第 6 章 （之 3）
第 28 次作业

教学内容：§6.1.3 不定积分的分部积分法

求下列不定积分 1—15：

**1. $\int \frac{\ln x}{\sqrt{x}} \mathrm{d}x$.

解：$\int \frac{\ln x}{\sqrt{x}} dx = \int \ln x d(2\sqrt{x}) = 2\sqrt{x} \cdot \ln x - \int 2\sqrt{x} \cdot \frac{1}{x} dx$

$$= 2\sqrt{x}\ln x - 2\int \frac{1}{\sqrt{x}} dx \quad = 2\sqrt{x}\ln x - 4\sqrt{x} + C .$$

**2. $\int x \sin x \cos x dx$.

解：原式 $= \frac{1}{2}\int x \cdot \sin 2x \cdot dx = -\frac{1}{4}\int x \cdot d\cos 2x$

$$= -\frac{1}{4}x\cos 2x + \frac{1}{4}\int \cos 2x \cdot dx = -\frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C .$$

**3. $\int \frac{x\sin x}{\cos^3 x} dx$.

解：原式 $= -\int \frac{xd\cos x}{\cos^3 x} dx \quad = \frac{1}{2}\int xd(\cos x)^{-2}$

$$= \frac{1}{2}x(\cos x)^{-2} - \frac{1}{2}\int \frac{1}{\cos^2 x} dx = \frac{1}{2}\frac{x}{\cos^2 x} - \frac{1}{2}\tan x + C .$$

**4. $\int x \cdot \tan x \cdot \sec^4 x \cdot dx$.

解：原式 $= \int x \cdot \sec^3 x \cdot d\sec x = \frac{1}{4}\int x \cdot d\sec^4 x$

$$= \frac{1}{4}x \cdot \sec^4 x - \frac{1}{4}\int \sec^4 x \cdot dx = \frac{x}{4}\sec^4 x - \frac{1}{4}\int (\tan^2 x + 1) d\tan x$$

$$= \frac{x}{4}\sec^4 x - \frac{1}{12}\tan^3 x - \frac{1}{4}\tan x + C .$$

***5. $\int e^x \cos^2 x dx$.

解：$\int e^x \cos^2 x dx = \int \cos^2 x \cdot de^x = e^x\cos^2 x + 2\int e^x \cos x \sin x dx$

$$= e^x \cos^2 x + \int \sin 2x \cdot de^x$$

$$\int \sin 2x \cdot de^x = e^x \sin 2x - 2\int e^x \cos 2x dx$$

$$=e^x\sin 2x-2\int\cos 2x de^x=e^x\sin 2x-2e^x\cos 2x-4\int e^x\sin 2x dx$$

$$\int\sin 2x\cdot e^x dx=\frac{1}{5}e^x\sin 2x-\frac{2}{5}e^x\cos 2x+C$$

原式$=e^x\cos^2 x+\frac{1}{5}e^x\sin 2x-\frac{2}{5}e^x\cos 2x+C$.

注：也可先将 $\cos^2 x$ 写成 $\frac{1+\cos 2x}{2}$.

答案也可以是 ： $\frac{1}{2}e^x+\frac{1}{10}e^x\cos 2x+\frac{1}{5}e^x\sin 2x+C$

**6. $\int\ln(\cos x)\cdot\cos 2x\cdot dx$.

解：$\int\ln(\cos x)\cdot\cos 2x\cdot dx=\frac{1}{2}\int\ln(\cos x)\cdot d\sin 2x$

$$=\frac{1}{2}\sin 2x\cdot\ln(\cos x)-\frac{1}{2}\int\sin 2x\cdot\frac{-\sin x}{\cos x}dx=\frac{1}{2}\sin 2x\cdot\ln(\cos x)+\int\frac{1-\cos 2x}{2}\cdot dx$$

$$=\frac{1}{2}\sin 2x\cdot\ln(\cos x)+\frac{1}{2}x-\frac{1}{4}\sin 2x+C.$$

**7. $\int\sin x\cdot\ln\tan x\,\mathrm{d}x$.

解：原式$=-\int\ln\tan x d(\cos x)$

$$=-\cos x\cdot\ln\tan x+\int\cos x\cdot\frac{1}{\tan x}\cdot\sec^2 x dx$$

$$=-\cos x\ln\tan x+\int\frac{1}{\sin x}dx=-\cos x\cdot\ln\tan x+\ln\left|\tan\frac{x}{2}\right|+C.$$

