Quantum Gates and Corresponding Channels

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CNOT Gate

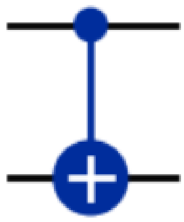

Figure 1: CNOT Gate Circuit

This process demonstrates how to analyze a quantum circuit using density matrix formalism. The steps include preparing the initial state, applying gates, and performing partial traces to observe the effects on specific qubits.

- The circuit starts with qubits in an initial mixed state.
- Apply a Controlled-Z (CZ) gate to entangle the qubits.
- Describe the new state as a density matrix in terms of probabilities
- We then perform partial trace over the control qubit.
- The reduced density matrix reveals the effect on the target qubit (i.e, phase flip channel).

$$
\begin{aligned}
\rho_{AB} &= (1-p)|0\rangle\langle 0| \otimes I_B + p|1\rangle\langle 1| \otimes X_B \\
\rho &= (1-p)|0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) + p|1\rangle\langle 1| \otimes (|0\rangle\langle 1| + |1\rangle\langle 0|) \\
\rho_{AB} &= (1-p)[|00\rangle\langle 00| + |01\rangle\langle 01|] + p[|10\rangle\langle 11| + |11\rangle\langle 10|] \\
\rho_B &= \mathrm{Tr}_A(\rho_{AB}) \\
\rho_B &= (1-p)(|0\rangle\langle 0| + |1\rangle\langle 1|) + p(|0\rangle\langle 1| + |1\rangle\langle 0|) \\
\rho_B &= (1-p)I + pX \quad \text{(Bit flip Channel)}
\end{aligned}
$$

Control Z Gate

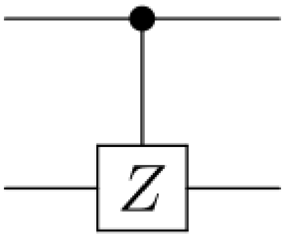

Figure 2: Control Z Gate Circuit

This process demonstrates how to analyze a quantum circuit using density matrix formalism. The steps include preparing the initial state, applying the Control Z gate, and performing partial traces to observe the effects on specific qubits. Finally, we are able to see the channel corresponding to the application of CZ gate to a single qubit.

- The circuit starts with qubits in an initial mixed state.
- Apply a Controlled-Z (CZ) gate to entangle the qubits such that the Z gate is only applied when the control qubit is in the state
$$|1\rangle$$
- Perform partial trace is then taken over the control qubit.
- The reduced density matrix shows the effect on the target qubit (e.g., phase flip channel).
- The qubit gets a phase with probability p

$$\begin{aligned}
\rho &= (1-p)|0\rangle\langle 0| \otimes I + p|1\rangle\langle 1| \otimes Z \\
\rho &= (1-p)|0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) + p|1\rangle\langle 1| \otimes (|0\rangle\langle 0| - |1\rangle\langle 1|) \\
\rho_{AB} &= (1-p)[|00\rangle\langle 00| + |01\rangle\langle 01|] + p[|10\rangle\langle 10| - |11\rangle\langle 11|] \\
\rho_B &= \mathrm{Tr}_A(\rho_{AB}) \\
\rho_B &= (1-p)(|0\rangle\langle 0| + |1\rangle\langle 1|) + p(|0\rangle\langle 0| - |1\rangle\langle 1|) \\
\rho_B &= (1-p)I + pZ \quad \text{(Phase flip Channel)}
\end{aligned}$$

Depolarizing Channel

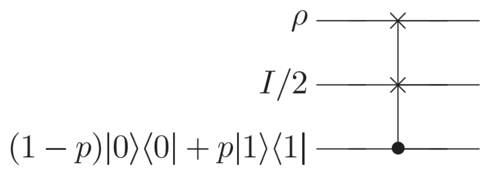

Figure 3: Depolarizing Channel

This process demonstrates how to analyze a depolarizing channel using density matrix formalism. The steps include preparing the initial state, applying the depolarizing operation, and performing partial traces to observe the effects of the depolarization channel on a single qubit.

