

华东理工大学

复变函数与积分变换作业 (第8册)

第十五次作业

教学内容: 7.4 拉普拉斯变换的应用; 8.1 共形映射的概念
8.2 分式线性映射

1. 求解下列微分方程

$$(1) \quad y'' - 2y' + y = e^t, y(0) = y'(0) = 0;$$

解 (1) 令 $Y(s) = \mathcal{L}[y(t)]$, 在方程两端取拉氏变换, 并代入初始条件, 将

$$\begin{aligned} s^2 Y(s) - 2sY(s) + Y(s) &= \frac{1}{s-1} \\ Y(s) &= \frac{1}{(s-1)^3} \\ y(t) &= \mathcal{L}^{-1}[Y(s)] = \operatorname{Res} \left[\frac{e^{st}}{(s-1)^3}, 1 \right] \\ &= \frac{1}{2!} (e^{st})'' \Big|_{s=1} \\ &= \frac{1}{2} t^2 e^t \end{aligned}$$

$$(2) \quad y'' - 2y' + 2y = 2e^t \cos t, y(0) = y'(0) = 0;$$

解: 同上题, 有

$$\begin{aligned} s^2 Y(s) - 2sY(s) + 2Y(s) &= 2 \frac{s-1}{(s-1)^2 + 1} \\ Y(s) &= \frac{2(s-1)}{(s^2 - 2s + 2)^2} = -\frac{d}{ds} \left(\frac{1}{(s-1)^2 + 1} \right) \end{aligned}$$

由象函数的微分性质: $\mathcal{L}^{-1}[F'(s)] = -t\mathcal{L}^{-1}[F(s)]$, 于是有

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] = -\mathcal{L}^{-1} \left[\frac{d}{ds} \left(\frac{1}{(s-1)^2 + 1} \right) \right] \\ &= t\mathcal{L}^{-1} \left[\frac{1}{(s-1)^2 + 1} \right] \end{aligned}$$

$$= te^t \sin t$$

$$(3) \quad y^{(4)} - y''' = \cos t, y(0) = y'(0) = y'''(0) = 0, y''(0) = 1;$$

同上题, 有

$$s^4 Y(s) - s - s^3 Y(s) + 1 = \frac{s}{s^2 + 1}$$

$$(s^4 - s^3)Y(s) = \frac{s}{s^2 + 1} + (s - 1)$$

$$Y(s) = \frac{1}{s^3(s-1)(s^2+1)} + \frac{1}{s^3}$$

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{1}{s^3(s-1)(s^2+1)}\right] + L^{-1}\left[\frac{1}{s^3}\right]$$

$$\begin{aligned} &= \lim_{s \rightarrow 6} \left[\frac{e^{st}}{(s-1)(s^2+1)} \right]' + \frac{e^{st}}{s^2(s^2+1)} \Big|_{s=-1} + \frac{e^{st}}{s^2(s+1)(s+i)} \Big|_{s=i} \\ &\quad + \frac{e^{st}}{s^2(s+1)(s^2-i)} \Big|_{s=-i} + \frac{1}{2} t^2 \\ &= t - 1 + \frac{1}{2} e^{-t} + \frac{1}{2} (\cos t - \sin t) + \frac{t^2}{2} \end{aligned}$$

$$(4) \quad y^{(4)} + 2y'' + y = 0, y(0) = y'(0) = y'''(0) = 0, y''(0) = 1;$$

同上题方法, 有

$$\begin{aligned} &s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) + 2s^2 Y(s) - 2s y(0) \\ &- 2y'(0) + Y(s) = 0 \end{aligned}$$

$$Y(s) = \frac{s}{(s^2+1)^2} = \frac{s}{s^2+1} \cdot \frac{1}{s^2+1}$$

从而方程解为:

$$y(t) = L^{-1}[Y(s)] = \cos t * \sin t = \frac{1}{2} t \sin t$$

2. 求解下列微积分方程

$$y' + 2y = \sin t - \int_0^t y(\tau) d\tau, y(0) = 0$$

解: 两端取拉氏变换, 并由微分和积分性质, 有

$$sY(s) + 2Y(s) = \frac{1}{s^2+1} - \frac{1}{s}Y(s)$$

$$\text{即 } (s+2+\frac{1}{s})Y(s) = \frac{1}{s^2+1}$$

$$Y(s) = \frac{s}{(s+1)^2(s^2+1)} = \frac{1}{2} \frac{1}{s^2+1} - \frac{1}{2} \frac{1}{(s+1)^2}$$

$$\begin{aligned} \text{因此 } y(t) &= \frac{1}{2} \sin t - \frac{1}{2} t e^{-t} \\ &= \frac{1}{2} (\sin t - t e^{-t}) \end{aligned}$$

3. 求解下列方程组

$$(1) \quad \begin{cases} x'' - x - 2y' = e^t & x(0) = -\frac{3}{2}, x'(0) = \frac{1}{2} \\ x' - y'' - 2y = t^2 & y(0) = 1, y'(0) = -\frac{1}{2} \end{cases}$$

解: 设 $\mathcal{L}[x(t)] = X(s)$, $\mathcal{L}[y(t)] = Y(s)$

在方程组两边取 Laplace 变换, 并应用初始条件得

$$\begin{cases} s^2 X(s) + \frac{3}{2}s - \frac{1}{2} - X(s) - 2sY(s) + 2 = \frac{1}{s-1} \\ sX(s) + \frac{3}{2} - s^2 Y(s) + s - \frac{1}{2} - 2Y(s) = \frac{2}{s^3} \end{cases}$$

解方程组, 得

$$\begin{cases} X(s) = -\frac{3}{2(s-1)} + \frac{2}{s^2} \\ Y(s) = -\frac{1}{2(s-1)} - \frac{1}{s^3} + \frac{3}{2s} \end{cases}$$

求逆变换得

$$\begin{cases} x(t) = -\frac{3}{2}e^t + 2t \\ y(t) = -\frac{1}{2}e^t - \frac{1}{2}t^2 + \frac{3}{2} \end{cases}$$

$$(2) \quad \begin{cases} y'' - x'' + x' - y = e^t - 2 & x(0) = x'(0) = 0 \\ 2y'' - x'' - 2y' + x = -t & y(0) = y'(0) = 0 \end{cases}$$

解：设 $L|x| = X(s)$, $L|y| = Y(s)$ 。对方程组的每个方程两边分别取拉氏变换，并考虑到

初始条件得：

$$\begin{cases} s^2 Y(s) - s^2 X(s) + sX(s) - Y(s) = \frac{1}{s-1} - \frac{2}{s}, \\ 2s^2 Y(s) - s^2 X(s) - 2sY(s) + X(s) = -\frac{1}{s^2}, \end{cases}$$

整理计算得：

$$\begin{cases} X(s) = \frac{2s-1}{s^2(s-1)^2} = \frac{2}{s(s-1)^2} - \frac{1}{s^2(s-1)^2}, \\ Y(s) = \frac{1}{s(s-1)^2} \end{cases}$$

以下求 $X(s)$ 的拉氏逆变换：因为

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1, \quad \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] = te^t, \quad \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t \text{ 故由卷积定理可得}$$

$$\mathcal{L}^{-1}[X(s)] = 2 \int_0^t \tau e^\tau d\tau - \int_0^t (t-\tau) \tau e^\tau d\tau = te^t - t,$$

同理可求

$$\mathcal{L}^{-1}[Y(s)] = te^t - e^t + 1,$$

所以方程组的解为 $\begin{cases} x = L^{-1}[X(s)] = te^t - t, \\ y = L^{-1}[Y(s)] = te^t - e^t + 1, \end{cases}$

$$(3) \begin{cases} (2x'' - x' + 9x) - (y'' + y' + 3y) = 0 & x(0) = x'(0) = 1 \\ (2x'' + x' + 7x) - (y'' - y' + 5y) = 0 & y(0) = y'(0) = 0 \end{cases}$$

