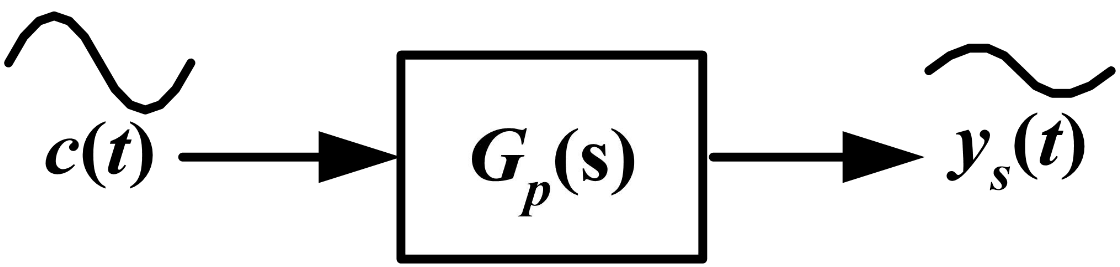
Chapter 13
Frequency Response Analysis and Control System Design

Process Exposed to a Sinusoidal Input



Force dynamic process with A sin ωt , $U(s) = \dfrac{A\omega}{s^2 + \omega^2}$

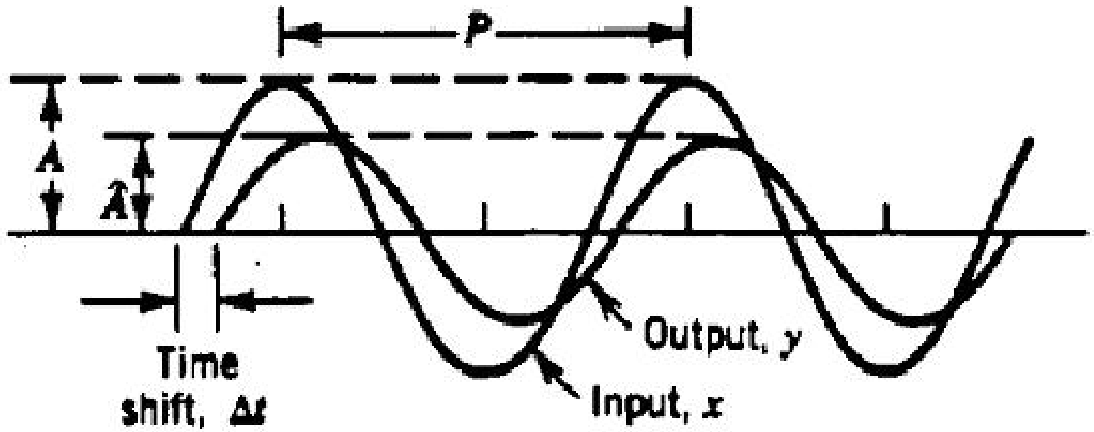
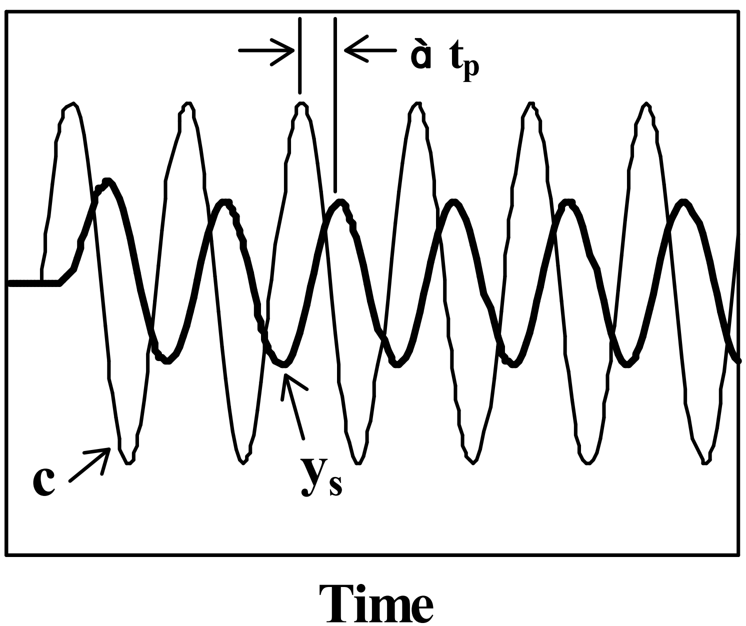
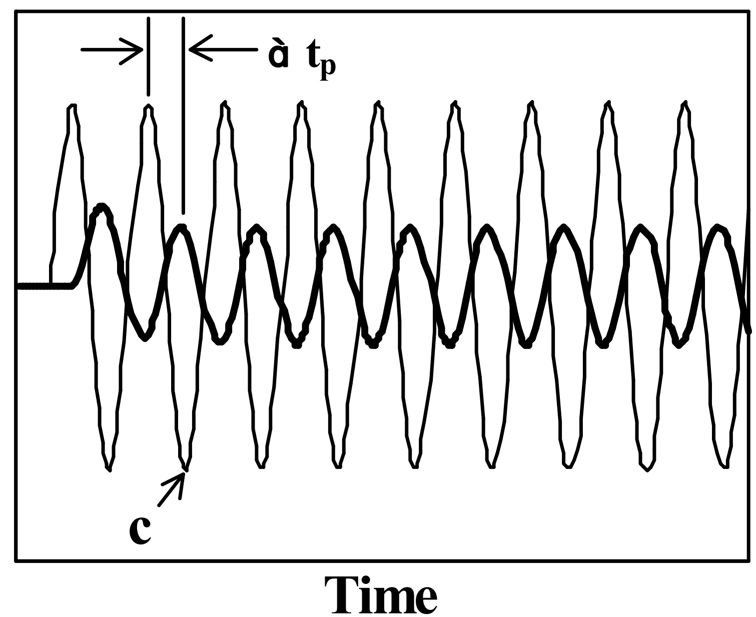


Figure 13.1.

Attenuation and time shift between input and output sine waves ($K = 1$). The phase angle ϕ of the output signal is given by $\phi = -\Delta t/P \times 360$, where Δt is the time (perioc 13.1 and P is the period of oscillation.

Effect of Frequency on A_r and ϕ





$$G(s) = \frac{k}{(s+s_1)(s+s_2)\cdots(s+s_n)} \qquad U(s) = \frac{A\omega}{s^2+\omega^2}$$

$$Y(s) = \frac{k}{(s+s_1)(s+s_2)\cdots(s+s_n)} \cdot \frac{A\omega}{(s^2+\omega^2)} = \sum_{i=1}^{n} \frac{\lambda_i}{s+s_i} + \frac{a}{s+jw} + \frac{b}{s-jw}$$

$$L^{-1}(Y(s)) = y(t) = \sum_{i=1}^{n} \lambda_i e^{-s_i t} + ae^{-jwt} + be^{jwt}$$

if $\quad -s_i = \alpha_i + \beta_i j, \quad \alpha_i < 0 \qquad$ **then** $\quad t \to \infty, \quad \sum_{i=1}^{n} \lambda_i e^{-s_i t} \to 0$

$$\lim_{t\to\infty} y(t) = ae^{-jwt} + be^{jwt}$$

$$a = G(s) \cdot \frac{A\omega}{s-j\omega}\bigg|_{s=-j\omega} \qquad b = G(s) \cdot \frac{A\omega}{s+j\omega}\bigg|_{s=j\omega}$$

$$= G(-j\omega) \cdot \frac{A\omega}{-2j\omega} \qquad = G(j\omega) \cdot \frac{A\omega}{2j\omega}$$