- The circuit starts with qubits in an initial mixed state.
- Apply a depolarizing channel operation to the qubits.
- Describe the new state using density matrix formalism.
- Perform partial trace over the environment(ancilla qubits).
- The reduced density matrix shows the effect of the depolarizing channel on the target qubit as a single quantum channel.

$$
\begin{aligned}
\rho &= \frac{1}{2}[I + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z] \\
\Rightarrow \frac{I}{2} &= \frac{1}{4}[I + X\rho X + Y\rho Y + Z\rho Z] \\
S &= |00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11| \\
\rho_{ABC} &= (1-p)|0\rangle\langle 0| \otimes (I_B \otimes \rho) + \frac{p}{2}|1\rangle\langle 1| \otimes [S(I_B \otimes \rho)S] \\
\rho_{ABC} &= (1-p)|0\rangle\langle 0| \otimes (I_B \otimes \rho) + \frac{p}{2}|1\rangle\langle 1| \otimes [(\rho_B \otimes I_A)] \\
\rho_{AB} &= \mathrm{Tr}_C(\rho_{ABC}) \\
\rho_{AB} &= (1-p)(I_B \otimes \rho) + \frac{p}{2}(\rho_B \otimes I_A) \\
\rho_{out} &= \mathrm{Tr}_B(\rho_{AB}) \\
\rho_{out} &= (1-p)\rho + \frac{p}{2}(I_A) \qquad \mathrm{Tr}(\rho) = 1 \\
\rho_{out} &= (1-p)\rho + \frac{p}{2}(I_A)
\end{aligned}
$$

Amplitude Damping

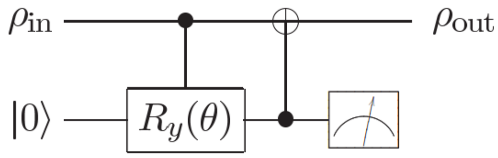

Figure 4: Amplitude Damping Channel

The amplitude damping channel models energy loss in a quantum system, essential for understanding quantum information degradation in computing and communication.

This circuit applies an amplitude damping effect on a qubit using the $R_y(\theta)$ gate, followed by environmental interaction. The process involves:

1. **Gate Action:** $R_y(\theta)$ rotates the qubit by θ, changing its state.
2. **Overall Action:** The transformation $U\rho U^\dagger$ modifies the state by applying unitary operations and tracing out environmental effects.
3. **Output State:** ρ_{out} reflects the qubit's state post-interaction, capturing decay effects.

$$
\begin{aligned}
R_y(\theta) &= \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix} \\
\rho_{out} &= |0\rangle\langle 0| \\
\rho_{initial} &= \rho_{in} \otimes \rho_0 \\
\rho_{rotate} &= (|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes R_y(\theta))\rho_{initial}(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes R_y^\dagger(\theta)) \\
\rho_{final} &= (I \otimes |0\rangle\langle 0| + X \otimes |1\rangle\langle 1|)\rho_{rotate}(I \otimes |0\rangle\langle 0| + X \otimes |1\rangle\langle 1|) \\
&= (I \otimes |0\rangle\langle 0| + X \otimes |1\rangle\langle 1|)(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1|R_y(\theta)) \\
&\quad \times \rho_{initial}(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1|R_y^\dagger(\theta))(I \otimes |0\rangle\langle 0| + X \otimes |1\rangle\langle 1|) \\
&= (|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|R_y(\theta)) + (|1\rangle\langle 0| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 1|R_y(\theta))\, \rho_{in} \otimes |0\rangle\langle 0| \\
&\quad \times (|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |1\rangle\langle 1| \; + |1\rangle\langle 1| \otimes R_y^\dagger(\theta)|0\rangle\langle 0| + |1\rangle\langle 0| \otimes R_y^\dagger(\theta)|1\rangle\langle 1|\Big) \\
&= (|0\rangle\langle 0| \otimes |0\rangle + |1\rangle\langle 1| \otimes |0\rangle\langle 0|R_y(\theta)|0\rangle) + (0 + |0\rangle\langle 1| \otimes |1\rangle\langle 1|R_y(\theta)|0\rangle)\, \rho_{in} \\
&\quad \times \Big(|0\rangle\langle 0| \otimes \langle 0| + 0 + |1\rangle\langle 1| \otimes \langle 0|R_y^\dagger(\theta)|0\rangle\langle 0| + |1\rangle\langle 0| \otimes \langle 0|R_y^\dagger(\theta)|1\rangle\langle 1|\Big) \\
&= (|0\rangle\langle 0| \otimes |0\rangle + |1\rangle\langle 1| \otimes \cos(\theta)|0\rangle) + (|0\rangle\langle 1| \otimes \sin(\theta)|1\rangle)\, \rho_{in} \\
&\quad \times (|0\rangle\langle 0| \otimes \langle 0| + |1\rangle\langle 1| \otimes \cos(\theta)\langle 0| + |1\rangle\langle 0| \otimes \sin(\theta)\langle 1|) \\
\rho_A &= \mathrm{Tr}_B(\rho_{AB}) \\
\rho_{out} &= (|0\rangle\langle 0| + \cos(\theta)|1\rangle\langle 1|)\, \rho_{in}\, (|0\rangle\langle 0| + \cos(\theta)|1\rangle\langle 1|) + (\sin(\theta)|0\rangle\langle 1|)\, \rho_{in}\, (\sin(\theta)|0\rangle\langle 1|)
\end{aligned}
$$

