

华东理工大学

复变函数与积分变换作业 (第2册)

第三次作业

教学内容: 2.1.2 柯西—黎曼方程

1. 填空:

(1) 函数 $f(z) = z \operatorname{Re} z$ 的导数 $f'(z) = \begin{cases} \text{不存在} & z \neq 0 \\ 0 & z = 0 \end{cases}$

(2) 函数 $f(z) = z^n$ 的导数 $f'(z) = \underline{nz^{n-1}}$

(3) 函数 $\frac{z-3}{(z+1)^2(z^2+1)}$ 的奇点为 $\underline{-1, \pm i}$

2. 下列函数何处可导? 何处解析?

(1) $f(z) = x^2 - yi$; (2) $f(z) = 2x^3 + 3y^3i$; (3) $f(z) = z^2 \bar{z}$

解: (1) $f(z) = x^2 - yi$, 则 $u=x^2$, $v=-y$,

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial v}{\partial y} = -1, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0, \text{ 令 } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ 得, } x = -\frac{1}{2}$$

即: 在直线 $x = -\frac{1}{2}$ 上可导, 复平面内处处不解析。

(2) $f(z) = 2x^3 + 3y^3i$, 则 $u=2x^3$, $v=3y$,

$$\frac{\partial u}{\partial x} = 6x^2, \frac{\partial v}{\partial y} = 9y^2, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0, \text{ 令 } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ 得, } \sqrt{2}x \pm \sqrt{3}y = 0$$

即: 在直线 $\sqrt{2}x \pm \sqrt{3}y = 0$ 上可导, 在复平面内处处不解析。

(3) $f(z) = z^2 \bar{z}$, $f(z) = (x^3 + xy^2) + i(x^2y + y^3)$, 则 $u=x^3 + xy^2$, $v=x^2y + y^3$,

$$\frac{\partial u}{\partial x} = 3x^2 + y^2, \frac{\partial v}{\partial y} = x^2 + 3y^2, \frac{\partial u}{\partial y} = 2xy, \frac{\partial v}{\partial x} = 2xy, \text{ 令 } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \text{ 得,}$$

$y = \pm x$, $x=0$, $y=0$, 即: 函数仅在 $(0,0)$ 上满足 C-R 方程, 该函数在 $(0,0)$ 可导,

在复平面内处处不解析。

3. 验证函数 $f(z) = \sin x \cosh y + i \cos x \sinh y$ 在复平面上解析，并求其导数。

解： $u = \sin x \cosh y$, $v = \cos x \sinh y$,

$$\frac{\partial u}{\partial x} = \cos x \cosh y, \frac{\partial v}{\partial y} = \cos x \cosh y, \frac{\partial u}{\partial y} = \sin x \sinh y, \frac{\partial v}{\partial x} = -\sin x \sinh y,$$

$$\text{即: } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \text{ 所以函数在复平面上解析。}$$

$$f'(z) = \cos x \cosh y - i \sin x \sinh y$$

4. 设函数 $f(z) = my^3 + nx^2y + i(x^3 + Lxy^2)$ 是复平面内解析函数，求 L, m, n 的值。

解： $u = my^3 + nx^2y$, $v = x^3 + Lxy^2$,

$$\frac{\partial u}{\partial x} = 2nxy, \frac{\partial v}{\partial y} = 2Lxy, \frac{\partial u}{\partial y} = 3my^2 + nx^2, \frac{\partial v}{\partial x} = 3x^2 + Ly^2$$

$$\text{由 } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ 得,}$$

$$L = -3, m = 1, n = -3$$

5. 设函数 $f(z) = u + iv$ 在区域 D 内解析，证明：如果 $f(z)$ 满足下列条件之一，那么它在 D 内为常数。

(1) $\overline{f(z)}$ 解析；(2) $2u + 3v = 1$ ；(3) $|f(z)|$ 在 D 内是一个常数。

证明：关键证明 u, v 的一阶偏导数皆为 0。

(1) $\overline{f(z)} = u - iv$ ，因其解析，故由柯西-黎曼方程得

$$u_x = -v_y, u_y = v_x \text{ --- (1)}$$

而由 $f(z)$ 的解析性，又有 $u_x = v_y, u_y = -v_x$ --- (2)