**8. $\int\frac{x\cdot\cos x}{\sin^2 x}\mathrm{d}x$.

解：原式$=-\int x d(\frac{1}{\sin x})=-\frac{x}{\sin x}+\int\frac{1}{\sin x}dx$

$$=-\frac{x}{\sin x}+\ln|\csc x-\cot x|+C.$$

**9. $\int\frac{\arctan x}{x^2(1+x^2)}dx$.

解：原式$=\int\frac{\arctan x}{x^2}dx-\int\frac{\arctan x}{1+x^2}dx$

$$=-\int\arctan x d\frac{1}{x}-\int\arctan x d\arctan x$$

$$=-\frac{1}{x}\arctan x+\int\frac{1}{x}\cdot\frac{1}{1+x^2}dx-\frac{1}{2}\arctan^2 x$$

$$=-\frac{1}{x}\arctan x+\ln|x|-\frac{1}{2}\ln\left|1+x^2\right|-\frac{1}{2}\arctan^2 x+C.$$

**10. $\int\frac{x\arcsin x}{\sqrt{1-x^2}}\mathrm{d}x$.

解：$\int\frac{x\arcsin x}{\sqrt{1-x^2}}dx=-\int\frac{-x}{\sqrt{1-x^2}}\arcsin xdx=-\int\arcsin xd\sqrt{1-x^2}$

$$=-\sqrt{1-x^2}\cdot\arcsin x+\int\sqrt{1-x^2}\cdot\frac{dx}{\sqrt{1-x^2}}$$

$$=-\sqrt{1-x^2}\arcsin x+x+C.$$

**11. $\int x\sin\sqrt{x}\,\mathrm{d}x$.

解：令 $\sqrt{x}=u$， 则 $x=u^2$， $dx=2udu$，

原式 $=2\int u^3\sin udu=2\left[-u^3\cos u+3\int u^2\cos udu\right]$

$$=2(-u^3\cos u+3u^2\sin u-6\int u\sin udu)$$

$$=2(-u^3\cos u+3u^2\sin u+6u\cos u-6\sin u)+C$$

$$=2u(6-u^2)\cos u+6(u^2-2)\sin u+C$$

$$=2\sqrt{x}(6-x)\cos\sqrt{x}+6(x-2)\sin\sqrt{x}+C.$$

***12. $\int\frac{xe^x}{\sqrt{e^x-2}}dx$.

解：$\int\frac{xe^x}{\sqrt{e^x-2}}dx$(令$t=\sqrt{e^x-2},x=\ln(t^2+2)$)

$$=\int\frac{(t^2+2)\cdot\ln(t^2+2)}{t}\cdot\frac{2t}{t^2+2}\cdot dt$$

$$=2\int\ln(t^2+2)dt=2t\cdot\ln(t^2+2)-2\int t\cdot\frac{2t}{t^2+2}\cdot dt$$

$$=2t\cdot\ln(t^2+2)-4\int(1-\frac{2}{t^2+2})\mathrm{d}t$$

$$=2t\cdot\ln(t^2+2)-4t+4\sqrt{2}\arctan\frac{t}{\sqrt{2}}+C$$

$$=2x\cdot\sqrt{e^x-2}-4\sqrt{e^x-2}+4\sqrt{2}\arctan\sqrt{\frac{e^x-2}{2}}+C.$$

**13. $\int e^{\arcsin x}dx$.

解：原式$=x\cdot e^{\arcsin x}-\int x\cdot e^{\arcsin x}\cdot\frac{1}{\sqrt{1-x^2}}dx$

$$=x\cdot e^{\arcsin x}+\int e^{\arcsin x}d\sqrt{1-x^2}$$

$$= x \cdot e^{\arcsin x} + \sqrt{1-x^2} \cdot e^{\arcsin x} - \int \sqrt{1-x^2} \cdot e^{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}} \cdot dx$$

原式$= \frac{1}{2}(x + \sqrt{1-x^2})e^{\arcsin x} + C$.

***14. $\int \cos(\ln x)dx$.

解： $\int \cos(\ln x)dx = x\cos(\ln x) - \int x[-\sin(\ln x)]\frac{1}{x}dx = x\cos(\ln x) + \int \sin(\ln x)dx$

$= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)dx$

所以 $\int \cos(\ln x)dx = \frac{x[\cos(\ln x) + \sin(\ln x)]}{2} + C$.

***15. $\int \sqrt{1-x^2} \arcsin x dx$.

解：令 $x = \sin t \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right)$ ，则 $\sqrt{1-x^2} = \cos t$, $dx = \cos t dt$ ，于是

$\int \sqrt{1-x^2} \arcsin x dx = \int t\cos^2 t dt = \frac{1}{2}\int t(1+\cos 2t)dt = \frac{1}{4}\int t d(2t + \sin 2t)$

$= \frac{t(2t+\sin 2t)}{4} - \frac{1}{4}\int (2t + \sin 2t)dt = \frac{t^2}{4} + \frac{t}{4}\sin 2t - \frac{1}{4}\sin^2 t + C$

$= \frac{(\arcsin x)^2}{4} + \frac{x}{2}\sqrt{1-x^2}\arcsin x - \frac{1}{4}x^2 + C$.

