Chapter 4

Dynamic behavior of First-order and Second-order Systems

- **Contents**

- **In this chapter, we learn how process respond to typical changes in some of input changes.**

Standard inputs

A number of standard types of input changes are widely used for two reasons:

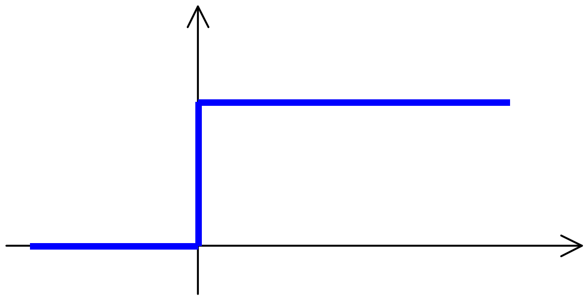
1. They are representative of the types of changes that occur in plants.

2. They are easy to analyze mathematically.

Standard inputs

- **Step Input**
  A sudden change in a process variable can be approximated by a step change of magnitude, *M*:

$$u_s = \begin{cases} 0 & t < 0 \\ M & t \geq 0 \end{cases}$$



$$u_s(s) = M/s$$

The step change occurs at an arbitrary time denoted as $t = 0$.

- *Special Case:* If $M = 1$, we have a “unit step change”. We give it the symbol, $S(t)$.

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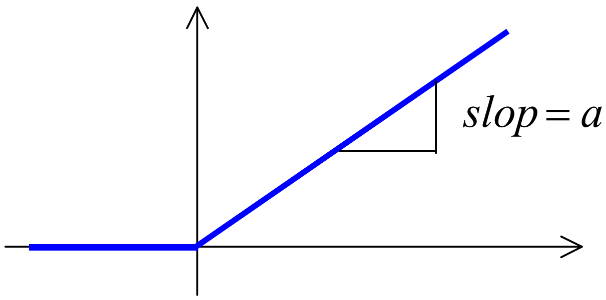
Standard inputs

2. Ramp Input

We can approximate a drifting disturbance by a *ramp input:*

$$u_R(t) = \begin{cases} 0 & t < 0 \\ \text{at} & t \geq 0 \end{cases}$$

$$u_R(s) = a/s^2$$



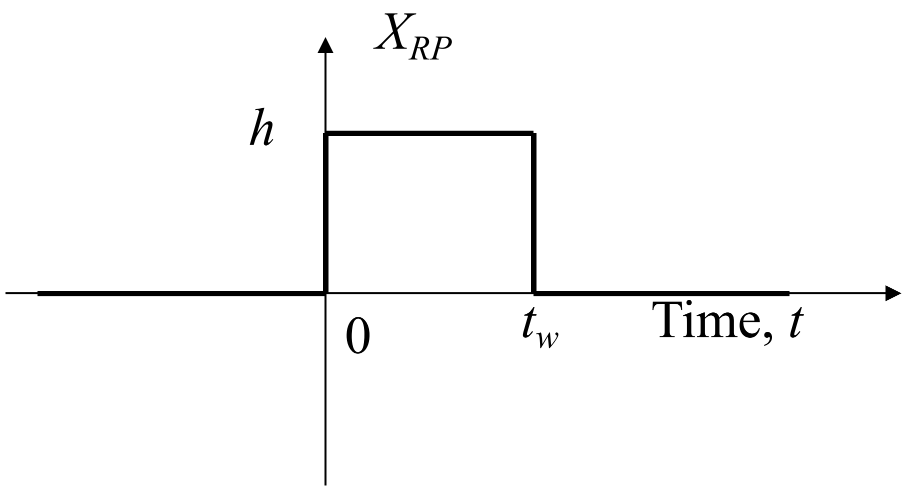
Standard inputs

3. Rectangular Pulse

It represents a brief, sudden change in a process variable:

$$u_{RP}(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ h & \text{for} \quad 0 \le t < t_w \\ 0 & \text{for} \quad t \ge t_w \end{cases}$$

$$u_{RP}(s) = \frac{h}{s}\left[1 - e^{-t_w s}\right]$$



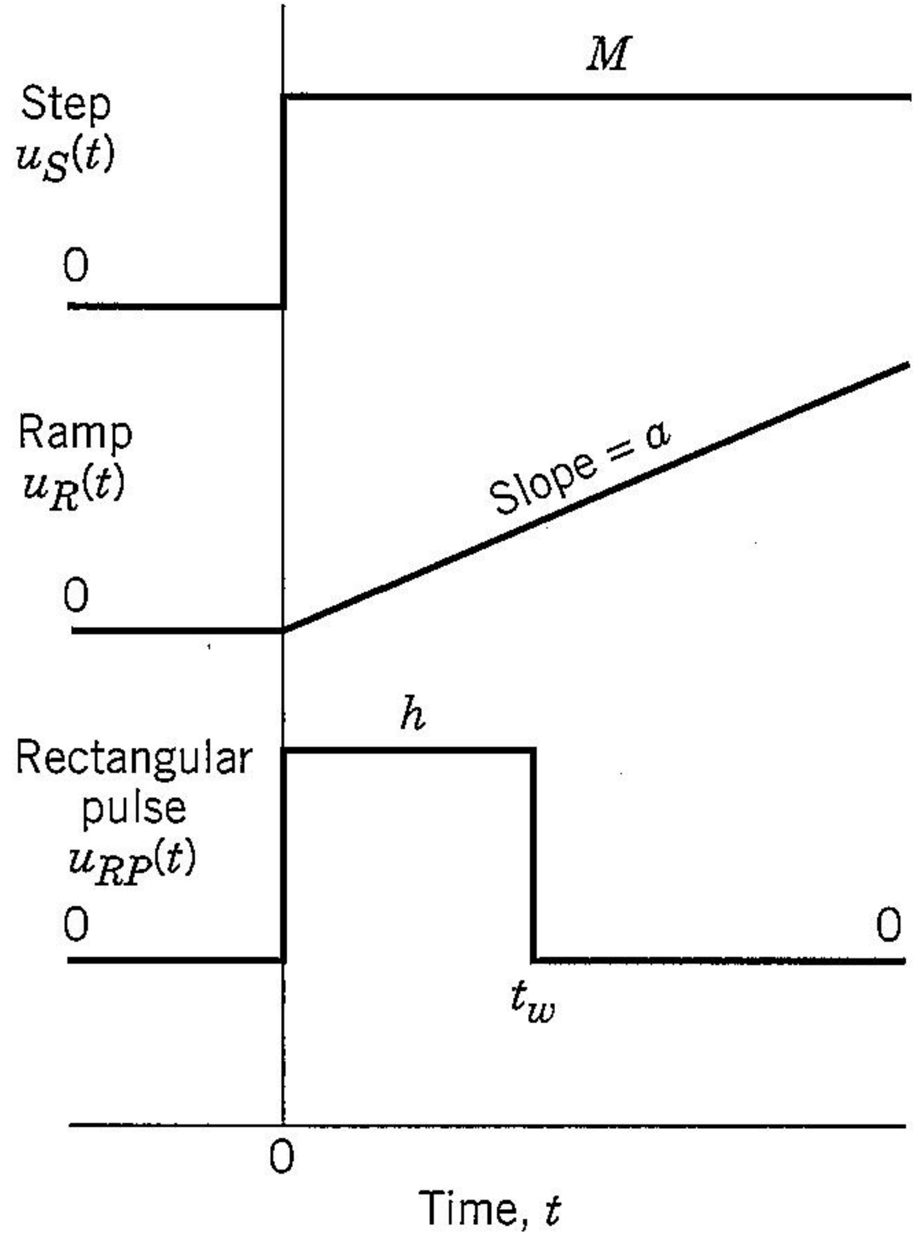


Figure 4.2 Three important examples of deterministic inputs.

Standard inputs

4. Sinusoidal Input

Processes are also subject to periodic, or cyclic, disturbances. They can be approximated by a sinusoidal disturbance:

$$U_{\sin}(t) \triangleq \begin{cases} 0 & \text{for} \quad t < 0 \\ A\sin(\omega t) & \text{for} \quad t \geq 0 \end{cases} \tag{5-14}$$

where: A = amplitude, ω = angular frequency

Standard inputs

5. Impulse Input

$$U_I(t) = \delta(t).$$

- Here,
- It represents a short, transient disturbance.

It has the simplest Laplace transform, but it is not a realistic input signal. Because to obtain an impulse input, it is necessary to inject amount of energy or material into a process in an infinitesimal length of time.

Response of First-Order System

The standard form for a first-order TF is:

$$\frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1} \qquad (5\text{-}16)$$

where:

$$K \triangleq \text{steady-state gain}$$

$$\tau \triangleq \text{time constant}$$

Consider the response of this system to a step of magnitude, *M*:

$$U(t) = M \text{ for } t \geq 0 \qquad \Rightarrow U(s) = \frac{M}{s}$$

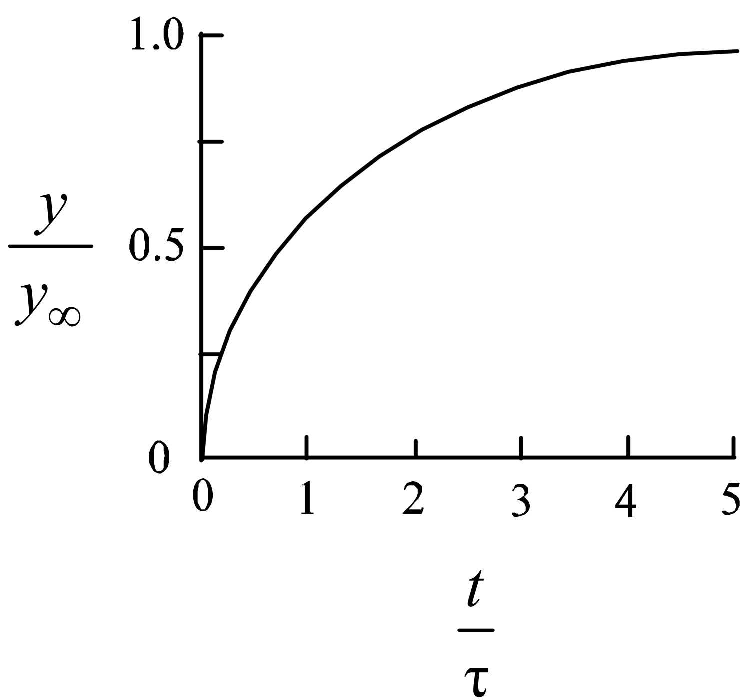
Substitute into (5-16) and rearrange,

$$Y(s) = \frac{KM}{s(\tau s + 1)} \qquad (5\text{-}17)$$

Take inverse Laplace transform, time –domain response is:

$$\boxed{y(t) = KM\left(1 - e^{-t/\tau}\right)} \qquad (5\text{-}18)$$

Let $y_\infty \triangleq$ steady-state value of $y(t)$. From (5-18), $y_\infty = KM$.



t	$\frac{y}{y_\infty}$
0	0
τ	0.632
2τ	0.865
3τ	0.950
4τ	0.982
5τ	0.993

Note: Large τ means a slow response.

