Laplace Transforms

- Important analytical method for solving *linear* ordinary differential equations.

  - Application to nonlinear ODEs? Must linearize first.

- Laplace transforms play a key role in important process control concepts and techniques.

  - Examples:

    - Transfer functions
    - Frequency response
    - Control system design
    - Stability analysis

Definition

The Laplace transform of a function, $f(t)$, is defined as

$$F(s) = \mathfrak{L}\left[f(t)\right] = \int_0^\infty f(t) e^{-st} dt \qquad (3\text{-}1)$$

where $F(s)$ is the symbol for the Laplace transform, $\mathfrak{L}$ is the Laplace transform operator, and $f(t)$ is some function of time, t.

Note: The $\mathfrak{L}$ operator transforms a time domain function $f(t)$ into an s domain function, $F(s)$. s is a *complex variable*:
$s = a + bj, \; j \doteq \sqrt{-1}$

Inverse Laplace Transform, $\mathfrak{L}^{-1}$:

By definition, the inverse Laplace transform operator, $\mathfrak{L}^{-1}$, converts an s-domain function back to the corresponding time domain function:

$$f(t) = \mathfrak{L}^{-1}\left[F(s)\right]$$

Important Properties:

Both $\mathfrak{L}$ and $\mathfrak{L}^{-1}$ are *linear operators*. Thus,

$$\mathfrak{L}\left[ax(t)+by(t)\right]=a\mathfrak{L}\left[x(t)\right]+b\mathfrak{L}\left[y(t)\right]$$
$$=aX(s)+bY(s) \qquad (3\text{-}3)$$

where:

- $x(t)$ and $y(t)$ are arbitrary functions
- a and b are constants
- $X(s)\triangleq\mathfrak{L}\left[x(t)\right]$ and $Y(s)\triangleq\mathfrak{L}\left[y(t)\right]$

Similarly,

$$\mathfrak{L}^{-1}\left[aX(s)+bY(s)\right]=ax(t)+by(t)$$

Laplace Transforms of Common Functions

1. Constant Function

Let $f(t) = a$ (a constant). Then from the definition of the Laplace transform in (3-1),

$$\mathfrak{L}(a) = \int_0^\infty a e^{-st} dt = -\frac{a}{s} e^{-st}\Bigg|_0^\infty = 0 - \left(-\frac{a}{s}\right) = \boxed{\frac{a}{s}} \qquad (3\text{-}4)$$

2. Step Function

The unit step function is widely used in the analysis of process control problems. It is defined as:

$$S(t) \triangleq \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases} \qquad (3\text{-}5)$$

Because the step function is a special case of a "constant", it follows from (3-4) that

$$\mathfrak{L}\left[S(t)\right] = \frac{1}{s} \qquad (3\text{-}6)$$

3. Derivatives

This is a very important transform because derivatives appear in the ODEs we wish to solve.

$$\mathfrak{L}\left[\frac{df}{dt}\right] = sF(s) - f(0) \qquad (3\text{-}9)$$

initial condition at $t = 0$

Similarly, for higher order derivatives:

$$\mathfrak{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) -$$

$$- \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0) \qquad (3\text{-}14)$$

where:

- n is an arbitrary positive integer

- $f^{(k)}(0) \triangleq \left. \dfrac{d^k f}{dt^k} \right|_{t=0}$

Special Case: All Initial Conditions are Zero

Suppose $f(0) = f^{(1)}(0) = \ldots = f^{(n-1)}(0)$. Then

$$\mathfrak{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s)$$

In process control problems, we usually assume zero initial conditions. *Reason:* This corresponds to the nominal steady state when "deviation variables" are used, as shown in Ch. 4.