解：方程组中每个方程两边取拉氏变换，得

$$\begin{cases} (2s^2 - s + 9)X(s) - (s^2 + s + 3)Y(s) = 1 + 2s \\ (2s^2 + s + 7)X(s) - (s^2 - s + 5)Y(s) = 3 + 2s \end{cases}$$

整理得

$$\begin{cases} 2X(s) - Y(s) = \frac{2s+2}{s^2+4} \\ X(s) + Y(s) = \frac{1}{s-1} \end{cases}$$

解之得

$$\begin{cases} X(s) = \frac{1}{3} \cdot \frac{1}{s-1} + \frac{2}{3} \cdot \frac{s}{s^2+4} + \frac{1}{3} \cdot \frac{2}{s^2+4} \\ Y(s) = \frac{2}{3} \cdot \frac{1}{s-1} - \frac{2}{3} \cdot \frac{s}{s^2+4} + \frac{1}{3} \cdot \frac{2}{s^2+4} \end{cases}$$

再取拉氏逆变换得到其解为：

$$\begin{cases} x(t) = \frac{1}{3}e^t + \frac{2}{3}\cos 2t + \frac{1}{3}\sin 2t \\ y(t) = \frac{2}{3}e^t - \frac{2}{3}\cos 2t - \frac{1}{3}\sin 2t \end{cases}$$

4. 填空题

(1) 分式线性映射 $w = \frac{z-i}{z+i}$ 在 $z=i$ 处的旋转角为 $-\frac{\pi}{2}$ 伸缩率为 $\frac{1}{2}$

(2) 在 $w = z^2$ 的映射下, $y = x+1$ 的像曲线为 $v = \frac{1}{2}(u^2-1)$, $y^2 = x^2+1$ 的像曲线 $u = -1$

(3) 在映射 $w = \frac{1}{z}$ 下, 区域 $x > 1, y > 0$. 映射为 $(u - \frac{1}{2})^2 + v^2 < (\frac{1}{2})^2, v < 0$.

(4) 在映射 $w = (1+i)z$ 下, 区域 $\text{Im } z > 0$ 像为 $\text{Im } w > \text{Re } w$

第十六次作业

教学内容：8.2 分式线性映射（续）;8.3 几种常见的分式线性映射
8.4 几个初等函数构成的映射

1. 填空

(1) 把 $z_1 = 2, z_2 = i, z_3 = -2$; 映射为 $w_1 = -1, w_2 = i, w_3 = 1$ 的分式线性映射为 $w = \frac{z-6i}{3iz-2}$

(2) 由三点 $z_1 = \infty, z_2 = i, z_3 = 0$ 到 $w_1 = 0, w_2 = i, w_3 = \infty$ 的分式线性映射为 $w = -\frac{1}{z}$

2 求把上半平面 $\text{Im } z > 0$ 映射成单位圆域 $|w| < 1$ 的分式线性映射 $w = f(z)$, 并满足条件:

$$(1) f(i) = 0, \quad \arg f'(i) = -\frac{\pi}{2}$$

$$\text{解: } f(z) = e^{i\theta} \frac{z-i}{z+i}, \text{ 则 } f'(z) = e^{i\theta} \frac{2i}{(z+i)^2}$$

$$f'(i) = e^{i\theta} \cdot \left(-\frac{i}{2}\right) = e^{i\theta} \cdot \frac{1}{2} e^{-\frac{\pi}{2}i}$$

$$\text{由于 } \arg f'(i) = -\frac{\pi}{2}$$

$$\text{所以 } \theta - \frac{\pi}{2} = -\frac{\pi}{2}, \quad \theta = 0$$

$$\text{所求映射为 } f(z) = \frac{z-i}{z+i}$$

$$(2) f(i) = 0, \quad f(-1) = 1;$$

解：因为 $f(i) = 0$ ，则 $f(-i) = \infty$ ，

$$f(z) = k \frac{z-i}{z+i}$$

$$\text{又 } f(-1) = k \frac{-1-i}{-1+i} = ki = 1$$

$$k = -i$$

$$\text{所求映射为 } f(z) = -i \frac{z-i}{z+i}$$

$$(3) f(2i) = 0, \quad \arg f'(2i) = 0;$$

$$f(z) = e^{i\theta} \frac{z-2i}{z+2i}$$

$$f'(2i) = e^{i\theta} \cdot \left(-\frac{2i}{9}\right) = e^{i\theta} \cdot \frac{2}{9} e^{-\frac{\pi}{2}i}$$