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Table 5.1 Response of a First-Order Process to a Step Input

t	$y(t)/KM = 1 - e^{-t/\tau}$
0	0
τ	0.6321
2τ	0.8647
3τ	0.9502
4τ	0.9817
5τ	0.9933

- A first-order system dose not respond instantaneously to a sudden change in its input and that after a time interval equal to the process time constant (τ), the process response is still only 63.2% complete.
- Theoretically the process output never reaches the new steady-state value; it dose approximate the new value when t equals 3 to 5 process time constants.

Response of Integrating Process Units

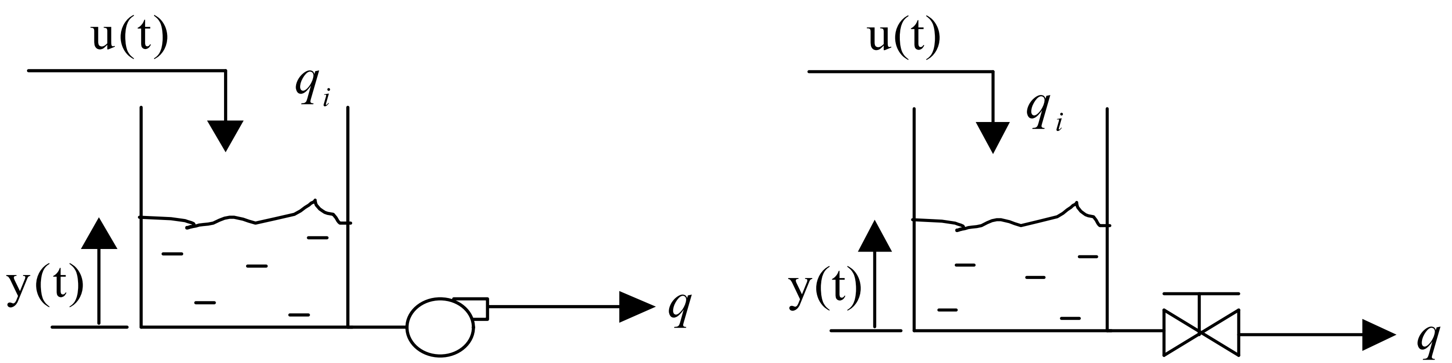
- **What is an '****Integrating Process****'?**

The process which has integrating unit($1/s$) in its transfer function.

– Open-loop unstable process(Non-self-regulating process).

A process that cannot reach a new steady state when subjected to step changes in inputs is called **'Open-loop unstable process'** or **'Non-self-regulating process'**.

? Which process is an integrating process?



Liquid level system with a pump(a) or valve(b).

Integrating Process

An "integrating process" or "integrator" has the transfer function:

$$\frac{Y(s)}{U(s)} = \frac{K}{s} \qquad (K = \text{constant})$$

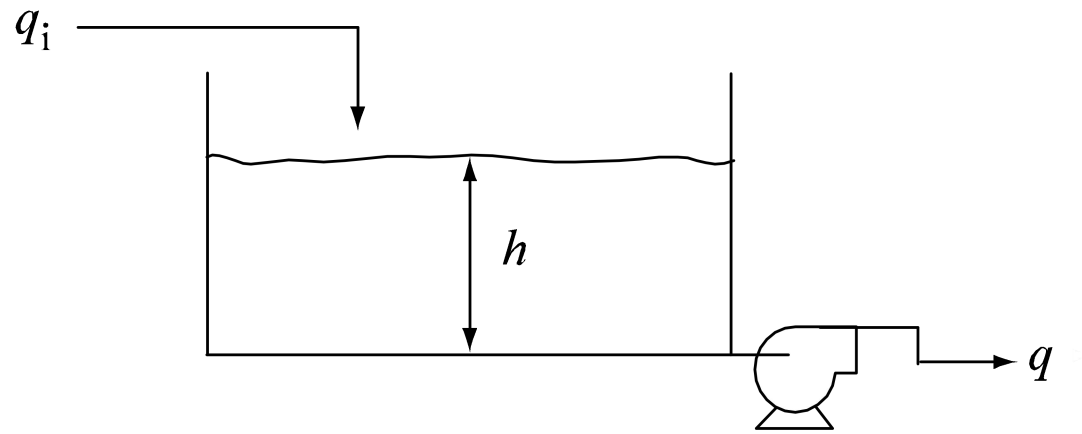
Consider a step change of magnitude M. Then $U(s) = M/s$ and,

$$Y(s) = \frac{KM}{s^2} \Rightarrow y(t) = KMt$$

Thus, $y(t)$ is unbounded and a new steady-state value does *not* exist.

Common Physical Example:

Consider a liquid storage tank with a pump on the exit line:



- Assume:
  1. Constant cross-sectional area, A.
  2. q is independent of h. $q \neq f(h)$
- Mass balance: $A\dfrac{dh}{dt} = q_i - q$ (1) $\Rightarrow$ $0 = \bar{q}_i - \bar{q}$ (2)
- Eq. (1) – Eq. (2), take L transfrom,
- $$H'(s) = \frac{1}{As}\left[Q_i'(s) - Q'(s)\right]$$
- For $Q'(s) = 0$ (constant q),

$$\boxed{\frac{H'(s)}{Q_i'(s)} = \frac{1}{As}}$$

Response of Second-Order Systems

- A second order transfer function can arise physically
  - Two first-order processes are connected in series.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K_1 K_2}{(\tau_1 s + 1)(\tau_2 s + 1)} = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

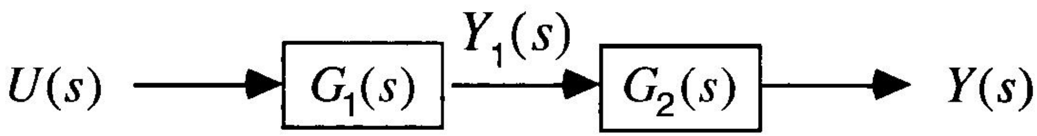


Figure 5.9. Two first-order systems in series yield an overall second-order system.

  - A second-order differential equation process model is transformed.

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- Standard form of the second-order transfer function.

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

Where

K is the **process gain**.

τ is the **time constant** which determines the speed of response of the system.

ζ is the **damping factor** which provides a measure of the amount of damping in the system, that is, the degree of oscillation in a process response after a perturbation.

$$\tau^2 s^2 + 2\zeta\tau s + 1 = \left(\frac{\tau s}{\zeta - \sqrt{\zeta^2 - 1}} + 1\right)\left(\frac{\tau s}{\zeta + \sqrt{\zeta^2 - 1}} + 1\right)$$

$$\tau_1 = \frac{\tau}{\zeta - \sqrt{\zeta^2 - 1}} \qquad \tau_2 = \frac{\tau}{\zeta + \sqrt{\zeta^2 - 1}}$$

- Three important subcases.

Table 5.2 The Three Forms of Second-Order Transfer Functions

Case	*Range of Damping Coefficient*	*Characterization of Response*	*Roots of Characteristic Equation*
a	$\zeta > 1$	Overdamped	Real and unequal
b	$\zeta = 1$	Critically damped	Real and equal
c	$0 \leq \zeta < 1$	Underdamped	Complex conjugates (of the form $a + jb$ and $a - jb$)

$\zeta < 0$; unstable second-order system that would have an unbounded response to any input.

Step response of second-order systems

$$U(s) = \frac{M}{s}, \quad Y(s) = \frac{KM}{(\tau^2 s^2 + 2\zeta\tau s + 1)s}$$

$$\tau_1 = \frac{\tau}{\zeta - \sqrt{\zeta^2 - 1}} \qquad \tau_2 = \frac{\tau}{\zeta + \sqrt{\zeta^2 - 1}}$$

Case a. $\zeta > 1$, root are real and distinct: **Overdamped.**

$$y(t) = KM\left(1 - \frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{\tau_1 - \tau_2}\right)$$

Case b. $\zeta = 1$, double root: **Critically damped.**

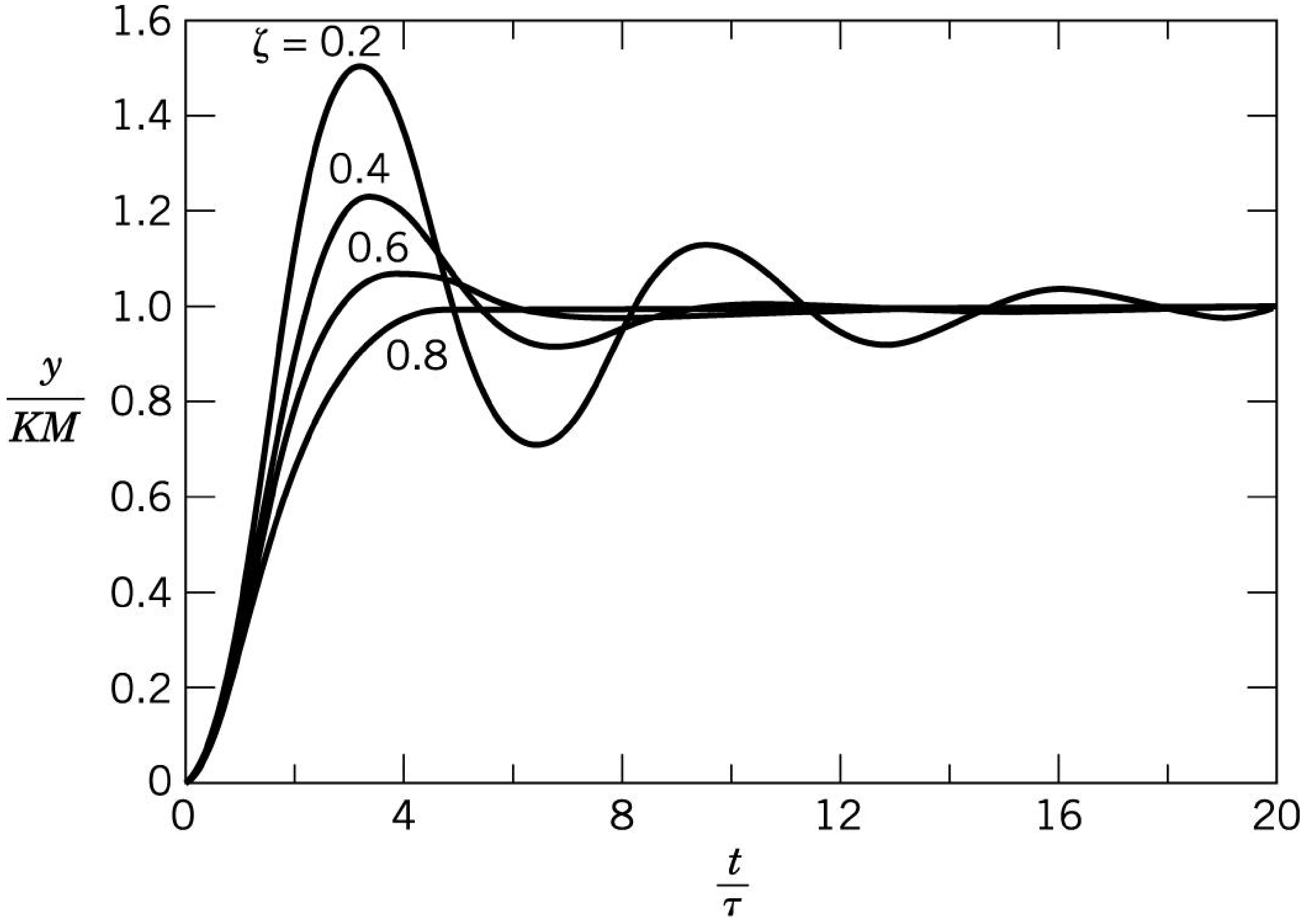
$$y(t) = KM\left[1 - \left(1 + \frac{t}{\tau}\right)\exp\left(-\frac{t}{\tau}\right)\right]$$

