+j+1+j) \\
&= 180° + 45° - 90° \\
&= 135°
\end{aligned}
$$

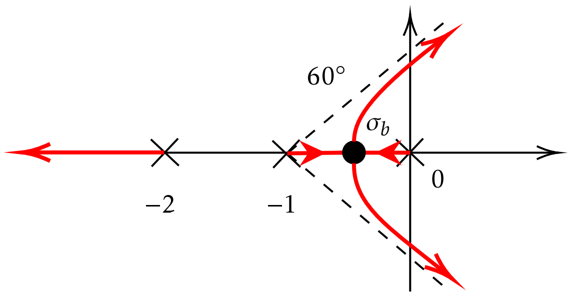


Similarly for $s = -1-j$, $\theta_D = -135°$.

13.8 Step 7: Root Locus Crossing Imaginary Axis

- Find out where root locus crosses the imaginary axis
- Substitute $s = j\omega$ into CE
- e.g. $\text{CE} = 1 + K\dfrac{1}{s(s+1)(s+2)} = 0$

$$
\begin{aligned}
s(s+1)(s+2) + K &= 0 \\
s^3 + 3s^2 + s + K &= 0 \\
\text{Let } s = j\omega: \quad (j\omega)^3 + 3(j\omega)^2 + 2j\omega + K &= 0 \\
(K - 3\omega^2) + j(2\omega - \omega^3) &= 0 \\
K - 3\omega^2 = 0 \quad \text{and} \quad 2\omega - \omega^3 &= 0
\end{aligned}
$$



$\therefore K = 0$ when $\omega = 0$, and $K = 6$ when $\omega = \pm\sqrt{2}$

- Root locus crosses imaginary axis at $K = 6$ when $\omega = \pm\sqrt{2}$
- Root locus touches imaginary axis at $K = 0$ when $\omega = 0$

13.9 Step 8: Closed Loop Poles and K

- Compute closed loop poles at $s = \bar{s}$ for a specific K
- Compute K for a set of closed loop poles at $s = \bar{s}$
- e.g. CE: $1 + K\dfrac{1}{s(s+1)(s+2)} = 0$

  We want the closed loop poles to have $\zeta = 0.5$.

  At the intersection, $s = -0.3337 \pm j0.5780$.

  Find value of K and the location of the third pole.

$$K = \left|\frac{D(s)}{N(s)}\right|_{s=-0.337+j0.5780} = 1.0383$$

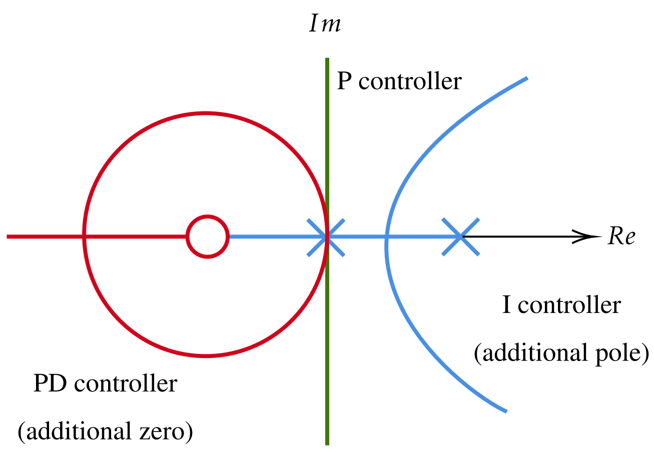
$$\Rightarrow \quad s^3 + 3s^2 + 2s + K = 0$$

$$(s + a_1)(s + 0.3337 + j0.5780)(s + 0.33737 - j0.5780) = 0$$

$$\Rightarrow \quad a_1 = 2.3326$$

- $\therefore K = 1.0383$, and the third pole is at $s = 2.3326$.