$$= G(-j\omega) \cdot \frac{A}{2} j \qquad = -G(j\omega) \cdot \frac{A}{2} j$$

$$y(t) = G(-j\omega)\frac{A}{2} j \cdot e^{-jwt} - G(j\omega) \cdot \frac{A}{2} j \cdot e^{jwt}$$

$$= \frac{A}{2} j[G(-j\omega)e^{-jwt} - G(j\omega)e^{jwt}]$$

$$G(s) = \frac{k}{(s+s_1)(s+s_2)\cdots(s+s_n)} \qquad U(s) = \frac{A\omega}{s^2+\omega^2}$$

$$\lim_{t\to\infty} y(t) = ae^{-jwt} + be^{jwt}$$

$$y(t) = G(-j\omega)\frac{A}{2}j\cdot e^{-jwt} - G(j\omega)\cdot\frac{A}{2}j\cdot e^{jwt}$$

$$= \frac{A}{2}j[G(-j\omega)e^{-jwt} - G(j\omega)e^{jwt}]$$

let $G(j\omega) = \left|G(j\omega)\right| e^{j\phi}, \quad G(-j\omega) = \left|G(j\omega)\right| e^{-j\phi}$

$$y(t) = \frac{A}{2}j[\left|G(j\omega)\right| e^{-j\phi}e^{-j\omega t} - \left|G(j\omega)\right| e^{j\phi}e^{j\omega t}]$$

$$= \frac{A}{2}j\left|G(j\omega)\right|(e^{-j(\phi+\omega t)} - e^{\mathrm{j}(\phi+\omega t)})$$

$$= \frac{A}{2}j\left|G(j\omega)\right|[-2j\sin(\phi+\omega t)]$$

$$= \mathrm{A}\left|G(j\omega)\right|\sin(\phi+\omega t)$$

Input: $A \sin \omega t$

Output: $\hat{A} \sin(\omega t + \phi)$

$\hat{A} / A$ is the normalized amplitude ratio (*AR*)

ϕ is the phase angle, response angle (*RA*)

AR and ϕ are functions of ω

Assume *G(s)* known and let

$$s = j\omega \qquad G(j\omega) = K_1 + K_2 j$$

$$|G| = AR = \sqrt{K_1^2 + K_2^2}$$

$$\phi = \angle G = \arctan \frac{K_2}{K_1}$$

Example

$$G(s)=\frac{1}{\tau s+1}$$

$$G(j\omega)=\frac{1}{1+\tau j\omega}\cdot\frac{1-\tau j\omega}{1-\tau j\omega} \qquad (j^2=-1)$$

$$G(j\omega)=\underbrace{\frac{1}{1+\omega^2\tau^2}}_{\mathbf{K_1}}-\underbrace{\frac{\omega\tau}{1+\omega^2\tau^2}}_{\mathbf{K_2}}\,j$$

$$|G|=\frac{1}{\sqrt{1+\omega^2\tau^2}}$$

$$\phi=-\arctan(\omega\tau)$$

as $\omega\to\infty,\quad \phi\to-90^0$

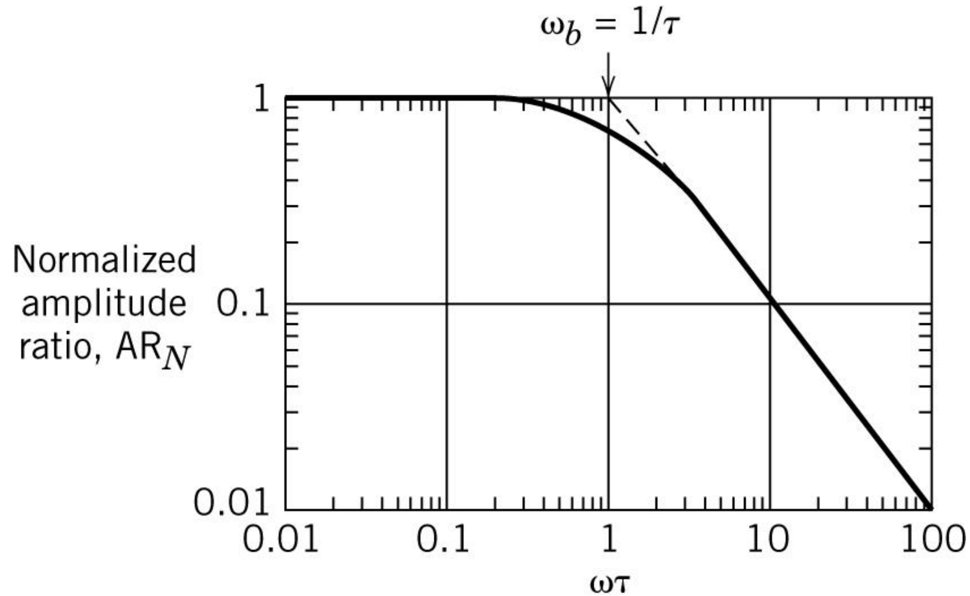
Use a Bode plot to illustrate frequency response

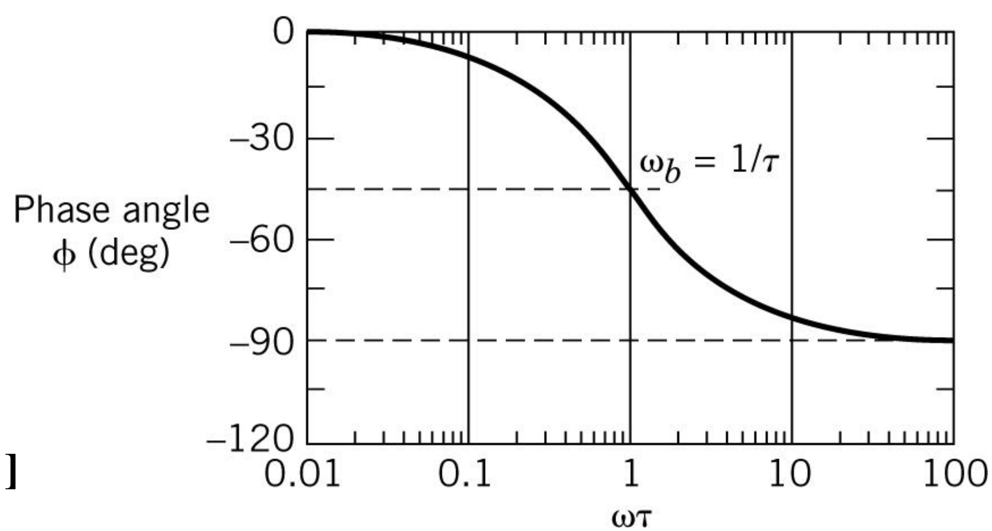
(plot of log $|G|$ vs. log ω and ϕ vs. log ω)

$$|G| = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

$$\phi = -\arctan(\omega\tau)$$

as $\omega \to \infty, \quad \phi \to -90^0$





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$$|G| = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

$$\phi = -\arctan(\omega\tau)$$

$$\text{as} \quad \omega \to \infty, \quad \phi \to -90^0$$

Use a Bode plot to illustrate frequency response

(plot of log $|G|$ vs. log ω and ϕ vs. log ω)

log coordinates:

$$G = G_1 \cdot G_2 \cdot G_3$$

$$|G| = |G_1| \cdot |G_2| \cdot |G_3|$$

$$\log|G| = \log|G_1| + \log|G_2| + \log|G_3|$$

$$\angle G = \angle G_1 + \angle G_2 + \angle G_3$$

$$G = \frac{G_1}{G_2}$$

$$\log|G| = \log|G_1| - \log|G_2|$$

$$\angle G = \angle G_1 - \angle G_2$$

Time Delay

Its frequency response characteristics can be obtained by substituting $s = j\omega$

$$G(j\omega) = e^{-j\omega\theta} \qquad (13\text{-}53)$$

which can be written in rational form by substitution of the Euler identity,

$$G(j\omega) = e^{-j\omega\theta} = \cos\omega\theta - j\sin\omega\theta \qquad (13\text{-}54)$$