Case c. $0 \le \zeta < 1$, complex root: **Underdamped.**

$$y(t) = KM\left\{1 - \exp\left(-\frac{\zeta t}{\tau}\right)\left[\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)\right]\right\}$$

$$= KM\left\{1 - \frac{1}{\sqrt{1-\zeta^2}}\exp\left(-\frac{\zeta t}{\tau}\right)\sin\left(\sqrt{1-\zeta^2}\,\frac{t}{\tau} + \psi\right)\right\}$$

Where $\psi = \tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)$



ζ = 0.2
0.4
0.6
0.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
$\frac{y}{KM}$
0
4
8
12
16
20
$\frac{t}{\tau}$

Figure 4.8

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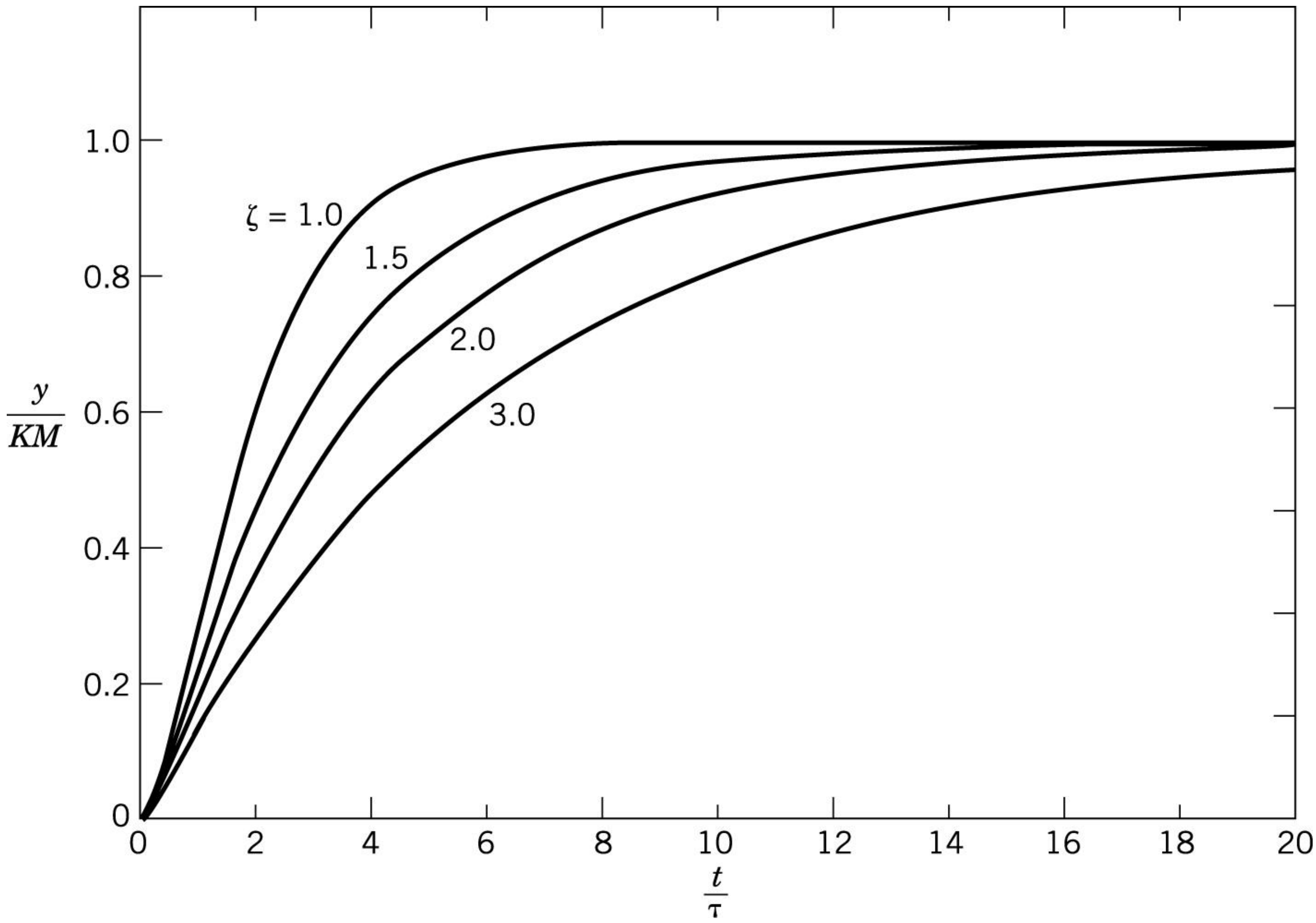


Figure 4.9

Several general remarks can be made concerning the responses show in Figs. 5.8 and 5.9:

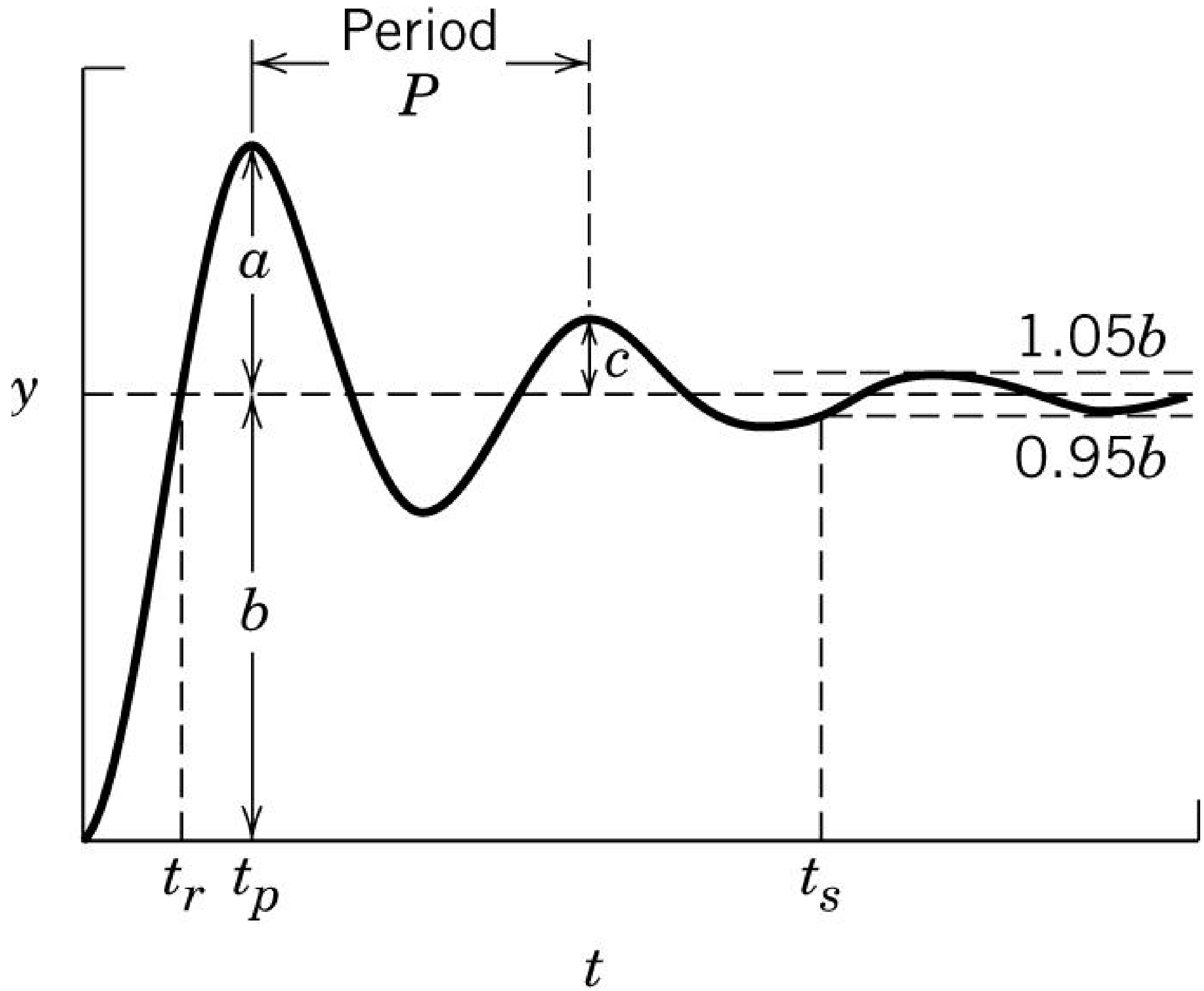
1. Responses exhibiting oscillation and overshoot ($y/KM > 1$) are obtained only for values of ζ less than one.