由 (1)、(2) 知， $u_x = u_y = v_y = v_x = 0$ ，因此 $u \equiv c_1, v \equiv c_2$ ，即

$$f(z) \equiv c_1 + ic_2 \text{ 为常数}$$

(2) 同前面一样， $2u + 3v = 1$ 两端分别对 x, y 求偏导数，得 $2u_x + 3v_x = 0, 2u_y + 3v_y = 0$

考虑到柯西-黎曼方程 $u_x = v_y, u_y = -v_x$ ，仍有 $u_x = u_y = v_y = v_x = 0$ ，证毕。

(3) 由已知， $|f(z)|^2 = u^2 + v^2 \equiv c_0$ 为常数，等式两端分别对 x, y 求偏导数，得

$$2uu_x + 2vv_x = 0,$$

$$2uu_y + 2vv_y = 0, \text{-----} (1)$$

$$\text{因 } f(z) \text{ 解析, 所以又有 } u_x = v_y, u_y = -v_x \text{-----} (2)$$

说明 u, v 皆与 x, y 无关, 因而为常数, 从而 $f(z)$ 也为常数。

$$6. \text{ 证明: 若 } f(z) \text{ 解析, 则有 } \left(\frac{\partial}{\partial x}|f(z)|\right)^2 + \left(\frac{\partial}{\partial y}|f(z)|\right)^2 = |f'(z)|^2$$

$$\text{证明: 由柯西-黎曼方程知, 左端 } \left(\frac{\partial}{\partial x}\sqrt{u^2 + v^2}\right)^2 + \left(\frac{\partial}{\partial y}\sqrt{u^2 + v^2}\right)^2$$

$$= \left(\frac{uu_x + vv_x}{\sqrt{u^2 + v^2}}\right)^2 + \left(\frac{uu_y + vv_y}{\sqrt{u^2 + v^2}}\right)^2 = \frac{(uu_x + vv_x)^2 + (uv_x - vu_x)^2}{u^2 + v^2}$$

$$= \frac{u^2(u_x^2 + v_x^2) + v^2(u_x^2 + v_x^2)}{u^2 + v^2} = (u_x^2 + v_x^2) = |u_x + iv_x|^2 = |f'(z)|^2 = \text{右端, 证毕。}$$

7. 试证下列函数在平面上任何点都不解析:

$$(1) f(z) = x + 2iy$$

由于 $\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 2$, 可见 $C-R$ 条件在 z 平面上处处不成立, 故

$f(z)$ 在 z 平面上任何点都不解析。

$$(2) f(z) = x + y$$

$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 1, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0$ 可见 $C-R$ 条件在 z 平面上处处不成立, 故

$f(z)$ 在 z 平面上任何点都不解析。

$$(3) f(z) = \operatorname{Re} z$$

$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$ 可见 $C-R$ 条件在 z 平面上处处不成立, 故

$f(z)$ 在 z 平面上任何点都不解析。

$$(4) f(z) = \frac{1}{|z|}$$

$$\frac{\partial u}{\partial x} = \frac{-x}{\sqrt{(x^2+y^2)^3}}, \quad \frac{\partial u}{\partial y} = \frac{-y}{\sqrt{(x^2+y^2)^3}}, \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \text{ 可见 } C-R \text{ 条件在 } z \neq 0 \text{ 处处}$$