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4. Exponential Functions

Consider $f(t) = e^{-bt}$ where $b > 0$. Then,

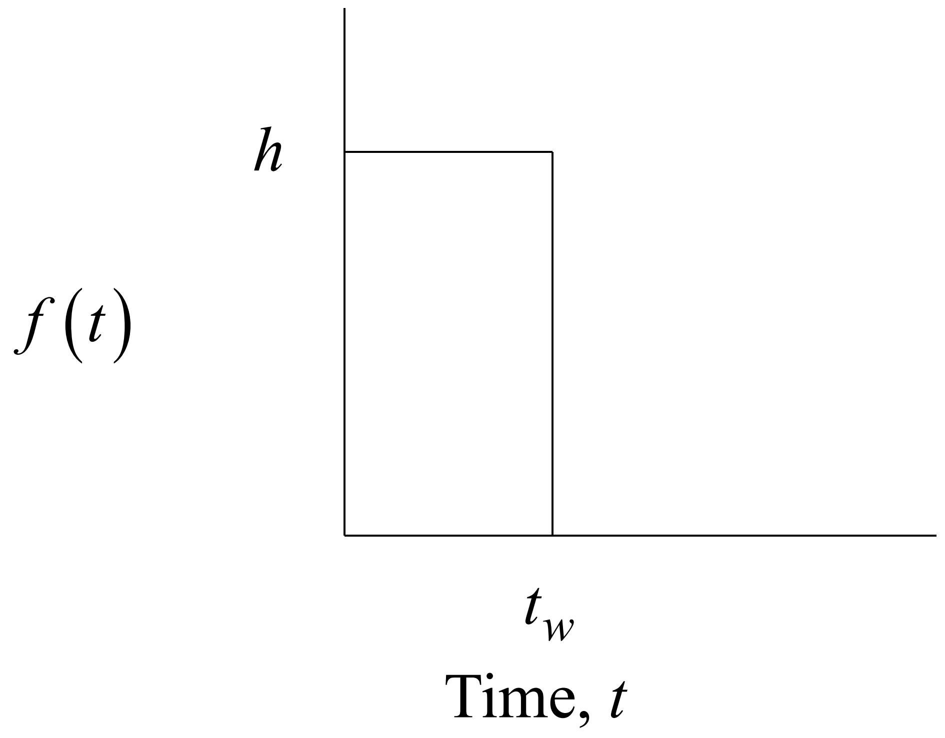
$$\mathfrak{L}\left[e^{-bt}\right] = \int_0^\infty e^{-bt} e^{-st} dt = \int_0^\infty e^{-(b+s)t} dt$$

$$= \frac{1}{b+s}\left[-e^{-(b+s)t}\right]_0^\infty = \boxed{\frac{1}{s+b}} \qquad (3\text{-}16)$$

5. Rectangular Pulse Function

It is defined by:

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ h & \text{for } 0 \le t < t_w \\ 0 & \text{for } t \ge t_w \end{cases} \qquad (3\text{-}20)$$



The Laplace transform of the rectangular pulse is given by

$$F(s) = \frac{h}{s}\left(1 - e^{-t_w s}\right) \qquad (3\text{-}22)$$

6. Impulse Function (or Dirac Delta Function)

The impulse function is obtained by taking the limit of the rectangular pulse as its width, t_w, goes to zero but holding the area under the pulse constant at one. (i.e., let $h = \dfrac{1}{t_w}$)

Let, $\delta(t) \triangleq$ impulse function

Then, $\mathfrak{L}\left[\delta(t)\right] = 1$

Appendix A

Some Commonly Used Laplace Transforms

$$Unit\ Step \Leftrightarrow 1/s$$

$$f(t-\theta) \Leftrightarrow F(s)e^{-\theta s}$$

$$t^n \Leftrightarrow \frac{n!}{s^{n+1}}$$

$$\frac{d\,f(t)}{dt} \Leftrightarrow s\,F(s) - f(0)$$

$$e^{-at} \Leftrightarrow \frac{1}{s+a}$$

$$\frac{d^2\,f(t)}{dt^2} \Leftrightarrow s^2\,F(s) - s\,f(0) - f'(0)$$

$$\sin(\omega t) \Leftrightarrow \frac{\omega}{s^2+\omega^2}$$

$$e^{-at}\sin(\omega t) \Leftrightarrow \frac{\omega}{(s+a)^2+\omega^2}$$

Table 3.1. Laplace Transforms

Table 3.1 Laplace Transforms for Various Time-Domain Functions*

$f(t)$	$F(s)$
1. $\delta(t)$ (unit impulse)	1
2. $S(t)$ (unit step)	$\dfrac{1}{s}$
3. t (ramp)	$\dfrac{1}{s^2}$
4. t^{n-1}	$\dfrac{(n-1)!}{s^n}$
5. e^{-bt}	$\dfrac{1}{s+b}$
6. $\dfrac{1}{\tau}e^{-t/\tau}$	$\dfrac{1}{\tau s+1}$
7. $\dfrac{t^{n-1}e^{-bt}}{(n-1)!}$ $(n>0)$	$\dfrac{1}{(s+b)^n}$
8. $\dfrac{1}{\tau^n(n-1)!}t^{n-1}e^{-t/\tau}$	$\dfrac{1}{(\tau s+1)^n}$
9. $\dfrac{1}{b_1-b_2}(e^{-b_2t}-e^{-b_1t})$	$\dfrac{1}{(s+b_1)(s+b_2)}$