2. Large values of ζ yield a sluggish (slow) response.

3. The fastest response without overshoot is obtained for the critically damped case $(\zeta = 1)$.

Performance characteristics of underdamped second-order systems

•Control systems are required to have performance of underdamped second –order systems:



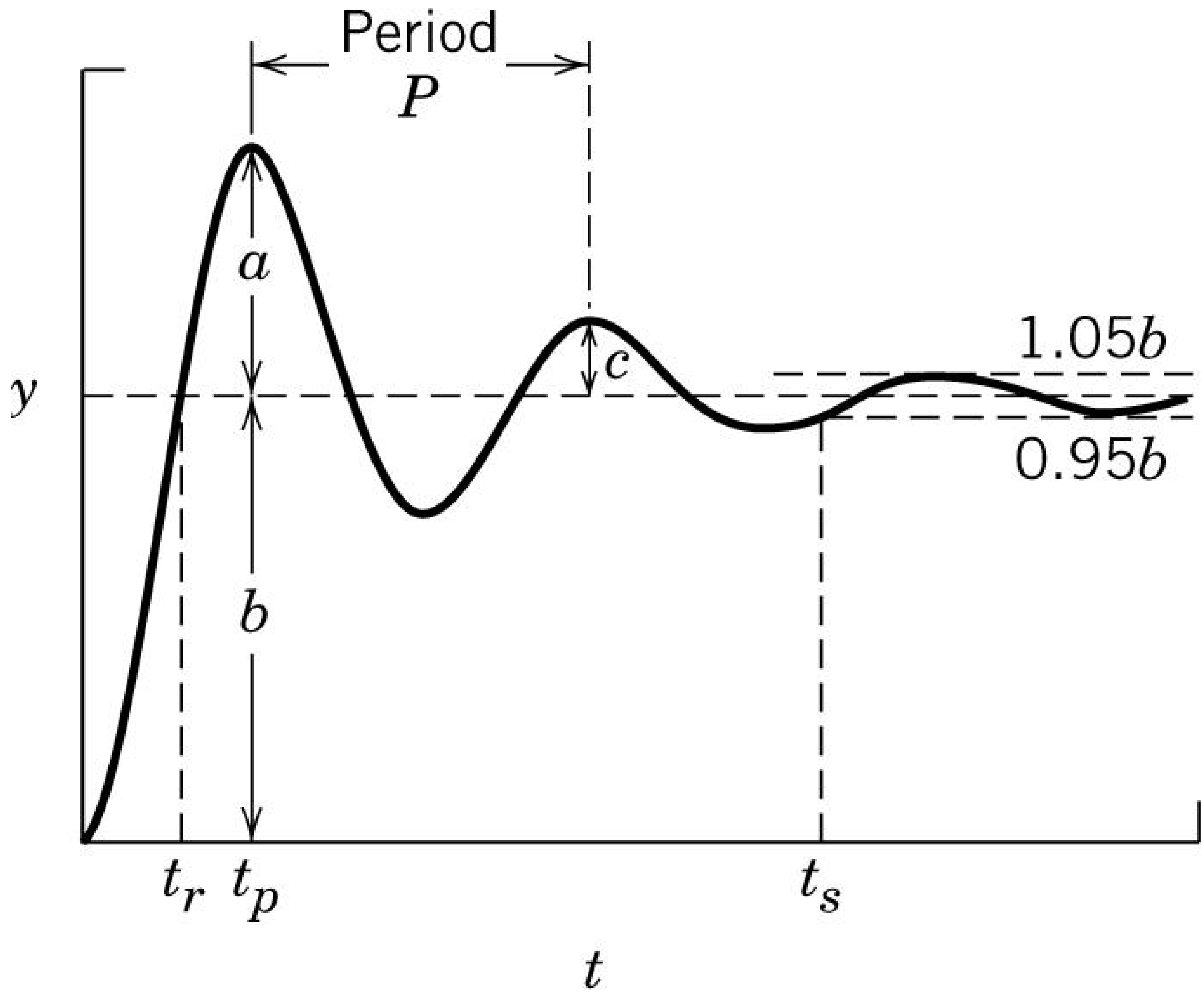
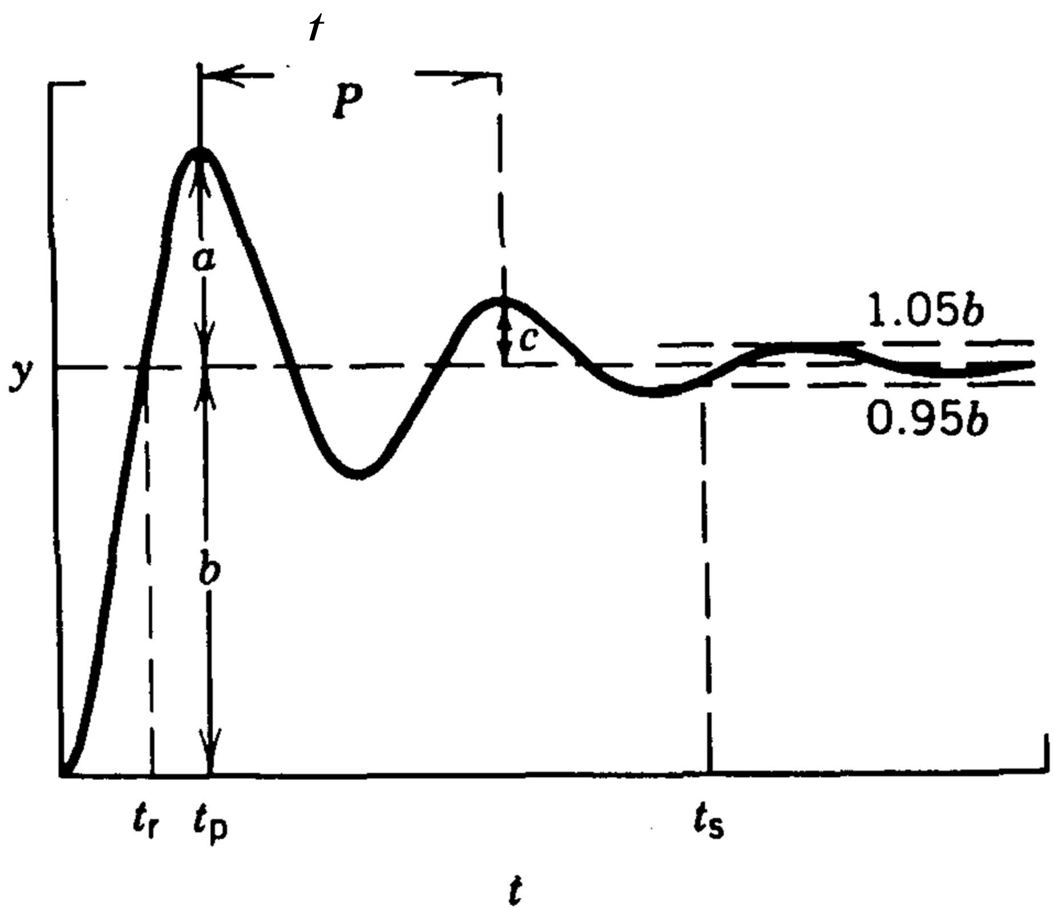


Figure 4.10

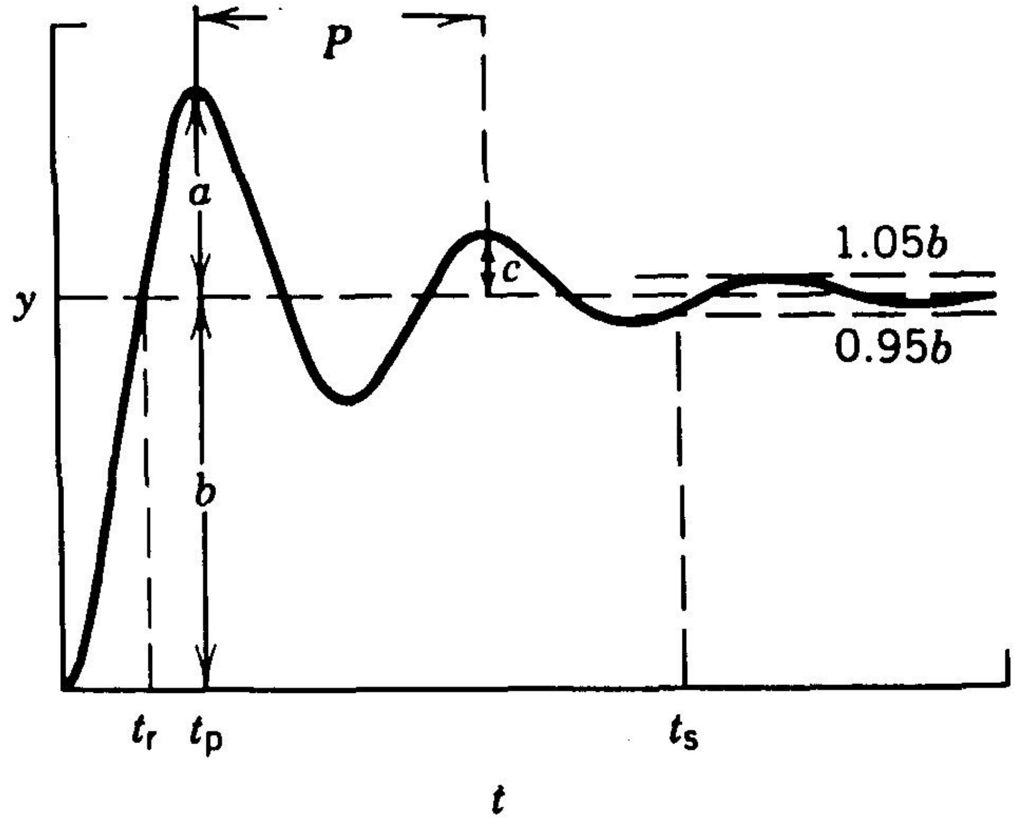
1. **Rise Time:** t_r is the time the process output takes to first reach the new steady-state value.

2. **Time to First Peak:** t_p is the time required for the output to reach its first maximum value.