**16 已知 $f'(e^x) = x,\ f(1) = 0$，求 $f(x)$.

解一：已知 $f'(e^x) = x$,即$\frac{df(e^x)}{de^x} = x$， 或 $df(e^x) = x \cdot de^x$，

两边积分，得 $f(e^x) = xe^x - e^x + C$，

由 $f(1) = 0$， 得 $C = 1$，

故 $f(e^x) = xe^x - e^x + 1$ 令 $e^x = u$，得

$f(u) = u \cdot \ln u - u + 1$， 即 $f(x) = x \cdot \ln x - x + 1$.

解二：已知 $f'(e^x) = x$，令 $e^x = u$，则有

$f'(u) = \ln(u)$， 两边积分，得 $f(u) = u \cdot \ln u - u + C$，

由 $f(1) = 0$，得$C = 1$.

所以 $f(u) = u \cdot \ln u - u + 1$，即 $f(x) = x \cdot \ln x - x + 1$。

***17. 若 $I_n \overset{def}{=\!=} \int \sec^n x\, dx$，试证降阶递推公式：

$$I_n = \frac{1}{n-1}(\tan x)(\sec^{n-2} x) + \frac{n-2}{n-1} I_{n-2}.$$

证明：$I_n = \int \sec^n x dx = \int \sec^{n-2} x\, d\tan x$

$$=\tan x\sec^{n-2}x-\int\tan^2 x(n-2)\sec^{n-3}x\sec x dx$$
$$=\tan x\sec^{n-2}x-(n-2)[\int\sec^n x dx-\int\sec^{n-2}x dx]$$
$$=\tan x\sec^{n-2}x-(n-2)[I_n-I_{n-2}]$$
$$\therefore (n-1)I_n=\tan x\sec^{n-2}x+(n-2)I_{n-2},$$
$$I_n=\frac{\tan x\sec^{n-2}x}{n-1}+\frac{n-2}{n-1}I_{n-2}.$$

***18. 导出计算积分$I_n=\int x^n\cos x\,\mathrm{d}\,x$的递推公式,其中$n$为自然数.

解: $I_n=\int x^n\cos xdx=\int x^n d\sin x=x^n\sin x-\int\sin x\cdot nx^{n-1}dx$
$$=x^n\sin x+n\int x^{n-1}d\cos x$$
$$=x^n\sin x+nx^{n-1}\cos x-n(n-1)I_{n-2},(n\ge 2),$$

为了能启动运算，还必须求出

$$I_1=\int x\cos xdx=\int xd\sin x=x\sin x-\int\sin xdx=x\sin x+\cos x+C,$$
$$I_0=\int\cos xdx=\sin x+C.$$

第 6 章 （之 4）
第 29 次作业

教学内容： §6.1.4 几种特殊类型函数的积分

求下列不定积分：

**1. $\int\frac{2x+3}{x^2+8x+16}dx$.

解：$\frac{2x+3}{x^2+8x+16}=\frac{2x+3}{(x+4)^2}=\frac{2(x+4)-5}{(x+4)^2}=\frac{2}{x+4}-\frac{5}{(x+4)^2}$,

$\therefore$ 原式$=2\ln|x+4|+\frac{5}{x+4}+C$.

***2. $\int\frac{8x-7}{9x^2-12x+5}dx$.

解：$\frac{8x-7}{9x^2-12x+5}=\frac{8x-7}{(3x-2)^2+1}=\frac{8(x-\frac{2}{3})-\frac{5}{3}}{(3x-2)^2+1}$

$\therefore$ 原式$=\int\frac{8x-\frac{16}{3}}{9x^2-12x+5}dx+\int\frac{-\frac{5}{3}}{(3x-2)^2+1}dx$

$$=\int \frac{\frac{8}{9}(9x-6)dx}{9x^2-12x+x}+\int \frac{-\frac{5}{3}}{(3x-2)^2+1}dx$$

$$=\frac{4}{9}\int \frac{d(9x^2-12x+5)}{9x^2-12x+5}-\frac{5}{9}\int \frac{d(3x-2)}{1+(3x-2)^2}$$

$$=\frac{4}{9}\ln(9x^2-12x+5)-\frac{5}{9}\arctan(3x-2)+C.$$

**3. $\int \frac{3x^4+3x^2+1}{x^2+1}\mathrm{d}x$.

解：原式$=\int\left(3x^2+\frac{1}{x^2+1}\right)\mathrm{d}x=x^3+\arctan x+C$.