From (13-54)

$$\text{AR} = |G(j\omega)| = \sqrt{\cos^2\omega\theta + \sin^2\omega\theta} = 1 \qquad (13\text{-}55)$$

$$\varphi = \angle G(j\omega) = \tan^{-1}\left(-\frac{\sin\omega\theta}{\cos\omega\theta}\right)$$

or

$$\varphi = -\omega\theta \qquad (13\text{-}56)$$

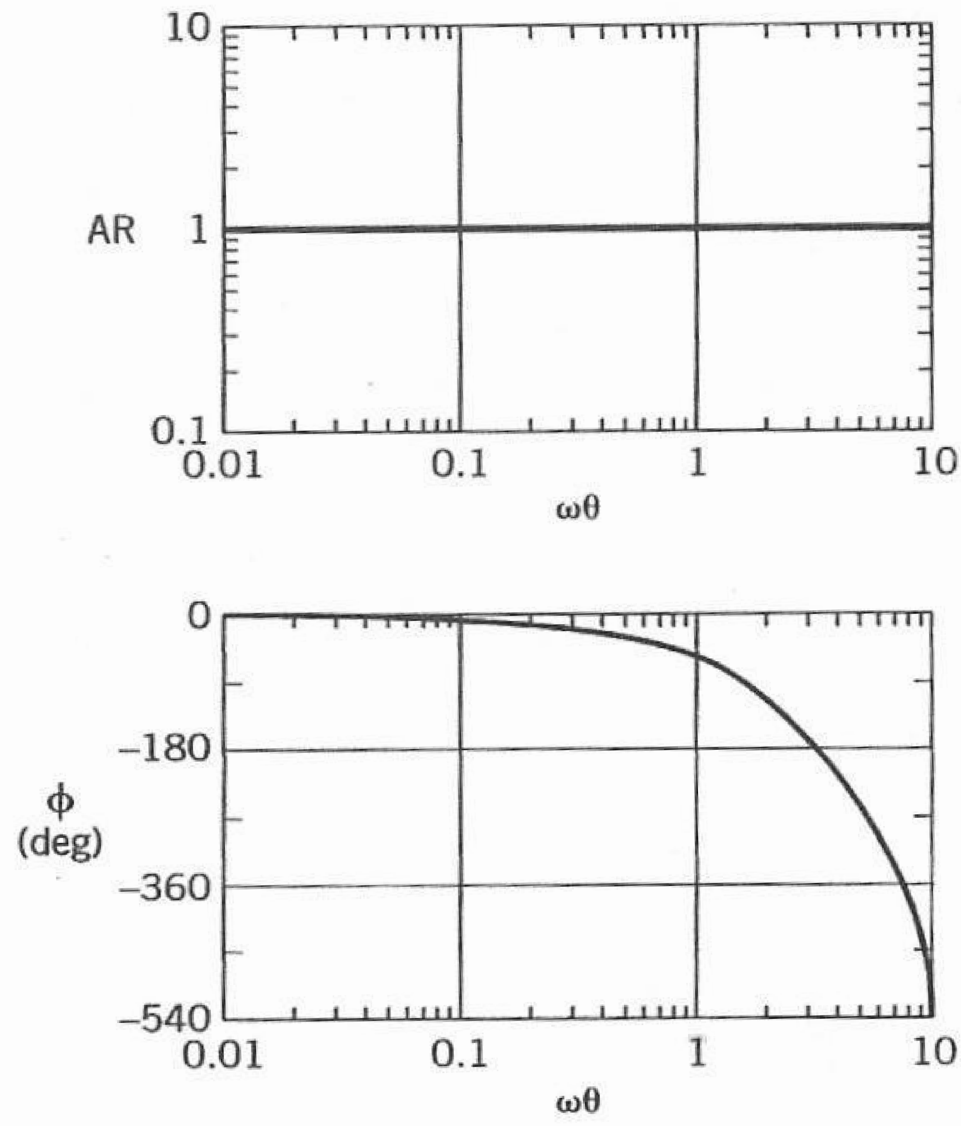
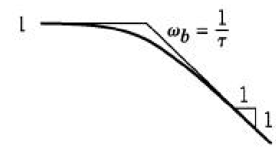
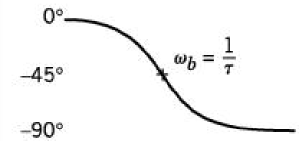
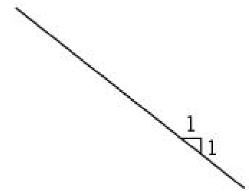
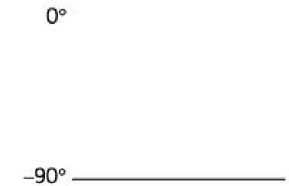
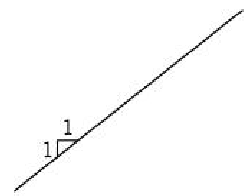
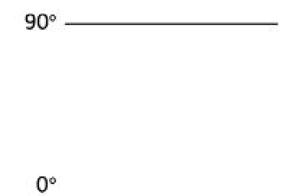
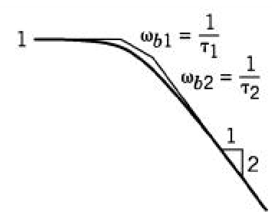
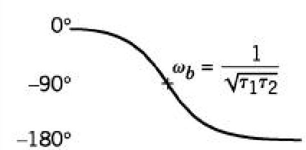
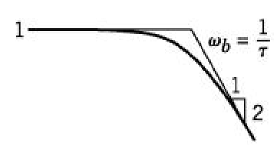
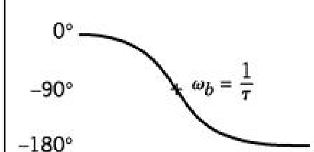
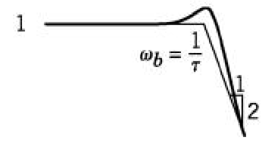
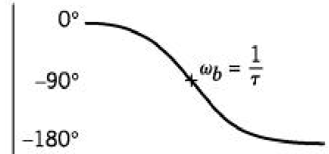
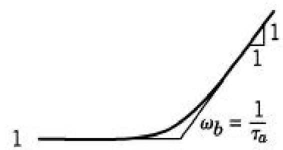
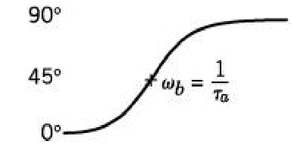
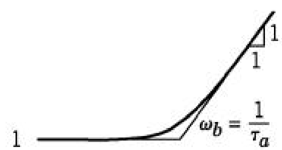
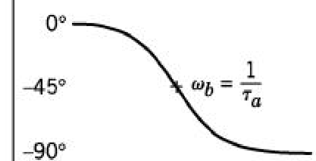
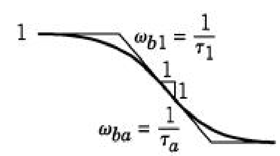
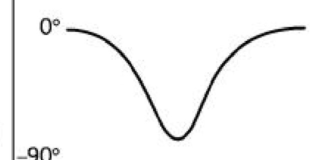
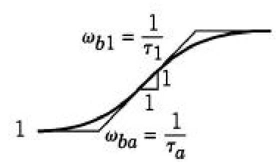
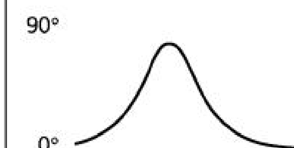
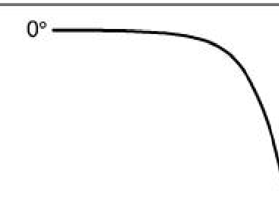


Figure 13.4 Bode diagram for a time delay, $e^{-\theta s}$.