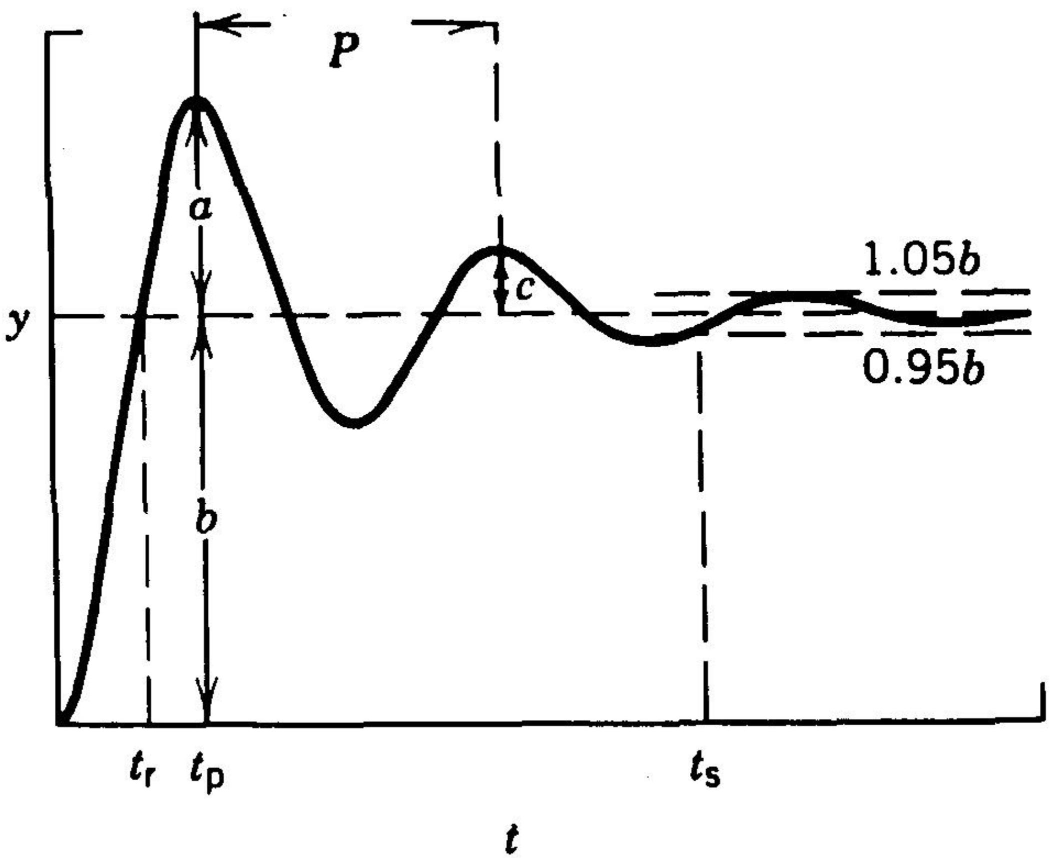
3 **Settling Time**: t_s is defined as the time required for the process output to reach and remain inside a band whose width is equal to $\pm 5\%$ of the total change in y. The term 95% response time sometimes is used to refer to this case. Also, values of $\pm 1\%$ sometimes are used.

4 **Overshoot:** OS = a/b (% overshoot is $100a/b$).



5. Decay Ratio: DR = *c*/*a* (where *c* is the height of the second peak).

6. Period of Oscillation: *P* is the time between two successive peaks or two successive valleys of the response.



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$$y(t) = KM\{1 - \frac{1}{\sqrt{1-\zeta^2}} \exp(-\frac{\zeta t}{\tau}) \sin(\sqrt{1-\zeta^2}\frac{t}{\tau} + \psi)\}$$

- **Rise time.**

$$t_r = \frac{\tau}{\sqrt{1-\zeta^2}}(\pi - \psi)$$

$$\left[\begin{array}{l} \because 1 = 1 - \frac{1}{\sqrt{1-\zeta^2}} \exp(-\frac{\zeta t}{\tau}) \sin(\sqrt{1-\zeta^2}\frac{t}{\tau} + \psi) \\ \sin(\sqrt{1-\zeta^2}\frac{t}{\tau} + \psi) = 0 \end{array}\right]$$

- **Time to first peak.**

$$t_p = \frac{\tau\pi}{\sqrt{1-\zeta^2}} \qquad [\because dy/dt = 0]$$

- **Overshoot.**

$$OS = \exp(-\pi\zeta / \sqrt{1-\zeta^2})$$

$$[\because a = y(t = t_p) - b = KM \exp(-\pi\zeta / \sqrt{1-\zeta^2})]$$

- **Decay ratio.** $DR = (OS)^2 = \exp(-2\pi\zeta \big/ \sqrt{1-\zeta^2})$

$[\because c = y(t = 3\tau\pi \big/ \sqrt{1-\zeta^2}) - b = KM\exp(-3\pi\zeta \big/ \sqrt{1-\zeta^2})\ ]$

- **Period of oscillation.** $P = \dfrac{2\tau\pi}{\sqrt{1-\zeta^2}}$

poles and zeros and their effect on process response

$$G(s) = \frac{K}{s(\tau_1 s+1)(\tau_2{}^2 s^2 + 2\zeta\tau_2 s+1)}$$

4 poles (denominator is 4[th] order polynomial)

Response to any input will contain : (response modes)

(1) A constant term resulting from the s factor

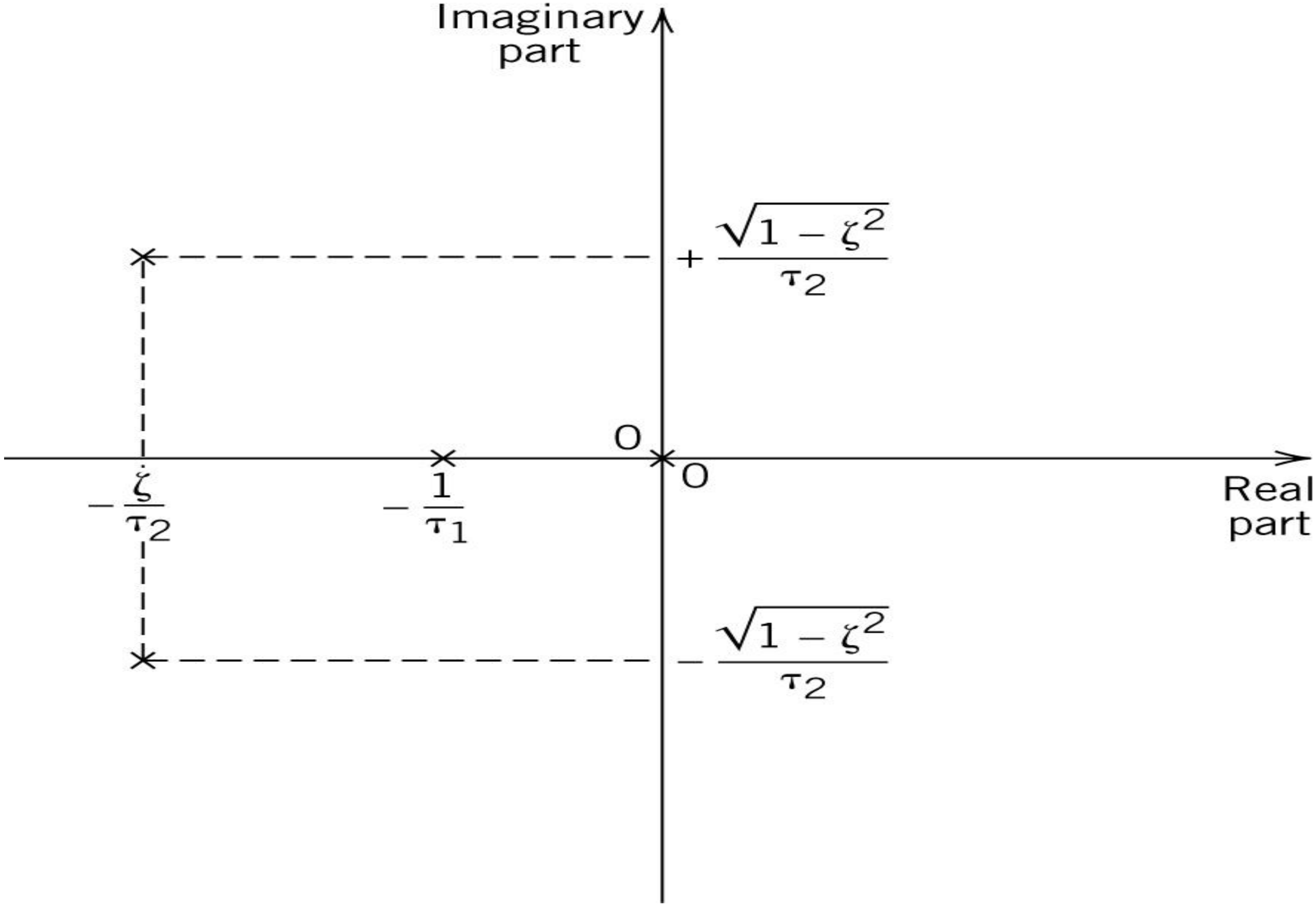
(2) An e^{-t/τ_1} term resulting from the factor $(\tau_1 s+1)$

(3) $e^{-\xi t/\tau_2} \sin \frac{\sqrt{1-\xi^2}}{\tau_2} t$

and

(4) $e^{-\xi t/\tau_2} \cos \frac{\sqrt{1-\xi^2}}{\tau_2} t$

Terms resulting from the $(\tau_2^2 s^2 + 2\xi\tau_2 s + 1)$



$$G(s) = \frac{K}{s(\tau_1 s + 1)(\tau^2 s^2 + 2\zeta\tau s + 1)}$$

Figure 5.1

4 poles (denominator is 4^{th} order polynomial)

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- **Zeros**

Value of s that cause the numerator of G(s) to become zero.

$$\tau_1 \frac{dy}{dt} + y = K(\mathrm{u} + \tau_a \frac{du}{dt}) \quad \Rightarrow \quad G(s) = \frac{K(\tau_a s + 1)}{(\tau_1 s + 1)}$$

$$\tau_1 \frac{dy}{dt} + y = K(\mathrm{u} + \frac{1}{\tau_a} \int_0^t u(t^*)\,\mathrm{dt}^*) \quad \Rightarrow \quad G(s) = \frac{K(\tau_a s + 1)}{\tau_a s(\tau_1 s + 1)}$$

More General Transfer Function Models

- **Poles and Zeros**

  The dynamic behavior of a transfer function model can be characterized by the numerical value of its poles and zeros.

- **General Representation of A TF**

There are two equivalent representations:

$$G(s) = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i}$$

$$G(s) = \frac{b_m(s - z_1)(s - z_2)\ldots(s - z_m)}{a_n(s - p_1)(s - p_2)\ldots(s - p_n)} \qquad (5\text{-}7)$$

where $\{z_i\}$ are the "zeros" and $\{p_i\}$ are the "poles".

- We will assume that there are no "pole-zero" calculations. That is, that no pole has the same numerical value as a zero.

- *Review:* $n \geq m$ in order to have a physically realizable system.