**4. $\int \frac{\mathrm{d}x}{(3+x^2)\cdot 2x^2}$.

解：原式 $=\frac{1}{2}\cdot\frac{1}{3}\int(\frac{1}{x^2}-\frac{1}{3+x^2})\mathrm{d}x$

$$=\frac{1}{6}(-\frac{1}{x})-\frac{1}{6\sqrt{3}}\arctan\frac{x}{\sqrt{3}}+C.$$

**5. $\int \frac{\mathrm{d}x}{x^3-2x^2+x}$.

解:$\int \frac{dx}{x^3-2x^2+x}=\int \frac{1}{x(x-1)^2}dx \quad =\int\left[\frac{1}{x}-\frac{1}{x-1}+\frac{1}{(x-1)^2}\right]dx$

$$=\int \frac{1}{x}dx-\int \frac{1}{x-1}dx+\int \frac{1}{(x-1)^2}dx=\ln|x|-\ln|x-1|-\frac{1}{x-1}+C.$$

**6. $\int \frac{\mathrm{d}x}{1+\sin x}$.

解：$\int \frac{dx}{1+\sin x}=\int \frac{1-\sin x}{(1+\sin x)(1-\sin x)}dx$

$$=\int \frac{1}{\cos^2 x}dx-\int \frac{\sin x}{\cos^2 x}dx$$

$$=\tan x+\int \frac{d\cos x}{\cos^2 x} \quad =\tan x-\frac{1}{\cos x}+C.$$

答案也可以是： $-\dfrac{2}{1+\tan\frac{x}{2}}+C$

**7. $\int \frac{1+\tan x}{\sin 2x}\mathrm{d}x$.

解:$\int \frac{1+\tan x}{\sin 2x}dx = \frac{1}{2}\int \frac{1+\tan x}{\sin x\cos x}dx = \frac{1}{2}\int \frac{1+\tan x}{\tan x}d\tan x$

$$= \frac{1}{2}\int \frac{1}{\tan x}d\tan x + \frac{1}{2}\int d\tan x = \frac{1}{2}\ln|\tan x| + \frac{1}{2}\tan x + C.$$

**8. $\int \frac{\sin x}{\sin x+\cos x}\mathrm{d}x$.

解：$\int \frac{\sin x}{\sin x+\cos x}\mathrm{d}x = \int \frac{\tan x}{\tan x+1}\mathrm{d}x$，设 $\tan x = t$，则 $x = \arctan t$，

$\int \frac{\tan x}{\tan x+1}\mathrm{d}x = \int \frac{t}{t+1}\cdot\frac{1}{1+t^2}dt = \frac{1}{2}\int(\frac{-1}{t+1}+\frac{t+1}{1+t^2})dt$

$= -\frac{1}{2}\ln|1+t| + \frac{1}{4}\ln(1+t^2) + \frac{1}{2}\arctan t + C$

$= -\frac{1}{2}\ln|1+\tan x| + \frac{1}{4}\ln(1+(\tan x)^2) + \frac{1}{2}x + C$.

**9. $\int \frac{x}{\sqrt{x+1}+\sqrt[4]{x+1}}dx$.

解：令 $\sqrt[4]{x+1} = t$， $\therefore x = t^4 - 1$，

$\therefore$ 原式$= \int \frac{t^4-1}{t+t^2}4t^3dt = 4\int \frac{t^2(t^2+1)(t-1)(t+1)}{1+t}dt$

$= 4\int t^2(t^2+1)(t-1)dt = 4\int(t^5-t^4+t^3-t^2)dt$

$= 4\left(\frac{t^6}{6}-\frac{t^5}{5}+\frac{t^4}{4}-\frac{t^3}{3}\right)+C = \frac{2(x+1)^{\frac{3}{2}}}{3} - \frac{4(x+1)^{\frac{5}{4}}}{5} + (x+1) - \frac{4(x+1)^{\frac{3}{4}}}{3} + C$.

**10. $\int \frac{1}{x^2}\sqrt[5]{\left(\frac{x}{x+1}\right)^3}dx$.

解：令 $\sqrt[5]{\frac{x}{x+1}} = t$ $\therefore x = \frac{t^5}{1-t^5}$ $dx = \frac{5t^4}{(1-t^5)^2}dt$

$\therefore$ 原式$= \int \frac{(1-t^5)^2}{t^{10}}t^3\cdot\frac{5t^4}{(1-t^5)^2}dt = \int 5t^{-3}dt = \frac{-5}{2t^2} + c = -\frac{5}{2}\sqrt[5]{\left(\frac{x+1}{x}\right)^2} + C$.

**11. $\int \frac{dx}{\sqrt{x}-2\sqrt[3]{x}-3\sqrt[6]{x}}$.

解：(令 $\sqrt[6]{x} = t$)

原式$= \int \frac{6t^5dt}{t^3-2t^2-3t} = 6\int[t^2+2t+7+\frac{1}{4}(\frac{81}{t-3}-\frac{1}{t+1})]\cdot dt$

$= 2t^3 + 6t^2 + 42t + \frac{243}{2}\ln|t-3| - \frac{3}{2}\ln|t+1| + C$

$$= 2\sqrt{x} + 6\sqrt[3]{x} + 42\sqrt[6]{x} + \frac{243}{2}\ln\left|\sqrt[6]{x} - 3\right| - \frac{3}{2}\ln\left|\sqrt[6]{x} + 1\right| + C.$$

