Semaine 11

Questions pour réviser

1/3) 1. On calcule

$$\hat{\vec{p}}\,\psi = -i\hbar\,\vec{\mathrm{grad}}\,\frac{1}{\sqrt{V}}\,e^{i\vec{k}.\vec{x}} = -\frac{i\hbar}{\sqrt{V}}\left(\frac{\partial}{\partial x}\left(e^{i\vec{k}\vec{x}}\right)\vec{e_x} + \frac{\partial}{\partial y}\left(e^{i\vec{k}\vec{x}}\right)\vec{e_y} + \frac{\partial}{\partial z}\left(e^{i\vec{k}\vec{x}}\right)\vec{e_z}\right)$$

$$= -\frac{i\hbar}{\sqrt{V}}\Big(\underbrace{i k_x\,\vec{e_x} + i k_y\,\vec{e_y} + i k_z\,\vec{e_z}}_{i\vec{k}}\Big)\,e^{i\vec{k}\vec{x}}$$

$\vec{k}.\vec{x} = k_x x + k_y y + k_z z$

$$\hat{\vec{p}}\,\psi = \hbar\vec{k}\,\frac{e^{i\vec{k}\vec{x}}}{\sqrt{V}}$$

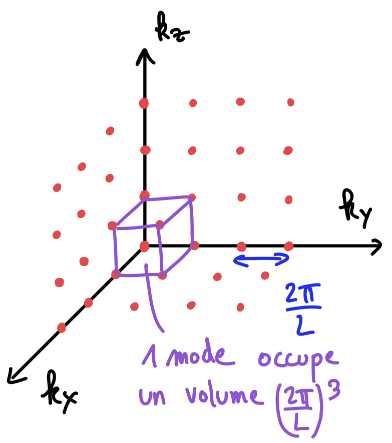
$$\boxed{\hat{\vec{p}}\,\psi = \hbar\vec{k}\,\psi}$$

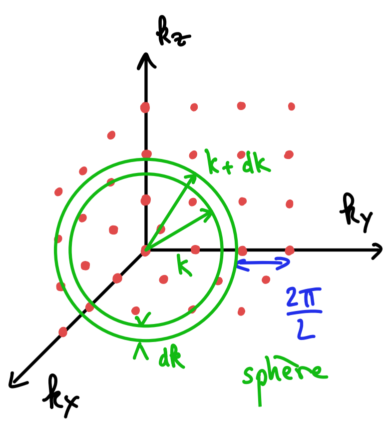
donc ψ est fonction propre de $\hat{\vec{p}}$ avec valeur propre $\vec{p} = \hbar\vec{k}$

2. On déduit $\hat{H}\psi = |\hat{\vec{p}}|\,c\,\psi = \underbrace{|\hbar\vec{k}|\,c}_{E = \hbar k c = \hbar\omega}\,.\,\psi$

donc ψ est fonction propre du Hamiltonien. L'énergie est $E = \hbar k c = \hbar\omega$.

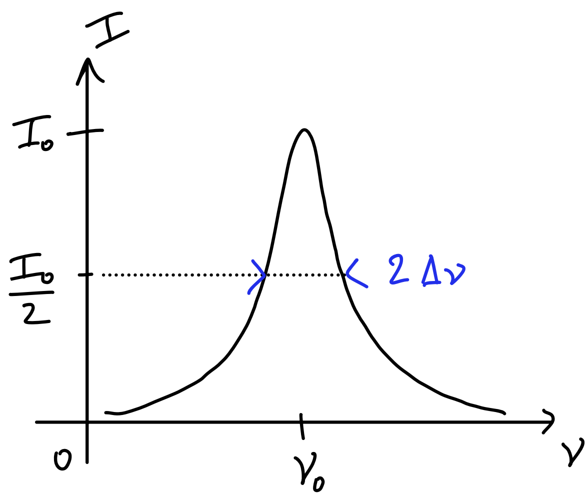
2/3) On compte le nombre d'états



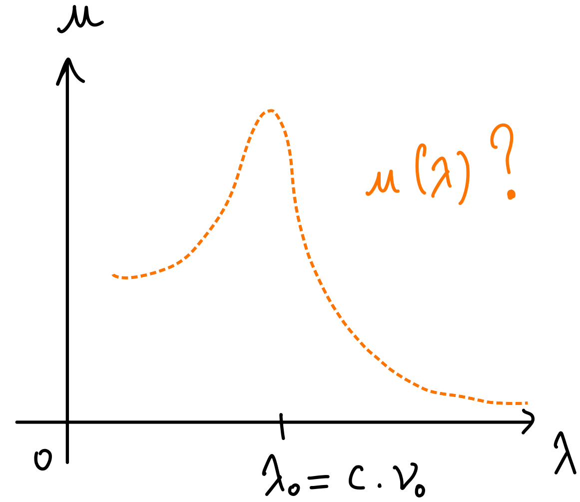


$$\underbrace{g(k)}_{\text{densité d'états en vecteur d'onde } k}\,dk \equiv \text{nombre de modes entre } k \text{ et } k+dk = \underbrace{\text{densité de modes}}_{\underbrace{2}_{\text{spin}} \times \frac{1}{\left(\frac{2\pi}{L}\right)^3}} \times \underbrace{\text{volume de la coquille}}_{4\pi k^2 dk} = L^3 \, \frac{8\pi k^2 dk}{(2\pi)^3}$$

3/3) Comment passer de $I(\nu)$ à $u(\lambda)$?