13.10 Root Locus with P, I and PD Controllers



- Adding an I controller: introduces a pole at origin
- Adding a D controller: introduces a zero
  - Location of zero depends on ratio of derivative to proportional gain

14 W11: Bode Diagrams

$$\text{Transfer function, } G(j\omega) = K\frac{N(j\omega)}{D(j\omega)}$$

- Transfer function of a LTI system can be represented by a Bode Diagram, made up of 2 diagrams:
  1. Magnitude of TF vs Frequency (log-log plot)
  2. Phase angle of TF vs Frequency (log-log plot)
- Magnitude of TF is measured in decibels (dB), where

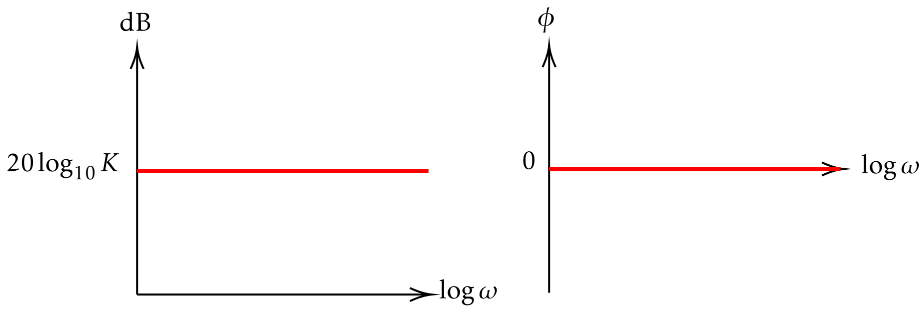
$$20\log_{10}|G(j\omega)| = 20\log_{10}K + 20\log_{10}|N(j\omega)| - 20\log_{10}|D(j\omega)|.$$

- Phase angle is measured in degrees, where $\phi = \angle N(j\omega) - \angle D(j\omega)$.

14.1 Bode Diagram of Constant

$$G(j\omega) = K,\ K > 0$$

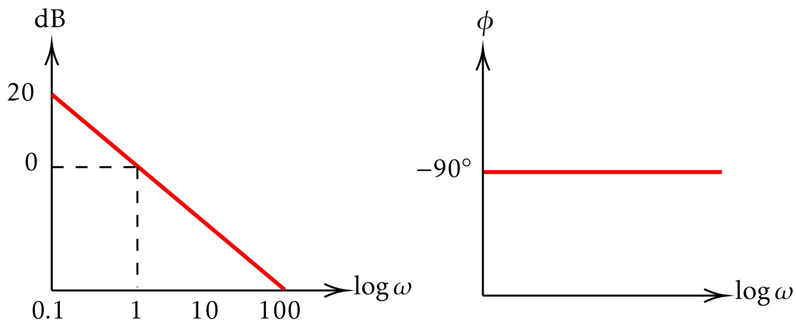
- Magnitude: dB $= 20\log_{10}K$
  - If $K > 1$, $20\log_{10}K > 0$.
  - If $0 < K < 1$, $20\log_{10}K < 0$.
- Phase: $\phi = 0$
  - Independent of K, ω



14.2 Bode Diagram of Integral Factor

$$G(j\omega) = \frac{1}{j\omega}$$

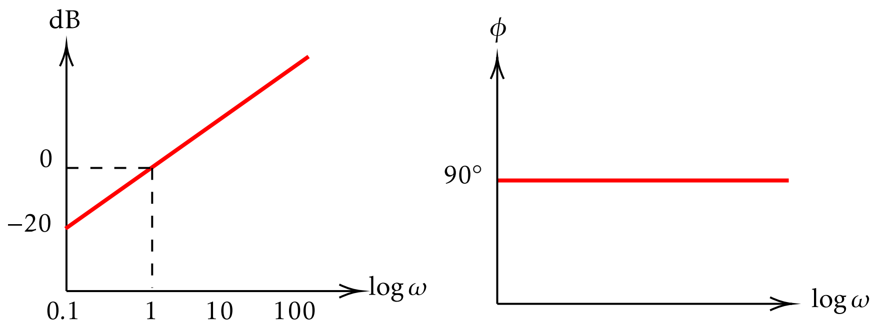
- Magnitude: dB $= 20\log_{10}\left|\frac{1}{j\omega}\right| = -20\log_{10}(\omega)$
  - Also known as -20 dB/decade
- Phase: $\phi = \angle\frac{1}{j\omega} = -90°$
  - Independent of ω