$$\theta - \frac{\pi}{2} = 0 \quad \theta = \frac{\pi}{2}$$

$$\text{所求映射为 } f(z) = i \frac{z-2i}{z+2i}$$

3. 求把单位圆 $|z| < 1$ 映射成单位圆 $|w| < 1$ 的分式线性映射 $w = f(z)$ ，并满足条件：

$$(1) f\left(\frac{1}{2}\right) = 0, \quad f(-1) = 1;$$

$$\text{解：令 } f(z) = k \frac{2z-1}{z-2}$$

$$\text{由 } f(-1) = 1 \text{ 得, } k = 1$$

$$\text{故 } f(z) = \frac{2z-1}{z-2}$$

$$(2) \quad f\left(\frac{1}{2}\right) = 0, \quad \arg f'\left(\frac{1}{2}\right) = \frac{\pi}{2}.$$

$$\text{解: } f\left(\frac{1}{2}\right) = 0 \text{ 则 } f(2) = \infty$$

$$\text{令 } f(z) = k \frac{z - \frac{1}{2}}{z - 2}$$

$$|f(1)| = \left| \frac{k}{2} \right| = 1, \quad |k| = 2$$

$$f(z) = 2e^{i\theta} \frac{z - \frac{1}{2}}{z - 2}$$

$$f'(z) = 2e^{i\theta} \frac{-3}{2(z-2)^2}, \quad f'\left(\frac{1}{2}\right) = \frac{4}{3}e^{i\theta+\pi i}$$

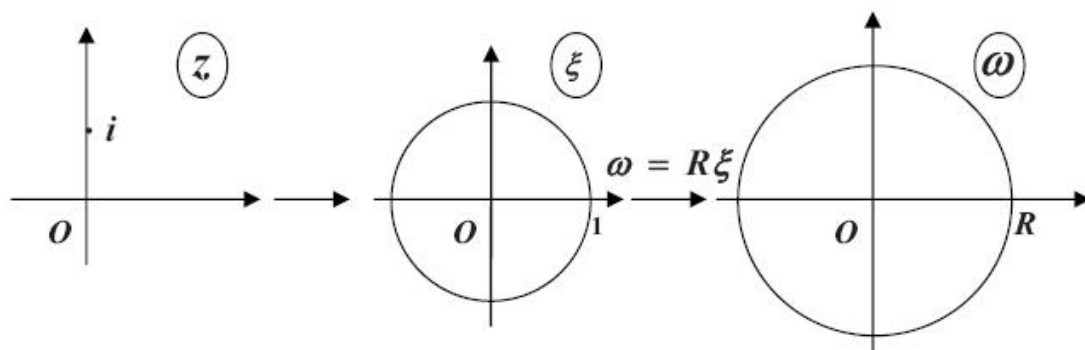
$$\theta + \pi = \frac{\pi}{2}, \quad \text{所以 } \theta = -\frac{\pi}{2}$$

$$f(z) = -2i \frac{z - \frac{1}{2}}{z - 2} = -i \frac{2z - 1}{z - 2}.$$

4. 求将上半平面 $\text{Im } z > 0$ 映射成圆 $|w| < R$ 的分式线性映射 $w = f(z)$ ，且满足 $f(i) = 0$ ，

$$f'(i) = 1.$$

解：



应用条件 $f(i) = 0$ 知 $w = Re^{i\theta} \frac{z-i}{z+i}$ ，再应用条件 $f'(i) = 1$ ，则可确定 $Re^{i\theta} = 2i$ ，所以变