$\longrightarrow$



On écrit la conservation de l'énergie

$$I(\nu)\, d\nu = u(\lambda)\,|d\lambda| \quad \text{avec} \quad \nu = \frac{c}{\lambda} \,, \quad d\nu = \frac{c}{\lambda^2}\,|d\lambda|$$

$$\parallel$$

$$I\left(\frac{c}{\lambda}\right) \cdot \frac{c}{\lambda^2}\,|d\lambda|$$

donc $\boxed{u(\lambda) = \frac{c}{\lambda^2}\, I\left(\frac{c}{\lambda}\right)}$

par exemple, avec une Lorentzienne $I(\nu) = \dfrac{I_0}{1 + \left(\frac{\nu - \nu_0}{\Delta\nu}\right)^2}$

on trouve $u(\lambda) = \dfrac{c}{\lambda^2} \, \dfrac{I_0}{1 + \left(\dfrac{\frac{c}{\lambda} - \frac{c}{\lambda_0}}{\Delta\nu}\right)^2}$

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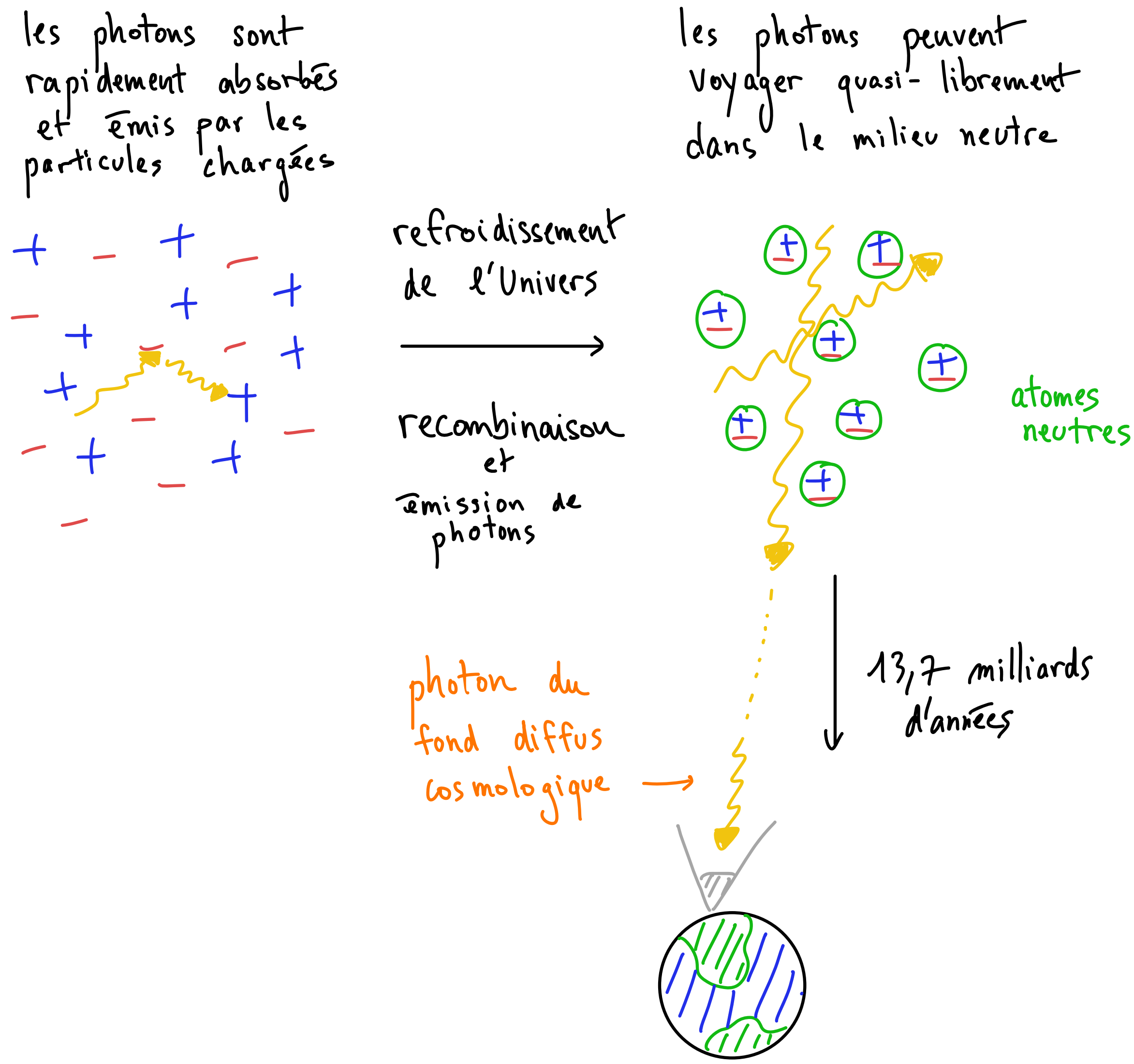
Big Bang
inflation
formation des noyaux



