Transfer Functions

- Convenient representation of a *linear*, dynamic model.
- A transfer function (TF) relates *one* input and *one* output:

$$\begin{matrix} x(t) \\ X(s) \end{matrix} \rightarrow \boxed{\text{system}} \rightarrow \begin{matrix} y(t) \\ Y(s) \end{matrix}$$

The following terminology is used:

$\underline{x}$	$\underline{y}$
input	output
forcing function	response
“cause”	“effect”

Definition of the transfer function:

Let $G(s)$ denote the transfer function between an input, x, and an output, y. Then, by definition

$$G(s) \triangleq \frac{Y(s)}{X(s)}$$

where:

$$Y(s) \triangleq \mathfrak{L}\left[y(t)\right]$$

$$X(s) \triangleq \mathfrak{L}\left[x(t)\right]$$

Development of Transfer Functions

Blending system

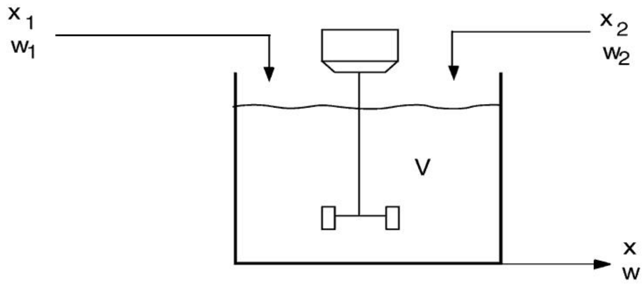


Figure 2.1. Stirred-tank blending process.

Assumptions:
(a) V is constant
(b) *w*1, w2,w are constant

x1 varies while x2 is constant

$$V\rho\frac{dx}{dt} = w_1 x_1 + w_2 x_2 - wx \quad (3-1)$$

Steady state:

$$0 = w_1\overline{x}_1 + w_2 x_2 - w\overline{x} \quad (3-2)$$

deviation variables: (3-1)-(3-2) , and let

$$x' = x - \overline{x}; x_1' = x_1 - \overline{x}_1;$$

$$V\rho\frac{dx'}{dt} = w_1 x_1' - wx' \quad (3)$$

Take Laplace transform of (3)

$$\Rightarrow \quad \rho VsX'(s) = w_1 X_1'(s) - wX'(s)$$

$$\Rightarrow \quad \frac{X'(s)}{X_1'(s)} = \frac{w_1}{\rho Vs + w}$$

Standard form:

$$G(s) = \frac{X'(s)}{X_1'(s)} = \frac{K_1}{\tau s + 1}$$

$$K_1 = \frac{w_1}{w}; \tau = \frac{\rho V}{w}$$

Chapter 3

Exercise: Derive a Transfer Function

For a level process:

$$A\frac{dh}{dt} = q_i - \frac{1}{R_v}h$$

q_i is the input variable

h is the output variable

Derive the transfer function for this process

Properties of Transfer Function Models

1. Steady-State Gain

The steady-state of a TF can be used to calculate the steady-state change in an output due to a steady-state change in the input. For example, suppose we know two steady states for an input, u, and an output, y. Then we can calculate the steady-state gain, K, from:

$$K = \frac{\bar{y}_2 - \bar{y}_1}{\bar{u}_2 - \bar{u}_1} \qquad (4\text{-}38)$$

For a linear system, K is a constant. But for a nonlinear system, K will depend on the operating condition $(\bar{u}, \bar{y})$.

Calculation of *K* from the TF Model:

If a TF model has a steady-state gain, then:

$$\boxed{K = \lim_{s \to 0} G(s)} \tag{14}$$

- This important result is a consequence of the Final Value Theorem

- *Note*: Some TF models do *not* have a steady-state gain (e.g., integrating process in Ch. 5)

Chapter 4

2. Order of a TF Model

Consider a general n-th order, linear ODE:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{dy^{n-1}}{dt^{n-1}} + \ldots a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} +$$

$$b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \ldots + b_1 \frac{du}{dt} + b_0 u \qquad (4\text{-}39)$$

Take $\mathfrak{L}$, assuming the initial conditions are all zero. Rearranging gives the TF:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i} \qquad (4\text{-}40)$$

Definition:

The order of the TF is defined to be the order of the denominator polynomial.

Note: The order of the TF is equal to the order of the ODE.

Physical Realizability:

For any physical system, $n \geq m$ in (4-38). Otherwise, the system response to a step input will be an impulse. This can't happen.

Example:

$$a_0 y = b_1 \frac{du}{dt} + b_0 u \quad \text{and step change in } u \qquad \text{(4-41)}$$

Chapter 4

3. Additive Property

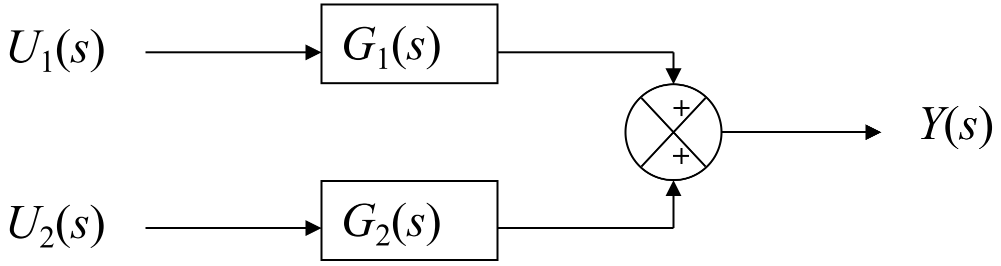
Suppose that an output is influenced by two inputs and that the transfer functions are known:

$$\frac{Y(s)}{U_1(s)} = G_1(s) \quad \text{and} \quad \frac{Y(s)}{U_2(s)} = G_2(s)$$

Then the response to changes in both U_1 and U_2 can be written as:

$$Y(s) = G_1(s)U_1(s) + G_2(s)U_2(s)$$

The graphical representation (or *block diagram*) is:



4. Multiplicative Property

Suppose that,

$$\frac{Y(s)}{U_2(s)} = G_2(s) \text{ and } \frac{U_2(s)}{U_3(s)} = G_3(s)$$

Then,

$$Y(s) = G_2(s)U_2(s) \;\; and \;\; U_2(s) = G_3(s)U_3(s)$$

Substitute,

$$Y(s) = G_2(s)G_3(s)U_3(s)$$

Or,

$$\frac{Y(s)}{U_3(s)} = G_2(s)G_3(s) \qquad U_3(s) \rightarrow \boxed{G_2(s)} \rightarrow \boxed{G_3(s)} \rightarrow Y(s)$$