Kraus Operators for Amplitude Damping

The Kraus operators $\{E_0, E_1\}$ for this channel are detailed as follows:

- E_0: Preserves the state if no decay occurs.

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \cos(\theta/2) \end{bmatrix}$$

- E_1: Represents decay from the excited state to the ground state.

$$E_1 = \begin{bmatrix} 0 & \sin(\theta/2) \\ 0 & 0 \end{bmatrix}$$

Single Qubit error correction Codes

In most cases where a quantum circuit is applied to qubits, we see that decoherence is introduced in the system due to environmental noise. This noise could include anything from phase flip, bit flip or a combination of the two errors. Assuming that the error effects a single qubit only and quantum gates work perfectly, we can regain the original state of the qubit using the error correction codes for bit flip, phase flip and the generalized shor's code.

In the previous section, we have seen how if there is a quantum circuit involving multiple qubits, we can see the overall action on a single qubit by tracing out the rest of the ancilla qubits. For this section, we will be finding the quantum channels corresponding to each of the error correction codes and then we will introduce errors to more than 1 qubit to see how the fidelity of our state changes as the probability 'p' of the error increases.

For each of the error correction codes, the quantum channel was calculated using python. The library used for drawing the circuits was qutip while the symbolic calculations were done using sympy. Wherever any numerical calculations were required, numpy and qutip were used in conjunction to ensure the code could execute as quickly as possible. The input state was assumed to be the generalized density matrix in all cases while the ancilla qubits are the state $|0\rangle$:

$$\rho_{in} = \rho \otimes |00\rangle\langle 00|, \quad \rho = \frac{1}{2}\left(I + \vec{r}\cdot\vec{\sigma}\right).$$

. The individual vector components of $\vec{r}$ are written as $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ in the rest of the report. Each of the ancillary qubits then was initialized in state $|0\rangle$ and encoded as required by the different error correction codes respectively. All of the codes have been uploaded on GitHub.

Bit flip Code

The bit flip code assumes that an error affects the qubits such that an X gate is applied on the original qubit. The qubit's phase remains unchanged.

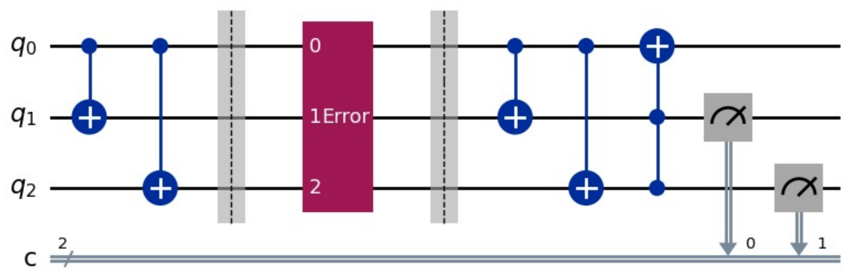

Figure 5: Bit Flip code

Single Qubit error

The error was simulated by the "Error" gate and was introduced as a Kraus operator where the probability of the error 'p' while the probability of no error was 1-p. The operators were:

$$K_0 = \sqrt{p}\,(X \otimes I \otimes I), \quad K_1 = \sqrt{1-p}\,(I \otimes I \otimes I)$$

After applying the circuit on the initial state and tracing out the ancilla qubits in the end, as one would expect, we got back the initial state ρ_{in} which means that the fidelity between the initial and final states was 1. This result was obtained for both cases of symbolic and numerical calculations.