***12. $\int \frac{dx}{\sqrt[4]{x(1+x)^7}}$.

解：原式$= \int \sqrt[4]{\frac{1+x}{x}} \cdot \frac{dx}{(1+x)^2}$ （令 $\sqrt[4]{\frac{1+x}{x}} = t,\ x = \frac{1}{t^4 - 1}$）

$$= \int t \cdot \frac{(t^4-1)^2}{t^8} \cdot \frac{-4t^3}{(t^4-1)^2}dt = -\int \frac{4}{t^4}dt = \frac{4}{3} \cdot \frac{1}{t^3} + C = \frac{4}{3}\sqrt[4]{\left(\frac{x}{1+x}\right)^3} + C.$$

***13. $\int \frac{dx}{1 + \sin x + \cos x}$.

解：令 $t = \tan\frac{x}{2}$ ， 则

$$\int \frac{dx}{1+\sin x+\cos x} = \int \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}dt$$

$$= \int \frac{dt}{1+t} = \ln|1+t| + C = \ln\left|1 + \tan\frac{x}{2}\right| + C.$$

第 6 章 （之 5）

第 30 次作业

教学内容：6. 2. 1 定积分的换元积分法　　6. 2. 2 定积分的分部积分法

计算定积分 1—10：

**1. $\int_0^{\frac{\pi}{4}} \cos^2 x dx$.

解：$\int_0^{\frac{\pi}{4}} \cos^2 x dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} (1 + \cos 2x)dx = \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} = \frac{\pi + 2}{8}$.

**2. $\int_1^{\sqrt{2}} \frac{2x \cdot e^{\arctan(x^2-1)}}{x^4 - 2x^2 + 2}dx$.

解：原式$= \int_1^{\sqrt{2}} \frac{e^{\arctan(x^2-1)}}{(x^2-1)^2+1}d(x^2-1) = \int_1^{\sqrt{2}} e^{\arctan(x^2-1)}d[\arctan(x^2-1)]$

$$= e^{\arctan(x^2-1)}\Big|_1^{\sqrt{2}} = e^{\frac{\pi}{4}} - 1 \ .$$

**3. $\int_{-1}^{0} \frac{2x+1}{x^2 - 3x + 2}dx$.

解：原式$=\int_{-1}^{0}(\frac{5}{x-2}+\frac{-3}{x-1})dx$

$=(5\ln|x-2|-3\ln|x-1|)\Big|_{-1}^{0}$

$=8\ln 2-5\ln 3$.

**4. $\int_{\frac{1}{2}}^{\frac{3}{4}}\frac{x+1}{\sqrt{x-x^2}}dx$.

解：原式$=\int_{\frac{1}{2}}^{\frac{3}{4}}\frac{x+1}{\sqrt{\frac{1}{4}-(x-\frac{1}{2})^2}}\mathrm{d}x \xlongequal{x-\frac{1}{2}=\frac{1}{2}\sin t} \int_{0}^{\frac{\pi}{6}}\frac{\frac{1}{2}\sin t+\frac{3}{2}}{\frac{1}{2}\cos t}\frac{1}{2}\cos t\mathrm{d}t$

$=\int_{0}^{\frac{\pi}{6}}(\frac{1}{2}\sin t+\frac{3}{2})dt=(-\frac{1}{2}\cos t+\frac{3}{2}t)\Big|_{0}^{\frac{\pi}{6}}=\frac{\pi}{4}+\frac{1}{2}-\frac{\sqrt{3}}{4}$.

*5. $\int_{1}^{\sqrt{3}}\frac{dx}{x^2\sqrt{1+x^2}}$.

解： $\int_{1}^{\sqrt{3}}\frac{dx}{x^2\sqrt{1+x^2}} \xlongequal{x=\frac{1}{t}} \int_{1}^{\frac{1}{\sqrt{3}}}\frac{-tdt}{\sqrt{1+t^2}}=\left[-\sqrt{1+t^2}\right]_{1}^{\frac{1}{\sqrt{3}}}=\sqrt{2}-\frac{2\sqrt{3}}{3}$

另解：令$x=\tan\theta$，原式$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sec^2\theta d\theta}{\tan^2\theta\cdot\sec\theta}=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos\theta\cdot d\theta}{\sin^2\theta}=-\frac{1}{\sin\theta}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}=\sqrt{2}-\frac{2\sqrt{3}}{3}$.

***6. $\int_{0}^{2-\sqrt{2}}\frac{dx}{(x+\sqrt{2})\sqrt{x^2+2\sqrt{2}x+1}}$.

解： 原式 $=\int_{0}^{2-\sqrt{2}}\frac{dx}{(x+\sqrt{2})\sqrt{(x+\sqrt{2})^2-1}} \xlongequal{x+\sqrt{2}=t} \int_{\sqrt{2}}^{2}\frac{dt}{t\sqrt{t^2-1}}$

$\xlongequal{t=\frac{1}{x}} \int_{\frac{1}{\sqrt{2}}}^{\frac{1}{2}}-\frac{1}{x^2}\frac{x^2}{\sqrt{1-x^2}}dx$

$=\arccos x\Big|_{\frac{1}{\sqrt{2}}}^{\frac{1}{2}}=\frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}$.