Table 14.2 Frequency Response Characteristics of Important Process Transfer Functions

Transfer Function	$G(s)$	$AR = \lvert G(j\omega)\rvert$	Plot of log AR_N vs. log ω	$\phi = \angle G(j\omega)$	Plot of ϕ vs. log ω
1. First-order	$\frac{K}{\tau s + 1}$	$\frac{K}{\sqrt{(\omega\tau)^2+1}}$		$-\tan^{-1}(\omega\tau)$	
2. Integrator	$\frac{K}{s}$	$\frac{K}{\omega}$		$-90°$	
3. Derivative	Ks	$K\omega$		$+90°$	
4. Overdamped second-order	$\frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}$	$\frac{K}{\sqrt{(\omega\tau_1)^2+1}\sqrt{(\omega\tau_2)^2+1}}$		$-\tan^{-1}(\omega\tau_1) - \tan^{-1}(\omega\tau_2)$	
5. Critically damped second-order	$\frac{K}{(\tau s + 1)^2}$	$\frac{K}{(\omega\tau)^2 + 1}$		$-2\tan^{-1}(\omega\tau)$	

6. Underdamped second-order	$\frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$	$\frac{K}{\sqrt{(1-(\omega\tau_1)^2)^2+(2\zeta\omega\tau)^2}}$		$-\tan^{-1}\left[\frac{2\zeta\omega\tau}{1-(\omega\tau)^2}\right]$	
7. Left-half plane (positive) zero	$K(\tau_a s + 1)$	$K\sqrt{(\omega\tau_a)^2 + 1}$		$+\tan^{-1}(\omega\tau_a)$	
8. Right-half plane (negative) zero	$-\tau_a s + 1$	$K\sqrt{(\omega\tau_a)^2 + 1}$		$-\tan^{-1}(\omega\tau_a)$	
9. Lead–lag unit ($\tau_a < \tau_1$)	$K\frac{\tau_a s+1}{\tau_1 s+1}$	$K\frac{\sqrt{(\omega\tau_a)^2+1}}{\sqrt{(\omega\tau_1)^2+1}}$		$+\tan^{-1}(\omega\tau_a) - \tan^{-1}(\omega\tau_1)$	
10. Lead–lag unit ($\tau_a > \tau_1$)	$K\frac{\tau_a s+1}{\tau_1 s+1}$	$K\frac{\sqrt{(\omega\tau_a)^2+1}}{\sqrt{(\omega\tau_1)^2+1}}$		$+\tan^{-1}(\omega\tau_a) - \tan^{-1}(\omega\tau_2)$	
11. Time delay	$Ke^{-\theta s}$	K		$-\omega_\theta$	

Frequency Response Characteristics of Feedback Controllers

Proportional Controller. **Consider a proportional controller with positive gain**

$$G_c(s) = K_c \qquad (13\text{-}57)$$

In this case $\left|G_c(j\omega)\right| = K_c$ **which is independent of ω. Therefore,**

$$\text{AR}_c = K_c \qquad (13\text{-}58)$$

and

$$\varphi_c = 0^\circ \qquad (13\text{-}59)$$

Proportional-Integral Controller. **A proportional-integral (PI) controller has the transfer function (cf. Eq. 8-9),**

$$G_c(s) = K_c\left(1+\frac{1}{\tau_I s}\right) = K_c\left(\frac{\tau_I s+1}{\tau_I s}\right) \qquad (13\text{-}60)$$

Substitute s=jω:

$$G_c(j\omega) = K_c\left(1+\frac{1}{\tau_I j\omega}\right) = K_c\left(\frac{j\omega\tau_I+1}{j\omega\tau_I}\right) = K_c\left(1-\frac{1}{\tau_I\omega}j\right)$$

Thus, the amplitude ratio and phase angle are:

$$\mathrm{AR}_c = \left|G_c(j\omega)\right| = K_c\sqrt{1+\frac{1}{(\omega\tau_I)^2}} = K_c\frac{\sqrt{(\omega\tau_I)^2+1}}{\omega\tau_I} \qquad (13\text{-}62)$$

$$\varphi_c = \angle G_c(j\omega) = \tan^{-1}(-1/\omega\tau_I) = \tan^{-1}(\omega\tau_I) - 90^\circ \qquad (13\text{-}63)$$

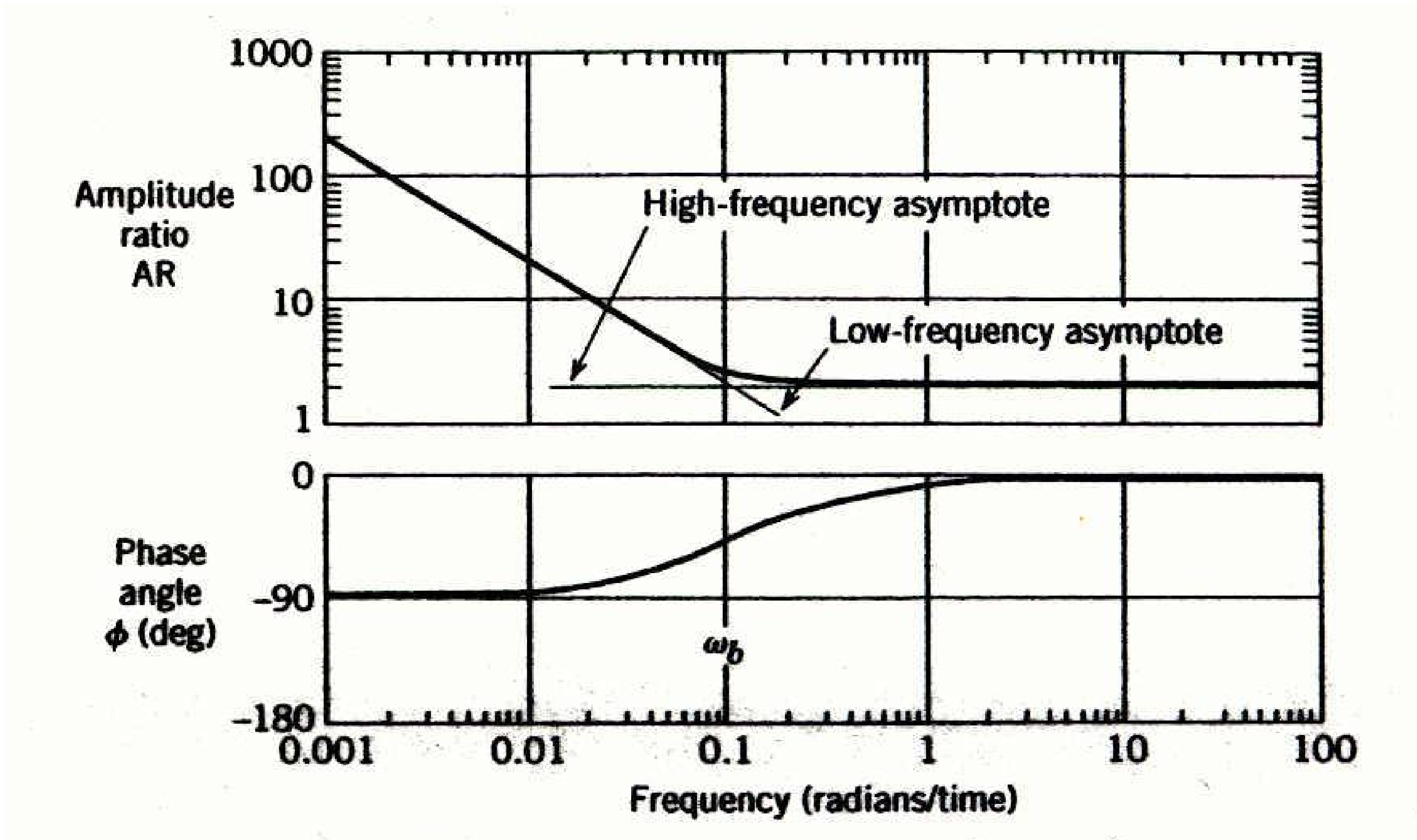


Figure 13.9 Bode plot of a PI controller, $G_c(s) = 2\left(\frac{10s+1}{10s}\right)$