Multiple Qubit error

We want to see how a quantum circuit can help increase coherence of a state. Noise increases this decoherence while the error correction codes decrease the decoherence. So, we try to see how the fidelity changes as a function of probability 'p' when the error is introduced on more than 1 qubit. The Kraus operators in this case are:

$$K_0 = \sqrt{p}\,(X \otimes X \otimes X), \quad K_1 = \sqrt{1-p}\,(I \otimes I \otimes I)$$

Applying the initial state into the bit flip correction code with U being defined as the Kraus operators above, we get the following ρ_{out}

$$\rho_{out} = \begin{bmatrix} -p\mathbf{z} + \frac{\mathbf{z}}{2} + \frac{1}{2} & ip\mathbf{y} + \frac{\mathbf{x}}{2} - \frac{i\mathbf{y}}{2} \\ -ip\mathbf{y} + \frac{\mathbf{x}}{2} + \frac{i\mathbf{y}}{2} & p\mathbf{z} - \frac{\mathbf{z}}{2} + \frac{1}{2} \end{bmatrix}$$

Calculating the fidelity between ρ_{out} and ρ_{in} became very complicated symobolically so instead, we chose to substitute for different values of p, x, y and z. A step of 0.1 was chosen where the value of p ranged from 0 to 1 and for each of $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ the values ranged from -1 to 1. Of course the restriction of the magnitude of these 3 components was set to be at most 1 such that the final state remains within the bloch sphere. The fidelity was calculated using qutip's built-in function after the substitution of these values. The average fidelity as a function of p can be seen in the following graph:

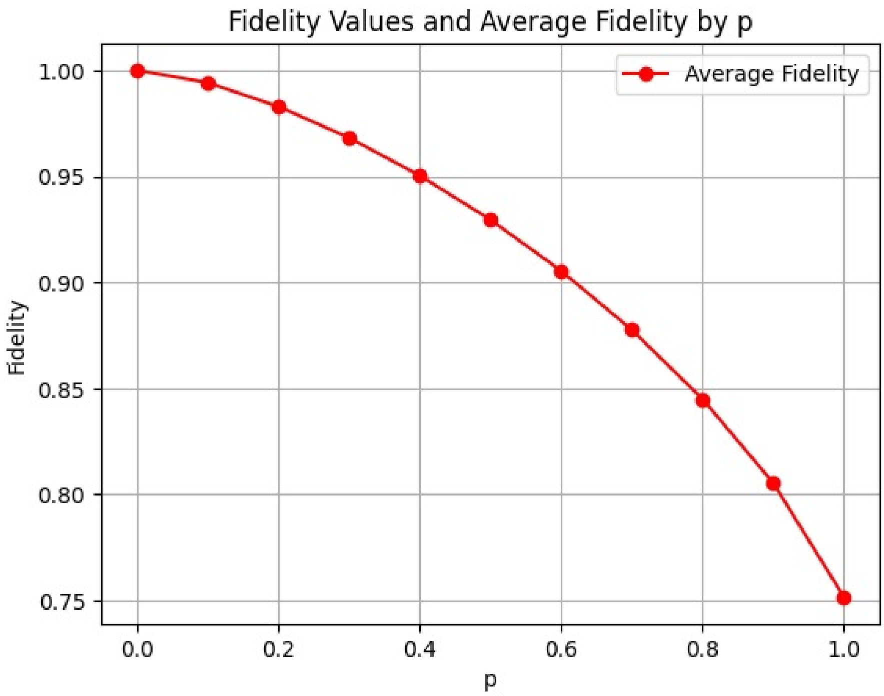

Figure 6: Fidelity Bit Flip

This needs to be pointed out that although we are introducing error to all the qubits, if instead we had introduced an error to just 2 of the 3 qubits, the final state $|\rho_{out}\rangle$ was found to be the same as for the circuit with error on all 3 qubits with equal probability and therefore the fidelity against p would also be same. This is something which would need further investigation as to why this is true.

Phase Flip code

The bit flip code assumes that an error affects the qubits such that an X gate is applied on the original qubit. The qubit's phase remains unchanged.