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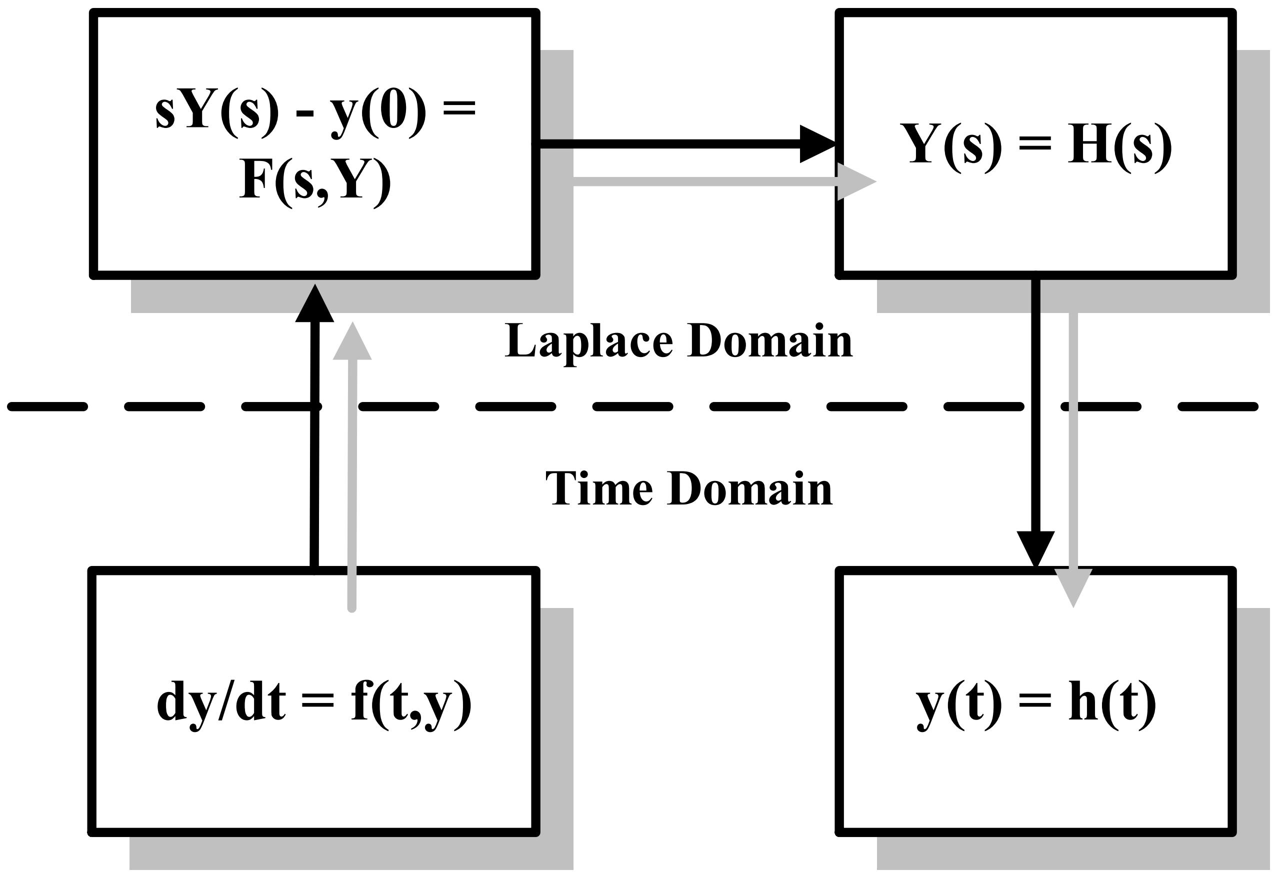
Table 3.1. Laplace Transforms