14.3 Bode Diagram of Derivative Factor

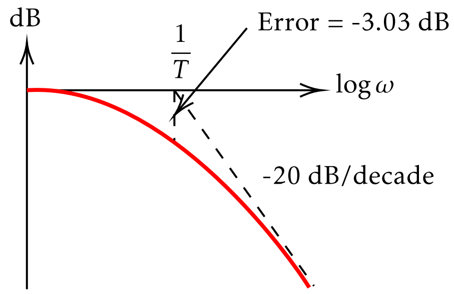
$$G(j\omega) = j\omega$$

- Magnitude: dB $= 20\log_{10}(j\omega)$
  $\Rightarrow$ 20 dB/decade
- Phase: $\phi = \angle j\omega = 90°$
  - Independent of ω

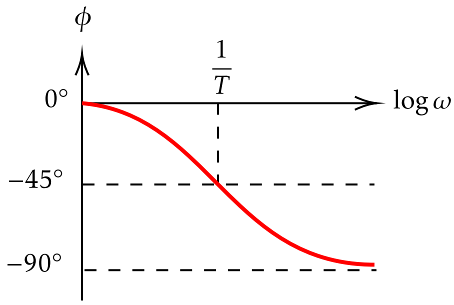


14.4 Bode Diagram of First Order System

$$G(j\omega) = \frac{1}{Tj\omega + 1}$$

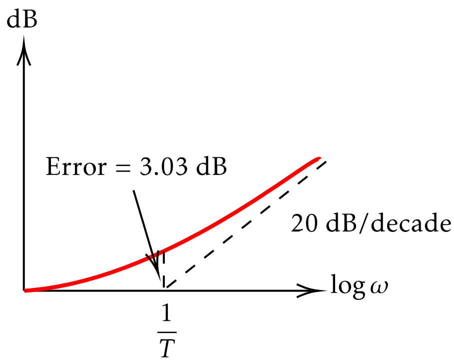


- Magnitude: $\text{dB} = 20\log_{10}\left|\frac{1}{Tj\omega+1}\right| = -20\log_{10}\sqrt{T^2\omega^2+1}$
  - At low frequency: $\omega \ll \frac{1}{T}$, $\text{dB} \approx 0$.
  - At high frequency: $\omega \gg \frac{1}{T}$, $\text{dB} \approx -20\log_{10} T\omega$.
    $\Rightarrow$ -20 dB/decade
- Phase: $\phi = \angle\frac{1}{Tj\omega+1} = -\tan^{-1}\omega T$
  - At low frequency: $\phi \approx -\tan^{-1} 0 = 0°$.
  - At high frequency: $\phi \approx -\tan^{-1}\infty = -90°$.
  - Asymptotes meet at corner frequency where $\omega = \frac{1}{T}$:
    $\phi = -\tan^{-1} 1 = -45°$