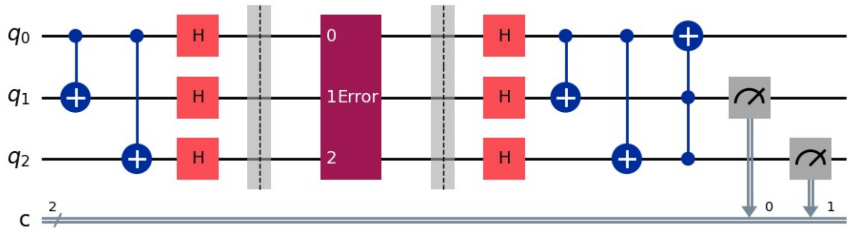

Figure 7: Phase Flip code

Single Qubit error

The phase flip error was simulated by the "Error" gate and was introduced as a Kraus operator where the probability of the error 'p' while the probability of no error was 1-p. The operators were:

$$K_0 = \sqrt{p}\,(Z \otimes I \otimes I), \quad K_1 = \sqrt{1-p}\,(I \otimes I \otimes I)$$

The result again was similar to what we would expect for a single qubit error after going through error correction code. The state ρ_{out} was similar to rho_{in} and therefore, the fidelity between the states is 1 for all possible initial states.

Multiple Qubit error

We want to see how a quantum circuit can help increase coherence of a state. Noise increases this decoherence while the error correction codes increase coherence. So, we try to see how the fidelity changes as a function of probability 'p' when the error is introduced on more than 1 qubit. The Kraus operators in this case are:

$$K_0 = \sqrt{p}\,(Z \otimes Z \otimes Z), \quad K_1 = \sqrt{1-p}\,(I \otimes I \otimes I)$$

Applying the initial state into the bit flip correction code with U being defined as the Kraus operators above, we get the following ρ_{out}

$$\rho_{out} = \begin{bmatrix} -p\mathbf{z} + \frac{\mathbf{z}}{2} + \frac{1}{2} & ip\mathbf{y} + \frac{\mathbf{x}}{2} - \frac{i\mathbf{y}}{2} \\ -ip\mathbf{y} + \frac{\mathbf{x}}{2} + \frac{i\mathbf{y}}{2} & p\mathbf{z} - \frac{\mathbf{z}}{2} + \frac{1}{2} \end{bmatrix}$$

Calculating the fidelity between ρ_{out} and ρ_{in} became very complicated symbolically so instead, we chose to substitute for different values of p, x, y and z. A step of 0.04 with similar restrictions on the value of p and the bloch vector components. The average fidelity as a function of p can be seen in the following graph:

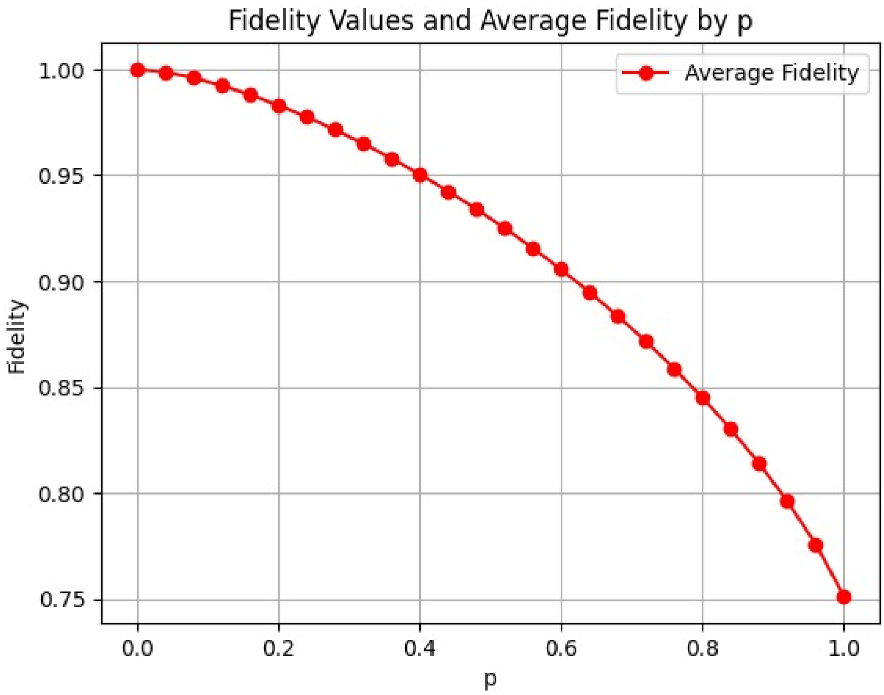

Figure 8: Fidelity Phase Flip

The result is very similar to the result for a bit flip, with the final state being independent of the number of qubits to which the error was introduced.