Ideal Proportional-Derivative Controller. **For the ideal proportional-derivative (PD) controller (cf. Eq. 8-11)**

$$G_c(s) = K_c(1 + \tau_D s) \qquad (13\text{-}64)$$

The frequency response characteristics are similar to those of a LHP zero:

$$\text{AR}_c = K_c\sqrt{(\omega\tau_D)^2 + 1} \qquad (13\text{-}65)$$

$$\varphi = \tan^{-1}(\omega\tau_D) \qquad (13\text{-}66)$$

Proportional-Derivative Controller with Filter. **The PD controller is most often realized by the transfer function**

$$G_c(s) = K_c\left(\frac{\tau_D s + 1}{\alpha\tau_D s + 1}\right) \qquad (13\text{-}67)$$

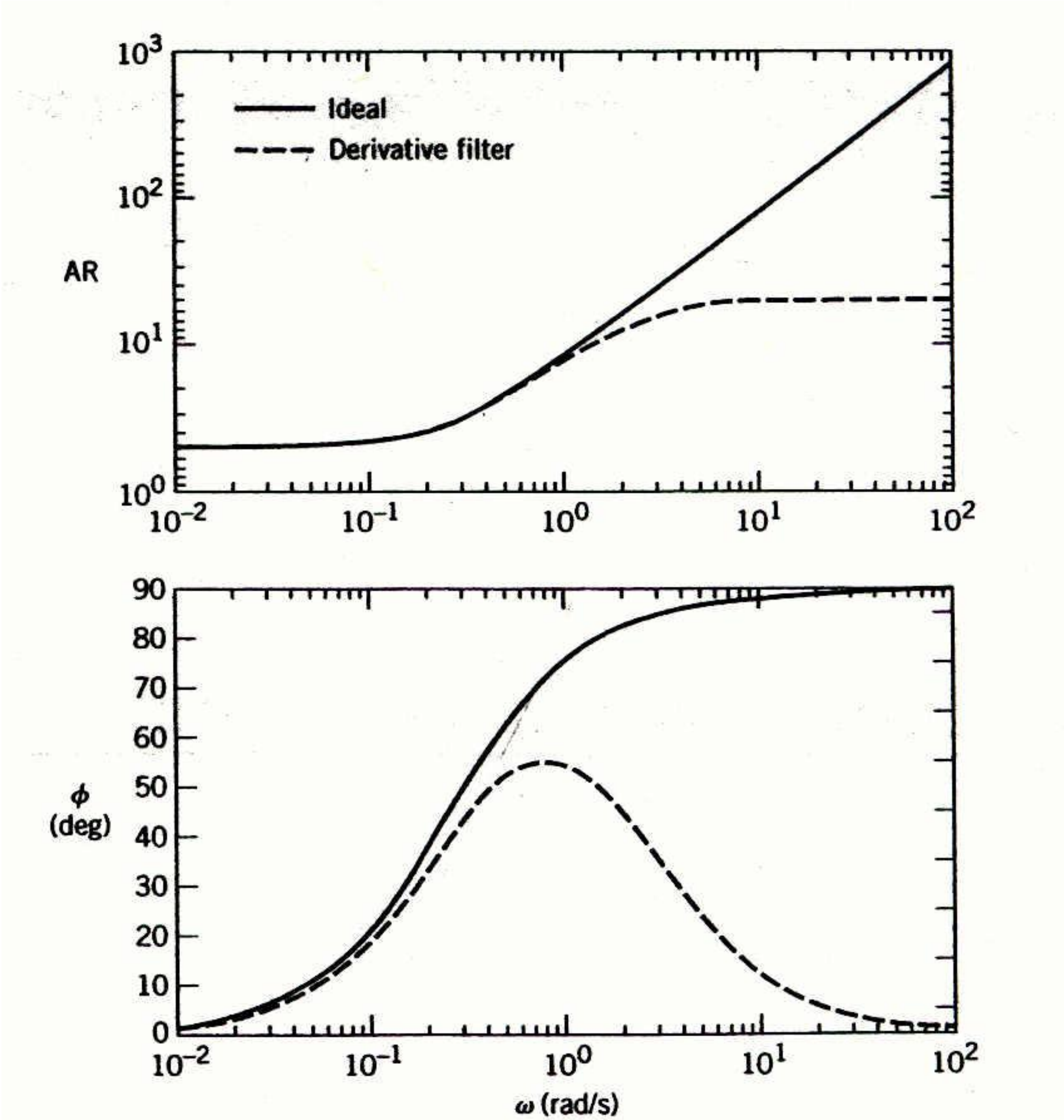


Figure 13.10 Bode plots of an ideal PD controller and a PD controller with derivative filter.

Ideal: $G_c(s) = 2(4s+1)$

With Derivative Filter:

$$G_c(s) = 2\left(\frac{4s+1}{0.4s+1}\right)$$

PID Controller Forms

•*Parallel PID Controller*. The simplest form in Ch. 8 is

$$G_c(s) = K_c\left(1 + \frac{1}{\tau_1 s} + \tau_D s\right)$$

***Series PID Controller*. The simplest version of the series PID controller is**

$$G_c(s) = K_c\left(\frac{\tau_1 s + 1}{\tau_1 s}\right)\left(\tau_D s + 1\right) \qquad (13\text{-}73)$$

Series PID Controller with a Derivative Filter.

$$G_c(s) = K_c\left(\frac{\tau_1 s + 1}{\tau_1 s}\right)\left(\frac{\tau_D s + 1}{\alpha \tau_D s + 1}\right)$$