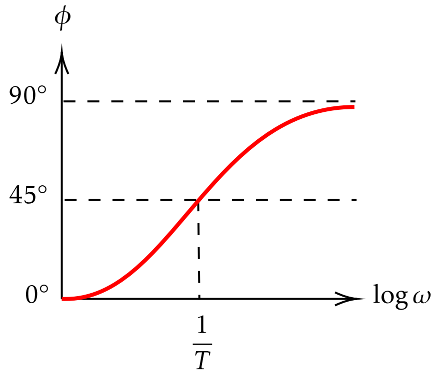


14.5 Bode Diagram of First Order Factor

$$G(j\omega) = Tj\omega + 1$$



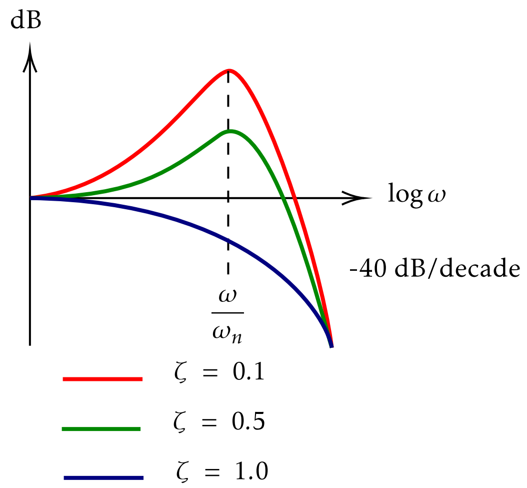
- Magnitude: $\text{dB} = 20\log_{10}\sqrt{T^2\omega^2+1}$
  - At low frequency: $\omega \ll \frac{1}{T}$, $\text{dB} \approx 0$.
  - At high frequency: $\omega \gg \frac{1}{T}$, $\text{dB} = 20\log_{10} T\omega$.
    $\Rightarrow$ 20 dB/decade
- Phase: $\phi = \angle Tj\omega + 1 = -\tan^{-1}\omega T$
  - At low frequency: $\phi \approx \tan^{-1} 0 = 0°$.
  - At high frequency: $\phi \approx \tan^{-1}\infty = 90°$.
  - Asymptotes meet at corner frequency where $\omega = \frac{1}{T}$:
    $\phi = \tan^{-1} 1 = 45°$

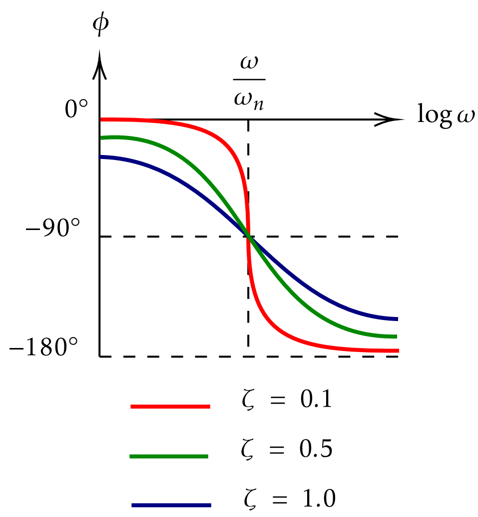


14.6 Bode Diagram of Second Order System

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{1}{\left(j\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + 1}$$

- If system is overdamped ($\zeta > 1$), $G(j\omega)$ is a product of two first order systems.
- Magnitude when $0 < \zeta < 1$: $\text{dB} = -20\log_{10}\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}$
  - At low frequency, $\omega \ll \omega_n$, $\text{dB} \approx 0$.
  - At high frequency, $\omega \gg \omega_n$, $\text{dB} \approx -20\log_{10}\frac{\omega^2}{\omega_n^2} = -40\log_{10}\frac{\omega}{\omega_n}$ (-40 dB/decade)
- Phase when $0 < \zeta < 1: \phi = \tan^{-1}\dfrac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}$
  - At low frequency, $\phi \approx 0°$.
  - At high frequency, $\phi \approx -\tan^{-1}\left(-\frac{\omega_n}{\omega}\right) = -180°$.
  - At corner frequency, $\phi = -\tan^{-1}\left(\frac{2\zeta}{0}\right) = -90°$.





- Maximum $|G(j\omega)|$ occurs when $g(\omega) = \left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2$ is a minimum.

	Damping ratio, ζ	
	$0 \le \zeta \le 0.707$	$\zeta > 0.707$
Resonant freq.	$\omega_n\sqrt{1-2\zeta^2}$	None
Magnitude at resonant freq.	$\dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$	1

14.7 General Procedure for Drawing Bode Diagrams

- Decompose function into Bode Form.

$$\text{e.g. } G(j\omega) = \frac{10(j\omega+3)}{j\omega(j\omega+2)[(j\omega)^2+j\omega+2]}$$
$$= \frac{7.5\left(\frac{j\omega}{3}+1\right)}{j\omega\left(\frac{j\omega}{2}+1\right)\left[\frac{(j\omega)^2}{2}+\frac{j\omega}{2}+1\right]}$$

- Identify corner frequency for each factor and construct asymptotes,
- Composite magnitude-frequency and phase-frequency plots are superposition of all individual magnitude-frequency and phase-frequency plots.
- e.g.