Shor's Code

The bit flip code assumes that an error affects the qubits such that an X gate is applied on the original qubit. The qubit's phase remains unchanged.

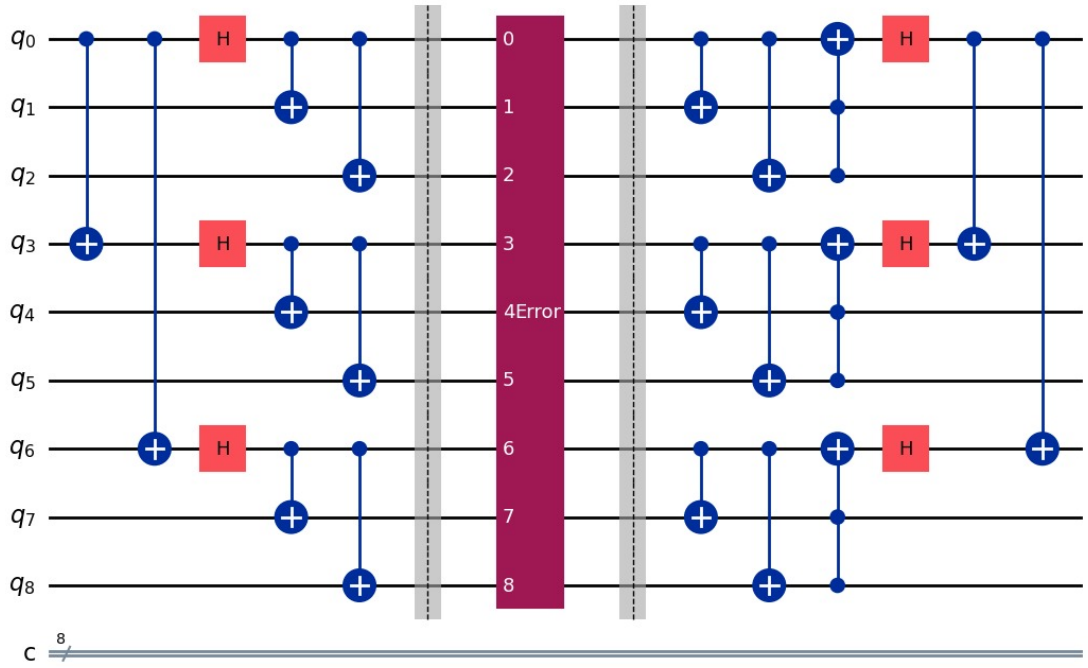

Figure 9: Shor's code

Single Qubit error

We can introduce any arbitrary phase or bit flip error to a single qubit and the shor code should be able to correct it. The Kraus operators were defined as:

$$K_0 = \sqrt{p}\,(XZ \otimes I^{\otimes 8}\ )$$

$$K_1 = \sqrt{1-p}\,(I^{\otimes 9})$$

Just like the bit flip and phase flip errors, any arbitrary as long as it is acted upon a single qubit can be corrected by the Shor's code. This error is not restricted to just a phase or a bit flip. Any arbitrary error on a single qubit can be corrected by shor's code. Therefore, the output state was same as the input state and we would thus always expect the fidelity between the states to be 1 regardless of the value of p or the initial state.

Multiple Qubit error

We want to see how a quantum circuit can help increase coherence of a state. Noise increases this decoherence while the error correction codes decrease the decoherence. So, we try to see how the fidelity changes as a function of probability 'p' when the error is introduced on more than 1 qubit. The Kraus operators in this case are:

$$K_0 = \sqrt{p}\,(XZ)^{\otimes 9} \qquad K_1 = \sqrt{1-p}\,(I^{\otimes 9})$$

Applying the initial state into the bit flip correction code with U being defined as the Kraus operators above, we get the following ρ_{out}

$$\rho_{out} = \begin{bmatrix} -p\mathbf{z} + \frac{\mathbf{z}}{2} + \frac{1}{2} & -p\mathbf{x} + \frac{\mathbf{x}}{2} - \frac{i\mathbf{y}}{2} \\ -p\mathbf{x} + \frac{\mathbf{x}}{2} + \frac{i\mathbf{y}}{2} & p\mathbf{z} - \frac{\mathbf{z}}{2} + \frac{1}{2} \end{bmatrix}$$