**7. $\int_{1}^{\sqrt{e}}\cos(\pi\ln x)\cdot dx$.

解：原式$=x\cos(\pi\cdot\ln x)\Big|_{1}^{\sqrt{e}}+\int_{1}^{\sqrt{e}}x\cdot\sin(\pi\cdot\ln x)\cdot\frac{\pi}{x}\cdot dx$

$=-1+\pi x\cdot\sin(\pi\cdot\ln x)\Big|_{1}^{\sqrt{e}}-\pi^2\int_{1}^{\sqrt{e}}\cos(\pi\cdot\ln x)dx$

$=-1+\sqrt{e}\cdot\pi-\pi^2\cdot\int_{1}^{\sqrt{e}}\cos(\pi\ln x)\cdot dx\cdot$

原式$=\dfrac{\pi\sqrt{e}-1}{1+\pi^2}$

**8. $\displaystyle\int_1^3 x^2\ln(3x)dx$.

解：原式$=\dfrac{1}{3}\displaystyle\int_1^3 \ln(3x)dx^3 \quad =\dfrac{1}{3}x^3\ln(3x)\Big|_1^3-\dfrac{1}{3}\int_1^3 x^2dx$

$=\dfrac{53}{3}\ln 3-\dfrac{1}{9}x^3\Big|_1^3=\dfrac{53}{3}\ln 3-\dfrac{26}{9}$.

***9. $\displaystyle\int_0^{\frac{\pi}{2}} e^{2x}\cos xdx$.

解： $\displaystyle\int_0^{\frac{\pi}{2}} e^{2x}\cos xdx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos xd\left(e^{2x}\right)=\frac{1}{2}\left[e^{2x}\cos x\right]_0^{\frac{\pi}{2}}+\frac{1}{2}\int_0^{\frac{\pi}{2}}e^{2x}\sin xdx$

$\displaystyle=-\frac{1}{2}+\frac{1}{4}\int_0^{\frac{\pi}{2}}\sin xd\left(e^{2x}\right)=-\frac{1}{2}+\frac{1}{4}\left[e^{2x}\sin x\right]_0^{\frac{\pi}{2}}-\frac{1}{4}\int_0^{\frac{\pi}{2}}e^{2x}\cos xdx$

$\displaystyle=-\frac{1}{2}+\frac{e^{\pi}}{4}-\frac{1}{4}\int_0^{\frac{\pi}{2}}e^{2x}\cos xdx$

所以 $\displaystyle\int_0^{\frac{\pi}{2}}e^{2x}\cos xdx=\frac{1}{5}\left(e^{\pi}-2\right)$

**10. $\displaystyle\int_{-\pi}^{\pi}|\cos x|\sin^2 xdx$.

解：原积分=2$\displaystyle\int_0^{\pi}|\cos x|\sin^2 xdx$ =2 $\displaystyle(\int_0^{\frac{\pi}{2}}\sin^2 x\cos xdx-\int_{\frac{\pi}{2}}^{\pi}\sin^2 x\cos xdx)$

=2 $\displaystyle(\frac{1}{3}\sin^3 x\Big|_0^{\frac{\pi}{2}}-\frac{1}{3}\sin^3 x\Big|_{\frac{\pi}{2}}^{\pi})$ =2 $(\dfrac{1}{3}+\dfrac{1}{3})=\dfrac{4}{3}$.

***11. 设 $f(3x+1)=xe^{\frac{x}{2}}$，求 $\displaystyle\int_0^1 f(t)dt$.

解：$\displaystyle\int_0^1 f(t)dt\underline{\underline{t=3x+1}}\int_{-\frac{1}{3}}^0 3f(3x+1)dx=3\int_{-\frac{1}{3}}^0 xe^{\frac{x}{2}}dx$

$\displaystyle=6\int_{-\frac{1}{3}}^0 xde^{\frac{x}{2}}=6\left[xe^{\frac{x}{2}}\Big|_{-\frac{1}{3}}^0-\int_{-\frac{1}{3}}^0 e^{\frac{x}{2}}dx\right]$

$=2e^{-\frac{1}{6}}-12e^{-\frac{x}{2}}\Big|_{-\frac{1}{3}}^0=14e^{-\frac{1}{6}}-12$.