Bode form	7.5	$(j\omega)^{-1}$	$1+j\dfrac{\omega}{3}$	$\left(1+j\dfrac{\omega}{2}\right)^{-1}$	$\left[1+j\dfrac{\omega}{2}+\dfrac{(j\omega)^2}{2}\right]^{-1}$
Corner frequency	-	-	$T=\dfrac{1}{3},\ \omega=3$	$T=\dfrac{1}{2},\ \omega=2$	$\omega_n^2=2,\ \omega=\omega_n=\sqrt{2}$

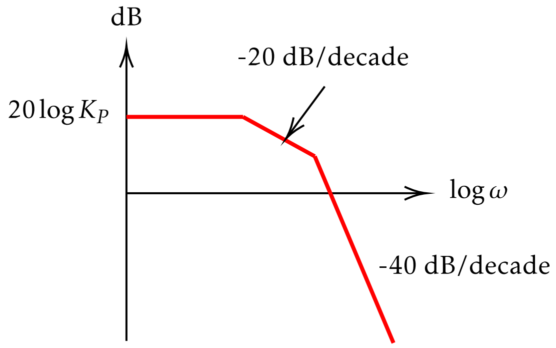
14.8 Minimum and Non-Minimum Phase Systems

	Minimum phase systems	**Non-minimum phase systems**
Poles and zeros in RHP	No	Yes
Range in phase angle is minimum	Yes	No
TF can be found from magnitude-frequency plot	Yes	No
Slope at $\omega=\infty$ of magnitude-frequency plot	$-20(p-q)$ dB/decade	
Phase angle at $\omega=0$	$-90°(q-p)$	NOT $-90°(q-p)$

where p is the degree of the numerator polynomial in the TF,
and q is the degree of the denominator polynomial in the TF.

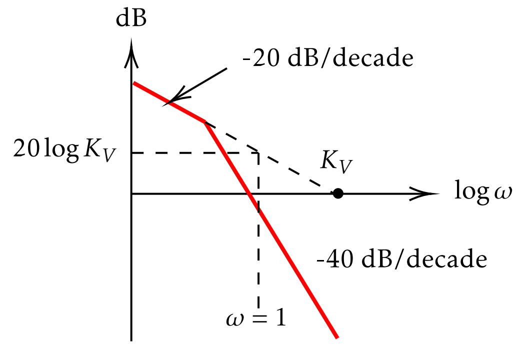
14.9 Interpreting Bode Diagrams

14.9.1 Type 0 System



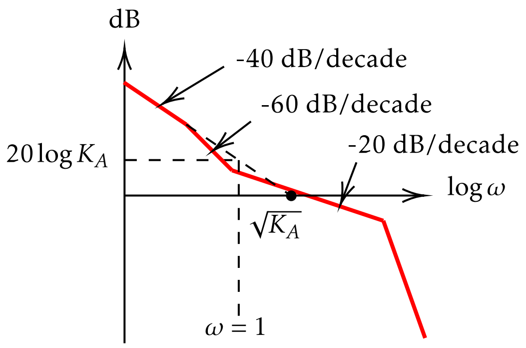
- Horizontal line at low frequencies with value $20\log K_P$ dB

14.9.2 Type 1 System



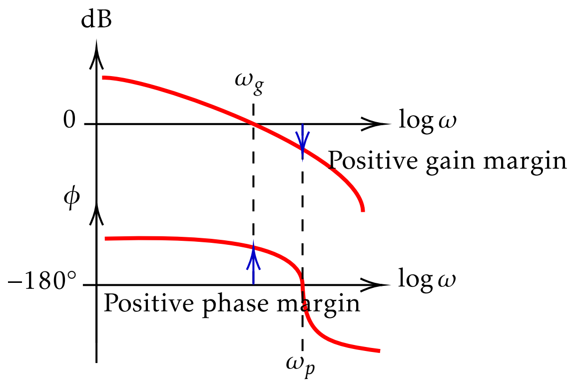
- Initial slope of -20 dB/decade
- Intersection with 0 dB line has frequency K_V

14.9.3 Type 2 System



- Initial slope of -40 dB/decade
- Intersection with 0 dB line has frequency $\sqrt{K_A}$

14.10 Stability Margins



- **Gain Margin**, K_g: amount of gain that can be raised before instability, measured from phase-frequency plot
- **Phase Margin**, γ: amount of additional phase lag before instability, measured from magnitude-frequency plot
- **Gain crossover frequency**, ω_g: frequency when dB = 0
- **Phase crossover frequency**, ω_p: frequency when $\phi = -180°$

- If phase plot does not intersect $-180°$, the gain margin is infinite.
- If magnitude plot does not intersect 0 dB, the phase margin is infinite.
- If both phase and gain margins are infinite, the system is absolutely stable.