换为 $w = 2i \frac{z-i}{z+i}$

5. 求分式线性映射 $w = f(z)$ ，它把 $|z| = 1$ 映射为 $|w| = 1$ ，并使 $1, 1+i$ 分别映射为 $1, \infty$

解： $1+i$ 关于 $|z| = 1$ 的对称点是 $\frac{1+i}{2}$ ， ∞ 关于 $|w| = 1$ 的对称点是 0 。

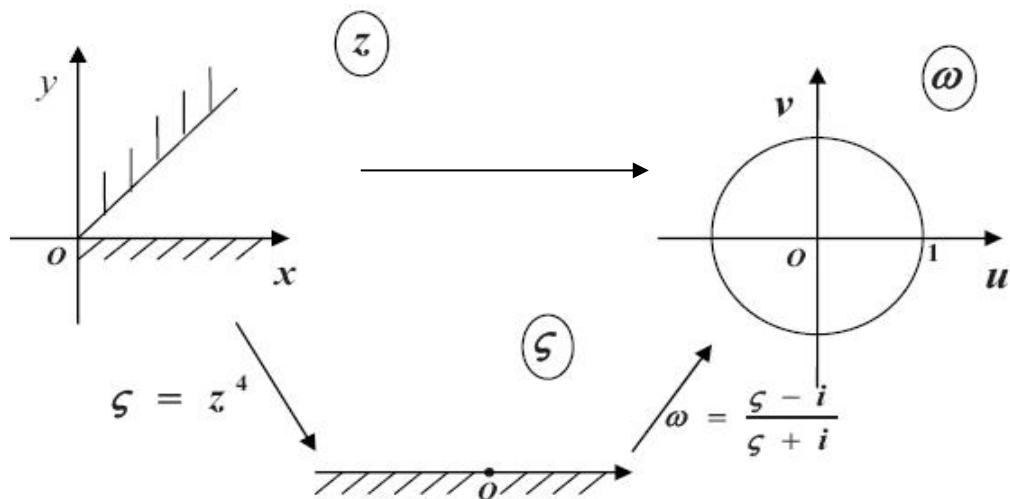
故分式线性映射 $w = f(z)$ 将单位圆内的点 $z = \frac{1+i}{2}$ 映为单位圆内的点 $w = 0$

$$\text{所以 } w = f(z) = e^{i\beta} \frac{z - \frac{1+i}{2}}{1 - \frac{1-i}{2}z},$$

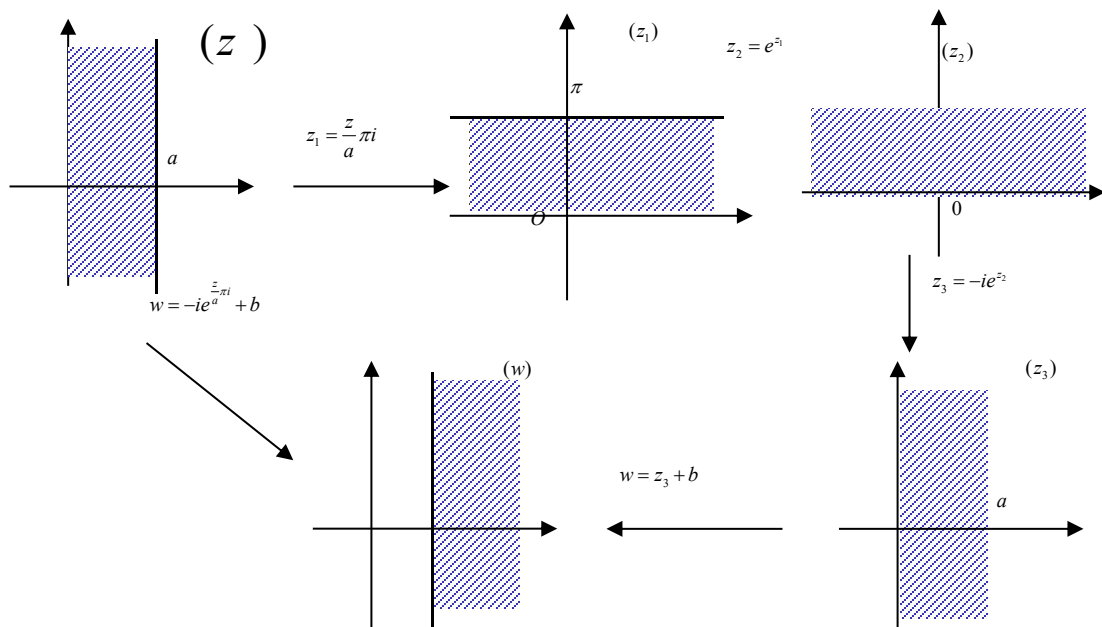
$$\text{又 } f(1) = 1, \text{ 所以 } e^{i\beta} = i \text{ 即所求变换为 } w = i \frac{2z - (1+i)}{2 - (1-i)z} = \frac{(i-1)z + 1}{-z + (1+i)}$$