不成立, $f(z)$ 在 $z=0$ 处无定义, 故 $f(z)$ 在 z 平面上任何点都不解析。

第四次作业

教学内容: 2.2 初等函数及其解析性 2.3 解析函数与调和函数的关系

1. 填空题

$$(1) \exp\left(\frac{2-\pi i}{3}\right) = \underline{\hspace{2cm}}$$

$$(2) (e^i)^i = \underline{\hspace{2cm}};$$

$$(3) \operatorname{Ln}(-3+4i) = \underline{\hspace{2cm}};$$

$$(4) \ln(ie) = \underline{\hspace{2cm}};$$

$$(5) \ln e^i = \underline{\hspace{2cm}}.$$

$$\text{解: (1) } \exp\left(\frac{2-\pi i}{3}\right) = e^{\frac{2}{3}} \left[\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right] = e^{\frac{2}{3}} \left(\frac{1}{2} - i \frac{\sqrt{3}}{2} \right)$$

$$(2) \operatorname{Lne}^i = i(1+2k\pi) \quad k=0, \pm 1, \text{ 等},$$

$$(e^i)^i = e^{i \operatorname{Lne}^i} = e^{-(1+2k\pi)} \quad k=0, \pm 1, \text{ 等}$$

$$(3) \operatorname{Ln}(-3+4i) = \ln|-3+4i| + i \left(\pi - \arctan \frac{4}{3} + 2k\pi \right)$$

$$= \ln 5 + i \left(\pi - \arctan \frac{4}{3} + 2k\pi \right) \quad k=0, \pm 1, \text{ 等} \quad (2) \ln(ie) = \ln|ie| + i \frac{\pi}{2} = 1 + i \frac{\pi}{2}$$

$$(4) \ln(ie) = \ln|ie| + i \frac{\pi}{2} = 1 + i \frac{\pi}{2}$$

$$(5) e^i = \cos 1 + i \sin 1, \quad \ln(e^i) = \ln|e^i| + i = i$$

2 求下列各式的值

$$(1) 3^i; (2) (1+i)^i; (3) \sin(1+2i); (4) |\cos z|^2$$

$$\text{解: (1) } 3^i = e^{i \operatorname{Ln} 3} = e^{i(\ln 3 + i 2k\pi)} = e^{-2k\pi + i \ln 3} \quad k=0, \pm 1, \text{ 等}$$

$$(2) (1+i)^i = e^{iLn(1+i)} = e^{i\left[\ln\sqrt{2}+i\left(\frac{\pi}{4}+2k\pi\right)\right]} = e^{-\left(\frac{\pi}{4}+2k\pi\right)+i\ln\sqrt{2}} \quad k=0, \pm 1, \text{等}$$

$$\begin{aligned}(3) \sin(1+2i) &= \frac{e^{i(1+2i)} - e^{-i(1+2i)}}{2i} = \frac{e^{-2+i} - e^{2-i}}{2i} \\ &= \frac{(e^{-2} - e^2)\cos 1 + i(e^{-2} + e^2)\sin 1}{2i} \\ &= \frac{(e^{-2} + e^2)\sin 1 - i(e^{-2} - e^2)\cos 1}{2}\end{aligned}$$

$$(4) \cos z = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y,$$

$$\begin{aligned}|\cos z|^2 &= (\cos x \cosh y)^2 + (\sin x \sinh y)^2 \\ &= \cos^2 x (1 + \sinh^2 y) + (1 - \cos^2 x) \sinh^2 y \\ &= \cos^2 x + \sinh^2 y\end{aligned}$$

3. 设 $z = re^{i\theta}$ 求 $\operatorname{Re}[Ln(z-1)]$

解: $Ln(z-1) = \ln|z-1| + i[\arg(z-1) + 2k\pi]$ 因此

$$\operatorname{Re}[Ln(z-1)] = \ln|z-1| = \ln\sqrt{(r\cos\theta-1)^2 + (r\sin\theta)^2} = \frac{1}{2}\ln(1-2r\cos\theta+r^2)$$

4. 解下列方程:

$$(1) e^x - 1 - \sqrt{3}i = 0; (2) \ln z = 2 - \frac{\pi}{6}i; (3) \cos z = 0; (4) \sin z + \cos z = 0$$