15 W12: State Space Representation

- System organized as a set of first-order DEs
- ODEs do not need to be linear or time-invariant
- Easily extended to MIMO systems
- Types of representations:
  - Non-linear, time varying $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t) \quad \mathbf{y} = g(\mathbf{x}, \mathbf{u}, t)$
  - Linear time invariant $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \quad \mathbf{y} = \mathbf{Cx} + \mathbf{Du}$
  - Matrices and vectors:

$$\text{State Vector, } \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{Output Vector, } \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \qquad \text{Input Vector, } \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nr} \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mn} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} d_{11} & \cdots & d_{1r} \\ \vdots & \ddots & \vdots \\ d_{m1} & \cdots & d_{mr} \end{bmatrix}$$

State Matrix — Input Matrix — Output Matrix — Direct Transmission Matrix

where n is the order of the system,

m is the no. of outputs,

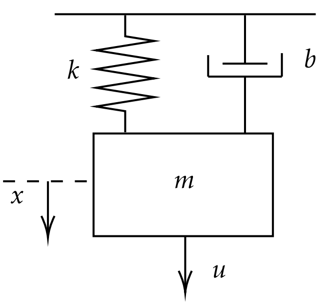
r is the no. of inputs.

- State variables and state space representation are not unique

15.1 Constructing State Space Models

- Define arbitrary state variables
  - Total order of system determines required number of state variables.

e.g.



Equation of motion : $m\ddot{x} + b\dot{x} + kx = u$

State variables : $x_1 = x, \quad x_2 = \dot{x}_1 = \dot{x}$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \ddot{x} = \frac{1}{m}(u - kx - b\dot{x}) = \frac{1}{m}(u - kx_1 - bx_2)$$

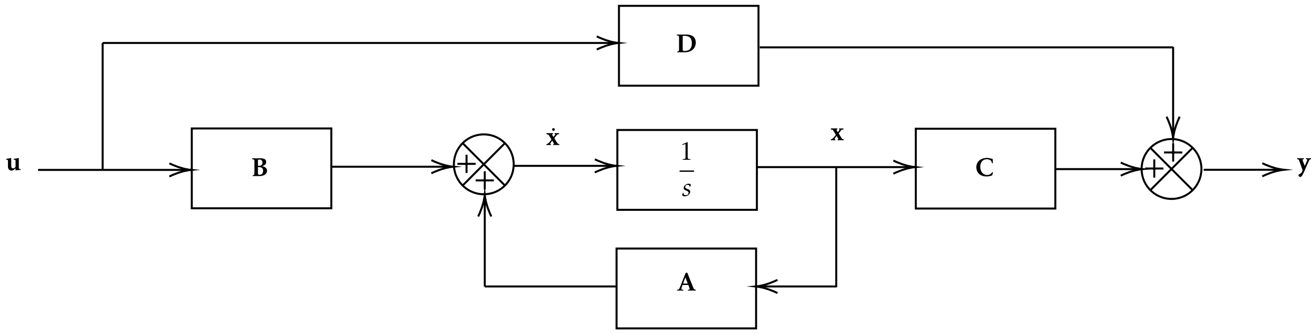
$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

15.2 Block Diagram Representation



$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

15.3 Transfer Matrix

Taking Laplace Transform:

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

Zero initial conditions:

$$\begin{aligned} s\mathbf{X}(s) &= \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) &= \mathbf{B}\mathbf{U}(s) \\ \mathbf{X}(\mathbf{s}) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \end{aligned}$$