解法二: 令 $x=\dfrac{t-1}{3}$, 则有 $f(t)=\dfrac{t-1}{3}e^{\frac{t-1}{6}}$, 所以

$$\int_0^1 f(t)dt = 2(t-1)e^{\frac{t-1}{6}}\Big|_0^1 - 2\int_0^1 e^{\frac{t-1}{6}}dt = 14e^{-\frac{1}{6}} - 12 \cdot$$

***12．设 $f(x)=\begin{cases}0, & |x|>2,\\ 4-x^2, & |x|\le 2,\end{cases}$ 求 $\int_{-2}^{2} xf(x-1)dx$.

解：$\int_{-2}^{2} xf(x-1)dx \overset{令x-1=t}{=\!=\!=} \int_{-3}^{1}(t+1)f(t)dt$

$$=\int_{-3}^{-2}(t+1)\cdot 0\cdot dt+\int_{-2}^{1}(t+1)(4-t^2)dt=\int_{-2}^{1}(4t-t^3-t^2+4)dt$$

$$=(2t^2-\frac{1}{4}t^4-\frac{1}{3}t^3+4t)\Big|_{-2}^{1}=\frac{27}{4}.$$

**13．求 $\int_{-1}^{1}[2\arctan x+\sqrt{\pi^2-4(\arctan x)^2}\,]^2dx$.

解： 原式 $=\int_{-1}^{1}[\pi^2+4\arctan x\cdot\sqrt{\pi^2-4(\arctan x)^2}\,]dx$

$$=\int_{-1}^{1}\pi^2\cdot dx(\because 4\arctan x\cdot\sqrt{\pi^2-4(\arctan x)^2}\text{为奇函数})=2\pi^2.$$

**14．设函数 $f(x)$ 在 $[0,a]$ 上连续，证明 $\int_0^a f(x)dx=\int_0^a f(a-x)dx$，并利用此式计算定积分

$\int_0^{\frac{\pi}{4}}\frac{1-\sin 2x}{1+\sin 2x}dx$.

证：证：(1) $\int_0^a f(a-x)dx \overset{令a - x = t}{=\!=\!=} -\int_a^0 f(t)dt=\int_0^a f(t)dt=\int_0^a f(x)dx$

(2) $$\int_0^{\frac{\pi}{4}}\frac{1-\sin 2x}{1+\sin 2x}dx=\int_0^{\frac{\pi}{4}}\frac{1-\sin\left[2(\frac{\pi}{4}-x)\right]}{1+\sin\left[2(\frac{\pi}{4}-x)\right]}dx$$

$$=\int_0^{\frac{\pi}{4}}\frac{1-\cos 2x}{1+\cos 2x}dx=\int_0^{\frac{\pi}{4}}\tan^2 xdx=\int_0^{\frac{\pi}{4}}(\sec^2 x-1)dx=(\tan x-x)\Big|_0^{\frac{\pi}{4}}=1-\frac{\pi}{4}\ .$$

***15．设函数 $f(x)$ 是区间[0，1]上的连续函数，试用分部积分法证明

$$\int_0^1\left[\int_x^1 f(u)du\right]dx=\int_0^1 f(u)udu.$$

证：$\int_0^1\left[\int_x^1 f(u)du\right]dx=(x\int_x^1 f(u)du|)\Big|_0^1+\int_0^1 xf(x)dx=\int_0^1 uf(u)du$.

****16．试证递推公式 $I_n \overset{def}{=\!=\!=} \int_0^{\pi} x\sin^n xdx=\frac{n-1}{n}I_{n-2}$.

证： $I_n=\int_0^{\pi} x\cdot\sin^n xdx=-\int_0^{\pi} x\sin^{n-1} x\,d\cos x$

$$=-x\cdot\sin^{n-1}x\cdot\cos x\Big|_0^{\pi}+\int_0^{\pi}\cos x[\sin^{n-1}x+(n-1)x\cdot\sin^{n-2}x\cdot\cos x]dx$$

$$=\int_0^{\pi}\sin^{n-1}x\,d\sin x+(n-1)\int_0^{\pi}x\cdot\sin^{n-2}x\cdot\cos^2x\cdot dx$$

$$=\frac{1}{n}\sin^n x\Big|_0^{\pi}+(n-1)\int_0^{\pi}x\cdot\sin^{n-2}x(1-\sin^2 x)\,dx$$

$$=-(n-1)\int_0^{\pi}x\cdot\sin^n x\cdot dx+(n-1)\int_0^{\pi}x\cdot\sin^{n-2}x\,dx$$

$$=-(n-1)I_n+(n-1)I_{n-2},$$

$\therefore I_n=\dfrac{n-1}{n}I_{n-2}$.