解: (1) $e^x = 1 + \sqrt{3}i$,

$$z = Ln(1 + \sqrt{3}i) = \ln 2 + i\left(\frac{\pi}{3} + 2k\pi\right) \quad k=0, \pm 1, \text{等}$$

$$(2) z = e^{2-\frac{\pi}{6}i} = e^2 \left(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6} \right) = e^2 \left(\frac{\sqrt{3}}{2} - \frac{i}{2} \right)$$

$$(3) \cos z = \frac{e^{iz} + e^{-iz}}{2} = 0, \quad e^{i2z} = -1,$$

$$i2z = Ln(-1) = i(\pi + 2k\pi)$$

$$z = \frac{\pi}{2} + k\pi \quad (k=0, \pm 1, \text{等})$$

(4) 由于 $\sin z = -\cos z$, $\frac{e^{iz} - e^{-iz}}{2i} = \frac{-1}{2}(e^{iz} + e^{-iz})$ 故

$$e^{2iz} - 1 = -i(e^{2iz} + 1) \quad e^{2iz} = \frac{1-i}{1+i}$$

$$z = \frac{1}{2i} \operatorname{Ln}\left(\frac{1-i}{1+i}\right) = \frac{1}{2i} \operatorname{Ln}(-i) = \frac{1}{2i} [\ln|-i| + i(\arg(-i) + 2k\pi)] = (k - \frac{1}{4})\pi, k = 0, \pm 1, \dots$$

5. 证明下列各式:

(1) $\cos iz = \cosh z$

证明: $\cos iz = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^z + e^{-z}}{2} = \cosh z$ 。

(2) $\cosh^2 z - \sinh^2 z = 1$;

证明: $\cosh^2 z - \sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2 = 1$ 。

6. 由下列各已知调和函数求解析函数 $f(z) = u + iv$:

(1) $u = (x - y)(x^2 + 4xy + y^2)$;

解: 则

$$u_x = 3x^2 + 6xy - 3y^2, u_y = 3x^2 - 6xy - 3y^2,$$

$$f'(z) = u_x - iu_y = 3x^2 + 6xy - 3y^2 - i(3x^2 - 6xy - 3y^2) = 3(1 - i)z^2,$$

故 $f(z) = (1 - i)z^3 + c, c \in R$

(2) $v = \arctan \frac{y}{x}, x > 0$;

$$f'(z) = v_y + iv_x = \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2} = \frac{x - iy}{x^2 + y^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}, \text{ 故 } f(z) = \ln z + c, c \in R$$

(3) $v = \frac{y}{x^2 + y^2}, f(2) = 0$ 。

$$f'(z) = u_x + iv_x = \frac{x^2 - y^2}{(x^2 + y^2)^2} - i \frac{2xy}{(x^2 + y^2)^2} = \left(\frac{\bar{z}}{z\bar{z}}\right)^2 = \frac{1}{z^2}, \text{ 故 } f(z) = \frac{-1}{z} + c, c \in R$$

由 $f(2) = 0$ 得, $c = \frac{1}{2}$ 故 $f(z) = \frac{-1}{z} + \frac{1}{2}, c \in R$

7. 设 $u(x, y) = e^{px} \sin y$, 求 p 的值使 $v(x, y)$ 为调和函数, 并求出解析函数 $f(z) = u + iv$ 。

解: $\frac{\partial u}{\partial x} = pe^{px} \sin y, \frac{\partial u}{\partial y} = e^{px} \cos y, \frac{\partial^2 u}{\partial x^2} = p^2 e^{px} \sin y, \frac{\partial^2 u}{\partial y^2} = -e^{px} \sin y,$

由拉普拉斯方程知 $p = \pm 1$

当 $p = 1$ 时,

$$u(x, y) = e^x \sin y$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x \sin y$$

$$v = \int e^x \sin y dy = -e^x \cos y + g(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow g(x) = C$$

$$f(z) = e^x \sin y + i(-e^x \cos y + C)$$

当 $p = -1$ 时,

$$u(x, y) = e^{-x} \sin y$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -e^{-x} \sin y$$

$$v = \int -e^{-x} \sin y dy = e^{-x} \cos y + g(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow g(x) = C$$

$$f(z) = e^{-x} \sin y + i(e^{-x} \cos y + C)$$

$$f(z) = -iz^3 + C$$