Similar problems arose when calculating fidelity as with bit flip and phase flip being introduced to more than 1 qubit so fidelity was calculated numerically by substituting the values of variables the same way as described for the previous code with the step being 0.1 this time. The function of fidelity against probability is shown in the following graph:

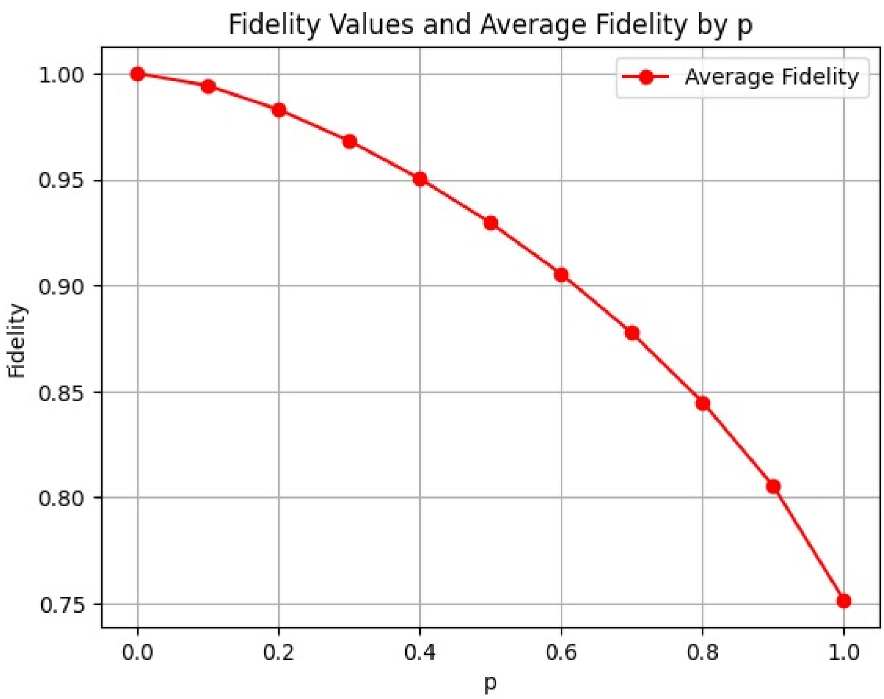

Figure 10: Fidelity Shor Code

The results were again very similar to bit flip and phase flip codes as shown in the graphs. A lower step size improved the accuracy of the plot but the trade off with time required to calculate this fidelity was very large. For example, with a step size of 0.1, all data points for value of fidelity took over 500 minutes to calculate. Just decreasing this step size to 0.05 would increase the time required for calculation 16 fold. Regardless, this step size was small enough to depict the relation between p and fidelity with the result being very similar to bit and phase flip codes.

It is interesting to note that all the error correction codes show a similar relation between p and fidelity which converges at fidelity = 0.75 as p approaches 1.

Pure State Fidelity after Shor's code

Instead of using the generalized density matrix, we instead choose a pure state and try to compute its average fidelity. The input state chosen was:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

while the ancilla qubits were again set to the state $|0\rangle$. We only compute the error for the Shor code as it can rectify any error on a **single** qubit. The corresponding density matrix for the input state was:

$$\rho_{in} = \rho \otimes I^{\otimes 8}, \qquad \rho = \begin{bmatrix} \cos^2\left(\frac{\theta}{2}\right) & \frac{e^{-i\phi}\sin(\theta)}{2} \\ \frac{e^{i\phi}\sin(\theta)}{2} & \sin^2\left(\frac{\theta}{2}\right) \end{bmatrix}$$

Single qubit error:

Just as in the case for single qubit error on the generalized state, the single qubit error was perfectly rectified by the simulated shor's code where the input state was pure. The error was introduced as a Kraus operator defined by:

$$K_0 = \sqrt{p}\,(XZ \otimes I^{\otimes 8}\ ) \quad K_1 = \sqrt{1-p}\,(I^{\otimes 9})$$

The density matrix after tracing out the ancilla qubits was found to be:

$$\rho_{out} = \begin{bmatrix} \cos^2\left(\frac{\theta}{2}\right) & \frac{e^{-i\phi}\sin(\theta)}{2} \\ \frac{e^{i\phi}\sin(\theta)}{2} & \sin^2\left(\frac{\theta}{2}\right) \end{bmatrix}$$

This matrix is equivalent to the matrix ρ and therefore, the fidelity calculated between ρ and ρ_{out} was 1. This result agrees with the fidelity found for a single qubit error when the input state was the generalized density matrix.