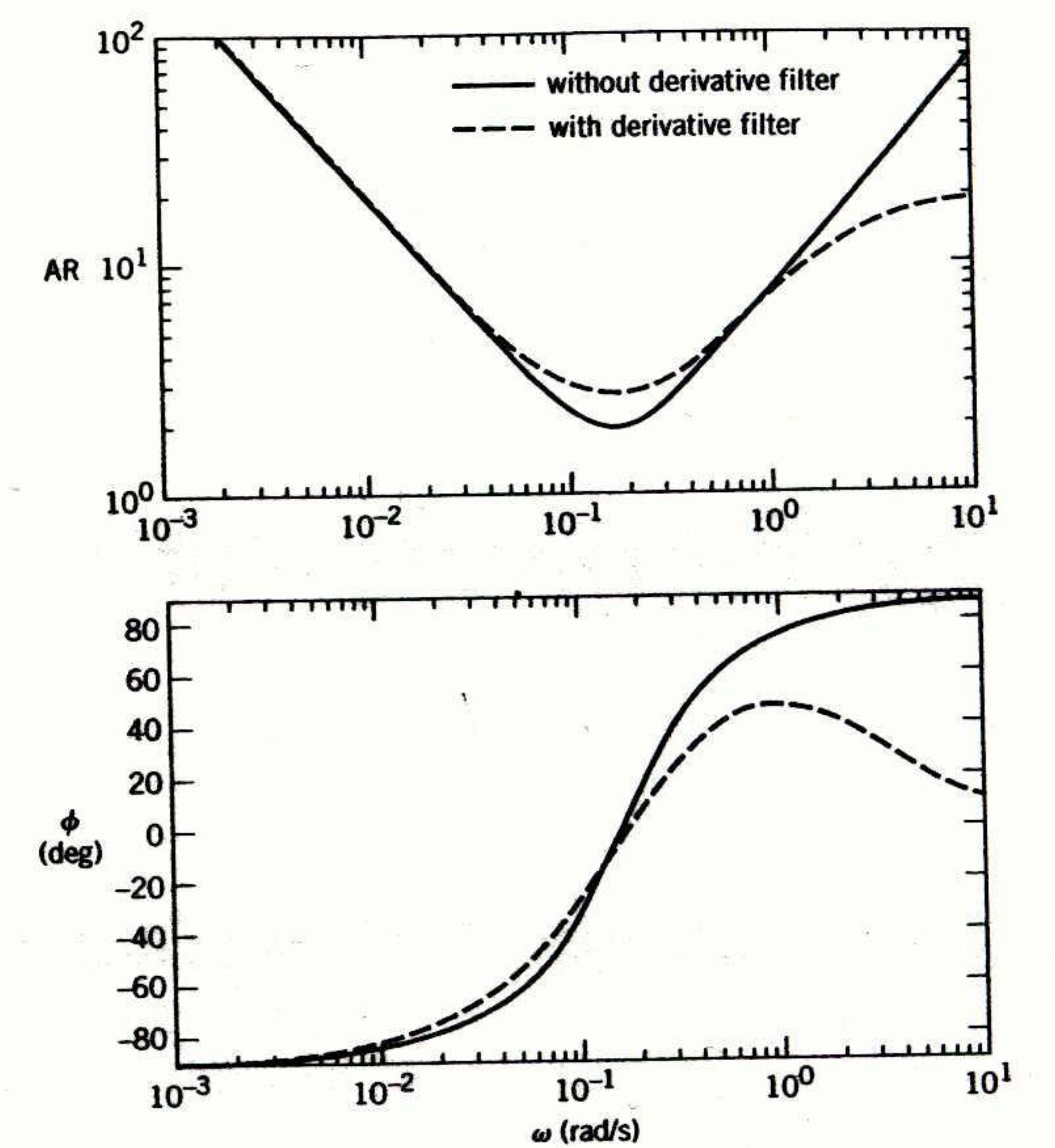


Figure 13.11 Bode plots of ideal parallel PID controller and series PID controller with derivative filter (α = 1).

Idea parallel:

$$G_c(s) = 2\left(1 + \frac{1}{10s} + 4s\right)$$

Series with Derivative Filter:

$$G_c(s) = 2\left(\frac{10s+1}{10s}\right)\left(\frac{4s+1}{0.4s+1}\right)$$

Controller Design by Frequency Response - Stability Margins

Analyze $G_{OL}(s) = G_C G_V G_P G_M$ (open loop gain)

Three methods in use:

(1) Bode plot $|G|$, ϕ vs. ω (open loop F.R.)

(2)Nyquist plot - polar plot of $G(j\omega)$ - Appendix J

(3)Nichols chart $|G|$, ϕ vs. $G/(1+G)$ (closed loop F.R.) - Appendix J

Advantages:

- do not need to compute roots of characteristic equation
- can be applied to time delay systems
- can identify stability margin, i.e., how close you are to instability.

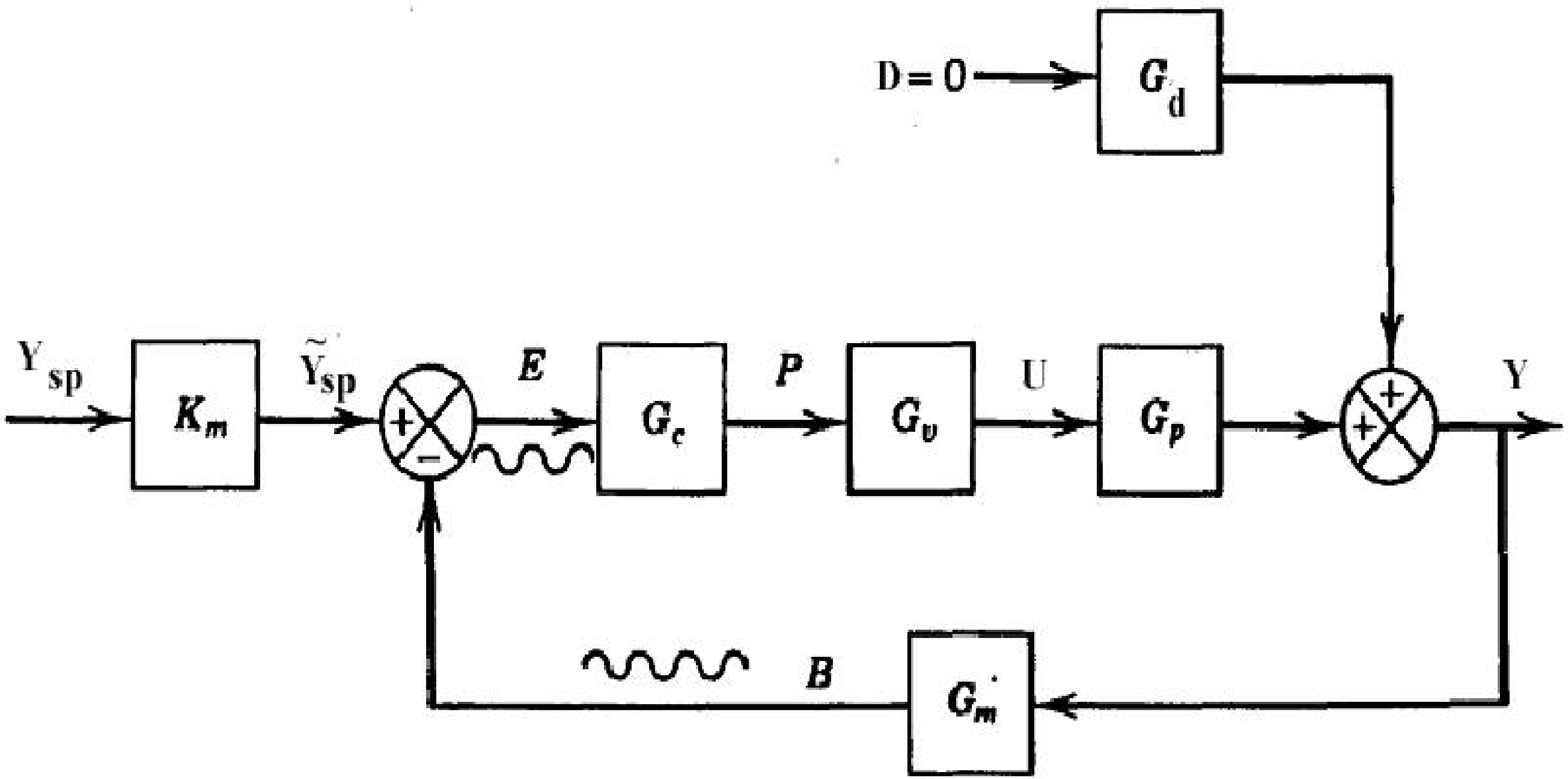


Figure 13.8 Sustained oscillation in a feedback control system.

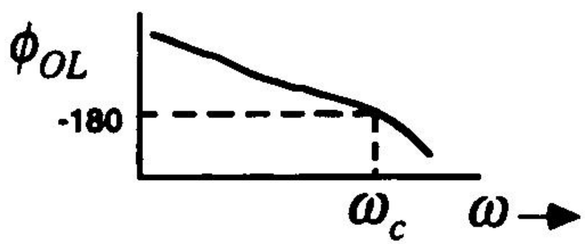
Frequency Response Stability Criteria

Two principal results:
1. Bode Stability Criterion
2. Nyquist Stability Criterion

I) Bode stability criterion

A closed-loop system is unstable if the FR of the open-loop T.F. $G_{OL}=G_C G_P G_V G_M$, has an amplitude ratio greater than one at the critical frequency, ω_C . Otherwise the closed-loop system is stable.