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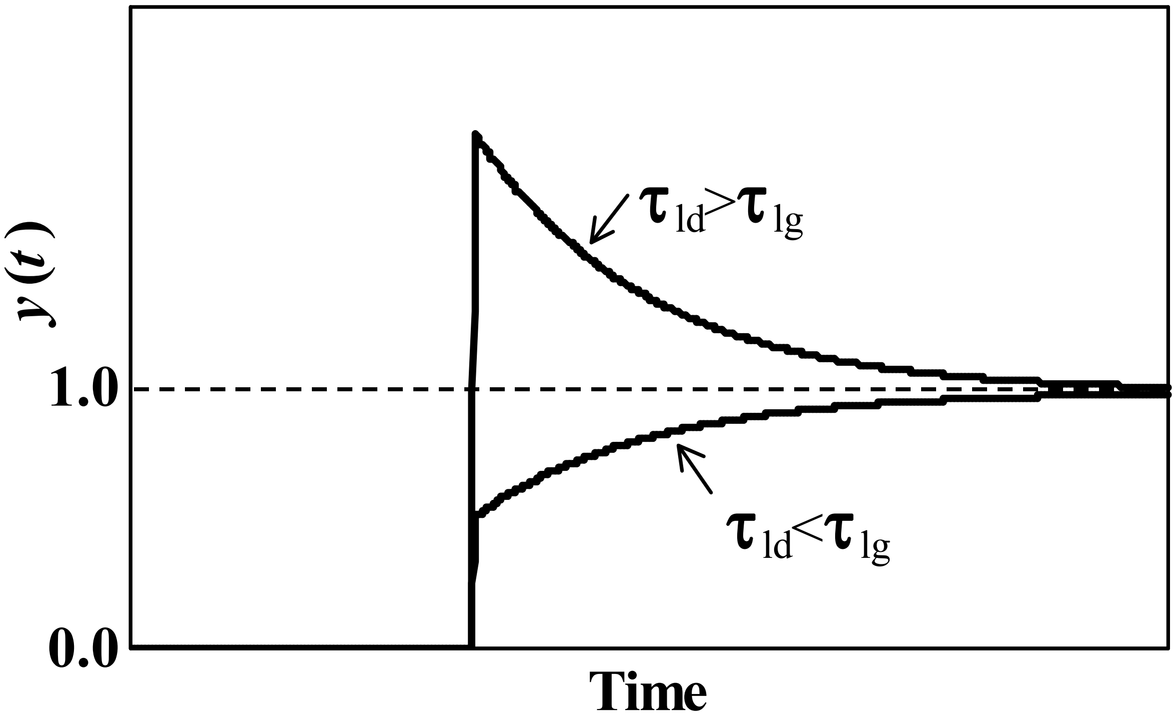
Lead-Lag Element

$$G(s) = \frac{\tau_{ld}\, s + 1}{\tau_{lg}\, s + 1}$$



Example 5.1

Calculate the response to the step input of magnitude *M* of the lead-lag element,

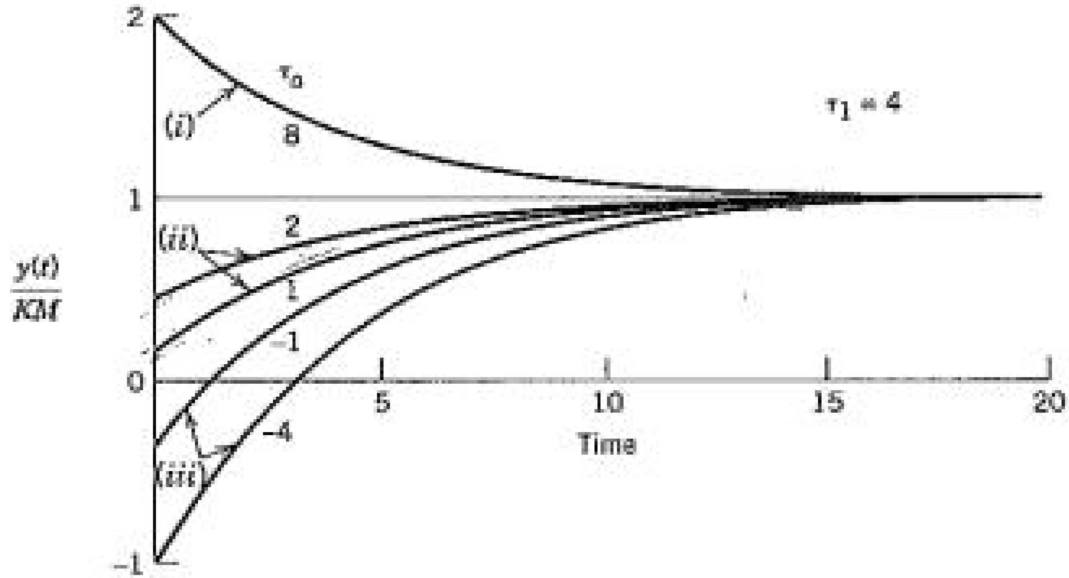
$$G(s)=\frac{K(\tau_a s+1)}{(\tau_1 s+1)}$$

Solution

The response of this system to a step change in input is

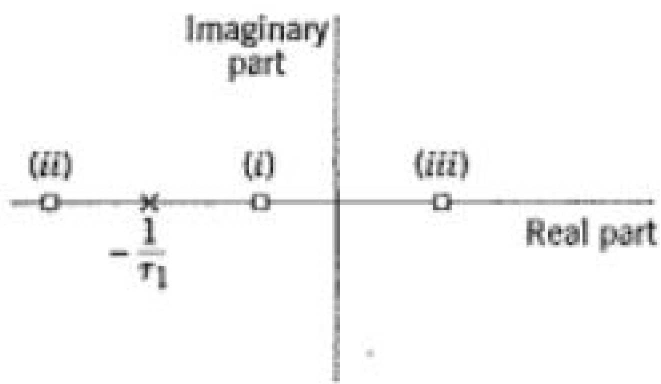
$$Y(s)=\frac{KM(\tau_a s+1)}{s(\tau_1 s+1)}$$

$$y(t)=KM\left(1-(1-\frac{\tau_a}{\tau_1})e^{-t/\tau_1}\right)$$



2
1
0
−1
y(t)/KM
(i)
(ii)
(iii)
τa
8
2
1
−1
−4
τ1 = 4
5
10
15
20
Time

(a)



Imaginary part
(ii)
(i)
(iii)
−1/τ1
Real part

(b)

Example 5.2

For the case of a single zero in an overdamped second-order transfer function,

$$G(s) = \frac{K(\tau_a s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} \tag{5-14}$$

calculate the response to the step input of magnitude *M* and plot the results qualitatively.

Solution

The response of this system to a step change in input is

$$y(t) = KM\left(1 + \frac{\tau_a - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_a - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2}\right) \tag{5-15}$$

Note that $y(t \to \infty) = KM$ as expected; hence, the effect of including the single zero does not change the final value nor does it change the number or location of the response modes. But the zero **does affect how the response modes (exponential terms) are weighted in the solution**, Eq. 5-15.

A certain amount of mathematical analysis will show that there are three types of responses involved here:

Case a: $\tau_a > \tau_1$

Case b: $0 < \tau_a \leq \tau_1$

Case c: $\tau_a < 0$

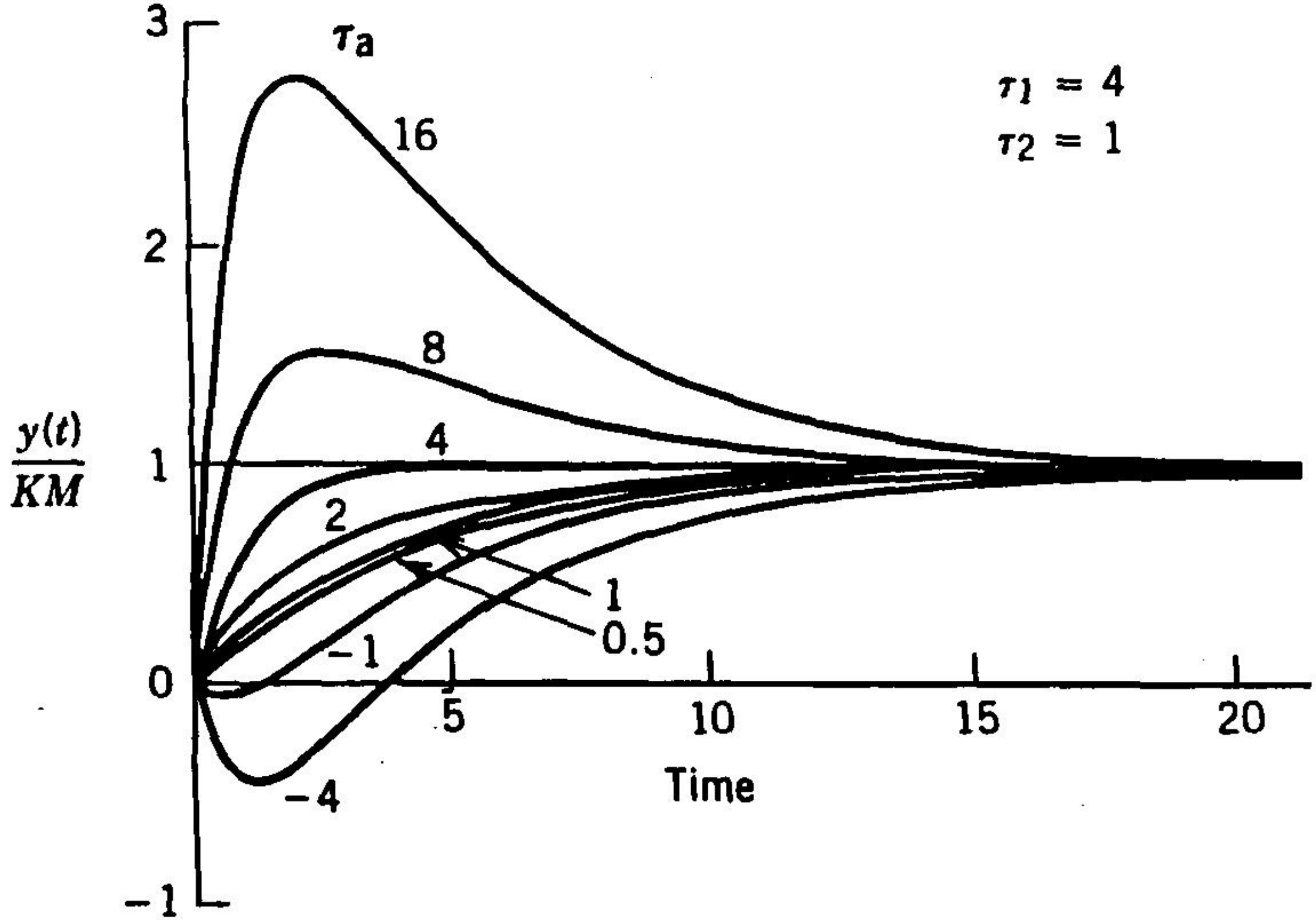


Figure 6.3. Step response of an overdamped second-order system (Eq. 6-14) with a single zero.