10. $\dfrac{1}{\tau_1 - \tau_2}(e^{-t/\tau_1} - e^{-t/\tau_2})$	$\dfrac{1}{(\tau_1 s + 1)(\tau_2 s + 1)}$
11. $\dfrac{b_3 - b_1}{b_2 - b_1} e^{-b_1 t} + \dfrac{b_3 - b_2}{b_1 - b_2} e^{-b_2 t}$	$\dfrac{s + b_3}{(s + b_1)(s + b_2)}$
12. $\dfrac{1}{\tau_1}\dfrac{\tau_1 - \tau_3}{\tau_1 - \tau_2} e^{-t/\tau_1} + \dfrac{1}{\tau_2}\dfrac{\tau_2 - \tau_3}{\tau_2 - \tau_1} e^{-t/\tau_2}$	$\dfrac{\tau_3 s + 1}{(\tau_1 s + 1)(\tau_2 s + 1)}$
13. $1 - e^{-t/\tau}$	$\dfrac{1}{s(\tau s + 1)}$
14. $\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
15. $\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
16. $\sin(\omega t + \phi)$	$\dfrac{\omega \cos \phi + s \sin \phi}{s^2 + \omega^2}$
17. $e^{-bt} \sin \omega t$ } b, ω real	$\dfrac{\omega}{(s + b)^2 + \omega^2}$
18. $e^{-bt} \sin \omega t$ } b, ω real	$\dfrac{s + b}{(s + b)^2 + \omega^2}$
19. $\dfrac{1}{\tau\sqrt{1 - \zeta^2}} e^{-\zeta t/\tau} \sin(\sqrt{1 - \zeta^2}\, t/\tau)$ $(0 \leq \lvert\zeta\rvert < 1)$	$\dfrac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$
20. $1 + \dfrac{1}{\tau_2 - \tau_1}(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2})$ $(\tau_1 \neq \tau_2)$	$\dfrac{1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
21. $1 - \dfrac{1}{\ldots} e^{-\zeta t/\tau} \sin[\sqrt{1 - \zeta^2}\, t/\tau + \psi]$	$\dfrac{1}{\ldots}$

Method for Solving Linear ODE's using Laplace Transforms



Solution of ODEs by Laplace Transforms

Procedure:

1. Take the $\mathfrak{L}$ of both sides of the ODE.
2. Rearrange the resulting algebraic equation in the *s* domain to solve for the $\mathfrak{L}$ of the output variable, e.g., $Y(s)$.
3. Perform a partial fraction expansion.
4. Use the $\mathfrak{L}^{-1}$ to find $y(t)$ from the expression for $Y(s)$.

Example 3.1

Solve the ODE,

$$5\frac{dy}{dt} + 4y = 2 \qquad\qquad y(0) = 1 \qquad\qquad (3\text{-}26)$$

First, take $\mathfrak{L}$ of both sides of (3-26),

$$5\left(sY(s) - 1\right) + 4Y(s) = \frac{2}{s}$$

Rearrange,

$$Y(s) = \frac{5s+2}{s(5s+4)} \qquad\qquad (3\text{-}34)$$

Take $\mathfrak{L}^{-1}$,

$$y(t) = \mathfrak{L}^{-1}\left[\frac{5s+2}{s(5s+4)}\right]$$

From Table 3.1,

$$\boxed{y(t) = 0.5 + 0.5e^{-0.8t}} \qquad\qquad (3\text{-}37)$$

Appendix A

Example of Solution of an ODE

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2 \quad y(0) = y'(0) = 0$$

ODE w/initial conditions

$$s^2 Y(s) + 6s\,Y(s) + 8Y(s) = 2/s$$

Apply Laplace transform to each term

$$Y(s) = \frac{2}{s(s+2)(s+4)}$$

Solve for Y(s)

$$Y(s) = \frac{1}{4s} + \frac{-1}{2(s+2)} + \frac{1}{4(s+4)}$$

$$y(t) = \frac{1}{4} - \frac{e^{-2t}}{2} + \frac{e^{-4t}}{4}$$

Apply partial fraction expansions w/Heaviside

Partial Fraction Expansions

Basic idea: Expand a complex expression for $Y(s)$ into simpler terms, each of which appears in the Laplace Transform table. Then you can take the $\mathfrak{L}^{-1}$ of both sides of the equation to obtain $y(t)$.

Example:

$$Y(s) = \frac{s+5}{(s+1)(s+4)} \tag{3-41}$$

Perform a partial fraction expansion (PFE)

$$\frac{s+5}{(s+1)(s+4)} = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{s+4} \tag{3-42}$$

where coefficients α_1 and α_2 have to be determined.

To find α_1*: Multiply both sides by s + 1 and let s = -1*

$$\therefore \quad \alpha_1 = \left.\frac{s+5}{s+4}\right|_{s=-1} = \frac{4}{3}$$

To find α_2*: Multiply both sides by s + 4 and let s = -4*

$$\therefore \quad \alpha_2 = \left.\frac{s+5}{s+1}\right|_{s=-4} = -\frac{1}{3}$$

A General PFE

Consider a general expression,

$$Y(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\overset{n}{\underset{i=1}{\pi}}(s+b_i)} \qquad \text{(3-46a)}$$

Here $D(s)$ is an n-th order polynomial with the roots $\left(s = -b_i\right)$ all being *real* numbers which are *distinct* so there are no repeated roots.