Multiple qubit error:

The setup for this simulation was same as for the single qubit error. However, the final density matrix that we got after the simulation was dependent on the error we introduced in the Kraus operators. A set of Kraus operators used was:

$$K_0 = \sqrt{p}\,((XZ)^{\otimes 4} \otimes I^{\otimes 5}) \quad K_1 = \sqrt{1-p}\,(I^{\otimes 9})$$

The evolved state that we got at the end was:

$$\rho_{out} = \begin{bmatrix} -2p\cos^2\left(\frac{\theta}{2}\right) + p + \cos^2\left(\frac{\theta}{2}\right) & \frac{\left(-pe^{i\phi} - pe^{-i\phi} + e^{-i\phi}\right)\sin(\theta)}{2} \\ \frac{\left(-pe^{-i\phi} - pe^{i\phi} + e^{i\phi}\right)\sin(\theta)}{2} & -2p\sin^2\left(\frac{\theta}{2}\right) + p + \sin^2\left(\frac{\theta}{2}\right) \end{bmatrix}$$

Clearly this matrix could have real and complex arguments depending on the values of the arguments p, θ and ϕ. The next section will be a short proof that any output state we get, it is guaranteed to be a positive matrix:

Positive Semidefinite Definition

A matrix ρ is positive semidefinite if for all vectors $|\alpha\rangle$,

$$\langle\alpha|\,\rho\,|\alpha\rangle \geq 0.$$

Our matrix ρ was defined as the outer product of a pure state $|\psi\rangle$ so, it is guaranteed to be a positive matrix.

Unitary Transformation

Given a unitary matrix U, we define the transformed matrix ρ' as:

$$\rho' = U\rho U^{\dagger}.$$

We need to show that ρ' is positive semidefinite.

Proof

Consider any vector $|\beta\rangle$. Let $|\alpha\rangle = U^\dagger |\beta\rangle$. Since U is unitary, the transformation preserves the norm of the vector, i.e.,

$$\|\alpha\| = \|\beta\|.$$

We need to show that

$$\langle\beta| \rho' |\beta\rangle \geq 0.$$

By substituting ρ' and using the unitary properties:

$$\langle\beta| \rho' |\beta\rangle = \langle\beta| (U\rho U^\dagger) |\beta\rangle .$$

Using the substitution $|\alpha\rangle = U^\dagger |\beta\rangle$:

$$\langle\beta| (U\rho U^\dagger) |\beta\rangle = \langle\beta| U\rho U^\dagger |\beta\rangle = \langle\alpha| \rho |\alpha\rangle .$$

Since ρ is positive semidefinite, we have:

$$\langle\alpha| \rho |\alpha\rangle \geq 0.$$

Therefore,

$$\langle\beta| \rho' |\beta\rangle = \langle\alpha| \rho |\alpha\rangle \geq 0.$$

This shows that $\rho' = U\rho U^\dagger$ is positive semidefinite.

Fidelity of Pure state for multiple errors

THe fidelity between 2 pure states ϕ and ψ is defined as:

$$F = \langle\phi| \rho_\psi |\phi\rangle , \quad \rho_\psi = |\psi\rangle \langle\psi|$$

For our input and output states, the fidelity was found to be:

$$F = \langle\psi| \rho_{out} |\psi\rangle$$

$$F(\theta, \phi, p) = \left[-\frac{pe^{2i\phi}\sin^2(\theta)}{4} + \frac{p\sin^2(\theta)}{2} - p - \frac{pe^{-2i\phi}\sin^2(\theta)}{4} + 1\right]$$

We were then able to evaluate the average fidelity by integrating over the surface of the bloch sphere and for p = 0 to 1 to solve for all values of p. We dont integrate over r as all pure states are on the surface of the bloch sphere. Additonally, a normalization factor of $\frac{1}{4\pi}$ was multiplied with integral:

$$\mathrm{I} = \frac{1}{4\pi}\int_0^1 \int_0^{2\pi} \int_0^{\pi} F(\theta, \phi, p)\sin(\theta)\, d\theta\, d\phi\, dp$$

Solving the integral with sympy gave us the result $\langle \mathrm{F} \rangle = \frac{2}{3}$ for this error.