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On cherche le maximum dans la loi de Planck

$$0 = \frac{\partial u}{\partial \lambda}(\lambda,T) = 8\pi hc\left[-\frac{5}{\lambda^6}\,\frac{1}{e^{\frac{hc}{\lambda k_BT}}-1} - \frac{1}{\lambda^5}\,\frac{\left(-\frac{hc}{\lambda^2 k_BT}\right)e^{\frac{hc}{\lambda k_BT}}}{\left(e^{\frac{hc}{\lambda k_BT}}-1\right)^2}\right]$$

$$0 = \frac{8\pi hc}{e^x-1}\cdot\frac{1}{\lambda^6}\cdot\left[-5+\frac{x\,e^x}{e^x-1}\right]$$

on note $x \equiv \frac{hc}{\lambda k_BT}$

x est solution de $5(e^x-1) = x\,e^x \Leftrightarrow 5(1-e^{-x}) = x$

$$\Leftrightarrow 1-\frac{x}{5} = e^{-x}$$

La solution (numérique) est $x \approx 4{,}965$.

D'où $x = 4{,}965 = \dfrac{hc}{k_B \lambda_{max} T}$ $\longrightarrow$ $\boxed{\lambda_{max} T = \dfrac{hc}{k_B x} = cte = 2{,}897 \times 10^{-3}\ K.m}$

loi de Wien

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On calcule l'énergie moyenne

$$\frac{\langle E \rangle}{V} = \frac{\hbar}{\pi^2 c^3} \int_0^{\infty} d\omega \frac{\omega^3}{e^{\beta\hbar\omega}-1} = \frac{\hbar}{\pi^2 c^3} \int_0^{\infty} \frac{dx}{(\beta\hbar)^4} \frac{x^3}{e^x-1}$$

$x = \beta\hbar\omega$

$dx = \beta\hbar\, d\omega$

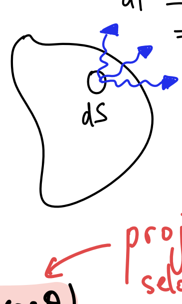
$$= \frac{1}{\pi^2 c^3 \hbar^3} (k_B T)^4 \underbrace{\int_0^{\infty} dx \frac{x^3}{e^x-1}}_{\frac{\pi^4}{15}} = \frac{\pi^2}{15} \cdot \frac{k_B^4}{c^3 \hbar^3} \underline{T^4}$$

$$\boxed{\frac{\langle E \rangle}{V} = \frac{\pi^2 k_B^4}{15 \hbar^3 c^3} T^4}$$

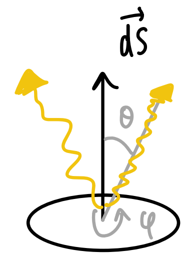
puissance en W

$dP = dS . M$

$= dS\, \sigma T^4$



Explication du facteur $\frac{1}{4}$



flux d'énergie à travers dS

$$= u \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \underbrace{\sin\theta\,(c\cos\theta)}_{c \cdot \frac{\sin 2\theta}{2}}$$

projection selon la normale $\vec{dS}$

$$= \frac{cu}{4\pi} \cdot 2\pi \cdot \left[-\frac{\cos 2\theta}{4}\right]_0^{\frac{\pi}{2}} = \frac{cu}{2}\left[\frac{1}{4} + \frac{1}{4}\right] = \frac{cu}{4}$$

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$$\langle n_i \rangle = \frac{1}{\beta} \frac{\partial}{\partial \mu} \left(\ln Z_{G,i} \right) = \frac{1}{\beta} \frac{1}{Z_{G,i}} \frac{\partial}{\partial \mu} \left(Z_{G,i} \right)$$

$$= \frac{1}{\cancel{\beta}} \cancel{\left(1 - e^{\beta(\mu-\varepsilon_i)}\right)} \cancel{(-1)} \frac{\left(\cancel{-} \cancel{\beta} e^{\beta(\mu-\varepsilon_i)}\right)}{\left(1 - e^{\beta(\mu-\varepsilon_i)}\right)^{\cancel{2}}} = \frac{e^{\beta(\mu-\varepsilon_i)}}{1 - e^{\beta(\mu-\varepsilon_i)}}$$

$$\boxed{\langle n_i \rangle = \frac{1}{e^{\beta(\varepsilon_i-\mu)} - 1}}$$

statistique de Bose-Einstein