6. 求把角形域 $0 < \arg z < \frac{\pi}{4}$ 映射成单位圆 $|w| < 1$ 的一个映射。

解：复合如图 4 所示的两个变换，即得所求的变换为 $w = \frac{z^4 - i}{z^4 + i}$ 。

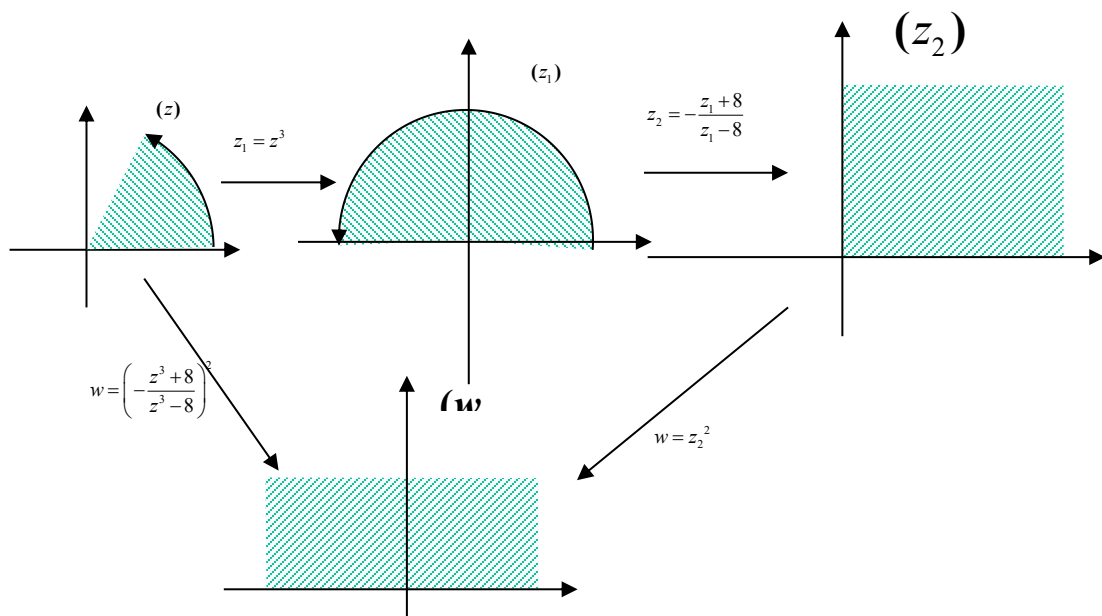


7. 求将带形域 $0 < \operatorname{Re} z < a$ 映射成 $\operatorname{Re} w > b$ 的共形映射。

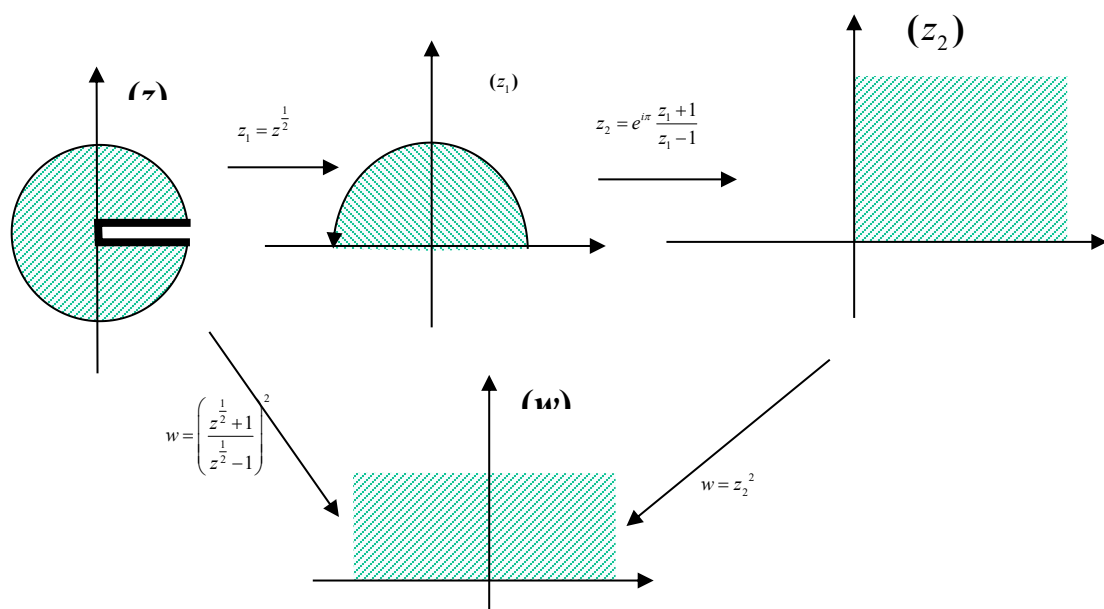


8. 求将下列区域映射为上半平面的一个共形映射.

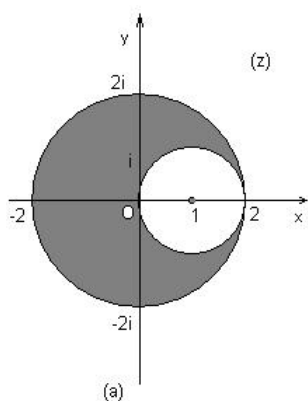
(1) $0 < \arg z < \frac{\pi}{3}$ 且 $|z| < 2$;



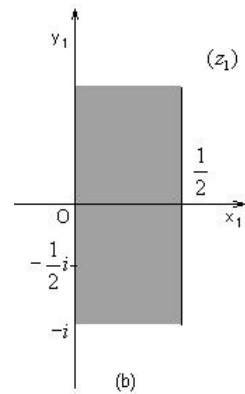