- Note: $\omega_C \equiv$ value of ω where the open-loop phase angle is -180^0. Thus,



- The Bode Stability Criterion provides info on closed-loop stability from open-loop FR info.

- Physical Analogy: Pushing a child on a swing or bouncing a ball.

Example 1:

A process has a T.F.,

$$G_p(s) = \frac{2}{(0.5s+1)^3}$$

And $G_V = 0.1$, $G_M = 10$. If proportional control is used, determine closed-loop stability for 3 values of K_c: 1, 4, and 20.

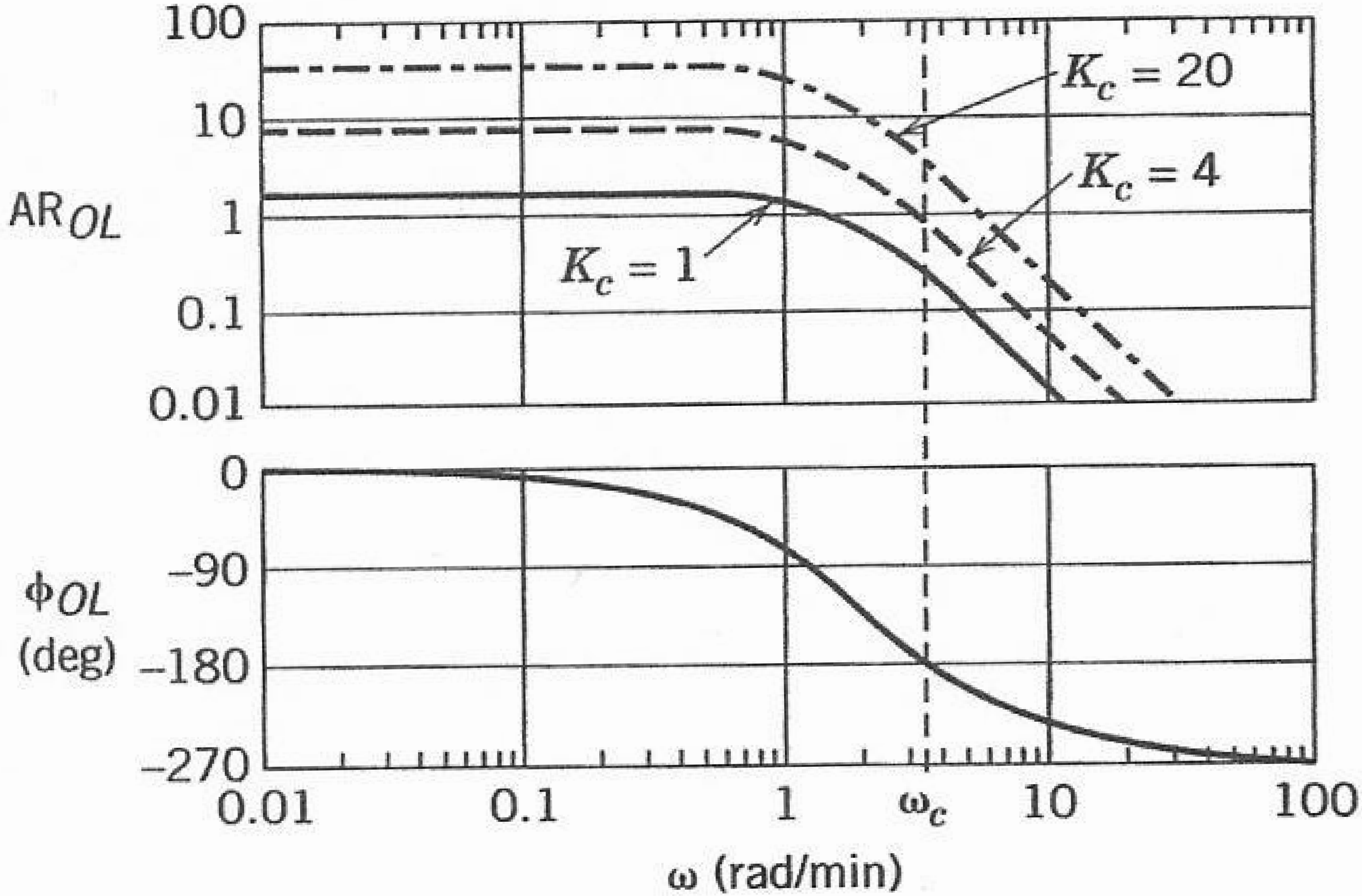
Solution:

The OLTF is $G_{OL} = G_C G_P G_V G_M$ or...

$$G_{OL}(s) = \frac{2K_C}{(0.5s+1)^3}$$

The Bode plots for the 3 values of K_c shown in Fig. 14.9. Note: the phase angle curves are identical. From the Bode diagram:

K_C	AR_{OL}	Stable?
1	0.25	Yes
4	1.0	Conditionally stable
20	5.0	No



Bode plots for $G_{OL} = 2K_c/(0.5s + 1)^3$.

- For proportional-only control, the ultimate gain K_{cu} is defined to be the largest value of K_c that results in a stable closed-loop system.

- For proportional-only control, $G_{OL} = K_c G$ *and* $G = G_v G_p G_m$.

$$AR_{OL}(\omega) = K_c\, AR_G(\omega) \qquad (14\text{-}58)$$

where AR_G denotes the amplitude ratio of G.

- At the stability limit, $\omega = \omega_c$, $AR_{OL}(\omega_c) = 1$ and $K_c = K_{cu}$.

$$K_{cu} = \frac{1}{AR_G(\omega_c)} \qquad (14\text{-}59)$$

Example:

Determine the closed-loop stability of the system,

$$G_p(s) = \frac{4e^{-s}}{5s+1}$$

Where $G_V = 2.0$, $G_M = 0.25$ and $G_C = K_C$. Find ω_C from the Bode Diagram. What is the maximum value of K_c for a stable system?

Solution:

The Bode plot for $K_c = 1$ is shown in Fig. 13.11.

Note that:

$$\omega_C = 1.69 \text{ rad/min}$$

$$AR_{\text{OL}}\Big|_{\omega=\omega_C} = 0.235$$

$$\therefore K_{C\max} = \frac{1}{AR_{OL}} = \frac{1}{0.235} = 4.25$$

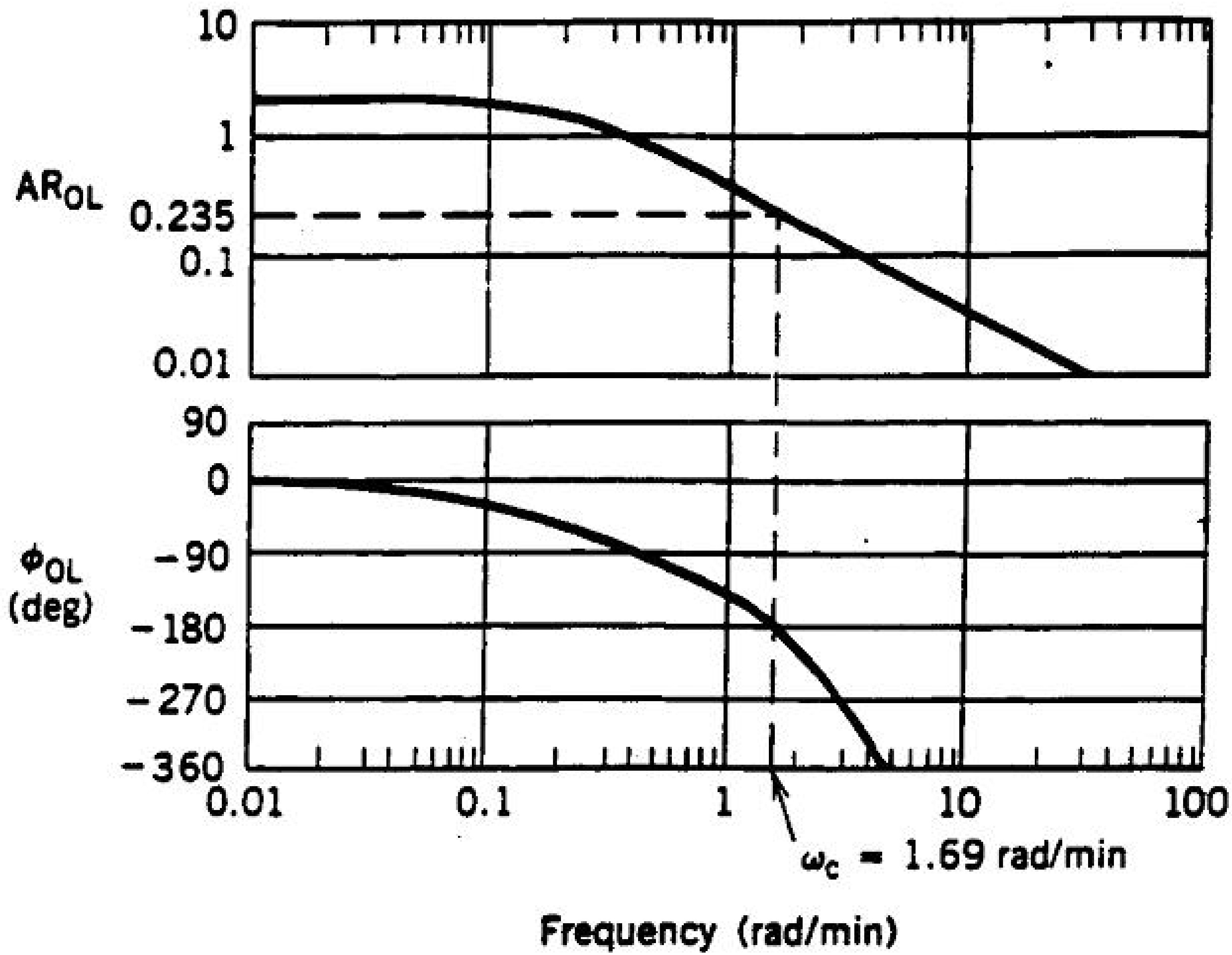


Figure 13.11 Bode plot for Example 14.6, $K_c = 1$.

Gain and Phase Margins

- The gain margin (GM) and phase margin (PM) provide measures of how close a system is to a stability limit.

- Gain Margin:

  Let $A_C = AR_{\mathrm{OL}}$ at $\omega = \omega_C$. Then the gain margin is defined as: $GM = 1/A_C$

  According to the Bode Stability Criterion, $GM > 1 \Leftrightarrow$ stability

- Phase Margin:

  Let ω_g = frequency at which $AR_{OL} = 1.0$ and the corresponding phase angle is ϕ_g. The phase margin is defined as: $PM = 180^\circ + \phi_g$

  According to the Bode Stability Criterion, $PM > 0 \Leftrightarrow$ stability

  See Figure 13.12.

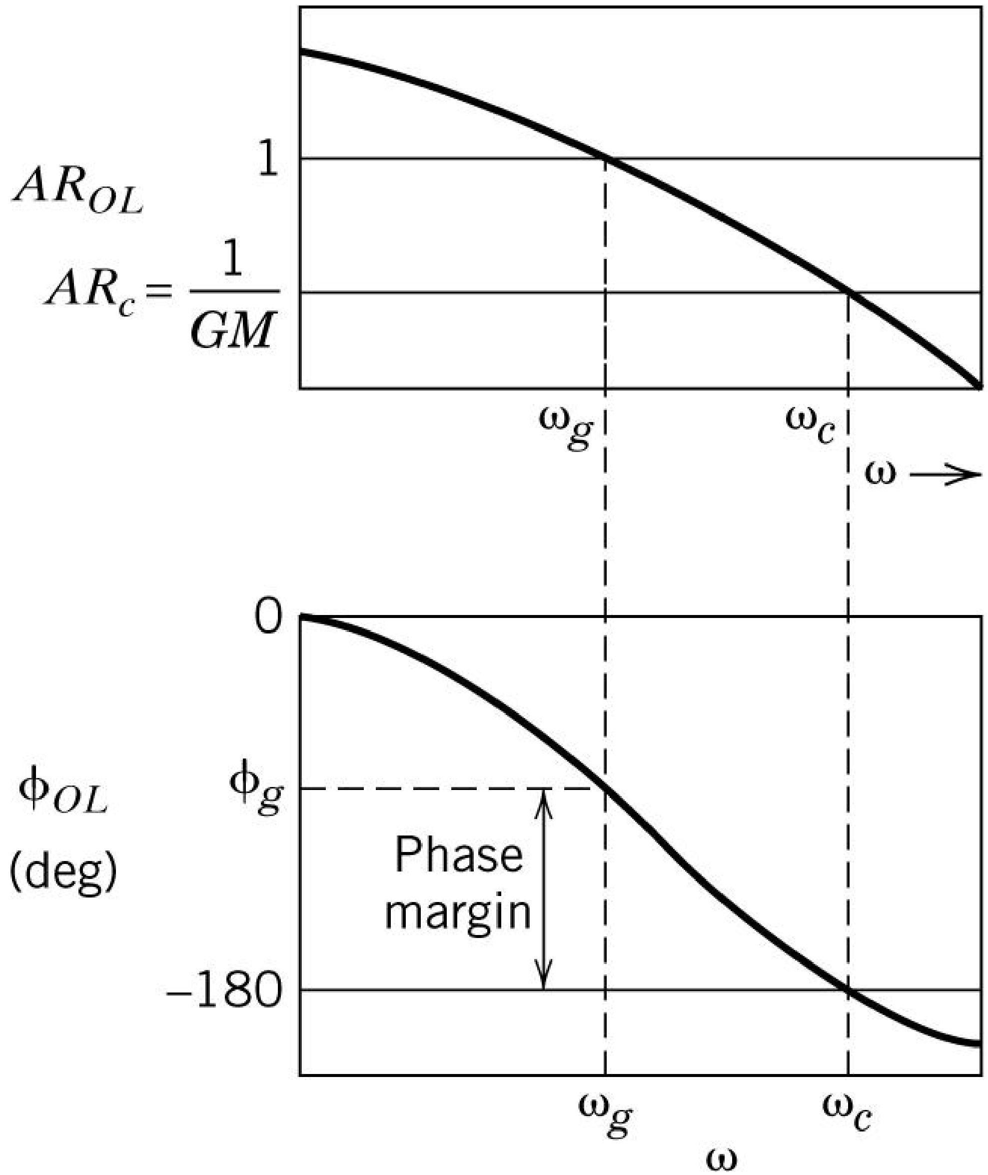

AR_{OL}
1
$AR_c = \frac{1}{GM}$
ω_g
ω_c
$\omega \rightarrow$
0
ϕ_{OL}
(deg)
ϕ_g
Phase margin
–180
ω_g
ω_c
ω

Figure 13.12

Rules of Thumb:

A well-designed FB control system will have:

$$1.7 \leq GM \leq 2.0 \qquad\qquad 30^\circ \leq PM \leq 45^\circ$$