***17. 设 $I_n=\int_1^e \ln^n xdx, n$ 为正整数，试导出 I_n 与 I_{n-1} 之间的关系式(递推公式).

解： $I_n=\int_1^e \ln^n xdx=x\ln^n x\Big|_1^e-\int_1^e xd\ln^n x$

$$=e-n\int_1^e\frac{x}{x}\ln^{n-1}xdx=e-nI_{n-1}$$

**18．设 $f(x)$ 是以 l 为周期的连续奇函数，试证明， $f(x)$ 的任意原函数都是以 l 为周期的周期函数.

证：设 $f(x)$ 的任意原函数为 $F(x)$，则 $F(x)=\int_0^x f(t)dt+C$ （C 为某一常数），

$$F(x+l)=\int_0^{x+l}f(t)dt+C$$

$$=\int_0^x f(t)dt+\int_x^{x+l}f(t)dt+C$$

$$=\int_0^x f(t)dt+\int_{-\frac{l}{2}}^{\frac{l}{2}}f(t)dt+C=\int_0^x f(t)dt+C=F(x)$$

$\therefore f(x)$ 的任意原函数都是以 l 为周期的周期函数.

***19. 设$f(x)$在 $(-\infty,+\infty)$ 上连续，且对任意 x 都有$\int_x^{x+l}f(t)dt=l, l$ 为非零常数.

试证：$f(x)$为周期函数.

证明：在$\int_x^{x+l}f(t)dt=l$等号两边对x求导，有$f(x+l)-f(x)=0$,

即$f(x+l)=f(x)$, 所以 $f(x)$是以l为周期的周期函数.

***20. 设$F(x)=\int_0^{\pi}\ln(1-2x\cos t+x^2)dt$ ，证明：$F(x)$为偶函数.

证：$F(-x)=\int_0^{\pi}\ln(x^2+2x\cos t+1)dt$， 令 $t=\pi-u$，

$$F(-x)=-\int_{\pi}^{0}\ln(x^2-2x\cos u+1)du$$

$$=\int_0^{\pi}\ln(x^2-2x\cos t+1)dt=F(x).$$

21．利用夹逼定理计算下列数列的极限：

***（1）$a_n=\int_0^1\frac{x^ne^x}{1+e^x}dx$；

解：当$0\le x\le 1$时，有 $0\le\frac{x^ne^x}{1+e^x}\le x^n$，即 $0\le a_n\le\int_0^1 x^n dx$

而 $\lim\limits_{n\to\infty}\int_0^1 x^n dx=\lim\limits_{n\to\infty}\frac{1}{n+1}=0$，

由夹逼定理知：$\lim\limits_{n\to\infty}a_n=0$，即 $\lim\limits_{n\to\infty}\int_0^1\frac{x^ne^x}{1+e^x}dx=0$.

****（2）（选作题）$\left\{\frac{\int_0^n|\sin x|dx}{n}\right\}$.

解：$\forall\, n>3$，$\exists\, k\in N$，使k满足 $k\pi\le n<(k+1)\pi$ 。再注意到：

$|\sin x|$ 的周期为π，且 $\int_0^{\pi}|\sin x|dx=2$.

$$\frac{2k}{(k+1)\pi}=\frac{\int_0^{k\pi}|\sin x|dx}{(k+1)\pi}\le\frac{\int_0^{n}|\sin x|dx}{n}\le\frac{\int_0^{(k+1)\pi}|\sin x|dx}{k\pi}=\frac{2(k+1)}{k\pi} \qquad (1)$$

显见，当$n\to\infty$ 时，必有$k\to\infty$，

而当$k\to\infty$时，$\frac{2k}{(k+1)\pi}\to\frac{2}{\pi}$， $\frac{2(k+1)}{k\pi}\to\frac{2}{\pi}$，

从而对（1）式用夹逼定理知$\lim\limits_{n\to\infty}\frac{\int_0^n|\sin x|dx}{n}=\frac{2}{\pi}$.

21. 利用定积分计算下列极限：若 $f(x)$在$[a,b]$上连续，则根据定积分定义有：

$$\lim_{n\to\infty}\sum_{k=1}^{n}f\left(a+k\frac{b-a}{n}\right)\cdot\frac{b-a}{n}=\int_a^b f(x)dx,$$

试用上式求极限：

***（1） $\lim\limits_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+n}\right]$.

解：原式=$\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\frac{1}{1+\frac{i}{n}}$ 。将区间［0，1］作 n 等分，并取 $\xi_i=x_i=\frac{i}{n}$（$i=1,2\cdots n$），

则 $\Delta x_i = x_i - x_{i-1} \equiv \dfrac{1}{n}$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\frac{i}{n}} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+\xi_i}\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)dx\,,$$

其中 $f(x)=\dfrac{1}{1+x}$, $a = x_0 = 0,\quad b = x_n = 1$,

则 $$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\frac{i}{n}} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+\xi_i}\Delta x_i = \int_0^1\frac{1}{1+x}dx\,.$$

而 $\displaystyle\int_0^1\frac{dx}{1+x} = \ln 2$, 所以原极限为 ln2.

***（2）$\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{e^{k/n}}{n+ne^{2k/n}}$.

解：原式= $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{e^{k/n}}{1+(e^{k/n})^2} = \int_0^1\frac{e^x}{1+(e^x)^2}dx = \arctan e^x\Big|_0^1 = \arctan e - \frac{\pi}{4}$.