(2) $|z| < 1$, 沿 0 到 1 有割痕的区域;



(3) $|z| < 2$ 且 $|z-1| > 1$;

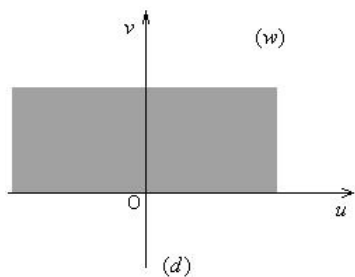


$$z_1 = \frac{z}{z-2}$$

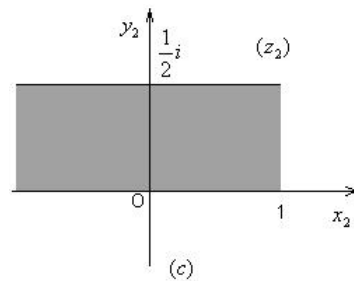


$$\omega = e^{2\pi i \frac{z}{z-2}}$$

$$z_2 = z_1 e^{\frac{\pi}{2}i} \quad (\text{旋转})$$



$$\omega = e^{2\pi i z_2}$$



$$(4) \quad |z+i| > \sqrt{2} \text{ 且 } |z-i| < \sqrt{2}.$$

$$\text{复合上述映射得 } w = \left(-ie^{-\pi i/4} \frac{z+1}{z-1}\right)^2 = i \left(\frac{z+1}{z-1}\right)^2$$