Substituting $\mathbf{X}$(s) into $\mathbf{Y}$(s):

$$\begin{aligned} \mathbf{Y}(s) &= \mathbf{C}\left[(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)\right] + \mathbf{D}\mathbf{U}(s) \\ &= \mathbf{C}\left[(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s) \\ \Rightarrow \mathbf{G}(s) &= \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} \\ &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \end{aligned}$$

15.4 Eigenvalues and Characteristic Equation

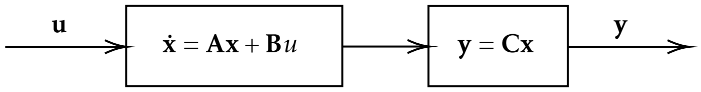
- Characteristic equation: $|s\mathbf{I} - \mathbf{A}| = 0$
- Eigenvalues of **A** are roots of the characteristic equation
- Eigenvalues are not affected by linear transformations applied to **A**

15.5 Stability Analysis in State-Space

- LTI State-Space System is stable if all eigenvalues have negative real parts
- Types of stability:
  - Absolute/ internal stability: LHP eigenvalues/poles
  - Neutral stability: Non-repeated $j\omega$ axis poles
  - Unstable: Repeated $j\omega$ axis poles, RHP eigenvalues/poles

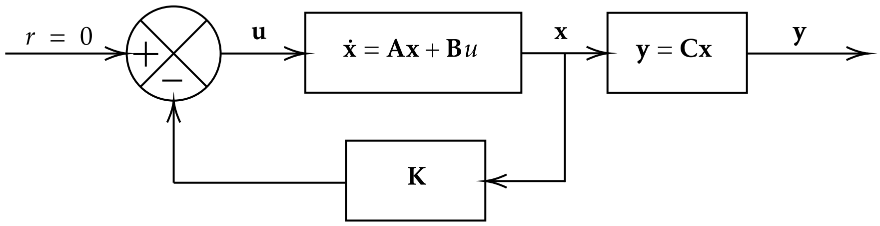
16 W13: Full State Feedback Control

- State space representation in W12 is an example of Open Loop Control, where $\mathbf{D} = 0$



$$\text{Characteristic equation: } |s\mathbf{I} - \mathbf{A}| = 0$$

- Full state feedback control can be achieved by adding feedback to the input:



$$\text{Characteristic equation: } |s\mathbf{I} - (\mathbf{A} - \mathbf{BK})| = 0$$

$$\text{Regulator control, } r = 0$$

$$\text{Control Law: } u = -\mathbf{Kx} = -k_1x_1 - k_2x_2 - k_3x_3 - \cdots - k_nx_n$$

16.1 Motivation

- Location of open loop poles/eigenvalues of state-space system dictate stability and transient response
- Using feedback control, the closed loop poles can be placed where we want them to be in the s-plane.

16.2 Pole Placement Controller Design

1. Check if possible to design such a controller.
   - Covered in 30.114
   - If system can be designed, gain matrix $\mathbf{K} = [k_1 \ k_2 \ \cdots \ k_n]$.
   - Control Law: $u = -\mathbf{Kx}$
2. Find desired closed loop performance using system parameters.
   - First Order System: Time constant T, where $p = -\dfrac{1}{T}$
   - Second Order System: Damping ratio ζ and natural frequency ω_n

$$(s - p_1)(s - p_2) = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\text{where } \zeta = \left[1 + \left(\frac{\pi}{M_p}\right)^2\right]^{-0.5}, \ \omega_n = \frac{\pi}{t_p\sqrt{1-\zeta^2}}.$$

   - Underdamped ($0 < \zeta < 1$), critically damped ($\zeta = 1$), overdamped ($\zeta > 1$)
3. Find required controller gain $\mathbf{K}$ by equating desired CE with closed loop CE.

$$\text{Desired CE} = |s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = 0$$

16.3 Additional Notes

- Regulator system: constant reference input
- Control system: time-varying reference input
- Placing poles increasingly far away from the $j\omega$ axis results in exponentially larger input signals
  - System may become linear
  - Require larger and heavier actuators
- Alternative method: Quadratic Optimal Control (covered in 30.114)
- Gain matrix $\mathbf{K}$ is not unique to systems, dependent on location of closed loop poles
- May be optimal to use computer simulations instead for higher order systems