Inverse Response Due to Two Competing Effects

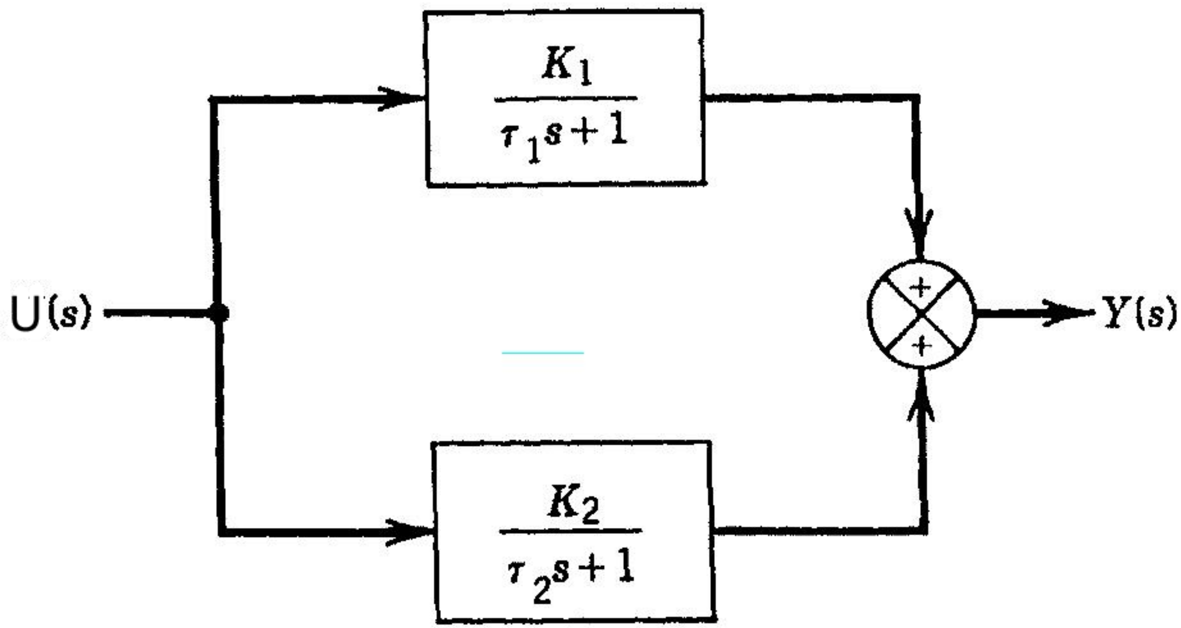


Figure 6.4. Two first-order process elements acting in parallel.

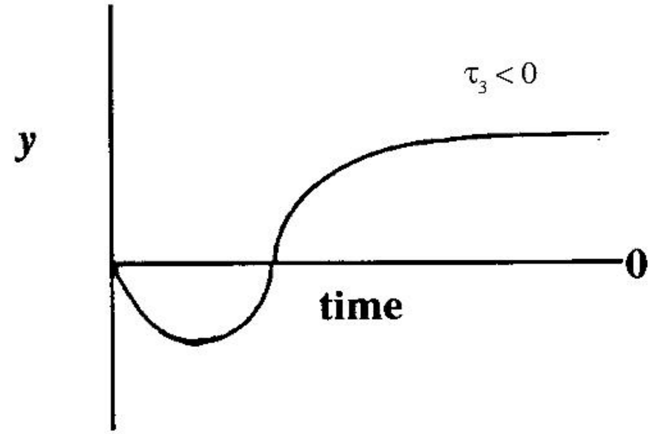
An inverse response occurs if:

$$-\frac{K_2}{K_1} > \frac{\tau_2}{\tau_1} \qquad (6\text{-}22)$$

Dynamic Response Characteristics of More Complicated Systems

Inverse Response

$$G(s) = \frac{K(1+\tau_3 s)}{(1+\tau_1 s)(1+\tau_2 s)}$$



If $\tau_3 > 0$ ….fast response

$\tau_3 < 0$ ....inverse response

τ_3 : zero of transfer function (see Fig. 5.3)

Use nonlinear regression for fitting data (graphical method not available)

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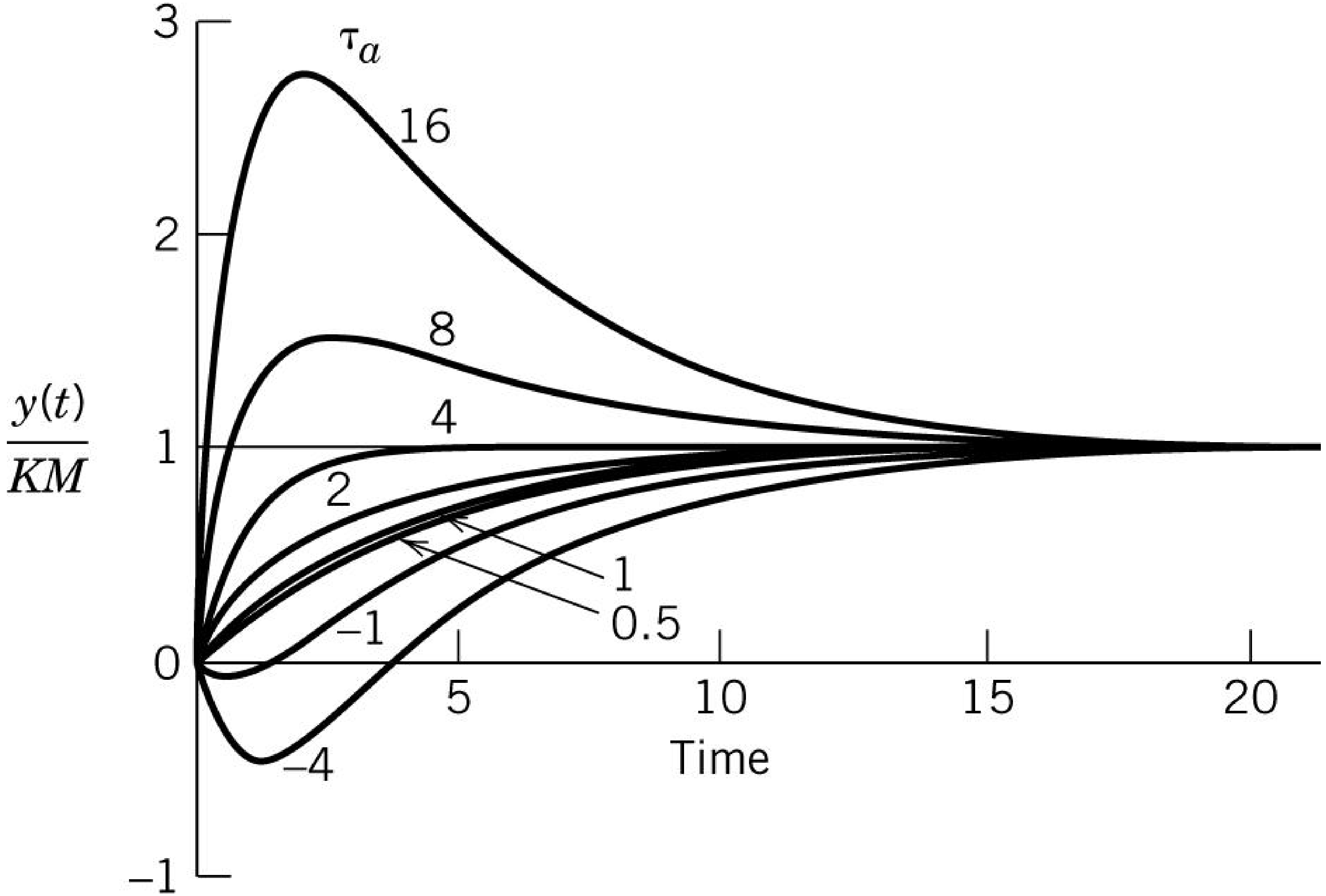


Figure 5.3

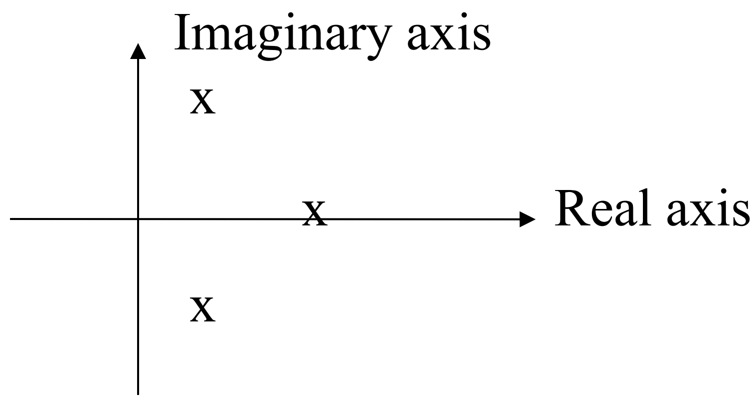
Summary: Effects of Pole and Zero Locations

1. Poles

- *Pole in "right half plane (RHP)"*: results in unstable system (i.e., unstable step responses)

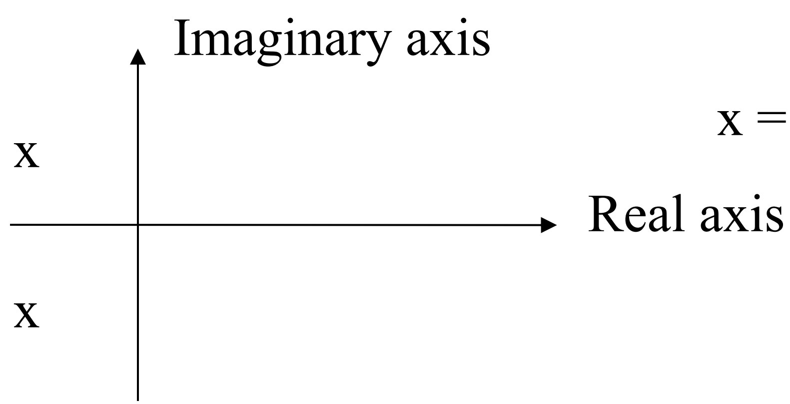
$$p = a + bj$$

$$\left(j = \sqrt{-1} \right)$$



x = unstable pole

- *Complex pole*: results in oscillatory responses



x = complex poles

Time Delays

Time delays occur due to:

1. Fluid flow in a pipe
2. Transport of solid material (e.g., conveyor belt)
3. Chemical analysis
   - Time required to do the analysis (e.g., on-line gas chromatograph)

Mathematical description:

A time delay, θ, between an input u and an output y results in the following expression:

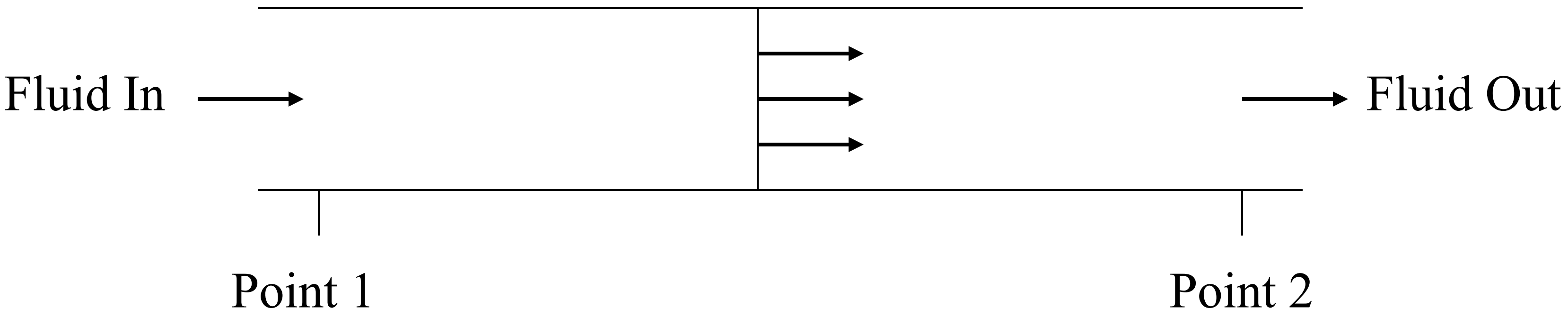
$$y(t) = \begin{cases} 0 & \text{for } t < \theta \\ u(t-\theta) & \text{for } t \geq \theta \end{cases}$$

Transfer Function Representation: $$\frac{Y(s)}{U(s)} = e^{-\theta s}$$

Example: Turbulent flow in a pipe

Let, $u \triangleq$ fluid property (e.g., temperature or composition) at point 1

$y \triangleq$ fluid property at point 2



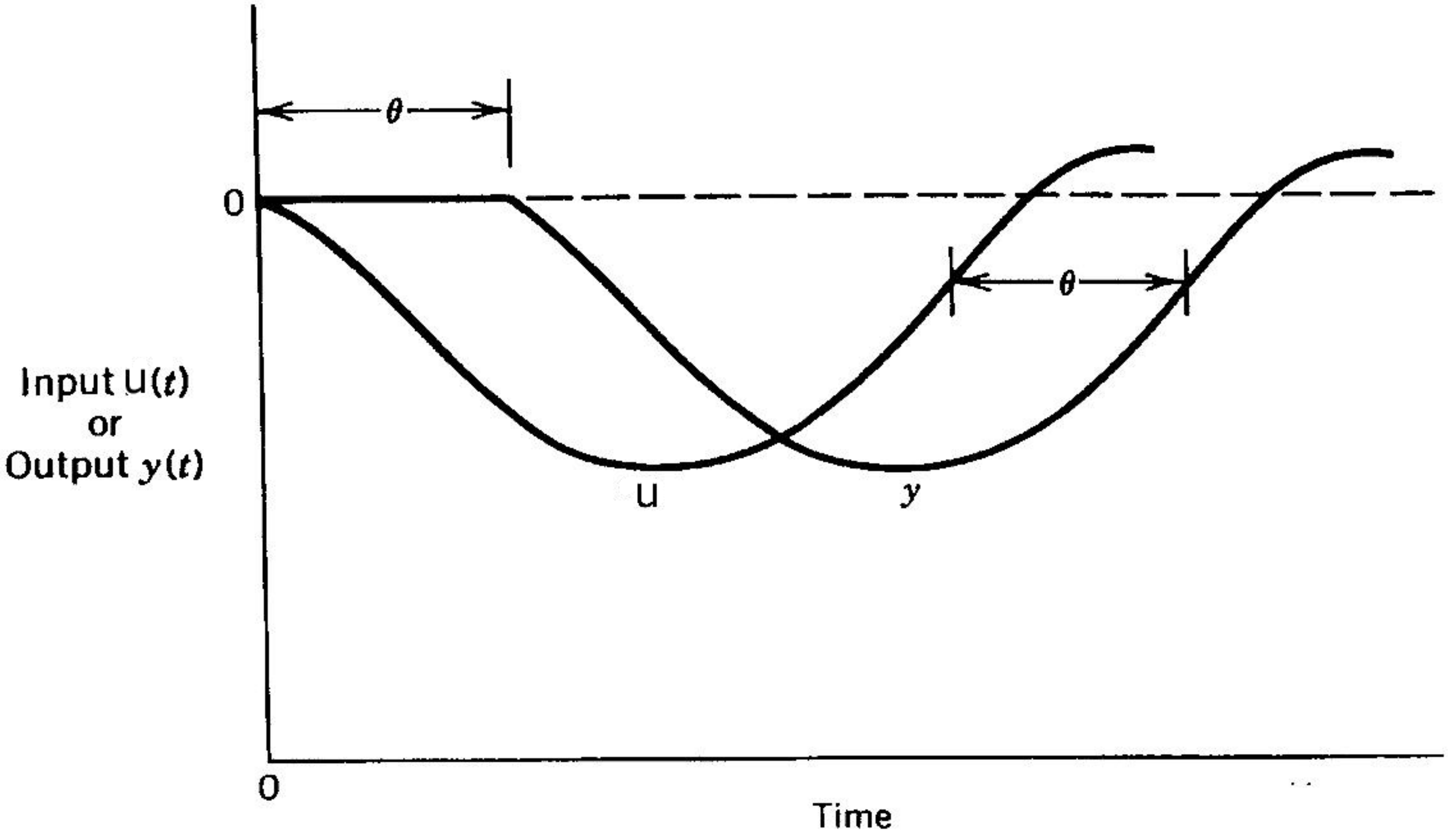
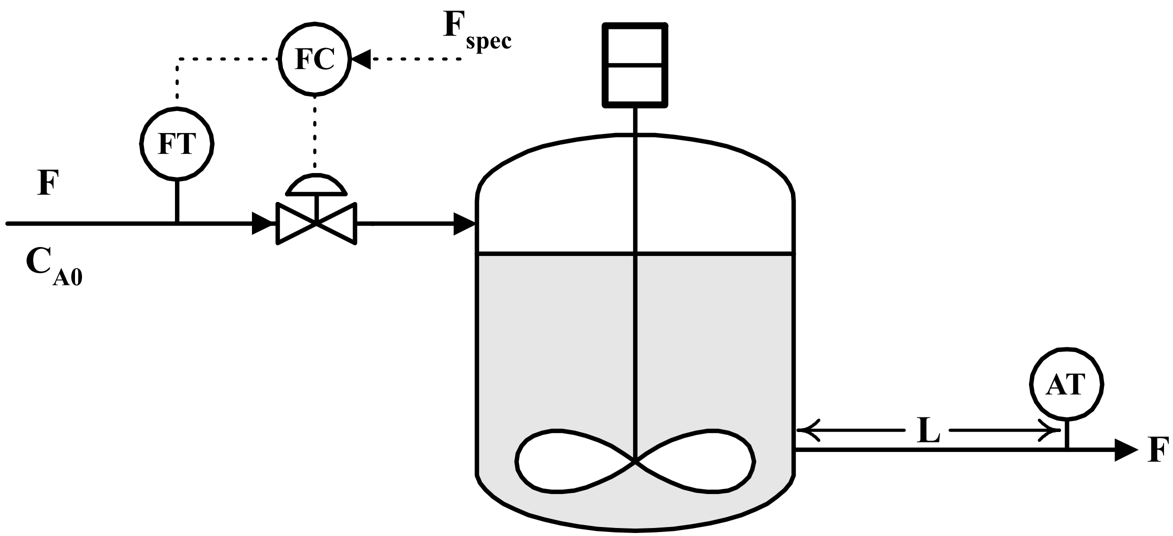


Figure 6.6. The effect of a pure time delay is a translation of the function in time.

Deadtime



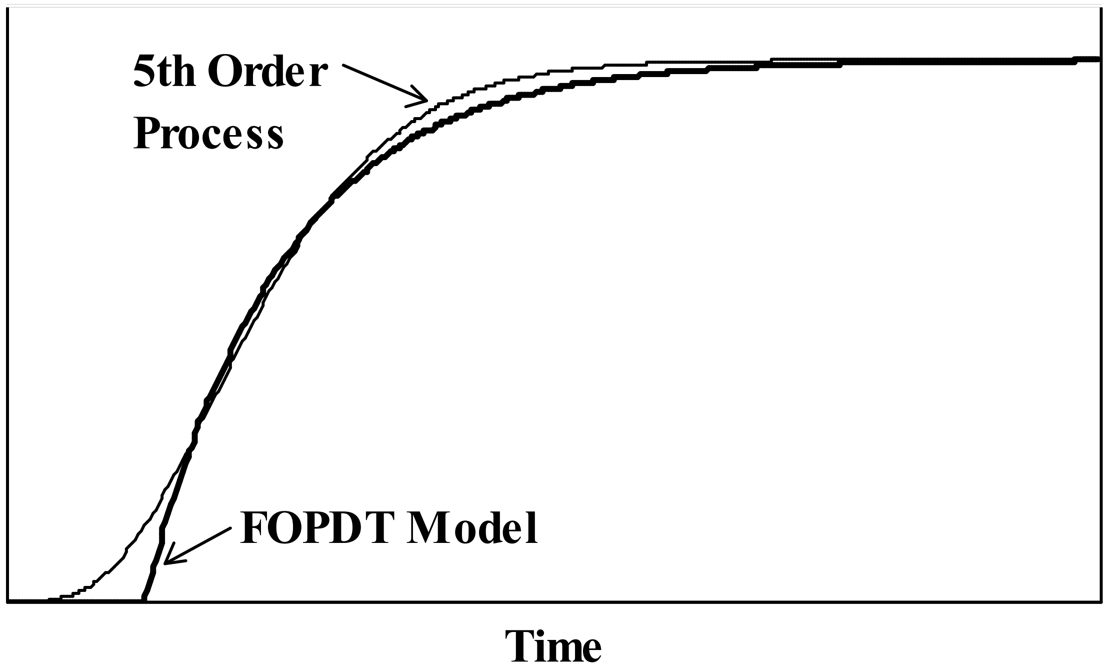
- Transport delay from reactor to analyzer:

$$C_s(t) = C(t-\theta) \quad \text{where} \quad \theta = \rho\, L\, A_c / F$$

- Transfer function:

$$G_p(s) = e^{-\theta s}$$

FOPDT Model



- High order processes are well represented by FOPDT models. As a result, FOPDT models do a better job of approximating industrial processes than other idealized dynamic models.