The PFE is:

$$Y(s) = \frac{N(s)}{\overset{n}{\underset{i=1}{\pi}}\left(s+b_i\right)} = \sum_{i=1}^{n} \frac{\alpha_i}{s+b_i} \qquad \text{(3-46b)}$$

Note: $D(s)$ is called the "characteristic polynomial".

Special Situations:

Two other types of situations commonly occur when $D(s)$ has:

i) Complex roots: e.g., $b_i = 3 \pm 4j \qquad \left(j \triangleq \sqrt{-1}\right)$

ii) Repeated roots (e.g., $b_1 = b_2 = -3$)

Example 3.2 (continued)

Recall that the ODE, $\dddot{y} + +6\ddot{y} + 11\dot{y} + 6y = 1$, with zero initial conditions resulted in the expression

$$Y(s) = \frac{1}{s\left(s^3 + 6s^2 + 11s + 6\right)} \tag{3-40}$$

The denominator can be factored as

$$s\left(s^3 + 6s^2 + 11s + 6\right) = s(s+1)(s+2)(s+3) \tag{3-50}$$

Note: Normally, numerical techniques are required in order to calculate the roots.

The PFE for (3-40) is

$$Y(s) = \frac{1}{s(s+1)(s+2)(s+3)} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+1} + \frac{\alpha_3}{s+2} + \frac{\alpha_4}{s+3} \tag{3-51}$$

Solve for coefficients to get

$$\alpha_1 = \frac{1}{6}, \quad \alpha_2 = -\frac{1}{2}, \quad \alpha_3 = \frac{1}{2}, \quad \alpha_4 = -\frac{1}{6}$$

(For example, find α, by multiplying both sides by s and then setting $s = 0$.)

Substitute numerical values into (3-51):

$$Y(s) = \frac{1/6}{s} - \frac{1/2}{s+1} + \frac{1/2}{s+2} + \frac{1/6}{s+3}$$

Take $\mathfrak{L}^{-1}$ of both sides:

$$\mathfrak{L}^{-1}\left[Y(s)\right] = \mathfrak{L}^{-1}\left[\frac{1/6}{s}\right] - \mathfrak{L}^{-1}\left[\frac{1/2}{s+1}\right] + \mathfrak{L}^{-1}\left[\frac{1/2}{s+2}\right] + \mathfrak{L}^{-1}\left[\frac{1/6}{s+3}\right]$$

From Table 3.1,

$$y(t) = \frac{1}{6} - \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t} - \frac{1}{6}e^{-3t} \qquad \text{(3-52)}$$

Important Properties of Laplace Transforms

1. Final Value Theorem

It can be used to find the steady-state value of a system (providing that a steady-state value exists.)

Statement of FVT:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0}\left[sY(s)\right]$$

providing that the limit exists (is finite) for all $\mathrm{Re}(s) \geq 0$, where Re (s) denotes the real part of complex variable, s.

2. Initial Value Theorem

Statement of IVT:

$$\lim_{t\to 0} y(t) = \lim_{s\to\infty}\left[sY(s)\right]$$

Important Properties of Laplace Transforms

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} \left[sY(s) \right]$$

$$\lim_{t \to 0} y(t) = \lim_{s \to \infty} [sY(s)]$$

Proof:

$$\mathfrak{L}\left[\frac{df}{dt}\right] = sF(s) - f(0) \qquad (3\text{-}9)$$

$$\int_0^\infty f'(t)e^{-st} = sF(s) - f(0)$$

when s->0

$$\int_0^\infty f'(t) = \lim_{s \to 0} sF(s) - f(0)$$

$$f(\infty) - f(0) = \lim_{s \to 0} sF(s) - f(0)$$

when s->∞

$$0 = \lim_{s -> \infty} sF(s) - f(0)$$

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Example:

Suppose,

$$Y(s) = \frac{5s+2}{s(5s+4)} \tag{3-34}$$

Then,

$$y(\infty) = \lim_{t\to\infty} y(t) = \lim_{s\to 0}\left[\frac{5s+2}{5s+4}\right] = 0.5$$

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3 . *Time Delay*

Time delays occur due to fluid flow, time required to do an analysis (e.g., gas chromatograph). The delayed signal can be represented as

$$y(t-\theta) \qquad \theta = \text{time delay}$$

Also,

$$\mathfrak{L}\left[y(t-\theta)\right] = e^{-\theta s}Y(s)$$